



UNIVERSITÀ  
DEGLI STUDI  
DI UDINE

Università degli studi di Udine

## Reciprocal Function Series Coefficients with Integer Partitions

*Original*

*Availability:*

This version is available <http://hdl.handle.net/11390/1123515> since 2021-03-27T11:00:05Z

*Publisher:*

*Published*

DOI:10.1007/s00009-018-1076-1

*Terms of use:*

The institutional repository of the University of Udine (<http://air.uniud.it>) is provided by ARIC services. The aim is to enable open access to all the world.

*Publisher copyright*

(Article begins on next page)

# RECIPROCAL FUNCTION SERIES COEFFICIENTS WITH INTEGER PARTITIONS

GIUSEPPE FERA AND VITTORINO TALAMINI

ABSTRACT. We obtain an explicit formula in terms of the partitions of the positive integer  $n$  to express the  $n$ -th coefficient of the formal series expansion of the reciprocal of a given function. A brief survey shows that our arithmetic proof differs from others, some obtained already in the XIX century. Examples are given to establish explicit formulas for Bernoulli, Euler and Fibonacci numbers.

Pre-print version of the article published in *Mediterranean Journal of Mathematics*, **15** 29, 1–12 (2018). Url: <https://doi.org/10.1007/s00009-018-1076-1>, DOI: 10.1007/s00009-018-1076-1.

## 1. INTRODUCTION AND MAIN RESULT

Consider two sequences of numbers  $a = \{a_0, a_1, \dots\}$  and  $b = \{b_0, b_1, \dots\}$  such that for all integers  $n \geq 0$  the following conditions hold:

$$(1.1) \quad \sum_{h=0}^n a_{n-h} b_h = \delta_{0,n}, \quad a_0 = b_0 = 1,$$

where  $\delta_{0,n}$  is 1, if  $n = 0$ , and 0 otherwise.

The conditions are equivalent to a recursive definition of the elements of one of the two sequences. In fact, we have:

$$(1.2) \quad b_n = - \sum_{h=0}^{n-1} a_{n-h} b_h, \quad \forall n > 0, \quad b_0 = 1,$$

that allows one to determine the  $n$ -th element of the sequence  $b$  from all the preceding ones.

By using the above sequences one can derive the formal power series

$$(1.3) \quad a(x) = \sum_{i=0}^{\infty} a_i x^i, \quad b(x) = \sum_{i=0}^{\infty} b_i x^i,$$

and then construct their “Cauchy product”, without requiring that both series are convergent, as described in Ch. 1 of [10]. Consequently, (1.1) is equivalent to

$$a(x) b(x) = 1,$$

showing that  $a(x)$  and  $b(x)$  are reciprocal functions.

---

2010 *Mathematics Subject Classification.* Primary 05A17; Secondary 11Bxx.  
*Key words and phrases.* reciprocal series, recursive sequences, partitions.

With the definition of convolution, that is used widely in many fields of applied mathematics, e.g. in probability theory or signal analysis, (1.1) means that the convolution of the sequences  $a$  and  $b$  is equal to the sequence  $u$  with elements  $u_n = \delta_{0,n}$ :  $a \star b = u$ .

Many sequences of numbers are defined by (1.1) and (1.2), some of which appear in Table 1. In order to calculate the  $n$ -th element of one of those sequences one has to know all the preceding elements of the sequence, that is, all those with index smaller than  $n$ .

Formulas that allow one to calculate the  $n$ -th element of the sequence only in terms of  $n$ , and not in terms of other elements of the sequence, are therefore useful and desirable, often more for theoretical than for practical interest. We call them *explicit* formulas.

We present in Theorem 1.1 of this article an explicit formula that allows one to calculate the  $n$ -th element of a sequence of numbers satisfying (1.1) in terms of the partitions of the integer  $n$ . As we shall see, this is not a new result, but we will prove it via an elementary method not previously seen in the literature. Later, we shall describe alternative methods of obtaining these results.

We recall some basic definitions concerning partitions and compositions.

Given a positive integer  $n$ , a *partition* of  $n$  is a sequence of  $l(p)$  integers  $p = \{n_1, n_2, \dots, n_{l(p)}\}$ , such that  $n = \sum_{i=1}^{l(p)} n_i$  and  $n_1 \geq n_2 \geq \dots, n_{l(p)} \geq 1$ . The number  $l(p)$  is called the *length* of the partition  $p$ . E.g. see Section 1.7 of [14].

Given a positive integer  $n$ , a *composition* of  $n$  is a sequence of  $l(c)$  positive integers  $c = \{n_1, n_2, \dots, n_{l(c)}\}$ , not necessarily different, such that  $n = \sum_{i=1}^{l(c)} n_i$ . The number  $l(c)$  is called the *length* of the composition  $c$ .

