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Abstract

In recent work Woodin has defined new axioms stronger than I0 (the existence of an elementary embedding j from $L(V_{\lambda+1})$ to itself), that involve elementary embeddings between slightly larger models. There is a natural correspondence between I0 and Determinacy, but to extend this correspondence in the new framework we must insist that these elementary embeddings are proper. Previous results validated the definition, showing that there exist elementary embeddings that are not proper, but it was still open whether properness was determined by the structure of the underlaying model or not. This paper proves that this is not the case, defining a model that generates both proper and non-proper elementary embeddings, and compare this new model to the older ones.

Keywords: Large Cardinals, Elementary Embeddings, Sharp, Relative Ordinal-Definability, Closed Games

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1 Introduction

Looking at any chart of large cardinal hypothesis, the dark space at the top of the hierarchy inevitably draws the reader's attention. In 1971, Kunen [4] proved a large cardinal hypothesis (the existence of an elementary embedding from V to itself) to be inconsistent with ZFC, casting a shadow of doubt on the whole structure. After that, much work has been done on refining and weakening already established large cardinals, in what was considered a "safe" setting. However, many other people bravely tried to analyze the virgin territory at the edge of inconsistency. This lead to the definition of the rank-into-rank axioms, usually indicated by I3, I2, I1 and I0. These axioms had a brief period of fame when they were used for proving consistency results of Determinacy axioms, but after some year the same results were obtained with much weaker hypotheses, so their usefulness for this purposes faded. Still, even if nowadays, as with many other very large cardinals, there are no known results of equiconsistency, there is an intrinsic interest in pursuing their investigation.

With time the focus shifted on the strongest of the rank-into-rank axioms, i.e., I0, that is the existence of an elementary embedding $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with critical point less than λ. Woodin, in fact, proved in [10] that $L(V_{\lambda+1})$ under I0 satisfies properties that are strikingly similar to the ones in $L(\mathbb{R})$ under AD, like the Coding Lemma, or the fact that λ^+ is measurable. This interesting outcome prompted an investigation on *indirect* connections with AD instead of direct connections. More on this can be found in [2] and [11].

In [11] Woodin pushes the research in still another direction, by considering axioms that are stronger than I0, with a double goal: to map the obscure ground between I0 and inconsistency, and to find an axiom that is similar to $\mathsf{AD}_{\mathbb{R}}$ in the same way I0 was similar to $\mathsf{AD}^{L(\mathbb{R})}$. These new axioms are of the form "There exists an elementary embedding $j : L(N) \prec L(N)$, with $V_{\lambda+1} \subset N \subset V_{\lambda+2}$ and $\mathrm{crt}(j) < \lambda$": generally the larger the set N, the stronger the axiom. He introduces a nicely absolute increasing sequence of such sets, in this paper called E^0-sequence, that in a certain sense can be considered canonical in the analysis of hypotheses stronger than I0, and that culminates in the $\mathsf{AD}_{\mathbb{R}}$-like axiom, called "E^0_∞ exists". In [11] one can find a captivating discussion on the similarities of E^0_∞ with $\mathsf{AD}_{\mathbb{R}}$ and on its credibility.

The main problem with these new axioms is in maintaining the tie with Determinacy. Since this tie was the driving force behind the exploration of I0, it is very desirable to have similar results: Woodin proved that this is true (for specific N's) if the elementary embedding considered is *proper*. Properness is a particular instance of the Axiom of Replacement that involves the elementary embedding and subsets of $V_{\lambda+1}$, and not only gives Determinacy-like results, but also iterability. Properness appears quite often among elementary embeddings, but in [1] there is an example of an N, part of the E^0-sequence, such that every elementary embedding $j : L(N) \prec L(N)$ is not proper. This raises a doubt: is the properness of the elementary embeddings always depending on the structure of the model? Is it always possible to see one model like $L(N)$ and say with certainty whether its elementary embeddings will be proper or not?

The answer is negative. Theorem 3.13 gives an example of an α that is partially non-proper, i.e., such that there exist elementary embeddings from $L(E^0_\alpha)$ to itself that are proper and that are not proper. It is also possible to localize it in a short initial segment of the E^0-sequence. The proof of this theorem takes almost the whole paper: Section 2 is dedicated to basic notations, definitions and the presentation (without proofs) of already known facts that are useful, while Theorem 3.13 and its proof will completely use up Section 3. Section 4 is a comparison of the results in [1] with the results in this paper, and it ends with a list of open problems.

2 Preliminaries

To avoid confusion or misunderstandings, all notations and standard basic results are collected here.

The double arrow (e.g. $f : a \twoheadrightarrow b$) denotes a surjection.

If M and N are sets or classes, $j : M \prec N$ denotes that j is an elementary embedding from M to N, that is an injective function whose range is an elementary submodel of N. The case in which j is the identity, i.e., if M is an elementary submodel of N, is simply written as $M \prec N$.

If $M \vDash \mathsf{AC}$ or $N \subseteq M$ and $j : M \prec N$ is not the identity, then it moves at least one ordinal. The *critical point*, $\mathrm{crt}(j)$, is the least ordinal moved by j.

Let j be an elementary embedding and $\kappa = \mathrm{crt}(j)$. Define $\kappa_0 = \kappa$ and $\kappa_{n+1} = j(\kappa_n)$. Then $\langle \kappa_n : n \in \omega \rangle$ is the *critical sequence* of j.

Kunen [4] proved that if $M = N = V_\eta$ for some ordinal η, and λ is the supremum of the critical sequence, then η cannot be bigger than $\lambda + 1$ (and of course cannot be smaller than λ).

If X is a set, then $L(X)$ denotes the smallest inner model that contains X; it is defined like L but starting with the transitive closure of $\{X\}$ as $L_0(X)$.

If X is a set, then OD_X denotes the class of the sets that are *ordinal-definable over* X, i.e., the sets that are definable using ordinals, X and elements of X as parameters. HOD_X denotes the class of the sets that are *hereditarily ordinal-definable over* X, i.e., the sets in OD_X such that all the elements of their transitive closure are in OD_X. For example, $L(X) \vDash V = \mathrm{HOD}_X$. One advantage in considering models of HOD_X is the possibility of defining partial Skolem functions. Let $\varphi(v_0, v_1, \ldots, v_n)$ be a formula with $n + 1$ free variables and let $a \in X$. Then:

$$h_{\varphi,a}(x_1,\ldots,x_n) = \begin{cases} y & \text{where } y \text{ is the least in } OD_{\{a\}} \text{ such that} \\ & \quad \varphi(y,x_1,\ldots,x_n) \\ \emptyset & \text{if } \forall x \neg\varphi(x,x_1,\ldots,x_n) \\ \text{not defined} & \text{otherwise} \end{cases}$$

are partial Skolem functions. For every set or class y, $H^{L(X)}(y)$ denotes the closure of y under partial Skolem functions for $L(X)$, and $H^{L(X)}(y) \prec L(X)$.

There are many definitions of the sharp operators: in this article, $X^\sharp$ is considered a complete theory in the language $\mathcal{L}_X^+$, that is the expansion of the language $\{\in\}$ obtained by adding a unary predicate $\mathring{X}$ and constant symbols $\mathring{x}$ and $\mathring{i}_n$, for all $x \in X$ and $n \in \omega$. The constants $\mathring{i}_n$ are used for the indiscernibles and the interpretations of $\mathring{X}$ and $\mathring{x}$ are, respectively, X and x, similarly to the original definition by Solovay [8]. Informally, $X^\sharp$ exists iff there is a class I of indiscernibles in $(L(X), \in, X, (x : x \in X))$ such that every cardinal bigger than $|X|$ is in I and $H^{L(X)}(I, X) = L(X)$. Then $X^\sharp$ is the set of formulas in $\mathcal{L}_X^+$ satisfied by finite sequences of indiscernibles. With the usual methods, $X^\sharp$ can be coded as a subset of $V_\omega \times X$ using Gödel numbers.

The starting point for the sequence of new large cardinal hypotheses that will be considered in this paper is I0:

I0 For some λ there exists a $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$, with $\mathrm{crt}(j) < \lambda$.

The elementary embeddings are considered with critical point less than λ to follow the thread of rank-into-rank axioms: in this case, in fact, I0 implies I1, the existence of an elementary embedding from $V_{\lambda+1}$ to itself. By Kunen's Theorem λ must be the supremum of the critical sequence of j. This means that λ is limit of inaccessible cardinals, so $|V_\lambda| = \lambda$ and V_λ is closed by finite sequences. Therefore every λ-sequence of elements of $V_{\lambda+1}$ can be codified in $V_{\lambda+1}$, and this fact will be used throughout the paper without notice.

Unfortunately there are few published results on I0. Most of the results are in [10] and [11], but it is possible to find something also in [2] and [5].

Lemma 2.1 ([10]). *Let $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ be such that $\mathrm{crt}(j) < \lambda$. Then there exists an $L(V_{\lambda+1})$-ultrafilter $U \subset L(V_{\lambda+1}) \cap V_{\lambda+2}$ such that $\mathrm{Ult}(L(V_{\lambda+1}), U)$ is well-founded. By condensation the collapse of $\mathrm{Ult}(L(V_{\lambda+1}), U)$ is $L(V_{\lambda+1})$, and $j_U : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$, the inverse of the collapse, is an elementary embedding. Moreover, there is an elementary embedding $k_U : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(k_U) > \Theta^{L(V_{\lambda+1})}$ such that $j = k_U \circ j_U$.*

An elementary embedding j is weakly proper if $j = j_U$. In this case, the behaviour of j depends only on a really small set.

