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ABSTRACT

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the written additively circle group and $\mathbf{u} = (u_n)$ be a sequence of integers. Many authors in various areas of Mathematics gave their attention to the following subgroups of \mathbb{T} and their subsets

$$t_{\mathbf{u}}(\mathbb{T}) = \{x \in \mathbb{T} \mid u_n x \rightarrow 0\}.$$

These subgroups are known with various names, here I refer to these subgroups as *topologically \mathbf{u} -torsion* subgroups, because of their strong connection with torsion subgroups. Here, besides these subgroups in the circle group, I consider their natural generalization for an arbitrary topological abelian group, with particular attention to the compact case: for a topological abelian group X and a sequence of characters $\mathbf{v} = (v_n)$ the following subgroup

$$s_{\mathbf{v}}(X) = \{x \in X \mid v_n(x) \rightarrow 0\}$$

is called *characterized* subgroup.

Here I present some of my research results. In particular, I give a complete description of the subgroups $t_{\mathbf{u}}(\mathbb{T})$ where \mathbf{u} is an *arithmetic sequence*, that is a strictly increasing sequence where $u_n \mid u_{n+1}$ for every $n \in \mathbb{N}$. I give also some new results on the study of the Borel complexity of these subgroups, both in the compact case and in the circle group. Moreover, I present a structure theorem for the subgroups that admit a finer locally compact Polish group topology. The latter is a sufficient condition for a subgroup to be characterized. Furthermore, I give a complete description of closed characterized subgroups in arbitrary topological abelian groups and various useful reductions to the metrizable case. Presenting these results, I take the opportunity to give an exhaustive description of the state of the art in this topic and to show some applications to other areas of

Mathematics, with the aim of providing a useful handbook to an expert audience and a starting point for potential research purposes to non-expert users.

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INTRODUCTION

MOTIVATIONS AND HISTORY

Motivations

Let (u_n) be a sequence of integers and let $x \in [0, 1]$ be a real number. The behaviour of the sequence of multiples $(u_n x)$, considered modulo 1 (see [67]), has deep roots in many branches of Mathematics as Number Theory, Analysis, Dynamical Systems and Topology.

Recall that a sequence of real numbers (x_n) is said to be *uniformly distributed mod 1* if for every $[a, b] \subseteq [0, 1)$ one has

$$\lim_{n \rightarrow \infty} \frac{|\{j \mid 0 \leq j < n, \{x_j\} \in [a, b]\}|}{n} = b - a,$$

where $\{x_j\}$ is the fractional part of x_j . For every $\mathbf{u} = (u_n) \in \mathbb{Z}^{\mathbb{N}}$ the set

$$\mathcal{W}_{\mathbf{u}} = \{x \in [0, 1] \mid (u_n x) \text{ is uniformly distributed mod } 1\}$$

is obviously contained in $[0, 1] \setminus \mathbb{Q}$. According to a celebrated theorem, proved by H. Weyl in 1916, $\mathcal{W}_{\mathbf{u}} = [0, 1] \setminus \mathbb{Q}$ if $\mathbf{u} = (p(n))$ and $p(x) \in \mathbb{Z}[x]$ is a polynomial, whereas $\mu(\mathcal{W}_{\mathbf{u}}) = 1$ for every one-to-one sequence $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$.

On the other hand, if $\alpha \in [0, 1] \setminus \mathbb{Q}$ then $\alpha \notin \mathcal{W}_{\mathbf{u}}$ for appropriate \mathbf{u} . Indeed, it is well known in Number Theory that if $\frac{r_n}{u_n}$ are the convergents of the continued fraction expansion of α , then $\|u_n \alpha\|_{\mathbb{Z}} \rightarrow 0$,¹ so $\alpha \notin \mathcal{W}_{\mathbf{u}}$. Moreover, Larcher proved in 1988 that if the continued fraction expansion of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is bounded, then

$$\{\beta \in \mathbb{R} \mid \|u_n \beta\|_{\mathbb{Z}} \rightarrow 0\}, \quad (\dagger)$$

¹ where $\|\cdot\|_{\mathbb{Z}}$ is the distance from the integers.

coincides with the subgroup $\langle \alpha \rangle + \mathbb{Z}$ of \mathbb{R} , generated by α and \mathbb{Z} , (for more details see Subsection 4.4). This aspect of the distribution of the multiples $(u_n x)$ (given by (†)), notoriously “complementary” to the uniform distribution, is the backbone of the present Thesis.

Instead of using the fractional part $\{x_j\}$ and working modulo 1, as done above, one can conveniently work in the circle group $\mathbb{R}/\mathbb{Z} = \mathbb{T}$ making use of the canonical projection $\pi : \mathbb{R} \rightarrow \mathbb{T}$ (in these terms, one has $u_n \pi(\beta) \rightarrow 0$ in (†)).

In Harmonic Analysis the behaviour of the sequence of multiples $(u_n x)$, (more precisely, the various ways the sequence may converge to 0) is related to various sets of convergence of trigonometric series (see Chapter 6 for more details).

In Topology, the behaviour of the sequence $(u_n x)$, for $x \in \mathbb{T}$, is closely related to two relevant issues. One is related to Hausdorff group topologies with or without non-trivial convergent sequences (for more detail see Chapter 7, where the connection to the relevant notion of T-sequences is given). As a precursor to the other aspect, the following fact proved by Markov in [74] can be pointed out: for every sequence \mathbf{u} in \mathbb{Z} there exists a Hausdorff group topology τ on \mathbb{Z} such that the set $U = \{u_n : n \in \mathbb{N}\}$ is dense in (\mathbb{Z}, τ) (i.e., U is *potentially dense* [74]). Apparently, Markov was unaware of Weyl’s theorem that easily provides \mathfrak{c} -many such topologies by dense embeddings in \mathbb{T} . Recent developments in this line (leading to a complete solution of Markov’s problem on the description of the potentially dense sets) can be found in [43].

In Topological Algebra the behaviour of the sequence of multiples $(u_n x)$ in a topological abelian group is related to topologically torsion subgroups, as described in the next subsection.

Towards the Definition of Characterized Subgroup

Another precursor of the notion of characterized subgroup is the generalization of the notion of torsion subgroup. Recall that an element x of an Abelian group is *torsion* if there exists $k \in \mathbb{N}$

such that $kx = 0$. Braconnier [21] and Vilenkin [84] introduced independently the notions of *topologically torsion* and *topologically p-torsion groups*, fundamental tools for the study of locally compact abelian groups.

Definition. A topological abelian group X is said to be

- *topologically torsion* if $n!x \rightarrow 0_X$, for every $x \in X$;
- *topologically p-torsion* if $p^n x \rightarrow 0_X$, for every $x \in X$.

Braconnier and Vilenkin refer to these groups by different names. The names used here were introduced for the first time by Robertson in [83].

Later, Armacost in [3] defined the following subgroups of a topological abelian group X ,

$$X_p = \{x \in X \mid p^n x \rightarrow 0\} \text{ and } X! = \{x \in X \mid n!x \rightarrow 0\}.$$

Armacost described \mathbb{T}_p and posed the problem to describe $\mathbb{T}!$. These notions were unified in the monograph [41], where the authors introduced, for a sequence $\mathbf{q} = (q_n)$ of natural numbers with $q_n \geq 2$ for every $n \geq 1$, the subgroup

$$X_{\mathbf{q}} = \{x \in X \mid u_n x \rightarrow 0\},$$

where $u_n = q_0 q_1 \cdots q_n$. Clearly, $X_p = X_{\mathbf{q}}$ for $\mathbf{q} = (p)_n$ and $X! = X_{\mathbf{q}}$ for $\mathbf{q} = (n+1)_n$. The first results on the groups $X_{\mathbf{q}}$ for $X = \mathbb{T}$ can be found already in [41, §4.4.2], providing, among others, a partial answer to Armacost problem. A complete solution of Armacost problem was given by Borel [20], and independently also in [31]. The latter paper contains also an attempt to describe the subgroup $\mathbb{T}_{\mathbf{q}}$ of \mathbb{T} for an *arbitrary* sequence $\mathbf{q} = (q_n)$. The gap left in [31] was filled in [36], where a complete description of $\mathbb{T}_{\mathbf{q}}$ can be found.

In [29] Dikranjan defined the *topologically u-torsion subgroup* $t_{\mathbf{u}}(X)$ of a topological abelian group X for an *arbitrary* sequence of integers $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$ in the following manner.

Definition ([29]). Let X be a topological abelian group and $\mathbf{u} = (u_n) \in \mathbb{Z}^{\mathbb{N}}$. The subgroup of X defined by

$$t_{\mathbf{u}}(X) = \{x \in X \mid u_n x \rightarrow 0\}.$$

is called *topologically \mathbf{u} -torsion subgroup* of X .

Let

$$t(H) = \bigcap \{t_{\mathbf{u}}(X) \mid \mathbf{u} \in \mathbb{Z}^{\mathbb{N}} \text{ and } H \leq t_{\mathbf{u}}(X)\}.$$

and call a subgroup H of a topological abelian group X *t -closed* (resp. *t -dense*) if $t(H) = H$ (resp., $t(H) = X$). Every topologically \mathbf{u} -torsion subgroup is obviously a t -closed subgroup. Some groups have only trivial t -closed subgroups ([29, Example 4.11]); in particular, this holds for the reals \mathbb{R} and the group \mathbb{J}_p of p -adic integers:

Example. If X is either \mathbb{R} or \mathbb{J}_p , then $H \leq X$ is t -closed if and only if $H = \{0\}$ or $H = X$.

Some results on t -closed subgroups and topologically \mathbf{u} -torsion subgroups can be found also in the Master Thesis of R. Di Santo [28]. Bíró, Deshouillers and Sós [17] proved that every countable subgroup of the circle group is a topologically \mathbf{u} -torsion subgroup for an appropriate \mathbf{u} . These authors conjectured that the result could be extended to all compact abelian groups, but they did not give any explicit hint for a suitable generalization of the notion of topologically \mathbf{u} -torsion subgroup. The necessity of an appropriate generalization became clear from the following surprising theorem.

Theorem ([32]). *The only non-discrete locally compact group such that every countable subgroup is t -closed is the circle group.*

The relevant generalization of the notion of topologically \mathbf{u} -torsion group for an arbitrary topological abelian group X was given in [40], making use of sequences of *characters* of X , i.e., sequences $\mathbf{v} = (v_n)$ in the *Pontryagin dual* \widehat{X} of X (see §A.3.2).

Definition ([40]). Let X be a topological abelian group. For a sequence of characters $\mathbf{v} = (v_n)$ in \widehat{X} let

$$s_{\mathbf{v}}(X) = \{x \in X \mid v_n(x) \rightarrow 0\}.$$

Then $s_{\mathbf{v}}(X)$ is a subgroup of X . A subgroup H of X is said to be *characterized* if $H = s_{\mathbf{v}}(X)$ for some sequence $\mathbf{v} = (v_n)$ of characters of X . We also say that \mathbf{v} *characterizes* H or that \mathbf{v} is a *characterizing sequence* for H . Furthermore, let $\mathcal{C}\text{har}(X)$ denote the class of all subgroups of X that are characterized.

As $\widehat{\mathbb{T}} = \mathbb{Z}$, for $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$ the subgroup $t_{\mathbf{u}}(\mathbb{T})$ of \mathbb{T} coincides with the subgroup $s_{\mathbf{v}}(\mathbb{T})$ where the sequence \mathbf{v} is formed by the characters of \mathbb{T} defined by $v_n : x \mapsto u_n x$.

This notion of characterized subgroup is the main topic of this thesis.

Some historical background on characterized subgroups

We recall here briefly some of the most relevant known results about characterized subgroups, others can be found in the main body of the thesis. In doing this we use the numeration of theorems, definitions, etc. from the main body of the thesis, so that the reader can easily find the theorem, definition, etc. in the main text.

Eggleston [45] found a remarkable connection between the behaviour of the sequence of ratios $\mathbf{q}^{\mathbf{u}} = (q_n) = (\frac{u_n}{u_{n-1}})$ and the cardinality of $t_{\mathbf{u}}(\mathbb{T})$. Call a sequence $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$ *\mathbf{q} -bounded* (resp. *\mathbf{q} -divergent*) whenever $\mathbf{q}^{\mathbf{u}}$ is bounded (resp. whenever $q_n \rightarrow \infty$). Then one can formulate the Eggleston Theorem in the following way:

Theorem 4.1.11. *Let $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$, then the following hold.*

- (i) *If \mathbf{u} is \mathbf{q} -bounded, then $|t_{\mathbf{u}}(\mathbb{T})| = \aleph_0$;*
- (ii) *if \mathbf{u} is \mathbf{q} -divergent, then $|t_{\mathbf{u}}(\mathbb{T})| = \mathfrak{c}$.*

A strong impact on this topic had the paper of Raczkowski [82], where she proved (unaware of the stronger result of Eggleston) that $|t_{\mathbf{u}}(\mathbb{T})| = \mathfrak{c}$ if the sequence \mathbf{u} satisfies $q_n \geq n$ for every n . She also asked if there exist (Haar-)measure-zero subgroups of \mathbb{T} which are t -dense in \mathbb{T} (according to a more general theorem due to Comfort, Trigos and Wu [26] if H is a non-(Haar)-measurable subgroup of \mathbb{T} , then H is t -dense in \mathbb{T}). A positive answer, assuming Martin Axiom, was given in [7], where $2^{\mathfrak{c}}$ -many such subgroups were constructed (these authors, unaware of Eggleston Theorem, provided also a new proof of that theorem). More generally, and without additional set-theoretic assumptions, Hart and Kunen, in [57], showed that every compact non-totally disconnected group contains a Haar-null subgroup that is t -dense. This result was generalized in [9], where every non-discrete locally compact abelian group was shown to contain a Haar-null subgroup that is t -dense.

As mentioned above, Bíró, Deshouillers and Sós proved that every countable subgroup of \mathbb{T} is characterized and asked for a generalization of this result to arbitrary compact metrizable abelian groups. It was first proved by Dikranjan, Milan and Tonolo [40] that every cyclic subgroup of a maximally almost periodic (briefly, MAP) topological abelian group is an intersection of characterized subgroups (and this property characterizes the MAP groups). This result was later extended by Lukács in [72] to all countable subgroups of compact metrizable abelian groups. Finally, Beiglböck, Steineder and Winkler [14] and Dikranjan and Kunen [39] independently proved the following theorem thereby resolving the problem of Bíró, Deshouillers and Sós:

Theorem 1.2.1. *Every countable subgroup of a compact metrizable abelian group is characterized.*

Moreover, inspired by Eggleston Theorem, Beiglböck, Steineder and Winkler proved that every countable subgroup of \mathbb{T} can be characterized by means of a \mathfrak{q} -bounded sequence (see [14]).

Bíró [16] and Gabrielyan [52] independently noticed that a necessary condition for a subgroup of a compact metrizable group to be characterized is the *polishability*. A polishable subgroup of a Polish group is a subgroup that admits a (unique) finer Polish group topology. Extending ideas already used by Bíró and Gabrielyan, the following metric $\rho_{\mathbf{v}}$ was introduced in [33] on a compact metrizable abelian group X . This metric, when restricted to the characterized subgroup $s_{\mathbf{v}}(X)$, gives rise to a finer Polish group topology of $s_{\mathbf{v}}(X)$ denoted by $\mathfrak{p}_{\mathbf{v}}$.

Definition 2.1.9. *Let X be a compact metrizable abelian group, δ be a compatible metric on X and $\mathbf{v} = (v_n)$ be a sequence of characters of X . Let $x, y \in X$ and*

$$\rho_{\mathbf{v}}(x, y) = \sup_{n \in \mathbb{N}} \{\delta(x, y), d(v_n(x), v_n(y))\}.$$

Gabrielyan [52] proved that for a characterized subgroup of a compact metrizable group the topology $\mathfrak{p}_{\mathbf{v}}$ is also *locally quasi-convex* (see Definition A.3.8). A natural class of locally quasi-convex groups is that of the locally compact ones. Answering a question of Gabrielyan [53], Negro [76] proved that every subgroup of a compact metrizable abelian group that admits a finer locally compact Polish group topology is characterized. This provides the only, so far, *sufficient* (but not necessary) condition for a subgroup to be characterized. For a compact metrizable group X , denote by $\mathfrak{Pol}_{lc}(X)$ the class of the subgroups of X that admits a finer locally compact Polish group topology.

Characterized subgroups in the non-compact case were studied by Borel [19], who proved, among others, that all countable subgroups of \mathbb{R} are characterized.

MAIN RESULTS AND CONTENTS OF THE THESIS

This Thesis is divided in three parts and an Appendix. The first part gives the main properties of characterized subgroups for arbitrary topological abelian groups with particular attention

to locally compact abelian groups. The second part is focused on the circle group, where these notions were introduced and where the research gave the most relevant results. The third part presents applications related to characterized subgroups. The Appendix contains the necessary background in general topology and topological groups theory.

The transversality of this topic led to the natural fragmentation of the research on this general topic in various distinct areas. As a result, many relevant facts were observed and results were obtained simultaneously (or at least independently) by several authors. One of the aims of this thesis is to try to fill this gap by providing an as complete global picture as possible. To this end we include, among others, many known related results providing a brief survey on various subtopics (this applies in full measure to Chapter 4 and to a certain extent to Chapters 6 and 7).

Part I: Characterized subgroups of topological abelian groups

Chapter 1 deals with basic properties of the characterized subgroups in the general case of arbitrary topological abelian groups. In §1.1.1 we isolate three special types of characterized subgroups $H = s_{\mathbf{v}}(X)$ of a topological abelian group X :

- T-characterized subgroups (when \mathbf{v} is a T-sequence in \widehat{X});
- K-characterized subgroups (when \mathbf{v} is a one-to-one sequence);
- N-characterized subgroups (when H has the form $H = \mathbf{n}_{\mathbf{v}}(X) = \bigcap_{n \in \mathbb{N}} \ker v_n$).

The K-characterized subgroups were studied by Kunen and coauthors [DK, HK₁, HK₂], while T-characterized were recently introduced by Gabrielyan [55]. Obviously, T-characterized subgroups are also K-characterized. The N-characterized subgroups, being annihilators of countable subsets (subgroups) have long been known and used in duality theory of abelian topological

groups. They are obviously closed. These are the immediate implications between these three classes. The remaining non-trivial connections are thoroughly examined in Chapters 1 and 7.

In §1.1.2 some general properties are given, e.g. characterized subgroups are $\mathcal{F}_{\sigma\delta}$ subgroups of index $\leq \mathfrak{c}$, as well as the related discussion of measurability with respect to the Haar measure when the environment group X is compact.

In §1.1.3 we study the K -characterized subgroups and the behaviour of characterized subgroups under continuous homomorphisms. In the latter direction we show that the inverse image of a characterized subgroup is a characterized subgroup.

As far as the former direction is concerned, we note that the dense characterized subgroups are K -characterized (see Chapter 7). Therefore, we focus on finding sufficient conditions for a closed characterized subgroup to be K -characterized. In Theorem 1.1.22 we show that if H is a dually closed subgroup of a topological abelian group X , such that X/H has infinite separable dual, then H is a K -characterized subgroup of X . Let us give a corollary of this general result covering two opposite cases: *if H is a closed characterized subgroup of an abelian topological group X of infinite index $[X : H]$, then H is K -characterized in case X is either compact or discrete* (corollaries 1.1.23 and 1.1.24). In particular,

- a closed characterized subgroup H of a compact abelian group is K -characterized if and only if H is not open.
- every closed non-open subgroup of a compact metrizable abelian group is K -characterized.

In §1.1.4 we describe the N -characterized subgroups of a topological abelian group. More precisely, we prove the following theorem.

Theorem 1.1.29. *Let X be a topological abelian group and $H \leq X$. Then the following are equivalent.*

- (i) *There exists a continuous injection from X/H into $\mathbb{T}^{\mathbb{N}}$;*
- (ii) *H is an N -characterized subgroup of X ;*

(iii) H is closed and \mathcal{G}_δ in $\tau_{\langle \mathbf{v} \rangle}$ for some $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$;

(iv) H is closed and \mathcal{G}_δ in the Bohr topology of X .

If these equivalent conditions hold, then $[X : H] \leq \mathfrak{c}$.

As a consequence of Theorem 1.1.29 one obtains the following theorem in the case X is locally compact showing, among others, that closed characterized subgroups of the LCA groups are N -characterized.

Theorem 1.2.4. *Let X be a locally compact abelian group and $H \leq X$. The following are equivalent:*

(i) H is a closed characterized subgroup of X ;

(ii) H is an N -characterized subgroup of X ;

(iii) H is a closed \mathcal{G}_δ subgroup and $[X : H] \leq \mathfrak{c}$.

Then we prove another relevant result namely, *the reduction to locally compact metrizable abelian group*. More precisely, we generalize some results from [33] previously proved for compact abelian groups. These generalizations allow for the following reduction:

Theorem 1.2.5. *A subgroup H of a locally compact abelian group X is characterized if and only if H contains a closed \mathcal{G}_δ subgroup K of X such that H/K is a characterized subgroup of the locally compact metrizable abelian group X/K .*

The last part of Chapter 1 introduces particular subgroups useful for the study of the structure of characterized subgroups. One of these kind of subgroups are the *countable modulo compact metrizable* (briefly CCM), i. e. subgroups H of X such that there exists a compact metrizable subgroup $K \leq H$ of X such that H/K is countable. In this chapter prevails the tendency to study characterized subgroups of not necessarily compact (or even compact-like) groups. Indeed, a great deal of the results are obtained in maximum generality.

The structure of subgroups in $\mathfrak{Pol}_c(X)$ is described in Chapter 2:

Theorem 2.2.17. *A subgroup H of a compact metrizable group X belongs to $\mathfrak{Pol}_{lc}(X)$ if and only if there exist a CCM subgroup H_0 of H and injective continuous images R_1, \dots, R_n of \mathbb{R} in X such that*

$$H = R_1 \oplus R_2 \oplus \cdots \oplus R_n \oplus H_0.$$

Moreover, the only infinite compact metrizable abelian group X , such that the only proper subgroups in $\mathfrak{Pol}_{lc}(X)$ are the countable ones, is the circle group:

Theorem 2.2.24. *Let X be an infinite compact metrizable abelian group. Then the following are equivalent:*

- (i) $X \cong \mathbb{T}$;
- (ii) whenever $H \in \mathfrak{Pol}_{lc}(X)$ and $H \neq X$, then H is countable.
- (iii) whenever H is a proper closed subgroup of X , then H is finite.

Chapter 3 discusses the Borel complexity of a characterized subgroup of a compact metrizable abelian group, i. e. the “lowest” class in the Borel Hierarchy where such a set belongs to. In particular, the following *test topology* is introduced.

Definition 3.2.2. *Let \mathbf{v} be a sequence of characters of a compact metrizable abelian group $X = (G, \tau)$ and $\tau_{\mathbf{v}}^{\sigma}$ be the group topology on G with neighbourhood filter at 0 generated by*

$$\left\{ W_n = \overline{B_{1/n}^{\rho_{\mathbf{v}}}(0)} \mid n \in \mathbb{N} \right\},$$

where \overline{M} denotes the closure in (G, τ) of a subset M of X and $\rho_{\mathbf{v}}$ is as in Definition 2.1.9.

We refer to $\tau_{\mathbf{v}}^{\sigma}$ as the \mathcal{F}_{σ} -test topology with respect to the sequence \mathbf{v} . This term is motivated by the following useful criterion:

Theorem 3.2.4. *Let $X = (G, \tau)$ be a compact metrizable abelian group and $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$. Then*

$$s_{\mathbf{v}}(X) \in \mathfrak{F}_{\sigma}(X) \iff s_{\mathbf{v}}(X) \in \tau_{\mathbf{v}}^{\sigma}.$$

By means of the \mathcal{F}_σ -test topology $\tau_{\mathbf{v}}^\sigma$ one can describe when $s_{\mathbf{v}}(X)$ is countable.

Theorem 3.2.5. *Let X be a metrizable compact abelian group and $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$. Then the following are equivalent:*

- (i) $s_{\mathbf{v}}(X)$ is countable;
- (ii) $\tau_{\mathbf{v}}$ is discrete;
- (iii) $\tau_{\mathbf{v}}^\sigma$ is discrete;
- (iv) $\tau_{\mathbf{v}}^\sigma \upharpoonright_{s_{\mathbf{v}}(X)}$ is discrete;
- (v) $\mathfrak{p}_{\mathbf{v}}$ is discrete.

Since $s_{\mathbf{v}}(X)$ is a Borel set, (i) is equivalent also to $|s_{\mathbf{v}}(X)| < \mathfrak{c}$.

Moreover, we describe when the metric topology $\tau_{\mathbf{v}}$ on X induced from the metric $\rho_{\mathbf{v}}$ defined in Definition 2.1.9 is Polish.

Corollary 3.2.8. *If $X = (G, \tau)$ is a compact metrizable abelian group and $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$, then the following are equivalent:*

- (i) $(X, \tau_{\mathbf{v}})$ is Polish;
- (ii) $\tau = \tau_{\mathbf{v}}$;
- (iii) $s_{\mathbf{v}}(X)$ is τ -open;
- (iv) \mathbf{v} has no faithfully indexed subsequences and every character that appears infinitely many times is torsion.

Part II: The circle group

This part is focused on characterized subgroups of the circle group where this notion coincides with the notion of *topologically \mathbf{u} -torsion subgroup*. Therefore, in the circle group one has an additional powerful tool for the study of characterized subgroups, namely the sequence of ratios $\mathbf{q}^{\mathbf{u}} = (q_n) = (\frac{u_n}{u_{n-1}})$. In Chapter 4 is a survey on known results on general sequences and recursive sequences of integers.

In Chapter 5 we consider strictly increasing sequences of integers $\mathbf{u} = (u_n)$ such that $u_n | u_{n+1}$ for every $n \in \mathbb{N}$, called *arithmetic* (or briefly, *a-sequences*). For these sequences it is possible to write an element of \mathbb{T} , identifying it with its unique preimage in $[0, 1)$ via canonical projection in the following manner. Let \mathbf{u} be an a-sequence, let $\mathbf{q} = (q_n)$ be its sequence of ratios and $\alpha \in [0, 1)$. Then there exists a unique sequence $(c_n) \in \mathbb{N}^{\mathbb{N}}$ such that $0 \leq c_n < q_n$ for every $n \in \mathbb{N}$,

$$\alpha = \sum_{n=0}^{\infty} \frac{c_n}{u_n}, \quad (5.1.1)$$

and $c_n < q_n - 1$ for infinitely many n . We call (5.1.1) *\mathbf{u} -representation* of x and $\text{supp}_{\mathbf{u}}(x) = \{n \in \mathbb{N} \mid c_n \neq 0\}$ the *\mathbf{u} -support* of x . Call a subset of $A \subseteq \mathbb{N}$ *\mathbf{q} -bounded* (resp. *\mathbf{q} -divergent*) if $\{q_n : n \in A\}$ is bounded (resp. $\{q_n : n \in A\}$ diverges).

For an a-sequence \mathbf{u} we provide in Theorem 5.1.6 a *complete description* of $t_{\mathbf{u}}(\mathbb{T})$ in terms of the \mathbf{u} -representation of its elements. In case the sequence \mathbf{u} has the *splitting property*, i. e. it can be partitioned in a \mathbf{q} -bounded part and a \mathbf{q} -divergent part, Theorem 5.1.6 can be stated in a more transparent way (see Corollary 5.2.13). As the theorem is somewhat heavy, we give below only some of its corollaries.

Corollary 5.2.2. *Let $x \in \mathbb{T}$. If $\text{supp}(x)$ is \mathbf{q} -bounded, then $x \in t_{\mathbf{u}}(\mathbb{T})$ if and only if $c_n = 0$ for almost all $n \in \mathbb{N}$.*

In other words, when $\text{supp}(x)$ is \mathbf{q} -bounded, $x \in t_{\mathbf{u}}(\mathbb{T})$ if and only if x has a finite \mathbf{u} -representation (in particular x is torsion and its order divides u_n eventually).

Corollary 5.2.4. *If $x \in \mathbb{T}$ has \mathbf{q} -divergent support, then $x \in t_{\mathbf{u}}(\mathbb{T})$ whenever $\lim_{n \in \text{supp}(x)} \frac{c_n}{q_n} = 0$ in \mathbb{R} .*

Corollary 5.2.6. *If $x \in \mathbb{T}$ and \mathbb{N} is \mathbf{q} -divergent, then $x \in t_{\mathbf{u}}(\mathbb{T})$ if and only if $\lim_{n \in \text{supp}(x)} \frac{c_n}{q_n} = 0$ in \mathbb{T} .*

This part contains some applications and related results about characterized subgroups.

The term “thin sets”, used in Chapter 6, refers to particular (measure zero) sets studied in Harmonic Analysis. Some of these sets are related to the *sets of uniqueness* of a trigonometric series, i. e. sets where a trigonometric expansion of a periodic function has a unique representation. The notions of *D-set* and *A-set* are closely related to the notion of characterized subgroup of \mathbb{T} . One can define these sets as follows.

Definition. A set $E \subseteq [0, 1]$ is

- an *Arbault set* (briefly, an *A-set*) if there is an increasing sequence of positive integers $\mathbf{u} = (u_n)_{n \in \mathbb{N}}$ such that

$$\lim \sin \pi u_n x = 0 \text{ for all } x \in E,$$

i. e. $\|u_n x\|_{\mathbb{Z}} \rightarrow 0$ pointwise on E .

- a *Dirichlet set* (briefly, a *D-set*) if there is an increasing sequence of positive integers $\mathbf{u} = (u_n)_{n \in \mathbb{N}}$ such that

$$(\sin \pi u_n x)_{n \in \mathbb{N}} \text{ converges uniformly to } 0 \text{ on } E,$$

i. e. $\|u_n x\|_{\mathbb{Z}} \rightarrow 0$ uniformly on E .

One can extend these notions for subsets E of the circle group \mathbb{T} , i. e. $E \subseteq \mathbb{T}$ is a *D-set* if there exists an increasing sequence of positive integers \mathbf{u} such that $u_n x \rightarrow 0$ uniformly on E .

Clearly, a set $E \subseteq \mathbb{T}$ is an *A-set* if and only if $E \subseteq t_{\mathbf{u}}(\mathbb{T})$ for some increasing sequence of positive integers \mathbf{u} . On the other hand every *D-set* is an *A-set*, but one can prove that *an infinite subgroup of \mathbb{T} is not a D-set*. In particular this chapter is focused on a particular kind of sets in $[0, 1)$ (as well as their counterparts in \mathbb{T}) defined as follow.

For a subset L of \mathbb{N} and an α -sequence \mathbf{u} , let $K_L^{\mathbf{u}} = \{x \in [0, 1) : \text{supp}_{\mathbf{u}}(x) \subseteq L\}$. According to a result of Marcinkiewicz [73], the subset $K_L^{\mathbf{u}}$ is a *D-set* whenever $\mathbf{u} = (2^n)$ and L is *non-large*, i. e.

for every finite $F \subseteq \mathbb{N}$ one has $L + F \neq \mathbb{N}$. We extend this result as follows:

Proposition 6.2.6. *Let \mathbf{u} be an a -sequence, and let L be a non-large set, then $K_L^{\mathbf{u}}$ is a D -set and $\pi(K_L^{\mathbf{u}}) \subseteq t_{\mathbf{u}^*}(\mathbb{T})$, where \mathbf{u}^* is a subsequence of \mathbf{u} .*

Moreover, we show that this statement can be inverted in case $\mathbf{u} = (q^n)$ for a positive integer q . In a forthcoming paper [6] a complete characterization of the sets of the form $K_L^{\mathbf{u}}$ that are D -sets is obtained when \mathbf{u} is an arbitrary a -sequence.

The study of the sets $K_L^{\mathbf{u}}$ gives also the possibility to answer a question posed in [7], namely:

Theorem 6.2.8. *If \mathbf{u} is an a -sequence, then there exist $\mathbf{v}, \mathbf{w} \subseteq \mathbf{u}$ such that $t_{\mathbf{v}}(\mathbb{T}) + t_{\mathbf{w}}(\mathbb{T}) = \mathbb{T}$.*

Chapter 7 is dedicated to another relevant application of characterized subgroup, namely a better understanding of the presence of non-trivial convergent sequences in precompact group topologies.

Definition. *Let G be an abelian group. A sequence \mathbf{v} in G is called a*

- *T-sequence if there exists a Hausdorff group topology τ such that $\mathbf{v} \xrightarrow{\tau} 0$ ([81]).*
- *TB-sequence if there exists a precompact group topology τ such that $\mathbf{v} \xrightarrow{\tau} 0$ ([40]).*

The notion of a T -sequence, introduced 25 years ago, turned out to be a formidable tool in the part of the topological group theory involving convergent sequences (see [5], [50]-[55],[71], [77]). The motivation to introduce TB -sequences comes from the fact that the TB -sequences are T -sequences and are much easier to come by (as the result below shows).

The relation between characterized subgroups and TB -sequences comes out by the following fact. If G_d is an abelian group equipped with the discrete topology and τ_H is the initial topology of $H \leq \widehat{G}_d$, then the following proposition holds.

Proposition 7.1.3. *Let $H \leq \widehat{G}_d$ and $\mathbf{v} \in G^{\mathbb{N}}$. The sequence \mathbf{v} converges to 0 in τ_H if and only if $H \leq s_{\mathbf{v}}(\widehat{G}_d)$.*

Moreover, if $\sigma_{b\mathbf{v}}$ is the finest totally bounded topology such that $\mathbf{v} \xrightarrow{\sigma_{b\mathbf{v}}} 0$, then

$$\sigma_{b\mathbf{v}} = \tau_{s_{\mathbf{v}}(\widehat{G}_d)}.$$

Hence, by Peter-Weyl Theorem A.3.10, $\sigma_{b\mathbf{v}}$ is Hausdorff (hence precompact) if and only if $s_{\mathbf{v}}(\widehat{G}_d)$ is dense in \widehat{G}_d if and only if \mathbf{v} is a TB-sequence. It can be easily deduced from this fact that every dense characterized subgroup of a compact abelian group is T-characterized. The question of whether a closed characterized subgroup H of a compact abelian group X is always T-characterized turned out to be quite subtle. It was proved by Gabrielyan [55] that such a subgroup H is T-characterized precisely when its annihilator H^\perp carries a MinAP group topology. We give a brief sketch of his proof and provide the following theorem describing when a closed characterized subgroup of a compact abelian group is not T-characterized in terms of functorial subgroups and cardinal invariants of the groups in question and their quotient groups ($T_p(X)$ denotes the closure of the subgroup X_p):

Theorem 7.2.13. *For a compact abelian group X and a \mathcal{G}_δ -subgroup H of X the following are equivalent:*

- (i) H is not a T-characterized subgroup of X ;
- (ii) H^\perp does not admit a MinAP group topology;
- (iii) there exists some $m > 0$ such that $m(X/H)$ is finite and non-trivial.
- (iv) $eo(X/H) < \exp(X/H)$;
- (v) there exists a finite set P of primes so that
 - (a) H contains the subgroups $T_q(X)$ for all primes $q \notin P$,
 - (b) for every $p \in P$ there exist $k_p \in \mathbb{N}$ with $p^{k_p} T_p(X) \subseteq H$

(c) *there exists $p \in P$ such that $p^{k_p-1}T_p(X) \not\subseteq H$ and $p^{k_p-1}T_p(X) \cap H$ has finite index in $p^{k_p}T_p(X)$.*

(f) *there exists a finite set P of primes so that $X/H \cong \prod_{p \in P} K_p$, where each K_p is a compact p -group and there exists some $p \in P$ and $k \in \mathbb{N}$ such that $p^k K_p$ is finite and non-trivial.*

Comfort, Raczkowski, Trigos-Arrieta, in [27], proved that every abelian group G admits families \mathcal{A}_1 and \mathcal{A}_2 , each consisting of $2^{2^{|G|}}$ -many pairwise non-homeomorphic precompact group topologies, such that no $\tau \in \mathcal{A}_1$ contains non-eventually null convergent sequences and such that every $\tau \in \mathcal{A}_2$ has a convergent sequence. To find precompact group topologies with no non-eventually null convergent sequences, the authors exploit an observation of Comfort, Trigos-Arrieta and Wu, if H is a non-measurable, dense subgroup of a compact group K , then the precompact group topology τ_H that H induces on the group of characters of K has no non-eventually null convergent sequences. The latter result is a corollary of Theorem 7.5.3.

The last chapter contains final comments and open questions.

BASIC NOTATION

Let \mathbb{N} , \mathbb{N}_+ , \mathbb{Z} , \mathbb{Q} and \mathbb{R} be respectively the natural, the positive natural, the integer, the rational and the real numbers. Denote by \mathbb{T} the circle group written additively. If \mathbb{P} denotes the set of all prime numbers, then for every $p \in \mathbb{P}$, $\mathbb{Z}(p^\infty)$ and $\mathbb{Z}(p^n)$ denote respectively the Prüfer p -group and the cyclic group of order p^n where $n \in \mathbb{N}$. Denote by \mathbb{J}_p the p -adic integers and by \mathbb{Q}_p the p -adic numbers. Let \aleph_0 and \mathfrak{c} be respectively the infinite countable cardinal and the cardinality of the continuum. If G is an abelian group and $m \in \mathbb{N}$, we will denote by $G[m] = \{g \in G \mid mg = 0\}$ the m -torsion subgroup of G and by $t(G) = \bigcup_m G[m]$ the torsion subgroup of G . For $E \subseteq G$, let $\langle E \rangle$ denote the subgroup of G generated by E .

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Part I

CHARACTERIZED SUBGROUPS OF TOPOLOGICAL ABELIAN GROUPS

In this part we establish some basic properties of the characterized subgroups of arbitrary topological abelian groups. A complete description of the closed characterized subgroups of locally compact abelian groups is obtained along with a useful reduction to the metrizable case. In doing so we obtain as corollaries most of the known relevant results for compact and locally compact abelian groups in the literature. The study of the characterized subgroups of compact metrizable abelian groups is relevant because of their applications to TB-sequences (see Chapter 7) and because of the close connection to and possible generalizations of relevant results in the circle group \mathbb{T} (see Part II). Also some new results from [35], [37], [38] and some others unpublished are given.

GENERAL PROPERTIES OF CHARACTERIZED SUBGROUPS

1.1 DEFINITIONS, BASIC FACTS AND SOME REDUCTIONS

1.1.1 Related definitions

Recall the definition of characterized subgroup of a topological abelian group.

Definition 1.1.1. Let X be a topological abelian group.

- (i) ([40]) For a sequence of characters $\mathbf{v} = (v_n)$ in \widehat{X} let

$$s_{\mathbf{v}}(X) = \{x \in X \mid v_n(x) \rightarrow 0\}.$$

Then $s_{\mathbf{v}}(X)$ is a subgroup of X . A subgroup $H \leq X$ is said to be *characterized* if $H = s_{\mathbf{v}}(X)$ for some sequence $\mathbf{v} = (v_n)$ of characters of X . We also say that \mathbf{v} *characterizes* H or \mathbf{v} is a *characterizing sequence* for H . Furthermore, let $\mathcal{C}\text{har}(X)$ denote the class of all subgroups of X that are characterized.

- (ii) ([57, 58]) For an infinite set V of characters, denote by

$$s_V(X) = \{x \in X \mid \forall U \in \mathcal{V}(0) \ |\{v \in V \mid v(x) \notin U\}| < \aleph_0\},$$

where $\mathcal{V}(0)$ is a filter of neighbourhoods of $0_{\mathbb{T}}$.

The following notion is the keystone of the reductions carried out in this chapter.

Definition 1.1.2. Let Γ be a set of characters of a topological abelian group X . The subgroup

$$\mathbf{n}_{\Gamma}(X) = \mathbf{n}_{\Gamma} = \bigcap_{\chi \in \Gamma} \ker \chi = \Gamma^{\perp}$$

is obviously closed.

- (i) In case $\Gamma = \emptyset$, $\mathbf{n}_\Gamma(X) = X$. In case $\Gamma = X^*$, $\mathbf{n}_\Gamma(X)$ is the *von Neumann radical* of X , denoted, for brevity, by $\mathbf{n}(X)$.
- (ii) If Γ is countable, then $\mathbf{n}_\Gamma(X)$ is obviously a characterized subgroup of X . Indeed, take as characterizing sequence a sequence where each character of Γ appears infinitely many times.

Notation 1.1.3. For a given sequence $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$, let $\Gamma_{\mathbf{v}} = \{v_n : n \in \mathbb{N}\}$ be the support of \mathbf{v} . Let $\mathbf{n}_{\mathbf{v}}(X) := \mathbf{n}_{\Gamma_{\mathbf{v}}}(X)$. The set $\Gamma_{\mathbf{v}}$ can be partitioned in the following two subsets.

- (i) $\Gamma_{\mathbf{v}}^{\infty} = \{v_n \in \Gamma_{\mathbf{v}} \mid n \in \mathbb{N} \text{ and } \exists^{\infty} m (v_n = v_m)\}$;
- (ii) $\Gamma_{\mathbf{v}}^0 = \Gamma_{\mathbf{v}} \setminus \Gamma_{\mathbf{v}}^{\infty}$.

Note that, if $\Gamma_{\mathbf{v}}^{\infty}$ is infinite, then $s_{\Gamma_{\mathbf{v}}^{\infty}}(X) = \mathbf{n}_{\Gamma_{\mathbf{v}}^{\infty}}(X)$.

The characterized subgroup $\mathbf{n}_{\mathbf{v}}(X)$ is described in the next subsection.

Remark 1.1.4. Let $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$. It is easy to see that $\mathbf{n}_{\mathbf{v}}(X) \leq s_{\mathbf{v}}(X)$, is closed. In particular, $\mathbf{n}(X) \leq s_{\mathbf{v}}(X)$ for every $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$.

If H is a subgroup of a topological group X and $\chi \in \widehat{X}$, then one may consider $\chi \upharpoonright_{H \in \widehat{H}}$, that is a character of H . Sometimes, to avoid heavy notation, one may write χ instead of $\chi \upharpoonright_H$ when no confusion is possible.

The following definition introduces three specific types of characterized subgroups.

Definition 1.1.5. Let X be a topological abelian group. A characterized subgroup H of X is

- (i) (Gabrielyan) T-characterized if $H = s_{\mathbf{v}}(X)$, where \mathbf{v} is T-sequence of \widehat{X} ;
- (ii) K-characterized if $H = s_{\mathbf{v}}(X)$, where \mathbf{v} is a finitely many-to-one sequence, i. e. $\Gamma_{\mathbf{v}}^{\infty} = \emptyset$;
- (iii) N-characterized if $H = \mathbf{n}_{\mathbf{v}}(X)$, where $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$.

