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# Non-trivial, non-negative periodic solutions of a system of singular-degenerate parabolic equations with nonlocal terms

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## Abstract

We study the existence of non-trivial, non-negative periodic solutions for systems of singular-degenerate parabolic equations with nonlocal terms and satisfying Dirichlet boundary conditions. The method employed in this paper is based on the Leray-Schauder topological degree theory. However, verifying the conditions under which such a theory applies is more involved due to the presence of the singularity. The system can be regarded as a possible model of the interactions of two biological species sharing the same isolated territory, and our results give conditions that ensure the coexistence of the two species.

Keywords: singular-degenerate parabolic equations, periodic solutions, a priori bounds, topological degree theory.

2000AMS Subject Classification: 35K65, 35B10, 47H11

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# 1 Introduction

In this paper we consider a periodic system of singular-degenerate parabolic equations with delayed nonlocal terms and Dirichlet boundary conditions of the form

$$\begin{cases} u_t - \operatorname{div}(|\nabla u^m|^{p-2} \nabla u^m) = \left( a(x, t) - \int_{\Omega} K_1(\xi, t) u^2(\xi, t - \tau_1) d\xi + \int_{\Omega} K_2(\xi, t) v^2(\xi, t - \tau_2) d\xi \right) u^{p-1}, & \text{in } Q_T \\ v_t - \operatorname{div}(|\nabla v^n|^{q-2} \nabla v^n) = \left( b(x, t) + \int_{\Omega} K_3(\xi, t) u^2(\xi, t - \tau_3) d\xi - \int_{\Omega} K_4(\xi, t) v^2(\xi, t - \tau_4) d\xi \right) v^{q-1}, & \text{in } Q_T \\ u(x, t) = v(x, t) = 0, & \text{for } (x, t) \in \partial\Omega \times (0, T), \\ u(\cdot, 0) = u(\cdot, T) \text{ and } v(\cdot, 0) = v(\cdot, T), \end{cases} \quad (1.1)$$

and we look for continuous weak solutions. Here  $\Omega$  is an open bounded domain of  $\mathbb{R}^N$  with smooth boundary  $\partial\Omega$ , satisfying the property of positive geometric density, see [39],  $Q_T := \Omega \times (0, T)$ ,  $T > 0$ ,  $\tau_i \in (0, +\infty)$ , the functions  $K_i, a$ , and  $b$  belong to  $L^\infty(Q_T)$ . The exponents  $p$  and  $q$  belong to the interval  $(1, 2)$ ,  $m > p$ ,  $n > q$  and  $s^m = |s|^{m-1}s$ . Setting  $Au := \operatorname{div}(|\nabla u^m|^{p-2} \nabla u^m)$  and  $l := (m-1)(p-1)$ , the operator  $Au$  becomes  $m^{p-1} \operatorname{div}(|u|^l |\nabla u|^{p-2} \nabla u)$ , which is the operator considered by Ivanov in [32], [33] and [34]. According to the classification proposed in these papers, we say that the first equation in (1.1) is of

1. *slow diffusion* type if  $m > \frac{1}{p-1}$ ,
2. *normal diffusion* type if  $m = \frac{1}{p-1}$ ,
3. *fast diffusion* type if  $m < \frac{1}{p-1}$ .

Of course, analogous definition in terms of  $n$  and  $q$  can be given for the second equation in (1.1).

The aim of this paper is to extend the results of [25] and [26], concerning the existence of non-negative, non-trivial periodic solutions, to a system of singular-degenerate parabolic equations. To the best of our knowledge, this is the first result for the case when  $1 < p, q < 2$ ,  $m > p$  and  $n > q$ , also in the case of a single equation. We recall that the cases  $p, q > 2$ ,  $m, n > 1$  and  $p = q = 2$ ,  $m, n > 1$  were treated respectively in [25] and [26], see also [21] and [22] for a system of anisotropic  $(p(x), q(x))$ -Laplacian parabolic equations, with  $p(x), q(x) > 2$  in  $\bar{\Omega}$ , and  $m = n = 1$ . In the very recent paper [60], the authors replace the nonlocal terms of (1.1) by  $\int_{\Omega} K_i(\xi, t) u(\xi, t) d\xi$ , for  $i = 1, 3$ , and  $\int_{\Omega} K_i(\xi, t) v(\xi, t) d\xi$ , for  $i = 2, 4$ . By means of local conditions, different from those proposed in [25, 26], the authors obtain the coexistence of the two species via a similar topological approach when  $p, q \geq 2$ ,  $m, n \geq 1$  and, thus, only the slow and normal diffusion occurs, i.e.  $m(p-1) \geq 1, n(q-1) \geq 1$ . More precisely, models for the interaction between two biological species sharing the same isolated territory, with the interactions represented by means of the kernels  $K_i, i = 1, 2, 3, 4$ , were considered in related systems of doubly degenerate parabolic equations in [25], [60] and in systems of porous medium equations in [26]. On the other hand, some previous biological models found in the literature, see e.g. [1], [2], [51], [52] involve the  $p$ -Laplacian with  $p > 1$  (and  $m = 1$ ). Furthermore, we observe that the equations of the system we consider treat all the possible types of diffusion: slow, normal and fast, while in [25], [60] and [26] only the slow and normal diffusions were presented. In fact, as it can be easily checked, if  $p \in \left(1, \frac{1+\sqrt{5}}{2}\right)$  one can have all the three types of diffusion, while if  $p \in \left[\frac{1+\sqrt{5}}{2}, 2\right)$  only the slow diffusion is possible under the condition  $m > p$ .

In the case of the fast diffusion and superlinear growth in  $u, v$  of the right hand sides, the solutions may blow up or vanish in some finite time depending on the initial conditions as illustrated in [11], [40], [41], [46] and the references therein for the simple equation obtained from the first of system (1.1) by letting  $p = 2$ ,  $a > 0$  constant and all the kernels  $K_i \equiv 0$  (observe that in the case when  $p = q = 2$  no restriction on  $m, n$  are required, see Remark 2.2). If  $\Omega = \mathbb{R}^N$  for such equation we have that the solution blows up for any initial condition in the case when the superlinear growth in  $u$  is less than a certain critical exponent, see [46], and the same occurs for doubly degenerate parabolic equations, see [47]. If

the growth is linear or sublinear we do not have blow up of any solution, see [41], hence solutions exist for all  $t \geq 0$  and in the linear case, depending on the initial condition, they may vanish in finite time or become unbounded as  $t \rightarrow +\infty$  and, thus, the considered initial conditions cannot give rise to a periodic solution.

The choice of the sublinear exponents  $p - 1$  and  $q - 1$ , respectively, for  $u$  and  $v$  in (1.1) is mainly technical since it depends on the topological method employed in the paper, which is based on a priori bounds of the solutions. Indeed, this choice enable us to establish the required a priori bounds on the solutions of the approximating problem (2.1) in a uniform way with respect to the perturbation parameter  $\epsilon > 0$ . We remark that, if the diffusion is slow and  $\Omega \subset \mathbb{R}^N$  is a bounded and open domain, then we can allow a superlinear growth in  $u, v$  in order to have both global existence solutions and periodic solutions, together with their  $L^\infty$ -estimates, see [8], [13], [14], [45], [48], [49], [57], [60], [61], [62], [63] and the references therein. Due to the singularity of the  $p, q$ -Laplacian, the way of proving the a priori bounds deeply differs from that employed in [25] and [26]. Moreover, in order to pass from the  $L^2$ - to the  $L^\infty$ -estimates, in Lemma 2.2 we have readapted Moser's technique to the case when  $1 < p, q < 2$ . Moreover we have to impose the technical restriction  $m > p$  and  $n > q$  in order to get the gradient estimates in Lemma 2.4.

Due to their importance in different physical and other natural sciences such as non-Newtonian fluid mechanics, flow in porous medium, nonlinear elasticity, glaciology, population biology etc., degenerate and singular parabolic equations have been the subject of extensive research in the last 25 years, with particular emphasis on the study of regularity for non-negative weak solutions. We mention here, among many others, the papers [32], [33], [34], [53] and the monographs [17], [54]. In particular, we refer to the very recent monograph [18] for a comprehensive treatment of the Harnack inequality for non-negative solutions to  $p$ -Laplacian and porous medium equations. Moreover, [18] provides an historical presentation of the achievements in this research field and many references to the applications concerning the topics mentioned above.

The regularity results for the singular  $p$ -Laplacian are crucial for the application of the topological degree approach used in this paper. Similar topological methods are also employed to a great extent for the existence of non-negative periodic solutions of degenerate and doubly degenerate parabolic equations, see [3], [9], [20], [30], [31], [38], [42], [44], [45], [48], [55], [56], [58], [59], [60], [61], [62], [63], [64], [67], [68]. Nonlocal models to study aggregation in biological systems with degenerate diffusion are proposed in several papers, see the recent [12], [43] and the references therein.

Moreover, we recall that the interest in studying the existence of periodic solutions for degenerate and non-degenerate parabolic equations modelling biological and physical phenomena, relies in the consideration that the periodic behavior of certain biological and physical non-negative quantities is the most natural and desirable one, see e.g. [3], [5], [6], [7], [21], [22], [25], [26], [29], [31], [35], [42], [50], [56], [57], [60], [67], [68]. We also recall the related problems faced in [23] and [24] also for higher order operators, and in [19] for  $p = 2$  and  $N = 1$ .

The paper is organized as follows. The goal of Section 2 is the proof of a coexistence result based on the explicit knowledge of suitable a priori bounds on the  $L^2$ -norms of the solutions. The search for such bounds is carried on in Section 3. The reason to split the argument in this way lies in the fact that our main coexistence conditions, namely Hypotheses 2.2, are applicable regardless of any other assumption on the terms of the equations. On the other hand, a priori bounds for the periodic solutions are more easily obtained when we focus on specific situations like the competitive (i.e.  $K_2, K_3 \leq 0$ ) and cooperative (i.e.  $K_2, K_3 \geq 0$ ) cases and those in which  $K_1, K_4$  are bounded away from zero or not and other restrictions on the exponents of the left hand sides are imposed.

More precisely, in order to deal with the singular-degenerate system (1.1), in Section 2 we introduce an approximating system (2.1) of nondegenerate-singular equations depending on a small parameter  $\epsilon > 0$ . Such equations satisfy structure conditions which, for any  $\epsilon > 0$ , allow the use of well-known regularity results, i.e. Hölder continuity, from e.g. [33], [34]; we will use this regularity to show that the map which associates to any couple of functions  $(f, g) \in L^\infty(Q_T) \times L^\infty(Q_T)$  the solution of the regularized system is a compact map from  $L^\infty(Q_T) \times L^\infty(Q_T) \rightarrow L^\infty(Q_T) \times L^\infty(Q_T)$ , see Lemma 2.1. Then, for  $\epsilon > 0$ , the problem of showing the existence of a non-negative solution  $(u_\epsilon, v_\epsilon)$  to (2.1) is equivalent to showing the

existence of a non-negative fixed point of such a solution map. The way we do this in Proposition 2.2 is based on the classical tools of the Leray-Schauder topological degree: first, we establish uniform (with respect to  $\varepsilon > 0$ ) a priori bounds, in this specific case in  $L^\infty(Q_T) \times L^\infty(Q_T)$ , for all possible non-negative solutions of (2.1). Then, by the homotopy invariance of the topological degree, Proposition 2.2 guarantees the existence of a solution  $(u_\varepsilon, v_\varepsilon)$  of (2.1) in a large ball  $B_R \subset L^\infty(Q_T) \times L^\infty(Q_T)$ . Moreover, by means of suitable conditions on the first positive eigenvalue of the  $p$ -Laplacian and on some estimates on the gradient of convenient powers of  $u_\varepsilon$  and  $v_\varepsilon$  established in Lemma 2.4, we are able to prove that  $\|u_\varepsilon\|_{L^\infty(Q_T)}$  and  $\|v_\varepsilon\|_{L^\infty(Q_T)}$  are bounded away from zero uniformly for  $\varepsilon > 0$  small enough, see Proposition 2.3. To conclude, by using the uniform bounds of  $(u_\varepsilon, v_\varepsilon)$  in  $L^\infty(Q_T) \times L^\infty(Q_T)$  and the consequent uniform Hölder continuity of  $(u_\varepsilon, v_\varepsilon)$  in  $\overline{Q_T}$ , we can pass to the limit as  $\varepsilon \rightarrow 0$  and show in Theorem 2.1 that  $(u_\varepsilon, v_\varepsilon)$ , by passing to a subsequence if necessary, converges to a solution  $(u, v)$  of (1.1) with  $u \neq 0$  and  $v \neq 0$ .

In Section 3, we give conditions on the kernels  $K_i, i = 1, 2, 3, 4$ , of the nonlocal terms that suffice for the existence of uniform a priori bounds in  $L^2(Q_T) \times L^2(Q_T)$  for the solutions  $(u_\varepsilon, v_\varepsilon)$  of (2.1). By Lemma 2.2 these a priori bounds imply uniform a priori bounds of  $(u_\varepsilon, v_\varepsilon)$  in  $L^\infty(Q_T) \times L^\infty(Q_T)$  and so, from now on, we can proceed as outlined in Section 2 in order to apply Theorem 2.1. In terms of the biological interpretations, system (1.1) is a model of the interactions of two biological species, with density  $u$  and  $v$  respectively, disliking crowding, i.e.  $m, n > 1$ , see [27], [28] and [50], and whose diffusion involves, as in [1], [2], [51] and [52], the singular  $p$ -Laplacian, i.e.  $1 < p < 2$ . The nonlocal terms  $\int_\Omega K_i(\xi, t)u^2(\xi, t - \tau_i)d\xi$  and  $\int_\Omega K_i(\xi, t)v^2(\xi, t - \tau_i)d\xi$  evaluate a weighted fraction of individuals that actually interact at time  $t > 0$ . Nonlocal terms in biological models were first introduced in [15] and [16]. The delayed densities  $u, v$  at time  $t - \tau_i$ , that appear in the nonlocal terms, take into account the time needed to an individual to become adult, and, thus to interact and to compete. The conditions on  $K_i, i = 1, 2, 3, 4$ , have the meaning of competitive systems if  $K_i \leq 0, i = 2, 3$ , or of cooperative systems if  $K_i \geq 0, i = 2, 3$ ; on the other hand, we always assume that  $K_i \geq 0, i = 1, 4$ , to take into account the intra species competition. The term on the right hand side of each equation in (1.1) denotes the actual increasing rate of the population at  $(x, t) \in Q_T$ . Related results are presented in the coexistence Theorem 3.1, which considers the coercive case, i.e.  $K_i \geq \underline{k}_i > 0, i = 1, 4$ , and its consequences: Corollary 3.1 and 3.2 for the coercive-cooperative and coercive-competitive cases respectively. In the non-coercive case we prove Theorem 3.2 for competitive systems when the diffusion is slow or normal for both the equations, and Theorems 3.3 and 3.4 under a stronger assumption on  $m, p, n, q$ , but without any conditions on the sign of  $K_2$  and  $K_3$ . Observe that these results concerning the slow and normal diffusion are relevant for the considered biological model, in fact the slow and normal diffusion are more realistic for the biological models as pointed out in [28], [33], [50], [52], [63]. Finally, in Section 4, for a generalization of system (1.1) which consists in having any power  $\alpha \geq 1$  of  $u$  and  $v$  in the nonlocal terms, we obtain, only in the competitive case, the coexistence Theorem 4.1 and the related Theorem 4.2 for the coercive case and Theorem 4.3 for the non-coercive case. Note that such a generalization of system (1.1) is a completely new contribution with respect to [25] and [26].

## 2 The approximating problem

Throughout the paper we will make the following assumptions:

**Hypotheses 2.1.** 1. The exponents  $p, q, m, n$  are such that  $p, q \in (1, 2)$ ,  $m > p$  and  $n > q$ .

2. The delays  $\tau_i \in (0, +\infty)$ ,  $i = 1, 2, 3, 4$ .

3. The functions  $a, b$  and  $K_i, i = 1, 2, 3, 4$ , belong to  $L^\infty(Q_T)$  and are extended to  $\Omega \times \mathbb{R}$  by  $T$ -periodicity. Moreover,  $a, b$  and  $K_i, i = 1, 4$ , are non-negative functions and there are constants  $\underline{k}_i, \bar{k}_i \geq 0, i = 2, 3$ , such that

$$-\underline{k}_i \leq K_i(x, t) \leq \bar{k}_i \text{ for } i = 2, 3,$$

for a.a.  $(x, t) \in Q_T$ .

We now recall the definition of weak solution to (1.1).

**Definition 2.1.** A pair of functions  $(u, v)$  is said to be a weak solution of (1.1) if  $u, v \in C(\overline{Q}_T)$ ,  $u^m \in L^p((0, T); W_0^{1,p}(\Omega))$ ,  $v^n \in L^q((0, T); W_0^{1,q}(\Omega))$  and  $(u, v)$  satisfies

$$\iint_{Q_T} \left( -u \frac{\partial \varphi}{\partial t} + |\nabla u^m|^{p-2} \nabla u^m \nabla \varphi - a u^{p-1} \varphi + u^{p-1} \varphi \int_{\Omega} [K_1(\xi, t) u^2(\xi, t - \tau_1) - K_2(\xi, t) v^2(\xi, t - \tau_2)] d\xi \right) dx dt = 0$$

and

$$\iint_{Q_T} \left( -v \frac{\partial \varphi}{\partial t} + |\nabla v^n|^{q-2} \nabla v^n \nabla \varphi - b v^{q-1} \varphi + v^{q-1} \varphi \int_{\Omega} [-K_3(\xi, t) u^2(\xi, t - \tau_3) + K_4(\xi, t) v^2(\xi, t - \tau_4)] d\xi \right) dx dt = 0,$$

for any  $\varphi \in C^1(\overline{Q}_T)$  such that  $\varphi(x, T) = \varphi(x, 0)$  for any  $x \in \Omega$  and  $\varphi(x, t) = 0$  for any  $(x, t) \in \partial\Omega \times [0, T]$ .

Here and in the following we assume that the functions  $t \mapsto u(\cdot, t)$  and  $t \mapsto v(\cdot, t)$  are extended from  $[0, T]$  to  $\mathbb{R}$  by  $T$ -periodicity so that  $(u, v)$  is a solution defined for all  $t \in \mathbb{R}^+$ .

In order to study system (1.1) we now consider the following nondegenerate-singular approximating  $p, q$ -Laplacian system

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}((\varepsilon + m^{p-1} u^{(m-1)(p-1)}) |\nabla u|^{p-2} \nabla u) = \left( a(x, t) - \int_{\Omega} K_1(\xi, t) u^2(\xi, t - \tau_1) d\xi + \int_{\Omega} K_2(\xi, t) v^2(\xi, t - \tau_2) d\xi \right) u^{p-1}, & \text{in } Q_T, \\ \frac{\partial v}{\partial t} - \operatorname{div}((\varepsilon + n^{q-1} v^{(n-1)(q-1)}) |\nabla v|^{q-2} \nabla v) = \left( b(x, t) + \int_{\Omega} K_3(\xi, t) u^2(\xi, t - \tau_3) d\xi - \int_{\Omega} K_4(\xi, t) v^2(\xi, t - \tau_4) d\xi \right) v^{q-1}, & \text{in } Q_T, \\ u(\cdot, t)|_{\partial\Omega} = v(\cdot, t)|_{\partial\Omega} = 0, & \text{for a.a. } t \in (0, T), \\ u(\cdot, 0) = u(\cdot, T) \text{ and } v(\cdot, 0) = v(\cdot, T), \end{cases} \quad (2.1)$$

where  $\varepsilon > 0$ . A solution  $(u, v)$  of (1.1) will be then obtained as the limit, for  $\varepsilon \rightarrow 0$ , of the solutions  $(u_\varepsilon, v_\varepsilon)$  of (2.1) with  $u_\varepsilon, v_\varepsilon \geq 0$ . For this we give the following definition:

**Definition 2.2.** A couple of functions  $(u_\varepsilon, v_\varepsilon)$  is said to be a generalized (weak) solution of (2.1) if

$$u_\varepsilon \in L^p(0, T; W_0^{1,p}(\Omega)) \cap C(\overline{Q}_T), \quad v_\varepsilon \in L^q(0, T; W_0^{1,q}(\Omega)) \cap C(\overline{Q}_T)$$

and  $(u_\varepsilon, v_\varepsilon)$  satisfies

$$\iint_{Q_T} \left( -u \frac{\partial \varphi}{\partial t} + \varepsilon |\nabla u|^{p-2} \nabla u \nabla \varphi + |\nabla u^m|^{p-2} \nabla u^m \nabla \varphi - a u_\varepsilon^{p-1} \varphi + u_\varepsilon^{p-1} \varphi \int_{\Omega} [K_1(\xi, t) u^2(\xi, t - \tau_1) - K_2(\xi, t) v^2(\xi, t - \tau_2)] d\xi \right) dx dt = 0$$

and

$$\iint_{Q_T} \left( -v \frac{\partial \varphi}{\partial t} + \varepsilon |\nabla v|^{q-2} \nabla v \nabla \varphi + |\nabla v^n|^{q-2} \nabla v^n \nabla \varphi - b v_\varepsilon^{q-1} \varphi + v_\varepsilon^{q-1} \varphi \int_{\Omega} [K_4(\xi, t) v^2(\xi, t - \tau_4) - K_3(\xi, t) u^2(\xi, t - \tau_3)] d\xi \right) dx dt = 0$$

for any  $\varphi \in C^1(\overline{Q}_T)$  such that  $\varphi(x, T) = \varphi(x, 0)$  for any  $x \in \Omega$  and  $\varphi(x, t) = 0$  for any  $(x, t) \in \partial\Omega \times [0, T]$ .