Lemma 2.2 ([10]). *Let $j, k : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ be weakly proper. If $j \upharpoonright V_\lambda = k \upharpoonright V_\lambda$, then $j = k$.*

I0 rose to prominence because of its similarities with $\mathsf{AD}^{L(\mathbb{R})}$. These similarities will be described now in a more general setting.

All the stronger large cardinal hypotheses will follow a common blueprint: "There exists an elementary embedding $j : L(N) \prec L(N)$ with $\mathrm{crt}(j) < \lambda$ where $V_{\lambda+1} \subseteq N \subset V_{\lambda+2}$". For clarity, it will always be assumed $N = L(N) \cap V_{\lambda+2}$. For example, I0 follows this blueprint, and also any $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $X \subset V_{\lambda+1}$.

Like in $L(\mathbb{R})$, it is possible to define a cardinal in $L(N)$ that "measures" the largeness of $V_{\lambda+1}$:

Definition 2.3. *Let M be a set or a class such that $V_{\lambda+1} \subseteq M$. Then Θ^M is the supremum of the ordinals α such that there exists $\pi : V_{\lambda+1} \twoheadrightarrow \alpha$ with $\{(a,b) \in V_{\lambda+1} \times V_{\lambda+1} : \pi(a) < \pi(b)\} \in M$. If M is a class, then this is equivalent to the more classical definition:*

$$\Theta^M = \sup\{\alpha : \exists \pi : V_{\lambda+1} \twoheadrightarrow \alpha,\ \pi \in M\}.$$

Note that $\Theta^{L(N)}$ is a cardinal in $L(N)$, and $\lambda^+ < \Theta^{L(N)} \leq (2^\lambda)^+$. Moreover, if $L(N) \cap V_{\lambda+2} = N$ then $\Theta^{L(N)} = \Theta^N$.

There is also a higher equivalent of DC:

Definition 2.4.

$$\mathsf{DC}_\lambda : \quad \forall X\ \forall F : X^{<\lambda} \to \mathcal{P}(X) \setminus \emptyset\ \exists g : \lambda \to X\ \forall \gamma < \lambda\ g(\gamma) \in F(g \upharpoonright \gamma).$$

In certain situations $L(N)$ has properties akin to $L(\mathbb{R})$:

Lemma 2.5 ([11]). *Let $V_{\lambda+1} \subset N \subset V_{\lambda+2}$ be such that $L(N) \vDash V = \mathrm{HOD}_{\{X\} \cup V_{\lambda+1}}$ for some $X \subseteq V_{\lambda+1}$. Then*

- *$\Theta^{L(N)}$ is regular;*
- *$L(N) \vDash \mathsf{DC}_\lambda$.*

Like already hinted, the introduction of an elementary embedding will produce characteristics similar to $\mathsf{AD}^{L(\mathbb{R})}$:

Lemma 2.6 ([11]). *Let $V_{\lambda+1} \subset N \subset V_{\lambda+2}$ be such that there exists $X \subseteq V_{\lambda+1}$ $L(N) \vDash V = \mathrm{HOD}_{\{X\} \cup V_{\lambda+1}}$. Let $j : L(N) \prec L(N)$ be such that $\mathrm{crt}(j) < \lambda$. Then*

- *λ^+ is measurable;*
- *a generalization of the Coding Lemma holds.*

For a description of the Coding Lemma see [6], and for a detailed enunciation of the generalization and the proof of the second part see [2]. One Corollary of the Coding Lemma will be most useful:

Corollary 2.7 ([11])**.** *Let $V_{\lambda+1} \subset N \subset V_{\lambda+2}$ be such that there exists $X \subseteq V_{\lambda+1}$ $L(N) \vDash V = \mathrm{HOD}_{\{X\}\cup V_{\lambda+1}}$, and there exists $j : L(N) \prec L(N)$ with $\mathrm{crt}(j) < \lambda$. Then for every $\gamma < \Theta^{L(N)}$ there exists a surjection $\pi : V_{\lambda+1} \twoheadrightarrow \mathcal{P}^{L(N)}(\gamma)$.*

This means that if γ it's "small", then there are "few" subsets of γ in $L(N)$, and it implies that $\Theta^{L(N)}$ is a weakly inaccessible cardinal in $L(N)$.

To complete the Theorem a generalization of the definition of weakly proper is needed:

Theorem 2.8 ([11])**.** *Let $V_{\lambda+1} \subset N \subset V_{\lambda+2}$ be such that $L(N) \cap V_{\lambda+2} = N$ and let $j : L(N) \prec L(N)$. Then there exists an ultrafilter $U \subset N$ such that $\mathrm{Ult}(L(N), U)$ is well-founded. By condensation the collapse of $\mathrm{Ult}(L(N), U)$ is $L(N)$ and $j_U : L(N) \prec L(N)$, the inverse of the collapse, is an elementary embedding with $\mathrm{crt}(j) < \lambda$. Moreover, there is an elementary embedding $k_U : L(N) \prec L(N)$ that is the identity on N and such that $j = j_U \circ k_U$.*

Definition 2.9. *Let $V_{\lambda+1} \subset N \subset V_{\lambda+2}$ be such that $L(N) \cap V_{\lambda+2} = N$ and let $j : L(N) \prec L(N)$. For every $a \in L(N)$, we will indicate with $\langle a_0, a_1, \dots \rangle$ the iteration of a under the action of j, i.e., $a_0 = a$ and $a_{i+1} = j(a_i)$ for all $i \in \omega$. Then*

- *j is* weakly proper *if $j = j_U$;*
- *j is* proper *if it is weakly proper and if for every $X \in N$, $\langle X_i : i < \omega \rangle \in L(N)$.*

By Theorem 2.8 any elementary embedding $j : L(N) \prec L(N)$ can be factored into two elementary embeddings, $j = j_U \circ k$. The first embedding, j_U, is obtained from an ultrafilter, and it is completely determined by its behaviour on N; the second one, k, is the identity on N and moves only larger cardinals, and hence can be generated by a shift of indiscernibles. In other words: every $j : L(N) \prec L(N)$ has a more important part, the weakly proper embedding j_U that controls the behaviour of j, and a less important part k that comes from a shift of indiscernibles.

Properness has important consequences that strengthen its role:

Lemma 2.10 ([11]). *Let* $V_{\lambda+1} \subset N \subset V_{\lambda+2}$ *be such that* $L(N) \cap V_{\lambda+2} = N$ *and let* $j : L(N) \prec L(N)$ *be proper. Then* j *is finitely iterable, i.e., it is possible to define* $j(j) = j^2$ *and* j^2 *is an elementary embedding from* $L(N)$ *to itself.*

Theorem 2.11 ([11]). *Let* $X \subseteq V_{\lambda+1}$. *Suppose that there exists* $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ *proper with* $\mathrm{crt}(j) < \lambda$. *Then* $\Theta^{L(X,V_{\lambda+1})}$ *is the supremum of ordinals* γ *such that:*

- γ *is weakly inaccessible in* $L(X, V_{\lambda+1})$*;*
- $\gamma = \Theta^{L_\gamma(X,V_{\lambda+1})}$ *and* $j(\gamma) = \gamma$*;*
- *for all* $\beta < \gamma$, $\mathcal{P}(\beta) \cap L(X, V_{\lambda+1}) \in L_\gamma(X, V_{\lambda+1})$*;*
- *for cofinally* $\kappa < \gamma$, κ *is a measurable cardinal in* $L(X, V_{\lambda+1})$ *and this is witnessed by the club filter on a stationary set;*
- $L_\gamma(X, V_{\lambda+1}) \prec L_\Theta(X, V_{\lambda+1})$.

For the equivalent of the theorem in $\mathsf{AD}^{L(\mathbb{R})}$ see [7]. Other consequences are less structural, but nonetheless very useful:

Lemma 2.12 ([11]). *Let* $V_{\lambda+1} \subset N \subset V_{\lambda+2}$ *be such that* $L(N) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$. *If* $j : L(N) \prec L(N)$ *is proper, then the fixed points of* j *are cofinal in* $\Theta^{L(N)}$.

It is immediate to see why: if $\beta < \Theta^{L(N)}$ then there exists a prewellorder Y that codes it, but then $\langle \beta_0, \beta_1, \dots \rangle$ is coded by $\langle Y_0, Y_1, \dots \rangle$ (defined as in Definition 2.9), that is in $L(N)$. So $\langle \beta_0, \beta_1, \dots \rangle \in L(N)$ and since $\Theta^{L(N)}$ is regular in $L(N)$, then $\sup_{i\in\omega} \beta_i < \Theta^{L(N)}$ and is a fixed point.

The next step consists in defining a "standard" sequence of such N's, that is called E^0-sequence. The purpose behind its definition is the attempt to define a new axiom that corresponds to $\mathsf{AD}_\mathbb{R}$ just like I0 corresponded to $\mathsf{AD}^{L(\mathbb{R})}$. The construction of the E^0-sequence, in fact, mimics the construction of the minimum model of $\mathsf{AD}_\mathbb{R}$ (that can be found in [9]), building a sequence of $E^0_\alpha(V_{\lambda+1})$ sets such that $V_{\lambda+1} \subseteq E^0_\alpha(V_{\lambda+1}) \subset V_{\lambda+2}$.

Definition 2.13 ([11]). *Suppose* $V_{\lambda+1} \subset N \subset V_{\lambda+2}$.

- $\mathcal{E}(N)$ *denotes the set of all the elementary embeddings* $k : N \prec N$.
- *Suppose that* $X \subseteq V_{\lambda+1}$. *Then* $N < X$ *if there exists a surjection* $\pi : V_{\lambda+1} \twoheadrightarrow N$ *such that* $\pi \in L(X, V_{\lambda+1})$.

The definition of the E^0_α-sequence is by induction with four steps: 0, limit, successor of a limit and successor of a successor.