If  $c$  is a composition obtained from a partition  $p$  of  $n$  by varying the order of its elements, then  $l(p) = l(c)$ .

Let  $\mu(p)$  be the number of different compositions formed from the elements of the partition  $p$ . It is trivial to verify that the number  $\mu(p)$  is given by the formula:

$$(1.4) \quad \mu(p) = l(p)! \left/ \prod_{n_i \in \cup(p)} m_p(n_i)! \right.,$$

where  $m_p(n_i)$  is the *multiplicity* of  $n_i$  in  $p$ , i.e. the number of times the number  $n_i$  appears in  $p$ , and the product in the denominator is evaluated over all values  $n_i$  appearing in the union  $\cup(p)$  of the elements of  $p$ . The multiplicities  $m_p(n_i)$  appearing in the denominator always yield a composition of the length  $l(p)$ , which appears in the numerator. The second member of (1.4) can also be written using the multinomial coefficient:

$$\mu(p) = \binom{l(p)}{m_p(n_1), m_p(n_2), \dots},$$

where the numbers  $l(p)$ ,  $m_p(n_1)$ ,  $m_p(n_2)$ ,  $\dots$  appear in (1.4).

Let  $C(n)$  and  $P(n)$  be the set of all compositions of  $n$  and the set of all partitions of  $n$ , respectively.

We now state our main result.

**Theorem 1.1.** *Let  $a = \{a_0, a_1, \dots\}$  and  $b = \{b_0, b_1, \dots\}$  be sequences such that (1.1) holds. Then one may calculate the elements  $b_n$ ,  $\forall n > 0$ , from the following explicit formulas involving the compositions or partitions of  $n$ :*

$$(1.5) \quad b_n = \sum_{c \in C(n)} \prod_{n_i \in c} (-a_{n_i})$$

$$(1.6) \quad b_n = \sum_{p \in P(n)} \mu(p) \prod_{n_i \in p} (-a_{n_i})$$

From (1.5) it can be seen that if the elements of  $a$  are integers then so are the elements of  $b$ .

Note that (1.5) and (1.6) differ only by the fact that the compositions have been replaced by the partitions. In Table 2 we list explicit formulas for the general terms of some well-known sequences of numbers in terms of partitions only, but there are also similar formulas using compositions. We shall begin by using compositions because they appear more naturally in the proof of the theorem.

The explicit formulas that use integer partitions to calculate the general  $n$ -th term of a sequence of numbers are often not convenient for practical computations, because the number of partitions  $P(n)$  experiences combinatorial explosion or grows exponentially with  $n$ . However, algorithms have been developed recently [10] to overcome this problem for values of  $n$  at least in the hundreds.

**Proof of Theorem 1.1.** Our aim is to solve the linear system expressed by (1.1) in the unknowns  $b_1, b_2, \dots$ . We do this by substituting equations for  $b_1, b_2, \dots$  obtained from (1.2) in terms of  $a_1, a_2, \dots$ . For the  $n$ -th value, we only require  $b_1$  up to  $b_n$ , which means the method is simple. For example, the first few equations become:

$$(1.7) \quad \begin{aligned} b_1 &= -a_1 \\ b_2 &= -a_2 - a_1 b_1 = -a_2 + a_1^2 \\ b_3 &= -a_3 - a_2 b_1 - a_1 b_2 = -a_3 + 2a_2 a_1 - a_1^3 \\ b_4 &= -a_4 - a_3 b_1 - a_2 b_2 - a_1 b_3 = -a_4 + 2a_3 a_1 + a_2^2 - 3a_2 a_1^2 + a_1^4 \\ b_5 &= -a_5 - a_4 b_1 - a_3 b_2 - a_2 b_3 - a_1 b_4 = \\ &= -a_5 + 2a_4 a_1 + 2a_3 a_2 - 3a_3 a_1^2 - 3a_2^2 a_1 + 4a_2 a_1^3 - a_1^5 \\ &\dots \end{aligned}$$

At this point one notes that for  $h = 1, 2, 3, 4, 5, \dots$ ,  $b_h$  is obtained as the sum of all products like

$$a_{n_1} a_{n_2} \dots a_{n_k},$$

whose indices satisfy the conditions

$$1 \leq k \leq h, \quad n_1 + n_2 + \dots + n_k = h.$$

In addition, each term is positive when  $k$  is even, and negative when  $k$  is odd. The  $k$  positive numbers  $n_1, \dots, n_k$  form then a composition of length  $k$  of the positive integer  $h$ . Therefore, we find that:

$$(1.8) \quad b_n = \sum_{c \in C(n)} \prod_{n_i \in c} (-a_{n_i}), \quad \forall n \geq 1,$$

