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On fractional Laplacians – 2

Roberta Musina^{*} and Alexander I. Nazarov[†]

Abstract. The present paper is the natural evolution of arXiv:1308.3606. For $s > -1$ we compare two natural types of fractional Laplacians $(-\Delta)^s$, namely, the “Navier” and the “Dirichlet” ones. As a main tool, we give the “dual” Caffarelli–Silvestre and Stinga–Torrea characterizations of these operators for $s \in (-1, 0)$.

1 Introduction

Recall that the Sobolev space $H^s(\mathbb{R}^n) = W_2^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, is the space of distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ with finite norm

$$\|u\|_s^2 = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\mathcal{F}u(\xi)|^2 d\xi,$$

see for instance Section 2.3.3 of the monograph [8]. Here \mathcal{F} denotes the Fourier transform

$$\mathcal{F}u(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} u(x) dx.$$

For arbitrary $s \in \mathbb{R}$ we define fractional Laplacian in \mathbb{R}^n by the quadratic form

$$Q_s[u] = ((-\Delta)^s u, u) := \int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi,$$

with domain

$$\text{Dom}(Q_s) = \{u \in \mathcal{S}'(\mathbb{R}^n) : Q_s[u] < \infty\}.$$

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Let Ω be a bounded and smooth domain in \mathbb{R}^n . We put

$$H^s(\Omega) = \{u|_{\Omega} : u \in H^s(\mathbb{R}^n)\},$$

see [8, Sec. 4.2.1] and the extension theorem in [8, Sec. 4.2.3].

Also we introduce the space

$$\tilde{H}^s(\Omega) = \{u \in H^s(\mathbb{R}^n) : \text{supp } u \subset \overline{\Omega}\}.$$

By Theorem 4.3.2/1 [8], for $s - \frac{1}{2} \notin \mathbb{Z}$ this space coincides with $H_0^s(\Omega)$ that is the closure of $\mathcal{C}_0^\infty(\Omega)$ in $H^s(\Omega)$ while for $s - \frac{1}{2} \in \mathbb{Z}$ one has $\tilde{H}^s(\Omega) \subsetneq H_0^s(\Omega)$. Moreover, $\mathcal{C}_0^\infty(\Omega)$ is dense in $u \in \tilde{H}^s(\Omega)$.

We introduce the “Dirichlet” fractional Laplacian in Ω (denoted by $(-\Delta_\Omega)_D^s$) as the restriction of $(-\Delta)^s$. The domain of its quadratic form is

$$\text{Dom}(Q_{s,\Omega}^D) = \{u \in \text{Dom}(Q_s) : \text{supp } u \subset \overline{\Omega}\}.$$

Also we define the “Navier” fractional Laplacian as s -th power of the conventional Dirichlet Laplacian in the sense of spectral theory. Its quadratic form reads

$$Q_{s,\Omega}^N[u] = ((-\Delta_\Omega)_N^s u, u) := \sum_j \lambda_j^s \cdot |(u, \varphi_j)|^2.$$

Here, λ_j and φ_j are eigenvalues and eigenfunctions of the Dirichlet Laplacian in Ω , respectively, and $\text{Dom}(Q_{s,\Omega}^N)$ consists of distributions in Ω such that $Q_{s,\Omega}^N[u] < \infty$.

It is well known that for $s = 1$ these operators coincide: $(-\Delta_\Omega)_N = (-\Delta_\Omega)_D$. We emphasize that, in contrast to $(-\Delta_\Omega)_N^s$, the operator $(-\Delta_\Omega)_D^s$ is not the s -th power of the Dirichlet Laplacian for $s \neq 1$. In particular, $(-\Delta_\Omega)_D^{-s}$ is not inverse to $(-\Delta_\Omega)_D^s$.

The present paper is the natural evolution of [6], where we compared the operators $(-\Delta_\Omega)_D^s$ and $(-\Delta_\Omega)_N^s$ for $0 < s < 1$. In the first result we extend Theorem 2 of [6].