Definition 2.14. *Let λ be a limit ordinal with cofinality ω. The sequence*

$$\langle E^0_\alpha(V_{\lambda+1}) : \alpha < \Upsilon_{V_{\lambda+1}}\rangle$$

is the maximum sequence such that the following hold:

1. $E^0_0(V_{\lambda+1}) = L(V_{\lambda+1}) \cap V_{\lambda+2}$;
2. *for* $\alpha < \Upsilon_{V_{\lambda+1}}$ *limit,* $E^0_\alpha(V_{\lambda+1}) = L(\bigcup_{\beta<\alpha} E^0_\beta(V_{\lambda+1})) \cap V_{\lambda+2}$;
3. *for* $\alpha < \Upsilon_{V_{\lambda+1}}$ *limit,*
   - *if* $L(E^0_\alpha(V_{\lambda+1})) \vDash \mathrm{cof}(\Theta^{E^0_\alpha(V_{\lambda+1})}) < \lambda$ *then*
   $$E^0_{\alpha+1}(V_{\lambda+1}) = L((E^0_\alpha(V_{\lambda+1}))^\lambda) \cap V_{\lambda+2};$$
   - *if* $L(E^0_\alpha(V_{\lambda+1})) \vDash \mathrm{cof}(\Theta^{E^0_\alpha(V_{\lambda+1})}) > \lambda$ *then*
   $$E^0_{\alpha+1}(V_{\lambda+1}) = L(\mathcal{E}(E^0_\alpha(V_{\lambda+1}))) \cap V_{\lambda+2};$$
4. *for* $\alpha = \beta + 2 < \Upsilon_{V_{\lambda+1}}$*, there exists* $X \subseteq V_{\lambda+1}$ *such that* $E^0_{\beta+1}(V_{\lambda+1}) = L(X, V_{\lambda+1}) \cap V_{\lambda+2}$ *and* $E^0_\beta(V_{\lambda+1}) < X$*, and*
$$E^0_{\beta+2} = L((X, V_{\lambda+1})^\sharp) \cap V_{\lambda+2}$$
5. $\forall\alpha < \Upsilon_{V_{\lambda+1}}$ $\exists X \subseteq V_{\lambda+1}$ *such that* $E^0_\alpha(V_{\lambda+1}) \subset L(X, V_{\lambda+1})$, $\exists j\colon L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1})$ *proper;*
6. $\forall\alpha$ *limit,* $\alpha + 1 < \Upsilon_{V_{\lambda+1}}$ *iff*
$$\text{if } L(E^0_\alpha(V_{\lambda+1})) \vDash \mathrm{cof}(\Theta^{E^0_\alpha(V_{\lambda+1})}) > \lambda \\ \text{then } \exists Z \in E^0_\alpha(V_{\lambda+1})\ L(E^0_\alpha(V_{\lambda+1})) \vDash V = \mathrm{HOD}_{V_{\lambda+1}\cup\{Z\}}.$$

The previous definition consists of two parts: the first four points give the real definition of the sequence, describing exactly what $E^0_\alpha(V_{\lambda+1}$ is, the last two points are conditions that guarantee a smooth application of the induction. For example, point 5 implies that indeed the sharp appearing in point 4 exists, and point 5 and 6 combined prove that $E^0_{\alpha+1}(V_{\lambda+1})$ in point 3 can be seen as $L(X, V_{\lambda+1})$ for some $X \subset V_{\lambda+1}$, justifing the inductive hypothesis for point 4. For more details about the balance of this construction, see [11].

For the rest of this article $V_{\lambda+1}$ will be omitted, and E^0_α and Υ will be written instead of $E^0_\alpha(V_{\lambda+1})$ and $\Upsilon_{V_{\lambda+1}}$. This is a slight abuse of notation, but since the λ is considered fixed, it will not create problems.

The complex nature of this definition is partly due to the necessity of keeping the E^0-sequence nicely absolute, and also having some condensation property. The following lemma, whose proof is implicit in [11], is a summary of both results:

Lemma 2.15. *Let $\beta < \Upsilon$, let M be a model of ZF such that $E^0_\beta \subseteq M$ and let $\bar{M}$ be M's transitive collapse. If M is an elementary substructure of $L(E^0_\eta)$ for some $\eta < \Upsilon$, then there exists $\beta \leq \gamma \leq \eta$ such that either $\bar{M} = L(E^0_\gamma)$ or else $\bar{M} = L_\zeta(E^0_\gamma)$ for some ζ. Moreover, if $j : \bar{M} \prec L(E^0_\eta)$ is the inverse of the Mostowski collapse, then $j(\gamma) = \eta$.*

The E^0-sequence has also the desired property of implying many elementary embeddings:

Lemma 2.16 ([11])**.** *Let $\alpha < \Upsilon$. Then there exists an elementary embedding $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ with $\mathrm{crt}(j) < \lambda$.*

It is clear from the definition that if $\gamma < \beta$ then $E^0_\gamma \subset E^0_\beta$, and one can prove that $\Theta^{E^0_\gamma} < \Theta^{E^0_\beta}$. Both sequences, however, are not necessarily continuous. It can be that $\bigcup_{\gamma<\beta} E^0_\gamma \neq E^0_\beta$ and $\sup_{\gamma<\beta} \Theta^{E^0_\gamma} < \Theta^{E^0_\beta}$, but in particular conditions:

Lemma 2.17 ([11])**.** *Let $\alpha < \Upsilon$ and suppose that $\Theta^{E^0_\alpha} > \sup_{\beta<\alpha} \Theta^{E^0_\beta}$. Then there exists $X \subset V_{\lambda+1}$ such that $L(E^0_\alpha) = L(X, V_{\lambda+1})$.*

Obviously the continuity in a limit point of the E^0-sequence implies the continuity of the Θ's sequence there.

Moreover, the Θ's sequence is important as a skeleton that construct (at least partially) the E^0-sequence:

Lemma 2.18 ([11])**.** *Suppose $\alpha < \Upsilon$ is a limit ordinal and $(\mathrm{cof}(\Theta^{E^0_\alpha}))^{L(E^0_\alpha)} > \lambda$. Then there exists $Z \in E^0_\alpha$ such that for each $Y \in E^0_\alpha$, Y is Σ_1-definable in $L(E^0_\alpha)$ with parameters from $\{Z\} \cup \{V_{\lambda+1}\} \cup V_{\lambda+1} \cup \Theta^{E^0_\alpha}$. Moreover, if $L(E^0_\alpha) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$, then $Z = \emptyset$.*

The last result on the E^0-sequence that will be useful is on the reflection of the sharps. For every $\alpha < \Upsilon$, by definition $(E^0_\alpha)^\sharp$ is a set of formulas in the language

$$\mathcal{L}^+_\alpha := \{\in\} \cup \{c_a\}_{a\in E^0_\alpha} \cup \{d_i\}_{i\in\omega} \cup \{C\},$$

where in $L(E^0_\alpha)$ every c_a is interpreted as a, every d_i is interpreted as an indiscernible and C is interpreted as E^0_α. The language

$$\mathcal{L}^+_{\alpha,n} := \{\in\} \cup \{c_a\}_{a\in E^0_\alpha} \cup \{d_1, \dots, d_n\} \cup \{C\}$$

is the restriction of $\mathcal{L}^+_\alpha$ to a language that uses at most n constants for indiscernibles.

Definition 2.19 ([1]). *For $\gamma, \alpha < \Upsilon$ define the* (γ,n)-fragment *of $(E^0_\alpha)^\sharp$ as $(E^0_\alpha)^\sharp \cap \mathcal{L}^+_{\gamma,n}$, and denote it as $(E^0_\alpha)^\sharp_{\gamma,n}$.*

Define the γ-fragment *of $(E^0_\alpha)^\sharp$ as $(E^0_\alpha)^\sharp \cap \mathcal{L}^+_\gamma$, and denote it as $(E^0_\alpha)^\sharp_\gamma$.*

Naturally $(E^0_\beta)^\sharp$ can be coded as a subset of $V_{\lambda+1}$ in $L(E^0_\alpha)$, i.e., as an element of E^0_α. This means that for every $\beta < \alpha < \Upsilon$ and every $n \in \omega$, $(E^0_\alpha)^\sharp \in E^0_{\alpha+1}$ and $(E^0_\alpha)^\sharp_{\beta,n} \in E^0_\alpha$. Then if $k : E^0_\alpha \prec E^0_\alpha$ it is possible to apply k to the sharp fragments.