To deal with the existence of  $T$ -periodic solutions  $(u_\varepsilon, v_\varepsilon)$  of system (2.1), with  $u_\varepsilon, v_\varepsilon \geq 0$  in  $Q_T$ , we introduce, for any  $\varepsilon > 0$ , the map  $G_\varepsilon : [0, 1] \times L^\infty(Q_T) \times L^\infty(Q_T) \rightarrow X$ , where  $X := L^p(0, T; W_0^{1,p}(\Omega) \cap L^2(\Omega)) \times L^q(0, T; W_0^{1,q}(\Omega) \cap L^2(\Omega))$ , as follows:

$$(\sigma, f, g) \mapsto (u_\varepsilon, v_\varepsilon) = G_\varepsilon(\sigma, f, g)$$

if and only if  $(u_\varepsilon, v_\varepsilon)$  solves the following uncoupled problem

$$\begin{cases} \frac{\partial u}{\partial t} - \varepsilon \operatorname{div}(|\nabla u|^{p-2} \nabla u) - \operatorname{div}(|\nabla(\sigma u^m)|^{p-2} \nabla(\sigma u^m)) = f, & \text{in } Q_T, \\ \frac{\partial v}{\partial t} - \varepsilon \operatorname{div}(|\nabla v|^{q-2} \nabla v) - \operatorname{div}(|\nabla(\sigma v^n)|^{q-2} \nabla(\sigma v^n)) = g, & \text{in } Q_T, \\ u(\cdot, t)|_{\partial\Omega} = v(\cdot, t)|_{\partial\Omega} = 0, & \text{for a.a. } t \in (0, T), \\ u(\cdot, 0) = u(\cdot, T) \text{ and } v(\cdot, 0) = v(\cdot, T). \end{cases} \quad (2.2)$$

For any fixed  $\sigma \in [0, 1]$  the operator  $A : \mathcal{X} = L^p(0, T; W_0^{1,p}(\Omega) \cap L^2(\Omega)) \rightarrow \mathcal{X}'$ ,  $u \mapsto \varepsilon \operatorname{div}(|\nabla u|^{p-2} \nabla u) + \operatorname{div}(|\nabla(\sigma u^m)|^{p-2} \nabla(\sigma u^m))$ , is hemicontinuous, strictly monotone (and hence pseudomonotone), coercive and bounded. Thus, by [66, Theorem 32.D], the map  $G_\varepsilon$  is well defined. Now, consider

$$f(\alpha, \beta) := \left( a - \int_{\Omega} K_1(\xi, \cdot) \alpha^2(\xi, \cdot - \tau_1) d\xi + \int_{\Omega} K_2(\xi, \cdot) \beta^2(\xi, \cdot - \tau_2) d\xi \right) \alpha^{p-1}$$

and

$$g(\alpha, \beta) := \left( b + \int_{\Omega} K_3(\xi, \cdot) \alpha^2(\xi, \cdot - \tau_3) d\xi - \int_{\Omega} K_4(\xi, \cdot) \beta^2(\xi, \cdot - \tau_4) d\xi \right) \beta^{q-1},$$

where  $\alpha$  and  $\beta$  belong to  $L^\infty(Q_T)$ . Clearly, if the nonnegative functions  $u_\varepsilon, v_\varepsilon \in L^\infty(Q_T)$  are such that  $(u_\varepsilon, v_\varepsilon) = G_\varepsilon(1, f(u_\varepsilon, v_\varepsilon), g(u_\varepsilon, v_\varepsilon))$ , then  $(u_\varepsilon, v_\varepsilon)$  is also a solution of (2.1) (with  $u_\varepsilon$  and  $v_\varepsilon \geq 0$ ) in  $Q_T$ . Hence, the existence of a nonnegative solution of (2.1) is equivalent to the existence of a fixed point  $(\alpha, \beta)$  of the map  $(\alpha, \beta) \mapsto G_\varepsilon(1, f(\alpha, \beta), g(\alpha, \beta))$  with  $\alpha$  and  $\beta \geq 0$ .

Let  $T_\varepsilon(\sigma, \alpha, \beta) := G_\varepsilon(\sigma, f(\alpha, \beta), g(\alpha, \beta))$ . By [33, Theorem 1.1] and [34, Theorem 1.3] we have the Hölder estimate in the interior of  $Q_T$  of the solutions to (2.2). Moreover, the property of positive geometric density of  $\partial\Omega$ , the fact that the Dirichlet data is Hölder continuous and the periodicity condition ensure that one can obtain the Hölder estimate up to the parabolic boundary of  $Q_T$ , see the comments to [33, Theorem 1.1] and [34, Theorem 7.1].

We will need the following result, which was proved in [26, Lemma 2.1] for the  $p, q = 2, m, n > 1$ , but whose proof is the same, so we omit it.

**Lemma 2.1.** *Let  $(\alpha, \beta) \in L^\infty(Q_T) \times L^\infty(Q_T)$  and let  $\varepsilon > 0$ . Then  $T_\varepsilon : [0, 1] \times L^\infty(Q_T) \times L^\infty(Q_T) \rightarrow L^\infty(Q_T) \times L^\infty(Q_T)$ ,  $(\sigma, \alpha, \beta) \mapsto T_\varepsilon(\sigma, \alpha, \beta) = (u_\varepsilon, v_\varepsilon)$  is compact. Moreover  $u_\varepsilon, v_\varepsilon \in C(\overline{Q_T})$ .*

Our aim is to prove the existence of  $T$ -periodic solutions  $u_\varepsilon, v_\varepsilon \in C(\overline{Q_T})$  of problem (2.1) with  $u_\varepsilon, v_\varepsilon > 0$  in  $Q_T$ , for all  $\varepsilon > 0$  small enough, as positive fixed points of the map  $(\alpha, \beta) \mapsto T_\varepsilon(1, \alpha, \beta)$ . As a first step we prove the following result.

**Proposition 2.1.** *If the non-trivial pair  $(u_\varepsilon, v_\varepsilon)$  solves*

$$(u, v) = G_\varepsilon(\sigma, f(u^+, v^+) + (1 - \sigma), g(u^+, v^+) + (1 - \sigma)) \quad (2.3)$$

for some  $\sigma \in [0, 1]$ , then

$$u_\varepsilon(x, t) \geq 0 \text{ and } v_\varepsilon(x, t) \geq 0 \text{ for any } (x, t) \in Q_T.$$

Moreover, if  $u_\varepsilon \neq 0$  or  $v_\varepsilon \neq 0$ , then  $u_\varepsilon > 0$  or  $v_\varepsilon > 0$  in  $Q_T$ , respectively.

*Proof.* Assume that  $(u_\varepsilon, v_\varepsilon)$  solves (2.3) with  $u_\varepsilon \neq 0$  for some  $\sigma \in [0, 1]$ . We first prove that  $u_\varepsilon \geq 0$ . Multiplying the first equation of (2.2), where  $f(\alpha, \beta)$  is replaced by  $f(u_\varepsilon^+, v_\varepsilon^+) + (1 - \sigma)$ , by  $u_\varepsilon^- := \min\{0, u_\varepsilon\}$ , integrating on  $Q_T$  and passing to the limit in the Steklov averages  $(u_\varepsilon)_h \in H^1(Q_{T-\delta})$ ,  $\delta, h > 0$ , in a standard way [39, p.85], we obtain

$$\varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla u_\varepsilon^- dx dt + \iint_{Q_T} |\nabla(\sigma u_\varepsilon^m)|^{p-2} \nabla(\sigma u_\varepsilon^m) \nabla u_\varepsilon^- dx dt = \iint_{Q_T} (1 - \sigma) u_\varepsilon^- dx dt \leq 0,$$

by the  $T$ -periodicity of  $u_\varepsilon$  and taking into account that  $u_\varepsilon^+ u_\varepsilon^- = 0$ . Hence we obtain

$$\varepsilon \iint_{Q_T} |\nabla u_\varepsilon^-|^p dx dt \leq \varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla u_\varepsilon^- dx dt + \iint_{Q_T} |\nabla(\sigma u_\varepsilon^m)|^{p-2} \nabla(\sigma u_\varepsilon^m) \nabla u_\varepsilon^- dx dt \leq 0.$$

Thus

$$\iint_{Q_T} |\nabla u_\varepsilon^-|^p dx dt = 0.$$

The Poincaré inequality gives

$$0 \leq \int_\Omega |u_\varepsilon^-|^p dx \leq \frac{1}{\mu_p} \int_\Omega |\nabla u_\varepsilon^-|^p dx \quad \text{for all } t,$$

where  $\mu_p$  is the first positive eigenvalue of the problem

$$\begin{cases} -\operatorname{div}(|\nabla z|^{p-2} \nabla z) = \mu |z|^{p-2} z, & x \in \Omega, \\ z = 0, & x \in \partial\Omega, \end{cases}$$

(see, for example, [37]). Integrating over  $(0, T)$ , we have

$$0 \leq \iint_{Q_T} |u_\varepsilon^-|^p dx dt \leq \frac{1}{\mu_p} \iint_{Q_T} |\nabla u_\varepsilon^-|^p dx dt = 0,$$

which, together with the boundary conditions and the fact that  $u_\varepsilon^- \in C(\overline{Q}_T)$ , implies  $u_\varepsilon^-(x, t) = 0$  for all  $(x, t) \in Q_T$ .

Now we prove that  $u_\varepsilon > 0$  in  $Q_T$ . Since  $u_\varepsilon$  is non-trivial, there exists  $(x_0, t_0) \in \Omega \times (0, T]$  such that  $u_\varepsilon(x_0, t_0) > 0$ . Hence  $u_\varepsilon(x, t) > 0$  for all  $(x, t) \in Q_T$  (see [10, p.3] and [36]). In the same way, one can prove that  $v_\varepsilon \neq 0$  implies  $v_\varepsilon(x, t) > 0$  for all  $(x, t) \in Q_T$ .  $\square$

The next lemma is crucial to prove Proposition 2.2 below.

**Lemma 2.2.** *Let  $K > 0$  and assume that  $u$  is a non-negative  $T$ -periodic continuous function such that  $x \mapsto u(x, t) \in W_0^{1,p}(\Omega)$  for all  $t \in [0, T]$  and which satisfies*

$$u_t - \varepsilon \operatorname{div}(|\nabla u|^{p-2} \nabla u) - \operatorname{div}(|\nabla u^m|^{p-2} \nabla u^m) \leq K u^{p-1} \quad \text{in } Q_T.$$

*Then there exists  $R > 0$  such that*

$$\|u\|_{L^\infty(Q_T)} \leq R \text{ for all } \varepsilon > 0.$$

*Proof.* We follow Moser's technique to show the stated a priori bounds. Multiplying

$$u_t - \varepsilon \operatorname{div}(|\nabla u|^{p-2} \nabla u) - \operatorname{div}(|\nabla u^m|^{p-2} \nabla u^m) \leq K u^{p-1}$$

by  $u^{s+1}$ , with  $s \geq 0$ , integrating over  $\Omega$  and passing to the limit as  $h \rightarrow 0$  in the Steklov averages  $u_h \in H^1(Q_{T-\delta})$ ,  $\delta, h > 0$ , we have

$$\frac{1}{s+2} \frac{d}{dt} \|u(t)\|_{L^{s+2}(\Omega)}^{s+2} + \int_\Omega (\varepsilon |\nabla u|^{p-2} \nabla u + |\nabla u^m|^{p-2} \nabla u^m) \nabla u^{s+1} dx \leq K \|u(t)\|_{L^{s+p}(\Omega)}^{s+p} \leq C_{|\Omega|} \|u(t)\|_{L^{s+2}(\Omega)}^{s+p},$$



and thus

$$\frac{d}{dt} \|u(t)\|_{L^{s+2}(\Omega)}^{s+2} + (s+1)(s+2)m^{p-1} \int_{\Omega} u^{(m-1)(p-1)+s} |\nabla u|^p dx \leq (s+2)C_{|\Omega|} \|u(t)\|_{L^{s+2}(\Omega)}^{s+p}$$

where  $C_{|\Omega|} := \sup_{s \geq 0} K|\Omega|^{1-\frac{s+p}{s+2}}$ . Since  $m, p > 1$ , it follows

$$\begin{aligned} & \frac{d}{dt} \|u(t)\|_{L^{s+2}(\Omega)}^{s+2} + \frac{s+2}{[m(p-1)+s+1]^p} \int_{\Omega} \left| \nabla u^{\frac{m(p-1)+s+1}{p}} \right|^p dx \\ & \leq \frac{d}{dt} \|u(t)\|_{L^{s+2}(\Omega)}^{s+2} + (s+2)m^{p-1} \left( \frac{p}{m(p-1)+s+1} \right)^p \int_{\Omega} \left| \nabla u^{\frac{m(p-1)+s+1}{p}} \right|^p dx \\ & \leq \frac{d}{dt} \|u(t)\|_{L^{s+2}(\Omega)}^{s+2} + (s+1)(s+2)m^{p-1} \int_{\Omega} u^{(m-1)(p-1)+s} |\nabla u|^p dx \\ & \leq C_{|\Omega|} (s+2) \|u(t)\|_{L^{s+2}(\Omega)}^{s+p}. \end{aligned} \quad (2.4)$$

For  $\varepsilon$  fixed and  $k = 1, 2, \dots$ , setting

$$s_k := 2p^k + \frac{p^k - p}{p-1} + m - 1, \quad \alpha_k := \frac{p(s_k + 2)}{m(p-1) + s_k + 1}, \quad w_k := u^{\frac{m(p-1)+s_k+1}{p}},$$

we obtain by (2.4)

$$\frac{d}{dt} \|w_k(t)\|_{L^{\alpha_k}(\Omega)}^{\alpha_k} + \frac{(s_k + 2)}{[m(p-1) + s_k + 1]^p} \|\nabla w_k(t)\|_{L^p(\Omega)}^p \leq C_{|\Omega|} (s_k + 2) \|w_k(t)\|_{L^{\alpha_k}(\Omega)}^{\alpha_k \frac{s_k+p}{s_k+2}}. \quad (2.5)$$

Now, let us fix  $s > p$  such that the continuous Sobolev immersion  $W_0^{1,p}(\Omega) \subset L^s(\Omega)$  holds and observe that since  $s_k \rightarrow +\infty$ , as  $k \rightarrow +\infty$ , there exists  $k_0$  such that  $\alpha_k \in (1, s)$  for all  $k \geq k_0$ . By the interpolation and the Sobolev inequalities, it results

$$\|w_k(t)\|_{L^{\alpha_k}(\Omega)} \leq \|w_k(t)\|_{L^1(\Omega)}^{\theta_k} \|w_k(t)\|_{L^s(\Omega)}^{1-\theta_k} \leq C \|w_k(t)\|_{L^1(\Omega)}^{\theta_k} \|\nabla w_k(t)\|_{L^p(\Omega)}^{1-\theta_k}$$

for all  $k \geq k_0$ . Here  $\theta_k = (s - \alpha_k)/[\alpha_k(s - 1)]$  and  $C$  is a positive constant. Using the fact that  $\|w_k(t)\|_{L^1(\Omega)} = \|w_{k-1}(t)\|_{L^{\alpha_{k-1}}(\Omega)}^{\alpha_{k-1}}$  and defining  $x_k := \sup_{t \in \mathbb{R}} \|w_k(t)\|_{L^{\alpha_k}(\Omega)}$ , one has

$$\begin{aligned} \|w_k(t)\|_{L^{\alpha_k}(\Omega)}^{\frac{p}{1-\theta_k}} & \leq C \|w_{k-1}(t)\|_{L^{\alpha_{k-1}}(\Omega)}^{p\alpha_{k-1} \frac{\theta_k}{1-\theta_k}} \|\nabla w_k(t)\|_{L^p(\Omega)}^p \\ & \leq C x_{k-1}^{p\alpha_{k-1} \frac{\theta_k}{1-\theta_k}} \|\nabla w_k(t)\|_{L^p(\Omega)}^p \end{aligned}$$

for all  $k \geq k_0$ . Thus, by (2.5),

$$\begin{aligned} \frac{d}{dt} \|w_k(t)\|_{L^{\alpha_k}(\Omega)}^{\alpha_k} & \leq C_{|\Omega|} (s_k + 2) \|w_k(t)\|_{L^{\alpha_k}(\Omega)}^{\alpha_k \frac{s_k+p}{s_k+2}} \\ & \quad - \frac{s_k + 2}{C[m(p-1) + s_k + 1]^p} \|w_k(t)\|_{L^{\alpha_k}(\Omega)}^{\frac{p}{1-\theta_k}} x_{k-1}^{p\alpha_{k-1} \frac{\theta_k}{\theta_k-1}} \\ & = \left( C_{|\Omega|} - \frac{1}{C[m(p-1) + s_k + 1]^p} \|w_k(t)\|_{L^{\alpha_k}(\Omega)}^{\frac{p}{1-\theta_k} - \alpha_k \frac{s_k+p}{s_k+2}} x_{k-1}^{p\alpha_{k-1} \frac{\theta_k}{\theta_k-1}} \right) (s_k + 2) \|w_k(t)\|_{L^{\alpha_k}(\Omega)}^{\alpha_k \frac{s_k+p}{s_k+2}} \end{aligned} \quad (2.6)$$

for all  $k \geq k_0$ . By Lemma 2.3 and (2.6), one has

$$\|w_k(t)\|_{L^{\alpha_k}(\Omega)} \leq \left( C_{|\Omega|} M_k x_{k-1}^{p\alpha_{k-1} \frac{\theta_k}{1-\theta_k}} \right)^{\eta_k} \quad (2.7)$$

for all  $k \geq k_0$ , where  $\eta_k := \frac{(1-\theta_k)(s_k+2)}{p(s_k+2)-\alpha_k(s_k+p)(1-\theta_k)}$  and  $M_k := C[m(p-1) + s_k + 1]^p$ . By definition of  $x_k$  and (2.7), we get

$$x_k \leq (C_{|\Omega|} M_k)^{\eta_k} x_{k-1}^{\nu_k}$$

for all  $k \geq k_0$ , with  $\nu_k := p\alpha_{k-1}\theta_k(s_k+2)/[p(s_k+2) - \alpha_k(s_k+p)(1-\theta_k)]$ .

