



UNIVERSITÀ
DEGLI STUDI
DI UDINE

Università degli studi di Udine

Variational Inequalities for the Fractional Laplacian

Original

Availability:

This version is available <http://hdl.handle.net/11390/1107309> since 2021-03-15T15:50:06Z

Publisher:

Published

DOI:10.1007/s11118-016-9591-9

Terms of use:

The institutional repository of the University of Udine (<http://air.uniud.it>) is provided by ARIC services. The aim is to enable open access to all the world.

Publisher copyright

(Article begins on next page)

Variational inequalities for the fractional Laplacian

Roberta Musina^{*}, Alexander I. Nazarov[†] and Konijeti Sreenadh[‡]

Abstract

In this paper we study the obstacle problems for the fractional Laplacian of order $s \in (0, 1)$ in a bounded domain $\Omega \subset \mathbb{R}^n$, under mild assumptions on the data.

1 Introduction

Let Ω be a bounded domain in \mathbb{R}^n , $n \geq 1$. Given $s \in (0, 1)$, a measurable function ψ and a distribution f on Ω , we consider the problem

$$\begin{cases} u \geq \psi & \text{in } \Omega \\ (-\Delta)^s u \geq f & \text{in } \Omega \\ (-\Delta)^s u = f & \text{in } \{u > \psi\} \\ u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{cases} \quad (1.1)$$

Our interest is motivated by the noticeable paper [19], where Louis E. Silvestre investigated (1.1) in case $\Omega = \mathbb{R}^n$, $f = 0$ and ψ smooth. His results apply also to Dirichlet's problems on balls, see [19, Section 1.3]. Besides remarkable results, in [19] the interested reader can find stimulating motivations for (1.1), arising from

^{*}Dipartimento di Matematica ed Informatica, Università di Udine, via delle Scienze, 206 – 33100 Udine, Italy. Email: roberta.musina@uniud.it. Partially supported by Miur-PRIN 201274FYK7_004.

[†]St.Petersburg Department of Steklov Institute, Fontanka 27, St.Petersburg, 191023, Russia, and St.Petersburg State University, Universitetskii pr. 28, St.Petersburg, 198504, Russia. E-mail: al.il.nazarov@gmail.com. Supported by RFBR grant 14-01-00534 and by St.Petersburg University grant 6.38.670.2013.

[‡]Department of Mathematics, Indian Institute of Technology Delhi, Hauz Khas, New Delhi-110016, India. Email:sreenadh@maths.iitd.ac.in

mathematical finance. In addition, Signorini's problem, also known as the lower dimensional obstacle problem for the classical Laplacian, can be recovered from (1.1) by taking $s = \frac{1}{2}$.

Among the papers dealing with (1.1) and related problems we cite also [1, 3, 4, 7, 15, 18] and references there-in, with no attempt to provide a complete reference list.

In the present paper we show that the free boundary problem (1.1) admits a solution under quite mild assumptions on the data, see Theorems 1.1 and 1.2 below. However, our starting interest included broader questions concerning the variational inequality

$$u \in K_\psi^s, \quad \langle (-\Delta)^s u - f, v - u \rangle \geq 0 \quad \forall v \in K_\psi^s, \quad (\mathcal{P}(\psi, f))$$

where $f \in \tilde{H}^s(\Omega)'$ and

$$K_\psi^s = \left\{ v \in \tilde{H}^s(\Omega) \mid v \geq \psi \text{ a.e. on } \Omega \right\}.$$

Notation and main definitions are listed at the end of this introduction. We will always assume that the closed and convex set K_ψ^s is not empty, also when not explicitly stated.

Problem $\mathcal{P}(\psi, f)$ admits a unique solution u , that can be characterized as the unique minimizer for

$$\inf_{v \in K_\psi^s} \frac{1}{2} \langle (-\Delta)^s v, v \rangle - \langle f, v \rangle. \quad (1.2)$$

The variational inequality $\mathcal{P}(\psi, f)$ and the free boundary problem (1.1) are naturally related. Any solution $u \in \tilde{H}^s(\Omega)$ to (1.1) coincides with the unique solution to $\mathcal{P}(\psi, f)$, see Remark 3.5. Conversely, if u solves $\mathcal{P}(\psi, f)$ then $(-\Delta)^s u - f$ is a nonnegative distribution on Ω , compare with Theorem 3.2. By analogy with the local case $s = 1$ one can guess that $(-\Delta)^s u = f$ outside the coincidence set $\{u = \psi\}$, at least when u is regular enough. This is essentially the content of Section 3 in [19], where $f = 0$ and ψ is a smooth, rapidly decreasing function on $\Omega = \mathbb{R}^n$, and of Theorems 1.1, 1.2 below.

To study the variational inequality $\mathcal{P}(\psi, f)$ we took inspiration from the classical theory about the local case $s = 1$. In particular, we refer to the fundamental monograph [9] by Kinderlehrer and Stampacchia, and to the pioneering papers [2, 10, 11, 12, 13, 20, 21], among others.

Standard techniques do not apply directly in the fractional case, mostly because of the different behavior of the truncation operator $v \mapsto v^+$, $H^s(\mathbb{R}^n) \rightarrow H^s(\mathbb{R}^n)$. Section 2 is entirely devoted to this subject; we collect there some lemmata that might have an independent interest.

We take advantage of the results in Section 2 to obtain equivalent and useful formulations for $\mathcal{P}(\psi, f)$, and to prove continuous dependence theorems upon the data f and ψ , see Sections 3 and 4, respectively.

Some extra difficulties arise from having settled a nonlocal problem on a bounded domain, producing at least, but not only, the same (partially solved) technical difficulties as for the unconstrained problem $(-\Delta)^s u = f$, $u \in \tilde{H}^s(\Omega)$ (see for instance [6], [16], [17] and references there-in, for regularity issues).

Our main results proved in Section 5. They involve the unique solution ω_f to

$$(-\Delta)^s \omega_f = f \quad \text{in } \Omega, \quad \omega_f \in \tilde{H}^s(\Omega). \quad (1.3)$$

Theorem 1.1 *Assume that ψ and $f \in \tilde{H}^s(\Omega)'$ satisfy the following conditions:*

- A1) $(\psi - \omega_f)^+ \in \tilde{H}^s(\Omega)$;
- A2) $(-\Delta)^s(\psi - \omega_f)^+ - f$ is a locally finite signed measure on Ω ;
- A3) $((-\Delta)^s(\psi - \omega_f)^+ - f)^+ \in L_{\text{loc}}^p(\Omega)$ for some $p \in [1, \infty]$.

Let $u \in \tilde{H}^s(\Omega)$ be the unique solution to $\mathcal{P}(\psi, f)$. Then the following facts hold.

- i) $(-\Delta)^s u - f \in L_{\text{loc}}^p(\Omega)$;
- ii) $0 \leq (-\Delta)^s u - f \leq ((-\Delta)^s(\psi - \omega_f)^+ - f)^+ \quad \text{a.e. on } \Omega$;
- iii) $(-\Delta)^s u = f \quad \text{a.e. on } \{u > \psi\}$.