Definition 2.20 ([1]). *A Σ_1-elementary embedding $k\colon E^0_\alpha \prec_1 E^0_\alpha$ is* $\sharp$-friendly *if for every $\gamma < \alpha$*

$$k((E^0_\alpha)^\sharp_{\gamma,n}) = (E^0_\alpha)^\sharp_{k(\gamma),n}.$$

More generally, given $\beta \leq \alpha < \Upsilon$, a Σ_1-elementary embedding $k\colon E^0_\beta \prec_1 E_\alpha$ is called $\sharp$-friendly *if for every $n \in \omega$ and $\gamma < \beta$*

$$k((E^0_\beta)^\sharp_{\gamma,n}) = (E^0_\alpha)^\sharp_{k(\gamma),n}.$$

Theorem 2.21 ([1]). *Let $\beta \leq \alpha < \Upsilon$ be limit ordinals, and let $k : E^0_\beta \prec E^0_\alpha$. Then k is $\sharp$-friendly iff it is extendible to $\hat{k} : L(E^0_\beta) \prec L(E^0_\alpha)$ such that $k \subset \hat{k}$.*

3 The Game

Theorems 2.10 and 2.11 witness the importance of properness, but not every elementary embedding is proper. There are two possible cases:

Definition 3.1. *Let $\alpha < \Upsilon$. Then*

- *α is* totally non-proper *if every weakly proper elementary embedding $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ is not proper;*

- *α is* partially non-proper *if there exist a weakly proper elementary embedding $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ that is not proper and a weakly proper elementary embedding $k : L(E^0_\alpha) \prec L(E^0_\alpha)$ that is proper.*

In [1] the existence of a totally non-proper ordinal is established under strong enough conditions:

Theorem 3.2 ([1])**.** *If there exists a* $\xi < \Upsilon$ *such that* $L(E^0_\xi) \not\models V = \mathrm{HOD}_{V_{\lambda+1}}$*, then there exists a totally non-proper ordinal.*

Under the same conditions, a partially non-proper ordinal exists:

Theorem 3.3. *If there exists a* $\xi < \Upsilon$ *such that* $L(E^0_\xi) \not\models V = \mathrm{HOD}_{V_{\lambda+1}}$*, then there exists a partially non-proper ordinal.*

Between all the ordinals less than Υ, there is one that has particular properties, that is the smallest ordinal α such that its sharp does not add new subsets of $V_{\lambda+1}$:

Definition 3.4. *Let* α *be the minimum ordinal such that*

1. $\alpha + \omega < \Upsilon_{V_{\lambda+1}}$*;*
2. $L((E^0_\alpha)^\sharp) \cap V_{\lambda+2} = E^0_\alpha$*.*

In fact the first requirement is slightly stronger than necessary, it is sufficient that the E^0-sequence is long enough to contain α and a finite number of its sharps (that depends on the proof).

In [1] the following Theorem was proved:

Theorem 3.5. *Let* $\xi < \Upsilon$ *be such that* $L(E^0_\xi) \models V \neq \mathrm{HOD}_{V_{\lambda+1}}$*, and let* $\eta < \xi$ *be the maximum ordinal such that* $E^0_\eta \subseteq (\mathrm{HOD}_{V_{\lambda+1}})^{L(E^0_\xi)}$*. Then* $L((E^0_\eta)^\sharp) \cap V_{\lambda+2} = E^0_\eta$*.*

This validates the definition of α: if there exists a ξ such that $L(E^0_\xi)$ is not a model for $\mathrm{HOD}_{V_{\lambda+1}}$ (in informal words, if the E^0-sequence is "long enough"), then such an α exists.

The game G_α is defined as such:

Definition 3.6. *Let* $\alpha < \Upsilon$*. The game* G_α *is defined as follows:*

I	$\langle k_0, \beta_0\rangle$		$\langle k_1, \beta_1\rangle$		$\langle k_2, \beta_2\rangle$	$\dots$
II		η_0		η_1		

with the following rules:

- $k_0 = \emptyset$*;*
- $k_{i+1} \colon E^0_{\beta_i} \prec E^0_{\beta_{i+1}}$ *is a* $\sharp$*-friendly elementary embedding;*

- *for every* $\gamma < \beta_i$, $k_{i+1}((E^0_\alpha)^\sharp_{\gamma,n}) = (E^0_\alpha)^\sharp_{k_{i+1}(\gamma),n}$*;*
- $\beta_i, \eta_i < \alpha$*;*
- $\beta_{i+1} > \eta_i$*;*
- $k_i \subseteq k_{i+1}$ *and* $k_{i+2}(\beta_i) = \beta_{i+1}$*;*
- *II wins if and only if I at a certain point cannot play anymore.*

Note that because of the third rule this game cannot be defined in $L(E^0_\alpha)$, it must be defined in a model that contains $(E^0_\alpha)^\sharp$. The arguments that follow take place in $L((E^0_\alpha)^\sharp)$ or in V.

If I wins G_α, it is possible to glue together all the k_i to form an elementary embedding $k = \bigcup_{i\in\omega} k_i$. If $\beta = \sup_{i\in\omega} \beta_i$ and $\eta = \sup_{i\in\omega} \eta_i$, then $\eta \leq \beta \leq \alpha$ and $k : E^0_\beta \prec E^0_\beta$ is an elementary embedding that preserves the sharp-fragments of E^0_α. Moreover, if $\gamma > \beta_0$, then there must exist i such that $\beta_i \leq \gamma < \beta_{i+1}$, and therefore $\gamma < \beta_{i+1} = k(\beta_i) \leq k(\gamma)$ is not a fixed point for k.

The strategy is to use Lemma 2.12, i.e., to construct an elementary embedding $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ such that the fixed points of j are not cofinal in $\Theta^{E^0_\alpha}$. If I wins with $\beta = \Theta^{E^0_\alpha}$, and for this to happen α must be necessarily equal to $\Theta^{E^0_\alpha}$, then the corresponding elementary embedding cannot be proper. To have such a β, II must push it up until $\alpha = \Theta^{E^0_\alpha}$, playing a cofinal sequence.

The first step is to prove that $\alpha = \Theta^{E^0_\alpha}$, and with it many other properties of α.

Lemma 3.7. *1. There is no* $X \subseteq V_{\lambda+1}$ *such that* $E^0_\alpha = L(X, V_{\lambda+1}) \cap V_{\lambda+2}$.

2. α *is a limit ordinal;*
3. $\Theta^{E^0_\alpha} = \sup_{\beta<\alpha} \Theta^{E^0_\beta}$*;*
4. *for every* $\beta \leq \alpha$ $L(E^0_\beta) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$*;*
5. $\alpha = \Theta^{E^0_\alpha}$ *and* α *is regular in* $L(E^0_\alpha)$*;*
6. $E^0_\alpha = \bigcup_{\beta<\alpha} E^0_\beta$.

Proof. 1. Suppose that there exists $X \subseteq V_{\lambda+1}$ such that

$$E^0_\alpha = L(X, V_{\lambda+1}) \cap V_{\lambda+2}.$$

Then

$$L((E^0_\alpha)^\sharp) \cap V_{\lambda+2} = L((X, V_{\lambda+1})^\sharp) \cap V_{\lambda+2}$$

and this, by definition, is equal to $E^0_\alpha = L(X, V_{\lambda+1}) \cap V_{\lambda+2}$. So

$$L((X, V_{\lambda+1})^\sharp) \cap V_{\lambda+2} = L(X, V_{\lambda+1}) \cap V_{\lambda+2}.$$

But $(X, V_{\lambda+1})^\sharp$ is by definition in $V_{\lambda+2}$, so $(X, V_{\lambda+1})^\sharp \in L(X, V_{\lambda+1})$, and this is a contradiction.

2. If α is a successor, then there exists $X \subseteq V_{\lambda+1}$ such that $E^0_\alpha = L(X, V_{\lambda+1}) \cap V_{\lambda+2}$. But this is a contradiction by the previous point.

3. Otherwise by Lemma 2.17 it would exist a X such that $E^0_\alpha = L(X, V_{\lambda+1}) \cap V_{\lambda+2}$, and this would be again a contradiction.

4. Suppose $\beta \leq \alpha$ and consider Theorem 3.5 with $\xi = \beta$. If $L(E^0_\beta) \vDash V \neq \mathrm{HOD}_{V_{\lambda+1}}$, then by the theorem there exists $\gamma < \beta$ such that $E^0_\gamma = L((E^0_\gamma)^\sharp) \cap V_{\lambda+2}$. But this is a contradiction, because α was the least one. So $L(E^0_\beta) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$.

5. Since $L(E^0_\alpha) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$, by Lemma 2.5 $\Theta^{E^0_\alpha}$ is regular in $L(E^0_\alpha)$. This implies by part 3 that $\alpha = \Theta^{E^0_\alpha}$ and α is regular in $L(E^0_\alpha)$.

6. The proof of this is in [1] with more details, as part of the proof of Theorem 3.5. Let $Y \in E^0_\alpha$. Since $L(E^0_\alpha) = L(\bigcup_{\eta<\Theta^{E^0_\alpha}} E^0_\eta)$, by definition and by part 5, the collapse of the closure $\mathcal{X}$ of $\{Y\} \cup V_{\lambda+1}$ has the form $L_\gamma(\bigcup_{\eta<\bar{\Theta}} E^0_\eta)$. By part 4 $L(E^0_\alpha)$ is a model for $\mathrm{HOD}_{V_{\lambda+1}}$, therefore it has "few" partial Skolem function, and there exists a surjection from $V_{\lambda+1}$ to $\mathcal{X}$. But then $\gamma, \bar{\Theta} < \Theta^{E^0_\alpha} = \alpha$, so

$$L_\gamma(\bigcup_{\eta<\Theta^{E^0_\alpha}} E^0_\eta) \subseteq \bigcup_{\beta<\alpha} L(E^0_\beta).$$

As Y is not collapsed, $Y \in \bigcup_{\beta<\alpha} L(E^0_\beta)$, and the Lemma is proved.

□

As useful as these properties are, they do not use the full potential of the definition of α. Since adding $(E^0_\alpha)^\sharp$ does not add new subsets of $V_{\lambda+1}$, α has some particular properties also in $L((E^0_\alpha)^\sharp)$, and this implies something of the structure of $L((E^0_\alpha)^\sharp)$ itself.

Lemma 3.8.
- *$\alpha = \Theta^{(E^0_\alpha)^\sharp}$;*
- *every element of E^0_α is definable in $L((E^0_\alpha)^\sharp)$ with parameters from $\Theta^{E^0_\alpha} \cup V_{\lambda+1}$;*

- $L((E^0_\alpha)^\sharp) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$.

Proof.