If  $x_{k_n} \leq 1$  along a sequence  $k_n \rightarrow +\infty$ , then one has  $\|u\|_{L^\infty(Q_T)} \leq 1$  by the very definition of  $x_k$  and the lemma is proved. On the other hand, assume  $x_k > 1$  for all  $k \geq k_0$  and observe that there exists  $\bar{k}_0$  such that, for all  $k \geq \bar{k}_0$ ,

$$\eta_k \leq 1/(p\theta) \quad \text{and} \quad \nu_k \leq \alpha_{k-1}.$$

Here  $\theta := (s-p)/[p(s-1)]$ . Without loss of generality, assume  $k_0 = \max\{\bar{k}_0, k_0\}$ . Then, there exists a positive constant  $A$  such that

$$\begin{aligned} x_k &\leq (C_{|\Omega|} C)^{\eta_k} [m(p-1) + s_k + 1]^{p\eta_k} x_{k-1}^{\nu_k} \\ &\leq (C_{|\Omega|} C)^{\eta_k} \left( mp + \frac{2p^{k+1}}{p-1} \right)^{p\eta_k} x_{k-1}^{\nu_k} \leq Ap^{\frac{k+1}{\theta}} x_{k-1}^{\alpha_{k-1}} \end{aligned}$$

for all  $k \geq k_0$ . Thus

$$\begin{aligned} \log x_k &\leq \log A + \frac{k+1}{\theta} \log p + \alpha_{k-1} \log x_{k-1} \\ &\leq \left( 1 + \sum_{i=1}^{k-k_0-1} \prod_{j=1}^i \alpha_{k-j} \right) \log A + \left( k+1 + \sum_{i=k_0+2}^k i \prod_{j=1}^{k+1-i} \alpha_{k-j} \right) \frac{\log p}{\theta} + \left( \prod_{j=1}^{k-k_0} \alpha_{k-j} \right) \log x_{k_0}. \end{aligned} \tag{2.8}$$

Since

$$\alpha_k = 1 + \frac{1-m(p-1)}{m(p-1) + s_k + 1} \leq 1 + \frac{|1-m(p-1)|}{2p^k},$$

for  $i \leq k$  we have that

$$\begin{aligned} \log \left( \frac{1}{p^i} \prod_{j=1}^i \alpha_{k-j} \right) &= \sum_{j=1}^i \log \frac{\alpha_{k-j}}{p} \\ &\leq \sum_{j=1}^i \log \left( 1 + \frac{|1-m(p-1)|}{2p^{k-j+1}} \right) \\ &\leq \frac{|1-m(p-1)|}{2p^k} \sum_{j=1}^i p^{j-1} \\ &\leq \frac{|1-m(p-1)|}{2(p-1)}, \end{aligned}$$

which means that

$$\prod_{j=1}^i \alpha_{k-j} \leq Mp^i \quad \text{with } M = \exp \frac{|1-m(p-1)|}{2(p-1)}.$$

Therefore, from (2.8) we obtain

$$\begin{aligned} \frac{\log x_k}{M} &\leq \log A \sum_{i=0}^{k-k_0-1} p^i + \frac{\log p}{\theta} \sum_{i=k_0+2}^{k+1} ip^{k+1-i} + p^{k-k_0} \log x_{k_0} \\ &\leq \frac{\log p}{\theta} \frac{p^{k-k_0}}{(p-1)^2} [k_0(p-1) + 2p-1] + \log A \frac{1-p^{k-k_0}}{1-p} + p^{k-k_0} \log x_{k_0}. \end{aligned} \tag{2.9}$$

In fact, taking  $x = \frac{1}{p}$  in  $x \frac{d}{dx} \sum_{i=0}^{k+1} x^i = x \frac{d}{dx} \left( \frac{1-x^{k+2}}{1-x} \right)$ , it results

$$\begin{aligned} \sum_{i=k_0+2}^{k+1} ip^{k+1-i} &= \frac{p^{k+3}}{(p-1)^2} \left[ \frac{1}{p^{k+2}} \left( \frac{k+1}{p} - k - 2 \right) - \frac{1}{p^{k_0+2}} \left( \frac{k_0+1}{p} - k_0 - 2 \right) \right] \\ &\leq \frac{p^{k+3}}{(p-1)^2} \frac{1}{p^{k_0+2}} \left( k_0 + 2 - \frac{k_0+1}{p} \right) = \frac{p^{k-k_0}}{(p-1)^2} [k_0(p-1) + 2p - 1]. \end{aligned}$$

Then, by (2.9), it follows

$$x_k^{1/M} \leq A^{\frac{1-p^{k-k_0}}{1-p}} p^{\frac{p^{k-k_0}}{\theta(p-1)^2} [k_0(p-1)+2p-1]} x_{k_0}^{p^{k-k_0}}.$$

Since  $x_k = \sup_{t \in \mathbb{R}} \|u(t)\|_{s_k+2}^{\frac{m(p-1)+s_k+1}{p}}$ , we obtain

$$\begin{aligned} \|u(t)\|_{L^\infty(\Omega)} &\leq \lim_{k \rightarrow \infty} \|u(t)\|_{s_k+2} \\ &\leq \limsup_{k \rightarrow \infty} \left\{ A^{\frac{p}{m(p-1)+s_k+1} \frac{1-p^{k-k_0}}{1-p}} x_{k_0}^{\frac{p^{k-k_0}+1}{m(p-1)+s_k+1}} p^{\frac{p^{k-k_0}+1}{\theta(p-1)^2} (k_0(p-1)+2p-1)} \right\}^M \\ &=: R \quad \forall t \in \mathbb{R}, \end{aligned}$$

where  $R$  is a positive constant. Hence  $\sup_{t \in \mathbb{R}} \|u(t)\|_{L^\infty(\Omega)} \leq R$ . It remains to prove that  $R$  is independent of  $\varepsilon$  as claimed. To this aim it is sufficient to prove that there exists  $C > 0$ , independent of  $\varepsilon$ , such that  $x_{k_0} \leq C$ . In fact, by (2.5) with  $s_0 := s_{k_0}$ , it follows

$$\frac{d}{dt} \|u(t)\|_{L^{s_0+2}(\Omega)}^{s_0+2} + \frac{s_0+2}{[m(p-1)+s_0+1]^p} \int_{\Omega} \left| \nabla u^{\frac{m(p-1)+s_0+1}{p}} \right|^p dx \leq C_{|\Omega|} (s_0+2) \|u(t)\|_{L^{s_0+2}(\Omega)}^{s_0+p}, \quad (2.10)$$

Moreover, we have

$$\|u(t)\|_{L^{s_0+2}(\Omega)}^{m(p-1)+s_0+1} \leq C_1 \left[ \int_{\Omega} \left( u^{\frac{m(p-1)+s_0+1}{p}} \right)^{p \frac{s_0+2}{s_0+2}} dx \right]^{\frac{s_0+p}{s_0+2}}$$

by Hölder's inequality with  $r = \frac{m(p-1)+s_0+1}{s_0+p}$ . Now, without loss of generality, we can assume that  $k_0$  is chosen large enough such that the continuous Sobolev immersion  $W_0^{1,p}(\Omega) \subset L^{p \frac{s_0+2}{s_0+p}}(\Omega)$  holds and, hence, we deduce that

$$\|u(t)\|_{L^{s_0+2}(\Omega)}^{m(p-1)+s_0+1} \leq C_2 \|\nabla u^{\frac{m(p-1)+s_0+1}{p}}\|_{L^p(\Omega)}^p,$$

for a positive constant  $C_2$ . Thus, using (2.10), one has

$$\begin{aligned} \frac{d}{dt} \|u(t)\|_{L^{s_0+2}(\Omega)}^{s_0+2} &+ \frac{s_0+2}{C_2[m(p-1)+s_0+1]^p} \|u(t)\|_{L^{s_0+2}(\Omega)}^{m(p-1)+s_0+1} \\ &\leq \frac{d}{dt} \|u(t)\|_{L^{s_0+2}(\Omega)}^{s_0+2} + \frac{s_0+2}{[m(p-1)+s_0+1]^p} \|\nabla u^{\frac{m(p-1)+s_0+1}{p}}\|_{L^p(\Omega)}^p \\ &\leq C_{|\Omega|} (s_0+2) \|u(t)\|_{L^{s_0+2}(\Omega)}^{s_0+p}. \end{aligned}$$

Hence

$$\frac{d}{dt} \|u(t)\|_{L^{s_0+2}(\Omega)}^{s_0+2} \leq \|u(t)\|_{L^{s_0+2}(\Omega)}^{s_0+p} \left( C_{|\Omega|} (s_0+2) - M \|u(t)\|_{L^{s_0+2}(\Omega)}^{(m-1)(p-1)} \right),$$

where  $M := \frac{s_0+2}{C_2[m(p-1)+s_0+1]^p}$ . This inequality and Lemma 2.3 below imply that

$$\|u(t)\|_{L^{s_0+2}(\Omega)} \leq \{C_2 C_{|\Omega|} [m(p-1)+s_0+1]^p\}^{\frac{1}{(m-1)(p-1)}} \quad \forall t \in \mathbb{R}.$$

Thus, there exists  $C > 0$ , independent of  $\varepsilon$ , such that  $x_{k_0} = \sup_{t \in \mathbb{R}} \|u(t)\|_{L^{s_0+2}(\Omega)}^{\frac{m(p-1)+s_0+1}{p}} \leq C$ , and this concludes the proof.  $\square$

**Lemma 2.3.** *Let  $f : \mathbb{R} \rightarrow (0, +\infty)$  be a differentiable and  $T$ -periodic function; suppose that there exist positive constants  $s, \alpha, \beta, \gamma$  such that*

$$f'(t) \leq f^s(t)(\beta - \gamma f^\alpha(t)),$$

for all  $t \in \mathbb{R}$ . Then  $\beta - \gamma f^\alpha(t) \geq 0$  for all  $t \in \mathbb{R}$ .

*Proof.* By the periodicity and continuity of  $f$ , let  $t_0$  be any point in which  $f$  attains its maximum. Then we have:

$$\beta - \gamma f^\alpha(t) \geq \beta - \gamma f^\alpha(t_0) \geq \frac{f'(t_0)}{f^s(t_0)} = 0 \quad \forall t \in \mathbb{R}. \quad \square$$

Next, we show that the map  $I - G_\varepsilon : \{1\} \times L^\infty(Q_T) \times L^\infty(Q_T) \rightarrow L^\infty(Q_T) \times L^\infty(Q_T)$  has the Leray-Schauder topological degree different from zero in the intersection of a sufficiently large ball centered at the origin with the cone of non-negative functions.

From now on we make the following assumption:

**Hypotheses 2.2.** There exist two positive constants  $C_1, C_2$  such that

1. for all  $\varepsilon > 0$  and all solution pairs  $(u_\varepsilon, v_\varepsilon)$  of

$$(u, v) = G_\varepsilon(1, f(u^+, v^+), g(u^+, v^+)), \quad (2.11)$$

it results

$$\|u_\varepsilon\|_{L^2(Q_T)}^2 \leq C_1 \text{ and } \|v_\varepsilon\|_{L^2(Q_T)}^2 \leq C_2, \quad (2.12)$$

2.  $\min \left\{ \frac{1}{T} \iint_{Q_T} a(x, t) e_p^p(x) dx dt - \frac{k_2 C_2}{T}, \frac{1}{T} \iint_{Q_T} b(x, t) e_q^q(x) dx dt - \frac{k_3 C_1}{T} \right\} > 0$ , where  $\underline{k}_2, \underline{k}_3$  are as in Hypotheses 2.1.3, and, for any  $r > 1$ ,  $e_r$  stands for the positive eigenvector associated to the first eigenvalue  $\mu_r$  of  $-\Delta_r$  with Dirichlet boundary conditions and normalized in such a way that  $\|e_r\|_{L^r(\Omega)} = 1$ .

**Remark 2.1.** The last assumptions, and in particular statement 2, will grant the periodic coexistence, that is the existence of a  $T$ -periodic solution couple  $(u, v)$  with  $u, v$  both non-negative and non-trivial (see Theorem 2.1). Generally speaking, the *explicit* knowledge of the a priori bounds  $C_1, C_2$  is required to check the validity of statement 2 of Hypotheses 2.2. This is the task we will devote ourselves to in the next Section. A notable exception is the case in which  $K_2, K_3$  are not negative (namely, the cooperative case), since one can choose  $\underline{k}_2 = \underline{k}_3 = 0$  and, thus, the second part of Hypotheses 2.2 is readily satisfied regardless of the values of  $C_1, C_2$ , if neither of the coefficients  $a, b$  is trivial.

The next result shows how we can pass from an  $L^2$ -estimate to an  $L^\infty$ -estimate.

**Proposition 2.2.** *There is a constant  $R > 0$  such that*

$$\|u_\varepsilon\|_{L^\infty(Q_T)}, \|v_\varepsilon\|_{L^\infty(Q_T)} < R$$

for all solution pairs  $(u_\varepsilon, v_\varepsilon)$  of (2.11) with  $\varepsilon > 0$  sufficiently small. In particular, one has that

$$\deg((u, v) - G_\varepsilon(1, f(u^+, v^+), g(u^+, v^+)), B_R, 0) = 1.$$

*Proof.* Assume  $u_\varepsilon \neq 0$ , thus  $u_\varepsilon > 0$  and  $v_\varepsilon \geq 0$  in  $Q_T$  by Proposition 2.1. Multiplying by  $u_\varepsilon$  the first equation of (2.1), integrating over  $\Omega$  and using the Steklov averages  $(u_\varepsilon)_h \in H^1(Q_{T-\delta})$ ,  $\delta, h > 0$ , see [39, p.85], we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} (u_\varepsilon)_h^2 dx + \varepsilon \int_{\Omega} |\nabla (u_\varepsilon)_h|^p dx + m^{p-1} \int_{\Omega} (u_\varepsilon)_h^{(m-1)(p-1)} |\nabla (u_\varepsilon)_h|^p dx \\ & \leq \left( \|a\|_{L^\infty(Q_T)} + \int_{\Omega} K_2(\xi, t) (v_\varepsilon)_h^2(\xi, t - \tau_2) d\xi \right) \int_{\Omega} (u_\varepsilon)_h^p dx \\ & \leq |\Omega|^{1-\frac{p}{2}} \left( \|a\|_{L^\infty(Q_T)} + \int_{\Omega} K_2(\xi, t) (v_\varepsilon)_h^2(\xi, t - \tau_2) d\xi \right) \left( \int_{\Omega} (u_\varepsilon)_h^2 dx \right)^{\frac{p}{2}}. \end{aligned}$$

Thus

$$\begin{aligned} & \frac{\frac{1}{2} \frac{d}{dt} \int_{\Omega} (u_\varepsilon)_h^2 dx + \varepsilon \int_{\Omega} |\nabla (u_\varepsilon)_h|^p dx + m^{p-1} \int_{\Omega} (u_\varepsilon)_h^{(m-1)(p-1)} |\nabla (u_\varepsilon)_h|^p dx}{\left( \int_{\Omega} (u_\varepsilon)_h^2 dx \right)^{\frac{p}{2}}} \\ & \leq |\Omega|^{1-\frac{p}{2}} \left( \|a\|_{L^\infty(Q_T)} + \|K_2\|_{L^\infty(Q_T)} \int_{\Omega} (v_\varepsilon)_h^2(\xi, t - \tau_2) d\xi \right). \end{aligned} \quad (2.13)$$

Since  $t \mapsto \|u(t)\|_{L^2(\Omega)}$  is continuous in  $[0, T]$ , there exist  $t_1$  and  $t_2$  in  $[0, T]$  such that

$$\int_{\Omega} u_\varepsilon^2(x, t_1) dx = \min_{t \in [0, T]} \int_{\Omega} u_\varepsilon^2(x, t) dx$$

and

$$\int_{\Omega} u_\varepsilon^2(x, t_2) dx = \max_{t \in [0, T]} \int_{\Omega} u_\varepsilon^2(x, t) dx.$$

Without loss of generality, by periodicity, we can assume that  $t_1 < t_2$ . Then, integrating (2.13) between  $t_1$  and  $t_2$  and passing to the limit as  $h \rightarrow 0$ , we find

$$\int_{t_1}^{t_2} \left( \int_{\Omega} u_\varepsilon^2 dx \right)^{-\frac{p}{2}} \frac{d}{dt} \left( \int_{\Omega} u_\varepsilon^2 dx \right) dt \leq 2|\Omega|^{1-\frac{p}{2}} \int_0^T \left( \|a\|_{L^\infty(Q_T)} + \|K_2\|_{L^\infty(Q_T)} \int_{\Omega} v_\varepsilon^2(\xi, t - \tau_2) d\xi \right) dt.$$

Thus

$$\left( \int_{\Omega} u_\varepsilon^2(x, t_2) dx \right)^{\frac{2-p}{2}} - \left( \int_{\Omega} u_\varepsilon^2(x, t_1) dx \right)^{\frac{2-p}{2}} \leq C(T \|a\|_{L^\infty(Q_T)} + \|K_2\|_{L^\infty(Q_T)} C_2),$$

where  $C := (2-p)|\Omega|^{1-\frac{p}{2}}$ . Hence

$$\left( \int_{\Omega} u_\varepsilon^2(x, t_2) dx \right)^{\frac{2-p}{2}} \leq \left( \int_{\Omega} u_\varepsilon^2(x, t_1) dx \right)^{\frac{2-p}{2}} + C(T \|a\|_{L^\infty(Q_T)} + \|K_2\|_{L^\infty(Q_T)} C_2),$$

or, equivalently,

$$\max_{t \in [0, T]} \int_{\Omega} u_\varepsilon^2(x, t) dx \leq \left\{ \left( \min_{t \in [0, T]} \int_{\Omega} u_\varepsilon^2(x, t) dx \right)^{\frac{2-p}{2}} + C(T \|a\|_{L^\infty(Q_T)} + \|K_2\|_{L^\infty(Q_T)} C_2) \right\}^{\frac{2}{2-p}}.$$

This implies that there exists a constant  $\gamma > 0$ , independent of  $\varepsilon$ , such that

$$\max_{t \in [0, T]} \int_{\Omega} u_\varepsilon^2(x, t) dx \leq \gamma.$$

Otherwise, for all  $\gamma > 0$  there would exist  $\varepsilon > 0$  such that the corresponding solution  $u_\varepsilon$  satisfies

$$\gamma < \max_{t \in [0, T]} \int_{\Omega} u_\varepsilon^2(x, t) dx \leq \left\{ \left( \min_{t \in [0, T]} \int_{\Omega} u_\varepsilon^2(x, t) dx \right)^{\frac{2-p}{2}} + C(T \|a\|_{L^\infty(Q_T)} + \|K_2\|_{L^\infty(Q_T)} C_2) \right\}^{\frac{2}{2-p}}.$$

Using the fact that  $\frac{2}{2-p} > 1$  and integrating the previous inequality on  $[0, T]$  for sufficiently large  $\gamma$ , one would have

$$\gamma T \leq \iint_{Q_T} u_\varepsilon^2(x, t) dx dt + CT(T\|a\|_{L^\infty(Q_T)} + \|K_2\|_{L^\infty(Q_T)}C_2),$$

that is  $u_\varepsilon$  is unbounded in  $L^2(Q_T)$ , in contradiction with Hypothesis 2.2.1. Of course, an analogous inequality holds for  $v_\varepsilon$ .

Now, we have

$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial t} - \varepsilon \operatorname{div}(|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon) - \operatorname{div}(|\nabla u_\varepsilon^m|^{p-2} \nabla u_\varepsilon) &\leq \\ &\leq \left( \|a\|_{L^\infty(Q_T)} + \|K_2\|_{L^\infty(Q_T)} \max_{t \in [0, T]} \int_\Omega v_\varepsilon^2(x, t) dx \right) u_\varepsilon^{p-1} \leq \\ &\leq (\|a\|_{L^\infty(Q_T)} + \|K_2\|_{L^\infty(Q_T)} \gamma) u_\varepsilon^{p-1}, \end{aligned} \quad (2.14)$$

i.e.

$$\frac{\partial u_\varepsilon}{\partial t} - \varepsilon \operatorname{div}(|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon) - \operatorname{div}(\nabla u_\varepsilon^m |\nabla u_\varepsilon^m|^{p-2}) \leq K u_\varepsilon^{p-1}, \quad \text{in } Q_T,$$

where  $K := \|a\|_{L^\infty(Q_T)} + \|K_2\|_{L^\infty(Q_T)} \gamma$ . By Lemma 2.2 we conclude that  $\|u_\varepsilon\|_{L^\infty(Q_T)} \leq R_1$  for some  $R_1 > 0$  independent of  $\varepsilon$ . Analogously,  $\|v_\varepsilon\|_{L^\infty(Q_T)} \leq R_2$  for some constant  $R_2 > 0$ . Therefore it is enough to choose  $R > \max\{R_1, R_2\}$ .

The previous calculations also show that any solution pair of

$$(u, v) = G_\varepsilon(1, \rho f(u^+, v^+), \rho g(u^+, v^+))$$

with  $\rho \in [0, 1]$  satisfies the same inequality (2.14). Therefore, the homotopy invariance property of the Leray-Schauder degree implies that

$$\begin{aligned} &\deg((u, v) - T_\varepsilon(1, u^+, v^+), B_R, 0) \\ &= \deg((u, v) - G_\varepsilon(1, \rho f(u^+, v^+), \rho g(u^+, v^+)), B_R, 0) \end{aligned}$$

for any  $\rho \in [0, 1]$ . If we take  $\rho = 0$ , using the fact that  $G_\varepsilon$  at  $\rho = 0$  is the zero map, we obtain

$$\deg((u, v) - T_\varepsilon(1, u^+, v^+), B_R, 0) = \deg((u, v), B_R, 0) = 1.$$

□

In order to prove that the solutions  $(u_\varepsilon, v_\varepsilon)$  of (2.1) we are going to find are not bifurcating from the trivial solution  $(0, 0)$ , the next estimate will be crucial.