Note that every T-characterized subgroup is also K-characterized. Indeed, every T-sequence contains no constant subsequences. The N-characterized subgroups are closed, while dense characterized subgroups are even T-characterized (we shall see in §7 that they are characterized even by a TB-sequence). In particular, dense characterized subgroups are K-characterized. We shall see below that closed (even open) subgroups need not be K-characterized in general. For more detail of T-characterized subgroups see Subsection 7.2.2.

Remark 1.1.6. Let X be a topological abelian group and $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$.

- (i) If $\Gamma_{\mathbf{v}}^0$ is infinite, then $s_{\mathbf{v}}(X) = s_{\Gamma_{\mathbf{v}}^0}(X) \cap \mathbf{n}_{\Gamma_{\mathbf{v}}^{\infty}}(X)$;
- (ii) If $\Gamma_{\mathbf{v}}^0$ is finite, then $s_{\mathbf{v}}(X) = \mathbf{n}_{\Gamma_{\mathbf{v}}^{\infty}}(X)$.

Therefore, every characterized subgroup is the intersection of an N-characterized subgroups and a K-characterized subgroup.

1.1.2 Basic properties

The following lemma lists some basic facts on characterized subgroups, that appear in [26, 33, 39, 40].

Lemma 1.1.7. *Let X be a topological group. If $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$, then the followings hold*

- (i) *if $J \leq X$, then $s_{\mathbf{v}}(J) = s_{\mathbf{v}}(X) \cap J$;*
- (ii) *if \mathbf{u} is any permutation of \mathbf{v} , then $s_{\mathbf{v}}(X) = s_{\mathbf{u}}(X)$;*
- (iii) *$\mathcal{C}\text{har}(X)$ is stable for finite intersection;*
- (iv) *if τ_1 and τ_2 are two compatible group topologies on an abelian group G , then $\mathcal{C}\text{har}((G, \tau_1)) = \mathcal{C}\text{har}((G, \tau_2))$;*
- (v) *$s_{\mathbf{v}}(X)$ is an $\mathcal{F}_{\sigma\delta}$ set.*

Proof. Items (i) and (ii) are obvious. To prove item (iii), if $\mathbf{u}, \mathbf{v} \in \widehat{X}^{\mathbb{N}}$, then take $\mathbf{w} = (w_n)$, where $w_{2n} = u_n$ and $w_{2n+1} = v_n$, as

characterizing sequence of $s_{\mathbf{u}}(X) \cap s_{\mathbf{v}}(X)$. To prove item (iv), just note that (G, τ_1) and (G, τ_2) have the same group of characters. To prove item (v), note that $s_{\mathbf{v}}(X)$ is the following subgroup of X

$$\left\{ x \in X \mid \forall m \exists k \forall n \left(n \geq k \rightarrow \|v_n(x)\| \leq \frac{1}{m} \right) \right\}.$$

Obviously, $S_{n,m} = \{x \in X \mid \|v_n(x)\| \leq \frac{1}{m}\}$ is a closed subset of X (indeed it is a counter-image of a closed set in \mathbb{T}). It is easy to see that,

$$s_{\mathbf{v}}(X) = \bigcap_m \bigcup_k \bigcap_{n \geq k} S_{n,m},$$

that is an $\mathcal{F}_{\sigma\delta}$ subset of X . □

Lemma 1.1.8. *Let X be a topological abelian group and $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$. Then for all compact subgroups K of $s_{\mathbf{v}}(X)$, the sequence $(v_n \upharpoonright_K)$ is eventually $0_{\mathbb{T}}$.*

Proof. To prove this lemma one can identify $v_n(x)$ with its unique counter-image $\widetilde{v_n(x)} \in [0, 1)$ (see §A.2.1). Let $K \leq s_{\mathbf{v}}(X)$ be compact and let λ_K be the normalized Haar measure on K . By the Lebesgue dominated convergence Theorem one has

$$\int_K e^{2\pi i \widetilde{v_n(x)}} d\lambda_K \longrightarrow \int_K 1 d\lambda_K = \lambda_K(K) = 1.$$

By Proposition A.3.6, $I_n = \int_K e^{2\pi i \widetilde{v_n(x)}} d\lambda_K$ is either 0 or 1 for every $n \in \mathbb{N}$ and $I_n = 1$ if and only if $v_n = 0$. Therefore, (I_n) is eventually 1 and hence \mathbf{v} is eventually 0. □

Corollary 1.1.9. *If X is compact, then $s_{\mathbf{v}}(X) = X$ if and only if the sequence \mathbf{v} is eventually null.*

Proof. Follows directly from Lemma 1.1.8. □

Note that, if one drops the compactness, then the above corollary may fail, as showed in the next example.

Example 1.1.10.

- (i) Let N be an infinite countable subgroup of \mathbb{T} . As mentioned in the introduction, N is characterized in \mathbb{T} . Therefore, $N = s_{\mathbf{v}}(\mathbb{T})$ for a non eventually null sequence $\mathbf{v} \in \mathbb{Z}^{\mathbb{N}}$. If $\mathbf{u} = \mathbf{v} \upharpoonright_N$, then \mathbf{u} is not eventually null (since N is dense in \mathbb{T}) and $s_{\mathbf{u}}(N) = N$.
- (ii) Let $X = \mathbb{R}$ and $v_n = 1/n \in \widehat{\mathbb{R}}$, so that $v_n(x) = \pi(\frac{x}{n}) \in \mathbb{T}$ for every $x \in \mathbb{R}$ and $n \in \mathbb{N}_+$. Obviously, $s_{\mathbf{v}}(\mathbb{R}) = \mathbb{R}$, even if \mathbf{v} is not eventually null.
- (iii) Let $X = \mathbb{Q}_p$, where p is a prime. Let $v_n = p^n \in \widehat{\mathbb{Q}_p}$. Obviously, $s_{\mathbf{v}}(\mathbb{Q}_p) = \mathbb{Q}_p$, even if \mathbf{v} is not eventually null.

Motivated by these examples and by Corollary 1.1.9, one can introduce the following notion.

Definition 1.1.11. Call a topological abelian group X *properly autocharacterized*, if $s_{\mathbf{v}}(X) = X$ for a sequence \mathbf{v} of non-zero characters \mathbf{v} of X .

Items (ii) and (iii) of Example 1.1.10 show that \mathbb{R} and \mathbb{Q}_p are autocharacterized.

One can prove the following easy general property:

Proposition 1.1.12. *If a topological abelian group X has a topological direct summand that is properly autocharacterized, then also X is properly autocharacterized.*

Proof. Let $X = Y \times Z$ and $Y = s_{\mathbf{v}}(Y)$ for some sequence \mathbf{v} of non-zero characters of Y . We shall consider Y and Z as subgroups of X in the natural way. For every $n \in \mathbb{N}$ let u_n be the unique character of X that extends v_n such that u_n vanishes on Z . Then $u_n \neq 0$ and the sequence \mathbf{u} characterizes X . \square

We established in Corollary 1.1.9 that the compact abelian groups are not properly autocharacterized. Now we prove that this property describes the compact groups within the larger class of all locally compact abelian groups. This follows easily from the above proposition for the locally compact abelian groups that contain a copy of \mathbb{R} . But the general cases requires the following deeper argument.

Theorem 1.1.13. *If X is a locally compact abelian group, then X is properly autocharacterized if and only if X is not compact.*

Proof. If X is properly autocharacterized, then it is not compact, according to Corollary 1.1.9.

Assume now that X is not compact. Then \widehat{X} is not discrete. Then \widehat{X} has a non-trivial null sequence $v_n \rightarrow 0$. We may assume, without loss of generality, that $v_n \neq 0$ for all n . Then $v_n \rightarrow 0$ in the Bohr topology of \widehat{X} , i.e. for every character χ of \widehat{X} , $\chi(v_n) \rightarrow 0$ in \mathbb{T} . By Pontryagin duality theorem, we can identify X with the second dual of X . Therefore, for every character $\chi \in \widehat{X}$ there exists $x \in X$ such that $\chi(u) = u(x)$ for all $u \in \widehat{X}$. Hence, $v_n(x) \rightarrow 0$ for all $x \in X$. This proves that $X = s_v(X)$. \square

Notation 1.1.14. Let $\mathbf{v} = (v_n) \in \widehat{X}^{\mathbb{N}}$ and denote by $\mathbf{v}_{(m)}$ the sequence $(v_n)_{n \geq m}$.

Proposition 1.1.15. *Let X be a topological abelian group, $F \leq X$ and $\mathbf{v} \in \widehat{X}$ such that $F \leq s_v(X)$. If the group F is not properly autocharacterized, then $F \leq \mathbf{n}_{\mathbf{v}_{(m)}}(X)$ for some $m \in \mathbb{N}$.*

Proof. Assume that $H = s_v(X)$. Let $\mathbf{u} \in \widehat{F}^{\mathbb{N}}$ denote the sequence of the restrictions of v_n to F . Then $F = s_{\mathbf{u}}(F)$, so the sequence \mathbf{u} must be eventually null. Let $m \in \mathbb{N}$ such that $u_n = 0$ for every $n \geq m$. Therefore, all $\mathbf{v}_{(m)}$ restricted to F vanish. Hence $F \leq \mathbf{n}_{\mathbf{v}_{(m)}}(X)$. \square

Corollary 1.1.16. *If a characterized subgroup H of X is not a properly autocharacterized group, then H is an \mathbb{N} -characterized subgroup of X .*

Proof. If $H = s_v(X)$, then $H \leq \mathbf{n}_{\mathbf{v}_{(m)}}(X)$ for some $m \in \mathbb{N}$ by Proposition 1.1.15. Since $H = s_v(X) = s_{\mathbf{v}_{(m)}}(X) \geq \mathbf{n}_{\mathbf{v}_{(m)}}(X)$, we deduce that $H = \mathbf{n}_{\mathbf{v}_{(m)}}(X)$ and hence it is \mathbb{N} -characterized. \square

Lemma 1.1.17 ([26, Lemma 3.10]). *Let X be a topological abelian group and let \mathbf{v} be a sequence in \widehat{X} . Then If X is compact and \mathbf{v} has a faithfully indexed subsequence, then $s_v(X)$ is a Haar measure zero subgroup of X (hence with infinite index).*

Proof. Let λ be the normalized Haar measure on X . Let $H = s_{\mathbf{v}}(X)$, by Lemma 1.1.7 (v) H is a Borel subgroup and hence a Haar-measurable subset of X . Suppose for a contradiction that $\lambda(H) > 0$. Therefore, by Theorem A.2.13 H is a clopen subgroup of X , so that $|X/H| < \aleph_0$. By Theorem A.3.14 $\widehat{X}/H = H^\perp$ that is the annihilator of H . Let us prove that having a finite H^\perp is impossible, obtaining the wanted contradiction. Indeed, since H is a clopen subgroup of a compact group, by Lemma 1.1.8 one has that $(v_n \upharpoonright_H)$ is eventually 0. Therefore, \mathbf{v} is eventually contained in H^\perp . Since, \mathbf{v} contains a faithfully indexed subsequence, one has that H^\perp is infinite, i. e. the wanted contradiction. □

Note that, if X is a connected compact abelian group, then Lemma 1.1.17 holds for all non-eventually null sequences in $\widehat{X}^{\mathbb{N}}$, since every measurable proper subgroup of a connected compact abelian group has measure 0 (see Corollary A.2.7).

Remark 1.1.18. As a consequence of Alexandroff-Hausdorff's Theorem (Theorem A.1.9) and Lemma 1.1.7 (v), one has that every characterized subgroup of a Polish group has cardinality \mathfrak{c} or \aleph_0 . Note also that item (ii) of the above lemma cannot be inverted, take for example the constant sequence $\mathbf{u} = (1)$ in $\widehat{\mathbb{T}}^{\mathbb{N}}$.

1.1.3 K -characterized subgroups

We start with a direct consequence of Lemma 1.1.17.

Corollary 1.1.19. *If X is a compact abelian group, then every K -characterized subgroup has measure 0. In particular, no open subgroup is K -characterized.*

This corollary shows that no subgroup of a finite (discrete) group is K -characterized

The next proposition provides an ample source of examples of K -characterized open subgroups of finite index showing that the above corollary may strongly fail in the non-compact case.

Proposition 1.1.20. *Let H be an abelian topological group and let F be a finite discrete abelian group. Then H is a K -characterized subgroup of the group $X = H \times F$ equipped with the product topology if and only if H is a properly autocharacterized, group.*

Proof. Assume that H is a properly autocharacterized and let $H = s_{\mathbf{v}}(H)$ for some sequence of characters $\mathbf{v} = (v_n)$. Obviously, we can assume without loss of generality that this is a one-to-one sequence. Let $k = |F| = |\widehat{F}|$ and let $\{\alpha_1, \dots, \alpha_k\}$ be a one-to-one enumeration of \widehat{F} . Split \mathbf{v} in k subsequences $\mathbf{v}^{(i)}$, $i = 1, \dots, k$. Now define $\mathbf{u} = (u_n)$ by $u_n(x, y) = v_n^{(i)}(x) + \alpha_i(y)$, where $n \equiv i \pmod{k}$. Clearly, all characters u_n are pairwise distinct. Moreover, $H \leq s_{\mathbf{v}}(X)$. To show that $s_{\mathbf{v}}(X) = H$ it suffices to prove that $(0, c) \notin s_{\mathbf{v}}(X)$ whenever $c \neq 0$ in F . Indeed, pick $\alpha_i \in \widehat{F}$ such that $\alpha_i(c) \neq 0$. Then $u_n(0, c) = \alpha_i(c) \not\rightarrow 0$ for all n with $n \equiv i \pmod{k}$.

Now assume that H is a K -characterized subgroup of X and let $H = s_{\mathbf{v}}(H)$ for some one-to-one sequence of characters $\mathbf{v} = (v_n)$ of X . Let $u_n := v_n \upharpoonright_H$ for $n \in \mathbb{N}$. Then the sequence $\mathbf{u} = (u_n)$ is finitely many to one, as F is finite. Obviously, $H = s_{\mathbf{u}}(H)$, so H is properly autocharacterized. \square

The next proposition was proved in [33] for characterized subgroups of compact groups. We generalize it in the following manner.

Proposition 1.1.21. *Let X be a topological abelian group, let F be a closed subgroup and let $\pi : X \rightarrow X/F$ be the canonical projection. If H is a characterized (resp. K -characterized, N -characterized, T -characterized) subgroup of X/F , then $\pi^{-1}(H)$ is a characterized (resp. K -characterized, N -characterized, T -characterized) subgroup of X .*

Proof. Let $H = s_{\widetilde{\mathbf{v}}}(X/F)$ where $\widetilde{\mathbf{v}} \in \widehat{X/F}^{\mathbb{N}}$. It is known that $\widehat{\pi}$ is an isomorphism (algebraically) from $\widehat{X/F}$ onto F^{\perp} . If $v_n = \widehat{\pi}(\widetilde{v}_n)$ for every $n \in \mathbb{N}$, then $\pi^{-1}(H) = s_{\mathbf{v}}(X)$ where $\mathbf{v} = (v_n)$. Indeed, $x \in s_{\mathbf{v}}(X)$ if and only if

$$v_n(x) \rightarrow 0 \Leftrightarrow \widetilde{v}_n(\pi(x)) \rightarrow 0 \Leftrightarrow \pi(x) \in H.$$

To prove that $\pi^{-1}(H)$ is K-characterized, whenever H is K-characterized note that if $\Gamma_{\tilde{\mathbf{v}}}^{\infty} = \emptyset$, then $\Gamma_{\mathbf{v}}^{\infty} = \emptyset$.

To prove that $\pi^{-1}(H)$ is N-characterized, whenever H is N-characterized note that if $H = \mathbf{n}_{\tilde{\mathbf{v}}}(X/F)$, then $\pi^{-1}(H) = \mathbf{n}_{\mathbf{v}}(X)$. Indeed,

$$v_n(x) = 0 \Leftrightarrow \tilde{v}_n(\pi(x)) = 0 \Leftrightarrow \pi(x) \in H.$$

To prove that $\pi^{-1}(H)$ is T-characterized, whenever H is T-characterized note that if $\tilde{\mathbf{v}}$ is a T-sequence of $\widehat{X/F}$, then \mathbf{v} is a T-sequence of \widehat{X} . Indeed, $\widehat{X/F}$ is isomorphic to F^{\perp} that is a subgroup of \widehat{X} . If τ is a Hausdorff topology on $\widehat{X/F}$, that makes $\tilde{\mathbf{v}}$ converging to 0, then by Proposition A.2.3 τ^* is a Hausdorff topology on \widehat{X} , that makes \mathbf{v} converging to 0. □

Theorem 1.1.22. *If X is a topological abelian group and H is a dually closed subgroup of X , such that X/H has infinite separable dual. Then H is a K-characterized subgroup of X .*

Proof. Let $Y = X/H$ equipped with the quotient topology. Then Y is MAP and $\widehat{Y} = H^{\perp}$ has a countable dense subset D . According to Proposition 1.1.21, applied to the quotient map $X \rightarrow Y$, it suffices to prove that $\{0\}$ is a K-characterized subgroup of Y . Let $\mathbf{v} = \{v_n : n \in \mathbb{N}\}$ be a one-to-one enumeration of D . We prove that $s_{\mathbf{v}}(Y) = \{0\}$. To this end we have to show that for every non-zero $y \in Y$ there exists a neighbourhood U of 0 in \mathbb{T} such that $v_n(y) \notin U$ for infinitely many $n \in \mathbb{N}$. Actually, we show that there is a single such U that works fine for all non-zero $y \in Y$. To this end pick a neighbourhood U of 0 that contains no non-trivial subgroups of \mathbb{T} , for example $U = \mathbb{T}_+$. Clearly, for every non-zero y in Y one has that $N_y = \{v_n(y) : n \in \mathbb{N}\}$ is a non-trivial subgroup of \mathbb{T} as Y is MAP and $y \neq 0$. If N_y is infinite, then N_y is dense, so $N_y \setminus U$ is infinite and we are done.

Now consider the case when N_y is finite. This implies in particular, that the order $k = o(y)$ of y is finite. As $N_y \neq \{0\}$, there exists $a \in \mathbb{N}$ such that $a \notin U$ by our choice of U . It is

enough to show that $V_y = \{v_n : v_n(y) = a\}$ is infinite. To this end we write $a = v_m(y)$ for some $m \in \mathbb{N}$. Then $L = V - v_m$ is a subgroup of \widehat{Y} , as $L = \{y\}^\perp$. Since $\{y\}$ is finite, $[\widehat{Y} : L]$ is finite as well. So L is infinite, as Y is infinite. Hence, V_y is infinite for every non zero y in Y and $v_n(y) = a \notin U$ for every $v_n \in V_y$, as required. \square

Corollary 1.1.23. *Let X be a discrete abelian group and $H \leq X$ such that $\omega \leq [X : H] \leq c$. Then H is K -characterized.*

Proof. This follows by the fact that \widehat{X}/H is a compact group of weight at most c and hence it is separable (see [41]). \square

Corollary 1.1.24. *If X is a compact metrizable abelian group, then every closed non-open subgroup of X is K -characterized.*

1.1.4 N -characterized subgroups

Lemma 1.1.25. *Let X be a topological abelian group and $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$. Then $\mathbf{n}_v(X)$ is a (closed) \mathcal{G}_δ subgroup of X with $[X : \mathbf{n}_v(X)] \leq c$.*

Proof. Let $\varphi : X \rightarrow \mathbb{T}^{\mathbb{N}}$ be such that

$$\varphi(x) = (v_0(x), \dots, v_n(x), \dots).$$

Hence, $\mathbf{n}_v(X) = \bigcap_n \ker v_n = \varphi^{-1}(\{0\})$. Therefore, $\mathbf{n}_v(X)$ is \mathcal{G}_δ , since it is a continuous preimage of $\{0\} \in \mathbb{T}^{\mathbb{N}}$, that is a closed subset of a metrizable group and hence \mathcal{G}_δ . Finally, observe that $X/\mathbf{n}_v(X)$ is isomorphic to a subgroup of $\mathbb{T}^{\mathbb{N}}$. \square

Remark 1.1.26.

- (i) $\mathbf{n}_v(X)$ is a closed \mathcal{G}_δ subgroup in every topology that makes v_n continuous for every $n \in \mathbb{N}$. In particular, this holds for any compatible topology.
- (ii) As a consequence of Lemma 1.1.25, one has that every characterized subgroup of a topological abelian group, has index at most c . Indeed, $\mathbf{n}_v(X) \leq s_v(X)$ for every $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$ and hence $[X : s_v(X)] \leq [X : \mathbf{n}_v(X)] \leq c$.

Theorem 1.1.27. *Let X be a topological abelian group and $H \leq X$. The following are equivalent:*

- (i) *there exists $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$ such that for every closed $F \leq \mathbf{n}_{\mathbf{v}}(X)$ one has that $H/F = \mathbf{s}_{\mathbf{u}}(X/F)$, where $\mathbf{u} = (\mathbf{u}_n)$ and each \mathbf{u}_n is the factorization of \mathbf{v}_n through the quotient map $X \rightarrow X/F$;*
- (ii) *there exists a closed subgroup F of X such that $F \leq H$ and H/F is a characterized subgroup of X/F ;*
- (iii) *H is a characterized subgroup of X .*

Proof. (i) \Rightarrow (ii). Take $F = \mathbf{n}_{\mathbf{v}}(X)$.

(ii) \Rightarrow (iii). Since $F \leq H$ one has $H = \pi^{-1}(H/F)$ and by Proposition 1.1.21 one can conclude.

(iii) \Rightarrow (i). Let $H = \mathbf{s}_{\mathbf{v}}(X)$ and $F \leq \mathbf{n}_{\mathbf{v}}(X)$. Let $\pi : X \rightarrow X/F$ be the canonical projection. Since F is closed, $\widehat{\pi} : \widehat{X}/F \rightarrow \widehat{X}$ is injective with image F^{\perp} . For every $n \in \mathbb{N}$, let $\widetilde{\mathbf{v}}_n$ be the character of X/F defined as follow:

$$\widetilde{\mathbf{v}}_n : \pi(x) \mapsto \mathbf{v}_n(x).$$

The character $\widetilde{\mathbf{v}}_n$ is well-defined since $F \leq \mathbf{n}_{\mathbf{v}}(X) \leq \ker \mathbf{v}_n$. Hence, $\widetilde{\mathbf{v}} = (\widetilde{\mathbf{v}}_n)_n$ is a sequence of characters of X/F such that $\widehat{\pi}(\widetilde{\mathbf{v}}_n) = \mathbf{v}_n$. Now, one has that $H/F = \mathbf{s}_{\widetilde{\mathbf{v}}}(X/F)$. Indeed, by construction, for every $\widetilde{\mathbf{h}} \in H/F$ the following holds $\widetilde{\mathbf{v}}_n(\widetilde{\mathbf{h}}) = \mathbf{v}_n(\mathbf{h}) \rightarrow 0$ and hence $H/F \subseteq \mathbf{s}_{\widetilde{\mathbf{v}}}(X/F)$. Conversely, if $\widetilde{\mathbf{x}} \in \mathbf{s}_{\widetilde{\mathbf{v}}}(X/F)$, then $\mathbf{v}_n(x) = \widetilde{\mathbf{v}}_n(\widetilde{\mathbf{x}}) \rightarrow 0$. Hence, $x \in H$ and so $\widetilde{\mathbf{x}} \in H/F$. \square

Lemma 1.1.28. *If F is a closed \mathcal{G}_{δ} subgroup of a precompact group X , then there exists a continuous injection from X/F into $\mathbb{T}^{\mathbb{N}}$. In this case $[X : F] \leq \mathfrak{c}$.*

Proof. Let $F = \bigcap_n U_n$, where U_n is an open set in X . Since X is precompact, for every $n \in \mathbb{N}$ there exist $k_n \in \mathbb{N}$, V_n open neighbourhood of 0 in \mathbb{T}^{k_n} and $f_n : X \rightarrow \mathbb{T}^{k_n}$ continuous such that $U_n = f_n^{-1}(V_n)$. It is possible to chose V_n such that V_n does not contain any non trivial subgroup of \mathbb{T}^{k_n} . Therefore, for every $H \subseteq f_n^{-1}(V_n)$ one has that $f_n(H) = \{0\}$ and hence $H \leq \ker f_n$, this holds in particular for F . Moreover, $\ker f_n \leq U_n$ for

every $\mathbf{n} \in \mathbb{N}$ and hence $F = \bigcap \ker f_n$. Let $\widetilde{f}_n : X/F \rightarrow \mathbb{T}^{k_n}$ be such that $\widetilde{f}_n(\widetilde{x}) = f_n(x)$, where $\widetilde{x} = x + F$. Since $F \leq \ker f_n$ one has that \widetilde{f}_n is well defined. Consider, the map j , from X/F into $\prod_n \mathbb{T}^{k_n} \cong \mathbb{T}^{\mathbb{N}}$, defined as follows $j(\widetilde{x}) = (\widetilde{f}_0(\widetilde{x}), \dots, \widetilde{f}_n(\widetilde{x}), \dots)$. By construction of \widetilde{f}_n , the map j is a continuous injection from X/F into $\mathbb{T}^{\mathbb{N}}$. The last part of the statement follows obviously from the first one. \square

Recall that, the Bohr topology of a topological abelian group X is the coarsest compatible precompact group topology on X , i. e. $\tau_{\widehat{X}}$ (see Subsection A.3.3).

Theorem 1.1.29. *Let X be a topological abelian group and $H \leq X$. Then the following are equivalent.*

- (i) *There exists a continuous injection from X/H into $\mathbb{T}^{\mathbb{N}}$;*
- (ii) *H is an N -characterized subgroup of X ;*
- (iii) *H is closed and \mathcal{G}_δ in $\tau_{\langle \mathbf{v} \rangle}$, where $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$;*
- (iv) *H is closed and \mathcal{G}_δ in the Bohr topology of X .*

In case these equivalent conditions hold, one has $[X : H] \leq \mathfrak{c}$.

Proof. **(i) \Rightarrow (ii).** Suppose that there exists a continuous injection j from X/H into $\mathbb{T}^{\mathbb{N}}$. If q is the quotient map of X onto X/H , p_n is the n -th projection of $\mathbb{T}^{\mathbb{N}}$ onto \mathbb{T} and $v_n = p_n \circ j \circ q$, then one has the following commutative diagram

$$\begin{array}{ccc} X/H & \xrightarrow{j} & \mathbb{T}^{\mathbb{N}} \\ \uparrow q & & \downarrow p_n \\ X & \xrightarrow{v_n} & \mathbb{T} \end{array}$$

Therefore, $v_n \in \widehat{X}^{\mathbb{N}}$ for every n and $H = \mathbf{n}_{\mathbf{v}}(X)$, where $\mathbf{v} = (v_n)$.

(ii) \Rightarrow (iii). This follows from Lemma 1.1.25 and Remark 1.1.26 (i).

(iii) \Rightarrow (iv). The Bohr topology is finer than $\tau_{\langle \mathbf{v} \rangle}$.

(iv) \Rightarrow (i). This is Lemma 1.1.28.

The last part of the statement is Lemma 1.1.25. □

Remark 1.1.30. Note that $\tau_{\langle \mathbf{v} \rangle}$ is metrizable since $\langle \mathbf{v} \rangle$ is countable (see Subsection A.3.3). Therefore, by Theorem 1.1.29 one may reduce the study of N -characterized subgroups only for precompact metrizable groups X .

Corollary 1.1.31. *Let H be an open subgroup of a topological abelian group X . Then the following are equivalent.*

- (i) $H \in \mathcal{C}\text{har}(X)$;
- (ii) $[X : H] \leq \mathfrak{c}$;
- (iii) H is N -characterized.

Proof. **(i) \Rightarrow (ii).** This is Remark 1.1.26 (ii).

(ii) \Rightarrow (iii). Let $[X : H] \leq \mathfrak{c}$. Since H is open, X/H is discrete and has cardinality at most \mathfrak{c} . Therefore, there exists a continuous injection j from X/H into $\mathbb{T}^{\mathbb{N}}$. Hence, by Theorem 1.1.29 one concludes.

(iii) \Rightarrow (i). This is obvious. □

Corollary 1.1.32. *For a subgroup H of a discrete abelian group X the following are equivalent.*

- (i) $H \in \mathcal{C}\text{har}(X)$;
- (ii) $[X : H] \leq \mathfrak{c}$;
- (iii) H is N -characterized.

Lemma 1.1.7 (iv) states that if G is an abelian group and τ_1, τ_2 two compatible group topologies on G , then

$$\mathcal{C}\text{har}((G, \tau_1)) = \mathcal{C}\text{har}((G, \tau_2)).$$

Therefore, if X is a topological abelian group, one may study characterized subgroups of X equipped with the Bohr topology. Hence, the study of characterized subgroups of precompact abelian groups will play a key-role and hence we prefer to

isolate the following result that is a consequence of Theorem 1.1.29. Here we are interested in subgroups of the form $\mathfrak{n}_v(X)$. Moreover, by Remark 1.1.30 one may reduce the study of this subgroups in the metrizable case.

Theorem 1.1.33. *Let X be a precompact metrizable abelian group and $H \leq X$. The following are equivalent:*

- (i) H is a closed characterized subgroup of X ;
- (ii) H is an N -characterized subgroup of X ;
- (iii) H is a closed subgroup.

Moreover, in case these equivalent conditions hold, one has $[X : H] \leq \mathfrak{c}$.

Proof. **(iii) \Rightarrow (ii).** Since X is metrizable H is obviously \mathcal{G}_δ . By Lemma 1.1.28 there exists a continuous injective homomorphism j from X/H into $\mathbb{T}^{\mathbb{N}}$. Therefore, one can conclude by Theorem 1.1.29.

(ii) \Rightarrow (i). This is obvious.

(i) \Rightarrow (iii). This is obvious. The last part of the statement follows from Lemma 1.1.25. \square

1.2 COMPACT AND LOCALLY COMPACT ABELIAN GROUPS

Here we prove some generalization of the results from [33], previously stated for compact abelian groups and now generalized to locally compact groups. One of the reasons to study characterized subgroups of compact abelian groups is their relation with TB-sequences reported in Chapter 7. Moreover, many authors studied and still study these subgroups in the circle group. Therefore one may expect to generalize some of the results obtained for the circle group.

1.2.1 Reduction to locally compact metrizable abelian groups

In [39, 14] the authors proved the following theorem.

Theorem 1.2.1 ([39, Theorem 1.4],[14]). *Let X be a compact metrizable abelian group and $H \leq X$. If $|H| = \aleph_0$, then H is characterized.*

It was pointed out, in [30, 14], that the metrizability of X in the Theorem 1.2.1 is necessary, as stated below.

Remark 1.2.2. Let X be a compact abelian group and $H \leq X$ be a countable subgroup. Then H is characterized subgroup of X if and only if X is metrizable. Indeed, if $H = s_v(X)$ for some $v \in X^{\mathbb{N}}$, then $\mathfrak{n}_v(X) \leq s_v(X)$ is a countable closed (compact and hence finite) \mathcal{G}_δ subgroup of X such that $X/\mathfrak{n}_v(X)$ is metrizable. Therefore, X is metrizable too. More precisely, the following are equivalent, for a compact abelian group X :

- X is metrizable;
- there exists a countable characterized subgroup of X ;
- every countable subgroup of X is characterized.

As mentioned in Definition 1.1.2, the subgroup of the form $\mathfrak{n}_v(X)$ are always closed and characterized. The next theorem describes the closed characterized subgroups of the locally compact abelian groups X by showing that these are precisely the subgroups of X of the form $\mathfrak{n}_v(X)$. But first we prove the following theorem that is also of independent interest.

Theorem 1.2.3. *Let X be a locally compact abelian group. Then X is a metrizable group of cardinality at most \mathfrak{c} if and only if there exists a continuous injective homomorphism from X into $\mathbb{T}^{\mathbb{N}}$.*

Proof. Suppose that X is metrizable and has cardinality at most \mathfrak{c} . By the principal structure theorem, $X = \mathbb{R}^n \times X_0$, where $n \in \mathbb{N}$ and X_0 has an open compact (metrizable) subgroup K . Clearly, there exists two continuous injective homomorphism $j_1 : \mathbb{R}^n \hookrightarrow \mathbb{T}^{\mathbb{N}}$ and $j_2 : K \hookrightarrow \mathbb{T}^{\mathbb{N}}$. Therefore, $j_3 = (j_1, j_2) : \mathbb{R}^n \times K \rightarrow \mathbb{T}^{\mathbb{N}} \times \mathbb{T}^{\mathbb{N}} \cong \mathbb{T}^{\mathbb{N}}$ is an injective continuous homomorphism too. Let \tilde{j}_3 be the continuous extension of j_3 on X . This extension exists since $\mathbb{T}^{\mathbb{N}}$ is divisible (by Lemma A.2.6) and $\mathbb{R}^n \times K$ is open. Let q_1 be the quotient map of X onto

$X/(\mathbb{R}^n \times K)$ and j_4 a continuous injective homomorphism from $X/(\mathbb{R}^n \times K)$ into $\mathbb{T}^{\mathbb{N}}$ (this injection exists since $X/(\mathbb{R}^n \times K)$ is discrete). If $\varphi = j_4 \circ q$ one has the following commutative diagram.

$$\begin{array}{ccc} X & & \\ \downarrow q & \searrow \varphi & \\ X/(\mathbb{R}^n \times K) & \xrightarrow{j_4} & \mathbb{T}^{\mathbb{N}} \end{array}$$

Note that $j = (\varphi, \tilde{j}_3)$ is an injective continuous homomorphism of X into $\mathbb{T}^{\mathbb{N}} \times \mathbb{T}^{\mathbb{N}} \cong \mathbb{T}^{\mathbb{N}}$. Indeed, if $j(x) = 0$, then $\varphi(x) = 0$ and $\tilde{j}_3(x) = 0$. By construction of φ , one has that $x \in \mathbb{R}^n \times K$. Since \tilde{j}_3 is an injection when restricted on $\mathbb{R}^n \times K$ one has that $x = 0$.

Conversely, if j is a continuous injective homomorphism from X into $\mathbb{T}^{\mathbb{N}}$, then X has cardinality at most \mathfrak{c} and $\{0_X\} = j^{-1}(\{0_{\mathbb{T}^{\mathbb{N}}}\})$. Therefore, $\{0_X\}$ is a \mathcal{G}_δ subgroup of X (since it is a continuous preimage of $\{0_{\mathbb{T}^{\mathbb{N}}}\}$ that is \mathcal{G}_δ in $\mathbb{T}^{\mathbb{N}}$). Hence, X is metrizable by Theorem A.2.9. \square

Theorem 1.2.4. *Let X be a locally compact abelian group and $H \leq X$. The following are equivalent:*

- (i) H is a closed characterized subgroup of X ;
- (ii) H is a N -characterized subgroup of X ;
- (iii) H is a closed \mathcal{G}_δ subgroup in the Bohr topology
- (iv) H is a closed \mathcal{G}_δ subgroup and $[X : H] \leq \mathfrak{c}$.

Proof. **(iv) \Rightarrow (iii).** X/H is a locally compact metrizable group of cardinality at most \mathfrak{c} , therefore by Theorem 1.2.3 there exists a continuous injective homomorphism j from X/H into $\mathbb{T}^{\mathbb{N}}$. Therefore, one can conclude by Theorem 1.1.29.

(iii) \Rightarrow (ii). This follows from Theorem 1.1.29.

(ii) \Rightarrow (i). This is obvious.

(i) \Rightarrow (iv). Suppose that $H = s_v(X)$ is a closed characterized subgroup of X . By Remark 1.1.26 (ii) $[X : H] \leq \mathfrak{c}$. By Remark 1.1.4 $\mathfrak{n}_v(X) \leq s_v(X)$ and by Lemma 1.1.25 $\mathfrak{n}_v(X)$ it is a closed \mathcal{G}_δ subgroup of X . By Theorem A.2.9 $X/\mathfrak{n}_v(X)$ is a locally compact metrizable abelian group and by hypothesis $s_v(X)$ is closed. Therefore $s_v(X)/\mathfrak{n}_v(X)$ is a closed and hence \mathcal{G}_δ subgroup of the locally compact metrizable group $X/\mathfrak{n}_v(X)$. Therefore, $s_v(X)$ is a counter-image via canonical projection of a closed \mathcal{G}_δ subgroup and hence closed and \mathcal{G}_δ in X . \square

The following theorem was proved in [33, Theorem B] for compact abelian groups, one can extend it to locally compact abelian groups.

Theorem 1.2.5. *A subgroup H of a locally compact abelian group X is characterized if and only if H contains a (closed) \mathcal{G}_δ subgroup K of X such that H/K is a characterized subgroup of the locally compact metrizable abelian group X/K .*

Proof. Follows directly from Theorem 1.1.27, since K is closed \mathcal{G}_δ and hence X/K is a locally compact metrizable group by Theorem A.2.9. \square

In this last part of the subsection, we prove two results that hold for compact groups. These results refine Theorem 1.2.4 and Proposition 1.1.21.

Theorem 1.2.6 ([33, Theorem A]). *Let X be a compact abelian group and let $H \leq X$ be closed. Then H is characterized if and only if it is \mathcal{G}_δ .*

Proof. Let H be a closed \mathcal{G}_δ subgroup of X . By Theorem A.2.9 X/H is compact and metrizable. Hence, $|X/H| \leq \mathfrak{c}$, so the equivalence of Theorem 1.2.4 applies.

Conversely, if H is characterized, then H is \mathcal{G}_δ , by Theorem 1.2.4. \square

Remark 1.2.7. In the case of locally compact metrizable groups, the \mathcal{G}_δ subgroups are precisely the closed subgroups by Theorem A.2.14 and the well-known fact that closed subset of metrizable spaces are \mathcal{G}_δ subset. In particular, if $X = \mathbb{T}$, then Theorem 1.2.6 is an easy consequence of the previous stated fact. Indeed, in the case of \mathbb{T} the closed subgroups are precisely the finite ones, i. e. $\mathbb{T}[m] = \{x \in \mathbb{T} : mx = 0\}$ where $m \in \mathbb{N}$, that are trivially characterized by the constant sequence $\mathbf{u} = (m)_{n \in \mathbb{N}}$.

Theorem 1.2.8. *If X is a compact abelian group, then for every closed subgroup H of X the following are equivalent.*

- (i) H is K -characterized;
- (ii) H is a non-open \mathcal{G}_δ -subgroup of X ;
- (iii) H is N -characterized and non-open.
- (iv) H is characterized and non-open.

Proof. (iv) \Rightarrow (iii). This follows by Theorem 1.2.4.

(iii) \Rightarrow (ii). This follows by Theorem 1.2.6.

(ii) \Rightarrow (i). X/H is a compact metrizable non discrete (hence infinite) group and hence $\{0_{X/H}\}$ is K -characterized by Corollary 1.1.24. By Proposition 1.1.21 H is K -characterized.

(i) \Rightarrow (iv). This follows by Corollary 1.1.19. □

Proposition 1.2.9 ([33]). *Let X be a compact abelian group, let F, H be subgroups of X such that F is closed and $F \leq H \leq X$. If $\pi : X \rightarrow X/F$ the canonical projection, then H/F is a characterized subgroup of X/F , if and only if $\pi^{-1}(H/F)$ is a characterized subgroup of X .*

Proof. The necessity is Proposition 1.1.21. To prove the sufficiency one can proceed as follow. Let $H = \pi^{-1}(H/F)$ be characterized by \mathbf{v} . Moreover, $F \leq H$ is a closed (and hence compact) subgroup of X . Therefore, by Corollary 1.1.9, F is not properly autocharacterized. By Proposition 1.1.15, F is contained in $\mathbf{n}_{\mathbf{v}(m)}$ for a sufficiently large $m \in \mathbb{N}$. If $\pi_{\mathbf{v}} : X/F \rightarrow X/\mathbf{n}_{\mathbf{v}(m)}$ and $q = \pi_{\mathbf{v}} \circ \pi$, then $q^{-1}(H/\mathbf{n}_{\mathbf{v}(m)}) = H = s_{\mathbf{v}}(X) = s_{\mathbf{v}(m)}(X)$. Therefore,

one can apply Theorem 1.1.27 to deduce that $H/\mathfrak{n}_{\mathbf{v}(m)}$ is a characterized subgroup of $X/\mathfrak{n}_{\mathbf{v}(m)}$. Hence, by Proposition 1.1.21 $H/F = \pi_{\mathbf{v}}^{-1}(H/\mathfrak{n}_{\mathbf{v}(m)})$ is a characterized subgroup of X/F . \square

Note that, the argument of the above proof fails in case X is not compact. Take for example $F = H = X = \mathbb{N}$, where \mathbb{N} is as in Example 1.1.10. Then one cannot conclude that $\mathbf{v} \upharpoonright F$ is eventually 0. Hence, one cannot conclude that F is contained in $\mathfrak{n}_{\mathbf{v}(m)}$.

1.2.2 Characterized groups

Note also that, the notion of characterized subgroup is relative to the environment group X , but one may also give a definition of *characterized group* that drops the environment group X .

Definition 1.2.10. An abelian group H is said to be *characterized* if there exists a sequence \mathbf{v} of characters of its completion \tilde{H} such that

$$H = s_{\mathbf{v}}(\tilde{H}) = \left\{ x \in \tilde{H} \mid v_n(x) \rightarrow 0_{\mathbb{T}} \right\}.$$

By the Theorem 1.2.5, one can reduce the problem of the study of characterized subgroups of locally compact abelian groups, to the study of these subgroups in locally compact metrizable abelian groups. If one drops the environment group X , then one has the following theorem.

Theorem 1.2.11. *A locally precompact group H is characterized if and only if there exists a closed \mathcal{G}_{δ} subgroup K of H such that H/K is a locally precompact metrizable characterized group.*

Proof. Let H be a characterized group, i.e. H is a characterized subgroup of its locally compact completion \tilde{H} . By Theorem 1.2.5 there exists a closed \mathcal{G}_{δ} (locally compact) subgroup K of \tilde{H} such that H/K is characterized subgroup of the locally compact metrizable group \tilde{H}/K . Therefore, H/K is metrizable and $H/K \in \mathcal{C}\text{har}(\tilde{H}/K)$. Since $\tilde{H}/K = \widetilde{H/K}$, one has $H/K \in$

$\mathcal{C}\text{har}(\widetilde{H/K})$, i. e. H/K is a locally precompact metrizable characterized group.