- By definition $\beta < \Theta^{E^0_\alpha}$ iff β is the order type of a prewellordering of $V_{\lambda+1}$ in $L(E^0_\alpha)$. But $L((E^0_\alpha)^\sharp)$ has the same prewellorders in $V_{\lambda+1}$ of $L(E^0_\alpha)$, so this happens iff β is the order type of a prewellordering of $V_{\lambda+1}$ in $L((E^0_\alpha)^\sharp)$, i.e. $\beta < \Theta^{(E^0_\alpha)^\sharp}$. So $\Theta^{E^0_\alpha} = \Theta^{(E^0_\alpha)^\sharp}$ and by Lemma 3.7(5) $\alpha = \Theta^{(E^0_\alpha)^\sharp}$.

- By Lemma 2.18 every element of E^0_α is definable in $L(E^0_\alpha)$ with parameters from $\Theta^{E^0_\alpha} \cup V_{\lambda+1}$. But in $L((E^0_\alpha)^\sharp)$, $L(E^0_\alpha)$ is a definable class, because $L(E^0_\alpha) = (L(V_{\lambda+2}))^{L((E^0_\alpha)^\sharp)}$, so every element of E^0_α can be defined in $L((E^0_\alpha)^\sharp)$ with parameters from $\Theta^{E^0_\alpha} \cup V_{\lambda+1}$.

- Since $\alpha+2 < \Upsilon$, $(E^0_\alpha)^{\sharp\sharp}$ exists, so every element of $L((E^0_\alpha)^\sharp)$ is definable in $L((E^0_\alpha)^\sharp)$ with parameters from the indiscernibles (of $L((E^0_\alpha)^\sharp)$) and $(E^0_\alpha)^\sharp$. The elements of $(E^0_\alpha)^\sharp$ are formulas in $\mathcal{L}^+_{E^0_\alpha}$, so they are definable in $L((E^0_\alpha)^\sharp)$ with parameters from the indiscernibles (of $L(E^0_\alpha)$) and E^0_α. By part 2, every element of E^0_α is definable in $L((E^0_\alpha)^\sharp)$ with parameters from $\Theta^{E^0_\alpha}$ and $V_{\lambda+1}$, so $L((E^0_\alpha)^\sharp) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$.

$\square$

Recalling Lemma 2.5 and Lemma 2.6, the previous Lemma has the following Corollary:

Corollary 3.9.

- *α is regular in $L((E^0_\alpha)^\sharp)$;*
- *$L((E^0_\alpha)^\sharp) \vDash \mathsf{DC}_\lambda$;*
- *$L((E^0_\alpha)^\sharp)$ satisfies the Coding Lemma.*

Proof. For the proof of part 3, consider the elementary embedding $j : L(E^0_{\alpha+2}) \prec L(E^0_{\alpha+2})$. As $(E^0_\alpha)^\sharp \in L(E^0_{\alpha+2})$ and it's therein definable, $j \restriction L((E^0_\alpha)^\sharp) : L((E^0_\alpha)^\sharp) \prec L((E^0_\alpha)^\sharp)$ is an elementary embedding. Then it suffices to apply Lemma 2.6. $\square$

So, α is not only "big" in $L(E^0_\alpha)$, but also in $L((E^0_\alpha)^\sharp)$, and is not only regular in $L(E^0_\alpha)$, but also in $L((E^0_\alpha)^\sharp)$. This is important because the game G_α is in $L((E^0_\alpha)^\sharp)$, and the proof that I has a winning strategy relies heavily on these characteristics.

Another key point is the fact that in G_α I has a limited amount of possible moves:

Lemma 3.10. *For every $\beta, \gamma < \alpha$ define*

$$\mathcal{E}^+(E^0_\beta, E^0_\gamma) = \{k : E^0_\beta \prec E^0_\gamma, k \text{ is } \sharp\text{-friendly}\}.$$

Then in $L((E^0_\alpha)^\sharp)$ there exists a surjection $\pi : V_{\lambda+1} \twoheadrightarrow \mathcal{E}^+(E^0_\beta, E^0_\gamma)$.

Proof. The proof takes place in $L((E^0_\alpha)^\sharp)$.

If there exist $X, Y \subset V_{\lambda+1}$ such that $L(E^0_\gamma) = L(Y, V_{\lambda+1})$ and $L(E^0_\beta) = L(X, V_{\lambda+1})$ the lemma is trivial, so by Lemma 2.17 we can assume that $\beta = \Theta^{E^0_\beta}$ and $\gamma = \Theta^{E^0_\gamma}$. Since by Lemma 3.7(4) both $L(E^0_\beta)$ and $L(E^0_\gamma)$ are models of $\mathrm{HOD}_{V_{\lambda+1}}$, by Lemma 2.18 every element in E^0_β is defined in $L(E^0_\beta)$ with parameters from $\Theta^{E^0_\beta} \cup V_{\lambda+1}$, and the same goes for E^0_γ. Let $k : E^0_\beta \prec E^0_\gamma$ be a $\sharp$-friendly elementary embedding. Since it can be extended to some $\hat{k} : L(E^0_\beta) \prec L(E^0_\gamma)$, its behaviour must be defined by $k \upharpoonright \Theta^{E^0_\beta} \cup V_{\lambda+1}$. But $k \upharpoonright V_{\lambda+1}$ is defined by a member of $V_{\lambda+1}$ (namely $k \upharpoonright V_\lambda$ by Lemma 2.2), and $k \upharpoonright \Theta^{E^0_\beta}$ can be codified as a subset of $\Theta^{E^0_\beta}$. Since $\Theta^{E^0_\beta} < \Theta^{(E^0_\alpha)^\sharp}$, the Coding Lemma proves the thesis. □

Theorem 3.11. *In $L((E^0_\alpha)^\sharp)$ II cannot have a winning strategy for the game G_α.*

Proof. Recall that the game G_α is

$$\begin{array}{llllll} I & \langle k_0, \beta_0 \rangle & & \langle k_1, \beta_1 \rangle & & \langle k_2, \beta_2 \rangle \quad \dots \\ II & & \eta_0 & & \eta_1 & \end{array}$$

with $k_0 = \emptyset$, $k_{i+1} \colon E^0_{\beta_i} \prec E^0_{\beta_{i+1}}$ $\sharp$-friendly elementary embeddings that preserve the fragments of $(E^0_\alpha)^\sharp$, $\beta_i, \eta_i < \alpha$, $\beta_{i+1} > \eta_i$, $k_i \subseteq k_{i+1}$, $k_{n+2}(\beta_n) = \beta_{n+1}$ and II wins if and only if I at a certain point can't play anymore.

Suppose that II has a winning strategy $\tau \in L((E^0_\alpha)^\sharp)$ and, since the game is open for II, with the usual analysis of open games we can suppose that τ is definable. By Lemma 2.16 there exists an elementary embedding from $L(E^0_{\alpha+2})$ to itself. Since $(E^0_\alpha)^\sharp \in L(E^0_{\alpha+2})$ and is definable, the restriction of the elementary embedding to $L((E^0_\alpha)^\sharp)$ is an elementary embedding; call it j. Define $\kappa_0 = \mathrm{crt}(j)$, and $\kappa_{i+1} = j(\kappa_i)$.

The rest of the proof is in $L((E^0_\alpha)^\sharp)$.

Let T_{G_α} be the tree of all the partial plays. Note that if p_n is a partial play of length $2n$, the sequence of the moves of I is definable from $\langle k_n, \beta_n, \beta_0 \rangle$. Moreover, we can suppose that II always plays within its strategy, so p_n can be written as $\langle k_n, \beta_n, \beta_0 \rangle$.

An ordinal $\eta < \alpha$ is closed under τ when for every $\langle k_n, \beta_n, \beta_0 \rangle \in T_{G_\alpha}$, if $\beta_n < \eta$ then $\tau(\langle k_n, \beta_n, \beta_0 \rangle) < \eta$. Let C be the set of the ordinals that are closed under τ.

Clearly C is closed. Let $\gamma_0 < \alpha$ and define

$$\gamma_1 = \sup\{\tau(\langle k_n, \beta_n, \beta_0 \rangle) : \langle k_n, \beta_n, \beta_0 \rangle \in T_{G_\alpha}, \beta_n < \gamma_0\}.$$

Since $\{k_n : \langle k_n, \beta_n, \beta_0 \rangle \in T_{G_\alpha}, \beta_n < \gamma_0\}$ is a subset of $\bigcup_{\beta_{n-1}, \beta_n < \gamma_0} \mathcal{E}^+(E^0_{\beta_{n-1}}, E^0_{\beta_n})$, $\gamma_1 < \Theta^{(E^0_\alpha)^\sharp} = \alpha$. The definition continues by induction

$$\gamma_{m+1} = \sup\{\tau(\langle k_n, \beta_n, \beta_0 \rangle) : \langle k_n, \beta_n, \beta_0 \rangle \in T_{G_\alpha}, \beta_n < \gamma_m\}.$$

As by Corollary 3.9(1) α is regular, $\sup_{i<\omega} \gamma_i < \alpha$, and $\sup_{i<\omega} \gamma_i \in C$. Thus C is not empty, and is unlimited in α. Therefore it has cardinality α.

Since τ is definable, C is definable, so $j(C) = C$. Now it is possible to show that I can play certain moves that counter the strategy τ. Let β_n be the κ_n-th element of C for every $n \in \omega$. I plays $\langle \emptyset, \beta_0 \rangle$ on his first turn, and $\langle j \upharpoonright E^0_{\beta_{n-1}}, \beta_n \rangle$ on his n-th turn. This moves follow the rules because:

- by Theorem 2.21 $j \upharpoonright E^0_{\beta_{n-1}}$ is a $\sharp$-friendly elementary embedding;
- $(E^0_\alpha)^\sharp$ is definable in $L((E^0_\alpha)^\sharp)$, so clearly j preserves its fragments;
- $k_{n+2}(\beta_n) = j(\beta_n) = \beta_{n+1}$ by the definability of C, and since $\beta_{n+1} \in C$, $\tau(\langle k_n, \beta_n, \beta_0 \rangle) < \beta_{n+1}$.