T:main1

Theorem 1 *Let $s > -1$, $s \notin \mathbb{N}_0$. Then for $u \in \text{Dom}(Q_{s,\Omega}^D)$, $u \neq 0$, the following relations hold:*

$$Q_{s,\Omega}^N[u] > Q_{s,\Omega}^D[u], \quad \text{if } 2k < s < 2k + 1, \quad k \in \mathbb{N}_0; \quad (1)$$

$$Q_{s,\Omega}^N[u] < Q_{s,\Omega}^D[u], \quad \text{if } 2k - 1 < s < 2k, \quad k \in \mathbb{N}_0. \quad (2)$$

Next, we take into account the role of dilations in \mathbb{R}^n . We denote by $F(\Omega)$ the class of smooth and bounded domains containing Ω . If $\Omega' \in F(\Omega)$, then any $u \in \text{Dom}(Q_{s,\Omega}^D)$ can be regarded as a function in $\text{Dom}(Q_{s,\Omega'}^D)$, and the corresponding form $Q_{s,\Omega'}^D[u]$ does not change. In contrast, the form $Q_{s,\Omega'}^N[u]$ does depend on $\Omega' \supset \Omega$. However, roughly speaking, the difference between these quadratic forms disappears as $\Omega' \rightarrow \mathbb{R}^n$.

T:new

Theorem 2 *Let $s > -1$. Then for $u \in \text{Dom}(Q_{s,\Omega}^D)$ the following facts hold:*

$$Q_{s,\Omega}^D[u] = \inf_{\Omega' \in F(\Omega)} Q_{s,\Omega'}^N[u], \quad \text{if } 2k < s < 2k + 1, \quad k \in \mathbb{N}_0; \quad (3)$$

$$Q_{s,\Omega}^D[u] = \sup_{\Omega' \in F(\Omega)} Q_{s,\Omega'}^N[u], \quad \text{if } 2k - 1 < s < 2k, \quad k \in \mathbb{N}_0. \quad (4)$$

For $-1 < s < 0$ we also obtain a pointwise comparison result reverse to the case $0 < s < 1$ (compare with [6, Theorem 1]).

T:main2 **Theorem 3** *Let $-1 < s < 0$, and let $f \in \text{Dom}(Q_{s,\Omega}^D)$, $f \geq 0$ in the sense of distributions, $f \not\equiv 0$. Then the following relation holds:*

$$(-\Delta_\Omega)_N^s f < (-\Delta_\Omega)_D^s f. \quad (5)$$

-pos_pres

Actually, fractional Laplacians of orders $s \in (-1, 0)$ play a crucial role in our arguments. In Section 2 we give a variational characterization of these operators, “dual” to variational characterization of fractional Laplacians of orders $s \in (0, 1)$ given in [4] and [7]. Theorems 1–3 are proved in Section 3.

Note that our statements hold in more general setting. Let Ω be a bounded and smooth domain in a complete smooth Riemannian manifold \mathcal{M} . Denote by $(-\Delta_\Omega)_N^s$ and $(-\Delta_\Omega)_D^s$, respectively, the s -th power of the Dirichlet Laplacian in Ω and the restriction of s -th power of the Dirichlet Laplacian in \mathcal{M} to the set of functions supported in Ω . Then proofs of Theorems 1–3 (and of Theorem 1 in [6] as well) run with minimal changes.

2 Fractional Laplacians of negative orders

negat

First, we recall some facts from the classical monograph [8] about the spaces $H^s(\Omega)$ and $\tilde{H}^s(\Omega)$.

Hs **Proposition 1** *(a particular case of [8, Theorem 4.3.2/1]).*

1. If $0 < \sigma < \frac{1}{2}$ then $\tilde{H}^\sigma(\Omega) = H_0^\sigma(\Omega) = H^\sigma(\Omega)$;
2. If $\sigma = \frac{1}{2}$ then $\tilde{H}^\sigma(\Omega)$ is dense in $H^\sigma(\Omega) = H_0^\sigma(\Omega)$;
3. If $\frac{1}{2} < \sigma < 1$ then $\tilde{H}^\sigma(\Omega) = H_0^\sigma(\Omega)$ is a subspace of $H^\sigma(\Omega)$.

duality **Proposition 2** *(a particular case of [8, Theorem 2.10.5/1]).*
For any $\sigma \in \mathbb{R}$ $(\tilde{H}^\sigma(\Omega))' = H^{-\sigma}(\Omega)$.