In particular, u solves the free boundary problem (1.1).

Theorem 1.2 *Assume that Ω is a bounded Lipschitz domain satisfying the exterior ball condition. Let $\psi \in C^0(\overline{\Omega})$ be a given obstacle, such that K_ψ^s is not empty, $\psi \leq 0$ on $\partial\Omega$ and $f \in L^p(\Omega)$, for some exponent $p > n/2s$.*

Then the unique solution u to $\mathcal{P}(\psi, f)$ is continuous on \mathbb{R}^n and solves the free boundary problem (1.1).

Our results plainly cover the non-homogeneous Dirichlet's free boundary problem

$$\begin{cases} u \geq \psi & \text{in } \Omega \\ (-\Delta)^s u \geq f & \text{in } \Omega \\ (-\Delta)^s u = f & \text{in } \{u > \psi\} \\ u = g & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \end{cases}$$

under appropriate assumptions on the datum g . Notice indeed that u solves the related variational inequality if and only if $u - g$ solves $\mathcal{P}(\psi - g, f + (-\Delta)^s g)$.

Free boundary problems for the operator $(-\Delta)^s u + u$ can be considered as well, with minor modifications in the statements and in the proofs.

Notation The definition of the fractional Laplacian $(-\Delta)^s$ involves the Fourier transform:

$$\mathcal{F}[(-\Delta)^s u] = |\xi|^{2s} \mathcal{F}[u], \quad \mathcal{F}[u](\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} u(x) dx.$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We adopt the standard notation

$$\begin{aligned} H^s(\mathbb{R}^n) &= \{u \in L^2(\mathbb{R}^n) \mid (-\Delta)^{\frac{s}{2}} u \in L^2(\mathbb{R}^n)\}, \\ \tilde{H}^s(\Omega) &= \{u \in H^s(\mathbb{R}^n) \mid u \equiv 0 \text{ on } \mathbb{R}^n \setminus \overline{\Omega}\}. \end{aligned}$$

We endow $H^s(\mathbb{R}^n)$ and $\tilde{H}^s(\Omega)$ with their natural Hilbertian structures. We recall that the norm of u in $\tilde{H}^s(\Omega)$ is given by the $L^2(\mathbb{R}^n)$ -norm of $(-\Delta)^{\frac{s}{2}} u$.

We do not make any assumption on Ω . Thus $\partial\Omega$ might be very irregular, even a fractal, and $C_0^\infty(\Omega)$ might be not dense in $\tilde{H}^s(\Omega)$. Notice that $\tilde{H}^s(\Omega)$ coincides with $\tilde{H}^s(\Omega')$, whenever $\overline{\Omega} = \overline{\Omega}'$.

We denote by $\langle \cdot, \cdot \rangle$ the duality product between $\tilde{H}^s(\Omega)$ and its dual $\tilde{H}^s(\Omega)'$. In particular, $(-\Delta)^s u \in \tilde{H}^s(\Omega)'$ for any $u \in \tilde{H}^s(\Omega)$, and

$$\langle (-\Delta)^s u, v \rangle = \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} u \cdot (-\Delta)^{\frac{s}{2}} v dx = \int_{\mathbb{R}^n} |\xi|^{2s} \mathcal{F}[u] \overline{\mathcal{F}[v]} d\xi.$$

2 Truncations

For measurable functions v, w we put, as usual,

$$v \vee w = \max\{v, w\}, \quad v \wedge w = \min\{v, w\}, \quad v^+ = v \vee 0, \quad v_- = -(v \wedge 0),$$

so that $v = v^+ - v_-$. It is well known that $v \vee w \in H^s(\mathbb{R}^n)$ and $v \wedge w \in H^s(\mathbb{R}^n)$ if $v, w \in H^s(\mathbb{R}^n)$.

Lemma 2.1 *Let $v \in H^s(\mathbb{R}^n)$. Then*

- i) $\langle (-\Delta)^s v^+, v^- \rangle = \langle (-\Delta)^s v^-, v^+ \rangle \leq 0;$
- ii) $\langle (-\Delta)^s v, v^- \rangle + \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v^-|^2 dx \leq 0;$
- iii) $\langle (-\Delta)^s v, v^+ \rangle - \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v^+|^2 dx \geq 0.$

In addition, if $v \in H^s(\mathbb{R}^n)$ does not have constant sign, then all the above inequalities are strict.

Proof. In [14, Theorem 6], the Caffarelli-Silvestre extension argument [5] has been used to check that

$$\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} |v||^2 dx < \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v|^2 dx,$$

whenever v changes sign. That is,

$$\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} (v^+ + v^-)|^2 dx < \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} (v^+ - v^-)|^2 dx.$$

The conclusion is immediate. □

Remark 2.2 *One can use ii) in Lemma 2.1 to get the well known weak maximum principle, that is, if $u \in \tilde{H}^s(\Omega)$ and $(-\Delta)^s u \geq 0$ in Ω then $u \geq 0$ in Ω .*

Corollary 2.3 *Let v_h be a sequence in $H^s(\mathbb{R}^n)$ such that v_h converges to a nonpositive function in $H^s(\mathbb{R}^n)$. Then $v_h^+ \rightarrow 0$ in $H^s(\mathbb{R}^n)$.*

Proof. Statement iii) in Lemma 2.1 provides the estimate

$$\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v_h^+|^2 dx \leq \langle (-\Delta)^s v_h, v_h^+ \rangle, \tag{2.1}$$

that gives us the boundedness of the sequence v_h^+ in $H^s(\mathbb{R}^n)$. Since $v_h^+ \rightarrow 0$ in $L^2(\mathbb{R}^n)$, we have $v_h^+ \rightarrow 0$ weakly in $H^s(\mathbb{R}^n)$. Thus $\langle (-\Delta)^s v_h, v_h^+ \rangle \rightarrow 0$, as $(-\Delta)^s v_h$ converges in $H^s(\mathbb{R}^n)'$, and the conclusion follows from (2.1). □

Lemma 2.4 *Let $v \in \tilde{H}^s(\Omega)$ and $m > 0$. Then $(v + m)^-, (v - m)^+, v \wedge m \in \tilde{H}^s(\Omega)$ and*

$$\begin{aligned} i) \quad & \langle (-\Delta)^s v, (v + m)^- \rangle + \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} (v + m)^-|^2 dx \leq 0; \\ ii) \quad & \langle (-\Delta)^s v, (v - m)^+ \rangle - \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} (v - m)^+|^2 dx \geq 0; \\ iii) \quad & \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} (v \wedge m)|^2 dx \leq \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v|^2 dx - \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} (v - m)^+|^2 dx. \end{aligned}$$

Proof. Clearly, $(v + m)^- \in L^2(\mathbb{R}^n)$ and $(v + m)^- \equiv 0$ outside Ω . Fix a cutoff function $\eta \in C_0^\infty(\mathbb{R}^n)$, with $0 \leq \eta \leq 1$, and such that $\eta \equiv 1$ in a ball containing $\bar{\Omega}$. Then $(v + m)^- = (v + m\eta)^- \in \tilde{H}^s(\Omega)$, as trivially $m\eta \in H^s(\mathbb{R}^n)$.