where  $C(n)$  is the set of all compositions of the integer  $n$ . We now prove by induction that (1.8) is true for all values of  $n \geq 1$ . From (1.7) we see that (1.8) is true for  $n = 1$ . Suppose (1.8) is true up to an arbitrary  $n \geq 1$ . We show that it is true for  $n$  substituted by  $n + 1$ . Starting from (1.2), and using (1.8), one finds that:

$$\begin{aligned} b_{n+1} &= -a_{n+1} - \sum_{h=1}^n a_{n+1-h} b_h = \\ &= -a_{n+1} - \sum_{h=1}^n \left( \sum_{c \in C(h)} \prod_{n_i \in c} (-a_{n_i}) \right) a_{n+1-h}. \end{aligned}$$

For any composition  $c = \{n_1, n_2, \dots, n_{l(c)}\} \in C(h)$ , consider the set obtained by appending to the  $l(c)$  numbers  $n_i$  in  $c$  the number  $n_0 = n + 1 - h$ , that appears in the index of the last factor  $a_{n+1-h}$  (so that it can be rewritten as  $a_{n_0}$ ). The set  $c' = \{n_0, n_1, n_2, \dots, n_{l(c)}\}$  is now a composition of  $(n + 1)$  of length  $l(c') = l(c) + 1$ . By considering all possible values of  $h$  from the sum of the last expression, we obtain all possible compositions of  $(n + 1)$  of length greater than 1. Moreover, since the index of the term outside the summation symbol corresponds to the unique composition  $\{n + 1\}$  of length 1 of  $(n + 1)$ , we can rewrite the last expression in the following form:

$$b_{n+1} = \sum_{c' \in C(n+1)} \prod_{n_i \in c'} (-a_{n_i}).$$

This is (1.8) expressed in terms of  $(n + 1)$  instead of  $n$ . Hence, (1.8) is true for all values of  $n \geq 1$ , which completes the proof.

Usually it is more convenient to use partitions instead of compositions. Thus, (1.8) can be expressed as

$$b_n = \sum_{p \in P(n)} \mu(p) \prod_{n_i \in p} (-a_{n_i}), \quad \forall n \geq 1,$$

which is (1.6), where  $\mu(p)$  is given by (1.4). □

## 2. A BRIEF SURVEY OF DIFFERENT APPROACHES

As mentioned already, one may find results similar to those expressed by Theorem 1.1 also in other papers. We present here a short historical review and with it we will stress how a simple proof of Theorem 1.1 is not available. In fact, the result expressed by Theorem 1.1 is usually seen in the context of the theory of formal power series or in relation with the Taylor series expansion of functions, but its proof needs a more advanced treatment.

On pp. 59–66 of [13], Scherk showed how to solve a triangular linear system, such as (1.2) or (1.7), with iterated substitutions. He correctly obtained the  $n$ -th unknown  $b_n$  by summing up  $2^{n-1}$  terms involving products of the sequence elements  $a_i$ ,  $\forall i = 1, \dots, n$ . Although  $2^{n-1}$  is the number of compositions of  $n$ , Scherk failed to observe that it represented the sum over compositions.

The triangular system studied by Scherk was expressed in matrix form by Brioschi [1], and solved by using Cramer's rule. The following expression for the  $n$ -th term

of the sequence was obtained:

$$b_n = (-1)^n \det \begin{pmatrix} a_1 & 1 & 0 & \dots & 0 \\ a_2 & a_1 & 1 & \dots & 0 \\ a_3 & a_2 & a_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & a_{n-2} & \dots & a_1 \end{pmatrix}.$$

By expanding the determinant, Brioschi obtained the following explicit formula:

$$(2.1) \quad b_n = \sum_{q_1+2q_2+3q_3+\dots+nq_n=n} (-1)^q \frac{q!}{q_1!q_2!\dots q_n!} a_1^{q_1} a_2^{q_2} \dots a_n^{q_n},$$

where  $q = q_1 + q_2 + \dots + q_n$ , and the sum is over all sets of non-negative integers  $\{q_1, q_2, \dots, q_n\}$  that are solutions of the Diophantine equation

$$(2.2) \quad q_1 + 2q_2 + 3q_3 + \dots + nq_n = n.$$

In his paper Brioschi also stated that (2.1) had already been obtained earlier by Fergola [4], by studying a different problem. In fact, Fergola found (2.1) by using a formula he derived in [3] to expand the  $n$ -th derivative of the inverse function  $f^{-1}(x)$  in terms of the derivatives of the function  $f(x)$  up to the  $n$ -th order, where  $f(x)$  is any differentiable function. At the end of his paper [4], using a result of Euler, Fergola recognizes that the number of solutions  $\{q_1, q_2, \dots, q_n\}$ ,  $q_i \geq 0$ , to the Diophantine equation (2.2), is equal to the number of partitions of  $n$ .