**Lemma 2.4.** *Let  $s > 0$  be such that*

$$s < \min \left\{ \frac{(p-1)(m-p)}{p}, \frac{(q-1)(n-q)}{q} \right\}.$$

*Then, there exist two positive constants  $M_1$  and  $M_2$  such that*

$$\left\| \nabla u_\varepsilon^{\frac{(p-1)(m-1)-s}{p-1}} \right\|_{L^p(Q_T)} \leq M_1 \quad \text{and} \quad \left\| \nabla v_\varepsilon^{\frac{(q-1)(n-1)-s}{q-1}} \right\|_{L^q(Q_T)} \leq M_2,$$

*for all solution pairs  $(u_\varepsilon, v_\varepsilon)$  of (2.11) and  $\varepsilon > 0$  sufficiently small.*

*Proof.* Let  $\gamma := \frac{(p-1)(m-p) - ps}{p-1} > 0$ . Multiplying the equation

$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial t} - \varepsilon \operatorname{div}(|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon) - \operatorname{div}(|\nabla u_\varepsilon^m|^{p-2} \nabla u_\varepsilon^m) &= \left( a(x, t) - \int_\Omega K_1(\xi, t) u_\varepsilon^2(\xi, t - \tau_1) d\xi + \right. \\ &\quad \left. + \int_\Omega K_2(\xi, t) v_\varepsilon^2(\xi, t - \tau_2) d\xi \right) u_\varepsilon^{p-1} \end{aligned}$$

by  $u_\varepsilon^\gamma$ , integrating over  $Q_T$  and passing to the limit in the Steklov averages, by the  $T$ -periodicity of  $u_\varepsilon$  we obtain

$$\begin{aligned} & \varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla u_\varepsilon^\gamma dx dt + \iint_{Q_T} |\nabla u_\varepsilon^m|^{p-2} \nabla u_\varepsilon^m \nabla u_\varepsilon^\gamma dx dt \\ &= \iint_{Q_T} a(x, t) u_\varepsilon^{\gamma+p-1}(x, t) dx dt - \int_0^T \left( \int_\Omega u_\varepsilon^{\gamma+p-1}(x, t) dx \right) \left( \int_\Omega K_1(\xi, t) u_\varepsilon^2(\xi, t - \tau_1) d\xi \right) dt \\ &+ \int_0^T \left( \int_\Omega u_\varepsilon^{\gamma+p-1}(x, t) dx \right) \left( \int_\Omega K_2(\xi, t) v_\varepsilon^2(\xi, t - \tau_2) d\xi \right) dt \end{aligned}$$

Now, since

$$\iint_{Q_T} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla u_\varepsilon^\gamma dx dt = \gamma \iint_{Q_T} u_\varepsilon^{\gamma-1} |\nabla u_\varepsilon|^p dx dt \geq 0,$$

then

$$\begin{aligned} & \gamma m^{p-1} \left( \frac{p-1}{(p-1)(m-1)-s} \right)^p \left\| \nabla u_\varepsilon^{\frac{(p-1)(m-1)-s}{p-1}} \right\|_{L^p(Q_T)}^p = \gamma m^{p-1} \iint_{Q_T} u_\varepsilon^{(p-1)(m-1)+\gamma-1} |\nabla u_\varepsilon|^p dx dt \\ & \leq \varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla u_\varepsilon^\gamma dx dt + \iint_{Q_T} |\nabla u_\varepsilon^m|^{p-2} \nabla u_\varepsilon^m \nabla u_\varepsilon^\gamma dx dt \\ & \leq \|a\|_{L^\infty(Q_T)} \iint_{Q_T} u_\varepsilon^{\gamma+p-1}(x, t) dx dt - \int_0^T \left( \int_\Omega u_\varepsilon^{\gamma+p-1}(x, t) dx \right) \left( \int_\Omega K_1(\xi, t) u_\varepsilon^2(\xi, t - \tau_1) d\xi \right) dt \\ & + \int_0^T \left( \int_\Omega u_\varepsilon^{\gamma+p-1}(x, t) dx \right) \left( \int_\Omega K_2(\xi, t) v_\varepsilon^2(\xi, t - \tau_2) d\xi \right) dt. \end{aligned} \tag{2.15}$$

Moreover, by the Hölder inequality with  $\beta := p \frac{(p-1)(m-1)-s}{(p-1)(m-1)-ps} > 1$  and the Poincaré inequality, one has

$$\begin{aligned} \iint_{Q_T} u_\varepsilon^{\gamma+p-1} dx dt & \leq |Q_T|^{\frac{1}{\beta'}} \left( \iint_{Q_T} u_\varepsilon^{\frac{(p-1)(m-1)-s}{p-1}} dx dt \right)^{\frac{1}{\beta}} = |Q_T|^{\frac{1}{\beta'}} \left\| u_\varepsilon^{\frac{(p-1)(m-1)-s}{p-1}} \right\|_{L^p(Q_T)}^{\frac{p}{\beta}} \\ & \leq |Q_T|^{\frac{1}{\beta'}} \left( \frac{1}{\mu_p} \right)^{\frac{1}{\beta}} \left\| \nabla u_\varepsilon^{\frac{(p-1)(m-1)-s}{p-1}} \right\|_{L^p(Q_T)}^{\frac{p}{\beta}}. \end{aligned} \tag{2.16}$$

Here  $\beta'$  is such that  $\frac{1}{\beta} + \frac{1}{\beta'} = 1$ . Thus (2.15) and (2.16) imply

$$\begin{aligned} & \gamma m^{p-1} \left( \frac{p-1}{(p-1)(m-1)-s} \right)^p \left\| \nabla u_\varepsilon^{\frac{(p-1)(m-1)-s}{p-1}} \right\|_{L^p(Q_T)}^p \leq \|a\|_{L^\infty(Q_T)} |Q_T|^{\frac{1}{\beta'}} \left( \frac{1}{\mu_p} \right)^{\frac{1}{\beta}} \left\| \nabla u_\varepsilon^{\frac{(p-1)(m-1)-s}{p-1}} \right\|_{L^p(Q_T)}^{\frac{p}{\beta}} \\ & - \int_0^T \left( \int_\Omega u_\varepsilon^{\gamma+p-1}(x, t) dx \right) \left( \int_\Omega K_1(\xi, t) u_\varepsilon^2(\xi, t - \tau_1) d\xi \right) dt \\ & + \int_0^T \left( \int_\Omega u_\varepsilon^{\gamma+p-1}(x, t) dx \right) \left( \int_\Omega K_2(\xi, t) v_\varepsilon^2(\xi, t - \tau_2) d\xi \right) dt. \end{aligned} \tag{2.17}$$

By assumptions, there are constants  $\bar{k}_i \geq 0$ ,  $i = 2, 3$ , such that  $K_i(x, t) \leq \bar{k}_i$  for a.a.  $(x, t) \in Q_T$ . Thus,

by (2.16), (2.17) and Proposition 2.2, we have

$$\begin{aligned}
& \gamma m^{p-1} \left( \frac{p-1}{(p-1)(m-1)-s} \right)^p \left\| \nabla u_\varepsilon^{\frac{(p-1)(m-1)-s}{p-1}} \right\|_{L^p(Q_T)}^p \\
& \leq \|a\|_{L^\infty(Q_T)} |Q_T|^{\frac{1}{\beta'}} \left( \frac{1}{\mu_p} \right)^{\frac{1}{\beta}} \left\| \nabla u_\varepsilon^{\frac{(p-1)(m-1)-s}{p-1}} \right\|_{L^p(Q_T)}^{\frac{p}{\beta}} + \bar{k}_2 |\Omega| R^2 \iint_{Q_T} u_\varepsilon^{\gamma+p-1} dx dt \\
& \leq (\|a\|_{L^\infty(Q_T)} + \bar{k}_2 |\Omega| R^2) |Q_T|^{\frac{1}{\beta'}} \left( \frac{1}{\mu_p} \right)^{\frac{1}{\beta}} \left\| \nabla u_\varepsilon^{\frac{(p-1)(m-1)-s}{p-1}} \right\|_{L^p(Q_T)}^{\frac{p}{\beta}}.
\end{aligned}$$

In particular,

$$\left\| \nabla u_\varepsilon^{\frac{(p-1)(m-1)-s}{p-1}} \right\|_{L^p(Q_T)} \leq M_1,$$

where

$$M_1 := \left( \frac{(\|a\|_{L^\infty(Q_T)} + \bar{k}_2 |\Omega| R^2) |Q_T|^{\frac{1}{\beta'}} \left( \frac{1}{\mu_p} \right)^{\frac{1}{\beta}} [(p-1)(m-1)-s]^p}{[m(p-1)]^{p-1} [(p-1)(m-p)-ps]} \right)^{\frac{\beta}{p(\beta-1)}}.$$

Analogously, one can prove that

$$\left\| \nabla v_\varepsilon^{\frac{(q-1)(n-1)-s}{q-1}} \right\|_{L^q(Q_T)} \leq M_2,$$

where

$$M_2 := \left( \frac{(\|b\|_{L^\infty(Q_T)} + \bar{k}_3 |\Omega| R^2) |Q_T|^{\frac{1}{\delta'}} \left( \frac{1}{\mu_q} \right)^{\frac{1}{\delta}} [(q-1)(n-1)-s]^q}{[n(q-1)]^{q-1} [(q-1)(n-q)-qs]} \right)^{\frac{\delta}{q(\delta-1)}}.$$

Here  $\delta$  and  $\delta'$  are such that  $\delta := q \frac{(q-1)(n-1)-s}{(q-1)(n-1)-qs}$  and  $\frac{1}{\delta} + \frac{1}{\delta'} = 1$ .  $\square$

**Remark 2.2.** Observe that in the case when  $p = q = 2$  a priori bounds for  $\|\nabla u_\varepsilon^m\|_{L^2(Q_T)}$ ,  $\|\nabla v_\varepsilon^n\|_{L^2(Q_T)}$  have been obtained in [25] for sufficiently small  $\varepsilon > 0$  under the conditions that  $m, n > 1$ , i. e. in the case of slow diffusion. Under the same condition  $m, n > 1$  a priori bounds for  $\|\nabla u_\varepsilon^m\|_{L^p(Q_T)}$ ,  $\|\nabla v_\varepsilon^n\|_{L^q(Q_T)}$  have been obtained in [26] when  $p, q > 2$ , which again corresponds to the case of slow diffusion. Therefore the assumptions  $m > p$  and  $n > q$  are required in Lemma 2.4 only for the singular case  $p, q \in (1, 2)$ , which, as already noticed, allows the fast diffusion if  $p, q \in \left(1, \frac{1+\sqrt{5}}{2}\right)$ . Finally, observe that, if  $p, q > 1$ , we have the fast diffusion when  $m < 1/(p-1)$  and  $n < 1/(q-1)$ : to the best of our knowledge, this case is not treated in the existing literature devoted to this problem.

The following result guarantees that the foreseen solutions  $(u_\varepsilon, v_\varepsilon)$  of (2.1) are not bifurcating from the trivial solution  $(0, 0)$  as  $\varepsilon$  ranges in  $(0, \varepsilon_0)$ , where  $\varepsilon_0$  is such that

$$\begin{aligned}
\theta(C_1, C_2) := \min & \left\{ \frac{1}{T} \iint_{Q_T} a(x, t) e_p^p(x) dx dt - \varepsilon_0 \mu_p - \frac{k_2 C_2}{T}, \right. \\
& \left. \frac{1}{T} \iint_{Q_T} b(x, t) e_q^q(x) dx dt - \varepsilon_0 \mu_q - \frac{k_3 C_1}{T} \right\} > 0,
\end{aligned} \tag{2.18}$$

where  $\mu_p, \mu_q, e_p, e_q, k_2$  and  $k_3$  are as in Hypotheses 2.2.



To this aim let

$$r_0 := \min \left\{ \left( \frac{\iint_{Q_T} a(x, t) e_p^p(x) dx dt - \varepsilon_0 T \mu_p}{D_1} \right)^{\frac{1}{2}}, \left( \frac{\iint_{Q_T} a(x, t) e_p^p(x) dx dt - \varepsilon_0 T \mu_p}{D_1} \right)^{\frac{1}{s}}, \right. \\ \left. \left( \frac{\iint_{Q_T} b(x, t) e_q^q(x) dx dt - \varepsilon_0 T \mu_q}{D_2} \right)^{\frac{1}{2}}, \left( \frac{\iint_{Q_T} b(x, t) e_q^q(x) dx dt - \varepsilon_0 T \mu_q}{D_2} \right)^{\frac{1}{s}} \right\},$$

where

$$D_1 := \|K_1\|_{L^1(Q_T)} + \|K_2\|_{L^1(Q_T)} + p \|e_p^{p-1}\|_{L^\infty(\Omega)} \|\nabla e_p\|_{L^\infty(\Omega)} \left( \frac{m(p-1)M_1}{(p-1)(m-1)-s} \right)^{p-1} |Q_T|^{\frac{1}{p}},$$

$$D_2 := \|K_3\|_{L^1(Q_T)} + \|K_4\|_{L^1(Q_T)} + q \|e_q^{q-1}\|_{L^\infty(\Omega)} \|\nabla e_q\|_{L^\infty(\Omega)} \left( \frac{n(q-1)M_2}{(q-1)(n-1)-s} \right)^{q-1} |Q_T|^{\frac{1}{q}}$$

and  $M_1, M_2, s$  are as in Lemma 2.4. By (2.18),  $r_0$  is obviously positive.

**Proposition 2.3.** *For all solution pairs  $(u_\varepsilon, v_\varepsilon)$  of (2.11) and all  $\varepsilon \in (0, \varepsilon_0)$ , it results*

$$\max\{\|u_\varepsilon\|_{L^\infty(Q_T)}, \|v_\varepsilon\|_{L^\infty(Q_T)}\} \geq r_0.$$

Moreover  $\deg((u, v) - T_\varepsilon(1, u^+, v^+), B_r, 0) = 0$  for all  $r \in (0, r_0)$ .

*Proof.* By contradiction, assume that for some  $r \in (0, r_0)$  there exists a pair  $(u_\varepsilon, v_\varepsilon) \neq (0, 0)$  such that  $(u_\varepsilon, v_\varepsilon) = G_\varepsilon(1, f(u_\varepsilon^+, v_\varepsilon^+), g(u_\varepsilon^+, v_\varepsilon^+))$  with  $\|u_\varepsilon\|_{L^\infty(Q_T)} \leq r$  and  $\|v_\varepsilon\|_{L^\infty(Q_T)} \leq r$ . Assume that  $u_\varepsilon \neq 0$  and take  $\phi \in C_c^\infty(\Omega)$ . Since by Proposition 2.1 we have  $u_\varepsilon > 0$  in  $Q_T$ , we can multiply the equation

$$\frac{\partial u_\varepsilon}{\partial t} - \varepsilon \operatorname{div}(|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon) - \operatorname{div}(|\nabla u_\varepsilon^m|^{p-2} \nabla u_\varepsilon^m) = \\ = \left( a(x, t) - \int_\Omega K_1(\xi, t) u_\varepsilon^2(\xi, t - \tau_1) d\xi + \int_\Omega K_2(\xi, t) v_\varepsilon^2(\xi, t - \tau_2) d\xi \right) u_\varepsilon^{p-1}$$

by  $\frac{\phi^p}{u_\varepsilon^{p-1}}$ , integrate over  $Q_T$  and pass to the limit in the Steklov averages in order to obtain

$$- \varepsilon \iint_{Q_T} \frac{\phi^p}{u_\varepsilon^{p-1}} \operatorname{div}(|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon) dx dt - \iint_{Q_T} \frac{\phi^p}{u_\varepsilon^{p-1}} \operatorname{div}(|\nabla u_\varepsilon^m|^{p-2} \nabla u_\varepsilon^m) dx dt = \iint_{Q_T} \phi^p(x) a(x, t) dx dt - \\ - \iint_{Q_T} \phi^p(x) \left( \int_\Omega K_1(\xi, t) u_\varepsilon^2(\xi, t - \tau_1) d\xi \right) dx dt + \iint_{Q_T} \phi^p(x) \left( \int_\Omega K_2(\xi, t) v_\varepsilon^2(\xi, t - \tau_2) d\xi \right) dx dt, \quad (2.19)$$

by the  $T$ -periodicity of  $u_\varepsilon$ . By the generalized Picone's identity due to Allegretto-Huan, see [4], one has

$$- \varepsilon \iint_{Q_T} \frac{\phi^p}{u_\varepsilon^{p-1}} \operatorname{div}(|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon) dx dt = \varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla \left( \frac{\phi^p}{u_\varepsilon^{p-1}} \right) dx dt \leq \varepsilon \iint_{Q_T} |\nabla \phi|^p dx dt. \quad (2.20)$$

Indeed, we have that

$$|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla \left( \frac{\phi^p}{u_\varepsilon^{p-1}} \right) \leq p |\nabla \phi| \left( \frac{\phi}{u_\varepsilon} |\nabla u_\varepsilon| \right)^{p-1} - (p-1) \left( \frac{\phi}{u_\varepsilon} |\nabla u_\varepsilon| \right)^p \\ = \left( \frac{\phi}{u_\varepsilon} |\nabla u_\varepsilon| \right)^p + p \left( \frac{\phi}{u_\varepsilon} |\nabla u_\varepsilon| \right)^{p-1} \left( |\nabla \phi| - \frac{\phi}{u_\varepsilon} |\nabla u_\varepsilon| \right) \\ \leq |\nabla \phi|^p,$$

since the function  $\mathbb{R} \ni \xi \mapsto |\xi|^p$  is convex for  $p > 1$ .

Moreover,

$$\begin{aligned}
& - \iint_{Q_T} \frac{\phi^p}{u_\varepsilon^{p-1}} \operatorname{div}(\nabla u_\varepsilon^m |\nabla u_\varepsilon^m|^{p-2}) dx dt = \iint_{Q_T} \nabla \left( \frac{\phi^p}{u_\varepsilon^{p-1}} \right) \nabla u_\varepsilon^m |\nabla u_\varepsilon^m|^{p-2} dx dt \\
& = m^{p-1} \iint_{Q_T} u_\varepsilon^{(m-1)(p-1)} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \frac{u_\varepsilon^{p-1} \nabla \phi^p - \phi^p \nabla u_\varepsilon^{p-1}}{u_\varepsilon^{2(p-1)}} dx dt \\
& = pm^{p-1} \iint_{Q_T} \phi^{p-1} u_\varepsilon^{(p-1)(m-2)} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla \phi dx dt - m^{p-1} (p-1) \iint_{Q_T} \phi^p u_\varepsilon^{(m-2)(p-1)-1} |\nabla u_\varepsilon|^p dx dt \\
& \leq p \|\phi^{p-1}\|_{L^\infty(\Omega)} \|\nabla \phi\|_{L^\infty(\Omega)} m^{p-1} \iint_{Q_T} |\nabla u_\varepsilon|^{p-1} u_\varepsilon^{(p-1)(m-2)} dx dt.
\end{aligned} \tag{2.21}$$

By (2.19)-(2.21), it follows

$$\begin{aligned}
& \iint_{Q_T} \phi^p(x) a(x, t) dx dt - \iint_{Q_T} \phi^p(x) \left( \int_\Omega K_1(\xi, t) u_\varepsilon^2(\xi, t - \tau_1) d\xi \right) dx dt \\
& + \iint_{Q_T} \phi^p(x) \left( \int_\Omega K_2(\xi, t) v_\varepsilon^2(\xi, t - \tau_2) d\xi \right) dx dt \\
& \leq p \|\phi^{p-1}\|_{L^\infty(\Omega)} \|\nabla \phi\|_{L^\infty(\Omega)} m^{p-1} \iint_{Q_T} |\nabla u_\varepsilon|^{p-1} u_\varepsilon^{(p-1)(m-2)} dx dt + \varepsilon \iint_{Q_T} |\nabla \phi|^p dx dt.
\end{aligned}$$

Taking  $\phi(x) = \phi_j(x) \rightarrow e_p(x)$  in  $C_0^1(\Omega)$  as  $j \rightarrow +\infty$  and since  $\varepsilon < \varepsilon_0$ , one has

$$\begin{aligned}
& \iint_{Q_T} e_p^p(x) a(x, t) dx dt - \iint_{Q_T} e_p^p(x) \left( \int_\Omega K_1(\xi, t) u_\varepsilon^2(\xi, t - \tau_1) d\xi \right) dx dt \\
& + \iint_{Q_T} e_p^p(x) \left( \int_\Omega K_2(\xi, t) v_\varepsilon^2(\xi, t - \tau_2) d\xi \right) dx dt \\
& \leq p \|e_p^{p-1}\|_{L^\infty(\Omega)} \|\nabla e_p\|_{L^\infty(\Omega)} m^{p-1} \iint_{Q_T} |\nabla u_\varepsilon|^{p-1} u_\varepsilon^{(p-1)(m-2)} dx dt + \varepsilon_0 \iint_{Q_T} |\nabla e_p|^p dx dt.
\end{aligned}$$

Taking into account that  $\|e_p\|_{L^p(\Omega)} = 1$ , the previous inequality imply

$$\begin{aligned}
& \iint_{Q_T} e_p^p(x) a(x, t) dx dt - \varepsilon_0 T \mu_p \leq \iint_{Q_T} K_1(\xi, t) u_\varepsilon^2(\xi, t - \tau_1) d\xi dt - \iint_{Q_T} K_2(\xi, t) v_\varepsilon^2(\xi, t - \tau_2) d\xi dt \\
& + p \|e_p^{p-1}\|_{L^\infty(\Omega)} \|\nabla e_p\|_{L^\infty(\Omega)} m^{p-1} \iint_{Q_T} |\nabla u_\varepsilon|^{p-1} u_\varepsilon^{(p-1)(m-2)} dx dt.
\end{aligned} \tag{2.22}$$

Now we estimate the term  $\iint_{Q_T} |\nabla u_\varepsilon|^{p-1} u_\varepsilon^{(p-1)(m-2)} dx dt$ . Since  $\|u_\varepsilon\|_{L^\infty(Q_T)} \leq r$ , using the Hölder inequality, one has

$$\begin{aligned}
& \iint_{Q_T} |\nabla u_\varepsilon|^{p-1} u_\varepsilon^{(p-1)(m-2)} dx dt \leq r^s \iint_{Q_T} |\nabla u_\varepsilon|^{p-1} u_\varepsilon^{(p-1)(m-2)-s} dx dt \\
& = r^s \left( \frac{p-1}{(p-1)(m-1)-s} \right)^{p-1} \iint_{Q_T} \left| \nabla u_\varepsilon^{\frac{(p-1)(m-1)-s}{p-1}} \right|^{p-1} dx dt \\
& \leq r^s \left( \frac{p-1}{(p-1)(m-1)-s} \right)^{p-1} |Q_T|^{\frac{1}{p}} \left\| \nabla u_\varepsilon^{\frac{(p-1)(m-1)-s}{p-1}} \right\|_{L^p(Q_T)}^{p-1}.
\end{aligned} \tag{2.23}$$

Observe that  $(p-1)(m-1)-s > 0$ , since, by assumption,  $s < \frac{(p-1)(m-p)}{p} < (p-1)(m-1)$ .

