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Abstract

Coarse spaces [26] and balleans [23] are known to be equivalent constructions ([25]). The main subject of this paper is the category, **Coarse**, having as objects these structures, and its quotient category **Coarse**/$_\sim$. We prove that the category **Coarse** is topological and hence **Coarse** is complete and co-complete and one has a complete description of its epimorphsims and monomorphisms. In particular, **Coarse** has products and coproducts, quotients, etc., and **Coarse** is not balanced. A special attention is paid to investigate quotients in **Coarse** by introducing some particular classes of maps, i.e. (weakly) soft maps which allow one to explicitly describe when the quotient ball structure of a ballean is a ballean. A particular type of quotients, namely the adjunction spaces, is considered in detail in order to obtain a description of the epimorphisms in **Coarse**/$_\sim$, shown to be the bornologous maps with large image. The monomorphisms in **Coarse**/$_\sim$ are the coarse embeddings; consequently, the bimorphims in **Coarse**/$_\sim$ are precisely the isomorphisms, i.e., **Coarse**/$_\sim$ is a balanced category.

Introduction

The origin of large scale geometry goes back to Milnor's problems and Gromov's ideas from geometric group theory and Mostow's rigidity theorem [16]. The basic notions of the theory, as asymptotic dimension and coarse embeddability, turned out to be relevant for a positive answer to Novikov conjecture, so largely motivated the interest to this new field.

While the large scale geometry of metric spaces enjoys a largely accepted setting of the main-stream notions and results, there is no universally accepted interpretation of the coarse category in the literature. The divergence starts already at the level of the *objects* of the category. Roe [26] introduced *coarse spaces* via entourages (see Definition 1.1), almost at the same time appeared also the book [23] by Protasov and Banakh, where ball structures and balleans were introduced (see §2.1 for the relevant definitions). In the monograph [25] by Protasov and Zarichnyi one can find both coarse spaces and balleans and they are shown to be equivalent constructions, although this book (as well as the long series of papers [3, 19, 22, 24] published by the Ukraine school of large scale geometry) makes use exclusively of ball structures. Coarse spaces and balleans are two faces of the same coin – while Roe's approach is based on the classical way to generalize metric spaces to uniform ones by means of entourages (imposing a slightly different axioms in the case of the coarse category), the ball structures and the balleans enhance the strong intuitive view of metric spaces based on balls. Somewhat later, Dydak and Hoffland [10] introduced the so called *large-scale structures* which take the other (alternative) road to get uniformity as a generalization of a metric space, namely based on the collections of "uniform covers" (although the families considered in [10] are not covers). Independently, Protasov [21] used appropriately covers to define a similar construction under the name of *asymptotic proximity* (as it was naturally leading to the coarse/asymptotic counterpart of proximity). The approach of [10] was recently developed and improved by Austin [1] by imposing covers in a bold way. These authors showed the close connection of large-scale structures to coarse spaces (see Remark 2.10), yet they didn't mention the alternative approach based on balleans from [23, 25], neither the paper [21]. As we show here, the large-scale structures are quite similar to balleans (see Remark 2.11 for the precise connection between these structures).

At the level of morphisms of the coarse category, several possible choices have been used (see for example [26], [15], [5] and [11]), but often the choice is so restrictive, that even products are not available (as the natural projections are not morphisms). For the same reason, pullbacks are not available either, while only very special maps (e.g., those with uniformly bounded fibers, see [25, 1]) admit quotients. This rules also out some standard

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constructions, as adjunction spaces. Recently coarse quotient mappings between metric spaces has been studied in Sheng Zhang, [29]

The aim of this paper is to provide a remedy for the above mentioned problems. In the first place, we try to reconcile the various approaches to large scale by using entourages or balleans depending on which of the two seems to be more appropriate in each case (we do not pretend the choice is always the most optimal one and this is why in some cases we adopt both). Roughly speaking, Roe's approach is more appropriate for a categorical treatment of coarse spaces, on the other hand, the balleans provide a very useful (at the intuitive level) tool in technically involved definitions and arguments (this is best visible in §§4, 5 and 6).

Secondly, we adopt a more relaxed condition on the morphisms compared to [26] and [25] (asking the maps to be only bornologous and not necessarily proper). This choice turns out to be quite fruitful, since the coarse category **Coarse** (with objects the coarse spaces and morphisms the bornologous maps) proves to be topological (i.e., admits initial and final structures), so arbitrary products, coproducts, as well as pullbacks and pushouts exist.

A particular emphasis is given to one of the basic construction, namely quotients. The difficulties with quotients of uniform spaces are well known, we shall cite Plaut [18]: *The notion that quotients of uniform spaces always have a uniform structure compatible with the quotient topology has been described not only as being false, but "horribly false" [13] and leading to "unavoidable difficulties" [14].* We study the counterpart of this problem in the realm of coarse spaces. More precisely, for a coarse space $(X, \mathcal{E})$ and a surjective map $q\colon X \to Y$, similarly to the case of uniformities, the "image" $\overline{\mathcal{E}}^q$ of the coarse structure $\mathcal{E}$ under the map q need not be a coarse structure on Y. In case q satisfies the quite restrictive condition of uniform boundedness of the fibers [25, 1], $\overline{\mathcal{E}}^q$ turns out to be a coarse structure (necessarily, the quotient coarse structure of Y). As mentioned above, the properness of q, usually imposed so far, was giving as a consequence the uniform boundedness of the fibers of q "for free", so the issue of when $\overline{\mathcal{E}}^q$ is a coarse structure never appeared before explicitily, as the maps were necessarily "too good". Here we characterize those maps q such that $\overline{\mathcal{E}}^q$ is a coarse structure, we call these map *weakly soft* and we define them by a technical condition involving the action of q on the entourages. We introduce also a stronger condition (soft maps) that is, on one side, easier to check and still ensures weak softness, but still weaker than the rather strong property of having uniformly bounded fibers.

The paper is organized as follows. In §1 we give the necessary background on coarse spaces and their morphisms, while §2 deals with balleans, as well as their connection to coarse structures and to large scale structures.

In §3 we define the coarse category **Coarse** and its quotient category **Coarse**$/_\sim$. In Theorem 3.5 we prove that **Coarse** is a topological category (the articulated definition of a topological category s given step-by-step throughout the proof of the theorem), so it has quotients, products, coproducts and pullbacks.

§4 is dedicated to the delicate issue of quotients. In §4.2 we give the theorem characterizing the weakly soft (Theorem 4.12) maps and we point out the similarity with a similar result of Himmelberg [12] about quotients of uniform spaces (Remark 4.13). To a special quotient, giving rise to the adjunction space with respect to a subspace (in categorical terms, the cokernel pair of an inclusion map), is given particular attention in §4.3. This adjunction space is used in the next §5 to describe the epimorphisms in the category **Coarse**$/_\sim$ as the maps with large image (Theorem 5.1). A description of the monomorphisms, as coarse embeddings, is given as well (Theorem 5.2). These two characterizations allow us to prove that the category **Coarse**$/_\sim$ is balanced (Corollary 5.4). The delicate question of preservation of epimorphisms under taking pullback is discussed in Corollary 5.4.

In §6 we shortly discuss coarse structures and balleans on groups, satisfying some natural compatibility condition between the large scale and the algebraic structures of groups. This provides a large source of examples of coarse spaces and soft maps.

The size of a subset (large, small, extra-large, etc.) in a coarse space was paid a particular attention in the monographs [23, 25] and in a series of papers (see, for example, [3, 19, 22]). The preservation of size under various kinds of maps in the category **Coarse** is studied in full detail in [7]. This provides a further application of soft and weakly soft maps introduced and studied here. Balleans will be used in the forthcoming paper [8] to define functorial coarse structures on (topological) groups (see also [28]).

1 Background on coarse spaces

1.1 Coarse spaces according to Roe

Definition 1.1. According to Roe [26], a *coarse space* is a pair $(X, \mathcal{E})$, where X is a set and $\mathcal{E} \subseteq \mathcal{P}(X \times X)$ a *coarse structure* on it, which means that

(i) $\Delta_X := \{(x,x) \mid x \in X\} \in \mathcal{E}$;

(ii) $\mathcal{E}$ is closed under passage to subsets;

(iii) $\mathcal{E}$ is closed under finite unions;
(iv) if $E \in \mathcal{E}$, then $E^{-1} := \{(y,x) \in X \times X \mid (x,y) \in E\} \in \mathcal{E}$;
(v) if $E, F \in \mathcal{E}$, then $E \circ F := \{(x,y) \in X \times X \mid \exists z \in X \text{ s.t. } (x,z) \in E, (z,y) \in F\} \in \mathcal{E}$.

For $E \subseteq X \times X$, $x \in X$ and $A \subseteq X$ let $E[x] = \{y \in X \mid (x,y) \in E\}$ and $E[A] = \{y \in Y \mid x \in A, (x,y) \in E\}$. The properties (ii) and (iii) say that $\mathcal{E}$ is an *ideal* of subsets of $X \times X$.

Slightly modifying the use made in [1], we call *entourage structure* any ideal of subsets of $X \times X$ that contains as a member the diagonal Δ_X. Hence, a coarse structure is simply an *entourage structure* satisfying (iv) and (v).

It is convenient to define also the notion of a *base of a coarse structure* as a family $\mathcal{B}$ of subsets of $X \times X$ satisfying (i) and

(iii*) if $B, B' \in \mathcal{B}$, then there exists $B'' \in \mathcal{B}$ containing $B \cup B'$;

(iv*) if $B \in \mathcal{B}$, then B^{-1} is contained in some member of $\mathcal{B}$; and

(v*) if $B, B' \in \mathcal{E}$, then $B \circ B'$ is contained in some member of $\mathcal{B}$.

Because of (iii*), $\mathcal{B}$ is an ideal base in $X \times X$. The ideal generated by $\mathcal{B}$ is a coarse structure on X. Note that if $\mathcal{B}$ is an ideal base in $X \times X$ with only (i), (iii*) and (iv*), then the ideal generated by $\mathcal{B}$ need not be a coarse structure as it may fail to satisfy (v).

The definition of coarse structure is quite similar to that of a uniformity. To ease the reader in the comparison of both notions, we recall that a family $\mathscr{U} \subseteq \mathcal{P}(X \times X)$ is a *uniformity* on X, if $\mathscr{U}$ satisfies (iv), $\Delta_X \subseteq U$ for each $U \in \mathscr{U}$ (this is a counterpart of (i)), $\mathscr{U}$ is stable under taking supersets and finite intersections (i.e., $\mathscr{U}$ is a *filter* of $X \times X$; this is a counterpart of (ii) and (iii)) and for any $U \in \mathscr{U}$, there exists $V \in \mathscr{U}$ such that $V \circ V \subseteq U$ (this is a counterpart of (v)).

Here is an outstanding example of a coarse structure. If (X,d) is a metric space, then the *bounded coarse structure over* (X,d) is the coarse structure whose a base is the family $\{(x,y) \in X \times X \mid d(x,y) \leq R\}_{R \in \mathbb{R}_{\geq 0}}$.

Definition 1.2. [26] For a non-empty set X let $\mathfrak{C}(X)$ denote the family of all coarse structures on X ordered by inclusion. For $\mathcal{E}, \mathcal{E}' \in \mathfrak{C}(X)$ say that $\mathcal{E}$ is *finer* than $\mathcal{E}'$ if $\mathcal{E} \subseteq \mathcal{E}'$.

With this order $\mathfrak{C}(X)$ becomes a complete lattice, with finest (bottom) element $\mathcal{T}_X$, the *trivial coarse structure* on X (having as a base $\{\Delta_X\}$) and with top element the *indiscrete* coarse structure $\mathcal{M}_X = P(X \times X)$ ([26]). The meet of a family $\{\mathcal{E}_i \mid i \in I\}$ in $\mathfrak{C}(X)$ is defined simply by the intersection $\bigcap_{i \in I} \mathcal{E}_i$.

Let us recall that a uniform space $(X, \mathscr{U})$ is *separated* if $\mathscr{U}$ satisfies $\bigcap \mathscr{U} = \Delta_X$. The large scale counterpart of this property sounds as follows. A coarse structure $\mathcal{E}$ on a set X is said to be *connected* if $\bigcup \mathcal{E} = X \times X$, i.e., for every $x, y \in X$, there is an entourage $E \in \mathcal{E}$ such that $(x,y) \in E$.

A subset B of a coarse space $(X, \mathcal{E})$ is said to be *bounded* ([26]) if there exists a point $x \in B$ such that $B \subseteq E[x]$. We say that the coarse space itself $(X, \mathcal{E})$ is *bounded* if X is bounded, i.e., $X \times X \in \mathcal{E}$.

1.2 Maps between coarse spaces

Now we focus on maps between coarse spaces.

First of all, we say that two maps $f, g \colon S \to (X, \mathcal{E})$ from a set to a coarse space are *close* (we write $f \sim g$) if $\{(f(x), g(x)) \mid x \in S\} \in \mathcal{E}$.

Definition 1.3. [26, 23] Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be two coarse spaces and $f \colon X \to Y$ a map. Then f is
(i) *bornologous* or *coarsely uniform* if $(f \times f)(E) \in \mathcal{E}_Y$ for all $E \in \mathcal{E}_X$;
(ii) *proper* if $f^{-1}(B)$ is bounded in $\mathcal{E}_X$ for every set B bounded in $\mathcal{E}_Y$;
(iii) *effectively proper* if $(f \times f)^{-1}(E) \in \mathcal{E}_X$ for all $E \in \mathcal{E}_Y$;
(iv) *coarse* if f is both bornologous and proper;
(v) a *coarse embedding* if f is bornologous and effectively proper;
(vi) an *asymorphism* if f is bijective and both f and f^{-1} are bornologous;
(vii) a *coarse equivalence* if f is bornologous and there exists a bornologous map $g \colon Y \to X$ such that $g \circ f \sim id_X$ and $f \circ g \sim id_Y$; in this case we call g to be the *coarse inverse* of f.

If $g, g' \colon Y \to Z$ is a pair of close maps such that g is bornologous, then g' is bornologous as well ([11, Proposition 1.20]).

Remark 1.4. A coarse structure $\mathcal{E}$ on a set X is finer than another coarse structure $\mathcal{E}'$ on it if and only if the map $id \colon (X, \mathcal{E}) \to (X, \mathcal{E}')$ is bornologous.

For a coarse space $(X,\mathcal{E})$ we say that $L\subseteq X$ is *large* if there is an $R\in\mathcal{E}$ such that $R[L]=X$. It is worth mentioning that, if X is a metric space endowed with its the bounded coarse structure, then the large subsets are precisely the *nets* (i.e. subsets N of X such that there exists $\varepsilon\geq 0$ such that for every point $x\in X$ there exists $y\in N$ with $d(x,y)\leq\varepsilon$).

Proposition 1.5. ([5, Proposition 2.7]) *Let $f\colon (X,\mathcal{E}_X)\to(Y,\mathcal{E}_Y)$ be a map between coarse spaces. Then f is a coarse equivalence if and only if f is a coarse embedding and $f(X)$ is large in Y.*

In the equivalent terms of balleans (and their maps, see Remark 2.6), this property was given much earlier in [23, p. 19].

2 Ball structures and balleans

2.1 Ball structures

Definition 2.1. ([25]) A *ball structure* is a triple $\mathfrak{B}=(X,P,B)$ where X and P are non-empty sets, they are called *support of the ball structure* and *set of radii* respectively, and, for every $x\in X$ and every radius $\alpha\in P$, a subset $B(x,\alpha)$ of X containing x is assigned, called *ball of center x and radius α*.