Later Sylvester [15] stated that the solutions of (2.2) are in a one-to-one correspondence with the partitions of  $n$ . This is also discussed in modern textbooks, such as on p. 95 of [2]. We prove it here, since it is needed to show that (2.1) is equivalent to (1.6).

**Proposition 2.1.** *The solutions  $\{q_1, q_2, \dots, q_n\}$  of the Diophantine equation (2.2), with  $q_i \geq 0$ , are in a one-to-one correspondence with the partitions  $p = \{n_1, n_2, \dots, n_k\}$  of  $n$ . The correspondence is obtained by taking  $q_i = m_p(n_i)$ , if  $n_i \in p$ , and 0 otherwise. It then follows that  $l(p) = \sum_{i=1}^n q_i$ .*

**Proof.** If  $\{q_1, q_2, \dots, q_n\}$  is a set of non negative integers that is a solution of (2.2), then  $p = \{1q_1, 2q_2, \dots, nq_n\} \setminus \{0\}$  is a partition of  $n$ , and vice-versa: given a partition  $p = \{n_1, n_2, \dots\}$  of  $n$ , with  $n_i \geq 1$  and multiplicity  $m_p(n_i)$ , the set  $\{q_1, q_2, \dots, q_n\}$ , where  $q_i = m_p(n_i)$ , if  $n_i \in p$ , and 0 otherwise, is a set of non-negative integers that is a solution of (2.2). According to the discussion after (1.4), one then has  $l(p) = \sum_{i=1}^n q_i$ .  $\square$

Proposition 2.1 is also the starting point for the explicit parametrization of the integer partitions given in [7]. Moreover, Proposition 2.1 is used in information theory for obtaining results about non-stationary renewal processes in terms of partitions [5].

We now use Proposition 2.1 to prove the following:

**Proposition 2.2.** *The results given by (2.1) and (1.6) are equivalent.*

**Proof.** Note that the sum in (2.1) is over all partitions  $p$  of  $n$ , where  $p = \{1q_1, 2q_2, \dots, nq_n\} \setminus \{0\}$ . Then the factor  $\frac{q!}{q_1!q_2!\dots q_n!}$  in (2.1) is equal to the factor  $\mu(p)$  as defined in (1.4), since  $q = q_1 + q_2 + \dots + q_n = l(p)$ . Moreover, the product

$(-1)^q a_1^{q_1} a_2^{q_2} \cdots a_n^{q_n}$  in (2.1) can be rewritten as  $(-a_1)^{q_1} (-a_2)^{q_2} \cdots (-a_n)^{q_n}$ , which is equal to the product  $\prod_{n_i \in p} (-a_{n_i})$  in (1.6).  $\square$

Usually, one treats recursive equations for sequences, such as (1.2), by expressing them as equations for the generating functions of formal power series. Links between the theory of formal power series and partitions are encountered in many modern textbooks. For example, see [2], [6], [10], [14] or [17].

We now present a proof of Theorem 1.1 in terms of formal power series.

**Proof of Theorem 1.1 (formal power series version).** Consider the formal power series (1.3), where  $b(x) = 1/a(x)$  and  $a_0 = 1$ . Since  $a(x) = 1 + \sum_{i=1}^{\infty} a_i x^i$ ,

$$b(x) = \frac{1}{a(x)} = \left(1 + \sum_{i=1}^{\infty} a_i x^i\right)^{-1} = 1 + \sum_{k=1}^{\infty} \left(-\sum_{i=1}^{\infty} a_i x^i\right)^k,$$

where the geometric series has been introduced into the last member. We shall not consider the convergence of the series here, since this is discussed at length in [10].

We expand the summand  $\left(-\sum_{i=1}^{\infty} a_i x^i\right)^k$  using the multinomial theorem

$$(x_1 + x_2 + \dots)^k = \sum_{r_1 + r_2 + \dots = k} \binom{k}{r_1, r_2, \dots} x_1^{r_1} x_2^{r_2} \cdots,$$

where  $r_1, r_2, \dots$  are non-negative integers. Then factoring out the same powers of  $x$ , yields:

$$\begin{aligned} b(x) &= 1 + \sum_{k=1}^{\infty} \sum_{r_1 + r_2 + \dots = k} \binom{k}{r_1, r_2, \dots} (-a_1 x)^{r_1} (-a_2 x^2)^{r_2} \cdots = \\ &= 1 + \sum_{k=1}^{\infty} \sum_{r_1 + r_2 + \dots = k} x^{r_1 + 2r_2 + \dots} \binom{k}{r_1, r_2, \dots} (-a_1)^{r_1} (-a_2)^{r_2} \cdots = \\ &= 1 + \sum_{n=1}^{\infty} x^n \sum_{r_1 + 2r_2 + \dots + nr_n = n} \binom{r_1 + r_2 + \dots + r_n}{r_1, r_2, \dots, r_n} (-a_1)^{r_1} (-a_2)^{r_2} \cdots (-a_n)^{r_n}. \end{aligned}$$