By Lemma 2.4, (2.22) and (2.23), it follows

$$\begin{aligned} \iint_{Q_T} e_p^p(x)a(x,t)dxdt - \varepsilon_0 T \mu_p &\leq (\|K_1\|_{L^1(Q_T)} + \|K_2\|_{L^1(Q_T)})r^2 \\ &+ p\|e_p^{p-1}\|_{L^\infty(\Omega)}\|\nabla e_p\|_{L^\infty(\Omega)} \left( \frac{m(p-1)M_1}{(p-1)(m-1)-s} \right)^{p-1} |Q_T|^{\frac{1}{p}} r^s \\ &\leq (\|K_1\|_{L^1(Q_T)} + \|K_2\|_{L^1(Q_T)} + C) \max\{r^2, r^s\}, \end{aligned}$$

where  $C := p\|e_p^{p-1}\|_{L^\infty(\Omega)}\|\nabla e_p\|_{L^\infty(\Omega)} \left( \frac{m(p-1)M_1}{(p-1)(m-1)-s} \right)^{p-1} |Q_T|^{\frac{1}{p}}$ .

Thus, if  $\max\{r^2, r^s\} = r^2$ , then

$$r_0 \leq \left( \frac{\iint_{Q_T} e_p^p(x)a(x,t)dxdt - \varepsilon_0 \mu_p T}{\|K_1\|_{L^1(Q_T)} + \|K_2\|_{L^1(Q_T)} + C} \right)^{\frac{1}{2}} \leq r,$$

that is a contradiction; analogously if  $\max\{r^2, r^s\} = r^s$ . The same argument applies if  $v_\varepsilon \neq 0$ . Fix any  $r \in (0, r_0)$ . The result above shows that

$$(u, v) \neq G_\varepsilon(\sigma, f(u^+, v^+) + (1 - \sigma), g(u^+, v^+) + (1 - \sigma)),$$

for all  $(u, v) \in \partial B_r$  and all  $\sigma \in [0, 1]$ . From the homotopy invariance of the Leray-Schauder degree, we have

$$\deg((u, v) - T_\varepsilon(1, u^+, v^+), B_r, 0) = \deg((u, v) - G_\varepsilon(0, f(u^+, v^+) + 1, g(u^+, v^+) + 1), B_r, 0).$$

The last degree is zero since the equation

$$(u, v) = G_\varepsilon(0, f(u^+, v^+) + 1, g(u^+, v^+) + 1)$$

admits neither trivial nor trivial solution in  $B_r$ , since  $r < r_0$ .  $\square$

The next result is our general coexistence result for (1.1).

**Theorem 2.1.** *Problem (1.1) has a  $T$ -periodic non-negative solution  $(u, v)$  with both non-trivial  $u, v$ .*

*Proof.* By Propositions 2.2 and 2.3 and the excision property of the topological degree, there are  $R > r > 0$ , independent of  $\varepsilon$ , such that

$$\deg((u_\varepsilon, v_\varepsilon) - G_\varepsilon(1, f(u_\varepsilon^+, v_\varepsilon^+), g(u_\varepsilon^+, v_\varepsilon^+)), B_R \setminus \overline{B}_r, 0) = 1,$$

for any  $\varepsilon \in (0, \varepsilon_0)$ .

Let us fix any  $\varepsilon \in (0, \varepsilon_0)$ . There is  $\sigma_0 = \sigma_0(\varepsilon) \in (0, 1)$  such that still

$$\deg((u_\varepsilon, v_\varepsilon) - G_\varepsilon(\sigma, f(u_\varepsilon^+, v_\varepsilon^+) + (1 - \sigma), g(u_\varepsilon^+, v_\varepsilon^+) + (1 - \sigma)), B_R \setminus \overline{B}_r, 0) = 1$$

for all  $\sigma \in [\sigma_0, 1]$ , by the continuity of Leray-Schauder degree. This implies that the set of solution triples  $(\sigma, u_\varepsilon, v_\varepsilon) \in [0, 1] \times (B_R \setminus \overline{B}_r)$  such that

$$(u_\varepsilon, v_\varepsilon) = G_{\varepsilon\eta}(\sigma, f(u_\varepsilon^+, v_\varepsilon^+) + (1 - \sigma), g(u_\varepsilon^+, v_\varepsilon^+) + (1 - \sigma)) \quad (2.24)$$

contains a continuum  $\mathcal{S}_\varepsilon$  with the property that

$$\mathcal{S}_\varepsilon \cap [\{\sigma\} \times (B_R \setminus \overline{B}_r)] \neq \emptyset \quad \text{for all } \sigma \in [\sigma_0, 1].$$

Now, all the pairs  $(u_\varepsilon, v_\varepsilon)$  such that  $(1, u_\varepsilon, v_\varepsilon) \in \mathcal{S}_\varepsilon$  are  $T$ -periodic solutions of (2.1) with  $(u_\varepsilon, v_\varepsilon) \neq (0, 0)$  and, hence, satisfy (2.12). Since the  $L^2$ -norm is continuous with respect to the  $L^\infty$ -norm and  $\mathcal{S}_\varepsilon$  is a continuum, for every  $\nu > 0$  there is  $\sigma_\nu \in [\sigma_0, 1)$  such that

$$\|u_\varepsilon\|_{L^2(Q_T)}^2 \leq C_1 + \nu \quad \text{and} \quad \|v_\varepsilon\|_{L^2(Q_T)}^2 \leq C_2 + \nu$$

for all  $(u_\varepsilon, v_\varepsilon)$  with  $(\sigma, u_\varepsilon, v_\varepsilon) \in \mathcal{S}_\varepsilon$  and  $\sigma \in [\sigma_\nu, 1]$ . Observe that, if  $(\sigma, u_\varepsilon, v_\varepsilon) \in \mathcal{S}_\varepsilon$  for  $\sigma < 1$ , then  $u_\varepsilon$  and  $v_\varepsilon$  are *positive* solutions of (2.24). Moreover, if  $\nu$  is sufficiently small, then we still have  $\theta(C_1 + \nu, C_2 + \nu) > 0$ .

Now, setting

$$K_p := \left[ \|K_1\|_{L^1(Q_T)} + p\|e_p^{p-1}\|_{L^\infty(\Omega)}\|\nabla e_p\|_{L^\infty(\Omega)} \left( \frac{m(p-1)M_1}{(p-1)(m-1)-s} \right)^{p-1} |Q_T|^{\frac{1}{p}} \right]$$

$$K_q := \left[ \|K_4\|_{L^1(Q_T)} + q\|e_q^{q-1}\|_{L^\infty(\Omega)}\|\nabla e_q\|_{L^\infty(\Omega)} \left( \frac{n(q-1)M_2}{(q-1)(n-1)-s} \right)^{q-1} |Q_T|^{\frac{1}{q}} \right],$$

we can prove that, if  $\nu$  is sufficiently small, then

$$\|u_\varepsilon\|_{L^\infty(Q_T)}, \|v_\varepsilon\|_{L^\infty(Q_T)} \geq \min \left\{ \left( \frac{T\theta(C_1 + \nu, C_2 + \nu)}{K_p} \right)^{\frac{1}{2}}, \left( \frac{T\theta(C_1 + \nu, C_2 + \nu)}{K_p} \right)^{\frac{1}{s}}, \right. \\ \left. \left( \frac{T\theta(C_1 + \nu, C_2 + \nu)}{K_q} \right)^{\frac{1}{2}}, \left( \frac{T\theta(C_1 + \nu, C_2 + \nu)}{K_q} \right)^{\frac{1}{s}} \right\} =: \lambda_\nu$$

for all  $u_\varepsilon, v_\varepsilon$  such that  $(\sigma, u_\varepsilon, v_\varepsilon) \in \mathcal{S}_\varepsilon$  and  $\sigma \in [\sigma_\nu, 1)$ . Indeed, let  $(u_\varepsilon, v_\varepsilon)$  be a solution of (2.24). Arguing by contradiction, assume that  $\|u_\varepsilon\|_{L^\infty(Q_T)} < \lambda_\nu$  and proceeding as in the proof of Proposition 2.3 (recall that  $u_\varepsilon > 0$  since  $(u_\varepsilon, v_\varepsilon)$  solves (2.24) with  $\sigma < 1$ ) we obtain the inequality

$$\iint_{Q_T} e_p^p(x)a(x,t)dxdt - \varepsilon_0\mu_p T < \max\{\lambda_\nu^2, \lambda_\nu^s\}K_p + k_2(C_2 + \nu).$$

Thus, if  $\max\{\lambda_\nu^2, \lambda_\nu^s\} = \lambda_\nu^2$ , using the definition of  $\theta$ , one has

$$T\theta(C_1 + \nu, C_2 + \nu) \leq \iint_{Q_T} e_p^p(x)a(x,t)dxdt - \varepsilon_0\mu_p T - k_2(C_2 + \nu) < \lambda_\nu^2 K_p,$$

that is

$$\left( \frac{T\theta(C_1 + \nu, C_2 + \nu)}{K_p} \right)^{\frac{1}{2}} < \lambda_\nu$$

which is a contradiction with the definition of  $\lambda_\nu$ ; analogously if  $\max\{\lambda_\nu^2, \lambda_\nu^s\} = \lambda_\nu^s$ . The same argument shows that  $\|v_\varepsilon\|_{L^\infty(Q_T)} \geq \lambda_\nu$ .

Now, if we let  $\sigma \rightarrow 1$  and  $\nu \rightarrow 0$ , we obtain that (2.1) has at least a solution  $(u_\varepsilon, v_\varepsilon)$  such that  $\|u_\varepsilon\|_{L^\infty(Q_T)}, \|v_\varepsilon\|_{L^\infty(Q_T)} \geq \lambda_0$ , since  $\mathcal{S}_\varepsilon$  is a continuum and  $\lambda_\nu \rightarrow \lambda_0$  as  $\nu \rightarrow 0$ .

Finally, we show that a solution  $(u, v)$  of (1.1) with both non-trivial  $u, v \geq 0$  is obtained as a limit of  $(u_\varepsilon, v_\varepsilon)$  as  $\varepsilon \rightarrow 0$ , since  $\lambda_0$  is independent of  $\varepsilon$ .

Since  $u_\varepsilon, v_\varepsilon$  are Hölder continuous in  $\overline{Q_T}$ , bounded in  $C(\overline{Q_T})$  uniformly in  $\varepsilon > 0$  and the structure conditions of [33] and [34] are satisfied for the equations of system (2.1), whenever  $\varepsilon \in (0, \varepsilon_0)$ , [33, Theorem 1.1] and [34, Theorem 1.3] apply to conclude that the inequality

$$|u_\varepsilon(x_1, t_1) - u_\varepsilon(x_2, t_2)| \leq \Gamma(|x_1 - x_2|^\beta + |t_1 - t_2|^{\frac{\beta}{p}})$$

holds for any  $(x_1, t_1), (x_2, t_2) \in \overline{Q}_T$ , where the constants  $\Gamma > 0$  and  $\beta \in (0, 1)$  are independent of  $\|u_\varepsilon\|_{L^\infty(Q_T)}$ . The same inequality holds for  $v_\varepsilon$ . Therefore, by the Ascoli-Arzelà Theorem, a subsequence of  $(u_\varepsilon, v_\varepsilon)$  converges uniformly in  $\overline{Q}_T$  to a pair  $(u, v)$  satisfying

$$\lambda_0 \leq \|u\|_{L^\infty(Q_T)}, \|v\|_{L^\infty(Q_T)} \leq R.$$

Moreover, from (2.14) we have that  $u_\varepsilon$  satisfies the inequality

$$\frac{\partial u_\varepsilon}{\partial t} - \varepsilon \operatorname{div}(|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon) - \operatorname{div}(|\nabla u_\varepsilon^m|^{p-2} \nabla u_\varepsilon^m) \leq K u_\varepsilon^{p-1}, \quad \text{in } Q_T, \quad (2.25)$$

where  $K$  is a positive constant independent of  $\varepsilon$ . Multiplying (2.25) by  $u_\varepsilon^m$ , integrating over  $Q_T$  and passing to the limit in the Steklov averages  $(u_\varepsilon)_h$ , one has

$$\begin{aligned} \iint_{Q_T} |\nabla u_\varepsilon^m|^p dx dt &\leq \varepsilon m \iint_{Q_T} u_\varepsilon^{m-1} |\nabla u_\varepsilon|^p dx dt + \iint_{Q_T} |\nabla u_\varepsilon^m|^p dx dt \\ &= \varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla u_\varepsilon^m dx dt + \iint_{Q_T} \nabla u_\varepsilon^m \nabla u_\varepsilon^m |\nabla u_\varepsilon^m|^{p-2} dx dt \\ &\leq K \iint_{Q_T} u_\varepsilon^{p+m-1} dx dt \leq M, \end{aligned} \quad (2.26)$$

by the  $T$ -periodicity of  $u_\varepsilon$ , its non-negativity and its boundedness in  $L^\infty(Q_T)$ . Here  $M$  is positive and independent of  $\varepsilon$ . An analogous estimate holds for  $v_\varepsilon$ . Thus the sequences  $u_\varepsilon^m, v_\varepsilon^n$  are uniformly bounded in  $L^p(0, T; W_0^{1,p}(\Omega))$  and in  $L^q(0, T; W_0^{1,q}(\Omega))$ , respectively. Thus, up to subsequence if necessary,  $(u_\varepsilon^m, v_\varepsilon^n)$  converges weakly in  $L^p(0, T; W_0^{1,p}(\Omega)) \times L^q(0, T; W_0^{1,q}(\Omega))$  and strongly in  $C(\overline{Q}_T) \times C(\overline{Q}_T)$  to  $(u^m, v^n)$ . In particular  $(u^m, v^n) \in L^p(0, T; W_0^{1,p}(\Omega)) \times L^q(0, T; W_0^{1,q}(\Omega))$ . We finally claim that the pair  $(u, v)$  satisfies the identities

$$\begin{aligned} 0 &= \iint_{Q_T} \left\{ -u \frac{\partial \varphi}{\partial t} + |\nabla u^m|^{p-2} \nabla u^m \cdot \nabla \varphi - a u^{p-1} \varphi \right. \\ &\quad \left. + u^{p-1} \varphi \int_{\Omega} [K_1(\xi, t) u^2(\xi, t - \tau_1) - K_2(\xi, t) v^2(\xi, t - \tau_2)] d\xi \right\} dx dt \end{aligned}$$

and

$$\begin{aligned} 0 &= \iint_{Q_T} \left\{ -v \frac{\partial \varphi}{\partial t} + |\nabla v^n|^{q-2} \nabla v^n \cdot \nabla \varphi - b v^{q-1} \varphi \right. \\ &\quad \left. + v^{q-1} \varphi \int_{\Omega} [-K_3(\xi, t) u^2(\xi, t - \tau_3) + K_4(\xi, t) v^2(\xi, t - \tau_4)] d\xi \right\} dx dt, \end{aligned}$$

for any  $\varphi \in C^1(\overline{Q}_T)$  such that  $\varphi(x, T) = \varphi(x, 0)$  for any  $x \in \Omega$  and  $\varphi(x, t) = 0$  for any  $(x, t) \in \partial\Omega \times [0, T]$ , that is  $(u, v)$  is a generalized solution of (1.1). The approach for doing this is standard, in the sequel we write it in detail for the reader's convenience. First of all, observe that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla \varphi dx dt = 0 \quad (2.27)$$

for all test functions  $\varphi$ . In fact, multiplying the equation

$$\begin{aligned} \frac{\partial u_\varepsilon}{\partial t} - \varepsilon \operatorname{div}(|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon) - \operatorname{div}(|\nabla u_\varepsilon^m|^{p-2} \nabla u_\varepsilon^m) &= \left( a(x, t) - \int_{\Omega} K_1(\xi, t) u_\varepsilon^2(\xi, t - \tau_1) d\xi + \right. \\ &\quad \left. + \int_{\Omega} K_2(\xi, t) v_\varepsilon^2(\xi, t - \tau_2) d\xi \right) u_\varepsilon^{p-1} \end{aligned}$$

by  $u_\varepsilon$ , integrating over  $Q_T$ , using the  $T$ -periodicity of  $u_\varepsilon$  and its non-negativity and passing, as  $h \rightarrow 0$ , to the limit in the Steklov averages  $(u_\varepsilon)_h$ , we obtain

$$\begin{aligned} \|\sqrt[p]{\varepsilon} \nabla u_\varepsilon\|_{L^p(Q_T)}^p &= \varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^p dx dt \leq \varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^p dx dt + m^{p-1} \iint_{Q_T} u_\varepsilon^{(m-1)(p-1)} |\nabla u_\varepsilon|^p dx dt \\ &\leq \varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^p dx dt + \iint_{Q_T} |\nabla u_\varepsilon^m|^{p-2} \nabla u_\varepsilon^m \nabla u_\varepsilon dx dt \leq C, \end{aligned}$$

where  $C := (\|a\|_{L^\infty(Q_T)} + \|K_2\|_{L^\infty(Q_T)} R^2) |Q_T|^{1-\frac{p}{2}} C_1^{\frac{p}{2}}$  (recall that, by assumption, being  $p < 2$ ,  $\|u_\varepsilon\|_{L^p(Q_T)} \leq |Q_T|^{\frac{1}{p}-\frac{1}{2}} \|u_\varepsilon\|_{L^2(Q_T)} \leq |Q_T|^{\frac{1}{p}-\frac{1}{2}} \sqrt{C_1}$ ). Thus, by the Hölder inequality,

$$\begin{aligned} \left| \varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla \varphi dx dt \right| &\leq \iint_{Q_T} \varepsilon^{\frac{1}{p'}} |\nabla u_\varepsilon|^{p-1} \varepsilon^{\frac{1}{p}} |\nabla \varphi| dx dt \\ &\leq \|\sqrt[p]{\varepsilon} \nabla u_\varepsilon\|_{L^p(Q_T)}^{\frac{p}{p'}} \varepsilon^{\frac{1}{p}} \|\nabla \varphi\|_{L^p(Q_T)} \leq \sqrt[p]{\varepsilon} \sqrt[p']{C} \|\nabla \varphi\|_{L^p(Q_T)} \rightarrow 0 \end{aligned}$$

as  $\varepsilon \rightarrow 0$ , for all test functions  $\varphi$ .