For a ball structure (X,P,B), $x\in X$, $\alpha\in P$ and a subset A of X, one puts

$$B^*(x,\alpha)=\{y\in X\mid x\in B(y,\alpha)\}\qquad B(A,\alpha)=\bigcup_{x\in A}B(x,\alpha).$$

A ball structure (X,P,B) is said to be

(i) *lower symmetric* if, for any $\alpha,\beta\in P$, there exist $\alpha',\beta'\in P$ such that, for every $x\in X$,

$$B^*(x,\alpha')\subseteq B(x,\alpha)\text{ and }B(x,\beta')\subseteq B^*(x,\beta);$$

(ii) *upper symmetric* if, for any $\alpha,\beta\in P$, there exist $\alpha',\beta'\in P$ such that, for every $x\in X$,

$$B(x,\alpha)\subseteq B^*(x,\alpha')\text{ and }B^*(x,\beta)\subseteq B(x,\beta');$$

(iii) *lower multiplicative* if, for any $\alpha,\beta\in P$, there exists $\gamma\in P$ such that, for every $x\in X$,

$$B(B(x,\gamma),\gamma)\subseteq B(x,\alpha)\cap B(x,\beta);$$

(iv) *upper multiplicative* if, for any $\alpha,\beta\in P$, there exists a $\gamma\in P$ such that for every $x\in X$,

$$B(B(x,\alpha),\beta)\subseteq B(x,\gamma).$$

The reader may have noticed that in the definition of ball structure (X,P,B) the set P of radii seems to be completely unrelated to the supporting set X. This "independence" is only apparent. Indeed, following [25] one can introduce a preorder $\leq$ in P by letting $\alpha\leq\beta$ when $B_X(x,\alpha)\subseteq B_X(x,\beta)$ for all $x\in X$. If $\sim$ is the equivalence relation defined by the conjunction of $\leq$ and $\geq$, then $\alpha\sim\beta$ precisely when $B_X(x,\alpha)=B_X(x,\beta)$ for all $x\in X$. We call the ball structure (X,P,B) *exact* if $\sim$ coincides with "=" (i.e., the assignment $P\ni\alpha\mapsto\{B_X(x,\alpha)\mid x\in X\}$ is injective). Obviously, every ball structure (X,P,B) allows for a "restriction" of the map $\alpha\mapsto\{B_X(x,\alpha)\mid x\in X\}$ to a smaller set $P'\subseteq P$ of radii, such that the restricted map is injective, having still the same image. So the ball structure (X,P',B) is exact and all such restrictions can be considered to give the same exact ball structure (X,P',B), that we shall call *associated exact ball structure*. What "same" means will shortly be clarified in (1).

Call a ball structure $\mathfrak{B}=(X,P,B)$ *symmetric*, if all its balls $B(x,\alpha)=B^*(x,\alpha)$ ($x\in X$, $\alpha\in P$) are symmetric. Clearly, every symmetric ball structure is both lower and upper symmetric.

2.2 Balleans: definition and examples

Definition 2.2. [23, 25]

(a) A *ballean* is an upper symmetric and upper multiplicative ball structure.
(b) If $\mathfrak{B}_X=(X,P_X,B_X)$ and $\mathfrak{B}_Y=(Y,P_Y,B_Y)$ are balleans, a map $f\colon(X,P_X,B_X)\to(Y,P_Y,B_Y)$ is a $\prec$-*mapping*, if for each $\alpha\in P_X$ there exists $\beta\in P_Y$ such that $f(B_X(x,\alpha))\subseteq B(f(x),\beta)$ for each $x\in X$.

Fix a non-empty set X and two ballean structures $\mathfrak{B} = (X, P, B)$ and $\mathfrak{B}' = (X, P', B')$ on X. We write $\mathfrak{B} \prec \mathfrak{B}'$ if the identity map $id\colon \mathfrak{B} \to \mathfrak{B}'$ is a $\prec$-mapping. We write $\mathfrak{B} = \mathfrak{B}'$, in case also $\mathfrak{B}' \prec \mathfrak{B}$ holds, i.e., when for every $\alpha \in P$ and $\alpha' \in P'$ there exist $\beta \in P'$ and $\beta' \in P$ such that

$$B(x,\alpha) \subseteq B'(x,\beta) \quad \text{and} \quad B'(x,\alpha') \subseteq B(x,\beta') \quad \text{for every} \quad x \in X. \tag{1}$$

In other words, we identify two ballean structures $\mathfrak{B}$ and $\mathfrak{B}'$ on X satisfying (1).

Now we can clarify what we anticipated in the final pat of §2.1 – every ballean $\mathfrak{B} = (X, P, B)$ admits an exact ballean $\mathfrak{B}_{ex} = (X, P', B')$ with $P' \subseteq P$ and $B'(x,\alpha) = B(x,\alpha)$ for all $\alpha \in P'$, so $\mathfrak{B}_{ex} = \mathfrak{B}$ in the sense of (1).

Every ballean $\mathfrak{B} = (X, P, B)$ admits an "equivalent", in the sense of (1), symmetric upper multiplicative ball structure $\mathfrak{B}_{sim} = (X, P, B_{sim})$, where $B_{sim}(x,\alpha) = B(x,\alpha) \cap B^*(x,\alpha)$ for every $x \in X$ and $\alpha \in P$ ([25]), since for every $\alpha \in P$ there exists a $\beta \in P$ such that $B_{sim}(x,\alpha) \subseteq B(x,\alpha) \subseteq B_{sim}(x,\beta)$.

Remark 2.3. (a) It is easy to deduce from the above axioms, that for every $x \in X$ the family $\mathcal{J}_x = \{B(x,\alpha) \mid \alpha \in P\}$ is an ideal base on X (i.e., $I_1, I_1 \in \mathcal{J}_x$ yields $I_1 \cup I_2 \subseteq I_3$ for some $I_3 \in \mathcal{J}_x$).

(b) For any $\alpha, \beta \in P$ there exists a $\gamma \in P$ such that for every $x, y, z \in X$ with $x \in B(z,\alpha)$ and $y \in B(z,\beta)$ one has $x \in B(y,\gamma)$ and $y \in B(x,\gamma)$.

Example 2.4. (1) Let (X, d) be a metric space. Then $\mathfrak{B}_d = (X, R_{\geq 0}, B_d)$, where $B_d(x, R)$ are the metric balls for every $x \in X$ and $R \in \mathbb{R}_{\geq 0}$, is the *metric ballean.*
A well-known result of metric topologies says that τ_d coincides with $\tau_{\overline{d}}$, where $\overline{d}(x,y) = \min\{d(x,y), 1\}$ for every pair of points $x, y \in X$. The reason why this fact happens is that the topology focus its attention on points which are "near" to each others. Conversely the large scale point of view takes care of the properties "at great distance" and we can imagine a similar result in this situations. This can be arranged by the new metric $\underline{d}$ on X, defined for each pair $x, y \in X$, by

$$\underline{d}(x,y) = \begin{cases} 0 & \text{if } x = y \\ \max\{d(x,y), 1\} & \text{otherwise,} \end{cases} .$$

For the pair $\mathfrak{B}_d$ and $\mathfrak{B}_{\underline{d}}$ one has the chain $B_d(x,R) \subseteq B_{\underline{d}}(x,R) = B_d(x, \max\{R,1\})$ for every $x \in X$ and $R \geq 0$. Hence, $\mathfrak{B}_d = \mathfrak{B}_{\underline{d}}$.

(2) Let X be a set and $\mathcal{J}$ be a base of an ideal on X. If $x \in X$ and $I \in \mathcal{J}$, we define the ball $B_{\mathcal{J}}(x, I)$ of center x and radius I to be

$$B_{\mathcal{J}}(x,I) := \begin{cases} I & \text{if } x \in I, \\ \{x\} & \text{otherwise.} \end{cases}$$

With this definition, it is not hard to see that $(X, \mathcal{J}, B_{\mathcal{J}})$ is actually a ballean. The above contruction can be carried out in the presence of a filter φ on the set X. Then the family $\mathcal{J}_\varphi$ of all complements of elements of φ is an ideal. The ballean $(X, \varphi, B_{\mathcal{J}_\varphi})$ is called also *filter ballean.* These balleans have been defined and widely studied in [19].

(3) There is a leading example tailored according to the above pattern. Let (X, τ) be a topological space. Denote by $\mathcal{C}(X)$ the family of all compact subsets of X and their subsets. Then $\mathcal{C}(X)$ is an ideal. The ballean $(X, \mathcal{C}(X), B_{\mathcal{C}(X)})$ is called *compact ballean.*

A ballean $\mathfrak{B} = (X, P, B)$ is said to be *connected* if for all $x, y \in X$ there exists a radius $\alpha \in P$ such that $y \in B(x,\alpha)$. The compact ballean, which is defined in Example 2.4, is connected.

2.3 Coarse spaces vs balleans

As we have anticipated in the introduction, coarse spaces and balleans are two faces of the same coin. In this section we recall this connection already revealed in [25].

Remark 2.5. (a) Let $(X, \mathcal{E})$ be a coarse space, then $\mathfrak{B}_{\mathcal{E}} = (X, \mathcal{E}_\Delta, B_{\mathcal{E}})$ is a ballean, where $\mathcal{E}_\Delta = \{E \in \mathcal{E} \mid \Delta_X \subseteq E\}$ and $B_{\mathcal{E}}(x, E) := E[x]$, for every $x \in E$ and $E \in \mathcal{E}_\Delta$. Clearly, for every coarse structure $\mathcal{E}$ on X the ballean $\mathfrak{B}_{\mathcal{E}}$ is exact and, moreover, for every bounded set $A \subseteq X$ and every one of its point $x \in A$ there exists a radius $E \in \mathcal{E}_\Delta$ such that $A = B_{\mathcal{E}}(x, E)$.

(b) Conversely, if $\mathfrak{B} = (X, P, B)$ is a ballean, then the family $\mathcal{B}_{\mathfrak{B}}$ of all the sets $E_\alpha := \bigcup_{x \in X} \{x\} \times B(x,\alpha)$, where $\alpha \in P$ is a base for a coarse structure $\mathcal{E}_{\mathfrak{B}}$ on X.

Actually, if we start with a coarse space $(X, \mathcal{E})$, the coarse structure $\mathcal{E}_{\mathfrak{B}_{\mathcal{E}}}$ precisely coincides with the original one, $\mathcal{E}$. Things are more complicated if we consider $\mathfrak{B}$ and $\mathfrak{B}_{\mathcal{E}_{\mathfrak{B}}}$, since they need not to be the same ballean (for example, the first one is not necessarily exact, while the second one has this property). Nevertheless, the equality the equality $\mathfrak{B} = \mathfrak{B}_{\mathcal{E}_{\mathfrak{B}}}$ holds in the broader sense (1), explained in §2.2

The correspondence defined in Remark 2.5 will allow us to freely pass from coarse structures to balleans, in the sense that if $\mathcal{E}$ and $\mathcal{E}'$ are coarse structures on a non-empty set X and $\mathfrak{B}$ and $\mathfrak{B}'$ are two balleans on X, then $\mathcal{E}_{\mathfrak{B}} = \mathcal{E}_{\mathfrak{B}'}$ if and only if $\mathfrak{B} = \mathfrak{B}'$; while $\mathfrak{B}_{\mathcal{E}} = \mathfrak{B}_{\mathcal{E}'}$ if and only if $\mathcal{E} = \mathcal{E}'$. Moreover, it works also when $\mathcal{E}$ and $\mathfrak{B}$ are not necessarily a coarse space or a ballean. In this case, $\mathfrak{B}_{\mathcal{E}}$ is a ball structure and $\mathcal{E}_{\mathfrak{B}}$ is an entourage structure.

Obviously, connectedness is preserved when we pass from balleans to coarse spaces and vice versa.

For the convenience of the reader, we recall here the terminology regarding morphisms in the framework of balleans (even if we are not using it in this paper), compared to that used in §1.2 for coarse spaces and using the correspondence between balleans and coarse spaces described above.

Remark 2.6. Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be two coarse spaces and $\mathfrak{B}_X = (X, P_X, B_X)$, $\mathfrak{B}_Y = (Y, P_Y, B_Y)$ the respective ballean structures on them as in in Remark 2.5. Clearly, the map $f\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ is bornologous if and only if $f\colon \mathfrak{B}_X \to \mathfrak{B}_Y$ is a $\prec$-mapping.

(a) The bornologous map $f\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ is a coarse embedding if and only if for every $\beta \in P_2$, there exists an $\alpha \in P_1$ such that for all $x_1, x_2 \in X$, $f(x_2) \in B_Y(f(x_1), \beta)$ implies $x_2 \in B_X(x_1, \alpha)$ (i.e., $f^{-1}(B_Y(f(x_1), \beta)) \subseteq B_X(x_1, \alpha)$) (*quasi-asymorphic embedding* is used for this property of balleans in [25, p.17]). In particular, *such a map has uniformly bounded fibers* $f^{-1}(y)$, $y \in Y$.

(b) If two maps $f, g\colon S \to (X, \mathcal{E}_X)$ are close, where S is a non-empty set, then there exists a radius $\alpha \in P_X$ such that f and g are *α-close*, i.e. $g(x) \in B_X(f(x), \alpha)$, for every $x \in S$.

(c) When the map $f\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ is a coarse equivalence, the term *quasi-asymorphism* is used for in [25] (and in such a case the balleans the balleans $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are said to be *quasi-asymorphic*).

To avoid confusion and useless proliferation of terminology, in the sequel we use only the terms "bornologous", "coarse embedding", "coarse equivalence", etc. (rather than their ballean counterparts) even when we work with balleans. Nevertheless, the readers who prefer to work with balleans and the terminology used in [25] may follow the guide from Remark 2.6 (bornologous $\leftrightarrow$ $\prec$-mapping, etc.) and freely choose the right name for the property of a map between balleans.

Remark 2.7. According to [23], a map $f\colon (X, P_X, B_X) \to (Y, P_Y, B_Y)$ is a *$\succ$-mapping* if for every $\alpha \in P_Y$ there exists $\beta \in P_X$ such that $B_Y(f(x), \alpha) \subseteq f(B_X(x, \beta))$ for every $x \in X$. This property is weaker than being effectively proper. In particular f is effectively proper if and only if f is a $\succ$-mapping with uniformly bounded fibers.

Remark 2.8. Uniformities and lower symmetric and lower multiplicative ball structures give rise to the same notion, similarly to the case of coarse structures and balleans we discussed above.

Let X be a space. If $\mathfrak{B} = (X, P, B)$ is a lower symmetric and lower multiplicative ball structure on it, then we define $\mathscr{U}_{\mathfrak{B}}$ to be the family of all subsets $U \subseteq X \times X$ such that there exists $\alpha \in P$ with $U \supseteq U_\alpha = \bigcup_{x \in X} \{x\} \times B(x, \alpha)$. Then $\mathscr{U}_{\mathfrak{B}}$ is a uniformity.

Conversely, if $(X, \mathscr{U})$ is an uniform space, then the ball structure $\mathfrak{B}_{\mathscr{U}} = (X, \mathscr{U}, B)$, where $B(x, U) = U[x]$ for all $x \in X$ and $U \in \mathscr{U}$, is lower symmetric and lower multiplicative.

One can argue also by using uniform covers generated by a ballean.

2.4 Large scale structures and their relation to coarse structures and ball structures

Dydak and Hoffland [10] provided an alternative approach to coarse structures introducing the so called large-scale structures. For a family $\mathscr{B}$ of subsets of a set X and $A \subseteq X$ let

$$\Delta(\mathscr{B}) = \bigcup_{B \in \mathscr{B}} B \times B, \quad St(A, \mathscr{B}) = \bigcup \{U \in \mathscr{B} \mid U \cap A \neq \emptyset\} \text{ and } e(\mathscr{B}) := \mathscr{B} \cup \{\{x\}\}_{x \in X},$$

calling $e(\mathscr{B})$ the *trivial extension* of $\mathscr{B}$. For another family $\mathscr{B}'$ of subsets of X let $St(\mathscr{B}, \mathscr{B}') = \{St(U, \mathscr{B}') \mid U \in \mathscr{B}\}$.

Definition 2.9. [10] A large-scale structure $\mathcal{LSS}_X$ on a set X is a non-empty set of families $\mathscr{B}$ of subsets of X (called *uniformly bounded*) with the following two properties:

(1) if $\mathscr{B} \in \mathcal{LSS}_X$, then every refinement of the family $e(\mathscr{B})$ also belongs to $\mathcal{LSS}_X$;

(2) if $\mathscr{B}, \mathscr{B}' \in \mathcal{LSS}_X$, then the star family $St(\mathscr{B}, \mathscr{B}')$ also belongs to $\mathcal{LSS}_X$.

A set $\mathfrak{G}$ of families in X is called a *base of a large-scale structure*, if for $\mathscr{B}_1, \mathscr{B}_2 \in \mathfrak{G}$ there exists $\mathscr{B}_3 \in \mathfrak{G}$ such that $\mathscr{B}_1 \cup \mathscr{B}_2 \cup St(\mathscr{B}_1, \mathscr{B}_2)$ refines $\mathscr{B}_3$. It is proved in [10, Proposition 1.6] that for such a set $\mathfrak{G}$ the family $\mathcal{LSS}_X$ of all refinements of trivial extensions of elements of $\mathfrak{G}$ forms a large scale structure on X.

In [10] the authors give several basic examples of large-scale structures induced by other structures on X. In particular, every group X has natural large-scale structures $\mathcal{LSS}_l(X)$ and $\mathcal{LSS}_r(X)$ determined by the left and the right shifts of finite subsets of X, respectively.

Remark 2.10. Comparing the notions of large-scale structures and coarse structures, the following is shown in [10].

(a) [10, Proposition 2.4] Every large-scale structure $\mathcal{LSS}_X$ on X induces a coarse structure $\mathscr{C}$ on X generated by the family $\{\Delta(\mathscr{B}) \mid \mathscr{B} \in \mathcal{LSS}_X\}$.