Using Proposition 2.1 and (1.4), we obtain:

$$b(x) = 1 + \sum_{n=1}^{\infty} x^n \sum_{p \in P(n)} \mu(p) \prod_{n_i \in p} (-a_{n_i}).$$

Finally, with the aid of (1.3), we obtain, for all  $n > 0$ :

$$b_n = \sum_{p \in P(n)} \mu(p) \prod_{n_i \in p} (-a_{n_i}),$$

which is identical to (1.6). This completes the proof.  $\square$

Earlier in this section we presented various methods for obtaining explicit formulas via the integer partitions for the elements of the sequences satisfying (1.1). None of these methods is elementary, because they involve either calculus or formal power series. Consequently, they can be difficult to understand without an advanced mathematical background. On the other hand, our proof of Theorem 1.1 is based only on elementary mathematics, so our method can be considered simple

to understand.

### 3. EXAMPLES

Table 1 lists some sequences of numbers satisfying (1.1). In [1] and [4] the aim was to derive explicit formulas for the Bernoulli and Euler numbers, and for this reason we report the first four entries in Table 1. The proofs of (1.1) for Entries 1, 3 and 4 are classical and can be found in the literature. For an arithmetic approach, one can follow the arguments reported in Sections 134 and 145 of [11]. Therefore, we shall only prove (1.1) for Entry 2 and for Entries 5–7, since they are not well-known. Before the proofs we make some comments.

Entry	Name	$a_n$	$b_n$
1	Bernoulli numbers	$\frac{1}{(n+1)!}$	$\frac{1}{n!} B_n$
2	Bernoulli numbers of even index	$\frac{2}{(2n+2)!}$	$-\frac{2n-1}{(2n)!} B_{2n}$
3	Euler numbers	$\frac{1+(-1)^n}{2 n!}$	$\frac{1}{n!} E_n$
4	Euler numbers of even index	$\frac{1}{(2n)!}$	$\frac{1}{(2n)!} E_{2n}$
5	Fibonacci numbers	$\frac{(-1)^n - 1}{2}$	$F_n$
6	Fibonacci numbers of even index	$-n$	$F_{2n}$
7	Fibonacci numbers of odd index	$-\max(1, 2^{n-2})$	$F_{2n-1}$

TABLE 1. Some sequences satisfying (1.1) and (1.2). In the table, the index  $n$  starts from 1, and  $a_0 = b_0 = 1$ .

We have already observed that, given the formal power series (1.3), the equation  $a(x)b(x) = 1$  is equivalent to (1.1). This applies in particular to the functions  $f(x)$  and  $1/f(x)$  for which a Taylor series expansion around  $x = x_0$  exists. Therefore, Theorem 1.1 yields the following expressions for the sequence elements in terms of integer partitions:

$$\begin{aligned}
 a_n &= \frac{1}{n!} \left. \frac{d^n}{dx^n} f(x) \right|_{x=x_0}, \\
 (3.1) \quad b_n &= \frac{1}{n!} \left. \frac{d^n}{dx^n} \frac{1}{f(x)} \right|_{x=x_0} = \sum_{p \in P(n)} \mu(p) \prod_{n_i \in p} \left( -\frac{1}{n_i!} \left. \frac{d^{n_i}}{dx^{n_i}} f(x) \right|_{x=x_0} \right)
 \end{aligned}$$

Many sequences of numbers can be defined as the coefficients of the Taylor series expansions of specific generating functions  $g(x)$ . In these cases, the sequence elements can be written in terms of integer partitions by using (3.1) for the reciprocal



$f(x) = 1/g(x)$  of the generating function  $g(x)$ .

By applying Theorem 1.1 to the sequences listed in Table 1, one obtains the expansions in terms of integer partitions displayed in Table 2. These expansions can also be derived by using (3.1) with the correct generating function. However, our approach allows us to obtain the same results purely arithmetically.