In what follows we will prove that

$$\lim_{\varepsilon \rightarrow 0} \iint_{Q_T} |\nabla u_\varepsilon^m|^{p-2} \nabla u_\varepsilon^m \cdot \nabla \varphi dx dt = \iint_{Q_T} |\nabla u^m|^{p-2} \nabla u^m \cdot \nabla \varphi dx dt, \quad (2.28)$$

for all test functions  $\varphi$ . To this aim, observe that  $|\nabla u_\varepsilon^m|^{p-2} \nabla u_\varepsilon^m$  is bounded in  $\left(L^{\frac{p}{p-1}}(Q_T)\right)^N$ . In fact,

$$\iint_{Q_T} \left| |\nabla u_\varepsilon^m|^{p-2} \nabla u_\varepsilon^m \right|^{\frac{p}{p-1}} dx dt = \iint_{Q_T} |\nabla u_\varepsilon^m|^p dx dt \leq M,$$

as proved in (2.26). Thus there exists  $H \in \left(L^{\frac{p}{p-1}}(Q_T)\right)^N$  such that  $|\nabla u_\varepsilon^m|^{p-2} \nabla u_\varepsilon^m$  weakly converges to  $H$  in  $\left(L^{\frac{p}{p-1}}(Q_T)\right)^N$  as  $\varepsilon \rightarrow 0$ . Now, using (2.27), it is easy to prove that

$$\begin{aligned} 0 &= \iint_{Q_T} \left\{ -u \frac{\partial \varphi}{\partial t} + H \cdot \nabla \varphi - a u^{p-1} \varphi \right. \\ &\quad \left. + u^{p-1} \varphi \int_{\Omega} [K_1(\xi, t) u^2(\xi, t - \tau_1) - K_2(\xi, t) v^2(\xi, t - \tau_2)] d\xi \right\} dx dt \end{aligned} \quad (2.29)$$

for any  $\varphi \in C^1(\overline{Q_T})$  such that  $\varphi(x, T) = \varphi(x, 0)$  for any  $x \in \Omega$  and  $\varphi(x, t) = 0$  for any  $(x, t) \in \partial\Omega \times [0, T]$  (and, by density, for any  $T$ -periodic  $\varphi \in L^p(0, T; W_0^{1,p}(\Omega)) \cap C(\overline{Q_T})$ ). For this it remains to prove that for every  $\varphi \in C^1(\overline{Q_T})$

$$\iint_{Q_T} |\nabla u^m|^{p-2} \nabla u^m \cdot \nabla \varphi dx dt = \iint_{Q_T} H \cdot \nabla \varphi dx dt. \quad (2.30)$$

Consider the vector function  $H(Y) := |Y|^{p-2} Y$ . Then

$$H'(Y) = |Y|^{p-2} I + (p-2)|Y|^{p-4} Y \otimes Y$$

is a positive definite matrix and, taken  $w \in L^p(0, T; W_0^{1,p}(\Omega))$ , there exists a vector  $Y$  such that

$$\begin{aligned} 0 &\leq \langle H'(Y)(\nabla u_\varepsilon^m - \nabla w), \nabla u_\varepsilon^m - \nabla w \rangle \\ &= \langle H(\nabla u_\varepsilon^m) - H(\nabla w), \nabla u_\varepsilon^m - \nabla w \rangle. \end{aligned}$$

The previous inequality implies

$$\iint_{Q_T} \{ |\nabla u_\varepsilon^m|^{p-2} \nabla u_\varepsilon^m - |\nabla w|^{p-2} \nabla w \} \cdot \nabla (u_\varepsilon^m - w) dx dt \geq 0,$$

for all  $w \in L^p(0, T; W_0^{1,p}(\Omega))$ , that is

$$\iint_{Q_T} |\nabla u_\varepsilon^m|^p dxdt - \iint_{Q_T} |\nabla u_\varepsilon^m|^{p-2} \nabla u_\varepsilon^m \cdot \nabla w dxdt - \iint_{Q_T} |\nabla w|^{p-2} \nabla w \cdot \nabla (u_\varepsilon^m - w) dxdt \geq 0,$$

for all  $w \in L^p(0, T; W_0^{1,p}(\Omega))$ . As in (2.26), one has

$$\begin{aligned} \iint_{Q_T} |\nabla u_\varepsilon^m|^p dxdt &\leq \varepsilon m \iint_{Q_T} u^{m-1} |\nabla u_\varepsilon|^p dxdt + \iint_{Q_T} |\nabla u_\varepsilon^m|^p dxdt \\ &\leq \iint_{Q_T} \left[ a - \int_\Omega K_1(\xi, t) u_\varepsilon^2(\xi, t - \tau_1) d\xi + \int_\Omega K_2(\xi, t) v_\varepsilon^2(\xi, t - \tau_2) d\xi \right] u_\varepsilon^{p+m-1} dxdt. \end{aligned}$$

Thus, from the previous two inequalities, we obtain

$$\begin{aligned} &\iint_{Q_T} |\nabla u_\varepsilon^m|^{p-2} \nabla u_\varepsilon^m \cdot \nabla w dxdt + \iint_{Q_T} |\nabla w|^{p-2} \nabla w \cdot \nabla (u_\varepsilon^m - w) dxdt \\ &\leq \iint_{Q_T} \left[ a - \int_\Omega K_1(\xi, t) u_\varepsilon^2(\xi, t - \tau_1) d\xi + \int_\Omega K_2(\xi, t) v_\varepsilon^2(\xi, t - \tau_2) d\xi \right] u_\varepsilon^{p+m-1} dxdt. \end{aligned}$$

Letting  $\varepsilon \rightarrow 0$  and using (2.26), we have

$$\begin{aligned} &\iint_{Q_T} [H \cdot \nabla w + |\nabla w|^{p-2} \nabla w \cdot \nabla (u^m - w)] dxdt \\ &\leq \iint_{Q_T} \left[ a - \int_\Omega K_1(\xi, t) u^2(\xi, t - \tau_1) d\xi + \int_\Omega K_2(\xi, t) v^2(\xi, t - \tau_2) d\xi \right] u^{p+m-1} dxdt. \end{aligned} \tag{2.31}$$

(Observe that, being  $p > 1$ ,  $\nabla u_\varepsilon^m$  is also bounded in  $L^1(Q_T)$ .)

On the other hand, by density we can take  $u^m = \varphi$  in (2.29) and obtain

$$\begin{aligned} &\iint_{Q_T} H \cdot \nabla u^m dxdt \\ &= \iint_{Q_T} \left[ a - \int_\Omega K_1(\xi, t) u^2(\xi, t - \tau_1) d\xi + \int_\Omega K_2(\xi, t) v^2(\xi, t - \tau_2) d\xi \right] u^{p+m-1} dxdt. \end{aligned}$$

This equality together with (2.31) implies

$$0 \leq \iint_{Q_T} (H - |\nabla w|^{p-2} \nabla w) \cdot \nabla (u^m - w) dxdt. \tag{2.32}$$

Taking  $w := u^m - \lambda\varphi$ , with  $\lambda > 0$  and  $\varphi \in C^1(\overline{Q_T})$ , we get

$$0 \leq \iint_{Q_T} (H - |\nabla(u^m - \lambda\varphi)|^{p-2} \nabla(u^m - \lambda\varphi)) \cdot \nabla \varphi dxdt.$$

Letting  $\lambda \rightarrow 0$  yields

$$0 \leq \iint_{Q_T} (H - |\nabla u^m|^{p-2} \nabla u^m) \cdot \nabla \varphi dxdt.$$

If in (2.32) we take  $w := u^m + \lambda\varphi$ , with  $\lambda > 0$ ,  $\varphi \in C^1(\overline{Q_T})$  and letting again  $\lambda \rightarrow 0$ , then

$$\iint_{Q_T} (H - |\nabla u^m|^{p-2} \nabla u^m) \cdot \nabla \varphi dxdt \leq 0.$$

Thus (2.30) holds and (2.28) is proved.  $\square$

Obviously, the previous result holds also for a single equation. In particular, we have the following corollary:

**Corollary 2.1.** *Consider the problem*

$$\begin{cases} u_t - \operatorname{div}(|\nabla u^m|^{p-2} \nabla u^m) = (a(x, t) - \int_{\Omega} K(\xi, t) u^2(\xi, t - \tau) d\xi) u^{p-1}, & \text{in } Q_T, \\ u(x, t) = 0, & \text{for } (x, t) \in \partial\Omega \times (0, T), \\ u(\cdot, 0) = u(\cdot, T), \end{cases} \quad (2.33)$$

and assume that

1. the exponents  $p, m$  are such that  $p \in (1, 2)$  and  $m > p$ ,
2. the delay  $\tau \in (0, +\infty)$ ,
3. the functions  $a$  and  $K$  belong to  $L^\infty(Q_T)$ , are extended to  $\Omega \times \mathbb{R}$  by  $T$ -periodicity and are non-negative for a.a.  $(x, t) \in Q_T$ ,
4. there exists a positive constant  $C$  such that for all  $\varepsilon > 0$  and all the non-negative solutions  $u_\varepsilon$  of

$$u = G_\varepsilon(1, f(u^+)),$$

it results

$$\|u_\varepsilon\|_{L^2(Q_T)}^2 \leq C.$$

Then problem (2.33) has a  $T$ -periodic non-negative and non-trivial solution.

Here  $G_\varepsilon(1, f(u^+))$  is defined as before.

### 3 A priori bounds in $L^2(Q_T)$

In this section we apply Theorem 2.1 by looking for explicit a priori bounds in  $L^2(Q_T)$  for the solutions of the approximating problem (2.1) in different situations. More precisely, under different assumptions on the kernels  $K_i, i = 1, 2, 3, 4$  which model the interactions between the quantities  $u, v$ , we determine the constants  $C_1, C_2$  of (2.12) in an explicit form. For this we consider two main different cases. In the first one, which we call the “coercive case”, we assume that  $K_i(x, t) \geq \underline{k}_i > 0$  a.a. in  $Q_T$  for  $i = 1, 4$ . In the second one, the “non-coercive case”, we allow the non-negative functions  $K_1, K_4$  to vanish on sets with positive measure. We distinguish also between cooperative and competitive situations by imposing sign conditions on  $K_2, K_3$  having in mind the biological interpretation of model (1.1).

#### 3.1 The coercive case

**Theorem 3.1.** *Assume that*

1. Hypotheses 2.1 are satisfied
2. there are constants  $\underline{k}_i > 0, i = 1, 4$ , such that

$$K_i(x, t) \geq \underline{k}_i \text{ for } i = 1, 4,$$

for a.a.  $(x, t) \in Q_T$ , and  $\underline{k}_1 \underline{k}_4 > \bar{k}_2 \bar{k}_3$ , where  $\bar{k}_2, \bar{k}_3$  are as in Hypothesis 2.1

3. Hypothesis 2.2.2 is satisfied with

$$\begin{aligned} C_1 &= \frac{T(\underline{k}_4 \|a\|_{L^\infty(Q_T)} + \bar{k}_2 \|b\|_{L^\infty(Q_T)})}{\underline{k}_1 \underline{k}_4 - \bar{k}_2 \bar{k}_3} \\ C_2 &= \frac{T(\bar{k}_3 \|a\|_{L^\infty(Q_T)} + \underline{k}_1 \|b\|_{L^\infty(Q_T)})}{\underline{k}_1 \underline{k}_4 - \bar{k}_2 \bar{k}_3}. \end{aligned} \quad (3.1)$$



Then problem (1.1) has a non-negative  $T$ -periodic solution  $(u, v)$  with non-trivial  $u, v$ .

*Proof.* We just need to show that  $\|u_\varepsilon\|_{L^2(Q_T)}^2 \leq C_1$  and  $\|v_\varepsilon\|_{L^2(Q_T)}^2 \leq C_2$  for any solution  $(u_\varepsilon, v_\varepsilon)$  of (2.11). Then, assume  $u_\varepsilon \neq 0$ , thus  $u_\varepsilon > 0$  and  $v_\varepsilon \geq 0$  in  $Q_T$  by Proposition 2.1. Multiplying the inequality

$$\begin{aligned} & \frac{\partial u_\varepsilon}{\partial t} - \varepsilon \operatorname{div}(|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon) - \operatorname{div}(|\nabla u_\varepsilon^m|^{p-2} \nabla u_\varepsilon^m) \\ & \leq \left[ \|a\|_{L^\infty(Q_T)} - \int_\Omega K_1(\xi, t) u_\varepsilon^2(\xi, t - \tau_1) d\xi + \int_\Omega K_2(\xi, t) v_\varepsilon^2(\xi, t - \tau_2) d\xi \right] u_\varepsilon^{p-1} \end{aligned}$$

by  $u_\varepsilon$ , integrating over  $\Omega$  and using the Steklov averages  $(u_\varepsilon)_h \in H^1(Q_{T-\delta})$ ,  $\delta, h > 0$ , we obtain

$$\begin{aligned} & \frac{\frac{1}{2} \frac{d}{dt} \int_\Omega (u_\varepsilon)_h^2 dx + \varepsilon \int_\Omega |\nabla (u_\varepsilon)_h|^p dx + m^{p-1} \int_\Omega (u_\varepsilon)_h^{(m-1)(p-1)} |\nabla (u_\varepsilon)_h|^p dx}{\int_\Omega (u_\varepsilon)_h^p dx} \\ & \leq \left( \|a\|_{L^\infty(Q_T)} - \int_\Omega K_1(\xi, t) (u_\varepsilon)_h^2(\xi, t - \tau_1) d\xi + \int_\Omega K_2(\xi, t) (v_\varepsilon)_h^2(\xi, t - \tau_2) d\xi \right). \end{aligned}$$

Integrating the previous inequality over  $[0, T]$ , and passing to the limit as  $h \rightarrow 0$ , by the  $T$ -periodicity of  $u_\varepsilon$ , we have that

$$0 < \left( T \|a\|_{L^\infty(Q_T)} - \underline{k}_1 \|u_\varepsilon\|_{L^2(Q_T)}^2 + \bar{k}_2 \|v_\varepsilon\|_{L^2(Q_T)}^2 \right). \quad (3.2)$$

The same procedure, when it is applied to the second equation of (2.1), leads to

$$0 < \left( T \|b\|_{L^\infty(Q_T)} - \underline{k}_4 \|v_\varepsilon\|_{L^2(Q_T)}^2 + \bar{k}_3 \|u_\varepsilon\|_{L^2(Q_T)}^2 \right). \quad (3.3)$$

Hence, if  $u_\varepsilon \neq 0$  and if  $v_\varepsilon \neq 0$ , by the positiveness of the right hand sides of (3.2) and (3.3), we have

$$\begin{aligned} \left( 1 - \frac{\bar{k}_2 \bar{k}_3}{\underline{k}_1 \underline{k}_4} \right) \|u_\varepsilon\|_{L^2(Q_T)}^2 & < \frac{T}{\underline{k}_1} \left( \|a\|_{L^\infty(Q_T)} + \frac{\bar{k}_2}{\underline{k}_4} \|b\|_{L^\infty(Q_T)} \right) \\ \left( 1 - \frac{\bar{k}_2 \bar{k}_3}{\underline{k}_1 \underline{k}_4} \right) \|v_\varepsilon\|_{L^2(Q_T)}^2 & < \frac{T}{\underline{k}_4} \left( \|b\|_{L^\infty(Q_T)} + \frac{\bar{k}_3}{\underline{k}_1} \|a\|_{L^\infty(Q_T)} \right) \end{aligned}$$

for any  $\varepsilon \in (0, \varepsilon_0)$  and the desired bounds follow since  $\bar{k}_2 \bar{k}_3 < \underline{k}_1 \underline{k}_4$ . Obviously, if  $v_\varepsilon \equiv 0$ , then

$$\|u_\varepsilon\|_{L^2(Q_T)}^2 \leq \frac{T}{\underline{k}_1} \|a\|_{L^\infty(Q_T)} \leq C_1.$$

or if  $u_\varepsilon \equiv 0$ , then

$$\|v_\varepsilon\|_{L^2(Q_T)}^2 \leq \frac{T}{\underline{k}_4} \|b\|_{L^\infty(Q_T)} \leq C_2. \quad \square$$

As an immediate consequence of the previous result we obtain the following corollaries for the coercive-cooperative (see Remark 2.1) and the coercive-competitive cases respectively.

**Corollary 3.1.** *Assume that*

1. *Hypotheses 2.1 are satisfied with nontrivial coefficients  $a, b$ ,*
2.  *$0 \leq K_i(x, t)$  for  $i = 2, 3$ , for a.a.  $(x, t) \in Q_T$ ,*
3. *there are constants  $\underline{k}_i > 0$ ,  $i = 1, 4$ , such that*

$$K_i(x, t) \geq \underline{k}_i \text{ for } i = 1, 4,$$

*for a.a.  $(x, t) \in Q_T$ , and  $\underline{k}_1 \underline{k}_4 > \bar{k}_2 \bar{k}_3$ , where  $\bar{k}_2, \bar{k}_3$  are as in Hypothesis 2.1.*

Then problem (1.1) has a non-negative  $T$ -periodic solution  $(u, v)$ .

**Corollary 3.2.** Assume that

1. Hypotheses 2.1 are satisfied,
2.  $K_i(x, t) \leq 0$  for  $i = 2, 3$ , for a.a.  $(x, t) \in Q_T$ ,
3. there are constants  $\underline{k}_i > 0$ ,  $i = 1, 4$ , such that

$$K_i(x, t) \geq \underline{k}_i \text{ for } i = 1, 4,$$

for a.a.  $(x, t) \in Q_T$ ,

4. Hypothesis 2.2.2 is satisfied with

$$C_1 = \frac{T}{\underline{k}_1} \|a\|_{L^\infty(Q_T)}$$

$$C_2 = \frac{T}{\underline{k}_4} \|b\|_{L^\infty(Q_T)}.$$

Then problem (1.1) has a non-negative  $T$ -periodic solution  $(u, v)$ .

We observe that the condition  $\bar{k}_2 \bar{k}_3 < \underline{k}_1 \underline{k}_4$  of Theorem 3.1 is crucial to establish the a priori  $L^2$ -bounds on the solution pairs  $(u_\varepsilon, v_\varepsilon)$  of (2.1). Roughly speaking this condition guarantees that the terms in the equations that contribute to the growth of the respective species do not prevail globally on those limiting the growth.

On the other hand, when the strict positivity of the functions  $K_1$  and  $K_4$  is relaxed, obtaining the needed a priori bounds becomes more difficult (at least with our approach). In fact, we are able to obtain simple a priori bounds in the non-coercive case when the system is competitive, provided that  $\min\{n(q-1), m(p-1)\} \geq 1$ , i.e. when each equation of (1.1) is of slow or normal diffusion type. Otherwise, we have to impose one more technical restriction, i.e.  $\min\left\{m \frac{p-1}{p+1}, n \frac{q-1}{q+1}\right\} \geq 1$  to obtain a result like Theorem 3.1 with no sign condition on the functions  $K_2$  and  $K_3$ .

Obviously, Theorem 3.1 holds also for a single equation. In particular, we have the following corollary:

**Corollary 3.3.** Consider the problem (2.33) and assume that

1. the exponents  $p, m$  are such that  $p \in (1, 2)$  and  $m > p$ ,
2. the delay  $\tau \in (0, +\infty)$ ,
3. the functions  $a$  and  $K$  belong to  $L^\infty(Q_T)$ , are extended to  $\Omega \times \mathbb{R}$  by  $T$ -periodicity and are non-negative for a.a.  $(x, t) \in Q_T$  and there exists a constant  $\underline{k} > 0$  such that

$$K(x, t) \geq \underline{k}$$

for a.a.  $(x, t) \in Q_T$ ,

4. hypothesis 4 of Corollary 2.1 is satisfied with

$$C = \frac{T}{\underline{k}} \|a\|_{L^\infty(Q_T)}.$$

Then problem (2.33) has a  $T$ -periodic non-negative and non-trivial solution.

### 3.2 The non-coercive case: the competitive system

**Theorem 3.2.** *Assume that*

1. *Hypotheses 2.1 are satisfied*
2.  $\bar{k}_2 = \bar{k}_3 = 0$ , *that is*

$$K_i(x, t) \leq 0 \text{ for } i = 2, 3,$$

*for a.a.  $(x, t) \in Q_T$ ,*

3.  $m, n, p$  and  $q$  are such that  $m \geq \frac{1}{p-1}$  and  $n \geq \frac{1}{q-1}$ , *i.e. both the equations of system (1.1) have slow or normal diffusion,*
4. *Hypothesis 2.2.2 is satisfied with*

$$C_1 = \frac{|Q_T|^{\frac{m(p-1)-1}{(p-1)(m-1)}}}{\mu_p^{\frac{2}{(p-1)(m-1)}}} \left( \frac{|\Omega|^{1-\frac{p}{2}} \|a\|_{L^\infty(Q_T)} (m(p-1)+1)^p}{m^{p-1} p^p} \right)^{\frac{2}{(p-1)(m-1)}}$$

*and*

$$C_2 = \frac{|Q_T|^{\frac{n(q-1)-1}{(q-1)(n-1)}}}{\mu_q^{\frac{2}{(q-1)(n-1)}}} \left( \frac{|\Omega|^{1-\frac{q}{2}} \|b\|_{L^\infty(Q_T)} (n(q-1)+1)^q}{n^{q-1} q^q} \right)^{\frac{2}{(q-1)(n-1)}}.$$

*Then problem (1.1) has a  $T$ -periodic non-negative solution  $(u, v)$  with non-trivial  $u, v$ .*

*Proof.* As a first step we find the bound for the non-negative solutions  $u_\varepsilon$ . Multiplying the first equation of (2.11) by  $u_\varepsilon$ , integrating in  $Q_T$  and passing to the limit, as  $h \rightarrow 0$ , in the Steklov averages  $(u_\varepsilon)_h$ , we obtain

$$\begin{aligned} m^{p-1} \left( \frac{p}{m(p-1)+1} \right)^p \iint_{Q_T} \left| \nabla u_\varepsilon^{\frac{m(p-1)+1}{p}} \right|^p dx dt &\leq \varepsilon \iint_{Q_T} |\nabla u_\varepsilon|^p dx dt + \iint_{Q_T} |\nabla u_\varepsilon^m|^{p-2} \nabla u_\varepsilon^m \nabla u_\varepsilon dx dt \\ &\leq \|a\|_{L^\infty(Q_T)} \|u_\varepsilon\|_{L^p(Q_T)}^p \leq |\Omega|^{1-\frac{p}{2}} \|a\|_{L^\infty(Q_T)} \|u_\varepsilon\|_{L^2(Q_T)}^p, \end{aligned} \quad (3.4)$$

by the  $T$ -periodicity of  $u_\varepsilon$  and the non-positivity of the function  $K_2$ . Using the Hölder inequality with  $r := \frac{m(p-1)+1}{2}$ , and the Poincaré inequality, one has:

$$\begin{aligned} \iint_{Q_T} u_\varepsilon^2 dx dt &\leq |Q_T|^{\frac{m(p-1)-1}{m(p-1)+1}} \left( \iint_{Q_T} u_\varepsilon^{m(p-1)+1} dx dt \right)^{\frac{2}{m(p-1)+1}} = |Q_T|^{\frac{m(p-1)-1}{m(p-1)+1}} \left\| u_\varepsilon^{\frac{m(p-1)+1}{p}} \right\|_{L^p(Q_T)}^{\frac{2p}{m(p-1)+1}} \\ &\leq |Q_T|^{\frac{m(p-1)-1}{m(p-1)+1}} \left( \frac{1}{\sqrt[p]{\mu_p}} \left\| \nabla u_\varepsilon^{\frac{m(p-1)+1}{p}} \right\|_{L^p(Q_T)} \right)^{\frac{2p}{m(p-1)+1}}. \end{aligned}$$