For any integer $h \geq 1$ we set

$$\eta_h(x) = \eta\left(\frac{x}{h}\right),$$

so that $\eta_h \rightarrow 1$ pointwise. A direct computation shows that

$$(-\Delta)^s \eta_h(x) = h^{-2s} \left((-\Delta)^s \eta \right) \left(\frac{x}{h} \right) \rightarrow 0 \quad \text{in } L_{\text{loc}}^2(\mathbb{R}^n). \quad (2.2)$$

By *ii)* in Lemma 2.1 we have that

$$\begin{aligned} 0 & \geq \langle (-\Delta)^s (v + m\eta_h), (v + m)^- \rangle + \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} (v + m)^-|^2 dx \\ & = \langle (-\Delta)^s v, (v + m)^- \rangle + \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} (v + m)^-|^2 dx + m \int_{\Omega} ((-\Delta)^s \eta_h) (v + m)^- dx \\ & = \langle (-\Delta)^s v, (v + m)^- \rangle + \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} (v + m)^-|^2 dx + o(1), \end{aligned}$$

by (2.2) and since $(v + m)^-$ has compact support in Ω . Claim *i)* is proved. To check *ii)* notice that $(v - m)^+ = ((-v) + m)^-$ and then use *i)* with $(-v)$ instead of v .

It remains to prove *iii)*. Notice that $v \wedge m = v - (v - m)^+$. Hence $v \wedge m \in \tilde{H}^s(\Omega)$. Using *ii)* we get

$$\begin{aligned} \|(-\Delta)^{\frac{s}{2}} (v \wedge m)\|^2 & = \|(-\Delta)^{\frac{s}{2}} v\|^2 - 2\langle (-\Delta)^s v, (v - m)^+ \rangle + \|(-\Delta)^{\frac{s}{2}} (v - m)^+\|^2 \\ & \leq \|(-\Delta)^{\frac{s}{2}} v\|^2 - \|(-\Delta)^{\frac{s}{2}} (v - m)^+\|^2. \end{aligned}$$

The proof is complete. \square

3 Equivalent formulations

We start this section by introducing a crucial notion.

Definition 3.1 *A function $\mathcal{U} \in \tilde{H}^s(\Omega)$ is a supersolution for $(-\Delta)^s v = f$ if*

$$\langle (-\Delta)^s \mathcal{U} - f, \varphi \rangle \geq 0 \quad \text{for any } \varphi \in \tilde{H}^s(\Omega), \varphi \geq 0.$$

The above definition extends the usually adopted one in the local case $s = 1$, see [9, Definition 6.3]. A different definition of supersolution is used in [19] for $f = 0$. We refer to [19, Subsection 2.10], for a stimulating discussion on this subject.

Theorem 3.2 *Let $u \in K_\psi^s$. The following sentences are equivalent.*

- a) *u is the solution to problem $\mathcal{P}(\psi, f)$;*
- b) *u is the smallest supersolution for $(-\Delta)^s v = f$ in the convex set K_ψ^s . That is, $\mathcal{U} \geq u$ almost everywhere in Ω , for any supersolution $\mathcal{U} \in K_\psi^s$;*
- c) *u is a supersolution for $(-\Delta)^s v = f$ and*

$$\langle (-\Delta)^s u - f, (v - u)^- \rangle = 0 \quad \text{for any } v \in K_\psi^s.$$

- d) *$\langle (-\Delta)^s v - f, v - u \rangle \geq 0$ for any $v \in K_\psi^s$.*

Proof. *a) \iff b). Assume that u solves $\mathcal{P}(\psi, f)$. Fix any nonnegative $\varphi \in \tilde{H}^s(\Omega)$. Testing $\mathcal{P}(\psi, f)$ with $u + \varphi \in K_\psi^s$ one gets $\langle (-\Delta)^s u - f, \varphi \rangle \geq 0$, that proves that u is a supersolution.*

Next, take any supersolution $\mathcal{U} \in K_\psi^s$. Then $u - (u - \mathcal{U})^+ = \mathcal{U} \wedge u \in K_\psi^s$. Thus

$$\langle (-\Delta)^s u - f, -(u - \mathcal{U})^+ \rangle \geq 0.$$

On the other hand, from $(-\Delta)^s \mathcal{U} - f \geq 0$ we get

$$\langle (-\Delta)^s \mathcal{U} - f, (u - \mathcal{U})^+ \rangle \geq 0.$$

Adding the above inequalities we arrive at

$$0 \geq \langle (-\Delta)^s (u - \mathcal{U}), (u - \mathcal{U})^+ \rangle \geq \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} (u - \mathcal{U})^+|^2 dx,$$

thanks to *iii*) in Lemma 2.1. Thus $(u - \mathcal{U})^+ = 0$ almost everywhere in Ω , that is, $u \leq \mathcal{U}$ and proves that *a*) implies *b*).

Conversely, assume that u satisfies *b*) and let \tilde{u} be the solution to $\mathcal{P}(\psi, f)$. We already know that $a) \Rightarrow b)$. Thus u and \tilde{u} must coincide, because both obey the condition of being the smallest supersolution to $(-\Delta)^s v = f$ in K_ψ^s . Hence, *a*) holds.

a) \iff c). Let u be the solution to $\mathcal{P}(\psi, f)$. We already know that u is supersolution. Fix any function $v \in K_\psi^s$. Notice that

$$u + (v - u)^- \geq u \geq \psi, \quad u - (v - u)^- = v \wedge u \geq \psi.$$

Thus, testing $\mathcal{P}(\psi, f)$ with $u \pm (v - u)^-$ we get $\langle (-\Delta)^s u - f, \pm (v - u)^- \rangle \geq 0$, that is, *c*) holds.