As an immediate consequence we obtain

H-s **Corollary 1** *1. If $0 < \sigma < \frac{1}{2}$ then $\tilde{H}^{-\sigma}(\Omega) = H^{-\sigma}(\Omega)$;*
2. If $\sigma = \frac{1}{2}$ then $\tilde{H}^{-\sigma}(\Omega)$ is dense in $H^{-\sigma}(\Omega)$;
3. If $\frac{1}{2} < \sigma < 1$ then $H^{-\sigma}(\Omega)$ is a subspace of $\tilde{H}^{-\sigma}(\Omega)$.

1D **Remark 1** *In the one-dimensional case, for $\frac{1}{2} < \sigma < 1$ the codimension of $H^{-\sigma}(\Omega)$ in $\tilde{H}^{-\sigma}(\Omega)$ equals 2 since the same is codimension of $\tilde{H}^\sigma(\Omega)$ in $H^\sigma(\Omega)$.*

The next statement gives explicit description of domains of quadratic forms under consideration.

domain

Lemma 1 *Let $0 < \sigma < 1$. Then*

1. $\text{Dom}(Q_{-\sigma,\Omega}^N) = H^{-\sigma}(\Omega);$
2. $\text{Dom}(Q_{-\sigma,\Omega}^D) = \tilde{H}^{-\sigma}(\Omega)$ if $n \geq 2$ or $\sigma < \frac{1}{2}$;
3. $\text{Dom}(Q_{-\sigma,\Omega}^D) = \{u \in \tilde{H}^{-\sigma}(\Omega) : \mathcal{F}u(0) = 0\}$ if $n = 1$ and $\sigma \geq \frac{1}{2}$.

Proof. The first statement follows from the relation $\text{Dom}(Q_{\sigma,\Omega}^N) = \tilde{H}^\sigma(\Omega)$, see, e.g., [8, Theorems 1.15.3 and 4.3.2/2], and from Proposition 2.

The second and the third statements follow directly from definition of $\tilde{H}^{-\sigma}(\Omega)$, if we take into account that $\mathcal{F}u$ is a smooth function. \square

By Lemma 1 and Corollary 1, for $0 < \sigma \leq \frac{1}{2}$ we have $\text{Dom}(Q_{-\sigma,\Omega}^D) \subseteq \text{Dom}(Q_{-\sigma,\Omega}^N)$ (even $\text{Dom}(Q_{-\sigma,\Omega}^D) = \text{Dom}(Q_{-\sigma,\Omega}^N)$ if $0 < \sigma < \frac{1}{2}$). In the case $\frac{1}{2} < \sigma < 1$, $\text{Dom}(Q_{-\sigma,\Omega}^N)$ is a subspace of $\text{Dom}(Q_{-\sigma,\Omega}^D)$ (for $n = 1$ this follows from Remark 1). However, for arbitrary $f \in \text{Dom}(Q_{-\sigma,\Omega}^D)$ we can consider f as a functional on $H^\sigma(\Omega)$, put $\tilde{f} = f|_{\tilde{H}^\sigma(\Omega)} \in \text{Dom}(Q_{-\sigma,\Omega}^N)$ and define $Q_{-\sigma,\Omega}^N[f] := Q_{-\sigma,\Omega}^N[\tilde{f}]$.

Next, we recall that in the paper [4] the fractional Laplacian of order $\sigma \in (0, 1)$ in \mathbb{R}^n was connected with the so-called *harmonic extension in $n + 2 - 2\sigma$ dimensions* and with generalized Dirichlet-to-Neumann map (see also [3] for the case $\sigma = \frac{1}{2}$). In particular, for any $u \in \tilde{H}^\sigma(\Omega)$ the function $w_\sigma^D(x, y)$ minimizing the weighted Dirichlet integral

$$\mathcal{E}_\sigma^D(w) = \int_0^\infty \int_{\mathbb{R}^n} y^{1-2\sigma} |\nabla w(x, y)|^2 dx dy$$

over the set

$$\mathcal{W}_\sigma^D(u) = \left\{ w(x, y) : \mathcal{E}_\sigma^D(w) < \infty, \quad w|_{y=0} = u \right\},$$

satisfies

$$Q_{\sigma,\Omega}^D[u] = \frac{C_\sigma}{2\sigma} \cdot \mathcal{E}_\sigma^D(w_\sigma^D), \tag{6} \quad \text{quad_D}$$

where the constant C_σ is given by

$$C_\sigma := \frac{4^\sigma \Gamma(1 + \sigma)}{\Gamma(1 - \sigma)}.$$

Moreover, $w_\sigma^D(x, y)$ is the solution of the BVP

$$-\text{div}(y^{1-2\sigma} \nabla w) = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+; \quad w|_{y=0} = u,$$

and for sufficiently smooth u

$$(-\Delta)^\sigma u(x) = -\frac{C_\sigma}{2\sigma} \cdot \lim_{y \rightarrow 0^+} y^{1-2\sigma} \partial_y w_\sigma^D(x, y), \quad x \in \mathbb{R}^n \tag{7} \quad \text{extension_D}$$