Thus by (3.4) we get

$$\begin{aligned} \|u_\varepsilon\|_{L^2(Q_T)}^2 &\leq |Q_T|^{\frac{m(p-1)-1}{m(p-1)+1}} \left( \frac{1}{\sqrt[p]{\mu_p}} \left\| \nabla u_\varepsilon^{\frac{m(p-1)+1}{p}} \right\|_{L^p(Q_T)} \right)^{\frac{2p}{m(p-1)+1}} \\ &\leq \frac{|Q_T|^{\frac{m(p-1)-1}{m(p-1)+1}}}{\mu_p^{\frac{2}{m(p-1)+1}}} \left( \frac{|\Omega|^{1-\frac{p}{2}} \|a\|_{L^\infty(Q_T)} (m(p-1)+1)^p}{m^{p-1} p^p} \right)^{\frac{2}{m(p-1)+1}} \|u_\varepsilon\|_{L^2(Q_T)}^{\frac{2p}{m(p-1)+1}}. \end{aligned}$$

This implies

$$\|u_\varepsilon\|_{L^2(Q_T)}^2 \leq \frac{|Q_T|^{\frac{m(p-1)-1}{(p-1)(m-1)}}}{\mu_p^{\frac{2}{(p-1)(m-1)}}} \left( \frac{|\Omega|^{1-\frac{p}{2}} \|a\|_{L^\infty(Q_T)} (m(p-1)+1)^p}{m^{p-1} p^p} \right)^{\frac{2}{(p-1)(m-1)}}.$$

In an analogous way we obtain that

$$\|v_\varepsilon\|_{L^2(Q_T)}^2 \leq \frac{|Q_T|^{\frac{n(q-1)-1}{(q-1)(n-1)}}}{\mu_q^{\frac{2}{(q-1)(n-1)}}} \left( \frac{|\Omega|^{1-\frac{q}{2}} \|b\|_{L^\infty(Q_T)} (n(q-1)+1)^q}{n^{q-1} q^q} \right)^{\frac{2}{(q-1)(n-1)}},$$

if  $v_\varepsilon$  is a solution of the second equation of (2.11).  $\square$

The previous result still holds for a single equation:

**Corollary 3.4.** *Consider problem (2.33) and assume that*

1. *the exponents  $p, m$  are such that  $p \in (1, 2)$  and  $m \geq \frac{1}{p-1}$ ,*
2. *the delay  $\tau \in (0, +\infty)$ ,*
3. *the functions  $a$  and  $K$  belong to  $L^\infty(Q_T)$ , are extended to  $\Omega \times \mathbb{R}$  by  $T$ -periodicity and are non-negative for a.a.  $(x, t) \in Q_T$ ,*
4. *hypothesis 4 of Corollary 2.1 is satisfied with*

$$C = \frac{|Q_T|^{\frac{m(p-1)-1}{(p-1)(m-1)}}}{\mu_p^{\frac{2}{(p-1)(m-1)}}} \left( \frac{|\Omega|^{1-\frac{p}{2}} \|a\|_{L^\infty(Q_T)} (m(p-1)+1)^p}{m^{p-1} p^p} \right)^{\frac{2}{(p-1)(m-1)}}.$$

Then problem (2.33) has a  $T$ -periodic non-negative and non-trivial solution.

### 3.3 The non-coercive case: $\min \left\{ m \frac{p-1}{p+1}, n \frac{q-1}{q+1} \right\} \geq 1$

In the case that  $\min \left\{ m \frac{p-1}{p+1}, n \frac{q-1}{q+1} \right\} \geq 1$ , we are able to find explicit bounds (although complicated) without any assumption on the sign of the functions  $K_2, K_3$ , as shown in the next result.

**Theorem 3.3.** *Assume*

1.  $\min \left\{ m \frac{p-1}{p+1}, n \frac{q-1}{q+1} \right\} > 1$ ,
2.  $K_i(x, t) \geq 0$ ,  $i = 1, 4$  and  $K_i(x, t) \leq \bar{k}_i$ ,  $i = 2, 3$  for a.a.  $(x, t) \in Q_T$  and for some positive constants  $\bar{k}_i$ ,  $i = 2, 3$ ,
3. Hypothesis 2.2.2 is satisfied with

$$\begin{aligned} C_1 &= \sqrt{T} \left\{ \frac{(q-1)(n-1)(p-1)(m-1)}{(q-1)(n-1)(p-1)(m-1)-4} \alpha_p + \beta_p^{\frac{(q-1)(n-1)(p-1)(m-1)}{(q-1)(n-1)(p-1)(m-1)-4}} \right\}^{1/2}, \\ C_2 &= \sqrt{T} \left\{ \frac{(q-1)(n-1)(p-1)(m-1)}{(q-1)(n-1)(p-1)(m-1)-4} \alpha_q + \beta_q^{\frac{(q-1)(n-1)(p-1)(m-1)}{(q-1)(n-1)(p-1)(m-1)-4}} \right\}^{1/2}, \end{aligned} \quad (3.5)$$

where

$$\begin{aligned} \alpha_p &:= C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} \left( 2T \|a\|_{L^\infty(Q_T)}^2 \right)^{\frac{2}{(p-1)(m-1)}} \\ &+ C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} \left( 2\bar{k}_2 C_q^{\frac{(q-1)(n-1)+2}{(q-1)(n-1)}} \right)^{\frac{2}{(p-1)(m-1)}} \left( 2T \|b\|_{L^\infty(Q_T)}^2 \right)^{\frac{4}{(q-1)(n-1)(p-1)(m-1)}}, \end{aligned}$$

$$\begin{aligned}
\alpha_q &:= C_q^{\frac{(q-1)(n-1)+2}{(q-1)(n-1)}} \left( 2T \|b\|_{L^\infty(Q_T)}^2 \right)^{\frac{2}{(q-1)(n-1)}} \\
&\quad + C_q^{\frac{(q-1)(n-1)+2}{(q-1)(n-1)}} \left( 2\bar{k}_3^2 C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} \right)^{\frac{2}{(q-1)(n-1)}} \left( 2T \|a\|_{L^\infty(Q_T)}^2 \right)^{\frac{4}{(q-1)(n-1)(p-1)(m-1)}}, \\
\beta_p &:= C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} \left( 2\bar{k}_2^2 C_q^{\frac{(q-1)(n-1)+2}{(q-1)(n-1)}} \right)^{\frac{2}{(p-1)(m-1)}} \left( 2\bar{k}_3^2 \right)^{\frac{4}{(q-1)(n-1)(p-1)(m-1)}},
\end{aligned}$$

and

$$\beta_q := C_q^{\frac{(q-1)(n-1)+2}{(q-1)(n-1)}} \left( 2\bar{k}_3^2 C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} \right)^{\frac{2}{(q-1)(n-1)}} \left( 2\bar{k}_2^2 \right)^{\frac{4}{(q-1)(n-1)(p-1)(m-1)}}.$$

Here

$$C_p := \left( \frac{[(p-1)(m-1)+2]^p |\Omega|^{\frac{(p-1)(m-1)}{2}}}{p^p m^{p-1} (3-p) \mu_p} \right)^{\frac{4}{(p-1)(m-1)+2}} T^{\frac{(p-1)(m-1)-2}{(p-1)(m-1)+2}} \quad (3.6)$$

and

$$C_q := \left( \frac{[(q-1)(n-1)+2]^q |\Omega|^{\frac{(q-1)(n-1)}{2}}}{q^q n^{q-1} (3-q) \mu_q} \right)^{\frac{4}{(q-1)(n-1)+2}} T^{\frac{(q-1)(n-1)-2}{(q-1)(n-1)+2}}. \quad (3.7)$$

Then problem (1.1) has a non-negative  $T$ -periodic solution  $(u, v)$  with non-trivial  $u, v$ .

*Proof.* Let  $(u_\varepsilon, v_\varepsilon)$  be a solution of (2.11). We have, by the Poincaré inequality and the Hölder inequality with  $r := \frac{(p-1)(m-1)+2}{2}$ ,

$$\left( \int_\Omega u_\varepsilon^2 dx \right)^{\frac{(p-1)(m-1)+2}{2}} \leq |\Omega|^{\frac{(p-1)(m-1)}{2}} \int_\Omega u_\varepsilon^{(p-1)(m-1)+2} dx \leq \frac{1}{\mu_p} |\Omega|^{\frac{(p-1)(m-1)}{2}} \int_\Omega \left| \nabla u_\varepsilon^{\frac{(p-1)(m-1)+2}{p}} \right|^p dx.$$

Integrating over  $[0, T]$ , we have:

$$\int_0^T \left( \int_\Omega u_\varepsilon^2 dx \right)^{\frac{(p-1)(m-1)+2}{2}} dt \leq \frac{1}{\mu_p} |\Omega|^{\frac{(p-1)(m-1)}{2}} \iint_{Q_T} \left| \nabla u_\varepsilon^{\frac{(p-1)(m-1)+2}{p}} \right|^p dx dt. \quad (3.8)$$

Now, we estimate the term  $\iint_{Q_T} \left| \nabla u_\varepsilon^{\frac{(p-1)(m-1)+2}{p}} \right|^p dx dt$ . Multiplying the first equation of (2.11) by  $u_\varepsilon^{3-p}$ , integrating in  $Q_T$  and passing to the limit in the Steklov averages  $(u_\varepsilon)_h$ , we obtain by the  $T$ -periodicity of  $u_\varepsilon$

$$\begin{aligned}
&m^{p-1} (3-p) \left[ \frac{p}{(p-1)(m-1)+2} \right]^p \iint_{Q_T} \left| \nabla u_\varepsilon^{\frac{(p-1)(m-1)+2}{p}} \right|^p dx dt \\
&\leq \int_0^T \left[ \|a\|_{L^\infty(Q_T)} + \bar{k}_2 \int_\Omega v_\varepsilon^2(\xi, t - \tau_2) d\xi \right] \left( \int_\Omega u_\varepsilon^2 dx \right) dt \\
&\leq \left[ \int_0^T \left( \|a\|_{L^\infty(Q_T)} + \bar{k}_2 \int_\Omega v_\varepsilon^2(\xi, t - \tau_2) d\xi \right)^2 dt \right]^{\frac{1}{2}} \left[ \int_0^T \left( \int_\Omega u_\varepsilon^2 dx \right)^2 dt \right]^{\frac{1}{2}}.
\end{aligned}$$

Thus

$$\iint_{Q_T} \left| \nabla u_\varepsilon^{\frac{(p-1)(m-1)+2}{p}} \right|^p dx dt \leq M_p \left[ \int_0^T \left( \|a\|_{L^\infty(Q_T)} + \bar{k}_2 \int_\Omega v_\varepsilon^2(\xi, t - \tau_2) d\xi \right)^2 dt \right]^{\frac{1}{2}} \left[ \int_0^T \left( \int_\Omega u_\varepsilon^2 dx \right)^2 dt \right]^{\frac{1}{2}}, \quad (3.9)$$

where  $M_p := \frac{1}{m^{p-1}(3-p)} \left[ \frac{(p-1)(m-1)+2}{p} \right]^p$ . By the Hölder inequality with  $s := \frac{(p-1)(m-1)+2}{4}$  (observe that  $s \geq 1$  by the assumption on  $m$  and  $p$ ) and by (3.8), (3.9), it follows

$$\begin{aligned} \int_0^T \left( \int_{\Omega} u_{\varepsilon}^2 dx \right)^2 dt &\leq T^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)+2}} \left( \int_0^T \left( \int_{\Omega} u_{\varepsilon}^2 dx \right)^{\frac{(p-1)(m-1)+2}{2}} dt \right)^{\frac{4}{(p-1)(m-1)+2}} \\ &\leq C_p \left[ \int_0^T \left( \|a\|_{L^{\infty}(Q_T)} + \bar{k}_2 \int_{\Omega} v_{\varepsilon}^2(\xi, t - \tau_2) d\xi \right)^2 dt \right]^{\frac{2}{(p-1)(m-1)+2}} \left[ \int_0^T \left( \int_{\Omega} u_{\varepsilon}^2 dx \right)^2 dt \right]^{\frac{2}{(p-1)(m-1)+2}}, \end{aligned}$$

where  $C_p$  is the constant defined in (3.6). Therefore, setting  $U = \int_0^T (\int_{\Omega} u_{\varepsilon}^2 dx)^2 dt$ ,  $V = \int_0^T (\int_{\Omega} v_{\varepsilon}^2 dx)^2 dt$ , and using the assumption  $m > \frac{p+1}{p-1}$ , the last inequality implies

$$\begin{aligned} U &\leq C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} \left[ \int_0^T \left( \|a\|_{L^{\infty}(Q_T)} + \bar{k}_2 \int_{\Omega} v_{\varepsilon}^2 dx \right)^2 dt \right]^{\frac{2}{(p-1)(m-1)}} \\ &\leq C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} \left[ 2T \|a\|_{L^{\infty}(Q_T)}^2 + 2\bar{k}_2^2 V \right]^{\frac{2}{(p-1)(m-1)}} \\ &\leq C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} \left[ \left( 2T \|a\|_{L^{\infty}(Q_T)}^2 \right)^{\frac{2}{(p-1)(m-1)}} + \left( 2\bar{k}_2^2 V \right)^{\frac{2}{(p-1)(m-1)}} \right]. \end{aligned}$$

In an analogous way, we can show that

$$V \leq C_q^{\frac{(q-1)(n-1)+2}{(q-1)(n-1)}} \left[ \left( 2T \|b\|_{L^{\infty}(Q_T)}^2 \right)^{\frac{2}{(q-1)(n-1)}} + \left( 2\bar{k}_3^2 U \right)^{\frac{2}{(q-1)(n-1)}} \right],$$

where  $C_q$  is the constant introduced in (3.7). Hence, it results

$$\begin{aligned} U &\leq C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} \left[ \left( 2T \|a\|_{L^{\infty}(Q_T)}^2 \right)^{\frac{2}{(p-1)(m-1)}} + \left( 2\bar{k}_2^2 V \right)^{\frac{2}{(p-1)(m-1)}} \right] \\ &\leq C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} \left( 2T \|a\|_{L^{\infty}(Q_T)}^2 \right)^{\frac{2}{(p-1)(m-1)}} \\ &\quad + C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} \left( 2\bar{k}_2^2 C_q^{\frac{(q-1)(n-1)+2}{(q-1)(n-1)}} \right)^{\frac{2}{(p-1)(m-1)}} \left( 2T \|b\|_{L^{\infty}(Q_T)}^2 \right)^{\frac{4}{(q-1)(n-1)(p-1)(m-1)}} \\ &\quad + C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} \left( 2\bar{k}_2^2 C_q^{\frac{(q-1)(n-1)+2}{(q-1)(n-1)}} \right)^{\frac{2}{(p-1)(m-1)}} \left( 2\bar{k}_3^2 U \right)^{\frac{4}{(q-1)(n-1)(p-1)(m-1)}}. \end{aligned}$$

The last inequality has the form:

$$U \leq \alpha + \beta U^{\frac{4}{(q-1)(n-1)(p-1)(m-1)}}, \quad (3.10)$$

with  $\alpha, \beta > 0$ . Since  $\min \left\{ m^{\frac{p-1}{p+1}}, n^{\frac{q-1}{q+1}} \right\} > 1$  the function  $f(U) := \alpha + \beta U^{\frac{4}{(q-1)(n-1)(p-1)(m-1)}}$  is strictly concave, and then

$$U \leq f(U) \leq f(U_0) + f'(U_0)(U - U_0), \quad (3.11)$$

where  $U_0 := \beta^{\frac{(q-1)(n-1)(p-1)(m-1)}{(q-1)(n-1)(p-1)(m-1)-4}}$ . Using the fact that  $f(U_0) = \alpha + U_0$  and (3.11), one has

$$U \leq \frac{(q-1)(n-1)(p-1)(m-1)}{(q-1)(n-1)(p-1)(m-1)-4} \alpha + \beta^{\frac{(q-1)(n-1)(p-1)(m-1)}{(q-1)(n-1)(p-1)(m-1)-4}}.$$

A final application of Hölder's inequality shows that  $\|u_{\varepsilon}\|_{L^2}^2 \leq T^{1/2} U^{1/2} = C_1$ . The argument for  $v_{\varepsilon}$  proceeds in a similar way.  $\square$

As a consequence of Theorem 3.3 one has the next corollaries for the cooperative and competitive cases, respectively.

**Corollary 3.5.** *Assume that*

1.  $\min \left\{ m \frac{p-1}{p+1}, n \frac{q-1}{q+1} \right\} > 1$ ,
2.  $K_i(x, t) \geq 0$  for  $i = 1, 4$ , for a.a.  $(x, t) \in Q_T$ , and there are positive constants  $\bar{k}_2, \bar{k}_3$  such that
$$0 \leq K_i(x, t) \leq \bar{k}_i \text{ for } i = 2, 3,$$

for a.a.  $(x, t) \in Q_T$ ,

3. Hypothesis 2.2.2 is satisfied with  $C_1$  and  $C_2$  as in (3.5).

Then problem (1.1) has a non-negative  $T$ -periodic solution  $(u, v)$  with non-trivial  $u, v$ .

**Corollary 3.6.** *Assume that*

1.  $\min \left\{ m \frac{p-1}{p+1}, n \frac{q-1}{q+1} \right\} > 1$ ,
2.  $K_i(x, t) \geq 0$  for  $i = 1, 4$ , for a.a.  $(x, t) \in Q_T$ , and there are non-negative constants  $\underline{k}_2, \underline{k}_3$  such that
$$-\underline{k}_i \leq K_i(x, t) \leq 0 \text{ for } i = 2, 3,$$

for a.a.  $(x, t) \in Q_T$ ,

3. Hypothesis 2.2.2 is satisfied with

$$C_1 = \left( TC_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} \left( 2T \|a\|_{L^\infty(Q_T)}^2 \right)^{\frac{2}{(p-1)(m-1)}} \right)^{\frac{1}{2}}$$

and

$$C_2 = \left( TC_q^{\frac{(q-1)(n-1)+2}{(q-1)(n-1)}} \left( 2T \|b\|_{L^\infty(Q_T)}^2 \right)^{\frac{2}{(q-1)(n-1)}} \right)^{\frac{1}{2}},$$

where  $C_p$  and  $C_q$  are as in (3.6) and in (3.7).

Then problem (1.1) has a non-negative  $T$ -periodic solution  $(u, v)$ .

The proof of Theorem 3.3 suggests the following result when  $\min \left\{ m \frac{p-1}{p+1}, n \frac{q-1}{q+1} \right\} = 1$ .

**Theorem 3.4.** *Suppose that Assumptions 2 and 3 of Theorem 3.3 hold true. If, in addition,*

$$\min \left\{ m \frac{p-1}{p+1}, n \frac{q-1}{q+1} \right\} = 1$$

and

$$C_p^{\frac{(p-1)(m-1)+2}{(p-1)(m-1)}} \left( 2\bar{k}_2 C_q^{\frac{(q-1)(n-1)+2}{(q-1)(n-1)}} \right)^{\frac{2}{(p-1)(m-1)}} \left( 2\bar{k}_3 \right)^{\frac{4}{(q-1)(n-1)(p-1)(m-1)}} < 1, \quad (3.12)$$

then problem (1.1) has a non-negative  $T$ -periodic solution  $(u, v)$ .

*Proof.* First note that, if, for instance  $n \frac{q-1}{q+1} = 1$ , then  $(q-1)(n-1) = 2$ , so that the expression in (3.12) can be simplified. Now the proof proceeds as the one of Theorem 3.3 up to inequality (3.10), which now reads

$$U \leq \alpha + \beta U,$$

where  $\beta$  is the left hand side of (3.12). Since  $\beta < 1$ , we obtain the desired upper bound on  $U$ .  $\square$

**Remark 3.1.** Observe that the technique used to prove Theorem 2.1 (or Corollary 2.1), and the a priori estimates in  $L^2(Q_T)$  can be adapted to prove analogous results if we consider system (1.1) with  $p, q \geq 2$ , that is if we consider a double degeneracy (or a single degenerate equation) as in [25], but with a  $p$ -linear term in the right hand side.