Conversely, let K be a closed \mathcal{G}_δ subgroup of H such that $H/K \in \mathcal{C}\text{har}(\widetilde{H/K})$, Theorem 1.2.5 ensures that $H \in \mathcal{C}\text{har}(\widetilde{H})$, i. e. H is a characterized group. \square

Characterized groups are related to characterized subgroups by the following proposition.

Proposition 1.2.12. *Let H be a locally precompact metrizable abelian group, then the following are equivalent:*

- (i) H is characterized;
- (ii) H is a characterized subgroup of \widetilde{H} ;
- (iii) if X is a locally compact metrizable abelian group such that H has index at most \mathfrak{c} in X , then H is a characterized subgroup of X .

Proof. **(i) \Leftrightarrow (ii).** This is the definition of characterized group.

(ii) \Rightarrow (iii). Let $\mathbf{v} \in \widehat{\widetilde{H}}^{\mathbb{N}}$ be such that $H = s_{\mathbf{v}}(\widetilde{H})$ and let X be a locally compact metrizable abelian group such that $[X : H] \leq \mathfrak{c}$. Note that \widetilde{H} is closed (hence \mathcal{G}_δ) and hence dually embedded (Lemma A.3.5) subgroup of X . Let $\mathbf{w} \in \widehat{X}^{\mathbb{N}}$ be such that $\mathbf{v} = \mathbf{w} \upharpoonright_{\widetilde{H}}$. Hence, $H \leq s_{\mathbf{w}}(X)$ and $H = s_{\mathbf{w}}(X) \cap \widetilde{H}$. Being closed (and \mathcal{G}_δ) in X , the subgroup \widetilde{H} is a characterized subgroup of X , by Theorem 1.2.4, since $[X : \widetilde{H}] \leq [X : H] \leq \mathfrak{c}$. Therefore, $H = s_{\mathbf{w}}(X) \cap \widetilde{H}$ is a finite intersection of characterized subgroups of X and hence characterized in X by Lemma 1.1.7(iii).

(iii) \Rightarrow (i). Take $X = \widetilde{H}$.

\square

As Remark 1.1.26 (ii) shows, the restraint $[X : H] \leq \mathfrak{c}$ in item (iii) of the above proposition is necessary.

1.2.3 CC subgroups

Definitions and related facts

From Theorems 1.2.1 and 1.2.5 one obtains the following corollary.

Corollary 1.2.13 ([33, Corollary B1]). *Let X be a compact abelian group and let H be a subgroup of X that contains a closed \mathcal{G}_δ subgroup K such that H/K is countable. Then H is characterized subgroup of X .*

The above corollary was motivated by the following theorem.

Theorem 1.2.14 ([39, Theorem 1.5]). *Let X be a compact abelian group and let $H = \bigcup_{n \in \mathbb{N}} F_n$ where for every $n \in \mathbb{N}$ one has $F_n \leq X$ is closed and $F_n \leq F_{n+1}$. Then H is characterized if and only if there exists $m \in \mathbb{N}$ such that X/F_m is metrizable and $[F_{n+1} : F_n] < \aleph_0$ for all $n \geq m$.*

Theorem 1.2.14 and Corollary 1.2.13 inspired the following definitions.

Definition 1.2.15. Let X be a topological abelian group. A subgroup H of X is called:

- *countable modulo compact* (briefly CC) if there exists a compact subgroup $K \leq H$ of X such that H/K is countable;
- *countable modulo compact \mathcal{G}_δ* (briefly CCG) if there exists a compact \mathcal{G}_δ subgroup $K \leq H$ of X such that H/K is countable;
- *countable modulo compact metrizable* (briefly CCM) if there exists a compact metrizable subgroup $K \leq H$ of X such that H/K is countable;
- *countable torsion modulo compact \mathcal{G}_δ* (briefly CTCG) if there exists a compact \mathcal{G}_δ subgroup $K \leq H$ of X such that H/K is countable and torsion.

Note that, if H is CTCG or CCG subgroup of a locally compact group X , then H/K is always metrizable in virtue of Theorem A.2.9. Indeed, being K a \mathcal{G}_δ subgroup of X , then X/K and hence $H/K \leq X/K$ is metrizable too. These subgroups appeared for the first time in [33] with one more restraint, i. e. the authors ask for their characterizability. But, as one can see in the next remark, these subgroups are characterized automatically.

Remark 1.2.16.

- (i) Clearly the subgroups of Corollary 1.2.13 are the CCG subgroups of X . Hence every CCG subgroups of compact abelian group is characterized.
- (ii) The subgroups treated in Theorem 1.2.14 are precisely the CTCG subgroups. Indeed, X/F_m is metrizable and hence, by Theorem A.2.9, F_m is also a \mathcal{G}_δ subgroup. Since $[F_{n+1} : F_n] < \aleph_0$ for all $n \geq m$, one has that $H/F_m = (\bigcup_{n \in \mathbb{N}} F_n)/F_m$ is countable and hence H is CTCG. Conversely, if H is a CTCG subgroup of X , then H/K can be written as an increasing union of finite torsion subgroup. If H_n/K are these finite torsion subgroup of H/K , then $H = \bigcup H_n$ and $[H_{n+1} : H_n] < \aleph_0$ for every $n \in \mathbb{N}$.

Moreover, in [33] the authors proved the following lemma, that provides a proper characterized subgroup for every infinite compact group.

Lemma 1.2.17 ([33, Lemma 3.6]). *Every infinite compact abelian group has a proper dense CCG (hence characterized) subgroup.*

Notation 1.2.18. Denote the collection of CC, CCG, CCM and CTCG subgroups of a topological abelian group X by $\mathfrak{cc}(X)$, $\mathfrak{cc}_g(X)$, $\mathfrak{cc}_m(X)$ and $\mathfrak{c}_{tcg}(X)$ respectively.

These classes of groups and subgroups are useful in the study of characterized subgroups as one can see in Chapters 2, where the class $\mathfrak{cc}_m(X)$ plays a key role. While, in Chapter 3, the class $\mathfrak{cc}(X)$ will be very useful. Moreover, with this

notation, Corollary 1.2.13 and Theorem 1.2.14, in view of Remark 1.2.16, can be reformulate as follows: Let X be a compact abelian group, then

$$\mathfrak{c}_t\mathfrak{c}_g(X) \subseteq \mathfrak{c}\mathfrak{c}_g(X) \subseteq \mathfrak{C}\mathfrak{h}\mathfrak{a}\mathfrak{r}(X). \quad (1.2.1)$$

Relations between classes

As stated in Remark 1.2.16 the subgroups appeared in Theorem 1.2.14 are precisely the CTCG subgroups of X , where X is compact abelian group. In relation with Theorem 1.2.14, in [33] the following theorem is proved.

Theorem 1.2.19 ([33, Theorem C]). *Let X be a compact abelian group, then the following are equivalent:*

- (i) X has finite exponent;
- (ii) $\mathfrak{c}_t\mathfrak{c}_g(X) = \mathfrak{C}\mathfrak{h}\mathfrak{a}\mathfrak{r}(X)$;
- (iii) $\mathfrak{c}\mathfrak{c}_g(X) = \mathfrak{C}\mathfrak{h}\mathfrak{a}\mathfrak{r}(X)$;

Being compact is an absolute property, i. e. independent from the environment group X , but being \mathfrak{G}_δ is not absolute in this sense. If $H \leq X$, H/K is countable and $K \leq H$ compact, then K is a \mathfrak{G}_δ subgroup of H , but not necessarily a \mathfrak{G}_δ subgroup of X . Therefore, $H \in \mathfrak{c}\mathfrak{c}(X)$ does not implies $H \in \mathfrak{c}\mathfrak{c}_g(X)$ in general. Here we present some further examples that, together with (1.2.2) and Remark 1.2.21 will point out the relations between the four classes from Definition 1.2.15.

Example 1.2.20.

- (i) Every countable non torsion subgroup of a topological group is CCM and CCG, but not CTCG;
- (ii) Let X be a compact non metrizable group. Then X is a subgroup of X obviously CTCG and CCG but not CCM;
- (iii) Let $X = K \times K$, where K is a compact non metrizable group. Then $H = K$ is a subgroup of X CC but neither CCM, nor CMCG.

(iv) Let X be a compact non metrizable. Then $H = \{0\}$ is a subgroup of X CCM but not CCG.

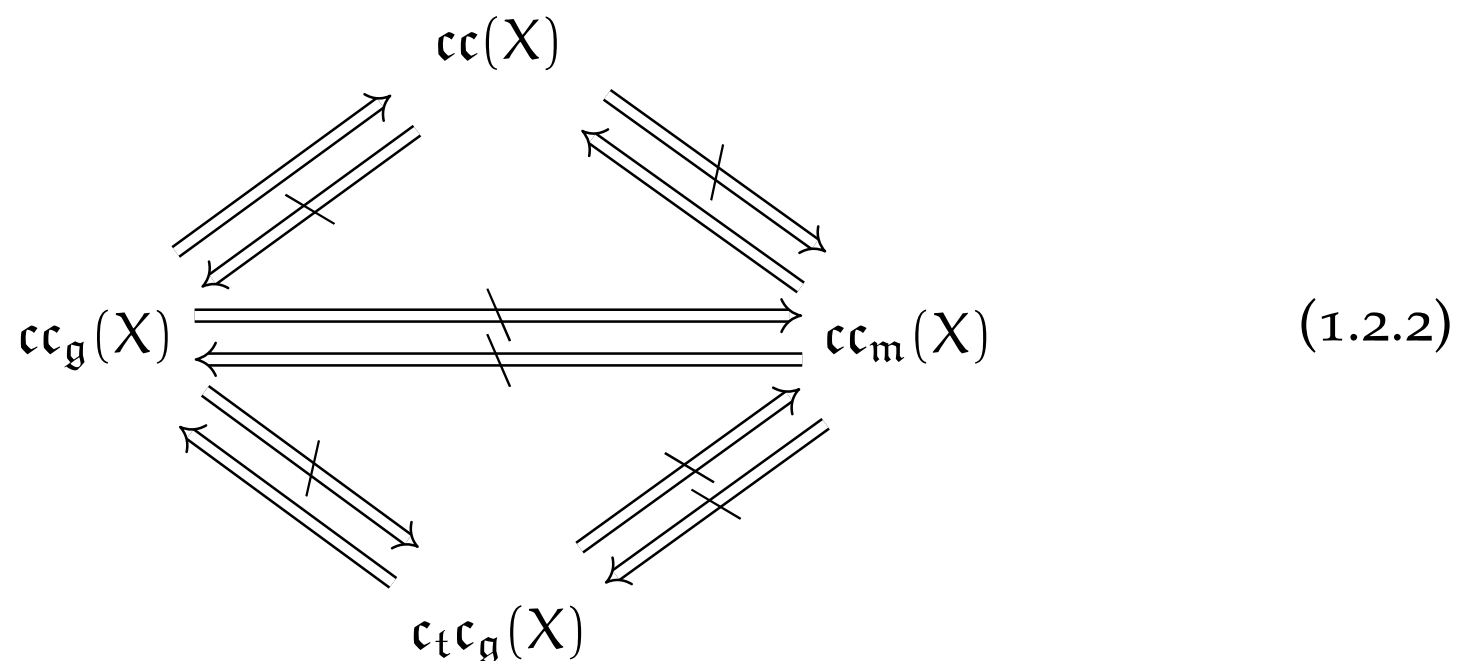
But, in the case of metrizable group the matter is easier, as shown in the next remark.

Remark 1.2.21 (Inclusions for metrizable groups). For a metrizable abelian group X , the following hold

$$\mathfrak{c}_t\mathfrak{c}_g(X) \subsetneq \mathfrak{c}\mathfrak{c}_g(X) = \mathfrak{c}\mathfrak{c}_m(X) = \mathfrak{c}\mathfrak{c}(X).$$

Indeed, Example 1.2.20(i) holds also if X is metrizable and this proves that $\mathfrak{c}_t\mathfrak{c}_g(X) \subsetneq \mathfrak{c}\mathfrak{c}_g(X)$. Moreover, every compact subgroup is also closed and hence \mathfrak{G}_δ since X is metrizable. Finally, every subgroup of a metrizable group is also metrizable.

Therefore, if one adds the assumption of metrizability in Theorem 1.2.19, one has the equivalence of all classes of subgroups seen until this point. But as one can see from example 1.2.20 if one drops metrizability, the relations between these classes are quite complicated. The following diagram will clarify the matter for an arbitrary topological abelian group X



Remark 1.2.22. Note also that if X is a compact non metrizable abelian group, then $\mathfrak{c}\mathfrak{c}_m(X) \not\subseteq \mathfrak{C}\mathfrak{h}\mathfrak{a}\mathfrak{r}(X)$ in contrast to (1.2.1) and so also $\mathfrak{c}\mathfrak{c}(X) \not\subseteq \mathfrak{C}\mathfrak{h}\mathfrak{a}\mathfrak{r}(X)$. Indeed, take $H = \{0\}$. Clearly $H \in \mathfrak{c}\mathfrak{c}_m(X)$, but $H = \{0\}$ is a countable non characterized subgroup of X , by Remark 1.2.2. Therefore, the following hold for an arbitrary topological abelian group X

$$\mathfrak{c}_t\mathfrak{c}_g(X) \subseteq \mathfrak{c}\mathfrak{c}_g(X) \subseteq \mathfrak{C}\mathfrak{h}\mathfrak{a}\mathfrak{r}(X) \not\subseteq \mathfrak{c}\mathfrak{c}_m(X) \subseteq \mathfrak{c}\mathfrak{c}(X).$$

CC subgroups of \mathbb{J}_p

Let $p \in \mathbb{P}$ and $\mathbf{c} = (c_n)$, where $c_n = \frac{1}{p^n} \in \mathbb{Z}(p^\infty)$ for every $n \in \mathbb{N}$. Clearly, $\langle \mathbf{c} \rangle = \mathbb{Z}(p^\infty) = \widehat{\mathbb{J}}_p$ and in [40] the authors proved that $s_{\mathbf{c}}(\mathbb{J}_p) = \mathbb{Z}$.

Remark 1.2.23. Recall that the \mathbb{J}_p is a totally disconnected compact metrizable abelian group and its closed (so compact) subgroups are $\{0\}$ and $p^n \mathbb{J}_p$, where $n \in \mathbb{N}$. Therefore, $H \in \mathfrak{cc}(\mathbb{J}_p)$ if and only if either H is countable or H contains the clopen finite index subgroup $p^n \mathbb{J}_p$ for some $n \in \mathbb{N}$, i. e. $H = p^m \mathbb{J}_p$ where $m \geq n$. In the latter case, $H \in \mathfrak{c}_t \mathfrak{c}_g(\mathbb{J}_p)$. Hence, one has the following

$$\begin{aligned} \mathfrak{cc}(\mathbb{J}_p) &= \{\text{countable subgroups}\} \cup \{\text{clopen subgroups}\} \\ &\neq \mathfrak{c}_t \mathfrak{c}_g(\mathbb{J}_p) = \{\text{clopen subgroups}\}. \end{aligned}$$

In [40], the authors characterized some countable subgroups of \mathbb{J}_p in the following manner.

Theorem 1.2.24 ([40, Theorem 3.3]). *Let $(n_k) \in \mathbb{N}^{\mathbb{N}}$ be strictly increasing and $\mathbf{v} = (c_{n_k})$, then the following are equivalent:*

- (i) $s_{\mathbf{v}}(\mathbb{J}_p) \cong \mathbb{Z}$;
- (ii) $s_{\mathbf{v}}(\mathbb{J}_p)$ is countable;
- (iii) $|s_{\mathbf{v}}(\mathbb{J}_p)| < \mathbf{c}$;
- (iv) the differences $n_{k+1} - n_k$ are bounded.

POLISHABILITY OF CHARACTERIZED SUBGROUPS OF COMPACT METRIZABLE GROUPS

2.1 POLISHABILITY OF $s_v(X)$

In this section we discuss one of the most useful properties of characterized subgroup of compact metrizable abelian groups, namely *the polishability*, introduced in [65]. As one can see in this section, the polishability is a necessary condition for a subgroup of a compact metrizable abelian group to be characterized.

2.1.1 *Definition and basic properties*

Definition 2.1.1. A *polishable subgroup* H of a Polish group X is a subgroup that satisfies one of the following equivalent conditions:

- (i) there exists a Polish group topology τ on H having the same Borel sets as H when considered as a topological subgroup of X ;
- (ii) there exists a continuous isomorphism from a Polish group P onto H ;
- (iii) there exists a continuous surjective homomorphism from a Polish group P onto H .

Remark 2.1.2. Clearly, one can add one more equivalent condition in the previous definition, i.e. $H \leq X$ admits a finer Polish group topology. Moreover, this topology is unique and follows from Corollary A.2.17.

Notation 2.1.3. Denote by $\mathfrak{Pol}(X)$ the class of all polishable subgroups of a Polish group X and if $H \in \mathfrak{Pol}(X)$, let \mathfrak{p}_H denote the unique Polish topology on H witnessing the polishability of H .

Not all Borel subgroups of compact metrizable abelian groups are polishable. A class of non polishable subgroups are those generated by an infinite *Kronecker set*.

Definition 2.1.4. A non empty compact subset K , of an infinite compact metrizable abelian group X , is called *Kronecker set*, if for every continuous function $f : K \rightarrow \mathbb{T}$ and $\varepsilon > 0$ there exists a character $\chi \in \widehat{X}$ such that

$$\max \{ \|f(x) - \chi(x)\| : x \in K \} < \varepsilon.$$

Example 2.1.5 ([70, pg. 8]). If $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha < \beta < 1$ and $\pi : \mathbb{R} \rightarrow \mathbb{T}$ is the canonical projection, then the following set is a Kronecker subset of $\mathbb{T}^{\mathbb{N}}$.

$$K_{\alpha, \beta} = \{ (\pi(t), \pi(t^2), \dots, \pi(t^n), \dots) \in \mathbb{T}^{\mathbb{N}} \mid t \in [\alpha, \beta] \subseteq \mathbb{R} \}$$

Theorem 2.1.6 ([16, Theorem 2]). *If K is an uncountable Kronecker set of \mathbb{T} , then $\langle K \rangle$ is not polishable.*

Unaware of this result of Biró, Gabrielyan generalized it as follows.

Theorem 2.1.7 ([52, Theorem 2]). *If K is an uncountable Kronecker set of a compact metrizable group, then $\langle K \rangle$ is not polishable.*

2.1.2 The metric ρ_v

Theorem 2.1.8 states that $\mathcal{Char}(X) \subseteq \mathfrak{Pol}(X)$ for a compact metrizable group X . The first step toward this fact was made by Biró in [16], where using a metric from [1], he proved that $\mathcal{Char}(\mathbb{T}) \subseteq \mathfrak{Pol}(\mathbb{T})$. In [52], this fact was extended to all compact metrizable abelian group.

Theorem 2.1.8 ([52, Theorem 1]). *Let X be a compact metrizable abelian group, then $s_v(X)$ is polishable.*

Then, Gabrielyan and Dikranjan proved that the following metric on X give rises to a finer Polish topology when restricted on $s_{\mathbf{v}}(X)$.

Definition 2.1.9 ([33]). Let X be a compact metrizable abelian group, δ be a compatible metric on X and $\mathbf{v} = (v_n)$ be a sequence of characters of X . Let $x, y \in X$ and

$$\rho_{\mathbf{v}}(x, y) = \sup_{n \in \mathbb{N}} \{\delta(x, y), d(v_n(x), v_n(y))\}.$$

Notation 2.1.10. Let $\tau_{\mathbf{v}}$ denote the topology on X generated by $\rho_{\mathbf{v}}$ and let $\mathfrak{p}_{\mathbf{v}}$ be the unique Polish topology on $s_{\mathbf{v}}(X)$, i. e. $\mathfrak{p}_{\mathbf{v}} = \mathfrak{p}_{s_{\mathbf{v}}(X)}$.

With this notation, in [33], the authors proved that

$$\mathfrak{p}_{\mathbf{v}} = \tau_{\mathbf{v}} \upharpoonright_{s_{\mathbf{v}}(X)}.$$

As an immediate corollary of Theorems 2.1.8 and 2.1.7 one has the following one.

Corollary 2.1.11. *If K is an uncountable Kronecker set of a compact metrizable group, then $\langle K \rangle$ is a σ -compact non characterized subgroup.*

One may ask if $\mathfrak{Pol}(X) \subseteq \mathcal{C}\text{har}(X)$. Unfortunately, this is not true. Indeed, Hjorth, answering in [60] a question of Farah and Solecki posed in [48], proved that there exist polishable subgroups of unbounded Borel complexity (see Section 3.1).

The assignments $\mathbf{v} \mapsto s_{\mathbf{v}}(X)$ and $\mathbf{v} \mapsto \tau_{\mathbf{v}}$ give rise to two natural equivalence relations between sequences of characters of X .

Definition 2.1.12. For two sequences of characters \mathbf{u} and \mathbf{v} of a compact metrizable abelian group X let write

- $\mathbf{u} \sim \mathbf{v}$, if $s_{\mathbf{u}}(X) = s_{\mathbf{v}}(X)$
- $\mathbf{u} \approx \mathbf{v}$, if $\tau_{\mathbf{u}} = \tau_{\mathbf{v}}$.

As mentioned in Remark 2.1.2, the Polish topology witnessing the polishability of characterized subgroup is unique, hence $\mathfrak{p}_u = \mathfrak{p}_v$ whenever $u \sim v$. That is $u \sim v$ always implies $\tau_u \upharpoonright_{s_u(X)} = \tau_v \upharpoonright_{s_v(X)}$, but we do not know if $u \sim v$ implies $u \approx v$ in general. In the next chapter, we prove that this happens whenever $s_v(X)$ is \mathcal{F}_σ moreover, we prove also that (X, τ_v) is Polish only in some trivial cases, although it is always completely metrizable.

2.1.3 Local quasi-convexity

The polishability is a necessary condition, but as reported in the previous subsection not a sufficient one. So one can reinforce the polishability condition. The following theorem, due to Gabrielyan, makes a step in this direction.

Theorem 2.1.13 ([52]). *If H is characterized subgroup of a compact metrizable abelian group, then \mathfrak{p}_H is also locally quasi-convex.*

Notation 2.1.14. Let $\mathfrak{Pol}_{lqc}(X)$ denote the class of all polishable subgroups H of a Polish group X such that \mathfrak{p}_H is also locally quasi-convex.

Clearly, for a compact metrizable abelian group X , by Theorem 2.1.13 one has that $\mathcal{Char}(X) \subseteq \mathfrak{Pol}_{lqc}(X)$. One may ask if $\mathfrak{Pol}_{lqc}(X) \subseteq \mathcal{Char}(X)$. Also this inclusion fails, as showed by Gabrielyan.

Example 2.1.15 ([51, 52]). Let H be the following subgroup of $\mathbb{T}^{\mathbb{N}}$.

$$H = \left\{ (x_n) \in \mathbb{T}^{\mathbb{N}} \mid \sum_{n=1}^{\infty} \|x_n\| < \infty \right\}.$$

Gabrielyan proved that H is not a characterized subgroup of $\mathbb{T}^{\mathbb{N}}$, but is a σ -compact, polishable subgroup of $\mathbb{T}^{\mathbb{N}}$ such that (H, \mathfrak{p}_H) is *reflexive*, i. e. $(H, \mathfrak{p}_H) \cong \widehat{\widehat{(H, \mathfrak{p}_H)}}$, hence by Remark A.3.9 locally quasi-convex.

2.2 FINER LOCALLY COMPACT POLISH GROUP TOPOLOGIES

2.2.1 The classes $\mathfrak{Pol}_{lc}(X)$ and \mathcal{PL}

A natural class of groups having both properties, being Polish and locally quasi convex, is the class of second countable locally compact abelian groups. Therefore, it makes sense to consider the subfamily $\mathfrak{Pol}_{lc}(X)$ of $\mathfrak{Pol}(X)$, consisting of those subgroups $H \in \mathfrak{Pol}(X)$ whose Polish topology \mathfrak{p}_H is locally compact. In this connection, Gabrielyan posed the following problem.

Problem 2.2.1 ([53, Problem 2.13]). Let Y be a second countable locally compact abelian and $\varphi : Y \rightarrow X$ be a continuous injective homomorphism into a compact metrizable abelian group X such that $\varphi(Y)$ is dense. Is $\varphi(Y)$ a characterized subgroup of X ?

In [76], Negro proved that the answer is affirmative, providing in this way the first *sufficient condition for a subgroup of a compact metrizable group to be characterized*. As an easy consequence of Lemma A.3.5, one has that requiring $\varphi(Y)$ to be dense is not restrictive. Therefore, one can obtain the following theorem.

Theorem 2.2.2 ([76, Theorem 1.7]). *If Y is a second countable locally compact abelian group and $\varphi : Y \rightarrow X$ is a continuous injective homomorphism into a compact metrizable abelian group X , then $\varphi(Y)$ is a characterized subgroup of X .*

Remark 2.2.3. Theorem 2.2.2 implies that $\mathfrak{Pol}_{lc}(X) \subseteq \mathcal{Char}(X)$ for every compact metrizable abelian group X . Indeed, if $H \in \mathfrak{Pol}_{lc}(X)$ and Y denotes the group H equipped with the finer locally compact group topology, then the inclusion map $i : Y \hookrightarrow X$ is a continuous injective homomorphism, so $i(Y)$ is a characterized subgroup of X by Theorem 2.2.2.

Therefore, the following inclusions hold for a compact metrizable abelian group X .

$$\mathfrak{Pol}_{lc}(X) \subseteq \mathcal{Char}(X) \subsetneq \mathfrak{Pol}_{lqc}(X) \subseteq \mathfrak{Pol}(X).$$

In this section, one can find a structure theorem (see Theorem 2.2.17) for the subgroups belonging to $\mathfrak{Pol}_{lc}(X)$, where X is a compact metrizable abelian group. More precisely, the structure theorem proved in this section concerns the more general class \mathcal{PL} , of all topological abelian group that admits a finer locally compact Polish group topology. Proposition 2.2.5 clarifies the connection between the two classes \mathcal{PL} and $\mathfrak{Pol}_{lc}(X)$, where X is a compact metrizable abelian group. It is easy to see that every countable group belongs to \mathcal{PL} and

$$\bigcup \{ \mathfrak{Pol}_{lc}(X) \mid X \text{ is compact and metrizable} \} \subseteq \mathcal{PL}. \quad (2.2.1)$$

but the inverse inclusion does not hold in general, as proved in the following remark.

Remark 2.2.4. If $H \in \mathcal{PL}$, then H is separable and σ -compact, i. e. respectively admits a countable dense subset and H is the union of countably many compact subsets. But H need not to be metrizable, e. g. H can be an arbitrary countable group, that is obviously in \mathcal{PL} .

If one considers only subgroups of compact metrizable abelian groups, i. e. H is a precompact metrizable group (Definition A.3.18), then (2.2.1) becomes an equality.

Proposition 2.2.5. *Let H be a precompact metrizable abelian group. Then $H \in \mathcal{PL}$ if and only if $H \in \mathfrak{Pol}_{lc}(\tilde{H})$, where \tilde{H} is the (compact) completion of H .*

Proof. If $H \in \mathfrak{Pol}_{lc}(\tilde{H})$, then by (2.2.1) $H \in \mathcal{PL}$. Conversely let $H \in \mathcal{PL}$, then H admits a locally compact Polish group topology finer than $\tau \upharpoonright_H$, where τ is the compact metrizable (hence Polish) topology on \tilde{H} . Therefore, H is a polishable subgroup of \tilde{H} , whose unique Polish topology is also locally compact, i. e. $H \in \mathfrak{Pol}_{lc}(\tilde{H})$. \square

Remark 2.2.6. $\mathfrak{cc}_m(X) \subseteq \mathcal{PL}$ for every topological abelian group X . Indeed, if $(H, \tau) \in \mathfrak{cc}_m(X)$ and K is a compact metrizable subgroup of countable index, then $(\tau \upharpoonright_H)^*$, i. e. the topology on H having as a local base in 0_H the $\tau \upharpoonright_K$ -open neighbourhoods

in K (see §A.2.2 for more details), is a locally compact Polish group topology finer than τ .

Remark 2.2.6 implies that $\mathfrak{cc}(X) \subseteq \mathfrak{Pol}_{lc}(X)$, for a compact metrizable group X . By the fact that

$$\mathfrak{cc}(X) = \mathfrak{cc}_g(X) = \mathfrak{cc}_m(X),$$

whenever X is metrizable, this refines the inclusion 1.2.1

$$\mathfrak{cc}(X) \subseteq \mathfrak{Pol}_{lc}(X) \subseteq \mathfrak{Char}(X). \quad (2.2.2)$$

Sometimes, $\mathfrak{Pol}_{lc}(X) = \mathfrak{cc}(X)$, as in the case of \mathbb{J}_p , i. e. the group of p -adic integers.

Example 2.2.7.

$$\mathfrak{Pol}_{lc}(\mathbb{J}_p) = \mathfrak{cc}(\mathbb{J}_p) = \{\text{countable subgr.}\} \cup \{\text{clopen subgr.}\}.$$

This situation is precisely described in Proposition 2.2.22. In general, not all subgroups in $\mathfrak{Pol}_{lc}(X)$ are in $\mathfrak{cc}(X)$ as proved in Subsection 2.2.3

2.2.2 Pseudolines

Definition and basic properties

In order to describe \mathcal{PL} , one can define the following notion for a topological abelian group.

Definition 2.2.8. Let X be a topological abelian group. A subgroup R of X is called *pseudoline* if R is a continuous isomorphic image of \mathbb{R} . Denote by $\mathfrak{R}(X)$ the class of all pseudolines of X . One can call *line* a topologically isomorphic image of \mathbb{R} .

Remark 2.2.9. Obviously, $\mathfrak{R}(X) \subseteq \mathcal{PL}$ where X is an arbitrary topological abelian group.

Remark 2.2.10.

- (i) A pseudoline R in a topological abelian group X is never compact. Indeed, let $f : \mathbb{R} \rightarrow X$ be a continuous injective

homomorphism with $f(\mathbb{R}) = R$. Suppose by contradiction that R is compact. Since both \mathbb{R} and R are locally compact and \mathbb{R} is σ -compact, the open mapping theorem yields that f is a topological isomorphism, a contradiction.

- (ii) Obviously, the pseudolines are separable, so have weight at most \mathfrak{c} .
- (iii) While lines are always closed, being isomorphic to the complete group \mathbb{R} , a pseudoline need not to be closed. The locally compact (necessarily compact abelian and connected) groups X having a dense pseudoline are known also under the name of *solenoids*. It turns out that these properties, along with the property from item (ii) completely determine the solenoids (see Theorem A.2.8).
- (iv) Every closed pseudoline of a locally compact group is a line. Indeed, if $f(\mathbb{R}) = R$, then R is open by the open mapping theorem (being R closed in a locally compact group and hence itself locally compact).

Item (i) of the above remark is also a direct consequence of the following deeper fact established by Prodanov.

Theorem 2.2.11 ([80]). *Every Hausdorff group topology τ on \mathbb{R} coarser than the Euclidean one, admits a strictly coarser Hausdorff group topology $\tau' < \tau$ on \mathbb{R}*

Since every compact Hausdorff topology is minimal, no pseudoline can be compact.

Pseudolines vs compact subgroups

Let $\mathcal{K}(X)$ denote the set of all compact subgroups of a topological group X . Then $\bigcup \mathcal{K}(X)$ is a subgroup of X .

Lemma 2.2.12 ([37, Lemma 2.17]). *Let R be a pseudoline in a topological abelian group X .*

- (i) *If K is a compact subgroup of G , then $R \cap K \neq \{0\}$ if and only if the subgroup $R + K$ of X is compact.*

- (ii) *The closure of R is not compact (that is R is not contained in any compact subgroup of X), if and only if $R \cap \bigcup \mathcal{K}(X) = \{0\}$.*
- (iii) *If R is a line, then $R \cap \bigcup \mathcal{K}(X) = \{0\}$.*

Proof. (i) Suppose that $R \cap K \neq \{0\}$. The closed non-trivial subgroup $R \cap K$ of R is either infinite cyclic, or just R (in this case $R \leq K$). In the latter case we are done. In case the subgroup $R \cap K$ of R is infinite cyclic, K is a compact subgroup of the group $R + K$ such that $(R + K)/K \cong R/(R \cap K)$. Since $R \cap K$ is an infinite cyclic subgroup and R is a continuous injective image of \mathbb{R} , the group $R/(R \cap K)$ is compact and isomorphic to \mathbb{T} . This yields that $R + K$ is compact as well.

It remains to see that, if $R \cap K = \{0\}$ then $H = R + K$ is not compact. Assume for contradiction that H is compact. The restriction of the canonical projection $q : H \rightarrow H/K$ to R is a continuous isomorphism as $H/K \cong R$ algebraically. By the compactness of H we deduce that H/K is compact as well. As R is a continuous isomorphic image of \mathbb{R} , this will produce a coarser compact group topology on \mathbb{R} , in contradiction to Prodanov's Theorem 2.2.11.

(ii) Assume that R is not contained in any compact subgroup. Take a compact subgroup K of X . Then $R + K$ contains R , so cannot be compact. Then (i) implies that $R \cap K = \{0\}$. Since this holds true for all compact subgroups K of X and since $\bigcup \mathcal{K}(X)$ is the union of all compact subgroups of X , one can conclude that $R \cap \bigcup \mathcal{K}(X) = \{0\}$. The converse is obvious.

(iii) It follows immediately from (ii) and Remark 2.2.10 (i), as any line, being locally compact, is a closed subgroup. \square

Remark 2.2.13. In general, pseudolines need not to be precompact (as for example \mathbb{R} and every line). On the other hand, in case a pseudoline non-trivially meets some compact subgroup, then that pseudoline is precompact by item (ii) of the above lemma. Moreover, pseudolines of compact groups are never lines by item (iii) of the above lemma.

Example 2.2.14.

- (i) In $X = \mathbb{T} \times \mathbb{T}$ every pseudoline meets a torus. Indeed, every pseudoline is dense. Therefore, it has a compact closure, i. e. the whole group X .
- (ii) In $X = \mathbb{R} \times \mathbb{T}$ every pseudoline is a line. Hence, all pseudolines (lines) trivially meet compact subgroups.
- (iii) In $X = \mathbb{T} \times \mathbb{T} \times \mathbb{R}$ (that is not compact) there are pseudolines with compact closure and lines (that are closed but not compact).

Corollary 2.2.15. *A pseudoline cannot contain a non-trivial compact subgroup, so cannot be a CC subgroup.*

Proof. Assume that, a pseudoline R , in a topological group X , contains a non-trivial compact subgroup K of R . Then $K \cap R = K \neq \{0\}$, so Lemma 2.2.12 implies that $R = R + K$ is compact. This contradicts Remark 2.2.10 (i). \square

2.2.3 Description of the class \mathcal{PL}

In this section, we prove that the groups in \mathcal{PL} can be written as a direct sum of finitely many pseudolines and a CCM subgroup. The next remark is a direct consequence of the structure Theorem for the LCA groups reported in §A.2.2.

Remark 2.2.16. A locally compact Polish group X has the form $\mathbb{R}^n \times H$, where $n \in \mathbb{N}$ and H contains an open compact metrizable subgroup K of countable index, i. e. $H \in \mathfrak{cc}_m(X)$ that is equal to $\mathfrak{cc}_g(X)$ and $\mathfrak{cc}(X)$ since X is metrizable.

Theorem 2.2.17 ([37, Theorem 4.6]). *A topological abelian group X belongs to \mathcal{PL} if and only if there exist $H \in \mathfrak{cc}_m(X)$ and $R_1, \dots, R_n \in \mathfrak{R}(X)$ such that*

$$X = R_1 \oplus R_2 \oplus \dots \oplus R_n \oplus H.$$

Proof. Let τ be the finer locally compact Polish group topology on X . By Remark 2.2.16 $(X, \tau) \cong \mathbb{R}^n \times H$ where $H \in \mathfrak{cc}_m((X, \tau))$ that is a CCM subgroup of X as well. One can take as R_i the

images of the coordinate lines in \mathbb{R}^n . The converse holds by Remarks 2.2.6 and 2.2.9 and the fact that finite products of groups in \mathcal{PL} belong to \mathcal{PL} . \square

As corollaries of Theorem 2.2.17 one obtains the following.

Corollary 2.2.18. *Let X be a topological abelian group. The following implications hold.*

$$X \in \mathcal{PL} \text{ and } \mathfrak{R}(X) = \emptyset \implies X \in \mathfrak{cc}_m(X) \implies X \in \mathcal{PL}.$$

Note that the above implications cannot be inverted as shown in the next example.

Example 2.2.19. For the first implication one can take $X = \mathbb{T}^2$ and for the second one $X = \mathbb{R}$.

Corollary 2.2.20. *Every precompact metrizable abelian group in \mathcal{PL} is characterized.*

Proof. Let H be a precompact metrizable abelian group in \mathcal{PL} and let τ be the finer Polish topology on H . Clearly, \tilde{H} is a compact metrizable abelian group. Hence, it suffices to apply Theorem 2.2.2 when $X = \tilde{H}$, $i = \text{id}$ and $Y = (H, \tau)$. \square

Example 2.2.21. Gabrielyan proved that every pseudoline in \mathbb{T}^2 is characterized, i. e. $\mathfrak{R}(\mathbb{T}^2) \subseteq \mathfrak{Char}(\mathbb{T}^2)$. This can be obtained also from Corollary 2.2.20. Indeed, such a group is a precompact metrizable group in \mathcal{PL} and by Proposition 2.2.5 one can conclude. More generally, every metrizable solenoid has a characterized pseudoline.

Proposition 2.2.22 ([37, Proposition 4.12]). *For an infinite compact metrizable abelian group X the following are equivalent:*

- (i) $\mathfrak{R}(X) = \emptyset$;
- (ii) $\mathfrak{Pol}_{lc}(X) = \mathfrak{cc}(X) = \mathfrak{cc}_m(X) = \mathfrak{cc}_g(X)$.

Proof. Since, by Proposition 2.2.5, $H \in \mathfrak{Pol}_{lc}(X)$ if and only if $H \in \mathcal{PL}$, in case $\mathfrak{R}(X) = \emptyset$, Corollary 2.2.18 yields $\mathfrak{Pol}_{lc}(X) = \mathfrak{cc}(X)$.

Conversely, if $H \in \mathfrak{R}(X) \neq \emptyset$, then $H \in \mathfrak{Pol}_{lc}(X)$. By Corollary 2.2.15, H is not a CC subgroup. \square

As a consequence of Proposition 2.2.22 one obtains Example 2.2.7.

2.2.4 When $\mathfrak{Pol}_{lc}(X) \setminus \{X\}$ consist only of countable subgroups

From Corollary 2.2.18 and the fact that the only proper compact subgroups of \mathbb{T} are the finite ones, one can deduce the following theorem.

Theorem 2.2.23 ([37, Theorem 4.10]). *Let $H \leq \mathbb{T}$. Then $H \in \mathfrak{Pol}_{lc}(\mathbb{T})$ if and only if either $H = \mathbb{T}$ or H is countable.*

Proof. If $H \in \mathfrak{Pol}_{lc}(\mathbb{T})$, since $\mathfrak{R}(\mathbb{T}) = \emptyset$, we conclude by Proposition 2.2.22 that $H \in \mathfrak{cc}(\mathbb{T})$. Let K be the compact subgroup of H of countable index. If $K = \mathbb{T}$, then $H = \mathbb{T}$. Otherwise, K is a finite subgroup of \mathbb{T} , hence H is a countable subgroup of \mathbb{T} . \square

The next theorem states that, the only infinite compact metrizable abelian group having the property proved in Theorem 2.2.23 is the circle group.

Theorem 2.2.24 ([37, Theorem 4.11]). *Let X be an infinite compact metrizable abelian group. Then the following are equivalent:*

- (i) $X \cong \mathbb{T}$;
- (ii) whenever $H \in \mathfrak{Pol}_{lc}(X)$ and $H \neq X$, then H is countable.
- (iii) whenever H is a proper closed subgroup of X , then H is finite.

Proof. (i) \Rightarrow (ii). This is Theorem 2.2.23.

(ii) \Rightarrow (iii). One has to see that if H is an infinite closed subgroup of X , then $H = X$. Clearly, $H \in \mathfrak{Pol}_{lc}(X)$. Indeed, one can take as \mathfrak{p}_H the restriction $\tau \upharpoonright_H$ of the topology τ on X . Since $H \in \mathfrak{Pol}_{lc}(X)$, one can deduce from (ii) that either $H = X$, or H is countable. Being H an infinite compact group, H is uncountable. Hence, $H = X$.

(iii) \Rightarrow (ii). This is a classical duality argument. Suppose now that all proper closed subgroups of X are finite. Let $G = \widehat{X}$ the

discrete dual group of X . If B is a non-zero (closed) subgroup of G , then $B = H^\perp$ for some proper closed (hence, finite) subgroup H of X . Moreover $\widehat{H} = G/H^\perp$. Since H is finite, $G/B = \widehat{H} \cong H$ is finite. Therefore, every non-zero subgroup of G has finite index and hence $G = \mathbb{Z}$. This proves that $X \cong \mathbb{T}$.

□

BOREL COMPLEXITY OF CHARACTERIZED SUBGROUPS OF COMPACT METRIZABLE GROUPS

The first part of this Chapter is devoted to the introduction of the Borel complexity of characterized subgroups. In the second part the study of the \mathcal{F}_σ subgroups is treated with particular attention.

3.1 BASIC DEFINITIONS AND KNOWN FACTS

3.1.1 *Basic definitions and notations*

The Borel complexity of a Borel set is the “lowest” class in the Borel Hierarchy (see §A.1) where such a set belongs to. Recall that the first six classes of the Borel Hierarchy are the following.

Notation 3.1.1.

$$\begin{array}{l} \{\text{open sets}\} \subseteq \{\mathcal{G}_\delta \text{ sets}\} \subseteq \{\mathcal{G}_{\delta\sigma} \text{ sets}\} \\ \{\text{closed sets}\} \subseteq \{\mathcal{F}_\sigma \text{ sets}\} \subseteq \{\mathcal{F}_{\sigma\delta} \text{-sets}\} \end{array} \quad '$$

where \mathcal{G}_δ -sets are countable intersections of open sets, \mathcal{F}_σ sets are countable unions of closed sets, $\mathcal{G}_{\delta\sigma}$ sets are countable unions of \mathcal{G}_δ sets and $\mathcal{F}_{\sigma\delta}$ sets are countable intersections of \mathcal{F}_σ sets.