(b) Every coarse structure $\mathscr{C}$ on X induces a large-scale structure $\mathcal{LSS}_X = \{\mathscr{B} \mid \Delta(\mathscr{B}) \in \mathscr{C}\}$ on X [10, Proposition 2.5].

Now we see that the large-scale structures are closely related to balleans.

Remark 2.11. (a) Every large-scale structure $\mathcal{LSS}_X$ on X induces a ballean on X with $P = \{\mathscr{B} \mid \mathscr{B} = e(\mathscr{B}) \in \mathcal{LSS}_X\}$ and $B(x, \mathscr{B}) = St(x, \mathscr{B})$ for $\mathscr{B} \in P$. Obviously, the balls $B(x, \mathscr{B})$ are symmetric.

(b) On the other hand, if $\mathfrak{B} = (X, P, B)$ is a ballean, then letting $\mathscr{B}_\alpha = \{B(x, \alpha) \mid x \in X\}$ for $\alpha \in P$, we get a family $\{\mathscr{B}_\alpha \mid \alpha \in P\}$ that is a base of a large-scale structure $\mathcal{LSS}_X$ on X.

Quite recently this approach has been improved by Austin [1], who is using *scales* (i.e. covers) instead of arbitrary families in Definition 2.9. *Scale structures*, which are families of scales, give a third unifying language one can use to deal with both small scale (uniformities, lower symmetric and lower multiplicative ball structures and small-scale structures) and large-scale geometry (coarse spaces, balleans, large-scale structures).

As kindly pointed out by the referee, this approach based on covers was independently developed much earlier by Protasov [21], who defined, among others, also the so-called *asymptotic proximities* providing a coarse counterpart of proximities.

3 The coarse category

As already mentioned in the introduction there are various choices for the coarse category. In [26, 25] the authors coin the category $\mathcal{R}$ with objects the coarse spaces and morphisms the coarse maps (in the sense of Definitions 1.1 and 1.3).

In this paper the *coarse category* **Coarse** will have as objects the coarse spaces (precisely as $\mathcal{R}$), but a larger supply of morphisms, namely *all bornologous maps*. Accordingly, for coarse spaces X and Y we denote by $\mathrm{Mor}_{\mathbf{Coarse}}(X, Y)$ the set of all bornologous maps $X \to Y$. As in [25], we pay special attention to the quotient category $\mathbf{Coarse}/_\sim$, having the same objects and having as morphisms the closeness classes of bornologous maps. In order to introduce $\mathbf{Coarse}/_\sim$, one need to check first that $\sim$ is a congruence:

Lemma 3.1. *If $(X, \mathcal{E}_X), (Y, \mathcal{E}_Y)$ and $(Z, \mathcal{E}_Z)$ are coarse spaces, and the pairs $f, f' \in \mathrm{Mor}_{\mathbf{Coarse}}(X, Y)$, $g, g' \in \mathrm{Mor}_{\mathbf{Coarse}}(Y, Z)$ satisfy $f \sim f'$ and $g \sim g'$, then $g \circ f \sim g' \circ f'$.*

Proof. Since $f \sim f'$, $\{(f(x), f'(x)) \mid x \in X\} \in \mathcal{E}_Y$ and then $M := \{(g(f(x)), g(f'(x))) \mid x \in X\} \in \mathcal{E}_Z$, because g is bornologous. Moreover, $g \sim g'$ and then $N := \{(g(f'(x)), g'(f'(x))) \mid x \in X\} \in \mathcal{E}_Z$. Finally we have

$$\{(g(f(x)), g'(f'(x))) \mid x \in S\} = M \circ N \in \mathcal{E}_Z.$$

□

Now the set of morphisms in $\mathbf{Coarse}/_\sim$ from X to Y can be defined by $\mathrm{Mor}_{\mathbf{Coarse}/_\sim}(X, Y) = \mathrm{Mor}_{\mathbf{Coarse}}(X, Y)/_\sim$. For the sake of simplicity, if $f \in \mathrm{Mor}_{\mathbf{Coarse}}(X, Y)$ is a representative of the equivalence class $[f]_\sim$, we often write simply f instead of $[f]_\sim$.

Remark 3.2. It will be useful to check that other properties of a map are shared by all maps in its equivalent class (see §5). In particular we want to focus on the property of having large image and being effectively proper.

Let $f, g \colon (X, \mathcal{E}) \to (Y, \mathcal{E}')$ be two close maps between two coarse spaces and let $f \sim g$ be witnessed by $F = \{(f(x), g(x)) \mid x \in X\} \in \mathcal{E}'$. Then $M := F^{-1} \in \mathcal{E}'$. If $f(X)$ is large in Y, then there exists $E \in \mathcal{E}'$ such that $E[f(X)] = Y$ and it is easy to check that $(M \circ E)[g(X)] = Y$.

Suppose now that f is effectively proper and let $E \in \mathcal{E}'$. Then for every pair $(x, y) \in (g \times g)^{-1}(E)$, one has

$$(f(x), f(y)) = (f(x), g(x)) \circ (g(x), g(y)) \circ (g(y), f(y)) \in M \circ E \circ M,$$

so $(g \times g)^{-1}(E) \subseteq (f \times f)^{-1}(M \circ E \circ M) \in \mathcal{E}$, which concludes the proof.

Definition 3.3. We call a coarse space $(X, \mathcal{E})$ *simply generated*, if it satisfies the following equivalent conditions:
(a) $\mathcal{E}$ is stable under arbitrary unions;
(b) $\mathcal{E}$ has a top element E_{max} with respect to inclusion.

It is easy to see that in case these conditions hold, the subset $R := E_{max}$ of $X \times X$ is simply an equivalence relation on X. Hence, the initial pair $(X, \mathcal{E})$ can be replaced by the pair (X, R) of a set provided with an equivalence relation. Clearly a map $f\colon (X, R) \to (Y, S)$ between two such pairs is bornologous precisely when the map respects the equivalence relations, i.e., xRy always implies $f(x)Sf(y)$. Therefore, the category of sets provided with an equivalence relation (and maps preserving the relation as morphisms) naturally embeds in the category **Coarse** as a full subcategory. It can be quite useful for testing various notions and properties to be introduced in **Coarse** and **Coarse**$/_\sim$.

3.1 The category Coarse is topological

Definition 3.4. A morphism $\alpha\colon X \to X'$, in a category $\mathcal{X}$, is called:

- an *isomorphism* if there exists a morphism $\beta\colon X' \to X$, called *inverse of* α, such that $\alpha \circ \beta = 1_X$ and $\beta \circ \alpha = 1_{X'}$;
- an *epimorphism* if every pair of morphisms $\beta, \gamma\colon X' \to X''$ such that $\beta \circ \alpha = \gamma \circ \alpha$ satisfies $\beta = \gamma$;
- a *monomorphism* if every pair of morphisms $\beta, \gamma\colon X'' \to X$ such that $\alpha \circ \beta = \alpha \circ \gamma$ satisfies $\beta = \gamma$;
- a *bimorphism* if is both epimorphism and monomorphism.

In any category $\mathcal{X}$, an isomorphism is, in particular, a bimorphism. The category $\mathcal{X}$ is called *balanced* if bimorphisms are exactly the isomorphisms. As the isomorphisms in **Coarse** are precisely the asymorphisms, it follows from Theorem 3.5, that **Coarse** is not balanced (see Remark 3.7). On the other hand, we shall see in §5, that the bimorphisms in the category **Coarse**$/_\sim$ are exactly the coarse equivalences, hence they coincide with the isomorphisms, i.e., **Coarse**$/_\sim$ is balanced.

The next result shows that the category **Coarse** is topological.

Theorem 3.5. *The category* **Coarse** *of coarse spaces is topological.*

Proof. The forgetful functor $U\colon$ **Coarse** $\to$ **Set** is *amnestic*, i.e., an isomorphism f in **Coarse** is an identity whenever Uf is an identity. Moreover, U is *transportable*, i.e., for any coarse space A and any bijection (i.e., **Set**-isomorphism) $h\colon UA \to X$ there exists a coarse space B and an isomorphism $f\colon A \to B$ in **Coarse** with $Uf = h$ (i.e., the coarse structure of A can be "transported" via the bijection h). Obviously, a singleton admits a unique coarse structure and the constant maps in **Coarse** are morphisms. The *fibers of U are small*, i.e., the collection $\mathfrak{C}(X)$ of coarse structures making a given set X a coarse space is a subset of $\mathcal{P}(\mathcal{P}(X \times X))$, so it is a set, not a proper class.

It remains to check that the functor $U\colon$ **Coarse** $\to$ **Set** allows for lifting initial sources [4]. Namely, if X is a set, $\{(Y_i, \mathcal{E}_i) \mid i \in I\}$ is a family of coarse spaces and $f_i\colon X \to Y_i$, $i \in I$, are maps, then the source (f_i) has an initial lift along U. Namely, a coarse structure $\mathcal{E}$ on X such that all maps $f_i\colon (X, \mathcal{E}) \to (Y_i, \mathcal{E}_i)$ are bornologous and for every map $g\colon UZ \to X$, such that $f_i \circ g\colon Z \to (Y_i, \mathcal{E}_i)$ is a bornologous map for every $i \in I$, the map $g\colon Z \to (X, \mathcal{E})$ is bornologous. For every $i \in I$ let

$$(f_i)_*\mathcal{E}_i = \{(f_i \times f_i)^{-1}(E) \mid E \in \mathcal{E}_i\} \tag{2}$$

and define $\mathcal{E}$ to be the intersection of all $(f_i)_*\mathcal{E}_i \in \mathfrak{C}(X)$, $i \in I$ (see the comment after Definition 1.2). □

The following corollary is an immediate consequence of Theorem 3.5 and the well known properties of topological categories [6].

Corollary 3.6. *Let f be a morphism in* **Coarse**.

(a) f is an epimorphism in **Coarse** *if and only if f is surjective.*

(b) f is a monomorphism in **Coarse** *if and only if f is injective.*

Remark 3.7. Here are some other consequences of Theorem 3.5.

(a) According to Corollary 3.6, a bimorphism in **Coarse** is a bijective bornologous map. Therefore, a bimorphism in **Coarse** need not to be an isomorphism (just take a non-singleton set X and two comparable non-coinciding coarse structure on it, e.g., $\mathcal{T}_X$ and $\mathcal{M}_X$).

(b) The initial coarse structure $f_*\mathcal{E}_Y$ of a single map $f\colon X \to (Y,\mathcal{E}_Y)$, defined as in (2) with $f_i = f$, has an additional remarkable property. Namely, the map $f\colon (X, f_*\mathcal{E}_Y) \to (Y,\mathcal{E}_Y)$ is also effectively proper, as $R_f = \{(x,y) \in X\times X \mid f(x) = f(y)\} \inf_* \mathcal{E}_Y$. Therefore, if f has large image in Y, then f is a coarse equivalence.

In terms of ball structures, if f is surjective, for every $y \in Y$ and $\alpha \in P_Y$, we have $B_Y(y,\alpha) = f(f_*B_Y(x,\alpha))$, where $x \in f^{-1}(y)$.

Since the coarse structures on a set X form a complete lattice $\mathfrak{C}(X)$, given two coarse structures $\mathcal{E}_1,\mathcal{E}_2 \in \mathfrak{C}(X)$, there exist the join $\max\{\mathcal{E}_1,\mathcal{E}_2\}$ and the meet $\min\{\mathcal{E}_1,\mathcal{E}_2\}$ of these two structures. While $\min\{\mathcal{E}_1,\mathcal{E}_2\}$ is simply the intersection $\mathcal{E}_1\cap\mathcal{E}_2$, the explicit description of the join $\max\{\mathcal{E}_1,\mathcal{E}_2\}$ requires a more substantial effort. Indeed, in terms of entourages, this is the coarse structure having as a base the family of all compositions of the form

$$E_1\circ E_2\circ\cdots\circ E_1\circ E_2, \text{ where the block } E_1\circ E_2 \text{ is repeated } n \text{ times}, n\in\mathbb{N},\ E_1\in\mathcal{E}_1 \text{ and } E_2\in\mathcal{E}_2.$$

Since we are going to use the ballean form of this meet, we give below also the ballean version of this computation in more detail.

For balleans $\mathfrak{B}_1 = (X,P_1,B_1)$ and $\mathfrak{B}_2 = (X,P_2,B_2)$ one can effectively describe the maximum as follows. Consider radii $\alpha_1 \in P_1$ and $\alpha_2 \in P_2$ and for $x \in X$ and $n \in \mathbb{N}$ let

$$B^\sharp(x,\alpha_1,\alpha_2,n) = B_1(B_2(\dots B_1(B_2(x,\alpha_2),\alpha_1)\dots)),$$

where the block B_1B_2 is repeated n times in the obvious way and in the obvious sense. Let $P^\sharp = P_1\times P_2\times\mathbb{N}$, and for $r := (\alpha_1,\alpha_2,n)$ let $B^\sharp(x,r) := B^\sharp(x,\alpha_1,\alpha_2,n)$. This defines a ball structure $\mathfrak{B}^\sharp = (X,P^\sharp,B^\sharp)$ on X.

Claim 3.8. *$\mathfrak{B}^\sharp = (X,P^\sharp,B^\sharp)$ is a ballean and it is the finest one such that $\mathfrak{B}_1 \prec \mathfrak{B}^\sharp$ and $\mathfrak{B}_2 \prec \mathfrak{B}^\sharp$.*

Proof. First we show that $\mathfrak{B}^\sharp$ is upper multiplicative. Let $(\alpha_1,\alpha_2,n),(\beta_1,\beta_2,m) \in P^\sharp$ be two radii of this ball structure. Pick, for each $\nu = 1,2$, a radius $\gamma_\nu \in P_\nu$ such that for every $x\in X$ the inclusion $B_\nu(x,\alpha_\nu)\cup B_\nu(x,\beta_\nu) \subseteq B_\nu(x,\gamma_\nu)$ holds. Then it is not hard to check that, for every $x \in X$, we have $B^\sharp(B^\sharp(x,\alpha_1,\alpha_2,n),\beta_1,\beta_2,m) \subseteq B^\sharp(x,\gamma_1,\gamma_2,n+m)$.

As usual, it is convenient to compute the sets $(B^\sharp)^*(x,\alpha_1,\alpha_2,n)$ for each $(\alpha_1,\alpha_2,n) \in P^\sharp$. We have the following equivalence chain

$$\begin{aligned}
y \in (B^\sharp)^*(x,\alpha_1,\alpha_2,n) &\Leftrightarrow x\in B^\sharp(y,\alpha_1,\alpha_2,n) = B_1(B_2(\cdots(B_1(B_2(y,\alpha_2),\alpha_1),\cdots),\alpha_2),\alpha_1) \Leftrightarrow\\
&\Leftrightarrow \exists\, y_2\in B_2(y,\alpha_2),\ y_3\in B_1(y_2,\alpha_1),\dots,y_{2n}\in B_2(y_{2n-1},\alpha_2) : x\in B_1(y_{2n},\alpha_1) \Leftrightarrow\\
&\Leftrightarrow \exists\, y_{2n}\in B_1^*(x,\alpha_1),\ y_{2n-1}\in B_2^*(y_{2n},\alpha_2),\dots,\ y_2\in B_1^*(y_3,\alpha_1) : y\in B_2^*(y_2,\alpha_2) \Leftrightarrow\\
&\Leftrightarrow y\in B_2^*(B_1^*(\cdots(B_2^*(B_1^*(x,\alpha_1),\alpha_2),\cdots),\alpha_1),\alpha_2),
\end{aligned}$$

which proves that $(B^\sharp)^*(x,\alpha_1,\alpha_2,n) = B_2^*(B_1^*(\cdots(B_2^*(B_1^*(x,\alpha_1),\alpha_2),\cdots),\alpha_1),\alpha_2)$, where the block $B_2^*B_1^*$ is repeated n times. Then, if $(\alpha_1,\alpha_2,n),(\beta_1,\beta_2,m) \in P^\sharp$ and we choose four radii $\alpha_1',\beta_1' \in P_1$ and $\alpha_2',\beta_2' \in P_2$ such that $B_\nu(x,\alpha_\nu) \subseteq B_\nu^*(x,\alpha_\nu')$ and $B_\nu^*(x,\beta_\nu) \subseteq B_\nu(x,\beta_\nu')$ for each $\nu = 1,2$ and each $x \in X$, we obtain the following inclusions

$$B^\sharp(x,\alpha_1,\alpha_2,n) \subseteq (B^\sharp)^*(x,\alpha_1',\alpha_2',n+1) \quad\text{and}\quad (B^\sharp)^*(x,\beta_1,\beta_2,m) \subseteq B^\sharp(x,\beta_1',\beta_2',m+1)$$

and we have finally proved that $\mathfrak{B}^\sharp$ is a ballean.