Similar results to the first four entries of Table 2 have been obtained by Vella, Malenfant and Kowalenko. In Theorem 11 of [16], Vella obtained explicit formulas for  $B_n$  and  $E_n$  using the formula of Faà di Bruno [8] for the  $n$ -th derivative of a composite generating function. Malenfant derived explicit formulas for  $B_n$ ,  $B_{2n}$  and  $E_{2n}$ , which appear respectively as Eqs. (36), (38) and (39) in [12], from the coefficients of the Taylor series of the generating functions, while Kowalenko derived them as (1.80), (1.82) and (1.27) in [10] by using his partition method for power series expansion [9, 10]. Although our explicit formulas for  $B_n$ ,  $B_{2n}$  and  $E_{2n}$  resemble those of Malenfant, our method to obtain them does not require calculus. On the other hand, our formulas for the Fibonacci numbers in terms of integer partitions (Entries 5–7 of Table 2) are, to the best of our knowledge, new.

**Proof of (1.1) for Entry 2.** To prove this result, we require (1.1) for the first entry of Table 1, that is,

$$(3.2) \quad \sum_{h=0}^n \frac{1}{(n+1-h)!} \frac{B_h}{h!} = 0.$$

It is well-known that  $B_{2n+1} = 0$ ,  $\forall n > 0$ . In the spirit of keeping everything elementary, we simply point out that an arithmetic proof of this property can be found on p. 238 of [11].

Since  $B_{2n+1} = 0$ ,  $\forall n > 0$ , we can separate (3.2) into even and odd indices. For convenience, we distinguish the cases  $n = 2m$  and  $n = 2m + 1$ . For  $n = 2m$ , we find that:

$$0 = \sum_{h=0}^{2m} \frac{1}{(2m+1-h)!} \frac{B_h}{h!} = \sum_{k=0}^m \frac{1}{(2m+1-2k)!} \frac{B_{2k}}{(2k)!} + \frac{B_1}{(2m)!},$$

or:

$$(3.3) \quad -\frac{B_1}{(2m)!} = \sum_{k=0}^m \frac{2m+2-2k}{(2m+2-2k)!} \frac{B_{2k}}{(2k)!}.$$

When  $n = 2m + 1$ , (3.2) gives:

$$0 = \sum_{h=0}^{2m+1} \frac{1}{(2m+2-h)!} \frac{B_h}{h!} = \sum_{k=0}^m \frac{1}{(2m+2-2k)!} \frac{B_{2k}}{(2k)!} + \frac{B_1}{(2m+1)!}.$$

Hence, we obtain:

$$(3.4) \quad -\frac{B_1}{(2m)!} = \sum_{k=0}^m \frac{2m+1}{(2m+2-2k)!} \frac{B_{2k}}{(2k)!}$$

The difference between (3.4) and (3.3) is:

$$\begin{aligned} 0 &= \sum_{k=0}^m \left( \frac{2m+1}{(2m+2-2k)!} - \frac{2m+2-2k}{(2m+2-2k)!} \right) \frac{B_{2k}}{(2k)!} = \\ &= \sum_{k=0}^m \frac{2k-1}{(2m+2-2k)!} \frac{B_{2k}}{(2k)!}. \end{aligned}$$

Thus this is equivalent to Entry 2 of Table 1. Note that a factor of 2 has been inserted so that  $a_0 = 1$ .  $\square$

**Proof of (1.1) for Entries 5–7.** First recall the standard definition of the Fibonacci numbers:  $F_1 = F_2 = 1$ ,  $F_n = F_{n-1} + F_{n-2}$ ,  $\forall n > 2$ . We now use induction to prove the following formulas are true:

$$(3.5) \quad F_{2n} = \sum_{h=1}^n F_{2h-1},$$

$$(3.6) \quad F_{2n+1} = 1 + \sum_{h=1}^n F_{2h},$$

$$(3.7) \quad F_{2n} = n + \sum_{h=1}^{n-1} F_{2h}(n-h).$$

$$(3.8) \quad F_{2n} = 2^{n-1} + \sum_{h=1}^{n-1} 2^{n-h-1} F_{2h-1}.$$

All these equations are true for  $n = 1$ . Suppose they are true for  $n (\geq 2)$ . In the case of (3.5), one has:

$$F_{2(n+1)} = F_{2n+2} = F_{2n+1} + F_{2n} = F_{2n+1} + \sum_{h=1}^n F_{2h-1} = \sum_{h=1}^{n+1} F_{2h-1}.$$

Thus (3.5) is true for all  $n$ . In the case of (3.6), one has

$$F_{2(n+1)+1} = F_{2n+3} = F_{2n+1} + F_{2n+2} = 1 + \sum_{h=1}^n F_{2h} + F_{2n+2} = 1 + \sum_{h=1}^{n+1} F_{2h}.$$