Note that if X is metrizable, then the diagonal inclusions hold too. More precisely,

$$\{\text{open sets}\} \subseteq \mathcal{F}_\sigma \subseteq \mathcal{G}_{\delta\sigma} \text{ and } \{\text{closed sets}\} \subseteq \mathcal{G}_\delta \subseteq \mathcal{F}_{\sigma\delta}.$$

Moreover, if X is an uncountable Polish space, then all the inclusions listed above are proper (see [63] for more details).

In this Thesis we treat, mainly, Borel subgroups instead of subsets, hence let $\mathfrak{G}_\delta(X)$, $\mathfrak{F}_\sigma(X)$, $\mathfrak{G}_{\delta\sigma}(X)$ and $\mathfrak{F}_{\sigma\delta}(X)$ denote respectively the class of \mathfrak{G}_δ subgroups, \mathfrak{F}_σ subgroups, $\mathfrak{G}_{\delta\sigma}$ subgroups and $\mathfrak{F}_{\sigma\delta}$ subgroups of a given topological group X (e. g. an $\mathfrak{F}_{\sigma\delta}$ subgroup means a subgroup that is an $\mathfrak{F}_{\sigma\delta}$ set as a subset, etc.).

3.1.2 *Known facts*

As proved in Lemma 1.1.7 (v) every characterized subgroup is an $\mathfrak{F}_{\sigma\delta}$ subgroup. That is, for every topological group X one has

$$\mathfrak{Char}(X) \subseteq \mathfrak{F}_{\sigma\delta}(X) \tag{3.1.1}$$

Farah and Solecki proved the following theorem.

Theorem 3.1.2 ([48, Theorem 2.1]). *For every uncountable Polish group there exists Borel subgroups of unbounded Borel complexity.*

As a consequence of the above Theorem and (3.1.1) one concludes that not all Borel subgroups are characterized. As recalled in §2.1.2, Hjorth improved Theorem 3.1.2 proving the following one.

Theorem 3.1.3 ([60]). *For every uncountable Polish group there exists Polishable subgroups of unbounded Borel complexity.*

As a consequence of the previous theorem, one has that neither all polishable subgroups are characterized.

Theorem 1.2.6 states that $\mathfrak{G}_\delta(X) \subseteq \mathfrak{Char}(X)$ whenever X is compact. If X is also infinite it is possible to prove that this inclusion is proper. Indeed, Lemma 1.2.17 proves that there exists $H \in \mathfrak{cc}_g(X)$ proper and dense. By Theorem A.2.14 H cannot be \mathfrak{G}_δ and it is characterized by (1.2.1).

Also the inclusion $\mathfrak{Char}(X) \subseteq \mathfrak{F}_{\sigma\delta}(X)$ is proper for an infinite compact abelian group X . Indeed, as stated in Theorem 2.1.7 every subgroup generated by an infinite Kronecker set is not Polishable and hence not characterized. This kind of subgroup

is obviously \mathcal{F}_σ , so neither $\mathfrak{F}_\sigma(X) \subseteq \mathcal{C}\text{har}(X)$ nor $\mathfrak{F}_{\sigma\delta}(X) \subseteq \mathcal{C}\text{har}(X)$.

Moreover, Theorem 1.2.19 states that if X has finite exponent $\mathcal{C}\text{har}(X) = \mathfrak{c}_t\mathfrak{c}_g(X)$ that are obviously \mathcal{F}_σ subgroups. The following theorem summarizes all inclusions and non-inclusions treated before.

Theorem 3.1.4 ([33]). *For every infinite compact abelian group X , the following inclusions hold:*

$$\mathfrak{G}_\delta(X) \subsetneq \mathcal{C}\text{har}(X) \subsetneq \mathfrak{F}_{\sigma\delta}(X) \quad \text{and} \quad \mathcal{F}_\sigma(X) \not\subseteq \mathcal{C}\text{har}(X). \quad (3.1.2)$$

If in addition X has finite exponent, then

$$\mathcal{C}\text{har}(X) \subseteq \mathcal{F}_\sigma(X). \quad (3.1.3)$$

Clearly, one has $\mathcal{C}\text{har}(X) \subsetneq \mathfrak{F}_\sigma(X)$ in (3.1.3), due to the second part of (3.1.2).

It was proved in [55], that the implication $\exp(X) < \infty \Rightarrow (3.1.3)$ in the final part of the above Theorem can be inverted.

Theorem 3.1.5 ([55]). *$\mathcal{C}\text{har}(X) \subseteq \mathfrak{F}_\sigma(X)$ for a compact abelian group X if and only if X has finite exponent.*

To prove Theorem 3.1.5 Gabrielyan produced a non \mathcal{F}_σ characterized subgroup for every compact abelian group X of infinite exponent.

Remark 3.1.6. More precisely, Gabrielyan produced a non \mathcal{F}_σ characterized subgroup for \mathbb{T} , \mathbb{J}_p and $\prod \mathbb{Z}(u_n)$ where $(u_n) \in \mathbb{N}^{\mathbb{N}}$ and $1 < u_0 < \dots < u_n < \dots$. Since every compact abelian group of infinite exponent has an open subgroup that is isomorphic either to \mathbb{T} or \mathbb{J}_p or $\prod \mathbb{Z}(u_n)$, this proves Theorem 3.1.5. A first step towards this direction was made in [2], where in different settings, Arbault proved that $t_{\mathbf{u}}(\mathbb{T})$ for $\mathbf{u} = (2^{2^n})$ is not \mathcal{F}_σ .

Remark 3.1.7. Moreover, in §2.2 the class $\mathfrak{Pol}_{lc}(X)$ is defined and described. It is easy to see that

$$\mathfrak{Pol}_{lc}(X) \subseteq \mathfrak{F}_\sigma(X),$$

for every Polish group X . Indeed, every locally compact Polish space is σ -compact (i.e. countable union of compact subset). That is (H, \mathfrak{p}_H) is σ -compact in the finer locally compact Polish group topology $\mathfrak{p}_H = \mathfrak{p}_v$, where $s_v(X) = H$ for a $v \in \widehat{X}^{\mathbb{N}}$ (since $\mathfrak{Pol}_{lc}(X) \subseteq \mathcal{Char}(X)$). Since $\mathfrak{p}_v = \tau_v \upharpoonright_{s_v(X)}$, one has that H is σ -compact also τ_v and hence in the coarser original topology of X .

Furthermore, every countable subgroup of an infinite compact metrizable group X belongs to $\mathfrak{Pol}_{lc}(X)$. Theorem 2.2.24 proves that $\mathfrak{Pol}_{lc}(X) = \{\text{countable subgroups}\} \cup \{X\}$ if and only if $X \cong \mathbb{T}$. Therefore, when $X \not\cong \mathbb{T}$ there are \mathcal{F}_σ subgroups other than the countable ones. As stated in (2.2.2) $\mathfrak{cc}(X) \subseteq \mathfrak{Pol}_{lc}(X)$ and if X has pseudolines then in $\mathfrak{cc}(X) \subsetneq \mathfrak{Pol}_{lc}(X)$.

Hence, the study of the \mathcal{F}_σ characterized subgroups is an interesting matter.

3.2 \mathcal{F}_σ SUBGROUPS

3.2.1 \mathcal{F}_σ -test topology

Notation 3.2.1. If x is an element of a metrizable group X , then let $B_r^d(x)$ denote the d -ball centred in x of radius r for a compatible metric d .

This subsection describes when a given characterized subgroup is \mathcal{F}_σ . In order to do this, the following *test topology* is useful.

Definition 3.2.2. Let v be a sequence of characters of a compact metrizable abelian group $X = (G, \tau)$ and τ_v^σ be the topology on G such that its filter of neighbourhoods of 0 in G is generated by

$$\left\{ W_n = \overline{B_{1/n}^{\rho_v}(0)} : n \in \mathbb{N} \right\},$$

where \overline{M} denotes the closure in (G, τ) of a subset M of X and ρ_v is as in Definition 2.1.9.

One can refer to τ_v^σ as the \mathcal{F}_σ -test topology with respect to the sequence \mathbf{v} (if no confusion is possible, one shall omit the index \mathbf{v}).

Remark 3.2.3. The \mathcal{F}_σ test topology is a metrizable group topology such that

$$\tau \subseteq \tau_v^\sigma \subseteq \tau_v.$$

The fact that $\{W_n \mid n \in \mathbb{N}\}$ of Definition 3.2.2 is a countable decreasing chain of symmetric sets such that $W_{2n} + W_{2n} \subseteq W_n$ proves that τ_v^σ is a metrizable group topology.

The inclusion $\tau \subseteq \tau_v^\sigma$ follows from the fact that for all $U \in \tau$ and every $x \in U$, there exists a closed ball in τ centred in x of radius $\frac{1}{n}$ for a certain $n \in \mathbb{N}$, denoted by $\overline{B_{1/n}^\delta(x)}$, where δ is a compatible metric with τ , such that $\overline{B_{1/n}^\delta(x)} = x + \overline{B_{1/n}^\delta(0)} \subseteq U$. Hence, by the fact that for every $x, y \in X$ one has $\rho_v(x, y) = \sup_{n \in \mathbb{N}} \{\delta(x, y), d(v_n(x), v_n(y))\} \geq \delta(x, y)$, the next inclusion holds $B_{1/n}^\tau(0) \supseteq B_{1/n}^{\rho_v}(0)$.

For the inclusion $\tau_v^\sigma \subseteq \tau_v$, note that for every $x \in W_n$, where W_n is as in Definition 3.2.2, there exists $m > n$ such that $x + B_{1/m}^{\rho_v}(0) \subseteq W_n$.

The topology τ_v^σ is called \mathcal{F}_σ test topology due to the following theorem.

Theorem 3.2.4 ([38, Theorem A]). *Let $X = (G, \tau)$ be a compact metrizable abelian group and $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$. Then*

$$s_v(X) \in \mathfrak{F}_\sigma(X) \iff s_v(X) \in \tau_v^\sigma.$$

Proof. Suppose that $s_v(X) = \bigcup_m F_m$ for some closed sets F_m of (G, τ) . As $s_v(X)$ is a subgroup, we can assume without loss of generality that $F_m = -F_m$ is symmetric for each $m \in \mathbb{N}$. Obviously, these F_m are ρ_v -closed as well. So, applying the Baire category Theorem to the Polish space $(s_v(X), \mathfrak{p}_v)$, one deduces that F_{m_0} has a non-empty interior for some m_0 . Since F_{m_0} is closed in (G, τ) , there exists some $x_0 \in F_{m_0}$ and $n \in \mathbb{N}$ such that

$$\overline{B_{1/n}^{\rho_v}(x_0)} = x_0 + \overline{B_{1/n}^{\rho_v}(0)} \subseteq F_{m_0} \subseteq s_v(X).$$

Hence $s_{\mathbf{v}}(X)$ is $\tau_{\mathbf{v}}^{\sigma}$ -open since it is a subgroup.

Conversely, let $s_{\mathbf{v}}(X) \in \tau_{\mathbf{v}}^{\sigma}$, then there exists $n \in \mathbb{N}$ such that $W_n \subseteq s_{\mathbf{v}}(X)$. Let $D \subseteq s_{\mathbf{v}}(X)$ be a $\mathfrak{p}_{\mathbf{v}}$ -dense countable subset. Then $D + W_n = s_{\mathbf{v}}(X)$ and hence $s_{\mathbf{v}}(X) = \bigcup_{d \in D} d + W_n$. Since W_n are τ -closed, then $d + W_n$ are τ -closed. \square

3.2.2 Countable subgroups vs \mathcal{F}_{σ} test topology

It follows immediately from the fact that $\tau_{\mathbf{v}} \upharpoonright_{s_{\mathbf{v}}(X)}$ is the finer Polish topology $\mathfrak{p}_{\mathbf{v}}$ on $s_{\mathbf{v}}(X)$, that the group $(s_{\mathbf{v}}(X), \mathfrak{p}_{\mathbf{v}})$ is discrete if and only if it is countable. Moreover, one can prove that $s_{\mathbf{v}}(X)$ is countable if and only if $\tau_{\mathbf{v}} = \tau_{\mathbf{v}}^{\sigma}$ is discrete in the whole X .

Theorem 3.2.5 ([35, Theorem 2.32]). *Let X be a metrizable compact abelian group and $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$. Then the following are equivalent:*

- (i) $s_{\mathbf{v}}(X)$ is countable;
- (ii) $|s_{\mathbf{v}}(X)| < \mathfrak{c}$;
- (iii) $\tau_{\mathbf{v}}$ is discrete;
- (iv) $\tau_{\mathbf{v}}^{\sigma}$ is discrete;
- (v) $\tau_{\mathbf{v}}^{\sigma} \upharpoonright_{s_{\mathbf{v}}(X)}$ is discrete;
- (vi) $\mathfrak{p}_{\mathbf{v}}$ is discrete.

Proof. (i) \Leftrightarrow (ii). This is a consequence of (3.1.1) (see Remark 1.1.18).

(i) \Rightarrow (iii). Indeed if $s_{\mathbf{v}}(X)$ is countable, then it is obviously \mathcal{F}_{σ} . Hence, $s_{\mathbf{v}}(X)$ is a $\tau_{\mathbf{v}}$ -open, by Theorem 3.2.4. Moreover, $\tau_{\mathbf{v}}$ is discrete when restricted to $s_{\mathbf{v}}(X)$ since the topology that witnesses the polishability of $s_{\mathbf{v}}(X)$ is unique. Thus, $s_{\mathbf{v}}(X)$ is both $\tau_{\mathbf{v}}$ -open and $\tau_{\mathbf{v}}$ -discrete. Consequently, $\tau_{\mathbf{v}}$ is discrete.

(iii) \Rightarrow (iv). This follows from the definition of $\tau_{\mathbf{v}}^{\sigma}$.

(iv) \Rightarrow (v). This is obvious.

(v) \Rightarrow (vi). It follows from the fact that $\mathfrak{p}_{\mathbf{v}} = \tau_{\mathbf{v}} \upharpoonright_{s_{\mathbf{v}}(X)}$ is finer than $\tau_{\mathbf{v}}^{\sigma} \upharpoonright_{s_{\mathbf{v}}(X)}$.

(vi) \Rightarrow (i). It follows from the fact that $(s_v(X), \mathfrak{p}_v)$ is separable. \square

3.2.3 Some consequences on the topology τ_v

Recall that the topology τ_v , where $v \in \widehat{X}^{\mathbb{N}}$ and X is a compact metrizable abelian group, is defined in the whole X and it is induced from the metric ρ_v defined in §2.1.2. Moreover, $\tau_v \upharpoonright_{s_v(X)} = \mathfrak{p}_v$. Furthermore, recall that for $u, v \in \widehat{X}^{\mathbb{N}}$, one has $u \sim v$ whenever $s_v(X) = s_u(X)$ and $u \approx v$ whenever $\tau_v = \tau_u$.

Remark 3.2.6. Theorem 3.2.5 yields that in general $u \approx v$ does not imply $u \sim v$. Indeed, every countable subgroup has the same associated topology, i. e. the discrete one.

The following corollary, in relation with Remark 3.2.6, proves that the topology τ_v , does not depend on the choice of v , whenever $s_v(X) \in \mathfrak{F}_\sigma(X)$.

Corollary 3.2.7 ([38, Corollary A1]). *Let $s_v(X) \in \mathfrak{F}_\sigma(X)$, where X is a compact metrizable abelian group and $v \in \widehat{X}^{\mathbb{N}}$. If $u \in \widehat{X}^{\mathbb{N}}$ such that $u \sim v$, then $u \approx v$.*

Proof. Let $H = s_v(X) = s_u(X)$. Recall that τ_v and τ_u coincide when restricted to H , since the topology witnessing the polishability of H is unique. Moreover, as $H \in \mathfrak{F}_\sigma(X)$, H is τ_v^σ -open and hence also τ_v -open. Analogously, H is τ_u -open. Hence, τ_v and τ_u coincide on a subgroup that is open in both topologies. Therefore, they coincide on the whole group X . \square

As previously, recalled $\tau_v \upharpoonright_{s_v(X)}$ is Polish. The following corollary answers question [35, Question 4.6], proving that τ_v is Polish only in some trivial cases.

Corollary 3.2.8 ([38, Corollary A2]). *If $X = (G, \tau)$ is a compact metrizable abelian group and $v \in \widehat{X}^{\mathbb{N}}$, then the following are equivalent:*

- (i) (G, τ_v) is Polish;
- (ii) $\tau = \tau_v$;

(iii) $s_{\mathbf{v}}(X)$ is τ -open;

(iv) \mathbf{v} has no faithfully indexed subsequences and every character that appears infinitely many times is torsion.

Proof. (i) \Leftrightarrow (ii) Since τ is Polish, by uniqueness of \mathfrak{p}_X , one has that $\tau = \tau_{\mathbf{v}}$.

(ii) \Rightarrow (iii) If $\tau = \tau_{\mathbf{v}}$, then $\mathfrak{p}_{\mathbf{v}} = \tau_{\mathbf{v}} \upharpoonright_{s_{\mathbf{v}}(X)} = \tau \upharpoonright_{s_{\mathbf{v}}(X)}$. By Lavrentieff Theorem (see Theorem A.1.8) $s_{\mathbf{v}}(X) \in \mathfrak{G}_{\delta}(X)$. Moreover, by Corollary A.2.15

$$\mathfrak{G}_{\delta}(X) = \{\text{closed subgroups of } X\}.$$

Therefore, $s_{\mathbf{v}}(X) \in \mathfrak{F}_{\sigma}(X)$, by Theorem 3.2.4 $s_{\mathbf{v}}(X)$ is $\tau_{\mathbf{v}}^{\sigma}$ -open and hence also τ -open.

(iii) \Rightarrow (iv) If $H = s_{\mathbf{v}}(X)$ is τ -open, then H has finite index. Thus, if μ is the normalized Haar measure on X , then $\mu(H) > 0$ and hence by Lemma 1.1.17 \mathbf{v} has no faithfully indexed subsequences. Let ν_n be a character that appears infinitely many times in \mathbf{v} . Since $\nu_n(H) = 0$, the following character of X/H is well defined

$$\tilde{\nu}_n : x + H \mapsto \nu_n(x).$$

Since X/H is finite, then $\tilde{\nu}_n(X/H) = \nu_n(X) \leq \mathbb{T}$ is finite. If $m = \exp(\nu_n(X))$, then $m\nu_n(x) = 0$ for all $x \in X$. Therefore, $m\nu_n = 0_{\hat{X}}$, i. e. ν_n is torsion.

(iv) \Rightarrow (ii) Let $\Gamma_{\mathbf{v}}^{\infty}$ be the finite set of all characters that appear infinitely many times in \mathbf{v} , i. e. the following

$$\Gamma_{\mathbf{v}}^{\infty} = \{\nu_n \mid n \in \mathbb{N} \text{ and } (\exists^{\infty} m\nu_n = \nu_m)\}.$$

By Remark 1.1.6(ii) one has that $s_{\mathbf{v}}(X) = \mathbf{n}_{\Gamma_{\mathbf{v}}^{\infty}}(X)$. Since each $\nu_n \in \Gamma_{\mathbf{v}}^{\infty}$ is torsion, one has that $\exp(\nu_n(X))$ is finite and hence $X/\ker \nu_n \cong \nu_n(X) \leq \mathbb{T}$ is finite. Therefore, since $\Gamma_{\mathbf{v}}^{\infty}$ is finite $\mathbf{n}_{\Gamma_{\mathbf{v}}^{\infty}}(X)$ has finite index in X . Hence $\mathbf{n}_{\Gamma_{\mathbf{v}}^{\infty}}(X) = s_{\mathbf{v}}(X)$ is a τ -clopen subgroup and hence also a $\tau_{\mathbf{v}}$ open. Therefore, $\tau = \tau_{\mathbf{v}}$ (since the topologies have a common open subgroup of finite index). \square

Part II

THE CIRCLE GROUP

Characterized subgroups were introduced for the first time in the circle group. Moreover, characterized subgroups are related also to Diophantine approximation (see [13, 15, 18, 45, 69, 66]), Dynamical System and Ergodic Theory (see [79] and [87]). In this part a theorem from [36] appears, where a complete description of $t_{\mathbf{u}}(\mathbb{T})$ is given, when \mathbf{u} is such that $u_n | u_{n+1}$. A panoramic on the main results obtained in the circle group is also given.

GENERAL AND RECURSIVE SEQUENCES

4.1 BASIC PROPERTIES AND NOTATIONS

4.1.1 Some basic facts

Lemma 4.1.1 ([10, Lemma2.1]). *If $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$ and $d \in \mathbb{N}$, then $\pi\left(\frac{1}{d}\right) \in \mathfrak{t}_{\mathbf{u}}(\mathbb{T})$ if and only if $d|u_n$ eventually.*

Notation 4.1.2. For $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$ and $p \in \mathbb{P}$, let v_p be the additive p -adic valuation and

$$n_p(\mathbf{u}) := \liminf_{n \rightarrow \infty} v_p(u_n) \in \mathbb{N} \cup \{\infty\}.$$

Since, the torsion subgroup of the torus $\mathfrak{t}(\mathbb{T}) = \mathbb{Q}/\mathbb{Z} = \bigoplus_{p \in \mathbb{P}} \mathbb{Z}(p^\infty)$, the following proposition holds.

Proposition 4.1.3 ([10, Proposition 2.3]). *For $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$ one has that*

$$\mathfrak{t}(\mathfrak{t}_{\mathbf{u}}(\mathbb{T})) = \bigoplus_{p \in \mathbb{P}} \mathbb{Z}(p^{n_p(\mathbf{u})}).$$

Notation 4.1.4. If $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^{\mathbb{N}}$, then one can write $\mathbf{u} \Delta \mathbf{v}$ instead of $\Gamma_{\mathbf{u}} \Delta \Gamma_{\mathbf{v}}$, where Δ is the symmetric difference and $\Gamma_{\mathbf{u}}$ and $\Gamma_{\mathbf{v}}$ are the supports (see Notation 1.1.3) respectively of \mathbf{u} and \mathbf{v} . One can write also $|\mathbf{u}|$ to denote the sequence of the absolute value of \mathbf{u} i. e. the sequence $(|u_n|)$.

Note that the order of the sequence is not relevant and in general if $\mathbf{v} = (|u_n|)$, then $\mathbf{u} \sim \mathbf{v}$ (i. e. $\mathfrak{t}_{\mathbf{u}}(\mathbb{T}) = \mathfrak{t}_{\mathbf{v}}(\mathbb{T})$). Therefore, in the case of integer sequences one may study only the sequences of natural numbers.

Proposition 4.1.5 ([7, Proposition 2.5]). *If $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^{\mathbb{N}}$ have no constant subsequences, then the following hold*

- (i) if $|\mathbf{u} \Delta \mathbf{v}| < \aleph_0$, then $\mathbf{u} \sim \mathbf{v}$;
- (ii) $\mathbf{u} \sim \mathbf{w}$, whenever $\mathbf{w} = |\mathbf{u}|$;
- (iii) there exists a strictly increasing sequence of positive integers \mathbf{w} such that $\mathbf{u} \sim \mathbf{w}$.

Proof. Item (i) is obvious since there exists $l \in \mathbb{N}$ such that $\{u_n : n \geq l\} = \{v_n : n \geq l\}$.

Item (ii) follows from the fact that $0 \in \mathbb{T}$ has a local base of symmetric sets.

To prove item (iii), take \mathbf{w} as an opportune enumeration of the set $\{|u_n| : u_n \neq \pm u_m \forall m, n \in \mathbb{N}\}$. Clearly since \mathbf{u} has no constant subsequences $|\mathbf{w} \Delta |\mathbf{u}|| < \aleph_0$ and hence by item (i) and (ii) $t_{\mathbf{u}}(\mathbb{T}) = t_{\mathbf{w}}(\mathbb{T})$. \square

Remark 4.1.6. Note that the implication in item (i) cannot be inverted. Indeed, let $\mathbf{u} = (p^n)$ and $\mathbf{v} = (p^{2^n})$, one can prove that $t_{\mathbf{u}}(\mathbb{T}) = t_{\mathbf{v}}(\mathbb{T})$ while $|\mathbf{u} \Delta \mathbf{v}| = \aleph_0$.

The next proposition is a direct consequence of Corollary 1.1.9, but for $X = \mathbb{T}$ one can prove it directly without using Lemma 1.1.8.

Proposition 4.1.7 ([7, Example 2.8]). *If $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$, then $t_{\mathbf{u}}(\mathbb{T}) = \mathbb{T}$ if and only if \mathbf{u} is eventually 0.*

Proof. Clearly, if \mathbf{u} is eventually 0, then $t_{\mathbf{u}}(\mathbb{T}) = \mathbb{T}$.

Conversely, suppose that \mathbf{u} is not eventually 0. One can distinguish two cases, namely \mathbf{u} contains a constant subsequence or not.

In the first case, if \mathbf{u} contains the constant subsequence $u_{k_n} = m$ for all n , then $t_{\mathbf{u}}(\mathbb{T}) \leq \mathbb{T}[m]$, hence $t_{\mathbf{u}}(\mathbb{T})$ is countable and therefore is not the whole \mathbb{T} .

In the second case, without loss of generality, one can suppose that \mathbf{u} is a strictly increasing sequence of positive integers, by Proposition 4.1.5. Therefore $u_n \rightarrow \infty$. Let $I_0 = [\varepsilon, 1 - \varepsilon]$ with $0 < \varepsilon < 1/2$ and for every $n > 0$ define $I_n \subseteq I_{n-1}$ such that

$\text{diam}(I_n) = \text{diam}(I_0)/u_n$. One can define such closed intervals as $I_n = [\varepsilon + \delta_n, 1 - \varepsilon - \delta_n]$ where

$$\begin{aligned} \delta_n &= \frac{\text{diam}(I_0) - \text{diam}(I_n)}{2} = \\ &= \frac{1 - 2\varepsilon - \frac{1-2\varepsilon}{u_n}}{2} = \frac{(1 - 2\varepsilon)(u_n - 1)}{2u_n}. \end{aligned}$$

Let $x \in \bigcap I_n$, clearly $u_n x \in I_0$ for all $n \in \mathbb{N}$ hence $\|u_n x\| \geq \varepsilon$. Therefore, $x \notin t_u(\mathbb{T})$. \square

4.1.2 The Eggleston Theorem

Notation 4.1.8. Let $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$. Denote by $\mathbf{q}^{\mathbf{u}} = \mathbf{q}$ the sequence $(q_n^{\mathbf{u}})_{n \in \mathbb{N}} = (q_n)_n$ where for every $n \in \mathbb{N}^*$ one has $q_n = u_n/u_{n-1}$ and $q_0 = u_0$. If no confusion is possible, one can omit the superscript \mathbf{u} .

Remark 4.1.9. With the above notation, $u_n = q_0 \cdots q_n$ for every $n \in \mathbb{N}$.

Definition 4.1.10. Call a sequence $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$ *\mathbf{q} -bounded* (*\mathbf{q} -divergent*) if $\mathbf{q}^{\mathbf{u}}$ is bounded ($q_n \rightarrow \infty$).

The following Theorem, first proved by Eggleston, underlines the relation between the behaviour of the sequence \mathbf{q} and the cardinality of $t_{\mathbf{u}}(\mathbb{T})$.

Theorem 4.1.11 ([45],[7, Theorem 3.1, Theorem 3.3]). *If $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$, then the following hold:*

(i) *If \mathbf{u} is \mathbf{q} -bounded, then $|t_{\mathbf{u}}(\mathbb{T})| = \aleph_0$;*

(ii) *if \mathbf{u} is \mathbf{q} -divergent, then $|t_{\mathbf{u}}(\mathbb{T})| = \mathfrak{c}$.*

Notation 4.1.12. If $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$, then denote by $q_- = q_-^{\mathbf{u}}$ and by $q_+ = q_+^{\mathbf{u}}$ respectively $\liminf q_n$ and $\limsup q_n$. Moreover, denote by $q_s = q_s^{\mathbf{u}}$ and by $q_i = q_i^{\mathbf{u}}$ respectively $\sup\{q_n \mid n \in \mathbb{N}\}$ and $\inf\{q_n \mid n \in \mathbb{N}\}$.

Remark 4.1.13. Clearly, $q_+ \leq q_s$ and $q_i \leq q_-$.

The next proposition shows that the implications of Theorem 4.1.11 cannot be inverted.

Proposition 4.1.14 ([7, Remark 3.5]). *If \mathbf{u} is a strictly increasing sequence of integers, then, for every $m \in \mathbb{N}$, there exists a strictly increasing sequence of integers \mathbf{v} such that $t_{\mathbf{u}}(\mathbb{T}) = t_{\mathbf{v}}(\mathbb{T})$, $q_{-}^{\mathbf{v}} = m$ and $q_{+}^{\mathbf{v}} = +\infty$.*

Proof. Let (u_{k_n}) be a subsequence of \mathbf{u} such that $u_{k_{n+1}}/u_{k_n} \rightarrow +\infty$ and $u_{k_n}/u_n \rightarrow +\infty$. Define \mathbf{v} as follows:

$$v_n = \begin{cases} u_{k_{\frac{n+1}{2}}} & \text{if } n \text{ is odd;} \\ mu_{k_{\frac{n}{2}}} + u_{\frac{n}{2}} & \text{if } n \text{ is even.} \end{cases}$$

That is $v_{2n-1} = u_{k_n}$ and $v_{2n} = mu_{k_n} + u_n$. Clearly, $q_{-}^{\mathbf{v}} = m$ and $q_{+}^{\mathbf{u}} = +\infty$. Moreover, if $u_n x \rightarrow 0$, then $u_{k_n} \rightarrow 0$, hence, $v_{2n-1} \rightarrow 0$ and $v_{2n} \rightarrow 0$. Therefore, $v_n x \rightarrow 0$. Conversely, if $v_n \rightarrow 0$, then $v_{2n} - v_{2n-1} = u_n x \rightarrow 0$. \square

Hence, for every sequence \mathbf{q} -bounded or \mathbf{q} -divergent, one can construct a sequence that is neither \mathbf{q} -bounded nor \mathbf{q} -divergent that characterizes the same subgroup.

4.2 $\tau_{\mathbf{u}}$ FOR A SEQUENCE OF INTEGERS \mathbf{u}

Proposition 4.2.1 ([38, Proposition B]). *If \mathbf{u} is a strictly increasing sequence of positive integers with $q_s^{\mathbf{u}} < \infty$, then $\tau_{\mathbf{u}}$ is discrete. More precisely, $B_{\frac{1}{2q_s^{\mathbf{u}}}}^{\rho_{\mathbf{u}}}(0) = \{0\}$ in \mathbb{T} .*

Proof. Let $\varepsilon = \frac{1}{2q_s}$ and $x \in B_{\varepsilon}^{\rho_{\mathbf{u}}}(0)$. Therefore, the following holds

$$\|x\| < \varepsilon \text{ and } \|u_n x\| < \varepsilon \text{ for every } n \in \mathbb{N}. \quad (4.2.1)$$

Here, we prov that $x = 0$. To do this one will need the following claim. In particular one has that

Claim: Let \tilde{x} be the unique pre-image in $[0, 1)$ of an element x of \mathbb{T} via canonical projection (see §A.2.1). If $\tilde{x} < \varepsilon$, then $u_n \tilde{x} = \|u_n x\| < \varepsilon$, for every $n \in \mathbb{N}$.

To prove the claim one can argue by induction. For $n = 0$ one has that $u_0 \leq q_s$ hence $u_0 \tilde{x} \leq q_s \tilde{x} < q_s \varepsilon < \frac{1}{2}$. Hence, $u_0 \tilde{x} = \|u_0 x\| < \varepsilon$ (this holds since if $n \tilde{x} < \frac{1}{2}$, then $\|n x\| = n \tilde{x}$). Assume that $n > 0$ and $u_n \tilde{x} < \varepsilon$ holds true. Then $u_{n+1} x \leq q_s u_n x < q_s \varepsilon < \frac{1}{2}$ and hence $u_{n+1} \tilde{x} < \frac{1}{2}$. Therefore, (4.2.1) yields $\varepsilon > \|u_{n+1} x\| = u_{n+1} \tilde{x}$. This concludes the inductive argument and the proof of the claim.

According to (4.2.1), one has that either $0 \leq \tilde{x} < \varepsilon$ or $1 - \varepsilon < \tilde{x} < 1$. In case $0 \leq \tilde{x} < \varepsilon$, the Claim proves, for all $n \in \mathbb{N}$, that $u_n \tilde{x} < \varepsilon$ and hence $\tilde{x} < \frac{\varepsilon}{u_n}$. Since \mathbf{u} is strictly increasing, one has that $\tilde{x} = 0_{\mathbb{R}}$ and hence $x = 0_{\mathbb{T}}$. The second case, i. e. $1 - \varepsilon < \tilde{x} < 1$ cannot occur. Indeed, if $1 - \varepsilon < \tilde{x} < 1$, then $y = 1 - \tilde{x} > 0$ and $y < \varepsilon$. Moreover, by Claim, $\pi(y) \in B_{\varepsilon}^{\rho_{\mathbf{u}}}(0)$, by previous argument, $y = 0_{\mathbb{R}}$, a contradiction. □

4.3 INCLUSIONS BETWEEN CHARACTERIZED SUBGROUPS

This section proposes a results from [46] and [47], inspired by a result of Arbault, that describes when $t_{\mathbf{u}}(\mathbb{T}) \leq t_{\mathbf{v}}(\mathbb{T})$ in case $\mathbf{q}^{\mathbf{u}} \rightarrow \infty$ and a recent result from [11] that extends to \mathbb{R} the result of Eliaš.

Notation 4.3.1. Let $\mathbf{Z} = (z_{i,j})_{i,j}$ be a row-column-(countably) infinite real valued matrix, i. e. $z_{i,j} \in \mathbb{R}$ and $i, j \in \mathbb{N}$. Denote by $\mathbf{z}_i = (z_{i,j})_j$ the i -th row of \mathbf{Z} and denote by $\mathbf{z}^j = (z_{i,j})_i$ the j -th column. Recall that, for $\mathbf{x} = (x_i)_i \in \mathbb{R}^{\mathbb{N}}$ the 1-norm is $\|\mathbf{x}\|_1 = \sum_i |x_i|$ and the ∞ -norm is $\|\mathbf{x}\|_{\infty} = \sup_i \{|x_i|\}$.

4.3.1 Good expansion

Definition 4.3.2. Let $m \in \mathbb{Z}$ and $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$ be a strictly increasing sequence such that $u_0 = 1$. A sequence $\mathbf{z} \in \bigoplus_{\mathbb{N}} \mathbb{Z}$ is a *good expansion* of m with respect to \mathbf{u} if

$$m = \sum_{n \in \mathbb{N}} z_n u_n \text{ and } \forall n \in \mathbb{N}, \left| \sum_{j < n} z_j u_j \right| \leq \frac{u_n}{2}.$$

Theorem 4.3.3 (Existence of good expansions). *For every strictly increasing sequence of integers \mathbf{u} with $u_0 = 1$ and every $m \in \mathbb{Z}$, there exists a good expansion \mathbf{z} of m w.r.t. \mathbf{u} .*

Proof. One can find $n_0 \in \mathbb{N}$ such that $|m| \leq u_{n_0}/2$. Put $z_n = 0$ for all $n > n_0$.

For every $n \leq n_0$ the element z_n is defined recursively from n_0 to 0. Let $m_{n_0+1} = m$. For $n \leq n_0$, let $z_n = [m_{n+1}/u_n]$, that is the nearest integer¹ to m_{n+1}/u_n , put $m_n = m_{n+1} - z_n u_n$. Note that since $u_0 = 1$ then $m_0 = 0$ and hence

$$m_{n+1} = \sum_{j \leq n} z_j u_j, \tag{4.3.1}$$

then, for $n = n_0$, the first part of the definition of good expansion is satisfied. Let us prove the second part; for $n \leq n_0$ one has $|m_{n+1}/u_n - z_n| \leq 1/2$, hence by (4.3.1) $\left| \sum_{j < n} z_j u_j \right| = |m_n| = |m_{n+1} - z_n u_n| \leq u_n/2$ for every $n \leq n_0$. On the other hand if $n > n_0$ one has $\left| \sum_{j < n} z_j u_j \right| = |m| \leq u_{n_0}/2 < u_n/2$ since \mathbf{u} is strictly increasing.

□

Definition 4.3.4. Let $\mathbf{Z} = (z_{i,j})_{i,j \in \mathbb{N}}$ a row-finite infinite integer matrix, i. e. $\mathbf{z}_i \in \bigoplus_{\mathbb{N}} \mathbb{Z}$, for every row \mathbf{z}_i , where $i \in \mathbb{N}$. Let \mathbf{z}^j denote the j -th column of \mathbf{Z} . Let \mathbf{u} be a strictly increasing sequence of integers with $u_0 = 1$ and let $\mathbf{v} \in \mathbb{Z}^{\mathbb{N}}$. One can say that \mathbf{Z} is a *good expansion matrix* (briefly *g.e.m.*) of \mathbf{v} with respect to \mathbf{u} if, for all $i \in \mathbb{N}$, \mathbf{z}_i is a good expansion of v_i with respect to \mathbf{u} . In this case $\mathbf{v} = \mathbf{Z}\mathbf{u}$.

¹ note that in some cases the nearest integer is not unique; Elias proved that these are the only cases such that the good expansion is not unique.

4.3.2 *Theorem [47, Theorem 1.2]*

Definition 4.3.5. A row-finite infinite integer matrix \mathbf{Z} satisfies the Arbault-Eliaš conditions (briefly A-E conditions) if the following conditions are satisfied.

- (1) for all $j \in \mathbb{N}$ there exists $k_j \in \mathbb{N}$ such that for all $i > k_j$, $z_{i,j} = 0$;
- (2) there exists $s \in \mathbb{N}$ such that for all $i \in \mathbb{N}$, $\sum_{j \in \mathbb{N}} |z_{i,j}| \leq s$.

These conditions can be stated also in the following way;

- (1*) \mathbf{Z} is a column-finite matrix;
- (2*) the sequence $(\|\mathbf{z}_i\|_1)_i$ is bounded.

Theorem 4.3.6 ([46, 47]). Let $\mathbf{u}, \mathbf{v} \in \mathbb{N}^{\mathbb{N}}$. If \mathbf{u} is \mathbf{q} -divergent, $u_0 = 1$, \mathbf{v} is strictly increasing and \mathbf{Z} is a g.e.m. of \mathbf{v} w.r.t. \mathbf{u} , then $t_{\mathbf{u}}(\mathbb{T}) \leq t_{\mathbf{v}}(\mathbb{T})$ if and only if \mathbf{Z} satisfies the A-E conditions.

Note that, the previous theorem, for $\mathbf{u} = 2^{2^n}$, was proved by Arbault in [2]. The sufficiency of the A-E conditions is easy and it can be proved as follows.

Proof. Let \mathbf{Z} be a g.e.m. of \mathbf{v} w.r.t. \mathbf{u} satisfying the A-E conditions and $x \in t_{\mathbf{u}}(\mathbb{T})$. Let $s \in \mathbb{N}$ satisfying the second A-E condition. For any $\varepsilon > 0$ there exists n_ε such that for every $n > n_\varepsilon$, $\|\mathbf{u}_n x\| \leq \frac{\varepsilon}{s}$. Let $k_\varepsilon \in \mathbb{N}$ such that, for all $n \leq n_\varepsilon$ and all $i > k_\varepsilon$, $z_{i,n} = 0$. Therefore, for every $i > k_\varepsilon$ one has

$$\|\mathbf{v}_i x\| \leq \sum_{n \in \mathbb{N}} |z_{i,n}| \|\mathbf{u}_n x\| \leq \frac{\varepsilon}{s} \sum_{n \in \mathbb{N}} |z_{i,n}| \leq \varepsilon.$$

Therefore, $x \in t_{\mathbf{v}}(\mathbb{T})$. □

4.3.3 *Extension of inclusion from \mathbb{T} to \mathbb{R}*

Inspired by Theorem 4.3.6 and using techniques from [86], Barbieri, Giordano Bruno and Weber generalized the result of Elias.

More precisely, they prove the following theorem for a sequence $\mathbf{u} \in (\mathbb{R} \setminus \{0\})^{\mathbb{N}}$ such that $|q_n| \rightarrow \infty$ and a row-finite integer infinite matrix \mathbf{Z} that is not necessarily a g.e.m..

Theorem 4.3.7 ([11, Theorem 3.10]). *Let $\mathbf{u} \in (\mathbb{R} \setminus \{0\})^{\mathbb{N}}$ be such that $|q_n| \rightarrow \infty$, let \mathbf{Z} be a row-finite integer infinite matrix and $\mathbf{v} = \mathbf{Z}\mathbf{u}$. Assume that:*

$$(1) \quad \forall i, n \in \mathbb{N}, \left(\sum_{j \leq n} z_{i,j} = 0 \rightarrow \forall j \leq n, z_{i,j} = 0 \right);$$

$$(2) \quad \mathbf{z}^j \text{ is bounded for every } j \in \mathbb{N};$$

$$(3) \quad \limsup_j \frac{\|\mathbf{z}^j\|_{\infty}}{q_{j+1}^{\mathbf{u}}} < 1.$$

Then $s_{\mathbf{u}}(\mathbb{R}) \subseteq s_{\mathbf{v}}(\mathbb{R})$ if and only if \mathbf{Z} satisfies the A-E conditions.

Note that in case $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$, $s_{\mathbf{u}}(\mathbb{R}) = \pi^{-1}(t_{\mathbf{u}}(\mathbb{T}))$ and $t_{\mathbf{u}}(\mathbb{T}) \leq t_{\mathbf{v}}(\mathbb{T})$ if and only if $s_{\mathbf{u}}(\mathbb{R}) \subseteq s_{\mathbf{v}}(\mathbb{R})$.