Let now $\overline{\mathfrak{B}} = (X,\overline{P},\overline{B})$ be a ballean such that both $\mathfrak{B}_1 \prec \overline{\mathfrak{B}}$ and $\mathfrak{B}_2 \prec \overline{\mathfrak{B}}$. We claim that $\mathfrak{B}^\sharp \prec \overline{\mathfrak{B}}$. Fix a radius $(\alpha_1,\alpha_2,n) \in P^\sharp$ and let $\beta_1,\beta_2 \in \overline{P}$ such that $B_\nu(x,\alpha_\nu) \subseteq \overline{B}(x,\beta_\nu)$ for each $\nu = 1,2$ and each $x \in X$. Thus there exists a radius $\gamma \in \overline{P}$ such that, for every $x \in X$,

$$B^\sharp(x,\alpha_1,\alpha_2,n) \subseteq \overline{B}(\overline{B}(\cdots(\overline{B}(x,\beta_2),\beta_1),\cdots),\beta_2),\beta_1) \subseteq \overline{B}(x,\gamma).$$

Hence $\mathfrak{B}^\sharp \prec \overline{\mathfrak{B}}$ and then $\mathfrak{B}^\sharp$ has the property we claim it has. □

Remark 3.9. According to §2.1, every ballean admits an asymorphic one with symmetric balls on the same support. In the sequel, we impose this symmetry property as a blanket condition on all balleans without specifying it explicitly.

3.2 Products, coproducts and pullbacks of balleans

Next we introduce some basic categorical constructions, by using the most convenient in the each situation between coarse spaces and balleans.

Let $\{(X_i,\mathcal{E}_i)\}_{i\in I}$ be a family of coarse spaces. One can describe the product coarse structure on $\Pi_i X_i$ and show that it coincides with the one defined in [5].

The following proposition, defining products in **Coarse** is an immediate corollary of Theorem 3.5 and its proof.

Proposition 3.10. *Let $\{(X_i, \mathcal{E}_i)\}_{i\in I}$ be a family of coarse spaces and let $X = \prod_{i\in I} X_i$ with canonical projections $p_i\colon X \to X_i$. Let $\mathcal{E}$ be the family of all subsets of $X\times X$ contained in intersections of the form $\bigcap_{i\in I}(p_i\times p_i)^{-1}(E_i)$ where E_i varies in $\mathcal{E}_i$. Then*

(1) $(X, \mathcal{E})$ is a coarse space;

(2) if $(Z, \mathcal{E}_Z)$ is a coarse space such that for each $i \in I$ there exists a bornologous map $f_i\colon (Z, \mathcal{E}_X) \to (X_i, \mathcal{E}_i)$, then the unique map $f\colon (Z, \mathcal{E}_Z) \to (X, \mathcal{E}_X)$ such that $p_i \circ f = f_i$ for every $i \in I$ is bornologous.

The definition given above agrees with that given by Roe ([26]) for of binary products.

Now we give also a description of the product via balleans. Let $\{\mathfrak{B}_i = (X_i, P_i, B_i)\}_{i\in I}$ be a family of balleans. In order to describe the ballean structure on $X = \Pi_{i\in I}X_i$, corresponding to the product of the respective coarse spaces, we write, for the sake of simplicity, $\Pi_i A_i$ instead of $\bigcap_i p^{-1}(A_i)$, where $A_i \subseteq X_i$ for every $i \in I$. The ball structure $\Pi_{i\in I}\mathfrak{B}_i = (X, P, \Pi_{i\in I}B_i)$ on X has radii set $P = \Pi_{i\in I}P_i$, and for each $x = (x_i)_{i\in I} \in X$ and each $\alpha = (\alpha_i)_i \in \Pi_i P_i$, one has the ball $(\Pi_i B_i)(x, \alpha) = \Pi_i B_i(x_i, \alpha_i)$.

Since **Coarse** is a topological category, it has equalizers of pairs of morphisms $f, g\colon X \to Y$ defined by $eq(f, g) := \{x \in X \mid f(x) = g(x)\}$ (more precisely, by the inclusion map $eq(f, g) \hookrightarrow X$). Since **Coarse** has also products, this yields the existence of pullbacks of pairs of morphisms $f\colon X \to Y$, $e\colon Z \to Y$ defined by the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle u}\uparrow & & \uparrow{\scriptstyle e} \\ P & \xrightarrow{\ v\ } & Z. \end{array} \qquad (3)$$

The *pullback* u, v of the morphisms f, e has the following two defining properties:

(a) $ev = fu$, i.e., the diagram is commutative;

(b) for every coarse space V and every pair of morphisms $u'\colon V \to X$, $v'\colon V \to Z$ with $ev' = fu'$ there exists a unique morphism $t\colon V \to P$ such that $u' = ut$ and $v' = vt$.

The pullback can be built as follows, using the product $X \times Z$ and the equalizer $P := eq(fp_1, ep_2) \to X \times Z$ of the pair of morphisms fp_1, ep_2, where $p_1\colon X \times Z \to X$ and $p_2\colon X \times Z \to Z$ are the projections of the product. The morphisms u, v of the pullback are obtained by $u := p_1 \restriction_P$ and $v := p_2 \restriction_P$.

Next we define the coproduct of two coarse spaces $(X_1, \mathcal{E}_1)$ and $(X_2, \mathcal{E}_2)$. Take as a supporting set the disjoint union $X = X_1 \sqcup X_2$ and let $i_\nu\colon X_\nu \to X$ be the canonical embeddings, $\nu = 1, 2$. The family

$$\mathcal{E} = \{(i_1 \times i_1)(E_1) \cup (i_2 \times i_2)(E_2) \mid E_1 \in \mathcal{E}_1, E_2 \in \mathcal{E}_2\}$$

is a coarse structure on X making it the coproduct of $(X_1, \mathcal{E}_1)$ and $(X_2, \mathcal{E}_2)$.

Now we define and consider in major detail the binary coproducts in the framework of balleans, according to [25, §2.2], where it is termed *disjoint union*.

Consider two balleans $\mathfrak{B}_1 = (X_1, P_1, B_1)$ and $\mathfrak{B}_2 = (X_2, P_2, B_2)$. Take as a supporting set the disjoint union $X = X_1 \sqcup X_2$ and let $P_X = P_1 \times P_2$. For $x = i_\nu(x_\nu)$, $x_\nu \in X_\nu$, where $\nu \in \{1, 2\}$ and $(\alpha_1, \alpha_2) \in P_1 \times P_2$ let

$$B_X(x, (\alpha_1, \alpha_2)) := i_\nu(B_\nu(x_\nu, \alpha_\nu)).$$

We denote by $\mathfrak{B}_1 \coprod \mathfrak{B}_2 = (X, P_X, B_X)$ to ball structure defined in this way.

Proposition 3.11. $\mathfrak{B}_1 \coprod \mathfrak{B}_2 = (X, P_X, B_X)$ *is a ballean having the universal property of coproduct.*

Coproducts of larger families of coarse spaces or balleans are defined similarly.

Let us see now that the radii set of a coproduct of two copies of the same ballean can be taken to be the same as that of its components.

Remark 3.12. Let $\mathfrak{B}_X = (X, P_X, B_X)$ be a ballean. There is an easier description of the coproduct ballean $\mathfrak{B}_X \coprod \mathfrak{B}_X = (X \sqcup X, P_X \times P_X, B_{X\sqcup X})$ as far as its radii set is concerned. We claim that $\mathfrak{B}_X \coprod \mathfrak{B}_X = (X \sqcup X, P_X, \widetilde{B_{X\sqcup X}})$, where $\widetilde{B_{X\sqcup X}}(i_\nu(x), \alpha) = i_\nu(B_X(x, \alpha))$ for every $\nu = 1, 2$, $x \in X$ and $\alpha \in P_X$. Trivially,

$$(X, \sqcup X, P_X, \widetilde{B_{X\sqcup X}}) \prec \mathfrak{B}_X \coprod \mathfrak{B}_X.$$

In the opposite direction, for every $(\alpha, \beta) \in P_X \times P_X$ there exists $\gamma \in P_X$ such that $B_X(x, \alpha) \cup B_X(x, \beta) \subseteq B_X(x, \gamma)$ for every $x \in X$, so $B(x, (\alpha, \beta)) \subseteq \widetilde{B_{X\sqcup X}}(x, \gamma)$ for every $x \in X \sqcup X$.

4 Quotients of coarse spaces

Since the category of coarse spaces is topological, it has quotients. Namely, if $(X,\mathcal{E})$ is a coarse space and $q\colon X\to Y$ a surjective map, the set Y admits a coarse structure $\widetilde{\mathcal{E}}^q$ that makes the map q bornologous and has one of the following two equivalent properties:

(a) $\widetilde{\mathcal{E}}^q$ is the finest coarse structure on Y such that $q\colon(X,\mathcal{E})\to(Y,\widetilde{\mathcal{E}}^q)$ is bornologous;

(b) for every coarse space Z and every map $f\colon Y\to Z$ such that $f\circ q\colon X\to Z$ is bornologous, also $f\colon Y\to Z$ is bornologous.

The existence of such a (final) coarse structure $\widetilde{\mathcal{E}}^q$ on Y is granted as **Coarse** is a topological category, but the *explicit description* of $\widetilde{\mathcal{E}}^q$ is somewhat complicated, (as it is sometimes the case of topological categories, e.g., the category of uniform spaces). The aim of this section is to describe explicitly this quotient, using also balleans, when appropriate.

As a first approximation one can form the "image of $\mathcal{E}$ under q", namely the entourage structure

$$\overline{\mathcal{E}}^q := \{(q\times q)(E)\mid E\in\mathcal{E}_X\}$$

which has the properties (i)–(iv) required for a coarse structure but may fail to satisfy (v). (Indeed, since the map $q\times q\colon X\times X\to Y\times Y$ is surjective, $\overline{\mathcal{E}}^q$ obviously contains Δ_Y and it is stable under taking finite unions and smaller subsets, this ensures (i)–(iii), similarly (iv) can be checked.) We call $\overline{\mathcal{E}}^q$ *quotient entourage structure*. According to (a), the quotient coarse structure $\widetilde{\mathcal{E}}^q$ on Y is the finest coarse structure on Y containing $\overline{\mathcal{E}}^q$, so it is generated by $\overline{\mathcal{E}}^q$. Hence, it is obtained by adding to $\widetilde{\mathcal{E}}^q$ all possible finite compositions $(q\times q)(E)\circ\cdots\circ(q\times q)(E)$ with $E\in\mathcal{E}_X$ to get a base of this coarse structure (see [11]).

4.1 The quotient of a ball structure or a coarse structure

In order to connect to the already known results about quotients [25, 11], it is convenient to translate the family $\overline{\mathcal{E}}^q$ in terms of a ballean.

Let $q\colon X\to Y$ be a surjective map from a coarse space $(X,\mathcal{E})$ to a set. As in Remark 3.7, we denote by

$$R_q=\{(x,y)\in X\times X\mid q(x)=q(y)\}$$

the equivalence relation associated to q. We want to describe the ball structure $\mathfrak{B}_{\overline{\mathcal{E}}^q}$ associated to the family $\overline{\mathcal{E}}^q$. Fix a radius $(q\times q)(E)$, where $\Delta_X\subseteq E\in\mathcal{E}$ and hence $\Delta_Y\subseteq(q\times q)(E)$. Then, for every point $x\in X$, one has the following chain of equalities:

$$B_{\overline{\mathcal{E}}^q}(q(x),(q\times q)(E))=((q\times q)(E))[q(x)]=\{q(z)\in Y\mid \exists w\in R_q[x]:\,(w,z)\in E\}=q(B_{\mathcal{E}}(R_q[x],E)).\qquad(4)$$

The equalities (4) suggest a possible definition of the quotient of a ball structure $\mathfrak{B}_X=(X,P_X,B_X)$ with respect to a surjective map $q\colon X\to Y$. To define the quotient ball structure on Y use the same radii set $P_Y=P_X$ and for every $y\in Y$ and $\alpha\in P_Y$ let

$$\overline{B}^q_Y(y,\alpha)=q(B_X(q^{-1}(y),\alpha)).$$

In other words, if $y=q(x)$, then $\overline{B}^q_Y(q(x),\alpha)=q(B_X(R_q[x],\alpha)))$. More generally, one has

$$q(B_X(R_q[A],\alpha))=\overline{B}^q_Y(q(A),\alpha)\qquad(5)$$

for arbitrary subsets A of X, not only singletons $\{x\}$ in X. This yields $q^{-1}(\overline{B}^q_Y(q(A),\alpha))=R_q[B_X(R_q[A],\alpha)]$ for $A\subseteq X$.

This defines a ball structure $\overline{\mathfrak{B}}^q=(Y,P_Y,\overline{B}^q_Y)$ on Y, which we call *quotient ball structure*. The chain of equalities (4) proves that actually $\overline{\mathfrak{B}}^q=\mathfrak{B}_{\overline{\mathcal{E}}^q}$. Obviously, this is the *finest ball structure* on Y making q bornologous.

Remark 4.1. Let $\mathfrak{B}_X=(X,P_X,B_X)$ be a ballean and let $q\colon X\to Y$ a surjective map. The quotient ball structure $\overline{\mathfrak{B}}^q$ is upper symmetric. This can be observed directly by using symmetric balls $B_X(-,-)$ in the definition of the balls $\overline{\mathfrak{B}}^q$ (as $q(z)\in\overline{B}^q(q(x),\alpha)$ if and only if $z\in R_q[B_X(R_q[x],\alpha)]$ and R_q is symmetric).

In general, $\overline{\mathfrak{B}}^q$ may fail to be upper multiplicative and hence a ballean (see Examples 4.2 and 4.21) as the quotient entourage structure $\overline{\mathcal{E}}^q$ need not contain all the possible finite compositions. Nevertheless, there are many cases in which $\overline{\mathfrak{B}}^q$ is a ballean and, equivalently, the quotient entourage structure is a coarse space (see Theorem 4.12).

Example 4.2. Let X be the euclidean metric ballean on $\mathbb{R}_{\geq 0}$ and consider the quotient map $q\colon X \to Y$ associated to a partition $\mathcal{P} = \{A_n \mid n \in \mathbb{N}\}$ whose elements are consecutive intervals A_n (i.e., $x < y$ for every $x \in A_i$ and $y \in A_j$ and $i < j$) with increasing length d_n that diverges to infinity. Then $\overline{\mathfrak{B}}^q$ is not upper multiplicative, since, for every $n \in \mathbb{N}$,

$$q(A_{n+2}) \subseteq \overline{B}^q_Y(\overline{B}^q_Y(q(A_n), 1), 1),$$

but there exists no radius $R \geq 0$ such that A_{n+2} meets $B_d(A_n, R)$ for every $n \in \mathbb{N}$.

Let us recall now the setting on quotients imposed in [25] (see also [1, Proposition 2.4.1]).

Remark 4.3. Let $\mathfrak{B}_X = (X, P, B_X)$ be a ballean and let $q\colon X \to Y$ be a surjective map with uniformly bounded fibers. Let $\mathcal{P} = \{q^{-1}(y) \mid y \in Y\}$ be the partition of X defined by q. Following [25], for $F \in \mathcal{P}$ and $\alpha \in P$, put

$$B_{\mathcal{P}}(F, \alpha) := \{F' \in \mathcal{P} \mid F' \subseteq B(F, \alpha)\},$$

which defines a ballean $\mathfrak{B}_{\mathcal{P}} = (X, \mathcal{P}, B_{\mathcal{P}})$, since $\mathcal{P}$ is a partition. The ballean $\mathfrak{B}_{Y,\mathcal{P}} = (Y, P, B_{Y,\mathcal{P}})$ on the quotient Y, defined by $B_{Y,\mathcal{P}}(y, \alpha) = q(B_{\mathcal{P}}(q^{-1}(y), \alpha))$ for every $y \in Y$ and $\alpha \in P$, is named *factor-ballean* of X in [25]. Moreover, $\mathfrak{B}_{Y,\mathcal{P}} = \overline{\mathfrak{B}}^q$. To check this equality note that for every $\alpha \in P$ and $y \in Y$ one has $F = q^{-1}(y) \in \mathcal{P}$ and $B_{Y,\mathcal{P}}(F, \alpha) \subseteq q(B_X(F, \alpha)) = \overline{B}^q_Y(y, \alpha)$. Hence, $\mathfrak{B}_{Y,\mathcal{P}} \prec \overline{\mathfrak{B}}^q$.