Therefore (3.6) is true for all  $n$ . Using this result one has

$$\begin{aligned} F_{2(n+1)} &= F_{2n+2} = F_{2n+1} + F_{2n} = 1 + \sum_{h=1}^n F_{2h} + n + \sum_{h=1}^{n-1} F_{2h}(n-h) = \\ &= n + 1 + \sum_{h=1}^n F_{2h} + \sum_{h=1}^n F_{2h}(n-h) = n + 1 + \sum_{h=1}^n F_{2h}(n+1-h), \end{aligned}$$

which proves that (3.7) is true for all  $n$ . In the case of (3.8), one has

$$\begin{aligned} F_{2(n+1)} &= F_{2n+2} = F_{2n+1} + F_{2n} = 2F_{2n} + F_{2n-1} = \\ &= 2 \left( 2^{n-1} + \sum_{h=1}^{n-1} 2^{n-h-1} F_{2h-1} \right) + F_{2n-1} = 2^n + \sum_{h=1}^{n-1} 2^{n-h} F_{2h-1} + F_{2n-1} = \end{aligned}$$

$$= 2^n + \sum_{h=1}^n 2^{n-h} F_{2h-1} = 2^{(n+1)-1} + \sum_{h=1}^{(n+1)-1} 2^{(n+1)-h-1} F_{2h-1},$$

which proves that (3.8) is true for all  $n$ .

We now form the sequences  $b$ , with elements  $b_n = F_{2n}$ ,  $\forall n > 0$ , and  $b_0 = 1$ , and  $a$ , with  $a_n = -n$ ,  $\forall n > 0$ , and  $a_0 = 1$ . Then (3.7) is in the form of (1.2), which is equivalent to (1.1). Hence we obtain Entry 6 of Table 1.

Both (3.5) and (3.6) can be summarized by the following result:

$$(3.9) \quad F_n = \frac{1 - (-1)^n}{2} + \sum_{h=1}^{n-1} F_h \frac{1 - (-1)^{n-h}}{2}.$$

We now form the sequences  $b$ , with elements  $b_n = F_n$ ,  $\forall n > 0$ , and  $b_0 = 1$ , and  $a$ , with  $a_n = \frac{(-1)^n - 1}{2}$ ,  $\forall n > 0$ , and  $a_0 = 1$ . Then (3.9) is in the form of (1.2), which is equivalent to (1.1). Hence we obtain Entry 5 of Table 1.

Using (3.8), one has, for  $n > 1$ :

$$(3.10) \quad F_{2n-1} = F_{2n-2} + F_{2n-3} = 2^{n-2} + \sum_{h=1}^{n-2} 2^{n-h-2} F_{2h-1} + F_{2n-3}.$$

If one defines the sequences  $a$  and  $b$ , such that  $a_n = -2^{n-2}$ ,  $\forall n > 1$ , and  $a_0 = -a_1 = 1$ , and  $b_n = F_{2n-1}$ ,  $\forall n > 0$ , and  $b_0 = 1$ , (3.10) is in the form of (1.2), which is again equivalent to (1.1), and hence is listed as Entry 7 of Table 1. A more compact notation has been introduced in both tables, where  $-a_n$ ,  $\forall n > 0$ , has been replaced by  $\max(1, 2^{n-2})$ .  $\square$

Finally, we present three explicit calculations based on the partitions that sum to 4, or  $P(4)$ . There are five partitions, which are:

$$P(4) = \{\{4\}, \{3, 1\}, \{2, 2\}, \{2, 1, 1\}, \{1, 1, 1, 1\}\}.$$

Then, according to the first, sixth and seventh rows of Table 2 one obtains:

$$B_4 = 4! \left[ \frac{1!}{1!} \cdot \frac{-1}{5!} + \frac{2!}{1!1!} \cdot \frac{(-1)^2}{4!2!} + \frac{2!}{2!} \cdot \frac{(-1)^2}{3!3!} + \frac{3!}{1!2!} \cdot \frac{(-1)^3}{3!2!2!} + \frac{4!}{4!} \cdot \frac{(-1)^4}{2!2!2!2!} \right] = -\frac{1}{30},$$

$$F_8 = \left[ \frac{1!}{1!} \cdot 4 + \frac{2!}{1!1!} \cdot 3 \cdot 1 + \frac{2!}{2!} \cdot 2^2 + \frac{3!}{1!2!} \cdot 2 \cdot 1^2 + \frac{4!}{4!} \cdot 1^4 \right] = 21,$$

$$F_7 = \left[ \frac{1!}{1!} \cdot 2^{4-2} + \frac{2!}{1!1!} \cdot 2^{3-2} \cdot 1 + \frac{2!}{2!} \cdot 2^{2(2-2)} + \frac{3!}{1!2!} \cdot 2^{2-2} \cdot 1^2 + \frac{4!}{4!} \cdot 1^4 \right] = 13.$$

## REFERENCES

- [1] F. Brioschi, *Sulle funzioni Bernoulliane ed Euleriane*, Annali di Matematica Pura ed Applicata (1858) 260–263 (in Italian).
- [2] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, 1974.
- [3] E. Fergola, *Dell'espressione di una derivata qualunque di una funzione in termini delle derivate della funzione inversa*, Memorie della Reale Accademia delle Scienze di Napoli, Vol. 1 (1856) 200–206 (in Italian).
- [4] E. Fergola, *Sopra lo sviluppo della funzione  $1/(e^x - 1)$ , e sopra una nuova espressione dei numeri di Bernoulli*, Memorie della Reale Accademia delle Scienze di Napoli, Vol. 2 (1857) 315–324 (in Italian).