Corollary 4.3.8 ([11, Corollary 3.11]). *Let $\mathbf{u} \in (\mathbb{R} \setminus \{0\})^{\mathbb{N}}$ be such that $|q_n| \rightarrow \infty$, let \mathbf{Z} be a row-finite integer infinite matrix and $\mathbf{v} = \mathbf{Z}\mathbf{u}$. Let $0 < \tau < 1$ be such that $\left| \sum_{j \leq n} z_{i,j} u_j \right| \leq \tau |u_{n+1}|$ for every $n, i \in \mathbb{N}$. Then $t_{\mathbf{u}}(\mathbb{T}) \leq t_{\mathbf{v}}(\mathbb{T})$ if and only if \mathbf{Z} satisfies the A-E conditions.*

Remark 4.3.9. Note that for a \mathbf{q} -divergent integer sequence \mathbf{u} and $\tau = \frac{1}{2}$, Corollary 4.3.8 is Theorem 4.3.6. Indeed, in this case \mathbf{Z} is a g.e.m. satisfying the A-E conditions.

4.4 CYCLIC SUBGROUPS

In this section we state a relevant result from [17] due to V.T. Sós. The author found a characterizing sequence for every infinite cyclic group, i.e. those ones generated by an irrational element of \mathbb{T} . This characterizing sequence is strictly related to the sequence of convergents of the continued fraction expansion of the generating element.

Definition 4.4.1. A *continued fraction* is an expression of the following form

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\ddots + \frac{1}{a_n}}}}}} = [a_0; a_1, \dots, a_n]$$

and *infinite continued fraction* an expression of the previous form where the sequence (a_n) is infinite, that one can denote by $[a_0; a_1, \dots, a_n, \dots]$.

It is well known that, $\rho \in \mathbb{R}$ is rational if and only if its continued fraction expansion is finite. Moreover, one can define the sequence of convergents for a real number ρ as follows.

Definition 4.4.2. Let $[a_0; a_1, \dots, a_n, \dots]$ be the continued fraction expansion of a real number ρ (in case ρ is rational consider the sequence (a_n) finite). Denote by $\frac{r_n}{s_n}$ the following rational number $[a_0; a_1, \dots, a_n]$, where $\gcd(r_n, s_n) = 1$ and call it the n -th convergent of ρ .

Example 4.4.3 (The Golden Ratio). Let $\phi = \frac{\sqrt{5}+1}{2}$ be the celebrated golden ratio. It is well known that $\phi^2 - \phi - 1 = 0$, hence $\phi = \frac{1}{\phi-1}$, that implies $\phi = \frac{1}{\frac{1}{\phi-1}-1}$ and so on. Therefore, $\phi = [1; 1, \dots, 1, \dots]$. Thus, the sequence of convergents is $\left(\frac{r_n}{s_n}\right) = \left(\frac{f_{n+1}}{f_n}\right)$, where $f_0 = f_1 = 1$ and $f_{n+2} = f_{n+1} + f_n$ for every n , i. e. $\mathbf{f} = (f_n)$ is the Fibonacci sequence. Moreover, the fractional part of the golden ratio $\{\phi\} = \phi - 1 = [0; 1, 1, \dots, 1, \dots]$ has as convergents the following sequence $\frac{r_n}{s_n} = \frac{f_n}{f_{n+1}}$, where $\mathbf{f} = (f_n)$ is the Fibonacci sequence.

Theorem 4.4.4 ([69, Larcher]). Let $\alpha \in \mathbb{T} \setminus \mathbb{Q}/\mathbb{Z}$ and let $\tilde{\alpha}$ be its unique pre-image in $[0, 1)$ via canonical projection. If $\mathbf{s} = (s_n)$, the sequence of denominators of the convergents of $\tilde{\alpha}$, is bounded, then $\langle \alpha \rangle = \mathbf{t}_{\mathbf{s}}(\mathbb{T})$.

Remark 4.4.5 (Quadratic irrational). Recall that a quadratic irrational is an irrational element $\alpha \in \mathbb{R}$ such that $a\alpha^2 + b\alpha + c = 0$

for some $a, b, c \in \mathbb{Z}$. It is a well known fact, due to Lagrange, that α is a quadratic irrational if and only if α has an infinite periodic continued fraction expansion. Therefore, such elements satisfy the hypothesis of Larcher Theorem.

Clearly, an example of quadratic irrational is the golden ratio.

Example 4.4.6 (The golden ratio and the Fibonacci sequence). A direct consequence of the Larcher Theorem is that $\langle \pi(\phi) \rangle$, where ϕ is the golden ratio and π the canonical projection, is characterized by the Fibonacci sequence. Indeed, by Example 4.4.3 the sequence of the denominators of the convergents of ϕ is the Fibonacci sequence \mathbf{f} .

For the general case of an arbitrary $\alpha \in \mathbb{T} \setminus \mathbb{Q}/\mathbb{Z}$, Sós proved the following theorem.

Theorem 4.4.7 (Sós [17]). *Let $\alpha \in \mathbb{T} \setminus \mathbb{Q}/\mathbb{Z}$, let $\tilde{\alpha}$ be its unique pre-image in $[0, 1)$ via canonical projection and*

$$\tilde{\alpha} = [0; a_1, \dots, a_n, \dots].$$

If (s_n) is the sequence of denominators of the convergents of $\tilde{\alpha}$ and \mathbf{u} is the following sequence

$$\{ks_n : 1 \leq k \leq a_{n+1}\},$$

then $\langle \alpha \rangle = t_{\mathbf{u}}(\mathbb{T})$.

4.5 A DESCRIPTION FOR RECURSIVE SEQUENCES

It is easy to prove that, if $[a_0; a_1, \dots]$ is an infinite continued fraction and $\frac{r_n}{s_n}$ is its sequence of convergents, then for every $n \in \mathbb{N}$ the following hold

$$r_{n+2} = a_{n+2}r_{n+1} + r_n;$$

$$s_{n+2} = a_{n+2}s_{n+1} + s_n.$$

Hence, by Theorem 4.4.4 one has that certain cyclic group are characterized by a sequence \mathbf{s} satisfying a second order recurrence linear relation.

In this section, we report a description of subgroups characterized by sequences satisfying a recurrence linear relation of finite order, with particular attention to those of second order.

4.5.1 Recurrence relations of finite order

Let $k \in \mathbb{N}^*$, $u_0, \dots, u_{k-1} \in \mathbb{Z}$ and $b_{n,i} \in \mathbb{Z}$ for every $n \in \mathbb{N}^*$ and $i \in \{1, \dots, k\}$. Here, we consider sequences $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$ of the following form:

$$u_n = b_{n,1}u_{n-1} + b_{n,2}u_{n-2} + \dots + b_{n,k}u_{n-k}, \quad (4.5.1)$$

where $n \geq k$.

Notation 4.5.1. Denote by $\mathcal{Z}_{\text{rec}_k}$ the set of all the sequences of the form (4.5.1) and by $\mathcal{B}_i = \{b_{n,i} \mid n \in \mathbb{N}^*\}$, for every $i \in \{1, \dots, k\}$ and by $\mathcal{B} = \bigcup_{i=1}^k \mathcal{B}_i$.

The next structure theorem for $t_{\mathbf{u}}(\mathbb{T})$ considers only $\mathbf{u} \in \mathcal{Z}_{\text{rec}_k}$ when \mathcal{B} is bounded.

Theorem 4.5.2 ([10, Theorem 3.12]). *Let $\mathbf{u} \in \mathcal{Z}_{\text{rec}_k}$. If \mathcal{B} is bounded and $\mathcal{P}_{\mathcal{B}} = \{p \in \mathbb{P} \mid \exists^\infty(n, i) \in \mathbb{N}^2, p \mid b_{n,i}\}$, then the following hold:*

- (i) $t_{\mathbf{u}}(\mathbb{T}) = t(t_{\mathbf{u}}(\mathbb{T})) \oplus H$, where H is torsion free. If $\mathcal{P}_\infty = \{p \in \mathcal{P} \mid n_p(\mathbf{u}) = \infty\}$, then $\mathcal{P}_\infty \subseteq \mathcal{P}_{\mathcal{B}}$;
- (ii) if \mathcal{B}_i is bounded for every $i \in \{1, \dots, k\}$, then torsion free rank $r_0(H)$ of H is $r_0(H) = r_0(t(t_{\mathbf{u}}(\mathbb{T}))) < k$;
- (iii) if $b_{n,k} \neq 0$, for every $n \in \mathbb{N}$, then there exists $H_0 \leq H$ such that $H_0 \cong \mathbb{Z}^{r_0(H)}$ and $H/H_0 \cong \bigoplus_{p \in \mathcal{P}_{\mathcal{B}}} \mathbb{Z}(p^\infty)^{j_p}$ where $0 \leq j_p \leq r_0(H)$ for all $p \in \mathcal{P}_{\mathcal{B}}$.

4.5.2 Recurrence relations of second order

A sequence \mathbf{u} in $\mathcal{Z}_{\text{rec}_2}$ is a sequence satisfying a relation of the following form:

$$u_n = a_n u_{n-1} + b_n u_{n-2}; \quad (4.5.2)$$

where $u_0, u_1, a_n, b_n \in \mathbb{Z}$, for every $n \geq 2$. In that case the elements of sequence \mathbf{q}^u have the following form:

$$q_n = \frac{u_n}{u_{n-1}} = \frac{a_n u_{n-1} + b_n u_{n-2}}{u_{n-1}} = a_n + \frac{b_n}{q_{n-1}}. \quad (4.5.3)$$

Here, we consider $\mathbf{u} \in \mathcal{Z}_{\text{rec}_2}$ where $u_0, u_1, a_n \in \mathbb{N}^*$ and $b_n \in \mathbb{N}$, let $\mathcal{Z}_{\text{rec}_2}^+$ denote the set of all sequences like these. Moreover, let z_n denote, for every natural $n \geq 2$, the following rational number $\frac{b_2 \cdots b_n}{u_n}$. In this way one has that

$$z_n = \frac{1}{u_1} \prod_{i=2}^n \frac{b_i}{q_i} \quad (4.5.4)$$

In this settings, one can refine Theorem 4.5.2, for $k = 2$, in the following manner.

Theorem 4.5.3 ([10, Corollary 4.5]). *If $\mathbf{u} \in \mathcal{Z}_{\text{rec}_2}^+$, then $t_{\mathbf{u}}(\mathbb{T})$ is not torsion whenever $z_n \rightarrow 0$. If, moreover, the sequences $(a_n), (b_n)$ are bounded, then the following are equivalent:*

- (i) $t_{\mathbf{u}}(\mathbb{T})$ is not torsion;
- (ii) $r_0(t_{\mathbf{u}}(\mathbb{T})) = 1$;
- (iii) $z_n \rightarrow 0$.

Corollary 4.5.4 ([10, Corollary 4.8]). *If $\mathbf{u} \in \mathcal{Z}_{\text{rec}_2}^+$, then the following hold.*

- (i) *If $a_n \geq b_n > 0$ for every $n \geq 2$, then $t_{\mathbf{u}}(\mathbb{T})$ is not torsion.*
- (ii) *If $a_n < b_n$ for every $n \in \mathbb{N}, n \geq 2$ and (b_n) is bounded, then $t_{\mathbf{u}}(\mathbb{T})$ is torsion.*

Proof. (i). If \mathbf{u} is not \mathbf{q} -divergent, then $\sum_{n \geq 2} \left(\frac{a_n - b_n}{b_n} + \frac{1}{q_n} \right)$ diverges. One can prove that this is equivalent to the fact that $z_n \rightarrow 0$. Therefore, Theorem 4.5.3 entails that $t_{\mathbf{u}}(\mathbb{T})$ is not torsion. If \mathbf{u} is \mathbf{q} -divergent, then Theorem 4.1.11 implies that $|t_{\mathbf{u}}(\mathbb{T})| = c$ and in particular $t_{\mathbf{u}}(\mathbb{T})$ is not torsion.

(ii). One can prove that z_n does not converge to 0. Indeed, by (4.5.3) one has that $q_n \leq b_n - 1 + \frac{b_n}{q_{n-1}}$ and therefore, $\frac{b_n}{q_n} \geq \frac{(q_n+1)q_{n-1}}{(q_{n-1}+1)q_n} = Q_n$. By (4.5.4) $z_n \geq \frac{1}{u_1} \prod_{i=2}^n Q_i = \frac{1}{u_1} Q_2 \geq \frac{q_1}{u_1(q_1+1)}$. \square

Theorem 4.5.5 ([10, Theorem 4.10]). *Let $(a_n) \in \mathbb{N}_+^{\mathbb{N}}$ be bounded, $\mathbf{u} \in \mathbb{N}_+^{\mathbb{N}}$ satisfy $u_n = a_n u_{n-1} + u_{n-2}$, $g = \gcd(u_0, u_1)$, $\lambda_0, \lambda_1 \in \mathbb{N}$ be such that $g = |\lambda_0 u_0 + \lambda_1 u_1|$. If (r_n) satisfies $r_n = a_n r_{n-1} + r_{n-2}$, where $r_0 = \lambda_0$ and $r_1 = \lambda_1$, then the sequence $\frac{r_n}{u_n}$ converges to an irrational number $\alpha \in \mathbb{R}$ and $t_{\mathbf{u}}(\mathbb{T}) = \langle \pi(\alpha) \rangle \oplus \langle \pi\left(\frac{1}{g}\right) \rangle$, where π is the canonical projection of \mathbb{R} onto \mathbb{T} .*

Remark 4.5.6. If $\{a_n \mid n \in \mathbb{N}\}$ are the coefficient of an infinite continued fraction, then $u_0 = s_0 = 1$ and hence $\gcd(u_0, u_1) = 1$. Therefore, Theorem 4.5.5 implies Theorem 4.4.4

Theorem 4.5.7 ([10, Theorem 4.12]). *Let $\mathbf{u} \in \mathcal{Z}_{rec_2}^+$ be such that $a_n \geq b_n \in \mathbb{N}$ for every $n \in \mathbb{N}$. Then $t_{\mathbf{u}}(\mathbb{T}) = \mathfrak{c}$ if and only if \mathbf{u} is not \mathbf{q} -bounded.*

Let \mathbf{u} satisfy (4.5.2), where the sequence $a_n = a \in \mathbb{N}^*$ and $b_n = b \in \mathbb{N}^*$ for every $n \in \mathbb{N}$. In this case Corollary 4.5.4 entails the following proposition

Proposition 4.5.8. *Let $\mathbf{u} \in \mathbb{N}_+^{\mathbb{N}}$ be such that $u_n = a u_{n-1} + b u_{n-2}$, where $a, b \in \mathbb{N}^*$. Then $a \geq b$ if and only if $t_{\mathbf{u}}(\mathbb{T})$ is not torsion.*

In particular, if $a \geq b$, then $t_{\mathbf{u}}(\mathbb{T})$ is infinite. In [8], the authors proved when the torsion subgroup of $t_{\mathbf{u}}(\mathbb{T})$ is infinite.

Proposition 4.5.9 ([8, Theorem 3.9]). *Let $\mathbf{u} \in \mathbb{N}_+^{\mathbb{N}}$ be such that $u_n = a u_{n-1} + b u_{n-2}$, where $a, b \in \mathbb{N}^*$. Then $t(t_{\mathbf{u}}(\mathbb{T}))$ is infinite if and only if $\gcd(a, b) > 1$ or \mathbf{u} is a geometric progression, i. e. $u_n = u_0 \lambda^{n-1}$ with $\lambda \in \mathbb{N}^*$.*

From the previous propositions one can obtain the following corollary.

Corollary 4.5.10 ([10, Corollary 5.4]). *Let $\mathbf{u} \in \mathbb{N}_+^{\mathbb{N}}$ be such that $u_n = au_{n-1} + bu_{n-2}$, where $a, b \in \mathbb{N}^*$. Then $t_{\mathbf{u}}(\mathbb{T})$ is infinite if and only if either $a \geq b$ or $\gcd(a, b) > 1$ or \mathbf{u} is a geometric progression.*

ARITHMETIC SEQUENCES

5.1 A DESCRIPTION OF $t_{\mathbf{u}}(\mathbb{T})$ FOR A-SEQUENCES

Definition 5.1.1. A strictly increasing sequence of integers $\mathbf{u} = (u_n)$ is called *arithmetic* (or briefly, an *a-sequence*) if $u_n | u_{n+1}$ for every $n \in \mathbb{N}$.

If \mathbf{u} is an a-sequence, then $\mathbf{q}^{\mathbf{u}}$ is a sequence of integers.

5.1.1 The canonical \mathbf{u} -representation

Proposition 5.1.2 (Existence of the \mathbf{u} -representation [78]). *Let \mathbf{u} be an a-sequence, let $\mathbf{q} = (q_n)$ be its sequence of ratios and $\alpha \in [0, 1)$. Then there exists a unique sequence $\mathbf{c}^{\mathbf{u}} = \mathbf{c} = (c_n) \in \mathbb{N}^{\mathbb{N}}$ such that $0 \leq c_n < q_n$ for every $n \in \mathbb{N}$,*

$$\alpha = \sum_{n=0}^{\infty} \frac{c_n}{u_n}, \quad (5.1.1)$$

and $c_n < q_n - 1$ for infinitely many n .

Proof. Indeed, let $c_0 = \lfloor u_0 \alpha \rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part. Hence,

$$\alpha - \frac{c_0}{u_0} < \frac{1}{u_0} = \frac{q_1}{u_1}.$$

Suppose that c_0, \dots, c_k are defined for some $k \geq 0$, where

$$\alpha_k = \sum_{n=0}^k \frac{c_n}{u_n},$$

and $\alpha - \alpha_k < \frac{1}{u_k}$. The $(k+1)$ -th element of \mathbf{c} is $c_{k+1} = \lfloor u_{k+1}(\alpha - \alpha_k) \rfloor$. \square

Notation 5.1.3. Call a representation as in (5.1.1) *the (canonical) \mathbf{u} -representation* of α and call \mathbf{c} the sequence of coefficients of the \mathbf{u} -representation (or briefly \mathbf{c} is the \mathbf{u} -representation of α). Very often, in this section, one can refer to an element $x \in \mathbb{T}$ as its unique preimage \tilde{x} in $[0, 1)$ (i. e. . with respect to π_1 see §A.2.1) without distinction.

For $x \in \mathbb{T}$ with \mathbf{u} -representation (5.1.1), let

- $\text{supp}(x) = \{n \in \mathbb{N} \mid c_n \neq 0\}$ and
- $\text{supp}_q(x) = \{n \in \mathbb{N} \mid c_n = q_n - 1\}$.

Call an infinite set A of naturals

- \mathbf{q} -*bounded* if the sequence $\{q_n : n \in A\}$ is bounded;
- \mathbf{q} -*divergent* if the sequence $\{q_n : n \in A\}$ diverges to infinity.

Clearly, \mathbf{u} is \mathbf{q} -divergent if and only if \mathbb{N} is \mathbf{q} -divergent.

Remark 5.1.4. If (5.1.1) is a canonical representation, then

$$\frac{c_{n+1}}{q_{n+1}} + \cdots + \frac{c_{n+t}}{q_{n+1} \cdots q_{n+t}} + \cdots < 1 \quad (5.1.2)$$

for all $t, n \in \mathbb{N}$. Indeed, (5.1.2) is a \mathbf{u}^* -representation for $\mathbf{u}^* = (\mathbf{u}_k^*)$ where $\mathbf{u}_k^* = q_{n+1} \cdots q_{n+k+1}$.

5.1.2 Statement of the theorem and lemmas

The theorem

The goal of this section is to prove the Theorem 5.1.6, that completely describes the elements of $t_{\mathbf{u}}(\mathbb{T})$, when \mathbf{u} is an \mathbf{a} -sequence, but first one can introduce the following notation.

Notation 5.1.5. Let A, B be two set. One can write $A \subseteq^* B$ if and only if $|A \setminus B| < \aleph_0$, while $A =^* B$ if and only if $|A \Delta B| < \aleph_0$. Moreover let denote the family of all infinite subset of \mathbb{N} by $[\mathbb{N}]^{\aleph_0}$.

Theorem 5.1.6 ([36] Theorem 2.3). *Let \mathbf{u} be an sequence, $x \in \mathbb{T}$ with \mathbf{u} -representation \mathbf{c} . Then $x \in t_{\mathbf{u}}(\mathbb{T})$ if and only if either $\text{supp}(x)$ is finite or if $\text{supp}(x)$ is infinite and for all $A \in [\mathbb{N}]^{\aleph_0}$ the following holds:*

(a) *If A is \mathbf{q} -bounded then*

(a1) *if $A \subseteq^* \text{supp}(x)$, then $A + 1 \subseteq^* \text{supp}(x)$, $A \subseteq^* \text{supp}_{\mathbf{q}}(x)$ and $\lim_{n \in A} \frac{c_{n+1}+1}{q_{n+1}} = 1$ in \mathbb{R} .*

Moreover, if $A + 1$ is \mathbf{q} -bounded, then $A + 1 \subseteq^ \text{supp}_{\mathbf{q}}(x)$ as well;*

(a2) *if $A \cap \text{supp}(x)$ is finite then $\lim_{n \in A} \frac{c_{n+1}}{q_{n+1}} = 0$ in \mathbb{R} .*

Moreover, if $A + 1$ is \mathbf{q} -bounded, then $(A + 1) \cap \text{supp}(x)$ is finite as well.

(b) *If A is \mathbf{q} -divergent, then $\lim_{n \in A} \frac{c_n}{q_n} = \lim_{n \in A} \frac{c_{n+1}}{q_n} = 0$ in \mathbb{T} .*

Remark 5.1.7. Obviously, item (b) imposes the restriction only on $A \cap \text{supp}(x)$ (since $c_n = 0$ for all $n \notin \text{supp}(x)$). Hence, one can consider only subsets A of $\text{supp}(x)$ in item (b).

Theorem 5.1.6 was obtained discovering a gap in [31, Theorem 2.2], where item (a2) of Theorem 5.1.6 is completely missing. The next example explains the necessity to add (a2).

Example 5.1.8. As usual, set $(2k + 1)!! = 1 \cdot 3 \cdots (2k + 1)$, consider the sequences of ratios $\mathbf{q}^{\mathbf{u}} = (q_n)$ and hence $u_n = q_1 \cdots q_n$ and $\mathbf{c} = (c_n)$ defined as follows for every $n > 0$ (consider $q_0 = 1$, $u_0 = 1$ and $c_0 = 0$):

$$\begin{aligned} q_n &= \begin{cases} 2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd.} \end{cases}, \\ u_n &= \begin{cases} 2^k(2k - 1)!! & \text{if } n = 2k \text{ is even} \\ 2^k(2k + 1)!! & \text{if } n = 2k + 1 \text{ is odd} \end{cases} \quad \text{and} \quad (5.1.3) \\ c_n &= \begin{cases} 0 & \text{if } n \text{ is even} \\ n - 1 & \text{if } n \text{ is odd} \end{cases}. \end{aligned}$$

Let $x = \sum_n^{\infty} \frac{c_n}{u_n}$.

With these data the hypothesis of [31, Theorem 2.2] is satisfied. Indeed, $\text{supp}(x) = \{\text{odds numbers}\} \setminus \{1\}$ and so it is infinite. If $A \subseteq \text{supp}(x)$ is infinite, then A is \mathbf{q} -divergent (i.e., $\lim_{n \in A} q_n = \infty$). From (5.1.3) one can deduce

$$\lim_{n \in A} \frac{c_n}{q_n} = \lim_{n \in A} \frac{n-1}{n} = 0 \quad \text{in } \mathbb{T}, \quad (5.1.4)$$

that is (b2) from [31, Theorem 2.2] holds true. Hence x verifies item (b) from [31, Theorem 2.2]. Yet $x \notin t_u(\mathbb{T})$ as $u_{2k-1}x \rightarrow \frac{1}{2} \neq 0$ in \mathbb{T} .

On the other hand, (a2) in Theorem 5.1.6 is not satisfied. Indeed, let A be the set of all even natural numbers. Then $A \cap \text{supp}(x) = \emptyset$ is finite and A is \mathbf{q} -bounded. Nevertheless,

$$\lim_{n \in A} \frac{c_{n+1}}{q_{n+1}} = \lim_{n \in A+1} \frac{c_n}{q_n} = \lim_{n \rightarrow \infty} \frac{n-1}{n} = 1 \quad \text{in } \mathbb{R},$$

while this limit must equal 0 according to (a2) of Theorem 5.1.6.

To prove Theorem 5.1.6 one need first the following lemmas.

Lemmas

Notation 5.1.9. Let $x \in [0, 1)$ with \mathbf{u} -representation (5.1.1). For $n \in \mathbb{N}^*$ and $t \in \mathbb{N}$ set

$$\sigma_{n,t}(x) = \frac{c_n}{q_n} + \dots + \frac{c_{n+t}}{q_n \cdots q_{n+t}}. \quad (5.1.5)$$

The motivation to introduce this “partial term” in the representation of $u_{n-1}x - \lfloor u_{n-1}x \rfloor$, that is the fractional part of $u_{n-1}x$, denoted by $\{u_{n-1}x\}$, comes from the following lemma.

Lemma 5.1.10. [31, Lemma 3.1] For $x \in [0, 1)$ represented as in (5.1.1), for every natural $n \in \mathbb{N}^*$ and $t \in \mathbb{N}$

$$\{u_{n-1}x\} = \sigma_{n,t}(x) + \frac{\{u_{n+t}x\}}{q_n \cdots q_{n+t}} \geq \sigma_{n,t}(x). \quad (5.1.6)$$

In particular, for $t = 0$ one has

$$\{u_{n-1}x\} = \frac{c_n}{q_n} + \frac{\{u_n x\}}{q_n}. \quad (5.1.7)$$

Proof. We prove first that

$$\{u_{n-1}x\} = \frac{c_n}{q_n} + \frac{c_{n+1}}{q_n q_{n+1}} + \frac{c_{n+2}}{q_n q_{n+1} q_{n+2}} + \cdots + \cdots + \frac{c_{n+t}}{q_n \cdots q_{n+t}} + \cdots \quad (5.1.8)$$

Indeed,

$$\begin{aligned} \{u_{n-1}x\} &= \left\{ u_{n-1} \left(\frac{c_0}{q_0} + \cdots + \frac{c_{n-1}}{q_0 \cdots q_{n-1}} + \sum_{k=n}^{\infty} \frac{c_k}{u_k} \right) \right\} \\ &= \left\{ q_0 \cdots q_{n-1} \sum_{k=n}^{\infty} \frac{c_k}{u_k} \right\} \\ &= \left\{ \frac{c_n}{q_n} + \frac{c_{n+1}}{q_n q_{n+1}} + \frac{c_{n+2}}{q_n q_{n+1} q_{n+2}} + \cdots + \cdots + \frac{c_{n+t}}{q_n \cdots q_{n+t}} + \cdots \right\} \\ &= \frac{c_n}{q_n} + \frac{c_{n+1}}{q_n q_{n+1}} + \frac{c_{n+2}}{q_n q_{n+1} q_{n+2}} + \cdots + \cdots + \frac{c_{n+t}}{q_n \cdots q_{n+t}} + \cdots \end{aligned}$$

since the last term is less than 1 (as $c_m = q_m - 1$ can not occur for all $m > n$ by the definition of \mathbf{u} -representation).

By (5.1.8)

$$\begin{aligned} \{u_{n-1}x\} &= \frac{c_n}{q_n} + \frac{c_{n+1}}{q_n q_{n+1}} + \frac{c_{n+2}}{q_n q_{n+1} q_{n+2}} + \cdots + \frac{c_{n+t}}{q_n \cdots q_{n+t}} + \\ &\quad + \frac{1}{q_n \cdots q_{n+t}} \left(\frac{c_{n+t+1}}{q_{n+t+1}} + \cdots + \frac{c_{n+t+h}}{q_{n+t+1} \cdots q_{n+t+h}} + \cdots \right). \end{aligned}$$

Analogously, (5.1.8) gives also

$$\{u_{n+t}x\} = \frac{c_{n+t+1}}{q_{n+t+1}} + \frac{c_{n+t+2}}{q_{n+t+1} q_{n+t+2}} + \cdots + \frac{c_{n+t+h}}{q_{n+t+1} \cdots q_{n+t+h}} + \cdots.$$

Replacing $\{u_{n+t}x\}$ in the above equality we obtain (5.1.6). \square

Notation 5.1.11. $t_u(\mathbb{T})$ is already defined. Let, for every $A \in [\mathbb{N}]^{\aleph_0}$, $t_u(\mathbb{T})_A = \{x \in \mathbb{T} \mid \lim_{n \in A} u_n x = 0\}$. For all $A \in [\mathbb{N}]^{\aleph_0}$ one has $t_u(\mathbb{T}) \subseteq t_u(\mathbb{T})_A$ and $t_u(\mathbb{T}) = \bigcap_{A \in [\mathbb{N}]^{\aleph_0}} t_u(\mathbb{T})_A$.

Furthermore, $\pi(u_n \alpha) = \pi(\{u_n \alpha\})$, and $\text{supp}_q(x) \subset \text{supp}(x)$.

Lemma 5.1.12. *If $A \in [\mathbb{N}]^{\aleph_0}$ and $x \in t_u(\mathbb{T})_{A-1}$, then*

(i) *if $A \subset^* \text{supp}(x)$ and q -bounded, then $\lim_{n \in A} \{u_{n-1}x\} = 1$ in \mathbb{R} and $A \subset^* \text{supp}_q(x)$*

(ii) *if $A \cap \text{supp}(x)$ is finite, then $\lim_{n \in A} \{u_{n-1}x\} = 0$ in \mathbb{R} .*

Proof. (i) Let $q := 1 + \max_{n \in A} \{q_n\}$. The hypothesis $A \subset^* \text{supp}(x)$ yields $c_n \geq 1$ for almost all $n \in A$. So by (5.1.7) we get

$$\{u_{n-1}x\} \geq \frac{c_n}{q_n} > \frac{1}{q} \text{ for almost all } n \in A. \quad (5.1.9)$$

Since $x \in t_u(\mathbb{T})_{A-1}$, one concludes that $\lim_{n \in A} \{u_{n-1}x\} = 1$ in \mathbb{R} .

Now since $\{u_{n-1}x\} \rightarrow 1$ in \mathbb{R} , by (5.1.7) and (5.1.9) we have

$$1 - \frac{1}{q_n} < 1 - \frac{1}{q} < \{u_{n-1}x\} = \frac{c_n}{q_n} + \frac{\{u_n x\}}{q_n} < \frac{c_n + 1}{q_n}$$

for almost all $n \in A$. That is, $q_n - 1 < c_n + 1$ and hence $c_n = q_n - 1$ (as $c_n > q_n - 2$), for almost all $n \in A$. This proves $A \subset^* \text{supp}_q(x)$.

(ii) As $A \cap \text{supp}(x)$ is finite by (5.1.7), $\{u_{n-1}x\} = 0 + \frac{\{u_n x\}}{q_n} < \frac{1}{2}$ for almost all $n \in A$. Since $x \in t_u(\mathbb{T})_{A-1}$, we conclude that $\lim_{n \in A} \{u_{n-1}x\} = 0$ in \mathbb{R} . \square

For the sake of convenience, sometimes, one can apply (5.1.6) with $t + 1$ and split $\sigma_{n,t+1}(x)$ in $\sigma_{n,t}(x) + \frac{c_{n+t+1}}{q_n \cdots q_{n+t+1}}$ to get also

$$\begin{aligned} \{u_{n-1}x\} &= \sigma_{n,t+1}(x) + \frac{\{u_{n+t+1}x\}}{q_n \cdots q_{n+t+1}} \\ &= \sigma_{n,t}(x) + \frac{c_{n+t+1}}{q_n \cdots q_{n+t+1}} + \frac{\{u_{n+t+1}x\}}{q_n \cdots q_{n+t+1}}. \end{aligned} \quad (5.1.10)$$

Along with (5.1.6), this gives the following obvious, but useful estimate:

$$\sigma_{n,t}(x) \leq \{u_{n-1}x\} < \sigma_{n,t}(x) + \frac{c_{n+t+1}}{q_n \cdots q_{n+t+1}} + 2^{-(t+2)}. \quad (5.1.11)$$

The following is a well-known fact that is useful for the proof.

Fact 5.1.13. Let $(a_n)_n$ be a sequence in a topological space X and $a \in X$. Then $a_n \rightarrow a$ if and only if for all infinite $A \subseteq \mathbb{N}$ there exists an infinite $A' \subseteq A$ such that $\lim_{n \in A'} a_n = a$.

Indeed, the necessity is obvious. For the sufficiency, suppose by contradiction that $a_n \not\rightarrow a$. Thence there exists an open neighbourhood U of a , such that there exists an infinite subset A of \mathbb{N} such that $a_n \notin U$ for every $n \in A$ and hence for all infinite $A' \subseteq A$ we get $\lim_{n \in A'} a_n \neq a$ in contradiction with our hypothesis.

5.1.3 The proof

Necessity

Suppose $\text{supp}(x)$ is infinite and let $x \in t_u(\mathbb{T})$ and $A \in [\mathbb{N}]^{\aleph_0}$.

(a) Suppose A is \mathbf{q} -bounded and consider two cases.

(a1) Let $A \subseteq^* \text{supp}(x)$. As $x \in t_u(\mathbb{T})$, we have $x \in t_u(\mathbb{T})_{A-1}$ as well. Since A is \mathbf{q} -bounded, Lemma 5.1.12(i) entails $A \subseteq^* \text{supp}_{\mathbf{q}}(x)$ and $1 = \lim_{n \in A} \{u_{n-1}x\}$ in \mathbb{R} . Hence, by (5.1.7),

$$\begin{aligned} 1 &= \lim_{n \in A} \left(\frac{c_n}{q_n} + \frac{\{u_n x\}}{q_n} \right) = \lim_{n \in A} \left(\frac{q_n - 1 + \{u_n x\}}{q_n} \right) \\ &= \lim_{n \in A} \left(1 - \frac{1 - \{u_n x\}}{q_n} \right). \end{aligned}$$

This yields $\lim_{n \in A} \left(\frac{1 - \{u_n x\}}{q_n} \right) = 0$ and therefore

$$\lim_{n \in A} \{u_n x\} = 1, \quad (5.1.12)$$

as A is \mathfrak{q} -bounded. By the definition of \mathbf{u} -representation, $c_{n+1} \leq q_{n+1} - 1$ for all $n \in \mathbb{N}$. By (5.1.7) one has that

$$\{u_n x\} = \frac{c_{n+1}}{q_{n+1}} + \frac{\{u_{n+1} x\}}{q_{n+1}} < \frac{c_{n+1} + 1}{q_{n+1}} \leq 1.$$

Hence, (5.1.12) entails $1 = \lim_{n \in A} \{u_n x\} \leq \lim_{n \in A} \frac{c_{n+1} + 1}{q_{n+1}} \leq 1$ i.e.

$$\lim_{n \in A} \frac{c_{n+1} + 1}{q_{n+1}} = 1 \quad (5.1.13)$$

From (5.1.13) and the fact that $q_{n+1} \geq 2$ for each $n \in A$ we deduce that $c_{n+1} + 1 > 1$ (i.e, $c_{n+1} \neq 0$) for almost all $n \in A$, i.e. $A + 1 \subseteq^* \text{supp}(x)$. By the first part of the proof, applied to $A + 1$, we conclude that in case $A + 1$ is \mathfrak{q} -bounded this gives $A + 1 \subseteq^* \text{supp}_{\mathfrak{q}}(x)$.

(a2) Let $A \cap \text{supp}(x)$ be finite. By item (ii) of Lemma 5.1.12, $\lim_{n \in A} \{u_{n-1} x\} = 0$ in \mathbb{R} , hence according to Lemma 5.1.10 (with $t = 1$)

$$\begin{aligned} 0 &= \lim_{n \in A} \{u_{n-1} x\} = \lim_{n \in A} \left(\frac{c_n}{q_n} + \frac{c_{n+1}}{q_n q_{n+1}} + \frac{\{u_{n+1} x\}}{q_n q_{n+1}} \right) \\ &= 0 + \lim_{n \in A} \left(\frac{c_{n+1}}{q_n q_{n+1}} + \frac{\{u_{n+1} x\}}{q_n q_{n+1}} \right) \end{aligned} \quad (5.1.14)$$

Hence, $\lim_{n \in A} \frac{\{u_{n+1} x\}}{q_n q_{n+1}} = \lim_{n \in A} \frac{c_{n+1}}{q_n q_{n+1}} = 0$ (for all $n \in \mathbb{N}$ we have $c_n, u_n \geq 0$ and $q_n > 0$). The \mathfrak{q} -boundedness of A yields $\lim_{n \in A} \frac{c_{n+1}}{q_{n+1}} = 0$.

If $A + 1$ is \mathfrak{q} -bounded, then the vanishing of the last limit implies that $(A + 1) \cap \text{supp}(x)$ is finite.

(b) Suppose A is \mathfrak{q} -divergent (i.e. $\lim_{n \in A} q_n = \infty$). By (5.1.7) we get

$$0 = \lim_{n \in A} \pi_1(\{u_{n-1} x\}) = \lim_{n \in A} \pi_1 \left(\frac{c_n}{q_n} + \frac{\{u_n x\}}{q_n} \right).$$

Along with $\{u_n x\} < 1$ and $\lim_{n \in A} q_n = \infty$, this yields

$$\lim_{n \in A} \pi_1 \left(\frac{c_n}{q_n} \right) = 0.$$

Some preliminaries for the sufficiency

Before starting the proof of the sufficiency, one can reformulate the necessary conditions in a stronger iterated version that will be frequently used in the sequel.

Notation 5.1.14. For any $A \in [\mathbb{N}]^{\aleph_0}$ and $t \in \mathbb{N}$ let

$$S_t(A) = \bigcup_{i=0}^t (A + i).$$

Since the aim is to compute

$$\begin{aligned} \lim_{n \in A} \{u_{n-1}x\} &= \lim_{n \in A} \sigma_{n,t}(x) + \lim_{n \in A} \frac{\{u_{n+t}x\}}{b_n \cdots b_{n+t}} \\ &= \lim_{n \in A} \sigma_{n,t}(x) + \lim_{n \in A} \frac{c_{n+t+1}}{b_n \cdots b_{n+t+1}} + \lim_{n \in A} \frac{\{u_{n+t+1}x\}}{b_n \cdots b_{n+t+1}}, \end{aligned} \quad (5.1.15)$$

one can use the second or the third part of (5.1.10) depending on whether there exists some t such that $S_t(A)$ is \mathbf{q} -bounded, but $S_{t+1}(A)$ is not \mathbf{q} -bounded. Note that in case such a t exists, one can assume without loss of generality that $A + t + 1$ is actually \mathbf{q} -divergent, by passing to an appropriate $A' \in [A]^{\aleph_0}$. One can fix this in the following claim.

Claim 5.1.15. Suppose $x \in [0, 1)$ with \mathbf{u} -representation (5.1.1) for an arithmetic sequence \mathbf{u} such that (a) and (b) of Theorem 5.1.6 hold. Let $A \in [\mathbb{N}]^{\aleph_0}$ be \mathbf{q} -bounded. If $S_t(A)$ is \mathbf{q} -bounded for some integer $t \geq 0$, then

1. if $A \subseteq^* \text{supp}(x)$, then $S_t(A) \subseteq^* \text{supp}_{\mathbf{q}}(x)$,

$$\lim_{n \in A+t+1} \frac{c_n + 1}{q_n} = 1 \text{ in } \mathbb{R}$$

and there exists $n_t \in \mathbb{N}$ such that for all $n \in A$ with $n \geq n_t$

$$\sigma_{n,t}(x) = 1 - \frac{1}{q_n \cdots q_{n+t}} \geq 1 - 2^{-(t+1)}. \quad (5.1.16)$$

Moreover, if $A + t + 1$ is \mathbf{q} -divergent, then

$$\lim_{n \in A+t+1} \frac{c_n}{q_n} = \lim_{n \in A} \frac{c_{n+t+1}}{q_{n+t+1}} = 1. \quad (5.1.17)$$

and

$$\lim_{n \in A} \frac{\{u_{n+t+1}x\}}{q_n \cdots q_{n+t+1}} = 0. \quad (5.1.18)$$

2. if $A \cap \text{supp}(x)$ is finite then $S_t(A) \cap \text{supp}(x)$ is finite as well (so there exists $n_t \in \mathbb{N}$ such that $\sigma_{n,t}(x) = 0$ for all $n \in A$ with $n \geq n_t$) and $\lim_{n \in A} \frac{c_{n+t+1}}{q_{n+t+1}} = 0$ in \mathbb{R} .

Moreover, if $A + t + 1$ is \mathbf{q} -divergent, then (5.1.18) holds true.

In the light of Fact 5.1.13, to prove the sufficiency it is enough to prove that, if (a) and (b) of the Theorem 5.1.6 hold, then for all infinite $A \subseteq \mathbb{N}$ there exists an infinite $A' \subseteq A$ such that $\lim_{n \in A'} \pi(u_{n-1}x) = 0$ and hence $\pi(x) \in t_{\mathbf{u}}(\mathbb{T})$.

Sufficiency

If $\text{supp}(x)$ is finite, let $n_0 := \max\{n \mid c_n \neq 0\}$. Then for all $n > n_0$ one gets $u_n x \in \mathbb{Z}$, so $x \in t_{\mathbf{u}}(\mathbb{T})$.

Suppose now that $\text{supp}(x)$ is infinite and satisfying conditions (a) and (b); one need to prove that $x \in t_{\mathbf{u}}(\mathbb{T})$. For this purpose, according to Notation 5.1.11 and Fact 5.1.13, one can check that, for all $A \in [\mathbb{N}]^{\aleph_0}$ there exists $A' \in [A]^{\aleph_0}$ such that $\lim_{n \in A'} \pi(\{u_{n-1}x\}) = 0$.

Indeed, one can assume without loss of generality that either $A \subseteq \text{supp}(x)$ or $A \cap \text{supp}(x) = \emptyset$.

Case 1. Assume that A is \mathbf{q} -bounded.

Subcase 1.1. Assume that there exists an integer $t \geq 0$ such that $A + t + 1$ is not \mathbf{q} -bounded, while $A + s$ is \mathbf{q} -bounded for all $s \in \{n \in \mathbb{N} \mid n \leq t\}$, i. e. $S_t(A)$ is \mathbf{q} -bounded. Choosing an $A' \in [A]^{\aleph_0}$ such that $A' + t + 1$ is \mathbf{q} -divergent, without loss of generality one can assume, $\lim_{n \in A} q_{n+t+1} = \infty$.