(we recall that $(-\Delta_\Omega)_D^\sigma u = (-\Delta)^\sigma u|_\Omega$).

In [7] this approach was developed in quite general situation. In particular, it was shown that for any $u \in \tilde{H}^\sigma(\Omega)$ the function $w_\sigma^N(x, y)$ minimizing the energy integral

$$\mathcal{E}_\sigma^N(w) = \int_0^\infty \int_\Omega y^{1-2\sigma} |\nabla w(x, y)|^2 dx dy$$

over the set

$$\mathcal{W}_{\sigma, \Omega}^N(u) = \{w(x, y) \in \mathcal{W}_\sigma^D(u) : w|_{x \in \partial\Omega} = 0\},$$

satisfies

$$Q_{\sigma, \Omega}^N[u] = \frac{C_\sigma}{2\sigma} \cdot \mathcal{E}_\sigma^N(w_\sigma^N). \quad (8) \quad \boxed{\text{quad_N}}$$

Moreover, $w_\sigma^N(x, y)$ is the solution of the BVP

$$-\operatorname{div}(y^{1-2\sigma} \nabla w) = 0 \quad \text{in } \Omega \times \mathbb{R}_+; \quad w|_{y=0} = u; \quad w|_{x \in \partial\Omega} = 0, \quad (9) \quad \boxed{\text{eq:ST}}$$

and for sufficiently smooth u it turns out that

$$(-\Delta_\Omega)_N^\sigma u(x) = -\frac{C_\sigma}{2\sigma} \cdot \lim_{y \rightarrow 0^+} y^{1-2\sigma} \partial_y w_\sigma^N(x, y). \quad (10) \quad \boxed{\text{extension_N}}$$

In a similar way, negative fractional Laplacians are connected with generalized Neumann-to-Dirichlet map. Namely, let $u \in \operatorname{Dom}(Q_{-\sigma, \Omega}^D)$. We consider the problem of minimizing the functional

$$\tilde{\mathcal{E}}_{-\sigma}^D(w) = \mathcal{E}_\sigma^D(w) - 2 \langle u, w|_{y=0} \rangle$$

over the set $\mathcal{W}_{-\sigma}^D$, that is closure of smooth functions on $\mathbb{R}^n \times \bar{\mathbb{R}}_+$ with bounded support, with respect to $\mathcal{E}_\sigma^D(\cdot)$. We recall that by Lemma 1 u can be considered as a compactly supported functional on $H^\sigma(\mathbb{R}^n)$, and thus the duality $\langle u, w|_{y=0} \rangle$ is well defined by the result of [4].

First, let $n > 2\sigma$ (this is a restriction only for $n = 1$). We claim that the Hardy type inequality

$$\mathcal{E}_\sigma^D(w) \geq \left(\frac{n-2\sigma}{2}\right)^2 \int_0^\infty \int_{\mathbb{R}^n} y^{1-2\sigma} \frac{w^2(x, y)}{r^2} dx dy \quad (11) \quad \boxed{\text{eq:H}}$$

holds for $w \in \mathcal{W}_{-\sigma}^D$ (here $r^2 = |x|^2 + y^2$). Indeed, for a smooth function w with bounded support we consider the restriction of w to arbitrary ray in $\mathbb{R}^n \times \mathbb{R}_+$ and write down the classical Hardy inequality

$$\int_0^\infty r^{n+1-2\sigma} w_r^2 dr \geq \left(\frac{n-2\sigma}{2}\right)^2 \int_0^\infty r^{n-1-2\sigma} w^2 dr.$$

We multiply it by $(\frac{y}{r})^{1-2\sigma}$, integrate over unit hemisphere in \mathbb{R}^{n+1} , and the claim follows.