On the other hand, there exists $\beta \in P$ such that $F' \subseteq B_X(x, \beta)$, for every fiber F' and every point $x \in F'$, as $\mathcal{P}$ is uniformly bounded. Now fix $\alpha \in P$ and pick $\gamma \in P$ with $B_X(B_X(z, \alpha), \beta) \subseteq B_X(z, \gamma)$ for every $z \in X$. Then $\overline{B}^q_Y(y, \alpha) \subseteq B_{Y,\mathcal{P}}(F, \gamma)$, so $\overline{\mathfrak{B}}^q \prec \mathfrak{B}_{Y,\mathcal{P}}$. In fact, each point of $\overline{B}^q_Y(y, \alpha)$ has the form $q(z)$ with $z \in B_X(F, \alpha)$. Moreover, $q^{-1}(q(z)) \subseteq B_X(z, \beta)$, by the choice of β. This yields

$$q^{-1}(q(z)) \subseteq B_X(z, \beta) \subseteq B_X(B_X(F, \alpha), \beta) \subseteq B_X(F, \gamma).$$

This yields $q^{-1}(q(z)) \subseteq B_{\mathcal{P}}(F, \gamma)$, by the definition of $B_{\mathcal{P}}$. Therefore, $q(z) = q(q^{-1}(q(z))) \in q(B_{\mathcal{P}}(F, \gamma)) = B_{Y,\mathcal{P}}(y, \gamma)$.

It should be noted that uniform boundedness of the fibers of q imposed by [25, 1] is a necessary condition when the map q must be effectively proper, as often imposed by many authors.

The following relations between entourages and the equivalence relation R_q will be needed in the sequel:

Proposition 4.4. *If R_q if the equivalence relation associated to a surjective map $q\colon X \to Y$ and E, A are entourages in $X \times X$, then*

$$(q \times q)(E) = (q \times q)(R_q \circ E) = (q \times q)(E \circ R_q) = (q \times q)(R_q \circ E \circ R_q), \tag{6}$$

$$(q \times q)(E) \circ (q \times q)(E) = (q \times q)(E \circ R_q \circ E). \tag{7}$$

and

$$(q \times q)(E) \circ (q \times q)(E) \circ (q \times q)(E) \circ (q \times q)(E) = (q \times q)(E \circ R_q \circ E \circ R_q \circ E \circ R_q \circ E). \tag{8}$$

Moreover, if $(q \times q)(A) \subseteq (q \times q)(E)$ then $A \subseteq R_q \circ E \circ R_q$. Consequently, $(q \times q)^{-1}((q \times q)[E]) = R_q \circ E \circ R_q$.

Proof. To prove (6) note first that $E \subseteq R_q \circ E \subseteq R_q \circ E \circ R_q$ and $E \subseteq E \circ R_q \subseteq R_q \circ E \circ R_q$, since $R_q \supseteq \Delta_X$. Therefore, it suffices to check the inclusion $(q \times q)(E) \supseteq (q \times q)(R_q \circ E \circ R_q)$. Pick $(x, y) \in R_q \circ E \circ R_q$. Then there exists $(z, u) \in E$ such that $q(x) = q(z)$ and $q(u) = q(y)$. Then, $(q(x), q(y)) = (q(z), q(u)) \in (q \times q)(E)$.

To prove (7), assume $(y, y') \in (q \times q)(E) \circ (q \times q)(E)$. Then there exist $x, x', z, z' \in X$ such that

$$y = q(x), \ y' = q(x'), \ (x, z), (z', x') \in E \ \text{ and } \ q(z) = q(z'),$$

consequently, $(z, z') \in R_q$. This yields $x' \in E \circ R_q \circ E[x]$, i.e., $(x, x') \in E \circ R_q \circ E$. Therefore, $(y, y') = (q(x), q(x')) \in (q \times q)(E \circ R_q \circ E)$. This proves the inclusion $\subseteq$ in (7).

Now assume that $(y, y') \in (q \times q)(E \circ R_q \circ E)$. Then $y = q(x)$ and $y' = q(x')$ for $(x, x') \in E \circ R_q \circ E$. So there exist $z, u \in X$ such that $(x, z), (u, x') \in E$ and $(z, u) \in R_q$, i.e., $q(z) = q(u)$. Then the pair $(q(x), q(x'))$ belongs to $(q \times q)(E) \circ (q \times q)(E)$, as $(q(x), q(z)) = (q(x), q(u)) \in (q \times q)(E)$, and $(q(z), q(x')) = (q(u), q(x')) \in (q \times q)(E)$. Therefore, $(y, y') = (q(x), q(x')) \in (q \times q)(E) \circ (q \times q)(E)$. This proves (7).

We deduce (8) from (7) as follows. Let $E_1 = (q \times q)(E)$. Then $E_1 \circ E_1 = (q \times q)(R_q \circ E \circ R_q)$ by (7). Applying once again (7) to

$$E_2 := E_1 \circ E_1 = (q \times q)(E \circ R_q \circ E) \circ (q \times q)(E \circ R_q \circ E)$$

we deduce that

$$E_1 \circ E_1 \circ E_1 \circ E_1 = E_2 \circ E_2 = (q \times q)(E \circ R_q \circ E \circ R_q \circ E \circ R_q \circ E).$$

This proves (8).

To prove $A \subseteq R_q \circ E \circ R_q$, under the assumption of $(q \times q)(A) \subseteq (q \times q)(E)$, pick $(x, y) \in A$. Then $(q(x), q(y)) \in (q \times q)(E)$, by our hypothesis. Thus, there exists $(u, v) \in E$, such that $(q(x), q(y)) = (q(u), q(v))$. Then $(x, u) \in R_q$ and $(v, y) \in R_q$ and this yields $(x, y) \in R_q \circ E \circ R_q$, as required.

The last assertion follows from the last proven inclusion and (6). □

Corollary 4.5. *The quotient entourage structure* $\overline{\mathcal{E}}^q$ *on* Y *is bounded if and only if there exists* $E \in \mathcal{E}_X$ *such that* $X \times X = R_q \circ E \circ R_q$. *In such a case,* $\overline{\mathcal{E}}^q$ *is a coarse structure.*

Proof. Clearly, $\overline{\mathcal{E}}^q$ on Y is bounded if and only if there exists $E \in \mathcal{E}_X$ such that $(q \times q)(E) = Y \times Y = (q \times q)(X \times X)$. According to the last assertion of Proposition 4.4, this occurs precisely when $X \times X = R_q \circ E \circ R_q$. □

This gives a nice criterion for boundedness in terms of the fibers of a quotient:

Corollary 4.6. *For a coarse space* $(X, \mathcal{E})$ *and a surjective map* $q \colon X \to Y$ *the following are equivalent:*

(a) X *is bounded,*

(b) q *has uniformly bounded fibers and the quotient space* $\overline{\mathfrak{B}}^q$ *is bounded.*

Proof. According to Remark 4.1(b), $\overline{\mathfrak{B}}^q$ is a ballean when q has uniformly bounded fibers.

The implication (a) $\to$ (b) is obvious.

(b) $\to$ (a) According to Corollary 4.5, $X \times X = R_q \circ E \circ R_q$ for some $E \in \mathcal{E}_X$ if $\overline{\mathfrak{B}}^q$ is bounded. The hypothesis that q has uniformly bounded fibers yields $R_q \in \mathcal{E}_X$. Along with the equality $X \times X = R_q \circ E \circ R_q$, this yields $X \times X \in \mathcal{E}_X$, i.e., $(X, \mathcal{E})$ is bounded. □

This corollary implies that for a quotient map q with uniformly bounded fibers $\overline{\mathfrak{B}}^q$ is bounded if and only if $(X, \mathcal{E})$ is bounded. This witnesses how restrictive is the hypothesis, usually imposed in the literature, of uniformly bounded fibers (one cannot have the quotient $\overline{\mathfrak{B}}^q$ bouned without imposing boundedness on X).

4.2 When the quotient ball structure of a ballean is a ballean

We shall see once again, by combining Lemma 4.9 and Proposition 4.10, that the quotient ball structure is a ballean when the fibers of the quotient map are uniformly bounded (see also Remark 4.3). Since this condition is rather strong, we propose now two (weaker) natural sufficient conditions ensuring that the quotient ball structure is a ballean.

Definition 4.7. Let X be a coarse space and $q \colon X \to Y$ be a surjective map. We say that

(1) q is *soft* if for all $E \in \mathcal{E}_X$ there exists a $F \in \mathcal{E}_X$ such that $R_q \circ E \subseteq F \circ R_q$.

(2) q is *weakly soft* if for all $E \in \mathcal{E}_X$ there exists a $F \in \mathcal{E}_X$ such that $E \circ R_q \circ E \subseteq R_q \circ F \circ R_q$.

(3) q is *2-soft* if for all $E \in \mathcal{E}_X$ there exists a $F \in \mathcal{E}_X$ such that

$$E \circ R_q \circ E \circ R_q \circ E \circ R_q \circ E \subseteq R_q \circ F \circ R_q \circ F \circ R_q.$$

Remark 4.8. The property $R_q \circ E \subseteq F \circ R_q$ in (1) reminds a (very) weak form of *commutativity* between $\mathcal{E}$ and R_q in the monoid of all entourages of $X \times X$ with respect to the composition law $\circ$, taken into account the fact that F can be chosen with $E \subseteq F$.

(a) Obviously, $R_q \circ E \subseteq F \circ R_q$ implies

$$R_q \circ E \circ R_q \subseteq F \circ R_q \circ R_q = F \circ R_q, \tag{9}$$

as $R_q \circ R_q = R_q$. On the other hand, (9) implies $R_q \circ E \subseteq F \circ R_q$ as $R_q \circ E \subseteq R_q \circ E \circ R_q$. Hence, q is soft if and only if for every $E \in \mathcal{E}_X$ there exists a $F \in E_X$ such that (9) holds.

Similarly, one can show that q is weakly soft (resp., 2-soft) if and only if for every $E \in \mathcal{E}_X$ there exists a $F \in \mathcal{E}_X$ such that

$$E \circ R_q \circ E \circ R_q \subseteq R_q \circ F \circ R_q \text{ (resp., } E \circ R_q \circ E \circ R_q \circ E \circ R_q \circ E \circ R_q \subseteq R_q \circ F \circ R_q \circ F \circ R_q). \tag{10}$$

(b) Now we reformulate the properties from Definition 4.7 in terms of balleans.

If we use the ballean form of X, q is soft if and only if for all $\alpha \in P_X$ there exists $\beta \in P_X$ such that $B_X(R_q[x], \alpha) \subseteq R_q[B_X(x, \beta)]$ for every $x \in X$. By applying q to both sides of the previous inclusion, one obtains the inclusion

$$\overline{B}^q_Y(q(x), \alpha) = q(B_X(R_q[x], \alpha)) \subseteq q(B_X(x, \beta)) \tag{11}$$

for every $x \in X$. Hence, if q is soft and Y is endowed with the quotient ball structure, then q is a $\succ$-mapping (see Remark 2.7). Conversely, if we take the preimages, (11) implies that

$$R_q[B_X(R_q[x], \alpha)] \subseteq R_q[B_X(x, \beta)],$$

for every $x \in X$, which is equivalent to (9). Thus a quotient map $q\colon X \to Y$ is soft if and only if q is a $\succ$-mapping, whenever Y is endowed with the quotient ball structure.

Focusing on weakly soft maps, q is weakly soft if and only if for every $\alpha \in P_X$ there exists $\beta \in P_X$ such that

$$B_X(R_q[B_X(x, \alpha)], \alpha) \subseteq R_q[B_X(R_q[x], \beta)]$$

for every $x \in X$. Thus we can apply q and obtain that

$$\overline{B}_Y^q(\overline{B}_Y^q(q(x), \alpha), \alpha) \subseteq \overline{B}_Y^q(q(x), \beta) \tag{12}$$

for every $x \in X$. (12) is equivalent to weak softness, since application of the preimage of q leads to

$$R_q[B_X(R_q[B_X(x, \alpha)], \alpha)] \subseteq R_q[B_X(R_q[B_X(R_q[x], \alpha)], \alpha)] \subseteq R_q[B_X(R_q[x], \beta)]$$

for every $x \in X$, which is equivalent to (10).

Similarly, by using (10), q is 2-soft if and only if for every $\alpha \in P_X$ there exists $\beta \in P_X$ such that

$$\overline{B}_Y^q(\overline{B}_Y^q(\overline{B}_Y^q(\overline{B}_Y^q(q(x), \alpha), \alpha), \alpha), \alpha) \subseteq \overline{B}_Y^q(\overline{B}_Y^q(q(x), \beta), \beta)$$

for every $x \in X$.

Lemma 4.9. *Let $q\colon X \to Y$ a surjective map from a coarse space $(X, \mathcal{E})$. Then the following implications hold:*
(a) if q has uniformly bounded fibers, then it is soft;
(b) if q is soft, then it is weakly soft;
(c) if q is weakly soft, then it is 2-soft.

Proof. (a) As $R_q \circ E \subseteq R_q \circ E \circ R_q$ for every $E \in \mathcal{E}$, our claim follows from $R_q \circ E \in \mathcal{E}$.

(b) If $E \in \mathcal{E}_X$ and $F \in E_X$ satisfies $R_q \circ E \subseteq F \circ R_q$, then

$$E \circ R_q \circ E \subseteq E \circ F \circ R_q \subseteq R_q \circ (E \circ F) \circ R_q.$$

(c) It is an easy application of the definition of weakly softness and of the fact that $R_q \circ R_q = R_q$. □

The above lemma gives the following implications between the above four properties of a map:

$$\text{uniformly bounded fibers} \longrightarrow \text{soft} \longrightarrow \text{weakly soft} \longrightarrow \text{2-soft}. \tag{13}$$

Counter-examples witnessing that none of these implications is reversible are given in Example 4.21.

Proposition 4.10. *Let $q\colon X \to Y$ be a surjective map from a coarse space $(X, \mathcal{E})$. Then:*
(a) if q is soft, then the quotient entourage structure $\overline{\mathcal{E}}^q$ on Y is a coarse structure;
(b) if the quotient entourage structure $\overline{\mathcal{E}}^q$ is bounded, then q is weakly soft.

Proof. (a) If there exists $F \in \mathcal{E}$, such that $R \circ E \subseteq F \circ R$, then $E \circ R \circ E \subseteq F \circ R \circ R = F \circ R$. Hence, $y' = q(x') \in (q \times q)(F)[y]$. This proves that $(q \times q)(E) \circ (q \times q)(E) \in \mathcal{E}$ if q is soft.

(b) Follows directly from Corollary 4.5. □

Item (b) cannot be reinforced to imply softness of q under the assumption that the quotient ball structure $\overline{\mathfrak{B}}^q$ is bounded as examples from [7] show.

Example 4.11. (a) An example of soft maps are the projections of a product of coarse spaces. It is enough to show that the projections from a product of two balleans are soft. Let (Y, P_Y, B_Y) and (Z, P_Z, B_Z) be two balleans, $X = Y \times Z$ and $q = p_1\colon X \to Y$. Fix $(\alpha, \beta) \in P_Y \times P_Z$. Then

$$B_X(R_q[x], (\alpha, \beta)) = B_X(\{y\} \times Z, (\alpha, \beta)) = B_Y(y, \alpha) \times Z = R_q[B_Y(y, \alpha) \times Z]$$

for every $x = (y, z) \in X$ and so $q(B_X(R_q[x], (\alpha, \beta))) = q(B_Y(y, \alpha) \times Z) = B_Y(y, \alpha)$, for every $x \in X$.

(b) According to the next theorem, the map q from Example 4.2 is not weakly soft. However, one can directly see that this map is not even 2-soft, so cannot be weakly soft according to Lemma 4.9.

The next theorem justifies our interest in the notion of weakly soft map.

Theorem 4.12. *Let $(X,\mathcal{E}_X)$ be a coarse space and $q\colon X\to Y$ be a surjective map.*

(a) The quotient entourage structure $\overline{\mathcal{E}}^q$ on Y is a coarse structure if and only if q is weakly soft.

(b) The family of entourages $\mathcal{E}^*_Y := \{(q\times q)(E)\circ(q\times q)(E)\mid E\in\mathcal{E}_X\}$ is a coarse structure precisely when q is 2-soft.

Proof. (a) In order to see that the family $\overline{\mathcal{E}}^q$ is a coarse structure, we need to only check the axiom (v). Namely, the property $(q\times q)(E)\circ(q\times q)(E)\in\overline{\mathcal{E}}^q$ whenever $E\in\mathcal{E}_X$.

Now suppose that q is weakly soft. Let us check that $(q\times q)(E)\circ(q\times q)(E)\in\overline{\mathcal{E}}^q$ whenever $E\in\mathcal{E}_X$. Pick $F\in\mathcal{E}$ such that $E\circ R_q\circ E\subseteq R_q\circ F\circ R_q$. Therefore, (7) and (6) imply

$$(q\times q)(E)\circ(q\times q)(E)=(q\times q)(E\circ R_q\circ E)\subseteq(q\times q)(R_q\circ F\circ R_q)=(q\times q)(F), \tag{14}$$

i.e., $(q\times q)(E)\circ(q\times q)(E)\in\overline{\mathcal{E}}^q$.