Then (5.1.18) holds, so (5.1.15) becomes

$$\lim_{n \in A} \{u_{n-1}x\} = \lim_{n \in A} \left(\sigma_{n,t}(x) + \frac{c_{n+t+1}}{q_n \cdots q_{n+t+1}} \right) \quad (5.1.19)$$

If $A \subseteq \text{supp}(x)$, then (5.1.17) and (5.1.16) of Claim 5.1.15(1) imply

$$\lim_{n \in A} \{u_{n-1}x\} = \lim_{n \in A} \left(1 - \frac{1}{q_n \cdots q_{n+t}} + \frac{c_{n+t+1}}{q_n \cdots q_{n+t+1}} \right) = 1. \quad (5.1.20)$$

If $A \cap \text{supp}(x) = \emptyset$, then by (5.1.15) and Claim 5.1.15(2) we get

$$\lim_{n \in A} \{u_{n-1}x\} = \lim_{n \in A} \frac{c_{n+t+1}}{q_n \cdots q_{n+t+1}} = 0. \quad (5.1.21)$$

In both these cases we have $\lim_{n \in A} \pi(\{u_{n-1}x\}) = 0$.

Subcase 1.2. Assume that $S_t(A)$ is \mathbf{q} -bounded for all integers $t \geq 0$.

Let $\varepsilon > 0$. Pick a $t \in \mathbb{N}$ such that $2^{-(t+1)} < \varepsilon$. According to Claim 5.1.15 we can choose $n_t \in \mathbb{N}$ such that

- (i) (5.1.16) holds for all $n \in A$ such that $n > n_t$, in case $A \subseteq \text{supp}(x)$; or
- (ii) $\sigma_{n,t} = 0$ and $\frac{c_{n+t+1}}{q_{n+t+1}} < \varepsilon$ hold for all $n \in A$ such that $n > n_t$, in case $A \cap \text{supp}(x) = \emptyset$.

In case (i), (5.1.11) and (5.1.16) imply

$$1 - \varepsilon < \{u_{n-1}x\} < 1 \text{ for } n > n_t \text{ in } A.$$

Hence $\lim_{n \in A} \{u_{n-1}x\} = 1$ in \mathbb{R} . Therefore, $\lim_{n \in A} \pi(\{u_{n-1}x\}) = 0$.

By (5.1.11) in case (ii) one has

$$\{u_{n-1}x\} \leq 2\varepsilon \text{ for } n > n_t \text{ in } A.$$

So $\lim_{n \in A} \{u_{n-1}x\} = 0$ in \mathbb{R} and hence $\lim_{n \in A} \pi(\{u_{n-1}x\}) = 0$.

Case 2 Assume that A is not \mathfrak{q} -bounded. Hence, there exists $A' \in [A]^{\aleph_0}$ such that $\lim_{n \in A'} q_n = \infty$. By (5.1.7) and hypothesis (b) of the theorem, one has

$$\lim_{n \in A'} \pi(\{u_{n-1}x\}) = \lim_{n \in A'} \pi\left(\frac{c_n}{q_n} + \frac{\{u_n x\}}{q_n}\right) = \lim_{n \in A'} \pi\left(\frac{\{u_n x\}}{q_n}\right). \quad (5.1.22)$$

Since $\{u_n x\} < 1$ and $\lim_{n \in A'} q_n = \infty$, we conclude that the last limit is 0.

5.2 COROLLARIES

The following simple claim will be needed in the proofs of the entire section.

Claim 5.2.1. If $A + 1 \subseteq^* A$ for an infinite subset A of \mathbb{N} , then A is a co-finite subset of \mathbb{N} .

Proof. Fix a one-to-one increasing enumeration $A = \{n_k : k \in \mathbb{N}\}$. By our assumption there exists k_0 such that for all $k \geq k_0$ one has $n_k + 1 \in A$. Hence, $n_{k_0} + 1 \in A$, so $n_{k_0} + 1 = n_{k_0+1} \in A$. Since $k_0 + 1 > k_0$, one has $n_{k_0+1} + 1 = (n_{k_0} + 1) + 1 \in A$. Analogously, $n_{k_0} + m \in A$ for all $m \in \mathbb{N}$, in other words A is co-finite. This proves the claim \square

5.2.1 Restriction on the support

Corollary 5.2.2. [31, Corollary 2.4] *Let $x \in \mathbb{T}$. If $\text{supp}(x)$ is \mathfrak{q} -bounded, then the following are equivalent:*

- (i) $x \in \mathfrak{t}_{\mathfrak{u}}(\mathbb{T})$;
- (ii) $c_n = 0$ for almost all $n \in \mathbb{N}$.

Proof. (ii) \rightarrow (i) See the sufficiency's proof of Theorem 5.1.6 in case $\text{supp}(x)$ is finite.

(i) \rightarrow (ii) Suppose $\text{supp}(x)$ is infinite. By (a1) of Theorem 5.1.6, setting $A = \text{supp}(x)$, we get $\text{supp}(x) + 1 \subseteq^* \text{supp}(x)$.

Hence $\text{supp}(x)$ is co-finite by Claim 5.2.1. Then the whole set \mathbb{N} is \mathbf{q} -bounded. Therefore, (a₁) applied to $A = \mathbb{N}$ implies that $\text{supp}_{\mathbf{q}}(x)$ is co-finite, a contradiction. \square

Corollary 5.2.3 ([36] Corollary 3.4). *Suppose that $x \in \mathbb{T}$ has \mathbf{q} -divergent support. Then $x \in \mathfrak{t}_{\mathbf{u}}(\mathbb{T})$ if and only if the following two conditions are satisfied:*

- (i) $\lim_{n \in \text{supp}(x)} \frac{c_n}{q_n} = 0$ in \mathbb{T} and
- (ii) $\lim_{n \in I'} \frac{c_n}{q_n} = 0$ in \mathbb{R} for every infinite $I' \subseteq \text{supp}(x)$ such that $I' - 1$ is \mathbf{q} -bounded.

Proof. Let $I = \text{supp}(x)$. If $x \in \mathfrak{t}_{\mathbf{u}}(\mathbb{T})$, then (i) holds true by item (b) of Theorem 5.1.6 applied to $A = I$. Assume that $A = I' - 1$ is \mathbf{q} -bounded for some infinite $I' \subseteq I$. Then $A \cap I$ is finite, as I is \mathbf{q} -divergent. Then by (a₂) applied to A , $\lim_{n \in I'} \frac{c_n}{q_n} = \lim_{n \in A} \frac{c_{n+1}}{q_{n+1}} = 0$ in \mathbb{R} . This proves (ii) and the necessity.

To establish the sufficiency, assume that (i) and (ii) hold true. According to Theorem 5.1.6, to prove that $x \in \mathfrak{t}_{\mathbf{u}}(\mathbb{T})$ one needs to check (a) and (b). Since (b) immediately follows from (i), one needs to prove only (a). Let A be an infinite \mathbf{q} -bounded set in \mathbb{N} . Then $A \cap I$ is finite, so we need to check only (a₂), i. e. $\lim_{n \in A} \frac{c_{n+1}}{q_{n+1}} = 0$ (since the final assertion of (a₂) follows from this equality, as mentioned in the final part of the proof of the necessity of (a₂)). Let $I' = (A + 1) \cap I$. If this set is infinite, then (ii) applies and this concludes. If I' is finite, one can conclude that $c_n = 0$ for almost all $n \in A + 1$ and hence $\lim_{n \in A+1} \frac{c_n}{q_n} = 0$. \square

The following results was established in [31]:

Corollary 5.2.4. *Suppose $x \in \mathbb{T}$ has \mathbf{q} -divergent support. Then $x \in \mathfrak{t}_{\mathbf{u}}(\mathbb{T})$ whenever $\lim_{n \in \text{supp}(x)} \frac{c_n}{q_n} = 0$ in \mathbb{R} .*

Proof. Follows immediately from Corollary 5.2.3 as the hypothesis implies both (i) and (ii) from that Corollary. \square

The next corollaries, still following obviously from Corollary 5.2.3, will be useful in the applications.

Corollary 5.2.5. *Suppose $x \in \mathbb{T}$ has \mathbf{q} -divergent support. Then $x \notin t_{\mathbf{u}}(\mathbb{T})$ whenever $\lim_{n \in \text{supp}(x)} \frac{c_n}{q_n} \neq 0$ in \mathbb{T} .*

Corollary 5.2.6. *Suppose that \mathbb{N} is \mathbf{q} -divergent. Then for $x \in \mathbb{T}$ $x \in t_{\mathbf{u}}(\mathbb{T})$ if and only if $\lim_{n \in \text{supp}(x)} \frac{c_n}{q_n} = 0$ in \mathbb{T} .*

Corollary 5.2.7. *For an a -sequence \mathbf{u} the following are equivalent:*

- (i) $|t_{\mathbf{u}}(\mathbb{T})| = \mathfrak{c}$;
- (ii) $t_{\mathbf{u}}(\mathbb{T})$ is uncountable;
- (iii) $t_{\mathbf{u}}(\mathbb{T})$ contains non-torsion elements;
- (iv) \mathbf{q} is not bounded.

Proof. The implications (i) \Rightarrow (ii) \Rightarrow (iii) are trivial. The implication (iii) \Rightarrow (iv) follows from Corollary 5.2.2.

The implication (iv) \Rightarrow (i) can be deduced directly from Theorem 4.5.7, but in this case, one can deduce this implication from Corollary 5.2.5.

Let $I \subseteq \mathbb{N}$ be a \mathbf{q} -divergent set witnessing our hypothesis (iv). For every infinite subset J of I let $x_J = \sum_{n \in J} \frac{1}{u_n}$. Then $x_J \neq x_{J'}$ whenever $J \neq J' \in [I]^{\aleph_0}$, so the set $M = \{x_J : J \in [I]^{\aleph_0}\}$ has size \mathfrak{c} . It remains to note that $M \subseteq t_{\mathbf{u}}(\mathbb{T})$ due to Corollary 5.2.4. \square

Proposition 5.2.8. *If \mathbf{u} is a \mathbf{q} -divergent a -sequence, then $t_{\mathbf{u}}(\mathbb{T})$ is not divisible.*

Proof. For every a -sequence there exists $k > 1$, $k \in \mathbb{N}$, such that $t := 1/k + \mathbb{Z} \in t_{\mathbf{u}}(\mathbb{T})$ (take for example $k = u_{n_0}$ for $n_0 \in \mathbb{N}^*$, then $u_n t = 0$ in \mathbb{T} for every $n \geq n_0$, since u_{n_0} divides u_n for every $n \geq n_0$). Our hypothesis about q_n ensures the existence of a sequence of non-negative integers \mathbf{c} such that $\lim_{n \rightarrow \infty} \pi(c_n/q_n) = 1/k$ and $kc_n < q_n$ for every $n \in \mathbb{N}$. Hence, for $x \in [0, 1)$ with \mathbf{u} -representation \mathbf{c} , i. e. $x = \sum_{n=1}^{\infty} \frac{c_n}{u_n}$, one has $x \notin t_{\mathbf{u}}(\mathbb{T})$ by Corollary 5.2.6. Then

$$y = kx = \sum_{n=1}^{\infty} \frac{kc_n}{u_n} \in t_{\mathbf{u}}(\mathbb{T})$$

by Corollary 5.2.6 since

$$\lim_{n \rightarrow \infty} \pi \left(\frac{kc_n}{q_n} \right) = 0.$$

Suppose now that there exists $z \in t_{\mathbf{u}}(\mathbb{T})$, such that $y = kz$. This means that $kx = kz$ and therefore $k(x - z) = 0$ in \mathbb{T} , that is $x - z = nt$ for some $n \in \mathbb{Z}$. Then $x = z + nt \in t_{\mathbf{u}}(\mathbb{T})$ since $nt \in t_{\mathbf{u}}(\mathbb{T})$ by the choice of t . This contradiction (with $x \notin t_{\mathbf{u}}(\mathbb{T})$) shows that $t_{\mathbf{u}}(\mathbb{T})$ is not divisible. \square

5.2.2 Restriction on the sequence of ratios

Here we consider sequences \mathbf{u} such that the sequence $\mathbf{q}^{\mathbf{u}}$ splits in to a \mathbf{q} -bounded parts and a \mathbf{q} -divergent part.

Definition 5.2.9. We say that the sequence $\mathbf{q}^{\mathbf{u}}$ of natural numbers has the *splitting property* if there exists a partition $\mathbb{N} = B^{\mathbf{u}} \cup I^{\mathbf{u}} = B \cup I$, such that

- (a) B and I are either empty or infinite
- (b) I is \mathbf{q} -divergent, in case I is infinite;
- (c) B is \mathbf{q} -bounded, in case B is infinite.

We say that B and I witness the splitting property for \mathbf{q} . Note the B and I are uniquely defined up to a finite set (i.e., if $B' \cup I'$ is another partition witnessing the splitting property for \mathbf{q} , then $B' =^* B$ and $I' =^* I$).

The following is an useful criterion to establish when a sequence is splitting.

Proposition 5.2.10. *A sequence \mathbf{q} has the splitting property if and only if there exists a natural number M such that the set $[M, m] \cap \{q_n \mid n \in \mathbb{N}\}$ is finite for every $m > M$.*

Proof. Assume that \mathbf{q} has the splitting property. Set $M = 1$ if $B = \emptyset$, and $M = \max\{q_n \mid n \in B\} + 1$ if $B \neq \emptyset$. Then $[M, m] \cap (q_n) = [M, m] \cap \{q_n \mid n \in I\}$ is finite (since $\lim_{n \in I} q_n = \infty$ in the case $I \neq \emptyset$).

Conversely, let $m \in \mathbb{N}$ be such that the set $[M, m] \cap (q_n)$ is finite for every $m > M$. Set $E = \{n \in \mathbb{N} \mid q_n \in [0, M)\}$ and

$S = \mathbb{N} \setminus E$. If E is infinite and S is finite, then one may set $B = \mathbb{N}$ and $I = \emptyset$. If E and S are infinite, then one may set $B = E$ and $I = S$. If E is finite, then S is infinite one may set $B = \emptyset$ and $I = \mathbb{N}$. Clearly, B and I witness the splitting property for \mathbf{q} . \square

Example 5.2.11. For every positive integer n write $n = 2^{q_n} n_1$, where n_1 is odd. Then, thanks to the last criterion, we get that the sequence $\mathbf{q} = (q_n)$ does not have the splitting property.

Our next aim is to simplify Theorem 5.1.6 in the case when $\mathbf{q}^{\mathbf{u}}$ has the splitting property. The simplification consists in reducing the number of the infinite sets A in the text of that theorem, by using, for a fixed $x \in \mathbb{T}$, only *three* infinite sets, $B_S^{\mathbf{u}}(x)$, $B_N^{\mathbf{u}}(x)$ and $I_S^{\mathbf{u}}(x)$, related to x and \mathbf{u} (if no confusion is possible one can omit the superscript \mathbf{u}).

Notation 5.2.12. Let \mathbf{u} be an a -sequence such that $\mathbf{q}^{\mathbf{u}}$ has the splitting property. For an element $x \in [0, 1)$, with \mathbf{u} -representation \mathbf{c} , one can define the following subsets of \mathbb{N}

$$\begin{aligned} B_S(x) &= B \cap \text{supp}(x), \\ B_N(x) &= B \setminus B_S(x) \text{ and} \\ I_S(x) &= I \cap \text{supp}(x). \end{aligned}$$

According to Remark 5.1.7 the set $I \setminus I_S(x)$ will play no relevant role in the sequel. Note that the sets $B_S(x)$ and $B_N(x)$ are \mathbf{q} -bounded, while $I_S(x)$ is \mathbf{q} -divergent whenever it is infinite.

The next corollary is a characterization of topologically torsion elements of \mathbb{T} , in the case when \mathbf{q} has the splitting property.

Corollary 5.2.13. *Suppose that \mathbf{u} is an a -sequence such that \mathbf{q} has the splitting property. If $x \in \mathbb{T}$ has \mathbf{u} -representation \mathbf{c} , then $x \in t_{\mathbf{u}}(\mathbb{T})$ if and only if the following conditions hold.*

- (i) $B_S(x) + 1 \subseteq^* \text{supp}(x)$, $B_S(x) \subseteq^* \text{supp}_{\mathbf{q}}(x)$ and if $B_S(x)$ is infinite, then $\lim_{n \in B_S(x)} \frac{c_{n+1} + 1}{q_{n+1}} = 1$ in \mathbb{R} ;
- (ii) if $B_N(x)$ is infinite, then $\lim_{n \in B_N(x)} \frac{c_{n+1}}{q_{n+1}} = 0$ in \mathbb{R} .

(iii) if $I_S(x)$ is infinite, then $\lim_{n \in I_S(x)} \pi \left(\frac{c_n}{q_n} \right) = 0$ in \mathbb{T} .

Proof. Necessity. Suppose $x \in t_u(\mathbb{T})$, hence, (a) and (b) of Theorem 5.1.6 hold true. One needs to check (i), (ii) and (iii).

(i) If $B_S(x)$ is finite there is nothing to prove. If $B_S(x)$ is infinite, then to get (i) it suffices to apply (a1) of Theorem 5.1.6 to $A = B_S(x)$.

(ii) Assume $B_N(x)$ is infinite. By (a2) of Theorem 5.1.6, applied to $A = B_N(x)$, one gets $\lim_{n \in A} \frac{c_{n+1}}{q_{n+1}} = 0$ in \mathbb{R} .

(iii) Suppose $I_S(x)$ infinite. By (b) of Theorem 5.1.6, applied to $A = I_S(x)$, one obtains $\lim_{n \in A} \pi \left(\frac{c_n}{q_n} \right) = 0$ in \mathbb{T} .

Sufficiency. Suppose now (i), (ii) and (iii) hold, we have to check that (a) and (b) of theorem 5.1.6 hold too. Let $A \in [\mathbb{N}]^{\aleph_0}$.

(a) Suppose that A is \mathbf{q} -bounded.

(a1) If $A \subseteq^* \text{supp}(x)$, then $A \subseteq^* B_S(x)$ by the \mathbf{q} -boundedness of A . Hence $B_S(x)$ is infinite. By (i), $B_S(x) + 1 \subseteq^* \text{supp}(x)$, $B_S(x) \subseteq^* \text{supp}_{\mathbf{q}}(x)$ and $\lim_{n \in B_S(x)} \frac{c_{n+1}+1}{q_{n+1}} = 1$ in \mathbb{R} . Since $A \subseteq^* B_S(x)$, one has $A + 1 \subseteq^* \text{supp}(x)$, $A \subseteq^* \text{supp}_{\mathbf{q}}(x)$ and $\lim_{n \in A} \frac{c_{n+1}+1}{q_{n+1}} = 1$ in \mathbb{R} .

(a2) If $A \cap \text{supp}(x)$ is finite, then $A \subseteq^* B_N(x)$ by the \mathbf{q} -boundedness of A , hence $B_N(x)$ is infinite and by (ii)

$$\lim_{n \in A} \frac{c_{n+1}}{q_{n+1}} = \lim_{n \in B_N(x)} \frac{c_{n+1}}{q_{n+1}} = 0$$

in \mathbb{R} therefore (a2) of the theorem holds.

(b) Suppose that A is \mathbf{q} -divergent. Then we can assume without loss of generality that $A \subseteq I = I_S(x) \cup I_N(x)$. On the other hand, according to Lemma 5.1.13, we can assume also that $A \subseteq \text{supp}(x)$. Hence $A \subseteq I_S(x)$. Since A is infinite, then also $I_S(x)$ is infinite. Hence $\lim_{n \in I_S(x)} \pi \left(\frac{c_n}{q_n} \right) = 0$ in \mathbb{T} due to (iii).

Consequently, $\lim_{n \in A} \pi \left(\frac{c_n}{q_n} \right) = 0$. \square

Corollary 5.2.14. *Suppose that u is an a -sequence such that \mathbf{q} has the splitting property and $x \in t_u(\mathbb{T})$. If $B_S(x)$ is infinite, then $I_S(x)$ is infinite.*

Proof. Let $A = B_S(x) + 1 \setminus B_S(x)$. By Corollary 5.2.13

$$B_S(x) + 1 \subseteq^* \text{supp}(x) \quad \text{and} \quad B_S(x) \subseteq^* \text{supp}_q(x) \quad (5.2.1)$$

Then the first inclusion in (5.2.1) implies that $A \subseteq^* \text{supp}(x) \setminus B_S(x) = I_S(x)$. So the corollary will be proved if one shows that A is infinite.

Arguing for a contradiction assume that A is finite, i.e., $B_S(x) + 1 \subseteq^* B_S(x)$. By Claim 5.2.1,

$B_S(x)$ is co-finite in \mathbb{N} . Now the second inclusion in (5.2.1) yields that $\text{supp}_q(x)$ is co-finite in \mathbb{N} as well, in contradiction with definition of \mathbf{u} -representation. \square

As an application of the above corollary, one can obtain a new proof of Corollary 5.2.2.

5.3 $\tau_{\mathbf{u}}$ FOR AN ARITHMETIC SEQUENCE OF INTEGERS \mathbf{u}

5.3.1 $t(t_{\mathbf{u}}(\mathbb{T}))$ is a dense subgroup of $(t_{\mathbf{u}}(\mathbb{T}), \mathfrak{p}_{\mathbf{u}})$

The polishability of $t_{\mathbf{u}}(\mathbb{T})$ holds for a general integer sequence, but in the case of an \mathbf{a} -sequence we can produce a simple subgroup witnessing the separability of $t_{\mathbf{u}}(\mathbb{T})$. If \mathbf{u} is an \mathbf{a} -sequence, it makes sense to consider the subgroup $\mathfrak{t}_{\mathbf{u}}(\mathbb{T})$ of $t_{\mathbf{u}}(\mathbb{T})$ formed by all $x \in \mathbb{T}$ having the sequence $\mathbf{c} = (c_n)$ of its \mathbf{u} -representation (5.1.1) definitely zero. It can be proved that $\mathfrak{t}_{\mathbf{u}}(\mathbb{T}) = t(t_{\mathbf{u}}(\mathbb{T}))$ when \mathbf{u} is an \mathbf{a} -sequence where $t(t_{\mathbf{u}}(\mathbb{T}))$ is the torsion subgroup of $t_{\mathbf{u}}(\mathbb{T})$. In general, if we consider an element not in $t_{\mathbf{u}}(\mathbb{T})$, it is possible that this element is torsion but its canonical representation is not finite. Take for example an element with a periodic representation (5.1.1), where $\mathbf{u} = ((10)^n)$.

Proposition 5.3.1. *If \mathbf{u} is an \mathbf{a} -sequence, then $\mathfrak{t}_{\mathbf{u}}(\mathbb{T}) = t(t_{\mathbf{u}}(\mathbb{T}))$.*

Proof. The inclusion $\mathfrak{t}_{\mathbf{u}}(\mathbb{T}) \subseteq t(t_{\mathbf{u}}(\mathbb{T}))$ is obvious. Let $x = \frac{a}{b} \in t(t_{\mathbf{u}}(\mathbb{T})) = t_{\mathbf{u}}(\mathbb{T}) \cap \mathbb{Q}/\mathbb{Z}$, where a, b are coprime and

$a < b$. Hence, there exists $m \in \mathbb{N}$ such that $\forall n \geq m, \|u_n x\| < \frac{1}{b}$. In particular,

$$\left\| u_m \frac{a}{b} \right\| < \frac{1}{b} \Leftrightarrow \left\| u_m \frac{a}{b} \right\| = 0 \Leftrightarrow b | u_m a.$$

Hence, $u_m a = lb$ for some $l \in \mathbb{N}$ and hence $x = \frac{a}{b} = \frac{l}{u_m}$. If $l < q_m$, then $\frac{l}{u_m}$ is a finite canonical representation of x . Otherwise let $l = q_m l_m + r_m$, with $r_m < q_m$ and put $c_m = r_m$. Let $k \leq m$, if $l_k < q_{k-1}$ put $c_{k-1} = l_k$, otherwise put $c_{k-1} = r_{k-1}$ where $l_k = q_{k-1} l_{k-1} + r_{k-1}$ and $r_{k-1} < q_{k-1}$. Clearly, since $a < b$ and hence $l < u_m = q_1 \cdots q_m$, this procedure stops in at most m steps. \square

Remark 5.3.2. In other words Proposition 5.3.1 is: *Let \mathbf{u} be an a -sequence and $x \in t_{\mathbf{u}}(\mathbb{T})$ with infinite support, then x is irrational. Hence, by Corollary 5.2.4, if (q_n) is not bounded, then there exists $x \in t_{\mathbf{u}}(\mathbb{T})$ that is irrational. In this case one can prove that there are \mathfrak{c} many irrational elements in $t_{\mathbf{u}}(\mathbb{T})$. Moreover, as another corollary, one can obtain the following: If \mathbf{u} is a q -divergent a -sequence, then every element with a periodic \mathbf{u} -representation is irrational*

Proposition 5.3.3 ([Proposition D][38]). *If \mathbf{u} is an a -sequence, then $t(t_{\mathbf{u}}(\mathbb{T}))$ is a dense subgroup of $(t_{\mathbf{u}}(\mathbb{T}), \mathfrak{p}_{\mathbf{u}})$.*

Proof. By Proposition 5.3.1 we have to prove that $\mathfrak{t}_{\mathbf{u}}(\mathbb{T})$ is a dense subgroup of $(t_{\mathbf{u}}(\mathbb{T}), \tau_{\rho})$.

If \mathbf{u} is q -bounded by Theorem 5.2.2 $\mathfrak{t}_{\mathbf{u}}(\mathbb{T}) = t_{\mathbf{u}}(\mathbb{T})$.

Suppose \mathbf{u} is not q -bounded, hence there exists $q_{n_k} \rightarrow \infty$. Let $x \in t_{\mathbf{u}}(\mathbb{T})$ with canonical representation (5.1.1) and $\varepsilon > 0$. It suffices to find $x' \in \mathfrak{t}_{\mathbf{u}}(\mathbb{T})$ such that $x - x' \in B_{\varepsilon}^{\rho}(0)$.

Let $k^* \in \mathbb{N}$ such that:

- (i) $\forall k \geq k^*, q_{n_k} > \frac{1}{\varepsilon}$;
- (ii) $\forall n \geq n_{k^*}, \|u_n x\| < \varepsilon$.

If

$$x' = \sum_{n \leq n_{k^*}} \frac{c_n}{u_n} \quad \text{and} \quad x'' = \sum_{n > n_{k^*}} \frac{c_n}{u_n},$$

then $x = x' + x''$ and $x' \in \mathfrak{t}_{\mathbf{u}}(\mathbb{T})$. Hence, it is enough to prove that $x'' \in B_{\varepsilon}^{\rho}(0)$, i.e., $\|u_s x''\| < \varepsilon$ holds for all $s \in \mathbb{N}$ (note that without loss of generality one may consider $u_0 = 1$, therefore we will check also that $\|x''\| < \varepsilon$).

If $s \geq n_{k^*}$, then $\|u_s x''\| = \|u_s(x - x')\| \leq \|u_s x\| + \|u_s x'\| < \varepsilon + 0$.

If $s < n_{k^*}$, then

$$\begin{aligned} \|u_s x''\| &\leq u_s x'' = \sum_{n > n_{k^*}} \frac{c_n u_s}{u_n} = \\ &= \frac{1}{q_{s+1} \cdots q_{n_{k^*}}} \left(\frac{c_{n_{k^*}+1}}{q_{n_{k^*}+1}} + \cdots + \frac{c_{n_{k^*}+t}}{q_{n_{k^*}+1} \cdots q_{n_{k^*}+t}} + \cdots \right). \end{aligned}$$

Since (c_n) is a \mathbf{u} -representation,

$$\left(\frac{c_{n_{k^*}+1}}{q_{n_{k^*}+1}} + \cdots + \frac{c_{n_{k^*}+t}}{q_{n_{k^*}+1} \cdots q_{n_{k^*}+t}} + \cdots \right) < 1$$

holds, according to Remark 5.1.4. By (i), $\frac{1}{q_{s+1} \cdots q_{n_{k^*}}} < \varepsilon$ and hence $\|u_s x''\| < \varepsilon$. \square

5.3.2 $\rho_{\mathbf{u}}$ -balls when \mathbf{u} is an a -sequence

By Theorem 4.1.11, the subgroup $\mathfrak{t}_{\mathbf{u}}(\mathbb{T})$ is countable, when $q_s^{\mathbf{u}} < \infty$. Hence, the topology $\tau_{\mathbf{u}}$ restricted to $\mathfrak{t}_{\mathbf{u}}(\mathbb{T})$ is discrete, and actually $(\mathbb{T}, \tau_{\mathbf{u}})$ is discrete, by Theorem 3.2.5. In addition one can specify which balls are trivial as showed in 4.2.1.

Notation 5.3.4. Let

$$S_+^{\mathbf{u}} = \{m \in \mathbb{N} \mid q_m = q_+^{\mathbf{u}}\}$$

and

$$S_s^{\mathbf{u}} = \{m \in \mathbb{N} \mid q_m = q_s^{\mathbf{u}}\}.$$

It turns out that the closed $\rho_{\mathbf{u}}$ -ball $\{x \in \mathbb{T} : \rho_{\mathbf{u}}(x, 0) \leq 1/q_+^{\mathbf{u}}\}$ centered at 0 has size \mathfrak{c} . Moreover, if $S_+^{\mathbf{u}}$ is a big set in \mathbb{N} (i.e., the gaps between two consecutive elements of $S_+^{\mathbf{u}}$ are bounded), then actually $|B_{\frac{1}{q_+^{\mathbf{u}}}}^{\rho_{\mathbf{u}}}(0)| = \mathfrak{c}$. To prove this, one needs the following lemma.

Lemma 5.3.5. *Let \mathbf{u} be a strictly increasing a -sequence with $q_+^{\mathbf{u}} < \infty$, let*

$$S = \{n_1 < n_2 \cdots < n_k < \cdots\} \subset S_+^{\mathbf{u}} \quad (5.3.1)$$

with $d_i = n_{i+1} - n_i \geq 2$ for $i = 1, 2, \dots$

and let

$$\chi_S = \frac{1}{u_{n_1}} - \frac{1}{u_{n_2}} + \frac{1}{u_{n_3}} - \frac{1}{u_{n_4}} + \cdots. \quad (5.3.2)$$

(i) *If $n_k \leq n < n_{k+1}$, then $\|u_{n-1}\chi_S\| \leq \|u_{n_k-1}\chi_S\|$ and*

$$\begin{aligned} \frac{1}{q_+^{\mathbf{u}}} - \frac{1}{2^{d_k-1}q_+^{\mathbf{u}}} &\leq \frac{1}{q_{n_k}} - \frac{1}{q_{n_k} \cdots q_{n_{k+1}}} \leq \|u_{n_k-1}\chi_S\| \leq \\ &\leq \frac{1}{q_{n_k}} - \frac{1}{q_{n_k} \cdots q_{n_{k+1}}} + \frac{1}{q_{n_k} \cdots q_{n_{k+2}}}. \end{aligned} \quad (5.3.3)$$

(ii) $\rho_{\mathbf{u}}(\chi_S, 0) = \max\{\|\chi_S\|, \sup\{\|u_{n_k-1}\chi_S\| : k \in \mathbb{N}\}\}$.

(iii) *If $S' \neq S$ is as in (5.3.1), then $\chi_S \neq \chi_{S'}$.*

Proof. (a) Clearly, one has

$$\|u_{n_k-1}\chi_S\| = \left\| \frac{1}{q_{n_k}} - \frac{1}{q_{n_k} \cdots q_{n_{k+1}}} + \frac{1}{q_{n_k} \cdots q_{n_{k+2}}} - \cdots \right\|,$$

so

$$\begin{aligned} \frac{1}{q_{n_k}} - \frac{1}{q_{n_k} \cdots q_{n_{k+1}}} &\leq \|u_{n_k-1}\chi_S\| \leq \\ &\leq \frac{1}{q_{n_k}} - \frac{1}{q_{n_k} \cdots q_{n_{k+1}}} + \frac{1}{q_{n_k} \cdots q_{n_{k+2}}}. \end{aligned} \quad (5.3.4)$$

This proves (5.3.3).

To prove the first assertion one has to show that, for all values of n satisfying $n_k \leq n < n_{k+1}$, the biggest value of $\|u_{n-1}\chi_S\|$ is obtained for $n = n_k$. Indeed, as $n < n_{k+1}$, one has that $\|u_{n-1}\chi_S\|$ is

$$\begin{aligned} \left\| \frac{1}{q_n \cdots q_{n_{k+1}}} - \frac{1}{q_n \cdots q_{n_{k+2}}} + \frac{1}{q_n \cdots q_{n_{k+3}}} - \cdots \right\| &\leq \\ &\leq \frac{1}{q_n \cdots q_{n_{k+1}}} \leq \frac{1}{2^{n_{k+1}-n}q_+^{\mathbf{u}}} \leq \frac{1}{2q_+^{\mathbf{u}}}. \end{aligned} \quad (5.3.5)$$

Obviously, $\frac{1}{2q_+^u} \leq \frac{1}{q_+^u} - \frac{1}{2^{d_k-1}(q_+^u)}$. Hence, (5.3.3) and (5.3.5) yield $\|u_{n-1}x_S\| \leq \|u_{n_k-1}x_S\|$.

(ii) Let us first prove that if $0 < n < n_1$, then $\|u_{n-1}x_S\| \leq \|u_{n_1-1}x_S\|$. Indeed,

$$\begin{aligned} \|u_{n-1}x_S\| &\leq \frac{1}{2^{n_1-n}q_+^u} \leq \frac{1}{2q_+^u} \leq \frac{1}{q_+^u} - \frac{1}{2^{d_1-1}(q_+^u)^2} \\ &\leq \frac{1}{q_{n_1}} - \frac{1}{q_{n_1} \cdots q_{n_2}} \leq \|u_{n_1-1}x_S\|, \end{aligned} \quad (5.3.6)$$

where the first inequality is obtained in analogy with (5.3.5), while the last one follows from (5.3.4) with $k = 1$. Now the assertion follows from (ii), (5.3.6), the definition of ρ_u and the obvious inequality $\rho_u(x_S, 0) \geq \max\{\|x_S\|, \sup\{\|u_{n_k-1}x_S\| \mid k \in \mathbb{N}\}\}$.

(iii) It is easy to see that

$$\begin{aligned} x_S &= \left(\frac{q_{n_1+1}-1}{u_{n_1+1}} + \frac{q_{n_1+2}-1}{u_{n_1+2}} + \dots + \frac{q_{n_2}-1}{u_{n_2}} \right) + \\ &+ \left(\frac{q_{n_3+1}-1}{u_{n_3+1}} + \dots + \frac{q_{n_4}-1}{u_{n_4}} \right) + \dots \end{aligned} \quad (5.3.7)$$

Since infinitely many coefficients $c_j = 0$ (e.g., those with $n_{2k} < j \leq n_{2k+1}$), (5.3.7) is the canonical representation of x_S . The same holds for $x_{S'}$. It is clear now, that $x_S \neq x_{S'}$ when $S \neq S'$. \square

Proposition 5.3.6 ([38, Proposition C]). *Let u be an a -sequence with bounded ratio sequence and let $\delta_k := m_{k+1} - m_k$, where $S_+^u = \{m_k \mid k \in \mathbb{N} \text{ and } m_k < m_{k+1}\}$.*

(i) *there are \mathfrak{c} many elements $x \in \mathbb{T}$ with $\rho_u(x, 0) = \frac{1}{q_+^u}$ when the sequence δ_k is unbounded;*

(ii) *if the sequence δ_k is bounded, then $\left| B_{\frac{1}{q_+^u}}^{\rho_u}(0) \right| = \mathfrak{c}$.*

Proof. Pick a subsequence S of $S_+^u = (m_k)$ as in (5.3.1), such that $n_1 > m_1$, so that $u_{n_1} \geq 2q_+^u$. Hence,

$$\|x_S\| \leq \frac{1}{u_{n_1}} \leq \frac{1}{2q_+^u}. \quad (5.3.8)$$

From the Lemma 5.3.5 we have

$$\begin{aligned}
 \frac{1}{q_+} - \frac{1}{2^{d_{k+1}}} &\leq \|u_{n_k-1} x_S\| \leq \\
 &\leq \frac{1}{q_{n_k}} - \frac{1}{q_{n_k} \cdots q_{n_{k+1}}} + \frac{1}{q_{n_k} \cdots q_{n_{k+2}}} \leq \\
 &\leq \frac{1}{q_+} - \frac{1}{q_+^{d_{k+1}}} + \frac{1}{2^{d_k+d_{k+1}}}.
 \end{aligned} \tag{5.3.9}$$

If the sequence of differences $\delta_k = m_{k+1} - m_k$ is unbounded, then we can choose S such that (d_k) is divergent, then (5.3.8), (5.3.9) and item (iii) of the lemma 5.3.5 ensure that $\rho_{\mathbf{u}}(0, x_S) = \frac{1}{q_+^{\mathbf{u}}}$. Since there are \mathfrak{c} many subsequences S of $S_+^{\mathbf{u}}$ with this property, this proves item (i) in view of Lemma 5.3.5(iii).

Now assume that the sequence (δ_k) is bounded. Pick a subsequence S of $S_+^{\mathbf{u}}$ as in (5.3.2), such that $d_k = n_{k+1} - n_k \leq d$ for some d and for all k . From the equality in (5.3.5) one has

$$\begin{aligned}
 \|u_{n-1} x_S\| &\leq \frac{1}{q_{n_k}} - \frac{1}{q_{n_k} \cdots q_{n_{k+1}}} + \frac{1}{q_{n_k} \cdots q_{n_{k+2}}} \\
 &= \frac{1}{q_+^{\mathbf{u}}} - \frac{1}{q_{n_k} \cdots q_{n_{k+1}}} \left(1 - \frac{1}{q_{n_{k+1}+1} \cdots q_{n_{k+2}}} \right) \\
 &\leq \frac{1}{q_{\mathbf{u}}} - \frac{1}{(q_+^{\mathbf{u}})^{d_{k+1}}} \left(1 - \frac{1}{2} \right) \\
 &\leq \frac{1}{q_+^{\mathbf{u}}} - \frac{1}{2(q_+^{\mathbf{u}})^d} < \frac{1}{q_+^{\mathbf{u}}}
 \end{aligned}$$

Hence, from (5.3.8) and in view of Lemma 5.3.5(b) one can deduce $x_S \in B_{\frac{1}{q_+^{\mathbf{u}}}}^{\rho_{\mathbf{u}}}(0)$. To get $\left| B_{\frac{1}{q_+^{\mathbf{u}}}}^{\rho_{\mathbf{u}}}(0) \right| = \mathfrak{c}$ take \mathfrak{c} many subsequences S of $S_{\mathbf{u}}$ as above and use item (iii) of Lemma 5.3.5. \square

Taking an \mathbf{a} -sequence with $q_+^{\mathbf{u}} = q_s^{\mathbf{u}}$ and $S_s^{\mathbf{u}} = S_+^{\mathbf{u}}$ big (in the sense that δ_n is bounded) in Proposition 5.3.6 will show that the radius $\frac{1}{2q_s^{\mathbf{u}}}$ in Proposition 4.2.1 cannot be increased to $\frac{1}{q_s^{\mathbf{u}}}$.

5.3.3 When $t_{\mathbf{u}}(\mathbb{T})$ is $\tau_{\mathbf{u}}$ -open

Proposition 5.3.7. *If \mathbf{u} is an a -sequence with $t_{\mathbf{u}}(\mathbb{T}) \in \tau_{\mathbf{u}}$, then the sequence of ratios (q_n) is bounded.*

Proof. Assume for a contradiction that

$$\lim_t q_{n_t} = \infty \quad (5.3.10)$$

for some subsequence $\{q_{n_t} \mid t \in \mathbb{N}\}$. One can assume without loss of generality that $q_{n_t} \geq 2$ for all t . We prove that $t_{\mathbf{u}}(\mathbb{T}) \notin \tau_{\mathbf{u}}$. To do that one can prove that $B_\varepsilon^p(0) \not\subseteq t_{\mathbf{u}}(\mathbb{T})$ for every $\varepsilon < \frac{1}{2}$. Hence one can find an element $x \in B_\varepsilon^p(0)$ such that $x \notin t_{\mathbf{u}}(\mathbb{T})$.

Let $\tilde{x} \in [0, 1)$ be such that

$$\tilde{x} = \sum_n \frac{c_n}{u_n}; \quad (5.3.11)$$

where $c_n = 0$ if $n \neq n_t$ for every t otherwise if $n = n_t$ $c_n = \lfloor q_n \varepsilon / 4 \rfloor$. Note that (5.3.11) is a \mathbf{u} -representation of \tilde{x} , as

$$c_{n_t} \leq \frac{q_{n_t} \varepsilon}{4} < \frac{q_{n_t}}{8} < q_{n_t} - 1 \quad (5.3.12)$$

for every t .

Let us prove that $\pi(\tilde{x}) = x \in B_\varepsilon^p(0)$. Pick $s \in \mathbb{N}$ and find a positive $r \in \mathbb{N}$ such that $n_{r-1} \leq s \leq n_r - 1$.

Hence, one has the following congruence modulo 1

$$\begin{aligned} u_s \tilde{x} \equiv_1 & \frac{c_{s+1}}{q_{s+1}} + \frac{c_{s+2}}{q_{s+1} q_{s+2}} + \cdots + \\ & + \frac{c_{n_r-1}}{q_{s+1} \cdots q_{n_r-1}} + \frac{c_{n_r}}{q_{s+1} \cdots q_{n_r}} + \cdots . \end{aligned}$$

By construction $c_{s+1} = \cdots = c_{n_r-1} = 0$ and $c_{n_j} \leq \frac{q_{n_j} \varepsilon}{4}$ for all $j \geq r$, according to (5.3.12). Therefore, $\frac{c_{n_j}}{q_{n_j}} \leq \frac{\varepsilon}{4}$, hence

$\frac{c_{n_j}}{q_{s+1} \cdots q_{n_j}} \leq \frac{\varepsilon}{2} \cdot \frac{1}{2^{n_j-s}}$. This yields

$$\begin{aligned} \|u_s x\| &= \left\| \frac{c_{n_r}}{q_{s+1} \cdots q_{n_r}} + \cdots \right\| \\ &\leq \frac{\varepsilon}{2} \sum_{j=r}^{\infty} \frac{1}{2^{n_j-s}} \leq \frac{\varepsilon}{2} \sum_{m=1}^{\infty} \frac{1}{2^m} = \frac{\varepsilon}{2}. \end{aligned}$$

Hence for every s one has $\|u_s x\| \leq \frac{\varepsilon}{2} < \varepsilon$, i. e. $x \in B_\varepsilon^p(0)$.
As $\frac{q_{n_t} \varepsilon}{4} - 1 \leq c_{n_t} \leq \frac{q_{n_t} \varepsilon}{4}$, the following inequality holds

$$\frac{\varepsilon}{4} - \frac{1}{q_{n_t}} \leq \frac{c_{n_t}}{q_{n_t}} \leq \frac{\varepsilon}{4}.$$

Therefore, (5.3.10) implies that $\frac{c_{n_t}}{q_{n_t}}$ converges to $\frac{\varepsilon}{4} \neq 0$. By Corollary 5.2.5, this yields $x \notin t_u(\mathbb{T})$. \square

The above proposition fails in the case of a general sequence of integers. Indeed, if H is a countable subgroup of \mathbb{T} , then H is characterized by some sequence \mathbf{u} that need not have bounded ratios (in case \mathbf{u} has bounded ratios, one can replace \mathbf{u} by some $\mathbf{u}^* \sim \mathbf{u}$ with $q_+^{\mathbf{u}^*} = \infty$, according to Proposition 4.1.14).