Conversely, assume that u satisfies *c*). Let $\tilde{u} \in K_\psi^s$ be the solution to $\mathcal{P}(\psi, f)$. We already proved that \tilde{u} is the smallest supersolution in K_ψ^s . In particular, $\tilde{u} \leq u$ and thus

$$\langle (-\Delta)^s u - f, u - \tilde{u} \rangle = \langle (-\Delta)^s u - f, (\tilde{u} - u)^- \rangle = 0$$

by the assumption *c*) on u . Since \tilde{u} solves $\mathcal{P}(\psi, f)$, we also get

$$\langle (-\Delta)^s \tilde{u} - f, u - \tilde{u} \rangle \geq 0.$$

Subtracting, we infer $\langle (-\Delta)^s (u - \tilde{u}), u - \tilde{u} \rangle \leq 0$, that is, $u = \tilde{u}$.

a) \iff d). Clearly *a*) implies *d*) because

$$\begin{aligned} & \langle (-\Delta)^s v - f, v - u \rangle \\ &= \langle (-\Delta)^s u - f, v - u \rangle + \langle (-\Delta)^s (v - u), v - u \rangle \geq \langle (-\Delta)^s u - f, v - u \rangle. \end{aligned}$$

Now assume that u satisfies *d*) and fix any $v \in K_\psi^s$. From $\frac{v+u}{2} \in K_\psi^s$ and *d*) we obtain

$$\begin{aligned} 0 &\leq 2 \langle (-\Delta)^s \left(\frac{v+u}{2} \right) - f, \frac{v+u}{2} - u \rangle = \frac{1}{2} \langle (-\Delta)^s (v+u), v-u \rangle - \langle f, v-u \rangle \\ &= \left(\frac{1}{2} \langle (-\Delta)^s v, v \rangle - \langle f, v \rangle \right) - \left(\frac{1}{2} \langle (-\Delta)^s u, u \rangle - \langle f, u \rangle \right). \end{aligned}$$

Thus u solves the minimization problem (1.2), that is, u solves $\mathcal{P}(\psi, f)$. \square

Remark 3.3 In the local case $s = 1$, the equivalence between a) and d) is commonly known as Minty's lemma, see [13].

Corollary 3.4 Let $f_1, f_2 \in \tilde{H}^s(\Omega)'$ and let u_i be the solution to $\mathcal{P}(\psi, f_i)$, $i = 1, 2$. If $f_1 \geq f_2$ in the sense of distributions, then $u_1 \geq u_2$ a.e. in Ω .

Proof. The function u_1 is a supersolution for $(-\Delta)^s v = f_2$ and $u_1 \in K_\psi^s$. Hence $u_1 \geq u_2$, by statement b) in Theorem 3.2. \square

Remark 3.5 Let $u \in \tilde{H}^s(\Omega)$ be a solution to (1.1). Then $(-\Delta)^s u - f$ can be identified with a nonnegative Radon measure on Ω having support in $\{u = \psi\}$. If $v \in K_\psi^s$, then $(v - u)^-$ vanishes on $\{u = \psi\}$. Thus $\langle (-\Delta)^s u - f, (v - u)^- \rangle = 0$, hence u solves $\mathcal{P}(\psi, f)$ by Theorem 3.2.

4 Continuous dependence results

Theorem 4.1 Let ψ_1, ψ_2 be given obstacles, $f \in \tilde{H}^s(\Omega)'$ and let u_i be the solution to $\mathcal{P}(\psi_i, f)$, $i = 1, 2$. If $\psi_1 - \psi_2 \in L^\infty(\Omega)$, then $u_1 - u_2$ is bounded, and

$$i) \quad \|(u_1 - u_2)^+\|_\infty \leq \|(\psi_1 - \psi_2)^+\|_\infty, \quad ii) \quad \|(u_1 - u_2)^-\|_\infty \leq \|(\psi_1 - \psi_2)^-\|_\infty.$$

Proof. Put $m := \|(\psi_1 - \psi_2)^+\|_\infty$. Since $(u_2 - u_1 + m)^- \in \tilde{H}^s(\Omega)$ by Lemma 2.4, then

$$v_1 := u_1 - (u_2 - u_1 + m)^- = (u_2 + m) \wedge u_1 \in K_{\psi_1}^s.$$

Hence we can use v_1 as test function in $\mathcal{P}(\psi_1, f)$ to get

$$\langle (-\Delta)^s u_1 - f, -(u_2 - u_1 + m)^- \rangle \geq 0.$$

On the other hand, we can test $\mathcal{P}(\psi_2, f)$ with $u_2 + (u_2 - u_1 + m)^- \in K_{\psi_2}^s$. Hence

$$\langle (-\Delta)^s u_2 - f, (u_2 - u_1 + m)^- \rangle \geq 0.$$

Adding and taking i) of Lemma 2.4 into account, we arrive at

$$-\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}(u_2 - u_1 + m)^-|^2 dx \geq \langle (-\Delta)^s(u_2 - u_1), (u_2 - u_1 + m)^- \rangle \geq 0.$$

Hence, $(u_2 - u_1 + m)^- = 0$. We have proved that $(u_1 - u_2)^+ \leq m$ a.e. in Ω , hence i) holds. Inequality ii) can be proved in the same way. \square

Corollary 4.2 *Let $\psi \in L^\infty(\Omega)$ and $f \in L^p(\Omega)$, with $p \in (1, \infty)$, $p > n/2s$. Let $u \in \tilde{H}^s(\Omega)$ be the unique solution to $\mathcal{P}(\psi, f)$. Then $u \in L^\infty(\Omega)$ and*

$$\psi \vee \omega_f \leq u \leq \|\psi^+\|_\infty + c\|f^+\|_p \quad \text{a.e. in } \Omega, \quad (4.1)$$

where ω_f solves (1.3) and c depends only on n, s, p and Ω . In particular, if $f = 0$ then

$$\psi^+ \leq u \leq \|\psi^+\|_\infty.$$

Proof. First of all, notice that $f \in \tilde{H}^s(\Omega)'$ by Sobolev embedding theorem. Since u is supersolution of (1.3), the first inequality in (4.1) follows by the maximum principle in Remark 2.2.

Denote by ω_{f^+} the unique solution to (1.3) with f replaced by f^+ . If $n > 2s$ we use convolution to define

$$U = c_1|x|^{2s-n} * (f^+ \cdot \chi_\Omega).$$

For proper choice of the constant c_1 , U solves $(-\Delta)^s U = f^+ \cdot \chi_\Omega$ in \mathbb{R}^n . Convolution estimates give $U \leq c\|f^+\|_p$ on \mathbb{R}^n . By the maximum principle, $\omega_{f^+} \leq U$ on Ω , hence $\omega_{f^+} \leq c\|f^+\|_p$. For $n = 1 \leq 2s$ this inequality also holds, see, e.g., [16, Remark 1.5].

Now let u_1 be the unique solution of $\mathcal{P}(\psi, f^+)$. Then $u_1 \geq u$ by Corollary 3.4. Finally, we can consider ω_{f^+} as the solution of the problem $\mathcal{P}(\omega_{f^+}, f^+)$. Theorem 4.1 gives

$$u \leq (u_1 - \omega_{f^+})^+ + \omega_{f^+} \leq \|(\psi - \omega_{f^+})^+\|_\infty + \omega_{f^+},$$

and the last inequality in (4.1) follows. \square

Roughly speaking, Theorem 4.1 concerns the continuity of $L^\infty \ni \psi \mapsto u \in L^\infty$. The next result gives the continuity of the arrow $L^\infty \ni \psi \mapsto u \in \tilde{H}^s(\Omega)$.