If I follows this strategy, then I wins. This is a contradiction, because τ was a winning strategy. □

Unfortunately this does not suffice to prove that there exists an elementary embedding from $L(E^0_\alpha)$ to itself that is not proper: even if I wins, α is regular in $L((E^0_\alpha)^\sharp)$, so it is not clear whether II can play a sequence cofinal in α. To prove Theorem 3.13 it is necessary to take a step back and consider V. In V, in fact, α has cofinality ω.

Lemma 3.12. $\mathrm{cof}(\alpha) = \omega$

Proof. Let $\mathcal{X}$ be the set of the elements in $L((E^0_\alpha)^\sharp)$ that are definable in $L((E^0_\alpha)^\sharp)$ using only elements of $V_{\lambda+1}$ and indiscernibles of $L((E^0_\alpha)^\sharp)$ as parameters.

Then $\mathcal{X} \prec L((E^0_\alpha)^\sharp)$. Therefore by Lemma 2.15 its collapse is $L((E^0_{\bar{\alpha}})^\sharp)$ for some $\bar{\alpha} \leq \alpha$. But since $L((E^0_\alpha)^\sharp) \vDash V_{\lambda+2} = E^0_\alpha$, because of the isomorphism $L((E^0_{\bar{\alpha}})^\sharp) \vDash V_{\lambda+2} = E^0_{\bar{\alpha}}$, i.e.

$$L(((E^0_{\bar{\alpha}})^\sharp)) \cap V_{\lambda+2} = E^0_{\bar{\alpha}},$$

and so $\bar{\alpha} = \alpha$ and the collapsing map is the identity.

Then every element of E^0_α is definable with parameters from $V_{\lambda+1}$ and from the indiscernibles of $L((E^0_\alpha)^\sharp)$. Let $i_1, \dots, i_n$ be the first n indiscernibles of $L((E^0_\alpha)^\sharp)$, and let

$$\alpha_n = \sup\{\gamma \in \mathrm{Ord} : \gamma \text{ is definable with parameters from } V_{\lambda+1} \cup \{i_1, \dots, i_n\}\}.$$

In $L((E^0_\alpha)^\sharp)$ there is a surjection from $V_{\lambda+1}$ to α_n, and as

$$\Theta^{L((E^0_\alpha)^\sharp)} = \Theta^{L(E^0_\alpha)} = \Theta^{E^0_\alpha} = \alpha,$$

it follows that $\alpha_n < \alpha$. But every ordinal $\beta < \alpha$ is definable using some m-uple of indiscernibles and elements of $V_{\lambda+1}$ as parameters, since $E^0_\beta \in E^0_\alpha$, therefore $\langle \alpha_n : n \in \omega \rangle$ is cofinal in α, and $\mathrm{cof}(\alpha) = \omega$. □

Theorem 3.13. *There exists an elementary embedding $k \colon L(E^0_\alpha) \prec L(E^0_\alpha)$ that is not proper.*

Proof. Let $\langle \eta_i : i \in \omega \rangle$ be a cofinal sequence of α. Consider the game G_α: the game is closed, so is quasidetermined. Suppose that II has a winning quasistrategy: since II plays only ordinals, the quasistrategy can be thinned out to a definable winning strategy for II in $L((E^0_\alpha)^\sharp)$. But this is impossible for Lemma 3.11, so I has a winning quasistrategy; call it σ. Suppose that II plays, against σ, the sequence $\langle \eta_i : i \in \omega \rangle$. Since σ is winning, player I can play according to it at every round. Consider $\langle k_i : i \in \omega \rangle$ the sequence of his moves. As $k_i \subseteq k_{i+1}$, define $k = \bigcup_{i \in \omega} k_i$. Then k is a $\sharp$-friendly elementary embedding, and, by Lemma 2.21, it is extendible to an elementary embedding $\hat{k} \colon L(E^0_\alpha) \prec L(E^0_\alpha)$.

Let γ be an ordinal greater than $\beta_0 = \sigma(\emptyset)$. Since for all $i \in \omega$, $\beta_i > \eta_i$, it follows that the β_i are cofinal in α, so there exists i such that $\beta_i \leq \gamma < \beta_{i+1}$. Then $\hat{k}(\gamma) \geq \hat{k}(\beta_i) = \beta_{i+1}$, but $\gamma < \beta_{i+1}$, therefore γ cannot be a fixed point of $\hat{k}$. So, by Lemma 2.12, $\hat{k}$ is not proper. □

The objective, however, was to prove that α was partially non-proper, so it is necessary to prove that there exists an elementary embedding from $L(E^0_\alpha)$ to itself that is proper. But this is quite easy:

Lemma 3.14. *There exists an elementary embedding $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ that is proper.*

Proof. Let $j : L(E^0_{\alpha+2}) \prec L(E^0_{\alpha+2})$. Then the restriction of j on $L(E^0_\alpha)$ is an elementary embedding, and we can assume that j is weakly proper, so is defined from an ultrafilter and the indiscernibles are fixed points for j.

Let $X \in E^0_\alpha$, we have to prove that $\langle X_i : i \in \omega \rangle \in L(E^0_\alpha)$, with $X_{i+1} = j(X_i)$. One of the points of the proof of Lemma 3.12 is that X is definable in $L((E^0_\alpha)^\sharp)$ with parameters from indiscernibles and $V_{\lambda+1}$. Let $a \in V_{\lambda+1}$ be the parameter that defines X. Therefore $\langle X_i : i \in \omega \rangle$ is definable in $L((E^0_\alpha)^\sharp$ from $\langle a_i : i \in \omega \rangle$, with $a_{i+1} = j(a_i)$, and indiscernibles for $L((E^0_\alpha)^\sharp)$. But then $\langle X_i : i \in \omega \rangle \in L((E^0_\alpha)^\sharp)$, and since it can be codified in $V_{\lambda+2}$, $\langle X_i : i \in \omega \rangle \in L(E^0_\alpha)$. □

4 Comparisons

The existence of a partially proper ordinal complements the results in [1]. Here is a brief recollection:

Definition 4.1 ([1])**.** *Let $\beta < \Upsilon$ be such that $L(E^0_\gamma) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$ for every $\gamma \leq \beta$. Then*

$$I_\beta = \{\gamma < \beta : (E^0_\beta)^\sharp_\gamma = (E^0_\gamma)^\sharp\}.$$

Lemma 4.2 ([1])**.** *If $\beta < \Upsilon$ and $I_\beta \neq \emptyset$, then there are no $X \subset V_{\lambda+1}$ such that $L(E^0_\beta) = L(X, V_{\lambda+1})$. In particular if $\Theta^{E^0_\beta}$ is regular in $L(E^0_\beta)$, then $\beta = \Theta^{E^0_\beta}$.*

Lemma 4.3 ([1])**.** *Let $\alpha < \Upsilon$ be a limit ordinal and $\gamma < \alpha$. The following are equivalent:*

- *$\gamma \in I_\alpha$;*
- *there exists an elementary embedding $j : L(E^0_\gamma) \prec L(E^0_\alpha)$ such that $j \upharpoonright E^0_\gamma$ is the identity.*

Lemma 4.4 ([1])**.** *Let $\beta < \Upsilon$ be a limit ordinal such that Θ^{E_β} is regular in $L(E_\beta)$ and $\mathrm{ot}(I_\beta) = \lambda$. Then β is totally non-proper., i.e., every elementary embedding $j : L(E^0_\beta) \prec L(E^0_\beta)$ is not proper.*

Lemma 4.5 ([1])**.** *Let $\gamma < \Upsilon$ be such that $L(E^0_\gamma) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$ and $L((E^0_\gamma)^\sharp) \cap V_{\lambda+2} = E^0_\gamma$. Then $\mathrm{ot}(I_\gamma) = \gamma$. In particular $\mathrm{ot}(I_\alpha) = \alpha$ and the λ-th element of I_α is totally non-proper.*

Theorem 3.2 proved for the first time the existence of non-proper elementary embeddings, but it raised a doubt. One could ask if being proper or non-proper depended directly on the structure of the underlaying model, since all the previous examples were of models where the elementary embedding were always proper or always non-proper. The appearance of a partially proper ordinal dismisses this doubt, proving that the situation is not black/white and that proper and non-proper elementary embeddings can cohexists.

Lemma 4.5 ties together the two kind of non-proper ordinals, stating that the existence of a totally non-proper ordinal is implied by the existence of the partially non-proper ordinal just discovered. It is interesting to investigate the differences between these two ordinals, for example in terms of numerosity of elementary embeddings.

Lemma 2.2 shows how every elementary embedding $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ is generated by $j \upharpoonright V_{\lambda+1}$. This last set is akin to a "seed" that generates the elementary embedding. One can ask: how many elementary embeddings can sprout from a seed?