By (11), a non-zero constant cannot be approximated by compactly supported functions. Thus, the minimizer of $\tilde{\mathcal{E}}_{-\sigma}^D$ is determined uniquely. Denote it by $w_{-\sigma}^D(x, y)$. Then formulae (6) and (7) imply relations

$$Q_{-\sigma, \Omega}^D[u] = -\frac{2\sigma}{C_\sigma} \cdot \tilde{\mathcal{E}}_{-\sigma}^D(w_{-\sigma}^D); \quad (-\Delta_\Omega)_D^{-\sigma} u(x) = \frac{2\sigma}{C_\sigma} w_{-\sigma}^D(x, 0), \quad x \in \Omega, \quad (12) \quad \boxed{-D}$$

that give the “dual” Caffarelli–Silvestre characterization of $(-\Delta_\Omega)_D^{-\sigma}$.

In case $n = 1 \leq 2\sigma$ the above argument needs some modification. Namely, the minimizer $w_{-\sigma}^D(x, y)$ in this case is defined up to an additive constant. However, by Lemma 1 we have

$$\mathcal{F}u(0) \equiv \langle u, \mathbf{1} \rangle = 0.$$

Therefore, $\tilde{\mathcal{E}}_{-\sigma}^D(w_{-\sigma}^D)$ does not depend on the choice of the constant, and the first relation in (12) holds. The second equality in (12) also holds if we choose the constant such that $w_{-\sigma}^D(x, 0) \rightarrow 0$ as $|x| \rightarrow \infty$.

Remark 2 *Note that for sufficiently smooth u the function $w_{-\sigma}^D$ solves the Neumann problem*

$$-\operatorname{div}(y^{1-2\sigma} \nabla w) = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+; \quad \lim_{y \rightarrow 0^+} y^{1-2\sigma} \partial_y w = -u. \quad (13) \quad \boxed{\text{eq: -CS}}$$

Analogously, formulae (8) and (10) imply the “dual” Stinga–Torrea characterization of $(-\Delta_\Omega)_N^{-\sigma}$. Namely, the function $w_{-\sigma}^N(x, y)$ minimizing the functional

$$\tilde{\mathcal{E}}_{-\sigma}^N(w) = \mathcal{E}_\sigma^N(w) - 2 \langle u, w|_{y=0} \rangle$$

over the set

$$\mathcal{W}_{-\sigma, \Omega}^N(u) = \{w(x, y) \in \mathcal{W}_{-\sigma}^D : w|_{x \notin \Omega} = 0\},$$

satisfies

$$Q_{-\sigma, \Omega}^N[u] = -\frac{2\sigma}{C_\sigma} \cdot \tilde{\mathcal{E}}_{-\sigma}^N(w_{-\sigma}^N); \quad (-\Delta_\Omega)_N^{-\sigma} u(x) = \frac{2\sigma}{C_\sigma} w_{-\sigma}^N(x, 0). \quad (14) \quad \boxed{-N}$$

ext **Remark 3** *Formula (14) shows that $w_{-\sigma}^N$ is the Stinga–Torrea extension of $\frac{C_\sigma}{2\sigma}(-\Delta_\Omega)_N^{-\sigma} u$. Similarly, from (12) we conclude that $w_{-\sigma}^D$ is the Caffarelli–Silvestre extension of $\frac{C_\sigma}{2\sigma}(-\Delta)_D^{-\sigma} u$ but not of $\frac{C_\sigma}{2\sigma}(-\Delta_\Omega)_D^{-\sigma} u$. This is due to the fact, already noticed in the introduction, that $(-\Delta_\Omega)_D^{-\sigma}$ is not the inverse of $(-\Delta_\Omega)_D^\sigma$.*

Remark 4 *The representation of $(-\Delta)^{-\sigma} u$ via solution of the problem (13) was used in [2]. Similar representation of $(-\Delta_\Omega)_N^{-\sigma} u$ via solution of corresponding mixed boundary value problem was used earlier in [5]. However, variational characterizations of negative fractional Laplacians (the first parts of formulae (12) and (14)) which play key role in what follows, are given for the first time.*

3 Proofs of main theorems

compar

We start by recalling an auxiliary result.

domain1

Lemma 2 *Let $s > 1$. Then*

$$\begin{aligned} \text{Dom}(Q_{s,\Omega}^D) &= \tilde{H}^s(\Omega) = \text{Dom}(Q_{s,\Omega}^N) \quad \text{for } s < 3/2; \\ \text{Dom}(Q_{s,\Omega}^D) &= \tilde{H}^s(\Omega) \subsetneq \text{Dom}(Q_{s,\Omega}^N) \quad \text{for } s \geq 3/2. \end{aligned}$$

Proof. For $Q_{s,\Omega}^D$ the conclusion follows directly from its definition. For $Q_{s,\Omega}^N$ this fact is well known for $s \in \mathbb{N}$; in general case it follows immediately from [8, Theorem 1.17.1/1] and [8, Theorem 4.3.2/1]. \square

Proof of Theorem 1. We split the proof in three parts.