Theorem 4.3 *Let $\psi_h \in L^\infty(\Omega)$ be a sequence of obstacles and let $f \in \tilde{H}^s(\Omega)'$ be given. Assume that there exists $v_0 \in \tilde{H}^s(\Omega)$, such that $v_0 \geq \psi_h$ for any h .*

Denote by u_h the solution to the obstacle problem $\mathcal{P}(\psi_h, f)$. If $\psi_h \rightarrow \psi$ in $L^\infty(\Omega)$, then $u_h \rightarrow u$ in $\tilde{H}^s(\Omega)$, where u is the solution to the limiting problem $\mathcal{P}(\psi, f)$.

Proof. Let u be the solution to $\mathcal{P}(\psi, f)$. We already know from Theorem 4.1 that $\|u - u_h\|_\infty \leq \|\psi - \psi_h\|_\infty$. Hence, in particular, $u_h \rightarrow u$ a.e. in Ω . Now, test $\mathcal{P}(\psi_h, f)$

with v_0 to obtain that

$$\langle (-\Delta)^s u_h, u_h \rangle \leq \langle (-\Delta)^s u_h - f, v_0 \rangle + \langle f, u_h \rangle.$$

Hence, the sequence u_h is bounded in $\tilde{H}^s(\Omega)$. Therefore, $u_h \rightarrow u$ weakly in $\tilde{H}^s(\Omega)$. To prove that $u_h \rightarrow u$ in the $\tilde{H}^s(\Omega)$ norm we only need to show that

$$\limsup_{h \rightarrow \infty} \| (-\Delta)^{\frac{s}{2}} u_h \|_2 \leq \| (-\Delta)^{\frac{s}{2}} u \|_2.$$

For any $\varepsilon > 0$ we introduce the function

$$v_\varepsilon = u + (v_0 - u) \wedge \varepsilon.$$

Since $\psi_h \rightarrow \psi$ in $L^\infty(\Omega)$, we have $v_\varepsilon \geq \psi_h$ for h large enough. Using v_ε as test function in $\mathcal{P}(\psi_h, f)$ we get

$$\langle (-\Delta)^s u_h - f, u + (v_0 - u) \wedge \varepsilon - u_h \rangle \geq 0,$$

and hence

$$\| (-\Delta)^{\frac{s}{2}} u_h \|_2^2 = \langle (-\Delta)^s u_h, u_h \rangle \leq \langle (-\Delta)^s u_h - f, u + (v_0 - u) \wedge \varepsilon \rangle + \langle f, u_h \rangle.$$

Letting $h \rightarrow \infty$ we infer

$$\begin{aligned} \limsup_{h \rightarrow \infty} \| (-\Delta)^{\frac{s}{2}} u_h \|_2^2 &\leq \langle (-\Delta)^s u - f, u + (v_0 - u) \wedge \varepsilon \rangle + \langle f, u \rangle \\ &= \| (-\Delta)^{\frac{s}{2}} u \|_2^2 + \langle (-\Delta)^s u - f, (v_0 - u) \wedge \varepsilon \rangle. \end{aligned} \quad (4.2)$$

Now we let $\varepsilon \rightarrow 0$. Clearly $(v_0 - u) \wedge \varepsilon \rightarrow -(v_0 - u)^-$ in $L^2(\Omega)$. In addition, the functions $(v_0 - u) \wedge \varepsilon$ are uniformly bounded in $\tilde{H}^s(\Omega)$ by *iii* in Lemma 2.4. Thus $(v_0 - u) \wedge \varepsilon \rightarrow -(v_0 - u)^-$ weakly in $\tilde{H}^s(\Omega)$. Thus, from (4.2) we get

$$\limsup_{h \rightarrow \infty} \| (-\Delta)^{\frac{s}{2}} u_h \|_2^2 \leq \| (-\Delta)^{\frac{s}{2}} u \|_2^2 - \langle (-\Delta)^s u - f, (v_0 - u)^- \rangle = \| (-\Delta)^{\frac{s}{2}} u \|_2^2$$

since u solves $\mathcal{P}(\psi, f)$, and therefore it satisfies condition *c*) in Theorem 3.2. Thus $u_h \rightarrow u$ in $\tilde{H}^s(\Omega)$. \square

Next we deal with the continuity of the arrow $H^s \ni \psi \mapsto u \in \tilde{H}^s$.

Theorem 4.4 *Let $\psi_h \in H^s(\mathbb{R}^n)$ be a sequence of obstacles such that $\psi_h^+ \in \tilde{H}^s(\Omega)$, and let f_h be a sequence in $\tilde{H}^s(\Omega)'$. Assume that*

$$\psi_h \rightarrow \psi \quad \text{in } H^s(\mathbb{R}^n), \text{ and } f_h \rightarrow f \quad \text{in } H^s(\Omega)'.$$

Denote by u_h the solution to the obstacle problem $\mathcal{P}(\psi_h, f_h)$. Then $u_h \rightarrow u$ in $\tilde{H}^s(\Omega)$, where u is the solution to the limiting obstacle problem $\mathcal{P}(\psi, f)$.

Proof. We can assume that $f_h, f = 0$. If not, replace the obstacles ψ_h and ψ with $\psi_h - \omega_{f_h}$ and $\psi - \omega_f$, respectively, see (1.3).

Let u_h solve $\mathcal{P}(\psi_h, 0)$ and let u be the solution to the limiting problem $\mathcal{P}(\psi, 0)$. Recall that u is the unique minimizer for

$$\inf_{v \in K_\psi^s} \langle (-\Delta)^s v, v \rangle. \quad (4.3)$$

Since $u \vee \psi_h = u + (\psi_h - u)^+$ and $\psi_h - u \rightarrow \psi - u \leq 0$, then

$$u \vee \psi_h \rightarrow u \quad \text{in } \tilde{H}^s(\Omega) \quad (4.4)$$

by Corollary 2.3. Moreover, $u \vee \psi_h \in K_{\psi_h}^s$ and thus from $\mathcal{P}(\psi_h, 0)$ we infer

$$\langle (-\Delta)^s u_h, u_h \rangle \leq \langle (-\Delta)^s u_h, u \vee \psi_h \rangle. \quad (4.5)$$

Inequality (4.5) guarantees the boundedness of the sequence u_h in $\tilde{H}^s(\Omega)$. Hence we can assume that $u_h \rightarrow \tilde{u}$ weakly in $\tilde{H}^s(\Omega)$. Since $\psi_h \rightarrow \psi$ and $u_h \rightarrow \tilde{u}$ a.e. in Ω , clearly $\tilde{u} \in K_\psi^s$.