Entry	$b_n$
1	$\frac{1}{n!} B_n = \sum_{p \in P(n)} \mu(p) \prod_{n_i \in p} \left( -\frac{1}{(n_i+1)!} \right)$
2	$-\frac{2n-1}{(2n)!} B_{2n} = \sum_{p \in P(n)} \mu(p) \prod_{n_i \in p} \left( -\frac{2}{(2n_i+2)!} \right)$
3	$\frac{1}{n!} E_n = \sum_{p \in P(n)} \mu(p) \prod_{n_i \in p} \left( -\frac{1+(-1)^{n_i}}{2 n_i!} \right)$
4	$\frac{1}{(2n)!} E_{2n} = \sum_{p \in P(n)} \mu(p) \prod_{n_i \in p} \left( -\frac{1}{(2n_i)!} \right)$
5	$F_n = \sum_{p \in P(n)} \mu(p) \prod_{n_i \in p} \left( -\frac{(-1)^{n_i}-1}{2} \right)$
6	$F_{2n} = \sum_{p \in P(n)} \mu(p) \prod_{n_i \in p} n_i$
7	$F_{2n-1} = \sum_{p \in P(n)} \mu(p) \prod_{n_i \in p} \max(1, 2^{n_i-2})$

TABLE 2. Explicit formulas using partitions for the sequences in Table 1.

- [5] P. Flajolet and W. Szpankowski, *Analytic Variations on Redundancy Rates of Renewal Processes*, IEEE Trans. Inf. Theory **48** (2002) 2911–2921.
- [6] P. Flajolet and R. Sedgewick, *Analytic combinatorics*, Cambridge University Press, Cambridge, 2009.
- [7] L. Hernández Encinas, A. Martín del Rey, J. Muñoz Masqué, *Faà di Bruno’s formula, lattices, and partitions*, Discrete Applied Mathematics **148** (2005) 246 – 255.
- [8] W. P. Johnson, *The curious history of Faà di Bruno’s formula*, Amer. Math. Monthly **109** (2002) 217–234.
- [9] V. Kowalenko, *Applications of the Cosecant and Related Numbers*, Acta Appl. Math. **114** (2011) 15–134.
- [10] V. Kowalenko, *The partition method for a power series expansion. Theory and applications*. Academic Press/Elsevier, London, 2017.
- [11] É. Lucas, *Théorie des nombres*, Gauthier-Villars, Paris, 1891.
- [12] J. Malenfant, *Finite closed-form expressions for the partition function and for Euler, Bernoulli, and Stirling numbers*, arXiv:1103.1585v6 [math.NT] (2011) 1–19.
- [13] H.F. Scherk, *Mathematische Abhandlungen*, G. Reimer, Berlin (1825) (in German). (Available at <https://hdl.handle.net/2027/hvd.hnxxhh>).
- [14] R.P. Stanley, *Enumerative Combinatorics*. Volume 1. Second edition. Cambridge Studies in Advanced Mathematics **49**. Cambridge University Press, Cambridge, 2012.
- [15] J.J. Sylvester, *A note on the theory of a point in partitions*, Edinburgh British Association Report (1871) 23–25.
- [16] D. C. Vella, *Explicit formulas for Bernoulli and Euler numbers*, Integers **8** (2008) 1–7.
- [17] H. S. Wilf, *Generatingfunctionology*. Third Edition. A. K. Peters, Wellesley, 2006.

DMIF, DIPARTIMENTO DI SCIENZE MATEMATICHE, INFORMATICHE E FISICHE, UNIVERSITÀ DI UDINE, VIA DELLE SCIENZE 206, 33100 UDINE, ITALY  
 Email address: [giuseppe.fera@uniud.it](mailto:giuseppe.fera@uniud.it)

DMIF, DIPARTIMENTO DI SCIENZE MATEMATICHE, INFORMATICHE E FISICHE, UNIVERSITÀ DI UDINE, VIA DELLE SCIENZE 206, 33100 UDINE, ITALY

INFN, SEZIONE DI TRIESTE, VIA VALERIO 2, 34127 TRIESTE, ITALY  
*Email address:* `vittorino.talamini@uniud.it`