1. Let $0 < s < 1$. Then the relation (1) is proved in [6, Theorem 2].
2. Let $-1 < s < 0$. We define $\sigma = -s \in (0, 1)$ and construct extensions $w_{-\sigma}^D$ and $w_{-\sigma}^N$ as described in Section 2.
We evidently have $\mathcal{W}_{-\sigma,\Omega}^N \subset \mathcal{W}_{-\sigma}^D$ and $\tilde{\mathcal{E}}_{-\sigma}^N = \tilde{\mathcal{E}}_{-\sigma}^D|_{\mathcal{W}_{-\sigma,\Omega}^N}$. Therefore, (12) and (14) provide

$$Q_{s,\Omega}^N[u] = -\frac{2\sigma}{C_\sigma} \cdot \inf_{w \in \mathcal{W}_{-\sigma,\Omega}^N} \tilde{\mathcal{E}}_{-\sigma}^N(w) \leq -\frac{2\sigma}{C_\sigma} \cdot \inf_{w \in \mathcal{W}_{-\sigma}^D} \tilde{\mathcal{E}}_{-\sigma}^D(w) = Q_{s,\Omega}^D[u].$$

To complete the proof, we observe that for $u \neq 0$ the function $w_{-\sigma}^N$ cannot be a solution of the homogeneous equation in (9) in the whole half-space, since such a solution is analytic in the half-space. Thus, it cannot provide $\inf_{w \in \mathcal{W}_{-\sigma}^D} \tilde{\mathcal{E}}_{-\sigma}^D(w)$, and (2) follows.

3. Now let $s > 1$, $s \notin \mathbb{N}$. We put $k = \lfloor \frac{s-1}{2} \rfloor$ and define for $u \in \tilde{H}^s(\Omega)$

$$v = (-\Delta)^k u \in \tilde{H}^{s-2k}(\Omega), \quad s - 2k \in (-1, 0) \cup (0, 1).$$

Note that $v \neq 0$ if $u \neq 0$. Then we have

$$Q_{s,\Omega}^N[u] = Q_{s-2k,\Omega}^N[v], \quad Q_{s,\Omega}^D[u] = Q_{s-2k,\Omega}^D[v],$$

and the conclusion follows from cases 1 and 2. \square

Proof of Theorem 2. Here we again distinguish three cases.

1. Let $0 < s < 1$. Then the relation (3) is proved in [6, Theorem 3].
2. Let $-1 < s < 0$. We define $\sigma = -s \in (0, 1)$ and proceed similarly to the proof of [6, Theorem 3]. It is sufficient to prove the statement for $u \in \mathcal{C}_0^\infty(\Omega)$.
For $\Omega' \supset \Omega$ we have $\mathcal{W}_{-\sigma,\Omega}^N \subset \mathcal{W}_{-\sigma,\Omega'}^N$. By (14), the quadratic form $Q_{s,\Omega}^N[u]$ is monotone increasing with respect to the domain inclusion. Taking (2) into account, we obtain

$$Q_{s,\Omega}^D[u] > Q_{s,\Omega'}^N[u] \geq Q_{s,\Omega}^N[u]. \quad (15) \quad \text{eq:monotone}$$

Denote by $w = w_{-\sigma}^D$ the Caffarelli–Silvestre extension of $\frac{C_\sigma}{2\sigma}(-\Delta)^{-\sigma}u$, described in Section 2. Next, for any $y \geq 0$ let $\phi_R(\cdot, y)$ be the harmonic extension of $w(\cdot, y)$ on the ball B_R , that is,

$$-\Delta \phi_R(\cdot, y) = 0 \quad \text{in } B_R; \quad \phi_R(\cdot, y) = w(\cdot, y) \quad \text{on } \partial B_R.$$