5.4 \mathcal{F}_σ SUBGROUPS CHARACTERIZED BY MEAN OF A-SEQUENCES

Theorem 5.4.1 ([38, Theorem E]). *The following are equivalent for an a-sequence \mathbf{u} in \mathbb{Z} :*

- (i) (q_n) is bounded;
- (ii) $t_u(\mathbb{T}) \leq \mathbb{Q}/\mathbb{Z}$;
- (iii) $t_u(\mathbb{T})$ is countable;
- (iv) $t_u(\mathbb{T}) \in \mathfrak{F}_\sigma(\mathbb{T})$;
- (v) $t_u(\mathbb{T})$ is $\tau_{\mathbf{u}}^\sigma$ -open;
- (vi) $t_u(\mathbb{T})$ is $\tau_{\mathbf{u}}$ -open.

Proof. The implication (i) \Rightarrow (ii) follows by Theorem 5.2.2; while (ii) \Rightarrow (iii) \Rightarrow (iv) are obvious.

(iv) \Rightarrow (v) follows by Theorem 3.2.4,

(v) \Rightarrow (vi) follows from the fact that $\tau_{\mathbf{u}}^* \subseteq \tau_{\mathbf{u}}$;

(vi) \Rightarrow (i) follows from Proposition 5.3.7. \square

Corollary 5.4.2 ([38, Corollary E]). *If \mathbf{u} is an a-sequence with unbounded sequence of ratios, then $t_u(\mathbb{T}) \notin \mathfrak{F}_\sigma(\mathbb{T})$.*

This corollary covers the results from [23] and [55] with $\mathbf{u} = (2^{2^n})$ and $\mathbf{u} = (n!)$.

Some implications of Theorem 5.4.1 hold for a general sequence of integers \mathbf{u} , while others are no more valid as showed in the next examples.

Example 5.4.3 ((i) $\not\Rightarrow$ (ii) \Leftarrow (iii)). If \mathbf{u} is the Fibonacci's sequence, then (q_n) is bounded, so $t_{\mathbf{u}}(\mathbb{T})$ is countable (by virtue of Theorem 4.1.11), i. e. (i) and (iii) of Theorem 5.4.1 holds. Indeed $u_n = u_{n-1} + u_{n-2}$ for all $n > 1$ and $u_0 = u_1 = 1$. Hence, $q_n = \frac{u_n}{u_{n-1}} = 1 + \frac{u_{n-2}}{u_{n-1}} \leq 2$ for all $n \in \mathbb{N}$.

On the other hand, $t_{\mathbf{u}}(\mathbb{T})$ is the infinite cyclic group generated by the fractional part of the golden ratio as mentioned in Example 4.4.6. In particular, $t_{\mathbf{u}}(\mathbb{T})$ is not torsion.

Remark 5.4.4 ((ii) $\not\Rightarrow$ (i) \Leftarrow (iii)). Take any infinite subgroup H of \mathbb{Q}/\mathbb{Z} . By Remark 4.1.14, H has a characterizing sequence such that its sequence of ratios is unbounded. This proves the non-implication (ii) $\not\Rightarrow$ (i) in Theorem 5.4.1. As (ii) implies (iii), this witnesses also the non-implication (iii) $\not\Rightarrow$ (i)

From Theorem 3.2.4 one has the equivalence between (iv) and (v).

Part III

APPLICATIONS, RELATED ARGUMENTS AND OPEN QUESTIONS

Characterized subgroups have several links with other branches of mathematics, one of these is Harmonic Analysis, where the sets of uniqueness for a trigonometric series are closely related to characterized subgroups (see [23, 47, 61, 62, 64]). These techniques give the possibility to answer positively [7, Problem 5.1(b)]. Moreover, we present one of the main application in topology, namely the study of precompact group topologies with and without convergent sequences.

TRIGONOMETRIC THIN SETS

In this chapter, after a brief historical introduction on thin sets of Harmonic Analysis, some original results from [6] are presented. In particular, special Dirichlet sets related to arithmetic sequences are studied, which allow for a solution of an open problem from [7].

6.1 THIN SETS

6.1.1 Sets of uniqueness and \mathbb{N} -sets

Let $S_{\mathbf{a},\mathbf{b}}(x)$ be the following series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi n x + b_n \sin 2\pi n x) \quad (6.1.1)$$

where $\mathbf{a} = (a_n), \mathbf{b} = (b_n) \in \mathbb{R}^{\mathbb{N}}$ ($b_0 = 0$ for simplicity) and $x \in \mathbb{R}$. The series $S_{\mathbf{a},\mathbf{b}}(x)$ is called a *trigonometric (or, Fourier) series*. From the well-known *Fourier formulas* one has that some “good enough” periodic real functions f admit a *trigonometric expansion*, i. e. there exists $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{\mathbb{N}}$ such that $f(x) = S_{\mathbf{a},\mathbf{b}}(x)$. In this case, the following is a natural problem to pose.

Problem 6.1.1 (Uniqueness Problem). In case a real function f admits a trigonometric expansion, is such an expansion unique?

Clearly, the trigonometric expansion of f is unique if and only if for every $\mathbf{a}, \mathbf{b}, \mathbf{a}', \mathbf{b}'$ such that $f(x) = S_{\mathbf{a},\mathbf{b}}(x) = S_{\mathbf{a}',\mathbf{b}'}(x)$ one has that $(a_n - a'_n) = (b_n - b'_n) = 0$ for every $n \in \mathbb{N}$. So Problem 6.1.1 is equivalent to the problem to find when, for a given trigonometric series $S_{\mathbf{a},\mathbf{b}}$, the sequences \mathbf{a}, \mathbf{b} are identically 0. In 1870 Cantor proved that this occurs if $S_{\mathbf{a},\mathbf{b}}(x)$ converges to 0 for all but finitely many $x \in [0, 1]$. Young extended

this by showing that countably many exceptions are still possible. The first notion of smallness related to trigonometric series comes from this aspect of uniqueness.

Definition 6.1.2. A set $A \subseteq [0, 1]$ is said to be a *set of uniqueness* (shortly *U-set*), if every trigonometric series $S_{\mathbf{a},\mathbf{b}}(x)$ converging to 0 outside A is identically 0.

The second notion of smallness comes from series $S_{\mathbf{a},\mathbf{b}}(x)$ that do not converge absolutely everywhere.

Theorem 6.1.3 (Denjoy-Lusin). *If $S_{\mathbf{a},\mathbf{b}}(x)$ converges absolutely on a set $A \subseteq [0, 1]$ that is either non-meager or has positive Lebesgue measure, then*

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|) < \infty, \quad (6.1.2)$$

i. e. , $S_{\mathbf{a},\mathbf{b}}(x)$ converges absolutely everywhere.

In order to get a more concise description of these sets of absolute convergence one can introduce some notation. For a pair of sequences $\mathbf{a} = (a_n)_{n \in \mathbb{N}}$ and $\mathbf{b} = (b_n)_{n \in \mathbb{N}}$, let

$$\mathcal{C}_{\mathbf{a},\mathbf{b}} = \{x \in [0, 1] : S_{\mathbf{a},\mathbf{b}}(x) \text{ is absolutely convergent}\}.$$

Hence, Denjoy-Lusin Theorem can be given also in the following counterpositive form:

Corollary 6.1.4 (Denjoy-Lusin). *For sequences \mathbf{a} and \mathbf{b} of real numbers the following conditions are equivalent:*

- (a) $\mathcal{C}_{\mathbf{a},\mathbf{b}}$ is a meager set of Lebesgue measure zero;
- (b) $\mathcal{C}_{\mathbf{a},\mathbf{b}} \neq [0, 1]$;
- (c) the sequences \mathbf{a} and \mathbf{b} fail (6.1.2).

The following notion (investigated first by Fatou) was named in honor of Niemytzki:

Definition 6.1.5. A set $A \subseteq [0, 1]$ is said to be an *N-set* if and only if $A \subseteq \mathcal{C}_{\mathbf{a},\mathbf{b}} \neq [0, 1]$ for some sequences \mathbf{a} and \mathbf{b} .

By Corollary 6.1.4 every N-set is a meager set of Lebesgue measure zero.

Inspired by a theorem of Salem, Arbault proved the following:

Theorem 6.1.6 ([2]). *The set $A \subseteq [0, 1]$ is an N-set if and only if there are reals $\rho_n \geq 0$, $n \in \mathbb{N}$ such that $\sum_{n=1}^{\infty} \rho_n = \infty$ and the series $\sum_{n=1}^{\infty} \rho_n \sin \pi n x$ converges absolutely on A .*

For a sequence $\rho = (\rho_n)_{n \in \mathbb{N}}$ of non negative real numbers let

$$\mathcal{N}_\rho = \{x \in [0, 1] : \sum_{n=1}^{\infty} \rho_n \sin \pi n x \text{ is absolutely convergent}\}.$$

Clearly, $\mathcal{N}_\rho = [0, 1]$ if and only if $\sum_{n=1}^{\infty} \rho_n < \infty$. This explains the restraint $\sum_{n=1}^{\infty} \rho_n = \infty$ in the above theorem, which can be announced also as follows:

$A \subseteq [0, 1]$ is an N-set if and only if there are reals $\rho_n \geq 0$ such that $A \subseteq \mathcal{N}_\rho$ for $\rho = (\rho_n)_{n \in \mathbb{N}}$ and $\sum_{n=1}^{\infty} \rho_n = \infty$.

6.1.2 A-sets, D-sets and wD-sets

Finally, we state the definitions with more connections to characterized subgroups of \mathbb{T} :

Definition 6.1.7 (Arbault). A set $A \subseteq [0, 1]$ is an A-set if there is an increasing sequence of positive integers $\mathbf{u} = (u_n)_{n \in \mathbb{N}}$ such that $\lim \sin \pi u_n x = 0$ for all $x \in A$.

Notation 6.1.8. Denote by $\|x\|_{\mathbb{Z}}$ the distance from the integers of a real number x . Moreover, denote by $\equiv_{\mathbb{Z}}$ the congruence modulo \mathbb{Z} . Hence, for every $x_1, x_2 \in \mathbb{R}$ one has that $\|x_1\|_{\mathbb{Z}} = \|x_2\|_{\mathbb{Z}}$ if and only if either $x_1 \equiv_{\mathbb{Z}} x_2$ or $x_1 \equiv_{\mathbb{Z}} -x_2$.

Remark 6.1.9. A set $A \subseteq [0, 1]$ is an A-set if and only if $\pi(A) \subseteq \mathfrak{t}_{\mathbf{u}}(\mathbb{T})$ for some increasing sequence of positive integers \mathbf{u} , i. e. $\|u_n x\|_{\mathbb{Z}} \rightarrow 0$ pointwise on A .

Definition 6.1.10. Let $A \subseteq [0, 1]$.

- (a) A is a *Dirichlet set* (briefly, *D-set*) if there is an increasing sequence of positive integers $\mathbf{u} = (u_n)_{n \in \mathbb{N}}$ such that $(\sin \pi u_n x)_{n \in \mathbb{N}}$ converges uniformly to 0 on A ;
- (b) A is a *weakly Dirichlet set* (briefly, *wD-set*) if A is Borel and for every positive Borel measure μ on $[0, 1]$ there is an increasing sequence of positive integers $\mathbf{u} = (u_n)_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow \infty} \int_A |e^{2\pi i u_n x} - 1| d\mu = 0$.

Lemma 6.1.11. *A subset A of $[0, 1]$ is a D-set if and only if there exists an increasing sequence of positive integers $(u_n)_{n \in \mathbb{N}}$ such that $(\|u_n x\|_{\mathbb{Z}})_{n \in \mathbb{N}}$ uniformly converges to 0 on A .*

Proof. Let $\alpha \in \mathbb{R}$ and $\tilde{\alpha} \in [-\frac{1}{2}, \frac{1}{2})$ such that $\alpha = \tilde{\alpha} + k$ for some $k \in \mathbb{Z}$. Then

$$\sin \pi u_n x \leq |\sin \pi u_n x| = |\sin \pi \widetilde{u_n x}| \leq |\pi \widetilde{u_n x}| = \pi |\widetilde{u_n x}| = \pi \|u_n x\|_{\mathbb{Z}}.$$

Therefore, if $\|u_n x\|_{\mathbb{Z}} \rightarrow 0$ uniformly also $\sin \pi u_n x \rightarrow 0$ uniformly.

Conversely,

$$\|u_n x\|_{\mathbb{Z}} = |\widetilde{u_n x}| \leq |\pi \widetilde{u_n x}| \leq |\tan \pi \widetilde{u_n x}| = |\tan \pi u_n x| = \left| \frac{\sin \pi u_n x}{\cos \pi u_n x} \right|.$$

Since $\sin(x) = \cos(x - \frac{\pi}{2})$, if $\sin \pi u_n x \rightarrow 0$ uniformly, then $\cos \pi u_n x \rightarrow 1$ uniformly, hence $\|u_n x\|_{\mathbb{Z}} \rightarrow 0$ uniformly. \square

Remark 6.1.12. Let $A \subseteq [0, 1]$ be a D-set and $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$ witnessing that, then:

- (i) A is also an A -set, in particular $\pi(A) \subseteq t_{\mathbf{u}}(\mathbb{T})$;
- (ii) if $B \subseteq A$, then B is a D-set too and \mathbf{u} witnesses that also for B ;
- (iii) as a consequence of Lemma A.1.3 one has that if A is a D-set, then \overline{A} is a D-set.

6.1.3 Thin sets of the circle group

Remark 6.1.13. The thin sets defined until this point are defined as subsets of $[0, 1]$. Since these notions hold for periodic functions, one can extend these notions for subsets A of the circle group \mathbb{T} applying these definitions to the unique preimages $\pi_1^{-1}(A)$ in $[0, 1)$, e. g. $A \subseteq \mathbb{T}$ is a D-set of \mathbb{T} if and only if $\pi_1^{-1}(A)$ is a D-set of $[0, 1]$, i. e. $A \subseteq \mathbb{T}$ is a D-set if and only if there exists an increasing sequence of positive integers \mathbf{u} such that $u_n x \rightarrow 0$ uniformly on A .

Notation 6.1.14. One can denote by $\mathfrak{N}(\mathbb{T})$, $\mathfrak{A}(\mathbb{T})$, $\mathfrak{D}(\mathbb{T})$ and $\mathfrak{wD}(\mathbb{T})$ respectively the class of all N-sets, A-sets, D-sets and wD-sets of \mathbb{T} .

Proposition 6.1.15. *Every characterized subgroup of \mathbb{T} is a wD-subgroup of \mathbb{T} .*

Proof. The proposition is a consequence of the Dominated Convergence Theorem. Indeed, consider $t_{\mathbf{u}}(\mathbb{T})$ where \mathbf{u} is a sequence of integers. Let $x \in t_{\mathbf{u}}(\mathbb{T})$ and let \tilde{x} be its unique preimage in $[0, 1)$. Then $e^{2\pi i u_n \tilde{x}} \rightarrow 1$ by definition of $t_{\mathbf{u}}(\mathbb{T})$. Since $|e^{2\pi i u_n \tilde{x}}| \leq 1$, the Dominated Convergence Theorem implies that

$$\lim_{n \rightarrow \infty} \int_A |e^{2\pi i u_n x} - 1| d\mu = 0, \text{ where } A = \pi_1^{-1}(t_{\mathbf{u}}(\mathbb{T})).$$

Therefore, A is a wD-set of $[0, 1]$ and hence $t_{\mathbf{u}}(\mathbb{T})$ is a wD-set of \mathbb{T} . \square

Theorem 6.1.16 ([61, pg. 49]). *A Borel set $A \subseteq \mathbb{T}$ is an N-set if and only if it is contained in a σ -compact wD-set (or equivalently, wD-subgroup)*

As a consequence of Proposition 6.1.15 and previous theorem one has the following chain of inclusion.

$$\text{Char}(\mathbb{T}) \cap \mathfrak{F}_{\sigma}(\mathbb{T}) \subseteq \mathfrak{wD}(\mathbb{T}) \cap \mathfrak{F}_{\sigma}(\mathbb{T}) \subseteq \mathfrak{N}(\mathbb{T}) \cap \mathfrak{F}_{\sigma}(\mathbb{T}).$$

Indeed, by Proposition 6.1.15 one has that $\mathcal{C}\text{har}(\mathbb{T}) \subseteq \mathfrak{wD}(\mathbb{T})$. For the second inclusion note that, an \mathcal{F}_σ \mathfrak{wD} -subgroup of \mathbb{T} (so σ -compact) is also an \mathbb{N} -set by Theorem 6.1.16. Theorem 5.4.1 proves that $t_{\mathbf{u}}(\mathbb{T}) \notin \mathfrak{F}_\sigma(\mathbb{T})$ if \mathbf{u} is a \mathbf{q} -divergent \mathbf{a} -sequence. In [55], Gabrielyan proved that $t_{\mathbf{u}}(\mathbb{T}) \notin \mathfrak{F}_\sigma(\mathbb{T})$, whenever $\mathbf{u} = (n!)$, but much earlier Arbault in [2] proved that $t_{\mathbf{u}}(\mathbb{T})$ is not an \mathbb{N} -set, for $\mathbf{u} = (2^{2^n})_n$ and hence, by previous inclusions, not in $\mathfrak{F}_\sigma(\mathbb{T})$.

An infinite characterized subgroup cannot be a \mathbb{D} -set as Proposition 6.1.17 shows.

Proposition 6.1.17. *If A is a dense subset of \mathbb{T} , then A is not a \mathbb{D} -set.*

Proof. Suppose that A is \mathbb{D} -set, by Remark 6.1.12(iii) one has that $\overline{A} = \mathbb{T}$ is a \mathbb{D} -set. But, this is impossible, since \mathbb{T} can be characterized only by an eventually null sequence by Proposition 4.1.7. \square

Corollary 6.1.18. *If H is an infinite subgroup of \mathbb{T} , then H is not a \mathbb{D} -set.*

Proof. Every infinite subgroup of \mathbb{T} is dense. \square

6.2 \mathbb{D} -SETS AND CHARACTERIZED SUBGROUPS OF \mathbb{T}

In this section, we study a special kind of \mathbb{D} -sets, introduced by Marcinkiewicz, defined starting from an \mathbf{a} -sequence \mathbf{u} and a subset L of \mathbb{N} , which properties allow for an answer to [7, Problem 5.1(b)].

6.2.1 Definition and properties of $K_L^{\mathbf{u}}$

Definition 6.2.1. Let $L \subseteq \mathbb{N}$ and \mathbf{u} be an \mathbf{a} -sequence, the following subset of $[0, 1)$ is defined:

$$K_L^{\mathbf{u}} = \{x \in [0, 1) : \text{supp}_{\mathbf{u}}(x) \subseteq L\},$$

where $\text{supp}_{\mathbf{u}}(x)$ is the support of the \mathbf{u} -representation of x (see Notation 5.1.3). Note that $K_{\mathbb{N}}^{\mathbf{u}} = [0, 1)$.

Lemma 6.2.2. *Let $L_1, L_2 \subseteq \mathbb{N}$ and \mathbf{u} be an a -sequence.*

- (i) *If $L_1 \subseteq L_2$, then $K_{L_1}^{\mathbf{u}} \subseteq K_{L_2}^{\mathbf{u}}$.*
- (ii) *If $L = L_1 \cup L_2$, then $K_{L_1}^{\mathbf{u}} + K_{L_2}^{\mathbf{u}} \supseteq K_L^{\mathbf{u}}$. If, in addition, $L_1 \cap L_2 = \emptyset$, then $K_{L_1}^{\mathbf{u}} + K_{L_2}^{\mathbf{u}} = K_L^{\mathbf{u}}$.*
- (iii) *If $L_1 \cup L_2 = \mathbb{N}$, then $K_{L_1}^{\mathbf{u}} + K_{L_2}^{\mathbf{u}} \supseteq [0, 1)$. If, in addition, L_1, L_2 is a partition of \mathbb{N} , $K_{L_1}^{\mathbf{u}} + K_{L_2}^{\mathbf{u}} = [0, 1)$.*

Proof. (i) is clear and (iii) follows from (ii).

(ii) If $x \in K_L^{\mathbf{u}}$, then $x = \sum_{i \in L_1 \cup L_2} \frac{a_i}{u_i}$ for some $a_i \in \mathbb{Z}$; therefore, $x = \sum_{i \in L_1} \frac{a_i}{u_i} + \sum_{i \in L_2 \setminus L_1} \frac{a_i}{u_i} \in K_{L_1}^{\mathbf{u}} + K_{L_2}^{\mathbf{u}}$. Conversely, if $L_1 \cap L_2 = \emptyset$ and $x \in K_{L_1}^{\mathbf{u}} + K_{L_2}^{\mathbf{u}}$, then $\text{supp}(x) \subseteq L_1 \cup L_2$ and so $x \in K_{L_1 \cup L_2}^{\mathbf{u}}$.

(iii) is a direct consequence of (ii). \square

Definition 6.2.3. Let $L \subseteq \mathbb{N}$ be infinite and not cofinite. Let $\mathcal{G}^L = \{G_i^L \mid i \in \mathbb{N}\}$ be the family of the *connected components* of $\mathbb{N} \setminus L$ (i.e. maximal subsets, with respect to the inclusion, of consecutive numbers). If

$$G_i^L = \{m_i^L \leq m_i^L + 1 \leq \dots \leq \underline{m}_i^L - 1 \leq \underline{m}_i^L\},$$

then $m_i^L - 1, \underline{m}_i^L + 1 \in L$. Call the elements of \mathcal{G}^L *the gaps* of L and denote by $(g_n^L)_n$ the sequence of the length of the gaps of L , i.e. $g_n^L = |G_n^L| = \underline{m}_n^L - m_n^L + 1$.

Remark 6.2.4. Let $L \subseteq \mathbb{N}$, recall that a large set is as in Definition A.3.17. Note that if L is finite, then L is obviously non-large, on the other hand if L is cofinite, then L is large. Therefore, one can consider only infinite non-cofinite $L \subseteq \mathbb{N}$. In the latter case the following are equivalent definitions of *non-large* set.

- (i) for every finite $F \subset \mathbb{N}$ one has $L + F \neq \mathbb{N}$ (or equivalently $(L - F) \cap \mathbb{N} \neq \mathbb{N}$);
- (ii) L has gaps of unbounded length, i.e. there exists a subsequence $(g_{n_k}^L)_k$ of $(g_n^L)_n$ such that $g_{n_k}^L \rightarrow \infty$.

(i) \Rightarrow (ii). Suppose that (ii) does not hold, i. e. $(g_n^L)_n$ is bounded. Therefore, there exists $M \geq |G_n^L|$ for every $n \in \mathbb{N}$. By definition of G_n^L , one has that $L + \{0, 1, \dots, M\} = \mathbb{N}$.

(ii) \Rightarrow (i). Suppose that (ii) holds. Let $F = \{0, \dots, M\}$ and $n \in \mathbb{N}$ such that $g_n^L > M$. Then there exists $t \in G_n^L$ such that $t \notin L + F$, hence $L + F \neq \mathbb{N}$.

Example 6.2.5. (i) If L is the set of elements of an arithmetic progression, then L is large. Indeed, the gaps of L are bounded by the common difference of the arithmetic progression;

(ii) If L is the set of elements of a geometric progression with common ratio strictly greater than 1, then L is non-large.

(iii) If $L = \{k^{2^n} \mid n \in \mathbb{N}\}$, the L is non-large.

In [23] the authors refers to non-large sets as colacunary sets, note also that $K_L^{\mathbf{u}}$ are defined only for $\mathbf{u} = (2^{2^n})_n$.

6.2.2 $K_L^{\mathbf{u}}$ is a D-set for an a-sequence \mathbf{u} and a non-large set L

The following results were proved in the case of $\mathbf{u} = (2^{2^n})$ by Marcinkiewicz in [73].

Proposition 6.2.6. *Let \mathbf{u} be an a-sequence, and let L be a non-large set, then $K_L^{\mathbf{u}}$ is a D-set and $\pi(K_L^{\mathbf{u}}) \subseteq \mathfrak{t}_{\mathbf{u}^*}(\mathbb{T})$, where \mathbf{u}^* is a subsequence of \mathbf{u} . More precisely, if $G_{n_k}^L = \{m_{n_k}, \dots, \underline{m}_{n_k}\}$ is a subsequence of gaps of L such that $g_{n_k}^L \rightarrow \infty$, then one can choose $\mathbf{u}^* = (\mathbf{u}_{m_{n_k}^L - 1})_k$.*

Proof. In this proof one can omit the superscript L . It suffices to prove that $\left\| \mathbf{u}_{m_{n_k} - 1} \chi \right\|_{\mathbb{Z}} \rightarrow 0$ uniformly on $K_L^{\mathbf{u}}$. For $\chi \in K_L^{\mathbf{u}}$, let $\chi = \sum_{i \in L} \frac{c_i}{u_i}$ be its \mathbf{u} -representation. Recall that $c_i \leq q_i - 1$ for every $i \in \mathbb{N}$ and hence

$$\begin{aligned}
\left\| \mathbf{u}_{m_{n_k}-1} \chi \right\|_{\mathbb{Z}} &= \left\| \mathbf{u}_{m_{n_k}-1} \sum_{i=\underline{m_{n_k}+1}}^{\infty} \frac{c_i}{u_i} \right\|_{\mathbb{Z}} = \\
&= \left\| \frac{\mathbf{u}_{m_{n_k}-1}}{\underline{u_{m_{n_k}}}} \left(\frac{c_{\underline{m_{n_k}+1}}}{\underline{q_{m_{n_k}+1}}} + \frac{c_{\underline{m_{n_k}+2}}}{\underline{q_{m_{n_k}+1} q_{m_{n_k}+2}}} + \dots \right) \right\|_{\mathbb{Z}} \\
&\leq \left| \frac{1}{\underline{q_{m_{n_k}}} \cdots \underline{q_{m_{n_k}}}} \left(\frac{\underline{q_{m_{n_k}+1} - 1}}{\underline{q_{m_{n_k}+1}}} + \frac{\underline{q_{m_{n_k}+2} - 1}}{\underline{q_{m_{n_k}+1} q_{m_{n_k}+2}}} + \dots \right) \right| \\
&\leq \frac{1}{2^{g_{n_k}}}.
\end{aligned}$$

The last inequality holds since $q_i \geq 2$ for every i and

$$\left(\frac{\underline{q_{m_{n_k}+1} - 1}}{\underline{q_{m_{n_k}+1}}} + \frac{\underline{q_{m_{n_k}+2} - 1}}{\underline{q_{m_{n_k}+1} q_{m_{n_k}+2}}} + \dots \right) \leq 1$$

by Remark 5.1.4. Let $\varepsilon > 0$, then there exists k_ε such that for every $k \geq k_\varepsilon$ one has that $\frac{1}{2^{g_{n_k}}} < \varepsilon$. Hence for every $\chi \in K_L^{\mathbf{u}}$ and every $k \geq k_\varepsilon$, $\left\| \mathbf{u}_{m_{n_k}-1} \chi \right\| < \varepsilon$. Hence, one can choose $\mathbf{u}^* = (\mathbf{u}_{m_{n_k}-1})_k$. □

The next corollary, that is a direct consequence of Proposition 6.2.6 and Lemma 6.2.2(iii), proves that there exist couples of D-sets (hence sets of measure 0) such that their *sums* are $[0, 1)$.

Corollary 6.2.7. *Let $L \subset \mathbb{N}$ be a non-large set with non-large complement $\mathbb{N} \setminus L$ and let \mathbf{u} be an a -sequence. Then $K_L^{\mathbf{u}}$ and $K_{\mathbb{N} \setminus L}^{\mathbf{u}}$ are two D-sets such that $K_L^{\mathbf{u}} + K_{\mathbb{N} \setminus L}^{\mathbf{u}} = [0, 1)$*

The next Theorem gives a positive answer to Problem [7, Problem 5.1(a)], asking whether there exist two non trivial sequences $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^{\mathbb{N}}$ such that $t_{\mathbf{v}}(\mathbb{T}) + t_{\mathbf{w}}(\mathbb{T}) = \mathbb{T}$. Recall that $t_{\mathbf{u}}(\mathbb{T}) = \mathbb{T}$ if and only if \mathbf{u} is eventually null (see Proposition 4.1.7). Therefore, the next theorem provides two proper characterized subgroups such that their sum is the whole torus.

Theorem 6.2.8. *If \mathbf{u} is an a-sequence, then there exist two sequences $\mathbf{v}, \mathbf{w} \subseteq \mathbf{u}$ such that $t_{\mathbf{v}}(\mathbb{T}) + t_{\mathbf{w}}(\mathbb{T}) = \mathbb{T}$.*

Proof. Consider L_1 and L_2 , two complementary non-large subset of \mathbb{N} , an example of such a pair of complementary non-large subsets of \mathbb{N} can be found in the next example. By Corollary 6.2.7 $K_{L_i}^{\mathbf{u}}$ for $i = 1, 2$ are two D-sets such that $\pi(K_{L_1}^{\mathbf{u}}) + \pi(K_{L_2}^{\mathbf{u}}) = \mathbb{T}$. By the second part of Proposition 6.2.6, one has that $\pi(K_{L_1}^{\mathbf{u}}) \subset t_{\mathbf{v}}(\mathbb{T})$ and $\pi(K_{L_2}^{\mathbf{u}}) \subset t_{\mathbf{w}}(\mathbb{T})$, where \mathbf{v} and \mathbf{w} are two suitable subsequence of \mathbf{u} . \square

Note that $t_{\mathbf{u}}(\mathbb{T}) \leq t_{\mathbf{v}}(\mathbb{T}) \cap t_{\mathbf{w}}(\mathbb{T})$, since \mathbf{v} and \mathbf{w} are subsequences of \mathbf{u} .

Example 6.2.9. Let \mathbf{u} be an a-sequence and let (n_k) be an increasing sequence of positive integers such that $(n_{k+1} - n_k) \rightarrow \infty$. If $\mathbf{v} = (u_{n_{2k}})_k$ and $\mathbf{w} = (u_{n_{2k+1}})_k$, then $t_{\mathbf{v}}(\mathbb{T}) + t_{\mathbf{w}}(\mathbb{T}) = \mathbb{T}$. Indeed,

$$L_1 = \bigcup_{k \in \mathbb{N}} [n_{2k-1}, n_{2k}) \text{ and } L_2 = \bigcup_{k \in \mathbb{N}} [n_{2k}, n_{2k+1}).$$

By the hypothesis L_1 and L_2 are two complementary non-large subset of \mathbb{N} , $\pi(K_{L_1}^{\mathbf{u}}) \subset t_{\mathbf{v}}(\mathbb{T})$ and $\pi(K_{L_2}^{\mathbf{u}}) \subset t_{\mathbf{w}}(\mathbb{T})$.

Remark 6.2.10. In the previous example \mathbf{v} and \mathbf{u} are two \mathbf{q} -divergent sequences. Note also that if $t_{\mathbf{u}}(\mathbb{T}) + t_{\mathbf{v}}(\mathbb{T}) = \mathbb{T}$ neither \mathbf{u} nor \mathbf{v} can be \mathbf{q} -bounded, since characterized subgroups are (Haar) measure-zero sets. Indeed, if \mathbf{u} is \mathbf{q} -bounded, then $t_{\mathbf{u}}(\mathbb{T})$ is countable. Therefore, $t_{\mathbf{u}}(\mathbb{T}) + t_{\mathbf{v}}(\mathbb{T})$ is countable union of measure-zero sets. By σ -additivity of the measure, $t_{\mathbf{u}}(\mathbb{T}) + t_{\mathbf{v}}(\mathbb{T})$ is a measure-zero subgroup and hence $t_{\mathbf{u}}(\mathbb{T}) + t_{\mathbf{v}}(\mathbb{T}) \neq \mathbb{T}$, since \mathbb{T} has positive measure.

6.2.3 When $K_L^{\mathbf{u}}$ is a D-set implies L is non-large

It is natural to ask when the implication of Proposition 6.2.6 can be inverted. Here, we consider sequences of the form $\mathbf{u} = (q^n)_n$ for $q \in \mathbb{N}^+$. In [6] the authors consider arbitrary \mathbf{q} -bounded a-sequences.

Proposition 6.2.11. *Let $A \subseteq \mathbb{T}$ be a D-set, $n_1, \dots, n_k \in \mathbb{Z}$, then $n_1A + n_2A + \dots + n_kA$ is a D-set.*

Proof. Let (u_n) witness that A is a D-set, then (u_n) proves also that $n_1A + n_2A + \dots + n_kA$ is a D-set too. Indeed, let $x_i \in A$ for $i \in \{1, \dots, k\}$. Then

$$\left\| u_n \sum_{i=1}^k n_i x_i \right\| \leq \sum_{i=1}^k |n_i| \|u_n x_i\| \leq m \sum_{i=1}^k \|u_n x_i\|,$$

where $m = \max\{|n_i| \mid i = 1, \dots, k\}$. Since $\|u_n x_i\| \rightarrow 0$ uniformly, for every $i \in \{1, \dots, k\}$ one can conclude that

$$\left\| u_n \sum_{i=1}^k n_i x_i \right\| \rightarrow 0 \text{ uniformly.}$$

□

Lemma 6.2.12. *If $\mathbf{u} = (q^n)$, where $q \in \mathbb{N}^+$, $L \subseteq \mathbb{N}$ and $m \in \mathbb{N}$, then $\pi(K_{(L-m) \cap \mathbb{N}}^{\mathbf{u}}) = \pi(q^m K_L^{\mathbf{u}})$.*

Proof. Without loss of generality and for the sake of simplicity, here one can consider $u_0 = q_0 = 1$ and $c_0 = 0$. Let $y \in K_L^{\mathbf{u}}$ then $y = \sum_{n \in L} \frac{c_n}{q^n}$ for some c_n . Hence,

$$\|q^m y\|_{\mathbb{Z}} = \left\| \sum_{\substack{n \in L \\ n > m}} \frac{c_n}{q^{n-m}} \right\|_{\mathbb{Z}} = \left\| \sum_{\substack{k > 0 \\ k \in L-m}} \frac{c_{k+m}}{q^k} \right\|_{\mathbb{Z}}.$$

Therefore, $\pi(q^m y) \in \pi(K_{(L-m) \cap \mathbb{N}}^{\mathbf{u}})$.

Conversely, let $x \in K_{(L-m) \cap \mathbb{N}}$. Hence

$$x = \sum_{\substack{k \in (L-m) \cap \mathbb{N}}} \frac{c_k}{q^k} = \sum_{\substack{n \in L \\ n > m}} \frac{c_{n-m}}{q^{n-m}} \equiv_{\mathbb{Z}} q^m \sum_{n \in L} \frac{c_{n-m}}{q^n}.$$

Thus, $\pi(x) \in \pi(q^m K_L^{\mathbf{u}})$. □

Proposition 6.2.13. *Let $\mathbf{u} = (q^n)$ for a natural positive number q . Then $K_L^{\mathbf{u}}$ is a D-set if and only if L is non-large.*

Proof. By Proposition 6.2.6 one has the sufficiency.

Necessity. Let L be a large set, hence there exist $F = \{1, 2, \dots, m\}$ such that $(L - F) \cap \mathbb{N} = \mathbb{N}$. Suppose by contradiction that K_L^u is a D-set. Then by Propositions 6.2.11 and Lemma 6.2.12 one has that $\mathbb{T} = \pi([0, 1)) = \pi(K_{\mathbb{N}}) = \pi(K_{(L-F) \cap \mathbb{N}}) \subseteq \pi(K_{L-1}) + \dots + \pi(K_{L-m}) = \pi(qK_L) + \pi(q^2K_L) + \dots + \pi(q^mK_L) = q\pi(K_L) + \dots + q^m\pi(K_L)$ is a D-set of \mathbb{T} , a contradiction. \square

CONVERGING SEQUENCES IN PRECOMPACT GROUP TOPOLOGIES

7.1 DEFINITIONS AND BASIC PROPERTIES

Notation 7.1.1. If \mathbf{v} is a sequence in an abelian group G and τ a group topology for G , one can denote by $\mathbf{v} \xrightarrow{\tau} 0$ the fact that \mathbf{v} converges to 0_G in (G, τ) . Moreover, let G_d denote the group G equipped with the discrete topology. As in Definition A.3.21, if $H \leq \widehat{G_d}$ one can denote by τ_H the *initial topology* of H (i. e. the coarsest topology that makes H continuous).

Recall that a precompact group topology is a Hausdorff totally bounded group topology (see Definition A.3.18). Moreover, a topological group X is precompact if and only if its completion \widetilde{X} is compact.

Definition 7.1.2. Let G be an abelian group. Call a sequence \mathbf{v} in G a *TB-sequence* if there exists a precompact group topology τ such that $\mathbf{v} \xrightarrow{\tau} 0$.

Note that, this chapter deals with sequences $\mathbf{v} \in G^{\mathbb{N}}$, sometimes it is convenient, for the purposes of the chapter, to identify a sequence $\mathbf{v} \in G^{\mathbb{N}}$ with its counterpart in $\widehat{\widehat{G_d}}^{\mathbb{N}}$. In this sense, we shall write $s_{\mathbf{v}}(\widehat{G_d})$.

Proposition 7.1.3 ([40, Proposition 3.1]). *Let $H \leq \widehat{G_d}$ and $\mathbf{v} \in G^{\mathbb{N}}$. The sequence \mathbf{v} converges to 0 in τ_H if and only if $H \leq s_{\mathbf{v}}(\widehat{G_d})$.*

Proof. By the definition of τ_H , $v_n \rightarrow 0$ in τ_H if and only if $\chi(v_n) \rightarrow 0$ in \mathbb{T} for all $\chi \in H$. If one identifies v_n with $v_n^* \in \widehat{\widehat{G_d}}$, then $\chi(v_n) = v_n^*(\chi)$. Therefore, $v_n^*(\chi) = \chi(v_n) \rightarrow 0$ in \mathbb{T} for all $\chi \in H$ if and only if $H \leq s_{\mathbf{v}}(\widehat{G_d})$. \square

Notation 7.1.4. Since the supremum of a family of totally bounded group topologies is still totally bounded, one can define the finest totally bounded topology $\sigma_{\mathbf{b}\mathbf{v}}$ such that $\mathbf{v} \xrightarrow{\sigma_{\mathbf{b}\mathbf{v}}} 0$.

Note that if \mathbf{v} is not a TB-sequence, then $\sigma_{\mathbf{b}\mathbf{v}}$ is not Hausdorff.

Remark 7.1.5. Since the assignment $H \mapsto \tau_H$ is monotonically increasing with respect to the inclusion, as an immediate consequence of Proposition 7.1.3 one has that

$$\sigma_{\mathbf{b}\mathbf{v}} = \tau_{s_{\mathbf{v}}(\widehat{G}_d)}.$$

By Comfort-Ross Theorem A.3.24, all the precompact group topology on G are of the form τ_H , where $H \leq \text{hom}(G, \mathbb{T})$ separates the points of G . Hence by Peter-Weyl Theorem A.3.10 $H \leq G_d^*$ separates the points of G if and only if H is dense in \widehat{G}_d . In particular, from Theorem A.3.24, one can deduce the following proposition.

Proposition 7.1.6 ([40, Proposition 3.2]). *Let G be an abelian group and $\mathbf{v} \in G^{\mathbb{N}}$. Then the following hold:*

- (i) $w(G, \sigma_{\mathbf{b}\mathbf{v}}) = |s_{\mathbf{v}}(\widehat{G}_d)|$;
- (ii) $\sigma_{\mathbf{b}\mathbf{v}}$ is Hausdorff (hence precompact) if and only if $s_{\mathbf{v}}(\widehat{G}_d)$ is dense in \widehat{G}_d if and only if \mathbf{v} is a TB-sequence;
- (iii) $\sigma_{\mathbf{b}\mathbf{v}}$ is metrizable if and only if $s_{\mathbf{v}}(\widehat{G}_d)$ is countable and dense in \widehat{G}_d ;
- (iv) $\sigma_{\mathbf{b}\mathbf{v}}$ is linear if and only if $s_{\mathbf{v}}(\widehat{G}_d)$ is torsion.

Remark 7.1.7. One can prove that for any abelian group G one has that $(G, \tau_H)^* = H$, where $H \leq G_d^*$. If \mathbf{v} is a sequence in G , then

$$\left(\widetilde{(G, \sigma_{\mathbf{b}\mathbf{v}})} \right)^* = (G, \sigma_{\mathbf{b}\mathbf{v}})^* = s_{\mathbf{v}}(\widehat{G}_d).$$

In general, if one considers these groups equipped with the relative compact-open topologies one has $\widetilde{\widetilde{(G, \sigma_{\mathbf{b}\mathbf{v}})}} \neq \widehat{(G, \sigma_{\mathbf{b}\mathbf{v}})}$. If

$(\widehat{G}, \sigma_{b\mathbf{v}}) = (\widehat{G}, \sigma_{b\mathbf{v}})$ one says that $(G, \sigma_{b\mathbf{v}})$ *determines* $(\widehat{G}, \sigma_{b\mathbf{v}})$ (see [27]). Independently, Außenhofer and Chasco proved that every dense subgroup of a metrizable abelian group determines the group (see [4] and [24]). So, in case $s_{\mathbf{v}}(\widehat{G}_d)$ is countable and dense in \widehat{G}_d , by Proposition 7.1.6 \mathbf{v} is a TB-sequence and $\sigma_{b\mathbf{v}}$ is metrizable and hence $(\widehat{G}, \sigma_{b\mathbf{v}}) = (\widehat{G}, \sigma_{b\mathbf{v}})$ and hence $(\widehat{G}, \sigma_{b\mathbf{v}})$ is discrete. As an immediate consequence one has that

$$(\widehat{G}, \sigma_{b\mathbf{v}}) \neq (s_{\mathbf{v}}(\widehat{G}_d), \tau \upharpoonright_{s_{\mathbf{v}}(\widehat{G}_d)}),$$

where τ is the compact topology of \widehat{G}_d .