Next, by weak lower semicontinuity, (4.5) and (4.4) we get

$$\langle (-\Delta)^s \tilde{u}, \tilde{u} \rangle \leq \liminf_{h \rightarrow \infty} \langle (-\Delta)^s u_h, u_h \rangle \leq \limsup_{h \rightarrow \infty} \langle (-\Delta)^s u_h, u_h \rangle \leq \langle (-\Delta)^s \tilde{u}, u \rangle. \quad (4.6)$$

Thus

$$\|(-\Delta)^{\frac{s}{2}} \tilde{u}\|_2^2 \leq \|(-\Delta)^{\frac{s}{2}} \tilde{u}\|_2 \|(-\Delta)^{\frac{s}{2}} u\|_2.$$

Hence, $\tilde{u} = u$, as the minimization problem (4.3) admits a unique solution, and (4.6) implies $\|(-\Delta)^{\frac{s}{2}} u_h\|_2 \rightarrow \|(-\Delta)^{\frac{s}{2}} u\|_2$. Hence $u_h \rightarrow u$ strongly in $\tilde{H}^s(\Omega)$. \square

5 Proof of the main results

We start with a preliminary theorem of independent interest, that gives distributional bounds on $(-\Delta)^s u - f$ under mild assumptions on the data.

Theorem 5.1 *Let ψ and $f \in \tilde{H}^s(\Omega)'$ satisfying assumptions A1) and A2) in Theorem 1.1. Let $u \in \tilde{H}^s(\Omega)$ be the unique solution to $\mathcal{P}(\psi, f)$. Then*

$$0 \leq (-\Delta)^s u - f \leq ((-\Delta)^s(\psi - \omega_f)^+ - f)^+ \quad \text{in the distributional sense on } \Omega.$$

Proof. The main tool was inspired by the penalty method by Lewy-Stampacchia [10] and already used for instance in [18] under smoothness assumptions on the data and on the solution.

In order to simplify notations we start the proof with some remarks. First, we can assume that $f = 0$, as we did in the proof of Theorem 4.4. Thus $(-\Delta)^s u \geq 0$ and $u \geq \psi$, that imply $u \geq \psi^+$, use the maximum principle in Remark 2.2. Clearly u is the smallest supersolution to $(-\Delta)^s v = 0$ in $K_{\psi^+}^s$, and hence it solves the obstacle problem $\mathcal{P}(\psi^+, 0)$. In conclusion, it suffices to prove Theorem 5.1 in case $f = 0$ and $\psi \geq 0$ in \mathbb{R}^n . Our aim is to show that

$$0 \leq (-\Delta)^s u \leq ((-\Delta)^s \psi)^+ \quad \text{in the distributional sense on } \Omega, \quad (5.1)$$

for $\psi \in \tilde{H}^s(\Omega)$, $\psi \geq 0$, such that $(-\Delta)^s \psi$ is a measure on Ω .

The proof of (5.1) will be achieved in few steps.

Step 1 *Assume $(-\Delta)^s \psi \in L^p(\Omega)$ for any large exponent $p > 1$. Then (5.1) holds.*

We take $p \geq \frac{2n}{n+2s}$, that is needed only if $n > 2s$. Then $\tilde{H}^s(\Omega) \hookrightarrow L^{p'}(\Omega)$ and $L^p(\Omega) \subset \tilde{H}^s(\Omega)'$ by Sobolev embeddings. In particular $((-\Delta)^s \psi)^+ \in \tilde{H}^s(\Omega)'$.

Take a function $\theta_\varepsilon \in C^\infty(\mathbb{R})$ such that $0 \leq \theta_\varepsilon \leq 1$, and

$$\theta_\varepsilon(t) = 1 \quad \text{for } t \leq 0, \quad \theta_\varepsilon(t) = 0 \quad \text{for } t \geq \varepsilon.$$

By standard variational methods we have that there exists a unique $u_\varepsilon \in \tilde{H}^s(\Omega)$ that weakly solves

$$(-\Delta)^s u_\varepsilon = \theta_\varepsilon(u_\varepsilon - \psi) ((-\Delta)^s \psi)^+ \quad \text{in } \Omega.$$

We claim that

$$u \leq u_\varepsilon \leq u + \varepsilon \quad \text{a.e. in } \Omega.$$

By *iii)* in Lemma 2.1 we can estimate

$$\begin{aligned} \|(-\Delta)^{\frac{s}{2}}(\psi - u_\varepsilon)^+\|_2^2 &\leq \langle (-\Delta)^s(\psi - u_\varepsilon), (\psi - u_\varepsilon)^+ \rangle \\ &\leq \int_{\Omega} ((-\Delta)^s \psi)^+ (1 - \theta_\varepsilon(u_\varepsilon - \psi)) (\psi - u_\varepsilon)^+ dx = 0. \end{aligned}$$

Hence, $u_\varepsilon \geq \psi$. Since $(-\Delta)^s u_\varepsilon \geq 0$, then $u_\varepsilon \geq u$ by *b)* in Theorem 3.2. Next, we use *iii)* in Lemma 2.4 and $(-\Delta)^s u \geq 0$ to estimate

$$\begin{aligned} \|(-\Delta)^{\frac{s}{2}}(u_\varepsilon - u - \varepsilon)^+\|_2^2 &\leq \langle (-\Delta)^s(u_\varepsilon - u), (u_\varepsilon - u - \varepsilon)^+ \rangle \\ &\leq \int_{\Omega} ((-\Delta)^s \psi)^+ \theta_\varepsilon(u_\varepsilon - \psi) (u_\varepsilon - u - \varepsilon)^+ dx = 0. \end{aligned}$$

Thus $u_\varepsilon \leq u + \varepsilon$, and the claim is proved. In particular, we have that $\|u_\varepsilon - u\|_\infty \rightarrow 0$ as $\varepsilon \rightarrow 0$. Therefore, for any nonnegative test function $\varphi \in C_0^\infty(\Omega)$ we have that

$$\begin{aligned} \langle (-\Delta)^s u, \varphi \rangle &= \int_{\Omega} u (-\Delta)^s \varphi dx = \int_{\Omega} u_\varepsilon (-\Delta)^s \varphi dx + o(1) \\ &= \langle (-\Delta)^s u_\varepsilon, \varphi \rangle + o(1) \leq \langle ((-\Delta)^s \psi)^+, \varphi \rangle + o(1), \end{aligned}$$

that readily gives $(-\Delta)^s u \leq ((-\Delta)^s \psi)^+$ in the distributional sense in Ω .