Theorem 4.6. *Let $\beta < \Upsilon$ be such that $\mathrm{ot}(I_\beta) = \lambda$. Let $j, k : L(E^0_\beta) \prec L(E^0_\beta)$ weakly proper. If $j \upharpoonright V_{\lambda+1} = k \upharpoonright V_{\lambda+1}$ then $j = k$.*

Theorem 4.7. *Let $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ be such that $\mathrm{crt}(j) < \lambda$. Then there are 2^λ non proper elementary embeddings $k : L(E^0_\alpha) \prec L(E^0_\alpha)$ with $k \upharpoonright V_{\lambda+1} = j \upharpoonright V_{\lambda+1}$.*

For the first theorem a technical lemma is needed:

Lemma 4.8. *Let $\beta < \Upsilon$ be a limit ordinal such that $L(E^0_\beta) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$, $E^0_\beta = \bigcup_{\gamma<\beta} E^0_\gamma$ and let $\langle \gamma_\xi : \xi < \eta \rangle$ be the enumeration of I_β. Then*

$$H^{L(E^0_\beta)}(Ind, V_{\lambda+1}, \bigcup_{\zeta<\xi} I_{\gamma_\zeta}) \cong L(E^0_{\gamma_\xi}),$$

where Ind is the class of the indiscernibles in $L(E^0_\beta)$. In particular

$$L(E^0_\beta) = H^{L(E^0_\beta)}(Ind, V_{\lambda+1}, I_\beta).$$

Proof. Let $\mathcal{X} = H^{L(E^0_\beta)}(Ind, V_{\lambda+1}, \bigcup_{\zeta<\xi} I_{\gamma_\zeta})$. Then $\mathcal{X} \prec L(E^0_\beta)$.

Let $\eta \in \Theta^{L(E^0_\beta)} \cap \mathcal{X}$. Since $\eta < \Theta^{L(E^0_\beta)}$ there exists a surjection $\pi : V_{\lambda+1} \twoheadrightarrow \eta$, and since $L(E^0_\beta) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$. Define

$$\bar{\pi}(\langle x, y_1, \dots, y_n \rangle) = \begin{cases} y & \text{if there exists } y \text{ such that } \varphi(y, x, y_1, \dots, y_n, \beta_1, \dots, \beta_m) \\ & \quad \text{and is unique;} \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $\bar{\pi} \in (\mathrm{HOD})^{L(E^0_\beta)}$, and minimizing the ordinal that defines it we can suppose it definable. Therefore $\pi \in \mathcal{X}$, so every $\mu < \eta$ is in $\mathcal{X}$. This means that $\mathcal{X} \cap \Theta^{L(E^0_\beta)}$ is an initial segment of the ordinals that contains every γ_ζ with $\zeta < \xi$.

There are two cases: $\Theta^{L(E^0_\beta)} \subset \mathcal{X}$ or $\mathcal{X} \cap \Theta^{L(E^0_\beta)} = \gamma$. By Corollary 2.18 in $L(E^0_\beta)$ every element of E^0_β is definable with parameters from $\Theta^{L(E^0_\beta)} \cup V_{\lambda+1}$ and in the first case $E^0_\beta \subseteq \mathcal{X}$, but then $\mathcal{X} \cong L(E^0_\beta)$.

Suppose then that the second case holds. Let M be the collapse of $\mathcal{X}$ and $j : M \prec L(E^0_\beta)$ the corresponding elementary embedding. Then the critical point of j is γ and $j(\gamma) = \Theta^{E^0_\beta} = \beta$ by Lemma 4.2, therefore by Lemma 2.15 $M = L(E^0_\gamma)$. Since $j \upharpoonright V_{\lambda+1}$ is the identity, for every $X \in E^0_\gamma$,

$$X = \{x \in V_{\lambda+1} : j(x) \in j(X)\} = \{x \in V_{\lambda+1} : x \in j(X)\} = j(X),$$

so $j \upharpoonright E^0_\gamma$ is the identity. Therefore there exists an elementary embedding $j : L(E^0_\gamma) \prec L(E^0_\beta)$ with critical point γ such that $j(E^0_\gamma) = E^0_\beta$, i.e., $\gamma \in I_\beta$, by Lemma 4.3.

So $\gamma_\xi \leq \gamma$. But

$$Ind \cup V_{\lambda+1} \cup \bigcup_{\zeta<\xi} I_{\gamma_\zeta} \subseteq L(E^0_{\gamma_\xi}),$$

then $\mathcal{X} \subseteq L(E^0_\xi)$ and $L(E^0_\gamma) \subseteq L(E^0_{\gamma_\xi})$, therefore $\gamma \leq \gamma_\xi$, i.e., $\gamma = \gamma_\xi$. □

Proof of Theorem 4.6. Every element of $L(E^0_\beta)$ is definable from indiscernibles and elements of E^0_β. By Lemma 4.8 every element of E^0_β is definable from elements of $V_{\lambda+1}$ and I_β. But $\mathrm{ot}(I_\beta) = \lambda$ and for every $\gamma \in I_\beta$, $j(I_\gamma) = I_{j(\gamma)}$ and $k(I_\gamma) = I_{k(\gamma)}$. So the behaviour of j, k on I_β depends only on their behaviour on λ, but that is the same, therefore $j = k$. □

Theorem 4.7 will be the result of two Lemmas:

Lemma 4.9. *Let $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ be such that $\mathrm{crt}(j) < \lambda$. Then there are at least 2^λ non proper elementary embeddings $k : L(E^0_\alpha) \prec L(E^0_\alpha)$ with $k \upharpoonright V_{\lambda+1} = j \upharpoonright V_{\lambda+1}$.*

Proof. Consider $\hat{G}_\alpha$, the variation of the game G_α with the rule "$k_0 = j \upharpoonright V_{\lambda+1}$" instead of $k_0 = \emptyset$. If II had a winning strategy, then there would exist a club of the ordinals closed under such strategy, and playing j on the κ_n-th element of that club (where κ_n is the n-th element of the critical sequence of j) I could win. So I has a winning quasistrategy, that is exactly the tree of the winning (or not-losing) positions for I. Call this WP. For notational clarity, fix the first move of I as $\langle k_0, \beta_0 \rangle$, where β_0 is the minimum such that $\langle k_0, \beta_0 \rangle \in WP$.

As in the proof of Lemma 3.11, if p_n is a partial play it is sufficient to consider $\langle k_n, \beta_n, \beta_0\rangle$ instead of all of it, and since β_0 is fixed, the notation is shortened to $\langle k_n, \beta_n\rangle$. Since WP is a winning quasistrategy, if $\langle k_n, \beta_n\rangle \in WP$ then for every $\eta < \alpha$ there exists $\langle k_{n+1}, \beta_{n+1}\rangle \in WP$ that extends $\langle k_n, \beta_n\rangle$ and such that $\beta_{n+1} > \eta$. This means that the set

$$\{\beta_{n+1} : \langle k_{n+1}, \beta_{n+1}\rangle \in WP, \langle k_{n+1}, \beta_{n+1}\rangle \text{ extends } \langle k_n, \beta_n\rangle\}$$

is cofinal in α. But this set is also definable in $L((E^0_\alpha)^\sharp)$, and since α is regular then the cardinality of this set in $L((E^0_\alpha)^\sharp)$ is α.

Therefore in V for every $\langle k_n, \beta_n\rangle \in WP$ the set of its immediate successors in WP has cardinality at least $|\alpha|$. Before describing the last step of the proof, WP must be trimmed a bit, leaving only branches that will generate elementary embeddings from E^0_α to itself. So fix a sequence $\langle \eta_n : n \in \omega\rangle$ cofinal in α and define

$$WP^* = \{\langle k_n, \beta_n\rangle = p_n \in WP : \forall n \in \omega \ \ \mathrm{lh}(p_n) = n \to \beta_n > \eta_n\}$$

where $\mathrm{lh}(p_n) = n$ indicates that p_n is a partial play at the n-th turn. Again, every element of WP^* has $|\alpha|$ successors and WP^* has $|\alpha|^{\aleph_0} = \lambda^{\aleph_0} = 2^\lambda$ branches.

It remains to prove that each branch of WP^* defines a different elementary embedding. Let $k, l : E^0_\alpha \prec E^0_\alpha$ defined from two different branches of WP^* and let $\langle k_n, \beta_n\rangle$ and $\langle l_n, \gamma_n\rangle$ be the first nodes that are not equal respectively in the two branches. Then either $k_n \neq l_n$, and therefore $k \neq l$, or $\beta_n \neq \gamma_n$. Since β_0 is fixed, $n \geq 1$, and therefore $k_{n+1}(\gamma_{n-1}) = k_{n+1}(\beta_{n-1}) = \beta_n \neq \gamma_n = l_{n+1}(\gamma_{n-1})$, so $k \neq l$. □

Lemma 4.10. *There are less than $(2^\lambda)^+$ $\sharp$-friendly elementary embeddings $j : E^0_\alpha \prec E^0_\alpha$.*

Proof. Let $j : E^0_\alpha \prec E^0_\alpha$ be $\sharp$-friendly. Then it is defined by its behaviour on $V_{\lambda+1} \cup \Theta^{E^0_\alpha}$. By Lemma 2.2 $j \upharpoonright V_{\lambda+1}$ is defined from $j \upharpoonright V_\lambda$, that is a subset of V_λ, so there are no more than 2^λ possibilities. By the Coding Lemma (for $L(E^0_{\alpha+1})$) there exists a surjection $\pi : V_{\lambda+1} \twoheadrightarrow \mathcal{P}(\Theta^{E^0_\alpha})$, so there are no more than 2^λ possibilities also for $\Theta^{E^0_\alpha}$. □

The proof of Lemma 4.9 gives the suggestion that it is possible to find new results on elementary embeddings in $L(E^0_\alpha)$ with appropriate changes in the game G_α. This is true, and the following Theorem is a first example:

Theorem 4.11. *Let $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ be such that $\mathrm{crt}(j) < \lambda$. Then there are 2^λ proper elementary embeddings $k : L(E^0_\alpha) \prec L(E^0_\alpha)$ with $k \upharpoonright V_{\lambda+1} = j \upharpoonright V_{\lambda+1}$.*

In order to prove Theorem 4.11 we need some sort of converse to Theorem 2.12.