Assume that $\overline{\mathcal{E}}^q$ on Y is a coarse structure. Then $(q\times q)(E)\circ(q\times q)(E)\in\overline{\mathcal{E}}^q$ for every $E\in\mathcal{E}$. By (7), $(q\times q)(E\circ R_q\circ E)\in\overline{\mathcal{E}}^q$, so there exists $F\in\mathcal{E}$ such that

$$(q\times q)(E\circ R_q\circ E)\subseteq(q\times q)(F). \tag{15}$$

In view of Proposition 4.4, (15) implies

$$E\circ R_q\circ E\subseteq R_q\circ F\circ R_q.$$

This proves that q is weakly soft.

(b) We have to check that the family of entourages $\mathcal{E}^*_Y$ satisfies (i), (iii*)-(v*) from §1.1. The argument is similar to that of the above case (a). Indeed, suppose that q is 2-soft. Let us check that $(q\times q)(E)\circ(q\times q)(E)\circ(q\times q)(E)\circ(q\times q)(E)\in\mathcal{E}^*_Y$ whenever $E\in\mathcal{E}_X$. Pick $F\in\mathcal{E}$ such that $E\circ R_q\circ E\circ R_q\circ E\circ R_q\circ E\subseteq R_q\circ F\circ R_q\circ F\circ R_q$. Therefore, (6), (7) and (8) imply

$$(q\times q)(E)\circ(q\times q)(E)\circ(q\times q)(E)\circ(q\times q)(E)=(q\times q)(E\circ R_q\circ E\circ R_q\circ E\circ R_q\circ E)\subseteq(q\times q)(R_q\circ F\circ R_q\circ F\circ R_q)=$$

$$(q\times q)(F\circ R_q\circ F)=(q\times q)(F)\circ(q\times q)(F)\in\mathcal{E}^*_Y.$$

□

Remark 4.13. (a) One can formulate item (b) of Theorem 4.12 in terms of balleans and ball structures as follows. As $\overline{\mathfrak{B}}^q$ is the ball structure corresponding to the entourage structure $\mathcal{E}^q$, the ball structure $\mathfrak{B}^*_Y$ corresponding to $\mathcal{E}^*_Y$ is given by the balls $\overline{B}^q_Y(\overline{B}^q_Y(y,\alpha),\alpha)$ $(\alpha\in P)$. According to item (b) of Theorem 4.12, q is 2-soft if and only if the ball structure $\mathfrak{B}^*_Y$ is a ballean (see also Remark 4.8(b)).

(b) The result of item (a) of Theorem 4.12 is closely related to a similar fact about uniformities (defined by means of a family of entourages) established in [12]: *if $(X,\mathcal{U})$ is a uniform space and $q\colon X\to Y$ is a surjective map, then the family of entourages $\mathcal{U}^*_Y := \{(q\times q)(U)\mid U\in\mathcal{U}\}$ is a uniformity precisely when for every $V\in\mathcal{U}$ there exists $U\in\mathcal{U}$ such that $U\circ R_q\circ U\subseteq(q\times q)^{-1}((q\times q)(V))$* (taking into account that $(q\times q)^{-1}((q\times q)(V))=R_q\circ V\circ R_q$, according to Proposition 4.4).

Theorem 4.12 provides an alternative proof of Proposition 4.10(a):

Corollary 4.14. *Let $\mathfrak{B}_X=(X,P_X,B_X)$ be a ballean and $q\colon X\to Y$ be a quotient map. If q is soft, then the quotient ball structure $\overline{\mathfrak{B}}^q=(Y,P_X,\overline{B}^q_Y)$ is a ballean.*

Corollary 4.15. *Let $\mathfrak{B}_X=(X,P_X,B_X)$ be a ballean and $q\colon X\to Y$ be a quotient map. If $q^{-1}(\overline{B}^q_Y(y,\alpha))=B_X(q^{-1}(y),\alpha)$ for all $\alpha\in P_X$ and for all $y\in Y$, then the quotient ball structure $\overline{\mathfrak{B}}^q=(Y,P_X,\overline{B}^q_Y)$ is a ballean.*

One can ask this natural question: when the restriction of a (weakly) soft map is (weakly) soft? Since injective maps are obviously soft (actually, with uniformly bounded fibers), every map has a soft restriction (e.g., an injective one).

We give now an explicit construction of quotients of coarse spaces and balleans in the general case.

Proposition 4.16. *Let $(X,\mathcal{E})$ be a coarse space and let $q\colon X\to Y$ be a surjective map. Denote by $\mathscr{R}_q$ the coarse structure on X having as a largest entourage the equivalence relation $R_q\subseteq X\times X$ generated by q and let $\mathcal{E}^\#=\max\{\mathcal{E},\mathscr{R}_q\}$. Then $\overline{\mathcal{E}^\#}^q=\widetilde{\mathcal{E}}^q$.*

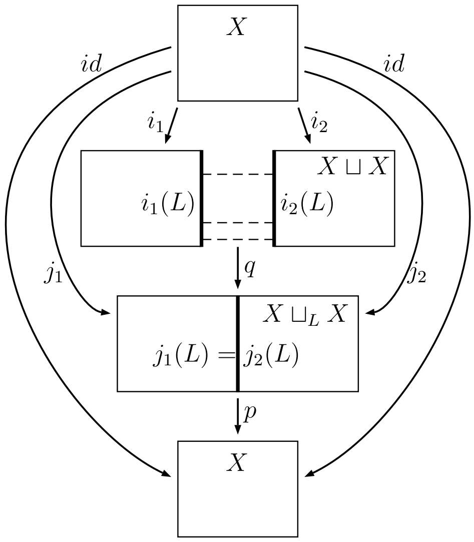

Figure 1: A representation of the adjunction space.

Proof. Applying Lemma 4.9 and Proposition 4.10 to $q\colon (X,\mathcal{E}^{\#}) \to Y$, we deduce that the quotient entourage structure $\overline{\mathcal{E}^{\#}}^{q}$ is a coarse structure (a ballean), as q has uniformly bounded fibers, in view of $R_q \in \mathcal{E}^{\#}$.

By easily adapting what we already observed in §3, $\mathcal{E}^{\#}$ is generated by the entourages of the form $W_n := E \circ R_q \circ E \circ \cdots \circ E \circ R_q \circ E$, where $E \in \mathcal{E}$ participates n-times, E runs over $\mathcal{E}$ and $n \in \mathbb{N}$. According to an obvious counterpart of (8) from Proposition 4.4, $(q\times q)(W_n) = (q\times q)(E) \circ \ldots \circ (q \times q)(E)$, where the composition on the right-hand side has n components. Since this is a typical entourage of $\widetilde{\mathcal{E}}^q$, the coarse structure generated by $\overline{\mathcal{E}}^q$ coincides with that generated by $(q\times q)(W_n)$, which in turn coincides with $\overline{\mathcal{E}^{\#}}^{q}$. □

4.3 The adjunction space $X \sqcup_L X$

Theorem 4.12 gives the description of the quotient ballean of a weakly soft map, namely this is the quotient ball structure (see Example 4.21 (c) for an example of a weakly soft map that is not soft). We aim to describe the quotient ballean (which always exists, as pointed out in Subsection 4.1), in a wider range of quotient maps. Here we do it in the case of the quotient map defining the adjunction space $X \sqcup_L X$ which will be substantially used in the sequel. As we show in Theorem 4.20 this map is very rarely weakly soft (the theorem provides a description of the cases when that quotient map can be weakly soft).

Definition 4.17. Let $\mathfrak{B}_X = (X, P_X, B_X)$ be a ballean and L a subset of X. Let $i_1, i_2\colon X \to X \sqcup X$ be the canonical inclusions of X into the disjoint union $X \sqcup X$. Let $X \sqcup_L X$ be the quotient space $(X\sqcup X)/\sim_L$ obtained from the equivalence relation

$$x \sim_L y \Leftrightarrow \begin{cases} x = i_1(l), y = i_2(l) \text{ with } l \in L, \\ y = i_1(l), x = i_2(l) \text{ with } l \in L, \\ x = y, \end{cases} .$$

If $L = \emptyset$, $X \sqcup_L X$ coincides with $X \sqcup X$, this is why we assume from now on that $L \neq \emptyset$. Our aim is to describe the quotient ballean structure $X \sqcup_L X$ of the quotient of coproduct ballean $\mathfrak{B}_X \coprod \mathfrak{B}_X$ under the canonical map $q\colon X \sqcup X \to X \sqcup_L X$ defined by the equivalence relation $\sim_L$. Put $j_\nu = q \circ i_\nu$, so that $X \sqcup_L X = j_1(X) \cup j_2(X)$.

Let $p\colon X \sqcup_L X \to X$ be the map defined by $p(j_\nu(x)) = x$ for all $x \in X$ (this definition is correct as both j_ν are injective, $j_1 \restriction_L = j_2 \restriction_L$ and $X \sqcup_L X = j_1(X) \cup j_2(X)$). Let σ be the obvious involution (symmetry) of the coproduct $X \sqcup X$ and σ' be the involution of $X \sqcup_L X$ induced by σ (so that $\sigma'(j_1(x)) = j_2(x)$ and $\sigma'(j_2(x)) = j_1(x)$ for every $x \in X$). All these maps are conveniently represented in Figure 1.

Example 4.21(a) shows that the quotient ball structure $\overline{\mathfrak{B}}^q$ on $X \sqcup_L X$ need not be a ballean in general. This is why we define a new ball structure $\mathfrak{B}^a_{X\sqcup_L X}$ on $X \sqcup_L X$ with radii set P_X and balls defined by

$$B_{X\sqcup_L X}(j_\nu(x), \alpha) = \begin{cases} j_\nu(B_X(x,\alpha)) & \text{if } B_X(x,\alpha) \cap L = \emptyset, \\ j_1(B_X(x,\alpha)) \cup j_2(B_X(x,\alpha)) & \text{otherwise,} \end{cases} \tag{16}$$

for every $x \in X$, $\nu = 1, 2$, $\alpha \in P_X$.

Theorem 4.18. $\mathfrak{B}^a_{X\sqcup_L X}$ *is the quotient ballean structure on* $X \sqcup_L X$.

Proof. We have to prove that $\mathfrak{B}^a_{X\sqcup_L X}$ is upper multiplicative and upper symmetric, q is bornologous and $\mathfrak{B}^a_{X\sqcup_L X}$ has quotient's universal property.

First we want to show that it is the upper mulplicative. Fix two radii $\alpha, \beta \in P_X$ and let $\gamma \in P_X$ be an element such that $B_X(B_X(x,\alpha),\beta) \subseteq B_X(x,\gamma)$ for every $x \in X$. Then it is easy to check that

$$B_{X\sqcup_L X}(B_{X\sqcup_L X}(j_\nu(x),\alpha),\beta) \subseteq B_{X\sqcup_L X}(j_\nu(x),\gamma) \quad \text{for} \quad \nu = 1,2 \quad \text{and} \quad x \in X,$$

since the property $B_X(B_X(x,\alpha),\beta) \cap L \neq \emptyset$ implies $B_X(x,\gamma) \cap L \neq \emptyset$.

The second thing we want to prove is the upper symmetry. Let us first note that for both embeddings $j_\nu\colon X \to X \sqcup_L X$ the ball structure induced on $j_\nu(X)$ coincides with the original ballean structure transported by j_ν. Without loss of generality we can assume the ballean $\mathfrak{B}_X$ to be symmetric (Remark 3.9).

Without loss of generality, fix a point $j_1(x) \in Y$, where $x \in X$, and a radius $\alpha \in P$. Let $j_\nu(x') \in B_{X\sqcup_L X}(j_1(x),\alpha)$ for some $x' \in X$ and $\nu = 1,2$.

If $B_X(x,\alpha) \cap L = \emptyset$, it is trivial to check that $j_1(x) \in B_{X\sqcup_L X}(j_\nu(x'),\alpha)$, since we have $B_{X\sqcup_L X}(j_1(x),\alpha) = j_1(B_X(x,\alpha))$ and $B_{X\sqcup_L X}(j_\nu(x'),\alpha) \supseteq j_1(B_X(x',\alpha))$.

We consider the case $B_X(x,\alpha)\cap L \neq \emptyset$ in the sequel. Note that one has $\sigma'(B_{X\sqcup_L X}(j_\mu(z),\alpha)) = B_{X\sqcup_L X}(j_\mu(z),\alpha)$ when $B_X(z,\alpha)\cap L \neq \emptyset$, for every $\mu = 1,2$, where $z \in X$. Applying p we obtain $x' = p(j_\nu(x')) \in p(B_Y(j_\nu(x),\alpha)) = B_X(x,\alpha)$. Hence, so $x \in B_X(x',\alpha)$ by the symmetry of the ball $B_X(x,\alpha)$. Thus, $j_1(x) \in j_1(B_X(x',\alpha)) \subseteq B_{X\sqcup_L X}(j_\nu(x'),\alpha)$, in case $B_X(x',\alpha) \cap L \neq \emptyset$ or $\nu = 1$. Otherwise, if $B_X(x',\alpha) \cap L = \emptyset$ and $\nu = 2$, we use the fact that $j_1(x) \in B_{X\sqcup_L X}(\sigma'(j_1(x)),\alpha)$ and $\sigma'(j_1(x)) \in j_2(B_X(x',\alpha)) \subseteq B_{X\sqcup_L X}(j_2(x'),\alpha)$. Therefore, $j_1(x) \in B_{X\sqcup_L X}(B_{X\sqcup_L X}(j_2(x'),\alpha),\alpha)$ and we conclude by upper multiplicativity.

So far we have checked that the ball structure $\mathfrak{B}^a_{X\sqcup_L X}$ is a ballean. Since $\overline{\mathfrak{B}}^q \prec \mathfrak{B}^a_{X\sqcup_L X}$, in order to conclude we only need to check that $\mathfrak{B}^a_{X\sqcup_L X} \prec \widetilde{\mathfrak{B}}^q$. As $\widetilde{\mathfrak{B}}^q$ is the finest coarse structure containing $\overline{\mathfrak{B}}^q$ (i.e., $\overline{\mathfrak{B}}^q \prec \widetilde{\mathfrak{B}}^q$), this will imply that $\mathfrak{B}^a_{X\sqcup_L X} = \widetilde{\mathfrak{B}}^q$.

In fact, assume that $z \in B_{X\sqcup_L X}(y,\alpha)$ for some $y \in X \sqcup_L X$ and $\alpha \in P$. Assume that $y = q(x)$ and $z = q(x')$ for some $x, x' \in X \sqcup X$. According to Proposition 4.16 (see also Claim 3.8 for the ballean version in a more general setting), it is enough to find a finite chain of points $x_0 = x', x_1, \ldots, x_n = x$ in $X \sqcup X$, such that each x_i is either contained in the ball $B_{X\sqcup X}(x_{i+1},\alpha)$, or $x_i \in R_q[x_{i+1}]$ (i.e., $q(x_i) = q(x_{i+1})$).

We can assume without loss of generality that $x = i_1(u) \in i_1(X)$ and $x' = i_\nu(u') \in i_\nu(X)$ for $u, u' \in X$ and $\nu = 1,2$ (so that $y = j_1(u), z = j_\nu(u')$). If $\nu = 1$ we deduce that $u' \in B_X(u,\alpha)$, so $x' \in B_{X\sqcup X}(x,\alpha)$, so we can simply take $n = 1$.

If $\nu = 2$, then $L \cap B_X(x,\alpha) \neq \emptyset$ so there exists $l \in L \cap B_X(u,\alpha)$, consequently, $i_1(l) \in B_{X\sqcup X}(x,\alpha)$. By the symmetry of the balls $u \in B_X(l,\alpha)$. Hence, $\sigma(x) \in B_{X\sqcup X}(i_2(l),\alpha)$. As $x' \in B_{X\sqcup X}(\sigma(x),\alpha)$, and $i_2(l) \in R_q[i_1(l)]$, we can put $n = 4$ and let $x_0 = x'$, $x_1 = \sigma(x)$, $x_2 = i_2(l)$, $x_3 = i_1(l)$, $x_4 = x$ to conclude that

$$x' \in B_{X\sqcup X}(B_{X\sqcup X}(R_q[B_{X\sqcup X}(x,\alpha)],\alpha),\alpha).$$

This concludes the proof of the equality $\mathfrak{B}^a_{X\sqcup_L X} = \widetilde{\mathfrak{B}}^q$, i.e., $\mathfrak{B}^a_{X\sqcup_L X}$ is the quotient ballean structure on $X \sqcup_L X$. ☐

Remark 4.19. (a) The pair of maps $j_1, j_2 : X \to Y := X \sqcup_L X$ associated to the subspace L of X is usually referred to as *cokernel pair* of the inclusion map $m : L \to X$ in category theory. In categorical terms, it means that $j_1, j_2 : X \to Y$ is the pushout of the pair $m, m : L \to X$ (in other words, it satisfies $j_1 \circ m = j_2 \circ m$ and for every pair of bornologous maps $u_1, u_2 : X \to Z$ with $u_1 \circ m = u_2 \circ m$ there exists a unique bornologous map $t : Y \to Z$ such that $u_\nu = t \circ j_\nu$ for $\nu = 1,2$). Certainly, cokernel pairs exist in **Coarse**, as it is co-complete (being a topological category, by Theorem 3.5). The knowledge of its concrete (simple) form described in Theorem 4.18, is the relevant issue in this case.