Step 2 *Approximation argument.*

Fix a small $\varepsilon > 0$ and put $\Omega_\varepsilon := \{x \in \Omega \mid \text{dist}(x, \Omega) < \varepsilon\}$. The convex set

$$K_\varepsilon = \{v \in \tilde{H}^s(\Omega_\varepsilon) \mid v \geq \psi \text{ a.e. on } \mathbb{R}^n\}$$

contains K_ψ^s , hence it is not empty. We denote by u_ε the unique solution to the variational inequality

$$u_\varepsilon \in K_\varepsilon, \quad \langle (-\Delta)^s u_\varepsilon, v - u_\varepsilon \rangle \geq 0 \quad \forall v \in K_\varepsilon, \quad (\mathcal{P}_\varepsilon)$$

so that $u_\varepsilon \in \tilde{H}^s(\Omega_\varepsilon)$ and is nonnegative. Next we prove that

$$0 \leq (-\Delta)^s u_\varepsilon \leq ((-\Delta)^s \psi)^+ \quad \text{in the distributional sense on } \Omega. \quad (5.2)$$

For, we approximate ψ in a standard way, via convolution. Let $(\rho_h)_h$ be a sequence of mollifiers such that $\text{supp}(\rho_h) \subset B_{\frac{1}{h}}$ and put $\psi_h = \psi * \rho_h$. Notice that for h large enough, $\psi_h = 0$ outside Ω_ε . Therefore

$$\psi_h \in \tilde{H}^s(\Omega_\varepsilon), \quad \psi_h \rightarrow \psi \quad \text{in } H^s(\mathbb{R}^n). \quad (5.3)$$

The convex set $K_{\varepsilon,h} := \{v \in \tilde{H}^s(\Omega_\varepsilon) \mid v \geq \psi_h\}$ is not empty, as it contains ψ_h . The variational inequality

$$u_h \in K_{\varepsilon,h}, \quad \langle (-\Delta)^s u_h, v - u_h \rangle \geq 0 \quad \forall v \in K_{\varepsilon,h}, \quad (\mathcal{P}_{\varepsilon,h})$$

has a unique solution $u_h \in \tilde{H}^s(\Omega_\varepsilon)$. Theorem 4.4 readily gives that $u_h \rightarrow u_\varepsilon$ in $\tilde{H}^s(\Omega_\varepsilon)$. Since $(-\Delta)^s \psi_h \in L^p(\mathbb{R}^n)$ for any $p \geq 1$, then Step 1 applies. In particular

$$0 \leq (-\Delta)^s u_h \leq ((-\Delta)^s \psi_h)^+ \quad \text{in the distributional sense on } \Omega. \quad (5.4)$$

Next, $((-\Delta)^s \psi)^+ * \rho_h$ is a nonnegative smooth function, and

$$((-\Delta)^s \psi)^+ * \rho_h \geq ((-\Delta)^s \psi) * \rho_h = (-\Delta)^s \psi_h.$$

Thus $((-\Delta)^s \psi)^+ * \rho_h \geq ((-\Delta)^s \psi_h)^+$, and (5.4) implies

$$0 \leq (-\Delta)^s u_h \leq ((-\Delta)^s \psi)^+ * \rho_h \quad \text{in the distributional sense on } \Omega.$$

Claim (5.2) follows, since $((-\Delta)^s \psi)^+ * \rho_h \rightarrow ((-\Delta)^s \psi)^+$ in the sense of measures, and $(-\Delta)^s u_h \rightarrow (-\Delta)^s u_\varepsilon$ in the sense of distributions.

Step 3 Conclusion of the proof.

The last step in the proof consists in passing to the limit along a sequence $\varepsilon \rightarrow 0$. First, we notice that $u \in \tilde{H}^s(\Omega_\varepsilon)$ and in particular $u \in K_\varepsilon$. Therefore, using the variational characterization of the unique solution u_ε to $(\mathcal{P}_\varepsilon)$ we find

$$\frac{1}{2} \langle (-\Delta)^s u_\varepsilon, u_\varepsilon \rangle \leq \frac{1}{2} \langle (-\Delta)^s u, u \rangle. \quad (5.5)$$

Now we fix $\varepsilon_0 > 0$. Thanks to (5.5), we get that the sequence u_ε is bounded in $\tilde{H}^s(\Omega_{\varepsilon_0})$, and therefore we can assume that $u_\varepsilon \rightarrow \tilde{u}$ weakly in $\tilde{H}^s(\Omega_{\varepsilon_0})$. From (5.5) we readily get

$$\frac{1}{2} \langle (-\Delta)^s \tilde{u}, \tilde{u} \rangle \leq \frac{1}{2} \langle (-\Delta)^s u, u \rangle. \quad (5.6)$$

On the other hand, $u_\varepsilon \rightarrow \tilde{u}$ almost everywhere. Hence $\tilde{u} \in \tilde{H}^s(\Omega)$ and $\tilde{u} \geq \psi$ on Ω , that is, $\tilde{u} \in K_\psi^s$. Using the characterization of u as the unique solution to the minimization problem (4.3), from (5.6), (5.5) we get that $\tilde{u} = u$ and $u_\varepsilon \rightarrow u$ in $\tilde{H}^s(\Omega_{\varepsilon_0})$. In particular, $\langle (-\Delta)^s u_\varepsilon, \varphi \rangle \rightarrow \langle (-\Delta)^s u, \varphi \rangle$ for any $\varphi \in C_0^\infty(\Omega)$. Now, from (5.2) we know that $((-\Delta)^s \psi)^+ - (-\Delta)^s u_\varepsilon$ is a nonnegative distribution on Ω . Thus $((-\Delta)^s \psi)^+ - (-\Delta)^s u$ is nonnegative as well, and (5.1) is proved. \square

Proof of Theorem 1.1

Statements *i*) and *ii*) hold by Theorem 5.1. It remains to prove the last claim.

It is not restrictive to assume $f \equiv 0$. Hence u solves $\mathcal{P}(\psi, 0)$, $(-\Delta)^s u \geq 0$ by Theorem 3.2, and u is nonnegative in Ω , see Remark 2.2. Actually u is lower semicontinuous and positive by the strong maximum principle, see for instance [8, Theorem 2.5]. Thus $u \geq \psi^+$ and $\{u > \psi\} = \{u > \psi^+\}$.

Next we use *c*) in Theorem 3.2 with $v = \psi^+ \in \tilde{H}^s(\Omega)$, to get

$$\langle (-\Delta)^s u, u - \psi^+ \rangle = 0.$$

Let Ω' be any domain compactly contained in Ω . We claim that

$$\int_{\Omega'} (-\Delta)^s u \cdot (u - \psi^+) dx = 0. \quad (5.7)$$

Since $(-\Delta)^s u \cdot (u - \psi^+)$ is a measurable nonnegative function then the integral in (5.7) is nonnegative. To prove the opposite inequality we put $g_m = (u - \psi^+) \wedge m$, $m \geq 1$. Let φ be any nonnegative cut off function, with $\varphi \in C_0^\infty(\Omega)$ and $\varphi \equiv 1$ on Ω' . Since $(-\Delta)^s u \geq 0$, $(-\Delta)^s u \in L_{\text{loc}}^1(\Omega)$, $u - \psi^+ \geq \varphi g_m$ and $\varphi g_m \in L^\infty(\Omega)$ has compact support in Ω , we have that

$$0 = \langle (-\Delta)^s u, u - \psi^+ \rangle \geq \langle (-\Delta)^s u, \varphi g_m \rangle = \int_{\Omega} (-\Delta)^s u \cdot (\varphi g_m) dx \geq \int_{\Omega'} (-\Delta)^s u \cdot g_m dx.$$

Next, use the monotone convergence theorem to get

$$0 \geq \lim_{m \rightarrow \infty} \int_{\Omega'} (-\Delta)^s u \cdot g_m dx = \int_{\Omega'} (-\Delta)^s u \cdot (u - \psi^+) dx,$$

that concludes the proof of (5.7).