## 4 A generalization in the competitive case

The techniques used in the previous sections allow us to prove the existence of a  $T$ -periodic non-negative solution  $(u, v)$  with non-trivial  $u, v$  for the following system:

$$\begin{cases} u_t - \operatorname{div}(|\nabla u^m|^{p-2} \nabla u^m) = \left( a(x, t) - \int_{\Omega} K_1(\xi, t) u^\alpha(\xi, t - \tau_1) d\xi + \int_{\Omega} K_2(\xi, t) v^\alpha(\xi, t - \tau_2) d\xi \right) u^{p-1}, & \text{in } Q_T, \\ v_t - \operatorname{div}(|\nabla v^n|^{q-2} \nabla v^n) = \left( b(x, t) + \int_{\Omega} K_3(\xi, t) u^\alpha(\xi, t - \tau_3) d\xi - \int_{\Omega} K_4(\xi, t) v^\alpha(\xi, t - \tau_4) d\xi \right) v^{q-1}, & \text{in } Q_T, \\ u(x, t) = v(x, t) = 0, & \text{for } (x, t) \in \partial\Omega \times (0, T), \\ u(\cdot, 0) = u(\cdot, T) \text{ and } v(\cdot, 0) = v(\cdot, T), \end{cases} \quad (4.1)$$

where  $\alpha \geq 1$ ,  $K_i(t, x) \leq 0$  ( $i = 2, 3$ ), and  $m, n, p, q, \tau_i$  ( $i = 1, 2, 3, 4$ ),  $a, b$  and  $K_i$  ( $i = 1, 4$ ), are as in Hypotheses 2.1.

### 4.1 The coexistence theorem

As before, one can prove that Lemma 2.1 and Proposition 2.1 still hold for the associated nondegenerate singular  $p$ -Laplacian problem

$$\begin{cases} \frac{\partial u}{\partial t} - \varepsilon \operatorname{div}(|\nabla u|^{p-2} \nabla u) - \operatorname{div}(|\nabla u^m|^{p-2} \nabla u^m) = \left( a(x, t) - \int_{\Omega} K_1(\xi, t) u^\alpha(\xi, t - \tau_1) d\xi + \right. \\ \quad \left. + \int_{\Omega} K_2(\xi, t) v^\alpha(\xi, t - \tau_2) d\xi \right) u^{p-1}, & \text{in } Q_T, \\ \frac{\partial v}{\partial t} - \varepsilon \operatorname{div}(|\nabla v|^{q-2} \nabla v) - \operatorname{div}(|\nabla v^n|^{q-2} \nabla v^n) = \left( b(x, t) + \int_{\Omega} K_3(\xi, t) u^\alpha(\xi, t - \tau_3) d\xi - \right. \\ \quad \left. - \int_{\Omega} K_4(\xi, t) v^\alpha(\xi, t - \tau_4) d\xi \right) v^{q-1}, & \text{in } Q_T, \\ u(\cdot, t)|_{\partial\Omega} = v(\cdot, t)|_{\partial\Omega} = 0, & \text{for a.a. } t \in (0, T), \\ u(\cdot, 0) = u(\cdot, T) \text{ and } v(\cdot, 0) = v(\cdot, T), \end{cases} \quad (4.2)$$

where  $\varepsilon > 0$  is small enough. Moreover the next result holds:

**Proposition 4.1.** *There is a constant  $R > 0$  such that*

$$\|u_\varepsilon\|_{L^\infty(Q_T)}, \|v_\varepsilon\|_{L^\infty(Q_T)} < R$$

for all solution pairs  $(u_\varepsilon, v_\varepsilon)$  of

$$(u, v) = G_\varepsilon(1, f(u^+, v^+), g(u^+, v^+)),$$

and all  $\varepsilon > 0$ . In particular, one has that

$$\deg((u, v) - G_\varepsilon(1, f(u^+, v^+), g(u^+, v^+)), B_R, 0) = 1.$$

Here  $G_\varepsilon$  is defined as in Section 2,

$$f(u^+, v^+) := \left( a - \int_{\Omega} K_1(\xi, \cdot) (u^+)^\alpha(\xi, \cdot - \tau_1) d\xi + \int_{\Omega} K_2(\xi, \cdot) (v^+)^\alpha(\xi, \cdot - \tau_2) d\xi \right) (u^+)^{p-1}$$

and

$$g(u^+, v^+) := \left( b + \int_{\Omega} K_3(\xi, \cdot) (u^+)^\alpha(\xi, \cdot - \tau_3) d\xi - \int_{\Omega} K_4(\xi, \cdot) (v^+)^\alpha(\xi, \cdot - \tau_4) d\xi \right) (v^+)^{q-1},$$



*Proof.* By the first equation of (4.2), we have

$$\frac{\partial u_\varepsilon}{\partial t} - \varepsilon \operatorname{div}(|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon) - \operatorname{div}(\nabla u_\varepsilon^m |\nabla u_\varepsilon^m|^{p-2}) \leq K u_\varepsilon^{p-1},$$

where  $K := \|a\|_{L^\infty(Q_T)}$ . By Lemma 2.2, using the Steklov averages  $(u_\varepsilon)_h \in H^1(Q_{T-\delta})$ ,  $\delta, h > 0$ , and the fact that  $(u_\varepsilon)_h$  converges to  $u_\varepsilon$  in  $L^\infty(Q_T)$ , we conclude that  $\|u_\varepsilon\|_{L^\infty(Q_T)} \leq R_1$  for some  $R_1 > 0$  independent of  $\varepsilon$ . Analogously,  $\|v_\varepsilon\|_{L^\infty(Q_T)} \leq R_2$  for some constant  $R_2 > 0$ . Therefore it is enough to choose  $R > \max\{R_1, R_2\}$ .

The second part of the Proposition follows as in the proof of Proposition 2.2  $\square$

From now on we make the following assumption:

**Hypothesis 4.1.** The functions  $a$  and  $b$  are such that

$$\min \left\{ \frac{1}{T} \iint_{Q_T} a(x, t) e_p^p(x) dx dt - \frac{k_2 C_2}{T}, \frac{1}{T} \iint_{Q_T} b(x, t) e_q^q(x) dx dt - \frac{k_3 C_1}{T} \right\} > 0.$$

Here  $k_2, k_3$  are as in Hypotheses 2.1.3,  $\mu_p, \mu_q, e_p$  and  $e_q$  are as in Section 2.

As before, take  $\varepsilon$  in  $(0, \varepsilon_0)$ , where  $\varepsilon_0$  is such that

$$\theta(C_1, C_2) := \min \left\{ \frac{1}{T} \iint_{Q_T} a(x, t) e_p^p(x) dx dt - \varepsilon_0 \mu_p - \frac{k_2 C_2}{T}, \right. \\ \left. \frac{1}{T} \iint_{Q_T} b(x, t) e_q^q(x) dx dt - \varepsilon_0 \mu_q - \frac{k_3 C_1}{T} \right\} > 0.$$

Then, Lemma 2.4 and Proposition 2.3 still hold with

$$M_1 := \left( \frac{\|a\|_{L^\infty(Q_T)} |Q_T|^{\frac{1}{\beta'}} \left(\frac{1}{\mu_p}\right)^{\frac{1}{\beta}} [(p-1)(m-1)-s]^p}{[m(p-1)]^{p-1} [(p-1)(m-p)-ps]} \right)^{\frac{\beta}{p(\beta-1)}}, \\ M_2 := \left( \frac{\|b\|_{L^\infty(Q_T)} |Q_T|^{\frac{1}{\delta'}} \left(\frac{1}{\mu_q}\right)^{\frac{1}{\delta}} [(q-1)(n-1)-s]^q}{[n(q-1)]^{q-1} [(q-1)(n-q)-qs]} \right)^{\frac{\delta}{q(\delta-1)}},$$

and

$$r_0 := \min \left\{ \left( \frac{\iint_{Q_T} a(x, t) e_p^p(x) dx dt - \varepsilon_0 T \mu_p}{D_1} \right)^{\frac{1}{\alpha}}, \left( \frac{\iint_{Q_T} a(x, t) e_p^p(x) dx dt - \varepsilon_0 T \mu_p}{D_1} \right)^{\frac{1}{s}}, \right. \\ \left. \left( \frac{\iint_{Q_T} b(x, t) e_q^q(x) dx dt - \varepsilon_0 T \mu_q}{D_2} \right)^{\frac{1}{\alpha}}, \left( \frac{\iint_{Q_T} b(x, t) e_q^q(x) dx dt - \varepsilon_0 T \mu_q}{D_2} \right)^{\frac{1}{s}} \right\},$$

where  $\beta, \beta', \delta, \delta', D_1$  and  $D_2$  are as in Section 2. Proceeding as in Theorem 2.1, one can prove that the next coexistence result holds:

**Theorem 4.1.** Assume that there exist two positive constants  $C_1, C_2$  such that for all  $\varepsilon > 0$  and all solution pairs  $(u_\varepsilon, v_\varepsilon)$  of (4.2) it results

$$\|u_\varepsilon\|_{L^\alpha(Q_T)}^\alpha \leq C_1 \text{ and } \|v_\varepsilon\|_{L^\alpha(Q_T)}^\alpha \leq C_2.$$

Then, problem (4.1) has a  $T$ -periodic non-negative solution  $(u, v)$  with non-trivial  $u, v$ .

Obviously, the previous result holds also for a single equation. In particular, we have the following corollary:

**Corollary 4.1.** *Consider the problem*

$$\begin{cases} u_t - \operatorname{div}(|\nabla u^m|^{p-2} \nabla u^m) = \left( a(x, t) - \int_{\Omega} K(\xi, t) u^\alpha(\xi, t - \tau) d\xi \right) u^{p-1}, & \text{in } Q_T, \\ u(x, t) = 0, & \text{for } (x, t) \in \partial\Omega \times (0, T), \\ u(\cdot, 0) = u(\cdot, T), \end{cases} \quad (4.3)$$

and assume that

1. the exponents  $p, m$  are such that  $p \in (1, 2)$  and  $m > p$ ,
2. the delay  $\tau \in (0, +\infty)$ ,
3. the functions  $a$  and  $K$  belong to  $L^\infty(Q_T)$ , are extended to  $\Omega \times \mathbb{R}$  by  $T$ -periodicity and are non-negative for a.a.  $(x, t) \in Q_T$ ,
4. there exists a positive constant  $C$  such that for all  $\varepsilon > 0$  and all solutions  $u_\varepsilon$  of

$$u = G_\varepsilon(1, f(u^+)),$$

it results

$$\|u_\varepsilon\|_{L^\alpha(Q_T)}^\alpha \leq C.$$

Then problem (4.3) has a  $T$ -periodic non-negative and non-trivial solution.

## 4.2 A priori bounds in $L^\alpha(Q_T)$ : the coercive and the non coercive cases

In this subsection we apply Theorem 4.1 by looking for explicit a priori bounds in  $L^\alpha(Q_T)$  for the solutions of (4.2) in the "coercive case" and in the "non-coercive case".

**Theorem 4.2.** *Assume that*

1. Hypotheses 2.1 are satisfied,
2. there exist constants  $\underline{k}_i > 0$ ,  $i = 1, 4$ , such that

$$K_i(x, t) \geq \underline{k}_i \text{ for } i = 1, 4,$$

for a.a.  $(x, t) \in Q_T$ ,

3.  $K_i(x, t) \leq 0$  for  $i = 2, 3$ , for a.a.  $(x, t) \in Q_T$ ,
4. Hypothesis 2.2.2 is satisfied with

$$\begin{aligned} C_1 &= \frac{T}{\underline{k}_1} \|a\|_{L^\infty(Q_T)} \\ C_2 &= \frac{T}{\underline{k}_4} \|b\|_{L^\infty(Q_T)}. \end{aligned}$$

Then problem (1.1) has a non-negative  $T$ -periodic solution  $(u, v)$  with non-trivial  $u, v$ .

*Proof.* We just need to show that  $\|u_\varepsilon\|_{L^\alpha(Q_T)}^\alpha \leq C_1$  and  $\|v_\varepsilon\|_{L^\alpha(Q_T)}^\alpha \leq C_2$  for any solution  $(u_\varepsilon, v_\varepsilon)$  of (4.2). Proceeding as in Theorem 3.1, one has

$$\varepsilon \int_0^T \frac{\int_\Omega |\nabla u_\varepsilon|^p dx}{\left(\int_\Omega u_\varepsilon^2 dx\right)^{\frac{p}{2}}} dt \leq |\Omega|^{1-\frac{p}{2}} \left( T \|a\|_{L^\infty(Q_T)} - \underline{k}_1 \|u_\varepsilon\|_{L^\alpha(Q_T)}^\alpha \right)$$

and again

$$\varepsilon \lambda_{p,2} T \leq |\Omega|^{1-\frac{p}{2}} \left( T \|a\|_{L^\infty(Q_T)} - \underline{k}_1 \|u_\varepsilon\|_{L^\alpha(Q_T)}^\alpha \right).$$

The same procedure, when applied to the second equation of (2.1), leads to

$$\varepsilon \lambda_{q,2} T \leq |\Omega|^{1-\frac{q}{2}} \left( T \|b\|_{L^\infty(Q_T)} - \underline{k}_4 \|v_\varepsilon\|_{L^\alpha(Q_T)}^\alpha \right),$$

Hence, we have

$$\begin{aligned} \|u_\varepsilon\|_{L^\alpha(Q_T)}^\alpha &\leq \frac{T \|a\|_{L^\infty(Q_T)}}{\underline{k}_1}, \\ \|v_\varepsilon\|_{L^\alpha(Q_T)}^\alpha &\leq \frac{T \|b\|_{L^\infty(Q_T)}}{\underline{k}_4}. \end{aligned}$$

□

**Theorem 4.3.** *Assume that*

1. *Hypotheses 2.1 are satisfied,*
2.  $K_i(x, t) \leq 0$  for  $i = 2, 3$ , for a.a.  $(x, t) \in Q_T$ ,
3. *Hypothesis 2.2.2 is satisfied with*

$$C_1 = \frac{|Q_T|}{\mu_p^{\frac{\alpha}{m(p-1)}}} \left( \frac{\|a\|_{L^\infty(Q_T)} (m(p-1) + \alpha)^p}{\alpha m^{p-1} p^p} \right)^{\frac{\alpha}{m(p-1)}}$$

and

$$C_2 = \frac{|Q_T|}{\mu_q^{\frac{\alpha}{n(q-1)}}} \left( \frac{\|b\|_{L^\infty(Q_T)} (n(q-1) + \alpha)^q}{\alpha n^{q-1} q^q} \right)^{\frac{\alpha}{n(q-1)}}.$$

Here  $\bar{k}_2, \bar{k}_3$ , are as in Hypotheses 2.1. Then problem (1.1) has a  $T$ -periodic non-negative solution  $(u, v)$  with non-trivial  $u, v$ .

*Proof.* We just need to show that  $\|u_\varepsilon\|_{L^\alpha(Q_T)}^\alpha \leq C_1$  and  $\|v_\varepsilon\|_{L^\alpha(Q_T)}^\alpha \leq C_2$  for any solution  $(u_\varepsilon, v_\varepsilon)$  of (4.2). Multiplying the first equation of (4.2) by  $u_\varepsilon^{\alpha-p+1}$ , integrating in  $Q_T$  and passing to the limit, as  $h \rightarrow 0$ , in the Steklov averages  $(u_\varepsilon)_h$ , we obtain, as in Proposition 2.2,

$$\begin{aligned} m^{p-1} \alpha \left( \frac{p}{m(p-1) + \alpha} \right)^p \iint_{Q_T} \left| \nabla u_\varepsilon^{\frac{m(p-1)+\alpha}{p}} \right|^p dx dt &\leq \varepsilon \alpha \iint_{Q_T} u^{\alpha-1} |\nabla u_\varepsilon|^p dx dt \\ &\quad + \iint_{Q_T} |\nabla u_\varepsilon^m|^{p-2} \nabla u_\varepsilon^m \nabla u_\varepsilon^\alpha dx dt \\ &\leq \|a\|_{L^\infty(Q_T)} \|u_\varepsilon\|_{L^\alpha(Q_T)}^\alpha, \end{aligned}$$

by the  $T$ -periodicity of  $u_\varepsilon$  and the non-positivity of the function  $K_2$ . Using the Hölder inequality, with  $r := \frac{m(p-1)+\alpha}{\alpha}$ , and the Poincaré inequality, one has:

$$\begin{aligned} \iint_{Q_T} u_\varepsilon^\alpha dx dt &\leq |Q_T|^{\frac{m(p-1)}{m(p-1)+\alpha}} \left( \iint_{Q_T} u_\varepsilon^{m(p-1)+\alpha} dx dt \right)^{\frac{\alpha}{m(p-1)+\alpha}} = |Q_T|^{\frac{m(p-1)}{m(p-1)+\alpha}} \left\| u_\varepsilon^{\frac{m(p-1)+\alpha}{p}} \right\|_{L^p(Q_T)}^{\frac{\alpha p}{m(p-1)+\alpha}} \\ &\leq |Q_T|^{\frac{m(p-1)}{m(p-1)+\alpha}} \left( \frac{1}{\sqrt[p]{\mu_p}} \left\| \nabla u_\varepsilon^{\frac{m(p-1)+\alpha}{p}} \right\|_{L^p(Q_T)} \right)^{\frac{\alpha p}{m(p-1)+\alpha}}. \end{aligned}$$

Thus

$$\begin{aligned} \|u_\varepsilon\|_{L^\alpha(Q_T)}^\alpha &\leq |Q_T|^{\frac{m(p-1)}{m(p-1)+\alpha}} \left( \frac{1}{\sqrt[p]{\mu_p}} \left\| \nabla u_\varepsilon^{\frac{m(p-1)+\alpha}{p}} \right\|_{L^p(Q_T)} \right)^{\frac{\alpha p}{m(p-1)+\alpha}} \\ &\leq \frac{|Q_T|^{\frac{m(p-1)}{m(p-1)+\alpha}}}{\mu_p^{\frac{\alpha}{m(p-1)+\alpha}}} \left( \frac{\|a\|_{L^\infty(Q_T)}(m(p-1)+\alpha)^p}{\alpha m^{p-1} p^p} \right)^{\frac{\alpha}{m(p-1)+\alpha}} \|u_\varepsilon\|_{L^\alpha(Q_T)}^{\frac{\alpha^2}{m(p-1)+\alpha}}. \end{aligned}$$

This implies

$$\|u_\varepsilon\|_{L^\alpha(Q_T)}^\alpha \leq \frac{|Q_T|}{\mu_p^{\frac{\alpha}{m(p-1)}}} \left( \frac{\|a\|_{L^\infty(Q_T)}(m(p-1)+\alpha)^p}{\alpha m^{p-1} p^p} \right)^{\frac{\alpha}{m(p-1)}}.$$

In an analogous way we obtain that

$$\|v_\varepsilon\|_{L^\alpha(Q_T)}^\alpha \leq \frac{|Q_T|}{\mu_q^{\frac{\alpha}{n(q-1)}}} \left( \frac{\|b\|_{L^\infty(Q_T)}(n(q-1)+\alpha)^q}{\alpha n^{q-1} q^q} \right)^{\frac{\alpha}{n(q-1)}},$$

if  $v_\varepsilon$  is a solution of the second equation of (4.2).  $\square$

An immediate consequence of Theorems 4.2 and 4.3 is the following existence result for a single equation:

**Corollary 4.2.** *Consider the problem (4.3) and assume that*

1. *the exponents  $p, m$  are such that  $p \in (1, 2)$  and  $m > p$ ,*
2. *the delay  $\tau \in (0, +\infty)$ ,*
3. *the functions  $a$  and  $K$  belong to  $L^\infty(Q_T)$ , are extended to  $\Omega \times \mathbb{R}$  by  $T$ -periodicity and are non-negative for a.a.  $(x, t) \in Q_T$ ,*
4. *hypothesis 4 of Corollary 4.1 is satisfied with*

$$C = \frac{T}{\underline{k}} \|a\|_{L^\infty(Q_T)}$$

*if  $K(t, x) \geq \underline{k} > 0$ , or*

$$C = \frac{|Q_T|}{\mu_p^{\frac{\alpha}{m(p-1)}}} \left( \frac{\|a\|_{L^\infty(Q_T)}(m(p-1)+\alpha)^p}{\alpha m^{p-1} p^p} \right)^{\frac{\alpha}{m(p-1)}}$$

*if  $K(t, x) \geq 0$ .*

*Then problem (4.3) has a  $T$ -periodic non-negative and non-trivial solution.*

*Proof.* We just need to show that  $\|u_\varepsilon\|_{L^\alpha(Q_T)}^\alpha \leq C$  for a positive constant  $C$ . Proceeding as in Theorem 4.2 if  $K(t, x) \geq \underline{k} > 0$  or in Theorem 4.3 if  $K(t, x) \geq 0$ , one has that  $\|u_\varepsilon\|_{L^\alpha(Q_T)}^\alpha \leq \frac{T}{\underline{k}_1} \|a\|_{L^\infty(Q_T)}$  or  $\|u_\varepsilon\|_{L^\alpha(Q_T)}^\alpha \leq \frac{|Q_T|}{\mu_p^{\frac{\alpha}{m(p-1)}}} \left( \frac{\|a\|_{L^\infty(Q_T)}(m(p-1)+\alpha)^p}{\alpha m^{p-1} p^p} \right)^{\frac{\alpha}{m(p-1)}}$ , respectively. The thesis follows by Corollary 4.1.  $\square$

**Remark 4.1.** We underline the fact that the generalization presented in this section can be extended to a single degenerate equation or to a double degenerate system, namely when  $p, q \geq 2$ , as considered in [25].

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