Finally, for $x \in B_R$ and $y \geq 0$ we put

$$w_R(x, y) = w(x, y) - \phi_R(x, y).$$

It is shown in the proof of [6, Theorem 3] that there exists a sequence $R_h \rightarrow \infty$ such that

$$\mathcal{E}_\sigma^N(w_{R_h}) \leq \mathcal{E}_\sigma^D(w) + o(1).$$

Further, since $(-\Delta_\Omega)^{-\sigma}u$ vanishes at infinity, for any multi-index β we evidently have $D^\beta \phi_{R_h}(\cdot, 0) \rightarrow 0$ locally uniformly as $R_h \rightarrow \infty$. This gives $\langle u, \phi_{R_h}(\cdot, 0) \rangle = o(1)$, and we obtain by (12) and (14)

$$\begin{aligned} Q_{s, B_{R_h}}^N[u] &\geq -\frac{2\sigma}{C_\sigma} \cdot \tilde{\mathcal{E}}_\sigma^N(w_{R_h}) \\ &\geq -\frac{2\sigma}{C_\sigma} \cdot \tilde{\mathcal{E}}_\sigma^D(w) - o(1) = Q_{s, \Omega}^D[u] - o(1). \end{aligned} \tag{16} \quad \boxed{\text{eq:tesi}}$$

The relation (3) readily follows by comparing (15) and (16).

3. For $s > 1$, $s \notin \mathbb{N}$, the conclusion follows from cases 1 and 2 just as in the proof of Theorem 1. \square

R:new **Remark 5** Assume that $0 \in \Omega$ and put $\alpha\Omega = \{\alpha x : x \in \Omega\}$. Thanks to (15), the proof above shows indeed that

$$Q_{s, \Omega}^D[u] = \lim_{\alpha \rightarrow \infty} Q_{s, \alpha\Omega}^N[u] \quad \text{for any } u \in \tilde{H}^s(\Omega).$$

Now put $u_\alpha(x) = \alpha^{\frac{n-2s}{2}}u(\alpha x)$. Then the scaling shows that

$$Q_{s, \Omega}^D[u_\alpha] \equiv Q_{s, \Omega}^D[u] = \lim_{\alpha \rightarrow \infty} Q_{s, \Omega}^N[u_\alpha] \quad \text{for any } u \in \tilde{H}^s(\Omega).$$

Proof of Theorem 3. First, let $f \in \mathcal{C}_0^\infty(\Omega)$. We define $\sigma = -s \in (0, 1)$ and construct extensions $w_{-\sigma}^D$ and $w_{-\sigma}^N$ described in Section 2. Making the change of the variable $t = y^{2\sigma}$, we rewrite the BVP (13) for $w_{-\sigma}^D(x, t)$ as follows:

$$\Delta_x w_{-\sigma}^D + 4\sigma^2 t^{\frac{2\sigma-1}{\sigma}} \partial_{tt}^2 w_{-\sigma}^D = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+; \quad \partial_t w_{-\sigma}^D|_{t=0} = -\frac{f}{2\sigma}. \tag{17} \quad \boxed{\text{BVP}}$$

Since $w_{-\sigma}^D$ vanishes at infinity, $w_{-\sigma}^D(x, t) > 0$ for $t > 0$ by the maximum principle. Moreover, by [1, Theorem 1.4] (the boundary point lemma) we have $w_{-\sigma}^D(x, 0) > 0$.

Further, the function $w_{-\sigma}^N$ satisfies the equalities (17) in $\Omega \times \mathbb{R}_+$. Since $w_{-\sigma}^N|_{x \notin \Omega} = 0$, we infer that the function

$$W(x, t) := w_{-\sigma}^D(x, t) - w_{-\sigma}^N(x, t)$$

meets the following relations:

$$\Delta_x W + 4\sigma^2 t^{\frac{2\sigma-1}{\sigma}} \partial_{tt}^2 W = 0 \quad \text{in } \Omega \times \mathbb{R}_+; \quad \partial_t W|_{t=0} = 0; \quad W|_{x \in \partial\Omega} > 0.$$

Again, [1, Theorem 1.4] gives $W(x, 0) > 0$, which gives (5) in view of (12) and (14).

For $f \in \widetilde{H}^s(\Omega)$ the statement holds by approximation argument. \square

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