Now, since Ω' was arbitrarily chosen and $(-\Delta)^s u \cdot (u - \psi^+) \geq 0$, equality (5.7) implies that $(-\Delta)^s u \cdot (u - \psi^+) = 0$ a.e. in Ω , and *iii*) is proved. \square

Remark 5.2 *Theorem 1.1 holds with the same proof also in the local case $s = 1$. Notice that no regularity assumptions on Ω are needed, and the cases $p = 1, p = \infty$ are included as well.*

Remark 5.3 *To obtain better regularity results for u , one can apply the regularity theory for*

$$(-\Delta)^s u = g \in L^p(\Omega) \quad \text{in } \Omega, \quad u \in \tilde{H}^s(\Omega).$$

In particular, if $p > \frac{n}{2s}$ and Ω is Lipschitz and satisfies the exterior ball condition, then u is Hölder continuous in Ω . See for example [16, Proposition 1.4] and [17, Proposition 1.1].

Proof of Theorem 1.2

As usual, we can assume $f = 0$. Fix a small $\varepsilon > 0$, and let ψ_h^ε be a mollification of $\psi - \varepsilon$. Then ψ_h^ε is smooth on $\overline{\Omega}$, $\psi_h^\varepsilon < 0$ on $\partial\Omega$ and $\psi_h^\varepsilon \rightarrow \psi - \varepsilon$ uniformly on $\overline{\Omega}$, as $h \rightarrow \infty$.

By Theorem 1.1, the solution $u_h \in \tilde{H}^s(\Omega)$ to $\mathcal{P}(\psi_h^\varepsilon, 0)$ satisfies $(-\Delta)^s u_h^\varepsilon \in L^p(\Omega)$ and therefore u_h^ε is Hölder continuous, see Remark 5.3. Moreover, the estimates in Theorem 4.1 imply that $u_h^\varepsilon \rightarrow u^\varepsilon$ uniformly on Ω , where u^ε solves $\mathcal{P}(\psi - \varepsilon, 0)$. In particular, $u^\varepsilon \in C^0(\overline{\Omega})$. Finally, use again Theorem 4.1 to get that $u^\varepsilon \rightarrow u$ uniformly, where u solves $\mathcal{P}(\psi, 0)$. In particular, u is continuous on \mathbb{R}^n .

To check the last statement notice that the set $\{u > \psi\} \subseteq \Omega$ is open; for any test function $\varphi \in C^\infty(\{u > \psi\})$ we have that $u \pm t\varphi \in K_\psi^s$ and therefore $t\langle (-\Delta)^s u, \pm\varphi \rangle \geq 0$ for $|t|$ small enough. The conclusion is immediate. \square

Acknowledgments. R. Musina wishes to thank the National Program on Differential equations (DST, Government of India) and IIT Delhi for supporting her visit in January, 2015. A.I. Nazarov is grateful to SISSA (Trieste) for the hospitality in October, 2015.

References

- [1] B. Barrios, A. Figalli and X. Ros-Oton, Global regularity for the free boundary in the obstacle problem for the fractional Laplacian arXiv preprint arXiv:1506.04684 (2015).

- [2] H. R. Brezis and G. Stampacchia, Sur la régularité de la solution d'inéquations elliptiques, *Bull. Soc. Math. France* **96** (1968), 153–180.
- [3] L. Caffarelli and A. Figalli, Regularity of solutions to the parabolic fractional obstacle problem, *J. Reine Angew. Math.* **680** (2013), 191–233.
- [4] L. A. Caffarelli, S. Salsa and L. Silvestre, Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian, *Invent. Math.* **171** (2008), no. 2, 425–461.
- [5] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. Part. Diff. Eqs.* **32** (2007), no. 7-9, 1245–1260.
- [6] M. Cozzi, Interior regularity of solutions of non-local equations in Sobolev and Nikol'skii spaces, preprint (2015).
- [7] N. Garofalo and A. Petrosyan, Some new monotonicity formulas and the singular set in the lower dimensional obstacle problem, *Invent. Math.* **177** (2009), no. 2, 415–461.
- [8] A. Iannizzotto, S. Mosconi and M. Squassina, H^s versus C^0 -weighted minimizers, *NoDEA Nonlinear Differential Equations Appl.* **22** (2015), no. 3, 477–497.
- [9] D. Kinderlehrer and G. Stampacchia, An introduction to variational inequalities and their applications, reprint of the 1980 original, *Classics in Applied Mathematics*, 31, SIAM, Philadelphia, PA, 2000.
- [10] H. Lewy and G. Stampacchia, On the regularity of the solution of a variational inequality, *Comm. Pure Appl. Math.* **22** (1969), 153–188.
- [11] H. Lewy and G. Stampacchia, On the smoothness of superharmonics which solve a minimum problem, *J. Analyse Math.* **23** (1970), 227–236.
- [12] J. L. Lions, Partial differential inequalities, *Uspehi Mat. Nauk* **26** (1971), no. 2(158), 205–263.
- [13] G. J. Minty, Monotone (nonlinear) operators in Hilbert space, *Duke Math. J.* **29** (1962), 341–346.
- [14] R. Musina and A. I. Nazarov, On the Sobolev and Hardy constants for the fractional Navier Laplacian, *Nonlinear Anal.* **121** (2015), 123–129.
- [15] A. Petrosyan and C. A. Pop, Optimal regularity of solutions to the obstacle problem for the fractional Laplacian with drift, *J. Funct. Anal.* **268** (2015), no. 2, 417–472.

- [16] X. Ros-Oton and J. Serra, The extremal solution for the fractional Laplacian, *Calc. Var. Partial Differential Equations* **50** (2014), no. 3-4, 723–750.
- [17] X. Ros-Oton and J. Serra, The Dirichlet problem for the fractional Laplacian: regularity up to the boundary, *J. Math. Pures Appl. (9)* **101** (2014), no. 3, 275–302.
- [18] R. Servadei and E. Valdinoci, Lewy-Stampacchia type estimates for variational inequalities driven by (non) local operators, *Rev. Mat. Iberoamericana*, to appear (2013).
- [19] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, *Comm. Pure Appl. Math.* **60** (2007), no. 1, 67–112.
- [20] N. N. Ural'tseva, The regularity of the solutions of variational inequalities, *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **27** (1972), 211–219.
- [21] N. N. Ural'tseva, On the regularity of solutions of variational inequalities, *Uspekhi Mat. Nauk* **42** (1987), no. 6(258), 151–174, 248. English transl. in *Russian Math. Surveys*, 42, no. 6 (1987), 191-219.