Lemma 4.12. *Let $\beta < \Upsilon$ be such that $L(E^0_\beta) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$, or $\beta = \Theta^{E^0_\beta}$ and $E^0_\beta = \bigcup_{\gamma<\beta} E^0_\gamma$. Then for every $j : L(E^0_\beta) \prec L(E^0_\beta)$, j is proper iff the set of fixed points of j is cofinal in $\Theta^{E^0_\beta}$.*

Proof. Suppose that the set of fixed points of j is cofinal in $\Theta^{E^0_\beta}$.

In the first case, let Γ_η be the set of the elements in E^0_β that are definable with parameters from $V_{\lambda+1} \cup \{V_{\lambda+1}\}$ and ordinals less than η. By Lemma 2.18 $E^0_\beta = \bigcup_{\eta<\Theta^{E^0_\beta}} \Gamma_\eta$. But in $L(E^0_\beta)$ for every $\eta < \Theta^{E^0_\beta}$ there exists a surjection from $V_{\lambda+1}$ to Γ_η, so Γ_η can be seen as a subset of $V_{\lambda+1}$, and $(\Gamma_\eta)^\omega \subset L(E^0_\beta)$. Let $X \in E^0_\beta$. Then there exists η such that $X \in \Gamma_\eta$, and we can suppose $j(\eta) = \eta$. But then $\langle X, j(X), j(j(X)), \dots \rangle \in (\Gamma_\eta)^\omega$ so it is in $L(E^0_\beta)$.

In the second case, let $X \in E^0_\beta$. Thus there exists an $\eta < \Theta^{E^0_\beta}$ such that $j(\eta) = \eta$ and $X \in E^0_\eta$. But there exists a surjection from $V_{\lambda+1}$ to E^0_η, so $(E^0_\eta)^\omega \subset L(E^0_\beta)$, and then, as above, j is proper. □

Proof of Theorem 4.11. Fix a $j : L(E^0_\alpha) \prec L(E^0_\alpha)$, with $\langle \kappa_i : i \in \omega \rangle$ its critical sequence. The variant of the game G_α is defined as follows:

I	$\langle k_0, \beta_0 \rangle$		$\langle k_1, \beta_1 \rangle$		$\langle k_2, \bar\beta_2, \beta_2 \rangle$		$\langle k_3, \beta_3, \rangle$	$\dots$
II		η_0		η_1		η_2		

with the following rules:

- $k_0 = j \upharpoonright V_{\lambda+1}$;
- $k_{2i+1} \colon E^0_{\beta_{2i}} \prec E^0_{\beta_{2i+1}}$ is a $\sharp$-friendly elementary embedding;
- $k_{2i} \colon E^0_{\bar\beta_{2i}} \prec E^0_{\bar\beta_{2i}}$ is a $\sharp$-friendly elementary embedding;
- for every $\gamma < \beta_i$, $k_{i+1}((E^0_\alpha)^\sharp_{\gamma,n}) = (E^0_\alpha)^\sharp_{k_{i+1}(\gamma),n}$;
- $\bar\beta_i, \beta_i, \eta_i < \alpha$;
- $\beta_{i+1} > \eta_i$, $\beta_{2i} > \bar\beta_{2i} > \eta_{2i}$;
- $k_i \subseteq k_{i+1}$, $k_{2i+2}(\beta_{2i}) = \beta_{2i+1}$ and $k_{2i+1}(\bar\beta_{2i}) = \bar\beta_{2i}$;
- II wins if and only if I at a certain point cannot play anymore.

The proof that II cannot have a winning strategy is almost the same as Lemma 3.11: suppose that II has a winning strategy σ, then we can suppose that it is definable; call C the set of the ordinals closed under σ: since by Lemma 3.10 I has "few" possible moves, C is a definable club and it has cardinality α in $L((E^0_\alpha)^\sharp)$; let β_{2n} be the $(n \cdot \lambda + \kappa_0)$-th element of C, β_{2n+1} the $(n \cdot \lambda + \kappa_1)$-th element of C and $\bar{\beta}_{2n}$ the $(n \cdot \lambda)$-th element of C; β_{i+1} is bigger than $\eta_i = \sigma(k_n, \beta_n)$ because $\beta_{i+1} \in C$, and the same works for $\bar{\beta}_i$, so with $k_{2n} = j \restriction E^0_{\bar{\beta}_{2n}}$ and $k_{2n+1} = j \restriction E^0_{\beta_{2n}}$ I wins, and II cannot have a winning strategy.

When II plays a sequence cofinal in α, the elementary embedding $k : L(E^0_\alpha) \prec L(E^0_\alpha)$ that results from glueing a successful play for I and extending it via Theorem 2.21 has the set of fixed points cofinal in $\Theta^{E^0_\alpha}$ (because it contains $\bar{\beta}_i$ for every $i \in \omega$), so by Lemma 4.12 is proper. Like in the proof of Lemma 4.9, it is possible to prove that every winning position for I has α winning positions as successors, so there are 2^λ possible plays where II plays a sequence cofinal in α and I wins. Unlikely Lemma 4.9, however, different plays can produce the same elementary embedding. The reason is that if two branches diverge at an odd step, they can generate the same elementary embedding. But if the branches are different at an even step, then they really generate a different elementary embedding, so limiting the quasistrategy for I with making him play just the smallest possible $\bar{\beta}_{2i}$ and β_{2i}, every play of I generates a different proper elementary embedding and there are 2^λ of them. □

The last variation of the game G_α will deal with the set of fixed points of j under $\Theta^{E^0_\alpha}$, when j is proper. Let $D_j = \{\gamma < \alpha : j(\gamma) = \gamma\}$. Theorem 4.11 shows that there are many proper elementary embeddings. How much the D_j's are different? Do all the elementary embeddings share the same D_j, or they vary? Since D_j is an ω-club, the intersection between two different D_j and D_k must be an ω-club, so $D_j \triangle D_k$ cannot be too much large. But it is possible to make it cofinal:

Theorem 4.13. *There exist $k, l : L(E^0_\alpha) \prec L(E^0_\alpha)$ such that $D_k \triangle D_l$ is cofinal in $\Theta^{E^0_\alpha}$.*

Proof. Consider the variation of the game G_α defined as follows:

I	$\langle k_0, l_0, \bar{\beta}_0, \beta_0 \rangle$	$\langle k_1, l_1, \bar{\beta}_1 \rangle$	$\langle k_2, l_2, \bar{\beta}_2, \beta_2 \rangle$	$\dots$
II	η_0	η_1		

with the following rules:

- $k_{2i+1}\colon E^0_{\bar\beta_{2i+1}} \prec E^0_{\bar\beta_{2i+1}}$, $k_{2i}\colon E^0_{\bar\beta_{2i}} \prec E^0_{\beta_{2i}}$, $l_i\colon E^0_{\bar\beta_i} \prec E^0_{\bar\beta_i}$ are $\sharp$-friendly elementary embeddings;
- for every $\gamma < \bar\beta_i$, $k_i((E^0_\alpha)^\sharp_{\gamma,n}) = (E^0_\alpha)^\sharp_{k_i(\gamma),n}$ and $l_i((E^0_\alpha)^\sharp_{\gamma,n}) = (E^0_\alpha)^\sharp_{l_i(\gamma),n}$;
- $\bar\beta_i, \beta_i, \eta_i < \alpha$;
- $\beta_{i+1} > \bar\beta_{i+1} > \eta_i$;
- $k_i \subseteq k_{i+1}$, $l_i \subseteq l_{i+1}$, $k_{2i+1}(\bar\beta_{2i+1}) = \bar\beta_{2i+1}$, $k_{2i}(\bar\beta_{2i}) = \beta_{2i}$ and $l_{i+1}(\bar\beta_i) = \bar\beta_i$;
- II wins if and only if I at a certain point cannot play anymore.

As usual, if II has a winning strategy σ then it is definable and C, the set of the ordinals closed under σ, is definable, cofinal in α and has cardinality α. Fix a $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ proper, let κ_0 be its critical point and $\kappa_1 = j(\kappa_0)$. By Theorem 2.10 j is finitely iterable, so $j(j) : L(E^0_\alpha) \prec L(E^0_\alpha)$ exists, and its critical point is κ_1. Then make I play:

- $\bar\beta_{2n} = (2n \cdot \lambda + \kappa_0)$-th element of C;
- $\bar\beta_{2n+1} = (2n \cdot \lambda)$;-th element of C;
- $\beta_n = (n \cdot \lambda + \kappa_1)$-th element of C;
- $k_n = j \upharpoonright E^0_{\beta_n}$;
- $l_n = j(j) \upharpoonright E^0_{\bar\beta_n}$.

With these moves I wins, so II cannot have a winning strategy.

The two elementary embeddings resulting from a succesful play of I (where II has played a cofinal sequence) are proper, because the sets of their fixed points (that contains all $\bar\beta_{2i+1}$) is cofinal in $\Theta^{E^0_\alpha}$. But $k(\bar\beta_{2i}) \neq \bar\beta_{2i}$, while $l(\bar\beta_{2i}) = \bar\beta_{2i}$ for every $i \in \omega$, so $\{\bar\beta_{2i} : i \in \omega\} \subset D_k \triangle D_k$, and it is cofinal.

□

Theorems 3.2 and 3.13 can be just the first steps of a larger analysis of the E^0_α-sequence. They answer to basic questions, but they also open new problems: are there totally or partially non-proper ordinals smaller that the ones already discovered? Does the existence of a partially non-proper ordinal imply the existence of a totally non-proper ordinal? Is it always the case that totally non-proper ordinals generate few elementary embeddings, while partially non-proper ordinals generate many?

Another fundamental problem is still open: all the examples of non-proper elementary embeddings discovered are in models that are not possible to describe as $L(X, V_{\lambda+1})$ with $X \subset V_{\lambda+1}$. In fact, this property is the key for both theorems. Must an elementary embedding $j : L(X, V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$ be proper?

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