(b) While for a non-empty space X the coproduct $X \sqcup X$ is never connected, the adjunction space $Y = X \sqcup_L X$ is connected precisely when X is connected and $L \neq \emptyset$. This follows from the fact that $X \sqcup_L X = j_1(X) \cup j_2(X)$, both $j_\nu(X)$ are connected and the union is not disjoint.

The next theorem will provide, among others, examples showing that the quotient ball structure of a ballean may fail to be a ballean. To this end the quotient map defining the adjunction space, as well as its restrictions, will be used.

Theorem 4.20. *For a ballean* X *and a subballean* Y *the restriction* q_1 *of the quotient map* $q\colon X \sqcup X \to X \sqcup_Y X$ *to* $X \sqcup Y$ *is weakly soft. Moreover, the following are equivalent:*

(a) $X = Y \sqcup X \setminus Y$;

(b) the quotient ball structure on $X \sqcup_Y X$ is a ballean;
(c) q_1 is soft.

Proof. It suffices to check that the quotient ball structure of $X \sqcup_Y Y$ coincides with the (ballean) structure of X, then Theorem 4.12 will imply that q is weakly soft. To check this we note that the map $j_1 \colon X \to X \sqcup_Y Y = j_1(X)$ is bijective. Moreover, for every $\alpha \in P$ one has

$$j_1(B_X(x,\alpha)) = \overline{B}^{q_1}_{j_1(X)}(j_1(x),\alpha) \tag{17}$$

This remains true also when $y \in Y$, then $j_1(y) = j_2(y)$, so again (17) holds true for $j_1(y) = j_2(y)$. Since these balls define the ball structure of both spaces, our claim is proved.

(a) $\to$ (b) To prove that the quotient ball structure on $X \sqcup_Y X$ is a ballean we need to check that it is upper multiplicative. Pick $\alpha, \gamma \in P$ and find a $\beta \in P$ such that $B_X(B_X(x,\alpha),\gamma) \subseteq B_X(x,\beta)$ for all $x \in X$. It is enough to show that for every $z \in Z := X \sqcup_Y X$ one has $\overline{B}^q_Z(\overline{B}^q_Z(z,\alpha),\gamma) \subseteq \overline{B}^q_Z(z,\beta)$. We can assume without loss of generality that $z = j_1(x)$ for some $x \in X$. If $x \in Y$, then $\overline{B}^q_Z(z,\alpha) = j_1(B_X(x,\alpha)) \cup j_2(B_X(x,\alpha))$. Hence,

$$\overline{B}^q_Z(\overline{B}^q_Z(z,\alpha),\gamma) = j_1(B_X(B_X(x,\alpha),\gamma)) \cup j_2(B_X(B_X(x,\alpha),\gamma)) \subseteq j_1(B_X(x,\beta)) \cup j_2(B_X(x,\beta)) = \overline{B}^q_Z(z,\beta).$$

In case $x \notin Y$, $\overline{B}^q_Z(z,\alpha) = j_1(B_X(x,\alpha))$ as $B_X(x,\alpha) \cap Y = \emptyset$. Hence, $\overline{B}^q_Z(\overline{B}^q_Z(z,\alpha),\gamma) = \overline{B}^q_Z(j_1(B_X(x,\alpha)),\gamma)$. Since, our assumption $x \notin Y$ yields $B_X(B_X(x,\alpha),\gamma) \cap Y = \emptyset$, one has

$$\overline{B}^q_Z(j_1(B_X(x,\alpha)),\gamma) = j_1(B_X(B_X(x,\alpha),\gamma)) \subseteq j_1(B_X(x,\beta)) \subseteq \overline{B}^q_Z(z,\beta).$$

(b) $\to$ (a) Assume that there exists $\alpha \in P$ and $y \in Y$, $x \in X \setminus Y$ with $y \in B(x,\alpha)$. Then

$$j_2(x) \in \overline{B}^q_{X \sqcup_Y X}(\overline{B}^q_{X \sqcup_Y X}(j_1(x),\alpha),\alpha),$$

but $j_2(x) \notin j_1(X) \supseteq \overline{B}^q_{X \sqcup_Y X}(j_1(x),\beta)$ for every $\beta \in P$, a contradiction.

(a) $\to$ (c) To check that q is soft pick an element $z \in Z := X \sqcup Y$. We have to check that

$$R_q[B_Z(R_q[z],\alpha)] \subseteq B_Z(R_q[z],\alpha) \tag{18}$$

for every $\alpha \in P$. If $z = (u,\nu)$ with $u \in X$ and $\nu = 1,2$, consider two cases. If $u \notin Y$, then necessarily $\nu = 1$ and $B_X(u,\alpha) \cap Y = \emptyset$. Therefore, $R_q[z] = \{z\}$ and $R_q[B_Z(R_q[z],\alpha)] = i_1(B_X(u,\alpha)) = B_Z(R_q[z],\alpha)$. Hence, (18) is proved in this case.

If $u \in Y$, then $R_q[z] = \{i_1(u), i_2(u)\}$, so $B_Z(R_q[z],\alpha) = i_1(B_X(u,\alpha)) \cup i_2(B_X(u,\alpha))$, therefore, $R_q[B_Z(R_q[z],\alpha)] = B_Z(R_q[z],\alpha)$. This proves again (18).

(c) $\to$ (a) Assume that $y \in Y \cap B_X(x,\alpha)$ for some $\alpha \in P$ and some $x \notin Y$. Then $R_q[x] = \{x\}$. To see that softness at x fails, note that $R_q[i_1(B_X(x,\alpha))] \not\subseteq i_1(B_X(x,\beta)) \subseteq i_1(X)$, since otherwise for $y \in B_X(x,\alpha)$ one would have

$$i_2(y) \in R_q[i_1(y)] \subseteq R_q[i_1(B_X(x,\alpha))] \subseteq i_1(X),$$

a contradiction. $\square$

The examples provided below show, among others, that none of the implications in (13) can be inverted.

Example 4.21. (a) Theorem 4.20 shows that the quotient ball structure on $X \sqcup_L X$ is not a ballean in general (choose L in such a way that X is not a coproduct of L and $X \setminus L$). Therefore, the map $q \colon X \sqcup X \to X \sqcup_L X$ is not weakly soft.

(b) Applying Theorem 4.20 in the extreme case when $Y = X$ we obtain an example of a soft map with unbounded fibers showing that the implication (b) of Proposition 4.10 cannot be inverted. Indeed, if $(X, \mathcal{E})$ is an unbounded coarse space, then the quotient map $q \colon X \coprod X \to X$ that glues together the two copies of X is soft, but its fibers are not bounded. This example shows also that the first implication in (13) cannot be inverted.

(c) Theorem 4.20 provides also an example of a weakly soft map that is not soft showing that the implication in Corollary 4.14 cannot be inverted (choose L in such a way that X is not a coproduct of L and $X \setminus L$ and consider the weakly soft map q_1). This shows also that the second implication in (13) cannot be inverted.

(d) Let us see now that the map $q \colon X \sqcup X \to Y := X \sqcup_L X$ is 2-soft. In conjunction with item (a) this will provide an example witnessing that the last implication in (13) cannot be inverted. According to Remark 4.13, the ball structure $\mathfrak{B}^*_Y$ of the quotient Y given by the "doubled" balls $\overline{B}^q_Y(\overline{B}^q_Y(y,\alpha),\alpha)$ ($\alpha \in P$) is a ballean precisely when the map q is 2-soft. On the other hand, it is not hard to realize that the ball structure $\mathfrak{B}^*_Y$ is asymorphic to $\mathfrak{B}^a_{X \sqcup_L X}$, shown to be a ball structure in Theorem 4.18. Therefore, $\mathfrak{B}^*_Y$ is itself is a ballean, so q is 2-soft.

5 Epimorphisms and monomorphisms in the coarse category $\mathbf{Coarse}/_\sim$

The morphisms in $\mathbf{Coarse}/_\sim$ are equivalence classes of morphisms $f\colon X \to Y$ in **Coarse**, nevertheless, we shall often speak of properties of morphisms of $\mathbf{Coarse}/_\sim$ having in mind some specific representative f in **Coarse** of the equivalence class $[f]$. In some cases, that property is available regardless of the choice of the representative f (see Remark 3.2), in other cases this may fail (Remark 5.5).

Theorem 5.1. *Let $\mathfrak{B}_X = (X, P_X, B_X)$ be a ballean and L be a subset of X. Then the following are equivalent.*
(1) L is large;
(2) every pair of bornologous maps $f, g\colon X \to Y$ with $f\restriction_L \sim g\restriction_L$ are close;
(3) every pair of bornologous maps $f, g\colon X \to Y$ with $f\restriction_L = g\restriction_L$ are close.
Consequently, a morphism $f\colon X \to Y$ in **Coarse** *is an epimorphism in* $\mathbf{Coarse}/_\sim$ *if and only if $f(X)$ is large in Y.*

Proof. (1)→(2) Assume that L is large in X and let $f, g\colon X \to Y$ be bornologous maps to a ballean $\mathfrak{B}_Y = (Y, P_Y, B_Y)$ with $f\restriction_L \sim g\restriction_L$. Pick $\alpha \in P_X$ such that $B_X(L, \alpha) = X$. Since the maps f, g are bornologous, there exist $\beta, \beta' \in P_Y$ be such that

$$f(B_X(y,\alpha)) \subseteq B_Y(f(y),\beta) \ \text{ and } \ g(B_X(y,\alpha)) \subseteq B_Y(f(y),\beta') \ \text{ for all } \ y \in X. \tag{19}$$

Since $f\restriction_L \sim g\restriction_L$, there exists $\gamma \in P_Y$ such that $g(l) \in B_Y(f(l), \gamma)$ for every $l \in L$. Then, according to Remark 2.3, there exists $\delta \in P_Y$ such that for all $u, v, w \in Y$, $u \in B_Y(w, \beta)$, $v \in B_Y(w, \gamma)$ imply $u \in B_Y(v, \delta)$. We apply once again Remark 2.3: there exists $\varepsilon \in P_Y$ such that for every $x, y, z \in Y$ with $x \in B_Y(y, \delta)$ and $z \in B_Y(y, \beta')$ we have $x \in B_Y(z, \varepsilon)$. We show that f, g are close.

Pick arbitrarily $x \in X$. As L is large, one can find $l \in L$ such that $x \in B_X(l, \alpha)$. Applying (19) to $y = l$ we deduce that $f(x) \in f(B_X(l, \alpha)) \subseteq B_Y(f(l), \beta)$. Hence, $f(x) \in B_Y(g(l), \delta)$, as $g(l) \in B_Y(f(l), \gamma)$. On the other hand, $g(x) \in B_Y(g(l), \beta')$, again by (19). Therefore, $f(x) \in B_Y(g(x), \varepsilon)$. This proves that f, g are close.

(2)→(3) This is trivial.

(3)→(1) Consider the canonical maps $j_\nu\colon X \to X \sqcup_L X$ associated to the adjunction space $Y = X \sqcup_L X$. As $j_1\restriction_L = j_2\restriction_L$, our hypothesis implies that j_1 and j_2 are close. Let this be witnessed by $\alpha \in P_X$. Let $\alpha' \in P_X$ be a radius such that $B_X^*(x, \alpha) \subseteq B_X(x, \alpha')$ for every $x \in X$. Now we show that $X = B_X(L, \alpha')$. Indeed, as $j_1(x)$ and $j_2(x)$ are α-close (see Remark 2.6), $j_2(x) \in B_Y(j_1(x), \alpha)$. This gives $j_2(x) \in j_2(B_X(x, \alpha))$ and $B_X(x, \alpha) \cap L \neq \emptyset$. This obviously implies $x \in B_X(L, \alpha')$.

The last assertion follows from Remark 3.2. □

Theorem 5.2. *Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be two coarse spaces and $h\colon X \to Y$ a bornologous map between them. Then the following are equivalent:*
(1) h is a coarse embedding, i.e., for every $E \in \mathcal{E}_Y$, $(h \times h)^{-1}(E) \in \mathcal{E}_X$;
(2) for every coarse space $(Z, \mathcal{E}_Z)$ and every pair of bornologous maps $f, g\colon Z \to X$, if $h \circ f \sim h \circ g$, then $f \sim g$.
Consequently, a morphism $h\colon X \to Y$ in **Coarse** *is a monomorphism in* $\mathbf{Coarse}/_\sim$ *if and only if h is a coarse embedding.*

Proof. (1) → (2) Assume that $f, g\colon Z \to X$ are bornologous maps with $h \circ f \sim h \circ g$. To establish $f \sim g$ we need to check that $M := \{(f(z), g(z)) \mid z \in Z\}$ belongs to $\mathcal{E}_X$. As $h \circ f \sim h \circ g$, one has $(h \times h)(M) = \{(h(f(z)), h(g(z))) \mid z \in Z\} \in \mathcal{E}_Y$. Consequently, $M \subseteq (h \times h)^{-1}((h \times h)(M)) \in \mathcal{E}_X$.

(2) → (1) Suppose for a contradiction that h is not a coarse embedding. This means that there exists an entourage $E \in \mathcal{E}_Y$ such that $E' = (h \times h)^{-1}(E) \notin \mathcal{E}_X$.

Let $Z := E'$ endowed with the discrete coarse structure $\mathcal{E}_Z = \{\Delta_Z\}$. Consider the maps $p_1, p_2\colon Z \to X$ defined by $p_1\colon (x, y) \mapsto x$ and $p_2\colon (x, y) \mapsto y$. These maps are bornologous, because $(Z, \mathcal{E}_Z)$ is discrete. Moreover, $\{(p_1(z), p_2(z)) \mid z \in Z\} = E' \notin \mathcal{E}_X$. This means that p_1 and p_2 are not close.

On the other hand,

$$\{((h \circ p_1)(z), (h \circ p_2)(z)) \mid z \in Z\} = \{(h(p_1(z)), h(p_2(z))) \mid z \in Z\} = \{(h \times h)(e) \mid e \in E'\} \subseteq E \in \mathcal{E}_Y$$

and so $\{(h \circ p_1)(z)), (h \circ p_2)(z))) \mid z \in Z\} \in \mathcal{E}_Y$. Therefore, $h \circ p_1 \sim h \circ p_2$. This contradicts our hypothesis (2).

As in the previous theorem, the last assertion follows from Remark 3.2. □

In particular Theorem 5.1 shows that morphisms with large image are epimorphisms in $\mathbf{Coarse}/_\sim$, while Theorem 5.2 implies that the monomorphisms are the coarse embeddings. If we apply Proposition 1.5, then we obtain the result we have announced in §3. Namely, the category $\mathbf{Coarse}/_\sim$ is *balanced*.

Corollary 5.3. *Let $f\colon X \to Y$ a morphism in the category* $\mathbf{Coarse}/_\sim$*. Then f is a bimorphism if and only if it is an isomorphism.*

Stability of epimorphisms under pullback is an important issue in category theory. This is why we are interested to determine here those morphisms $f\colon X \to Y$ in **Coarse** such that $[f]$ is an epimorphism in $\mathbf{Coarse}/_\sim$ and for every morphism $e\colon Z \to Y$ in **Coarse** such that $[e]$ is an epimorphisms in $\mathbf{Coarse}/_\sim$ the class $[u]$ of the pullback $u\colon P \to X$ in (3) is an epimorphism in **Coarse**. We shall shortly refer to this property in the sequel by simply saying "epimorphisms are preserved under taking pullback along f". As we shall see, this property is not invariant under $\sim$ (see Remark 5.5).

A morphism $f\colon X \to Y$ in the category **Coarse** is said to be *L-reflecting*, if $f^{-1}(L)$ is large in X for every large set L of Y. The properties of maps to preserve or to reflect size properties (for example largeness) will be studied in a forthcoming paper [7].

A subset A of a ballean X is called *extra-large* if, for every large subset L of X, the intersection $A \cap L$ is still large in X ([23]). The relevance of this notion from categorical point of view is revealed in the following corollary.

Corollary 5.4. *Let $f\colon X \to Y$ a representative of an epimorphism in the category* $\mathbf{Coarse}/_\sim$*. Then the following are equivalent:*

(a) the epimorphisms are preserved under taking pullback along f;
(b) the co-restriction map *$f\colon X \to f(X)$ is L-reflecting and $f(X)$ is extra-large in Y.*

Proof. We shall simplify the proof by reducing the argument to the case of epimorphisms that are simply inclusions. To this end consider a pullback diagram (3), put $Z_1 := f^{-1}(e(Z))$ and let $e_1\colon Z_1 \hookrightarrow X$ be the inclusion map. Let us see next that

$$Z_1 = u(P). \tag{20}$$

If $u(p) \in u(P)$ for some $p \in P$, then obviously $f(u(p)) = e(v(p)) \in e(X)$, so $u(p) \in Z_1$. On the other hand, if $x \in Z_1$, then $f(x) = e(z)$ for some $z \in Z$, hence $(x, z) \in P$ (see the construction of P as an equalizer in §3). Then $x = u(x, z) \in u(P)$. This proves (20).

Let $j\colon e(Z) \hookrightarrow Y$ be the inclusion map. Then one can easily see that

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle e_1}\big\uparrow & & \big\uparrow{\scriptstyle j} \\ Z_1 & \xrightarrow{\ f\restriction_Z\ } & e(Z). \end{array} \tag{21}$$

is a pullback diagram.

It easily follows from Theorem 5.1 that

- u is an epimorphisms if and only if $e_1\colon u(P) = Z_1 \hookrightarrow X$ is an epimorphism.
- e is an epimorphism precisely when j is an epimorphism.

This makes it clear that the epimorphisms are preserved under taking pullback along f precisely when pullbacks along f of epimorphisms that are inclusions in Y are preserved and the general pullback diagram (3) can be replaced by the pullback diagram (21), where the vertical arrows are inclusions.

(a) $\to$ (b) Assume that epimorphisms are preserved under taking pullback along f. To check that $f(X)$ is extra-large in Y pick a large subset L of Y. Then the inclusion map $j\colon L \hookrightarrow Y$ is an epimorphism in $\mathbf{Coarse}/_\sim$ by Theorem 5.1. Hence, the pullback $j_1\colon f^{-1}(L) \to X$ must be an epimorphism on $\mathbf{Coarse}/_\sim$. Hence, $f^{-1}(L)$ is large in X by Theorem 5.1. It easily follows from the definition of largeness (see [23, Lemma 11.3]), that $f(f^{-1}(L)) = f(X) \cap L$ is large in $f(X)$. As $f(X)$ is large in Y (again by Theorem 5.1, as f is an epimorphism), we deduce that $f(X) \cap L$ is large in Y. This proves that $f(X)$ is extra-large in Y.

The fact that $f\colon X \to f(X)$ is L-reflecting follows directly from the definitions.

(b) $\to$ (a) Suppose that $f(X)$ is extra-large in Y and let $e\colon Z \to Y$ be an epimorphism. Let us prove that $e_1 := f^{-1}(e)\colon Z_1 \to X$ is an epimorphism in $\mathbf{Coarse}/_\sim$. By Theorem 5.1, $L = e(Z)$ is large in Y. Then $L \cap f(X)$ is large in Y. Consequently, $L \cap f(X)$ is large in $f(X)$. Hence, $f^{-1}(L) = f^{-1}(L \cap f(X))$ is large in X, by hypothesis. As $f^{-1}(L) = e_1(Z_1)$ is large in X, by Theorem 5.1 we conclude that e_1 is an epimorphism in $\mathbf{Coarse}/_\sim$. □

Remark 5.5. Unlike Theorems 5.1 and 5.2, where the characterizing property of the morphism in **Coarse** is available *for all* representatives of the $\sim$-equivalence class (see Remark 3.2), the property of item (b) from the above corollary fails to be invariant under closeness. Indeed, the map $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \lfloor x \rfloor$ does not satisfy (b), as $\mathbb{Z} = f(\mathbb{R})$ is not extra-large in $\mathbb{R}$. Nevertheless, $f \sim id_{\mathbb{R}}$ and $id_{\mathbb{R}}$ obviously satisfies (b). This example shows that the property (a) is also "fragile" in this sense. This is explained by the fact that while epimorphisms are taken in **Coarse**$/_\sim$, the pullbacks are taken in **Coarse**.

One can prove that a surjective map that is either effectively proper or soft is L-reflecting, while a surjective weakly soft map need not be L-reflecting ([7]). This gives the following corollary:

Corollary 5.6. *Let $f\colon X \to Y$ a morphism in the category* **Coarse** *such that $f(X)$ is extra-large in Y. If the co-restriction map $f\colon X \to f(X)$ is soft, then the epimorphisms are preserved under taking pullback along f.*

6 Compatible coarse structures on groups

In this section we shortly discuss coarse structures and balleans on groups, for which we ask some compatibility between the large sale and the algebraic structures of groups. There are many papers on this topic (see for example [3], [17], [22], [24], [25]), where the authors provide a way to generalize the well-known theory of finitely generated groups and the more recent development in the framework of countable groups (see, for example, [9], [28] and [2]). The coarse groups (to be defined below) will provide a rich supply of examples of coarse spaces and soft maps.

If G is a group, we consider the action of G on $G \times G$ defined by $g \cdot (x, u) \mapsto (gx, gy)$. Consequently, we call a subset E of $G \times G$ *invariant*, if $GE := \{(gx, gy) \mid g \in G, (x, y) \in E\} = E$.

A coarse structure $\mathcal{E}$ on a group G is said to be *compatible* ([17]) if $\mathcal{E}$ has a base consisting of invariant entourages.

One can characterize this property using a notion for general coarse spaces. A family of maps $f_i\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ $(i \in I)$ in **Coarse** is *uniformly bornologous*, if for every $E \in \mathcal{E}_X$ there exists $E' \in \mathcal{E}_Y$ such that $(f_i \times f_i)(E) \subseteq E'$ for all $i \in I$. In these terms, a coarse structure $\mathcal{E}$ on a group G is compatible if and only if the family of all *left shifts* (i.e. the maps $h \mapsto gh$, $h \in G$) is a uniformly bornologous family.

We call a group G endowed with a compatible coarse structure $\mathcal{E}$ a *coarse group*.

Definition 6.1. ([24]) Let G be a group. A *group ideal* for G is a family $\mathcal{F} \subseteq \mathcal{P}(G)$ of subsets of the group G which satisfies the following properties:

(i) there exits a non-empty element $F \in \mathcal{F}$;
(ii) $\mathcal{F}$ is closed under finite unions;
(iii) $\mathcal{F}$ is closed by taking subsets;
(iv) for every $F_1, F_2 \in \mathcal{F}$, $F_1F_2 := \{gh \in G \mid g \in F_1, h \in F_2\} \in \mathcal{F}$;
(v) for each $F \in \mathcal{F}$, $F^{-1} := \{g^{-1} \in G \mid g \in F\} \in \mathcal{F}$.

If $\mathcal{F}$ is a group ideal, then it gives a compatible coarse structure on G by $\mathcal{E}_{\mathcal{F}} := \{E \subseteq G \times G \mid \exists F \in \mathcal{F} : E \subseteq G(F \times F)\}$ ([17, Proposition 2.4]). If $\mathcal{E}$ is a compatible coarse structure on G, then $\mathcal{F}(\mathcal{E}) = \{\pi_G(E) \mid E \in \mathcal{E}\}$ is a group ideal, where $\pi_G\colon G \times G \to G$ is the *shear map* defined by $\pi_G(x, y) = y^{-1}x$ for $x, y \in G$ ([17, Proposition 2.5]).

Remark 6.2. In paper [17], we can find a definition which is slightly different from the one of 6.1. The authors consider *generating families*, which are families of subsets of the group G satisfying (i), (ii), (iv) and (v) from the definition above. If a generating family satisfies also (iii) (so its is group ideals), it is said to be a *complete* generating family.

Let $\mathcal{F} \subseteq \mathcal{P}(G)$ be a family of subsets of a group G. Then we can construct its *completion* $\widehat{\mathcal{F}} := \{A \subseteq G \mid \exists F \in \mathcal{F} : A \subseteq F\}$. If $\mathcal{F}$ is a generating family, then its completion is still a generating family which is also complete and so a group ideal. One can use group ideals without loss of generality, as the passage from a generating family to a group ideal via the operation of completion has no impact on the generated coarse structure, i.e., $\mathcal{E}_{\mathcal{F}} = \mathcal{E}_{\widehat{\mathcal{F}}}$ for a compatible coarse structure $\mathcal{F}$ ([17, Proposition 2.7]).

Note that
$$\pi_G(G(F \times F)) = F^{-1}F \text{ and } \pi_G^{-1}(F^{-1}F) \supseteq G(F \times F).$$
If $\mathcal{E}$ is a compatible coarse structure on G, then $\mathcal{F}_{\mathcal{E}} := \{\pi_G(E) \subseteq G \mid E \in \mathcal{E}\}$ is a group ideal which generates $\mathcal{E}$ ([17, Proposition 2.5]).

For every group G, there is a one to one correspondence between group ideals $\mathcal{F} \subseteq \mathcal{P}(G)$ and compatible coarse structures on G.

Note that, if $\mathcal{F}$ is a group ideal on G, there always exists an element $F_e \in \mathcal{F}$ such that $e_G \in F_e$, as there exits a non-empty element $F \in \mathcal{F}$ and $F^{-1} \in \mathcal{F}$. Now pick any $x \in F$ to get
$$e_G = x \cdot x^{-1} \in F \cdot F^{-1} =: F_e \in \mathcal{F}.$$

As a consequence, we have that the singleton $\{e_G\} \in \mathcal{F}$. This means in particular that $\mathcal{F}_0 := \{e_G\}$ is the smallest group ideal and $\mathcal{E}_{\mathcal{F}_0} = \{\Delta_G\}$.

A *base* of a group ideal $\mathcal{F}$ is a family $\mathcal{B} \subseteq \mathcal{F}$, such that every K from $\mathcal{F}$ is contained in some B of $\mathcal{B}$. Hence, in the previous notation, $\mathcal{B}$ is a base of $\mathcal{F}$ if and only if $\widehat{\mathcal{B}}$ is a group ideal (in this case $\mathcal{F} = \widehat{\mathcal{B}}$).

Let G be a group and $\mathcal{F}$ a group ideal. Then $\mathcal{E}_{\mathcal{F}}$ is connected if and only if $\bigcup \mathcal{F} = G$. In this case, $\mathcal{F}$ is also stable under left and right shifts by arbitrary elements of $g \in G$.

Example 6.3. In the sequel we give examples of connected group ideals.

(1) $\mathcal{F}_{fin}$ is the collection of all finite subsets of G, which we name *finitary group ideal*. It generates the *group-finite coarse structure*. It is the finest connected group coarse structure.
(2) If G is a topological group, $\mathcal{C}(G) := \{K \subseteq G \mid K \text{ is compact}\}$ is a base of a group ideal which generates $\mathcal{E}_{C(G)}$, the *group-compact coarse structure*.
(3) If d is a left-invariant metric on a group G, then the base $\mathcal{F}_d := \{B_d(e_G, R) \mid R > 0\}$ generates $\mathcal{E}_d$, the d-bounded coarse structure.

It is possible to nicely unify items (a) and (c) in the case of a countably infinite group G. It was proved by Smith [27] that every such group G admits a left invariant proper metric d and every pair of such metrics are coarsely equivalent (actually asymorphic). Here proper means that all balls are finite, hence the d-bounded coarse structure coincides with the group-finite one.

If $\mathcal{B}$ is a base for a group ideal $\mathcal{F}$, then we can construct the compatible coarse structure $\mathcal{E}_{\mathcal{B}} := \{E \subseteq G \times G \mid \exists B \in \mathcal{B} : B \subseteq G(B \times B)\}$ and it coincides with $\mathcal{E}_{\mathcal{F}}$. Because of this observation, we can say that $\mathcal{B}$ is actually a base for a coarse structure.

Let G be a group and $\mathcal{F}$ a group ideal on it. Then we define a ballean $\mathfrak{B}_{\mathcal{F}} = (G, \mathcal{F}, B)$ where $B(g, A) := gA \cup \{g\} = g(A \cup \{e\})$ for every $g \in G$ and $A \in \mathcal{F}$. It is also possible to define $B^r(g, A) := Ag \cup \{g\}$, but it is not hard to see that these balleans are actually asymorphic.

Remark 6.4. Given a group ideal $\mathcal{F}$ on a group G, we have two, a priori different, coarse structures on G: the coarse structure generated by $\mathcal{F}$, $\mathcal{E}_{\mathcal{F}}$, and the one obtained by the ballean $\mathfrak{B}_{\mathcal{F}}$, $\mathcal{E}_{\mathfrak{B}_{\mathcal{F}}}$. We claim that they are actually the same coarse structure.

Let us first compute the sets $G(F \times F)[g]$, for every $g \in G$ and $F \in \mathcal{F}$. We prove that these are equal to $gF^{-1}F$, or gF^2 when $F = F^{-1}$ is symmetric. We have the following chain:

$$G(F \times F)[g] = \{h \in G \mid (g, h) \in G(F \times F)\} = \{h \in G \mid \exists k_1, k_2 \in F, \exists l \in G : g = lk_1, h = lk_2\};$$

therefore, deducing from $g = lk_1, h = lk_2$ the equality $g^{-1}h = k_1^{-1}k_2$, we conclude that

$$G(F \times F)[g] = \{h = gk_1^{-1}k_2 \mid k_1, k_2 \in F\} = gF^{-1}I.$$

We are now ready to prove that $\mathcal{E}_{\mathcal{F}} = \mathcal{E}_{\mathfrak{B}_{\mathcal{F}}}$.

($\subseteq$) Let $E \in \mathcal{E}_{\mathcal{F}}$. Then there is an element $F \in \mathcal{F}$, which we can assume without loss of generality containing the neutral element, such that $E \subseteq G(F \times F)$. We have

$$E \subseteq \bigcup_{g \in G} \{g\} \times G(F \times F)[g] = \bigcup_{g \in G} \{g\} \times (gF^{-1}F) = \bigcup_{g \in G} \{g\} \times B_{\mathcal{F}}(g, F^{-1}F) \in \mathcal{E}_{\mathfrak{B}_{\mathcal{F}}}$$

and so $E \in \mathcal{E}_{\mathfrak{B}_{\mathcal{F}}}$.

($\supseteq$) Conversely, let $E \in \mathcal{E}_{\mathfrak{B}_{\mathcal{F}}}$ and let $F \in \mathcal{F}$ be an element which contains e and such that $E \subseteq \bigcup_g \{g\} \times B_{\mathcal{F}}(g, F)$. Then

$$E \subseteq \bigcup_{g \in G} \{g\} \times B_{\mathcal{F}}(g, F) = \bigcup_{g \in G} \{g\} \times gF \subseteq \bigcup_{g \in G} \{g\} \times gF^{-1}F = G(F \times F) \in \mathcal{E}_{\mathcal{F}}$$

and so $E \in \mathcal{E}_{\mathcal{F}}$.

In [20], it is proved that every ballean X can be represented as a ballean of a G-space X, i.e., a set X with an action of the group G (that is a group of permutations of X) that provides a natural ballean structure on X with resect to some group ideal $\mathcal{F}$ of G.

Now our aim is to prove that quotient homomorphisms between groups provide an ample source of examples of soft quotient map. Moreover, by using this statement, we can give an application of Corollary 4.15: an alternative proof of a result due to Nicas and Rosenthal ([17, Proposition 2.15]).

Proposition 6.5. *Let G be a group, N be a normal subgroup and $q \colon G \to G/N$ the associated quotient map. If $\mathfrak{B}_G = (G, \mathcal{F}, B_G)$ is a ballean, then the quotient map is soft and so the principal quotient ball structure $\overline{\mathfrak{B}}^q$ on G/N is a ballean. Moreover, $\overline{\mathfrak{B}}^q = (G/N, q(\mathcal{F}), B_{G/N})$, where $q(\mathcal{F})$ is the ideal $q(\mathcal{F}) = \{q(A) \mid A \in \mathcal{F}\}$.*

Proof. In order to apply Corollary 4.15, fix an element $hN \in G/N$ and $A \in \mathcal{F}$. Then

$$\overline{B}^q_{G/N}(hN, A) = \bigcup_{g \in q^{-1}(hN)} q(B_G(g, A)) = \bigcup_{g \in hN} g(A \cup \{e\})N = (hN)((A \cup \{e\})N) = (h(A \cup \{e\}))N \qquad (22)$$

and so $q^{-1}(\overline{B}^q_{G/N}(gN, A)) = (h(A \cup \{e\}))N = hN(A \cup \{e\})N = \bigcup_{g \in hN} B_G(g, A)$.

Moreover, from (22), it follows that $\overline{B}_{G/N}(hN, A) = B_{G/N}(hN, q(A))$. □

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