Remark 7.1.8. Every characterized precompact group (see Definition 1.2.10) gives rise to a TB-sequence in the discrete dual of its compact completion. Indeed, by Proposition 2.2.5 if H is a characterized precompact group, then $H = s_{\mathbf{v}}(\widetilde{H})$ and it is obviously dense in its compact completion \widetilde{H} . Therefore, \mathbf{v} is a TB-sequence of \widetilde{H}^* .

Lemma 1.2.17 and Corollary 2.2.20 provide some examples of precompact characterized proper subgroups and hence of non-trivial TB-sequence.

Example 7.1.9. (i) As a consequence of Lemma 1.2.17 one has that every infinite compact abelian group has a non eventually null TB-sequence. Indeed, a proper CCG dense subgroup is characterized by a non eventually null sequence, that is a TB-sequence.

(ii) As a consequence of Corollary 2.2.20 and Example 2.2.21 one has that if $G \cong \mathbf{S}^*$, where \mathbf{S} is a solenoid, then G has a non trivial TB-sequence \mathbf{v} such that $(G, \sigma_{b\mathbf{v}})$ is not metrizable. Indeed, every dense pseudoline is characterized and has cardinality \mathfrak{c} .

7.2 TB-SEQUENCES VS T-SEQUENCES

7.2.1 Precompact topologies and Hausdorff topologies

Definition 7.2.1 ([81]). Let G be an abelian group and let $\mathbf{v} \in G^{\mathbb{N}}$. If there exists a Hausdorff group topology that makes \mathbf{v} converging to 0 , then \mathbf{v} is said to be a *T-sequence*.

Clearly, every TB-sequence is a T-sequence but the converse is not true. As a corollary of Proposition 7.1.6 one has the following.

Corollary 7.2.2. *If H is a dense characterized subgroup of a compact abelian group X , then H is a T-characterized subgroup of X .*

The problem of finding a T-sequence that is not a TB-sequence is quite hard. An example of such a sequence can be found in [71, Corollary 4.5], where a T-sequence that is not a TB-sequence of $\mathbb{Z}(p^\infty)$ is studied.

Protasov and Zeleniuk proved that if \mathbf{v} is a T-sequence, then there exists the finest group topology witnessing that \mathbf{v} is a T-sequence. Let $\sigma_{\mathbf{v}}$ denote that topology. In [81], the authors proved the following deep properties for $\sigma_{\mathbf{v}}$.

- $\sigma_{\mathbf{v}}$ is never metrizable;
- $\sigma_{\mathbf{v}}$ is never precompact.

Therefore,

$$\sigma_{b\mathbf{v}} \neq \sigma_{\mathbf{v}}.$$

The following fact was established in [40, 52].

Fact 7.2.3. If \mathbf{v} is a TB-sequence of an infinite abelian group G , then for every group topology σ on G such that

$$\sigma_{b\mathbf{v}} \subseteq \sigma \subseteq \sigma_{\mathbf{v}},$$

one has that

$$s_{\mathbf{v}}(\widehat{G}_d) = (G, \sigma_{b\mathbf{v}})^* = (G, \sigma)^* = (G, \sigma_{\mathbf{v}})^*. \quad (7.2.1)$$

In particular, $\sigma_{\mathbf{b}\mathbf{v}}$, σ and $\sigma_{\mathbf{v}}$ are *compatible*, i. e. they have the same characters group. More precisely, $\sigma_{\mathbf{b}\mathbf{v}}$ is the Bohr topology of $\sigma_{\mathbf{v}}$, i. e. $\sigma_{\mathbf{b}\mathbf{v}} = \tau_{(G, \sigma_{\mathbf{v}})^*}$.

Theorem 7.2.4 ([52, Theorem 4]). *If \mathbf{v} is a T-sequence of an infinite countable abelian group G , then*

$$\widehat{(G, \sigma_{\mathbf{v}})} = \left(s_{\mathbf{v}}(\widehat{G}_{\mathbf{d}}), \mathfrak{p}_{\mathbf{v}} \right).$$

Recall that a topological group X is *reflexive* if and only if α_X is a topological isomorphism. In particular $\widehat{\widehat{X}} \cong X$.

Remark 7.2.5. Note that in general $(G, \sigma_{\mathbf{v}})$ is not reflexive, since $\widehat{\widehat{(G, \sigma_{\mathbf{v}})}} \neq (G, \sigma_{\mathbf{v}})$. Indeed, in case \mathbf{u} is \mathbf{q} -bounded, then $t_{\mathbf{u}}(\mathbb{T})$ is countable and hence $\widehat{(\mathbb{Z}, \sigma_{\mathbf{u}})} = (t_{\mathbf{u}}(\mathbb{T}), \mathfrak{p}_{\mathbf{u}})$ is discrete, by Theorem 3.2.5. Therefore, its dual is compact. On the other hand $\sigma_{\mathbf{u}}$ is never precompact so $(\mathbb{Z}, \sigma_{\mathbf{u}})$ is not reflexive. This holds for every sequence \mathbf{v} of a countable group G that characterizes a countable dense subgroup of the compact metrizable group $\widehat{G}_{\mathbf{d}}$ as stated in Corollary 7.2.7.

The following theorem, proved in [52], describes when $(G, \sigma_{\mathbf{v}})$ is reflexive. Recall that $c_0(\mathbb{T}) = \{\mathbf{x} \in \mathbb{T}^{\mathbb{N}} \mid \mathbf{x} \rightarrow 0\}$.

Theorem 7.2.6 ([52, Theorem 5]). *Let \mathbf{v} be a T-sequence of a countably infinite abelian group G . Then $(G, \sigma_{\mathbf{v}})$ is reflexive if and only if the following three conditions hold:*

- (i) \mathbf{v} is a TB-sequence;
- (ii) $(s_{\mathbf{v}}(\widehat{G}_{\mathbf{d}}), \mathfrak{p}_{\mathbf{v}})$ is dually embedded¹ in $\widehat{G}_{\mathbf{d}} \times c_0(\mathbb{T})$;
- (iii) $(G, \sigma_{\mathbf{v}})$ is locally quasi-convex.

The following corollary, that follows from Theorem 3.2.5, Proposition 7.1.6 and Remark 7.2.5 underlines the relations between the topologies that appear in this Thesis.

Corollary 7.2.7. *Let \mathbf{v} be a TB-sequence of a countably infinite abelian group G . Then the following are equivalent:*

¹ Considered in the natural way as a subgroup of $\widehat{G}_{\mathbf{d}} \times c_0(\mathbb{T})$.

- (i) $\sigma_{\mathfrak{b}\mathfrak{v}}$ is metrizable;
- (ii) $s_{\mathfrak{v}}(X)$ is countable;
- (iii) $\tau_{\mathfrak{v}}$ is discrete;
- (iv) $\tau_{\mathfrak{v}}^{\sigma}$ is discrete;
- (v) $\mathfrak{p}_{\mathfrak{v}}$ is discrete;
- (vi) $(\widehat{G}, \sigma_{\mathfrak{v}})$ is countable and discrete;
- (vii) $(\widehat{G}, \sigma_{\mathfrak{b}\mathfrak{v}})$ is countable and discrete.

If the previous equivalent conditions hold, then $\sigma_{\mathfrak{v}}$ is not reflexive.

Proof. **(i) \Rightarrow (ii).** This follows by Proposition 7.1.6(iii).

(ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v). This is Theorem 3.2.5.

(v) \Rightarrow (vi). This follows by Theorem 7.2.4.

(vi) \Rightarrow (vii). This follows by the fact that $\sigma_{\mathfrak{b}\mathfrak{v}} \subsetneq \sigma_{\mathfrak{v}}$.

(vii) \Rightarrow (i). This follows by Proposition 7.1.6(iii) and the facts $\sigma_{\mathfrak{b}\mathfrak{v}} = \tau_{s_{\mathfrak{v}}(\widehat{G}_{\mathfrak{d}})}$ and $s_{\mathfrak{v}}(\widehat{G}_{\mathfrak{d}}) = (G, \sigma_{\mathfrak{b}\mathfrak{v}})^*$.

If the conditions (i)-(vii) hold, Remark 7.2.5 implies that $\sigma_{\mathfrak{v}}$ is not reflexive. \square

7.2.2 MAP, AMAP and MinAP groups

For a topological group X , recall that $\mathfrak{n}(X)$ is the von Neumann radical of X (see Definition 1.1.2).

Definition 7.2.8. A group $X = (G, \sigma)$ is said to be

- (i) *maximally almost periodic* (briefly MAP) if $\mathfrak{n}(X) = \{0\}$;
- (ii) *almost maximally almost periodic* (briefly AMAP) if $\mathfrak{n}(X)$ is finite;
- (iii) *minimally almost periodic* (briefly MinAP) if $\mathfrak{n}(X) = X$.

Remark 7.2.9. A group $X = (G, \sigma)$ is

- (i) MAP if and only if the Bohr topology is Hausdorff, i. e. \widehat{X} separates the points of X ;

- (ii) MinAP if and only if the Bohr topology is indiscrete, i. e. $\widehat{X} = \{0\}$.

In [52, Theorem 4] Gabrielyan observed that if \mathbf{u} is a T-sequence of an infinite countable abelian group G , then

$$\mathbf{n}(G, \sigma_{\mathbf{v}}) \cong s_{\mathbf{v}}(\widehat{G}_d)^\perp \text{ algebraically.} \quad (7.2.2)$$

Therefore, the following fact is an immediate consequence of (7.2.2).

Fact 7.2.10. Let \mathbf{v} be a T-sequence of an infinite countable abelian group G . Then the following hold.

- (i) $(G, \sigma_{\mathbf{v}})$ is MAP if and only if \mathbf{v} is a TB-sequence. Indeed, $\mathbf{n}(G, \sigma_{\mathbf{v}}) = s_{\mathbf{v}}(\widehat{G}_d)^\perp$ is $\{0\}$ if and only if $s_{\mathbf{v}}(\widehat{G}_d)$ is dense in \widehat{G}_d .
- (ii) $(G, \sigma_{\mathbf{v}})$ is MinAP if and only if $s_{\mathbf{v}}(\widehat{G}_d) = \{0\}$ and $G = \langle \mathbf{v} \rangle$.

In relation with fact 7.2.10, in [71], Lukács found a T-sequence in $\mathbb{Z}(p^\infty)$ that is not a TB-sequence, providing in this way an example of an AMAP group. More precisely, he found a characterizing sequence \mathbf{v} for $p^m\mathbb{J}_p \leq \mathbb{J}_p$ for a fixed $m \in \mathbb{N}_+$, i. e. $s_{\mathbf{v}}(\mathbb{J}_p) = p^m\mathbb{J}_p$. In this way, being $\mathbb{J}_p/p^m\mathbb{J}_p$ finite, then

$$s_{\mathbf{v}}(\mathbb{J}_p)^\perp = \mathbf{n}(\mathbb{Z}(p^\infty), \sigma_{\mathbf{v}}) \neq \{0\} \text{ is finite.}$$

Therefore, $(\mathbb{J}_p, \sigma_{\mathbf{v}})$ is AMAP. Further results in this direction were obtained by Nguyen [77]. Finally, Gabrielyan [50] proved that an abelian group G admits an AMAP group topology if and only if G has non trivial torsion elements.

The following theorem, due to Gabrielyan, links the notions of T-characterized subgroup and MinAP topology.

Theorem 7.2.11 ([55]). *Let X be a compact abelian group and let H be a closed subgroup of X . Then, H is a T-characterized subgroup of X if and only if H is \mathcal{G}_δ and H^\perp carries a MinAP topology.*

Following [41, §4], for a topological abelian group X and a prime number p we denote by $T_p(X)$ the closure of the subgroup X_p . In case X is compact, one can prove that

$$T_p(X) = \{mX : m \in \mathbb{N}_+, (m, p) = 1\}. \quad (7.2.3)$$

In particular, $T_p(X)$ contains the connected component $c(X)$ of X . More precisely, if $X/c(X) = \prod_p (X/c(X))_p$ is the topologically primary decomposition of the totally disconnected compact group $X/c(X)$, then

$$T_p(X)/c(X) \cong (X/c(X))_p = T_p(X/c(X)).$$

Following [42], we say that $d \in \mathbb{N}$ is a proper divisor of $n \in \mathbb{N}$ provided that $d \notin \{0, n\}$ and $dm = n$ for some $m \in \mathbb{N}$. Note that, according to our definition, each $d \in \mathbb{N} \setminus \{0\}$ is a proper divisor of 0.

Definition 7.2.12. Let G be an abelian group.

- (i) For $n \in \mathbb{N}$ the group G is said to be of exponent n if $nG = \{0\}$, but $dG \neq \{0\}$ for every proper divisor d of n . In this case, we call n the *exponent* of G and denote it by $\exp(G)$. If $\exp(G) > 0$, G is called *bounded*, otherwise, *unbounded*.
- (ii) ([56]) If G is bounded, then the essential order $eo(G)$ of G is the smallest positive integer n such that nG is finite. If G is unbounded, we define $eo(G) = 0$.

Let us recall that a closed subgroup X of a compact abelian group X is characterized if and only if H is a \mathcal{G}_δ subgroup, i. e. the quotient group X/H is metrizable. In the next theorem we aim to give a detailed description of the closed characterized subgroups H of X that are not T -characterized in terms of various functorial subgroups or appropriate cardinal invariants of X and its quotients. This explains the blanket condition imposed on H to be a \mathcal{G}_δ subgroup of X .

Theorem 7.2.13. For a compact abelian group X and a \mathcal{G}_δ -subgroup H of X the following are equivalent:

- (i) H is not a T -characterized subgroup of X ;
- (ii) H^\perp does not admit a MinAP group topology;

- (iii) *there exists some $m > 0$ such that $m(X/H)$ is finite and non-trivial.*
- (iv) $\text{eo}(X/H) < \text{exp}(X/H)$;
- (v) *there exists a finite set P of primes so that*
 - (a) *H contains the subgroups $T_q(X)$ for all primes $q \notin P$,*
 - (b) *for every $p \in P$ there exist $k_p \in \mathbb{N}$ with $p^{k_p} T_p(X) \subseteq H$*
 - (c) *there exists $p \in P$ such that $p^{k_p-1} T_p(X) \not\subseteq H$ and $p^{k_p-1} T_p(X) \cap H$ has finite index in $p^{k_p} T_p(X)$.*
- (vi) *there exists a finite set P of primes so that $X/H \cong \prod_{p \in P} K_p$, where each K_p is a compact p -group and there exists some $p \in P$ and $k \in \mathbb{N}$ such that $p^k K_p$ is finite and non-trivial.*

Proof. The equivalence of (i) and (ii) was proved in Theorem 7.2.11. For reader's convenience and for the sake of completeness we give a short sketch of the proof closely following Gabrielyan's proof.

According to the reduction results from Chapter 1, to prove that H is a T-characterized subgroup of X it is equivalent to prove that $\{0\}$ is a T-characterized subgroup of X/H . In other words, we can assume that $H = \{0\}$ and X is a metrizable compact abelian group (so \widehat{X} is a countable abelian group). If \mathbf{v} is a T-sequence in \widehat{X} with $s_{\mathbf{v}}(X) = \{0\}$, then $\sigma_{\mathbf{v}}$ is a MinAP topology on \widehat{X} , by Fact 7.2.10 (ii).

The other implication is based on the following subtle observation in [55]: *if a countable abelian group G admits a MinAP group topology, then it admits also a MinAP group topology of the form $\sigma_{\mathbf{v}}$, where \mathbf{v} is a T-sequence in G .* Applying this result to the dual \widehat{X} , we deduce that there exists a T-sequence \mathbf{v} in G such that $\sigma_{\mathbf{v}}$ is a MinAP topology on \widehat{X} . By Fact 7.2.10 (ii), we deduce that $s_{\mathbf{v}}(X) = \{0\}$.

To prove the equivalence of (ii) and (iii) we recall the main theorem in [44]: a group G does not admit a MinAP group topology precisely when there exists some $m > 0$ such that mG is finite and non-trivial. In our case $G = H^{\perp}$ is the dual of X/H .

Since $G = \widehat{X/H}$, we deduce that $mG = m\widehat{X/M} = \widehat{m(X/H)}$. As $\widehat{m(X/H)}$ is finite precisely when $m(X/M)$ is finite, we deduce that $m(X/H)$ is finite and non-trivial.

Let us check the equivalence of (iii) and (iv). To this end let $G = X/H$. If (iii) holds, then $\exp(G) > m \geq \text{eo}(G)$, according to Definition 7.2.12. This proves the implication (iii) \rightarrow (iv). If (iv) holds, then $m := \text{eo}(G) > 0$ satisfies (iii), as mG is finite and non-trivial.

According to (iii), $K = X/H$ is a bounded group. Hence we can write $K = \prod_{p \in P} K_p$, where each K_p is a compact p -group. According to the rest of (iii), $m(X/H)$ is finite and non-trivial. Therefore, there exists $p \in P$ such that $mK_p \neq 0$ is finite. Let p^k be the highest power of p dividing m . Then $mK_p = p^k K_p \neq 0$ is finite. This proves the implication (iii) \rightarrow (vi).

(vi) \rightarrow (v) Write $X/H \cong \prod_{p \in P} K_p$ as in (f), where each K_p is a compact p -group. Let $p^{k_p} = \exp(K_p)$ and let $p_0^k K_{p_0}$ be finite and non-trivial for some $p_0 \in P$ and $k \in \mathbb{N}$.

Obviously, all primes $q \notin P$ are coprime to the exponent $m = \exp(X/H)$ of X/H . As $mX \leq H$, we deduce from (7.2.3) that

$$T_q(X) \leq H, \quad \text{for all primes } q \notin P. \quad (7.2.4)$$

This proves (a).

Next, we deduce from (7.2.4) that $c(X) \leq H$.

The quotient groups $X' = X/c(X)$ and $H' = H/c(X)$ are totally disconnected, hence $X' = \prod_p X'_p$ and $H' = \prod_p H'_p$. Here

$$X'_p = T_p(X)/c(X) \quad \text{and} \quad H'_p = T_p(H)/c(X) \quad \text{for every prime } p. \quad (7.2.5)$$

Furthermore, $X' = \prod_{p \in P} X'_p \times \prod_{q \notin P} X'_q$. From (7.2.4) we deduce that $X'_q \leq H'_q$ for all $q \notin P$. Therefore, $\prod_{q \notin P} X'_q \leq H'$ and $H' = \prod_{p \in P} H'_p \times \prod_{q \notin P} X'_q$. Hence, $X/H = \prod_{p \in P} X'_p/H'_p$ and consequently, $X'_p/H'_p \cong K_p$ for all $p \in P$. Hence, $p^{k_p} X'_p \leq H'_p$ for all $p \in P$. Equivalently, $p^{k_p} T_p(X) \leq H$ for $p \in P$. This proves (b).

As $p_0^{p_0}K_{p_0}$ is finite and non-trivial, we deduce that $k < k_{p_0}$. Therefore, $p_0^{k_{p_0}-1}K_{p_0}$ is still finite and non-trivial. Hence

$$p_0^{k_{p_0}-1}X'_{p_0} \not\subseteq H'_{p_0},$$

so

$$p_0^{k_{p_0}-1}T_{p_0} \not\subseteq H.$$

To prove the second assertion in (c) note that the finiteness of $p_0^{p_0}K_{p_0}$ yields that

$$\begin{aligned} p_0^{k_{p_0}-1}(X'_{p_0}/H'_{p_0}) &\cong (p_0^{k_{p_0}-1}X'_{p_0} + H'_{p_0})/H'_{p_0} \\ &\cong p_0^{k_{p_0}-1}X'_{p_0}/(H'_{p_0} \cap p_0^{k_{p_0}-1}X'_{p_0}) \end{aligned}$$

is finite. Hence, from (7.2.5) we deduce that

$$p_0^{k_{p_0}-1}T_{p_0}(X) \cap T_{p_0}(H)$$

has finite index in $p_0^{k_{p_0}-1}T_{p_0}(X)$. Therefore,

$$p_0^{k_{p_0}-1}T_{p_0}(X) \cap H = p_0^{k_{p_0}-1}T_{p_0}(X) \cap T_{p_0}(H)$$

has finite index in $p_0^{k_{p_0}-1}T_{p_0}(X)$.

(v) \rightarrow (iii) Let m' be the product of all q^{k_q} when q runs over P and let $m = m'/p$. Then an argument similar to the above argument shows that $m(X/H) \neq \{0\}$ is finite. \square

Corollary 7.2.14. *If H is a closed subgroup of a compact abelian group X that does not contain the connected component $c(X)$ of X , then H is T-characterized if and only if it is characterized.*

Proof. Assume that H is characterized. Then H is a G_δ -subgroup of X by Theorem 1.2.6. Let $q : X \rightarrow X/H$ be the quotient homomorphism. Then $q(c(G))$ is a non-trivial connected subgroup of X/H , hence the group X/H is unbounded as its connected component $c(X/H)$ is a non-trivial divisible group subgroup of X/H by Lemma A.2.6. According to Theorem 7.2.13, H is T-characterized. \square

The following theorem trivially follows by Corollary 7.2.14.

Theorem 7.2.15 ([55]). *If H is a subgroup of a compact connected abelian group, then H is a T -characterized subgroup if and only if H is a characterized subgroup.*

Corollary 7.2.16 ([55, Theorem 1.14]). *Let X be a compact abelian. All \mathcal{G}_δ subgroups of X are T -characterized if and only if X is connected.*

Proof. If X is connected, then one can apply Corollary 7.2.14 to deduce that all \mathcal{G}_δ subgroups of X are T -characterized. In case X is not connected, then X has a proper open subgroup H , as the connected component of X is an intersection of clopen subgroups [59]. By Corollary 1.1.19, H is not even K -characterized. \square

7.3 SS-PRECOMPACT TOPOLOGIES

Definition 7.3.1 ([34]). A precompact group $X = (G, \sigma)$ (as well its precompact topology σ) is said to be *singular sequence precompact* (briefly *ss-precompact*) if $\sigma = \sigma_{\mathbf{b}\mathbf{v}}$, where $\mathbf{v} \in G^{\mathbb{N}}$ (and hence \mathbf{v} is a TB -sequence).

Remark 7.3.2. It is known that every precompact group topology on G is of the form τ_H , where $H \leq \widehat{G}_d$ is dense. Clearly, τ_H is *ss-precompact* if and only if H is a dense characterized subgroup of \widehat{G}_d . Since $(G, \tau_H)^* = H$, by Fact 7.2.3, one has that a precompact group (G, σ) is *ss-precompact* if and only if $(G, \sigma)^*$ is a dense characterized subgroup of \widehat{G}_d .

The description of these groups is important, since it gives a description of all metrizable precompact countably infinite abelian groups, as stated in the next theorem proved in [34].

Theorem 7.3.3 ([34, Theorem A]). *Let X be a metrizable precompact topological abelian group. Then X is *ss-precompact* if and only if X is countable. Moreover, every TB -sequence witnessing the *ss-precompactness* generates a finite index subgroup of X .*

As a corollary one has the following.

Corollary 7.3.4 ([34, Corollary A]). *Let (G, σ) be a countably infinite precompact metrizable abelian group. Then G admits a group topology σ' which is strictly finer than σ and $\widehat{(G, \sigma')} = \widehat{(G, \sigma)}$.*

More precisely, σ' is $\sigma_{\mathbf{v}}$, where \mathbf{v} is the TB-sequence such that $\sigma = \sigma_{\mathbf{b}\mathbf{v}}$.

Moreover, the countable ss-precompact groups are important since they are closely related to characterized subgroups.

Problem 7.3.5. The following problems are equivalent due to Remark 7.3.2 and Proposition 7.1.6:

- Describe all countable ss-precompact groups.
- Describe all dense characterized subgroups of compact metrizable groups.

Clearly, the density condition is not restrictive, since a subgroup of a compact group is characterized if and only if it is a characterized subgroup of its closure.

Theorem 7.3.6 ([34, Theorem C]). *For a precompact group (G, σ) the following are equivalent:*

- (i) (G, σ) is ss-precompact;
- (ii) (G, σ) has a countable B-embedded (see Definition A.3.2) ss-precompact subgroup.

Recall that a topological space is said to be *pseudocompact* if its image via real-valued continuous functions is bounded.

Corollary 7.3.7 ([34, Corollary C1]). *A pseudocompact Hausdorff group is ss-precompact if and only if it is finite.*

Recall also that the *tightness* of a topological space X is the minimal cardinal $\kappa \geq \aleph_0$ such that, for every $x \in X$ and every $S \subseteq X$ with $x \in \overline{S}$ there exists $Q \subseteq S$ such that $x \in \overline{Q}$ and $|Q| \leq \kappa$. Moreover, a topological group is *hereditarily disconnected* if its connected component (i. e. the maximal connected subgroup) is trivial.

Theorem 7.3.8 ([34, Theorem D]). *Let X be an ss-precompact group. Then*

- (i) X is hereditarily disconnected;
- (ii) X has countable tightness.

7.4 TB-SEQUENCES OF \mathbb{Z}

From Proposition 7.1.6 for $G = \mathbb{Z}$ and the fact that every infinite subgroup of \mathbb{T} is dense one obtains the following corollary.

Corollary 7.4.1 ([7, Proposition 2.4]). *Let $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$. Then the following hold:*

- (i) $\sigma_{\mathbf{b}\mathbf{u}}$ is Hausdorff (hence precompact) if and only if $t_{\mathbf{u}}(\mathbb{T})$ is infinite;
- (ii) $\sigma_{\mathbf{b}\mathbf{u}}$ is metrizable if and only if $|t_{\mathbf{u}}(\mathbb{T})| = \aleph_0$.

Corollary 7.4.2. *Let $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$ such that $t_{\mathbf{u}}(\mathbb{T})$ is infinite. Then the following hold*

- (i) If \mathbf{u} is \mathbf{q} -bounded, then $\sigma_{\mathbf{b}\mathbf{u}}$ is precompact and metrizable;
- (ii) If \mathbf{u} is \mathbf{q} -divergent, then $\sigma_{\mathbf{b}\mathbf{u}}$ is precompact and $w(\mathbb{Z}, \sigma_{\mathbf{b}\mathbf{u}}) = \mathfrak{c}$.

Example 7.4.3. (i) if $\mathbf{u} = (p^n)$, then $\sigma_{\mathbf{b}\mathbf{u}}$ is a precompact linear metrizable group topology on \mathbb{Z} such that $\mathbf{u} \xrightarrow{\sigma_{\mathbf{b}\mathbf{u}}} 0$;

(ii) if $\mathbf{u} = (n!)$, then $\sigma_{\mathbf{b}\mathbf{u}}$ is a precompact non linear non metrizable group topology on \mathbb{Z} such that $\mathbf{u} \xrightarrow{\sigma_{\mathbf{b}\mathbf{u}}} 0$;

(iii) [7] if \mathbf{u} is the Fibonacci sequence, then $\sigma_{\mathbf{b}\mathbf{u}}$ is a precompact metrizable non-linear group topology on \mathbb{Z} such that $\mathbf{u} \xrightarrow{\sigma_{\mathbf{b}\mathbf{u}}} 0$.

As a consequence of the results of Subsection 4.5.2 from [10] one can obtain the following theorem.

Theorem 7.4.4 ([8, Theorem 4.6]). *Let $\mathbf{u} \in \mathcal{Z}_{rec_2}^+$ and $a_n \geq b_n$ for every $n \in \mathbb{N}$. Then*

- (i) \mathbf{u} is a TB-sequence;
- (ii) $w(\mathbb{Z}, \sigma_{\mathbf{b}\mathbf{u}}) = \mathfrak{c}$ if and only if \mathbf{u} is not \mathbf{q} -bounded;
- (iii) $\sigma_{\mathbf{b}\mathbf{u}}$ is metrizable if and only if \mathbf{u} is \mathbf{q} -bounded;
- (iv) $\sigma_{\mathbf{b}\mathbf{u}}$ is linear if and only if \mathbf{u} is \mathbf{q} -bounded and $b_n = 0$ infinitely many times.

Moreover, from results of Subsection 4.5.2, Theorem 4.5.2 and Remark 7.1.7 one can deduce the following theorem.

Theorem 7.4.5 ([8, Theorem 4.11]). *Let $\mathbf{u} \in \mathcal{Z}_{rec_2}^+$ be such that $a_n = a$ and $b_n = b \in \mathbb{N}^*$ for every $n \in \mathbb{N}$. Then the following hold:*

- (i) $\sigma_{\mathbf{b}\mathbf{u}}$ has a countable 0-neighbourhood basis;
- (ii) The following are equivalent:
 - (a) $\sigma_{\mathbf{b}\mathbf{u}}$ is metrizable;
 - (b) \mathbf{u} is a TB-sequence;
 - (c) $a \geq b$ or $\gcd(a, b) > 1$ or \mathbf{u} is a geometric progression.
- (iii) $\sigma_{\mathbf{b}\mathbf{v}}$ is linear if and only if $a < b$. If \mathbf{u} is a TB-sequence and $a < b$, then the completion of $(\mathbb{Z}, \sigma_{\mathbf{b}\mathbf{v}})$ is a compact totally disconnected group isomorphic to $\mathbb{Z}(\mathfrak{m}) \times \prod_{\mathfrak{p} \in \mathcal{P}_\infty} \mathbb{Z}(\mathfrak{p}^\infty)$, where $\mathcal{P}_\infty = \{\mathfrak{p} \in \mathcal{P} \mid n_{\mathfrak{p}}(\mathbf{u}) = \infty\}$ and $\mathfrak{p} \nmid \mathfrak{m}$ for every $\mathfrak{p} \in \mathcal{P}_\infty$.

For a -sequences, the following corollary holds.

Corollary 7.4.6. *If \mathbf{u} is an a -sequence, then \mathbf{u} is a TB-sequence. Moreover, the following are equivalent:*

- (i) \mathbf{u} is \mathbf{q} -bounded;
- (ii) $t_{\mathbf{u}}(\mathbb{T})$ is countable
- (iii) $\sigma_{\mathbf{b}\mathbf{u}}$ is metrizable;
- (iv) $\sigma_{\mathbf{b}\mathbf{u}}$ is linear;
- (v) $t_{\mathbf{u}}(\mathbb{T}) \in \mathfrak{F}_\sigma(\mathbb{T})$;

- (vi) $t_{\mathbf{u}}(\mathbb{T})$ is $\tau_{\mathbf{u}}^{\sigma}$ -open;
- (vii) $\tau_{\mathbf{u}}^{\sigma}$ is discrete;
- (viii) $t_{\mathbf{u}}(\mathbb{T})$ is $\tau_{\mathbf{u}}$ -open;
- (ix) $\tau_{\mathbf{u}}$ is discrete;
- (x) $\mathfrak{p}_{\mathbf{u}}$ is discrete;
- (xi) $(\widehat{\mathbb{Z}}, \sigma_{\mathbf{u}})$ is discrete;
- (xii) $(\widehat{\mathbb{Z}}, \sigma_{b_{\mathbf{u}}})$ is discrete.

If the previous equivalent conditions hold, then $\sigma_{\mathbf{u}}$ is not reflexive.

Proof. (i) \Leftrightarrow (ii) \Leftrightarrow (v)-(x). This is Theorem 5.4.1.

(ii) \Leftrightarrow (iii) \Leftrightarrow (iv). This is Corollary 7.4.1.

(ii) \Leftrightarrow (xi) \Leftrightarrow (xii). This is Corollary 7.2.7.

If the conditions (i)-(xii) hold, then Remark 7.2.5 implies that $\sigma_{\mathbf{u}}$ is not reflexive. \square

7.5 PRECOMPACT GROUP TOPOLOGIES WITHOUT CONVERGING SEQUENCES

The following notion, introduced in [40] is a counterpart of t -density and t -closedness.

Definition 7.5.1 ([40]). Let X be a topological group. For a subgroup H of X one can define the following subgroup of X

$$\mathfrak{g}(H) = \bigcap \{s_{\mathbf{v}}(X) \mid H \leq s_{\mathbf{v}}(X)\}.$$

A subgroup H of X is said to be

- \mathfrak{g} -dense if $\mathfrak{g}(H) = X$;
- \mathfrak{g} -closed if $\mathfrak{g}(H) = H$.

Remark 7.5.2. H is a \mathfrak{g} -dense subgroup of a topological abelian group X if and only if for every $K \in \mathcal{C}\text{har}(X) \setminus \{X\}$ one has that $H \not\leq K$.

Theorem 7.5.3. *Let G be an abelian group and let H be subgroup of $X = \widehat{G}_d$. Then H is dense and \mathfrak{g} -dense if and only if (G, τ_H) is a precompact group without non-eventually null convergent sequences.*

Proof. Suppose that H is dense and \mathfrak{g} -dense. Since H is dense, then τ_H is a precompact group topology. Since H is \mathfrak{g} -dense, one has that for every $K \in \mathcal{C}\text{har}(X) \setminus \{X\}$ one has that $H \not\subseteq K$, as stated in Remark 7.5.2. Therefore, by Proposition 7.1.3 (G, τ_H) is a precompact group without non-eventually null convergent sequences.

Conversely, if (G, τ_H) is a precompact, then H is dense. By Proposition 7.1.3, if (G, τ_H) has no convergent sequences one has that for every $K \in \mathcal{C}\text{har}(X) \setminus \{X\}$ one has that $H \not\subseteq K$, i. e. by Remark 7.5.2 H is \mathfrak{g} -dense. \square

Corollary 7.5.4 ([27]). *Let G be a torsion free abelian group and let H be a dense subgroup of $X = \widehat{G}_d$. If H is non-(Haar)-measurable subgroup of X , then (G, τ_H) is a precompact group without non-eventually null convergent sequences.*

Proof. Note that $X = \widehat{G}_d$ is connected by Proposition A.3.16(i). Hence, the corollary follows by the fact that every proper characterized (hence measurable) subgroup of a compact connected abelian group has measure 0 (see Lemma A.2.7) and the fact that every subgroup of a measure zero subgroup is measurable (and it is measure zero too). Therefore, for every $K \in \mathcal{C}\text{har}(X) \setminus \{X\}$ one has that $H \not\subseteq K$. Consequently, H is \mathfrak{g} -dense. Hence, by Theorem 7.5.3 one can conclude. \square

In [27], the authors find $2^{|X|}$ -many non-measurable, dense subgroups, proving that there exists $2^{|X|}$ -many precompact group topologies on G without non-eventually null convergent sequences.

OPEN QUESTIONS

8.1 CHARACTERIZED SUBGROUPS OF TOPOLOGICAL ABELIAN GROUPS

8.1.1 *N-characterized, K-characterized and T-characterized subgroups*

Example 1.1.10 shows that the argument of the proof of Proposition 1.2.9 does not hold. Nevertheless the following question still remains open.

Question 8.1.1. Does Proposition 1.2.9 holds also for non-compact groups?

Some steps towards answering the above question are made in Proposition 1.1.15.

Corollary 1.1.23 shows that every subgroup H of a discrete abelian group X such that $\omega \leq [X : H] \leq \mathfrak{c}$ is K -characterized. But the following examples shows that this is not a necessary condition to be a K -characterized subgroup of a discrete abelian group (see also Proposition 1.1.20).

Example 8.1.2. Consider $X = \mathbb{Z}$ and $H = 2\mathbb{Z}$. Let $\mathbf{v} = (v_n) \in \widehat{\mathbb{Z}} = \mathbb{T}$ be such that $v_n = \frac{1}{2^n} + \frac{1}{2}$. Hence $H = s_{\mathbf{v}}(X)$ and $[X : H] = 2$.

Therefore, one may pose the following question.

Question 8.1.3. Are the finite index subgroups of discrete abelian group K -characterized?

In relation to the same Corollary 1.1.23, one may ask to describe the T -characterized subgroups of a discrete abelian group.

Problem 8.1.4. Describe the T -characterized subgroups of the discrete abelian groups.

8.1.2 Polishability

In Chapter 2 we see that $\mathcal{C}\text{har}(X) \subseteq \mathfrak{P}\text{ol}_{\text{lqc}}(X)$ for every compact metrizable group X . But unfortunately, the inverse inclusion does not hold. More precisely, for $X = \mathbb{T}^{\mathbb{N}}$ Example 2.1.15 proves the following non-inclusion.

$$\mathfrak{P}\text{ol}_{\text{lqc}}(X) \cap \mathfrak{F}_{\sigma}(X) \not\subseteq \mathcal{C}\text{har}(X). \quad (8.1.1)$$

It is natural to ask if this non-inclusion holds for all compact metrizable abelian groups. Recently Gabrielyan proved that *the following related non-inclusion holds for every compact metrizable non-totally disconnected abelian group.*

$$\mathfrak{P}\text{ol}(X) \cap \mathfrak{F}_{\sigma}(X) \not\subseteq \mathcal{C}\text{har}(X). \quad (8.1.2)$$

Question 8.1.5 ([54]). Does (8.1.2) holds also for totally disconnected compact metrizable abelian groups?

The following question remains open also for non-totally disconnected compact abelian groups.

Question 8.1.6 ([54]). Does (8.1.1) holds also for all compact metrizable abelian groups? Does it holds for $X = \mathbb{T}$?

Corollary 3.2.7 proves that $\mathbf{v} \approx \mathbf{u}$ whenever $s_{\mathbf{v}}(X) = s_{\mathbf{u}}(X) \in \mathfrak{F}_{\sigma}(X)$. Therefore, the following question arises naturally.

Question 8.1.7 ([35, Question 4.5]). If $s_{\mathbf{v}}(X) = s_{\mathbf{u}}(X) \notin \mathfrak{F}_{\sigma}(X)$, is $\mathbf{v} \approx \mathbf{u}$?

8.2 THE CIRCLE GROUP

8.2.1 General Sequences

Theorem 6.2.8 proves that there exists $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^{\mathbb{N}}$ such that $t_{\mathbf{u}}(\mathbb{T}) + t_{\mathbf{v}}(\mathbb{T}) = \mathbb{T}$. This answers to Question [7, Question 5.1 (b)], but remains unanswered the following question.

Question 8.2.1 ([7, Question 5.1 (a)]). When does there exists $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^{\mathbb{N}}$ such that $t_{\mathbf{u}}(\mathbb{T}) + t_{\mathbf{v}}(\mathbb{T}) = t_{\mathbf{w}}(\mathbb{T})$ for a given $\mathbf{w} \in \mathbb{Z}^{\mathbb{N}}$?

We proved in Theorem 5.4.1 that the \mathcal{F}_σ subgroups characterized by means of \mathfrak{a} -sequences are countable. Moreover, Theorem 2.2.24 proves that $\mathfrak{B}ol_{lc}(\mathbb{T})$, that is obviously contained in $\mathfrak{F}_\sigma(\mathbb{T})$, consist of \mathbb{T} and the countable subgroups. Hence, it is natural to pose the following question.

Question 8.2.2. Are the countable subgroups the only proper characterized \mathcal{F}_σ subgroups of \mathbb{T} ?

8.2.2 Arithmetic Sequences

Remark 8.2.3. If $\mathfrak{a}\text{-Char}(\mathbb{T})$ denotes the set of all subgroups of \mathbb{T} characterized by an \mathfrak{a} -sequence, then

$$\{\text{closed subgroups}\} = \mathfrak{G}_\delta(\mathbb{T}) \subsetneq \mathfrak{a}\text{-Char}(\mathbb{T}) \cap \mathfrak{F}_\sigma(\mathbb{T}).$$

An example witnessing the above proper inclusion is $t_{\mathbf{u}}(\mathbb{T}) \in \mathfrak{F}_\sigma(\mathbb{T}) \setminus \mathfrak{G}_\delta(\mathbb{T})$ where $\mathbf{u} = (p^n)$. More generally all countably infinite $t_{\mathbf{u}}(\mathbb{T}) \in \mathfrak{a}\text{-Char}(\mathbb{T})$ witness that proper inclusion, where \mathbf{u} is an \mathfrak{a} -sequence.

To give an exhaustive description of $\mathfrak{Char}(\mathbb{T})$ and $\mathfrak{a}\text{-Char}(\mathbb{T})$, in terms of Borel complexity, it is needed to establish if there exists a sequence \mathbf{u} such that $t_{\mathbf{u}}(\mathbb{T}) \notin \mathfrak{G}_{\delta\sigma}(\mathbb{T})$. This kind of sets is called $F_{\sigma\delta}$ -complete (or Π_3^0 -complete using the Descriptive Set-Theoretic terminology, see [63]), that is, in plain words, a set among the most complex in $\mathcal{F}_{\sigma\delta}(\mathbb{T})$. Hence question of whether $\mathfrak{Char}(\mathbb{T}) \subseteq \mathfrak{G}_{\delta\sigma}(\mathbb{T})$ remains open.

Question 8.2.4. (i) ([55]) Does there exists a sequence of integers \mathbf{u} such that $t_{\mathbf{u}}(\mathbb{T}) \notin \mathfrak{G}_{\delta\sigma}(\mathbb{T})$?

(ii) What about \mathfrak{a} -sequences \mathbf{u} ?

8.2.3 Thin Sets

In Chapter 6 we prove that $\pi(K_L^{\mathbf{u}})$ is a D-set contained in $t_{\mathbf{u}^*}(\mathbb{T})$ for a suitable subsequence \mathbf{u}^* of \mathbf{u} , when L is non-large. Therefore, $\langle \pi(K_L^{\mathbf{u}}) \rangle \leq t_{\mathbf{u}^*}(\mathbb{T})$. Obviously, every finite set of \mathbb{T} is a D-set

and therefore $\langle F \rangle$ is characterized, whenever F is finite. On the other hand in [70] the authors prove that every Kronecker set is a D -set, but Biró proved that for every uncountable Kronecker set K of \mathbb{T} one has $\langle K \rangle$ is not polishable and hence not characterized. So it make sense to pose the following questions.

- Question 8.2.5.* (i) If L is non-large, is $\langle \pi(K_L^u) \rangle$ characterized?
- (ii) Under what conditions, for a D -set E of \mathbb{T} $\langle E \rangle$ is characterized?

Part IV

APPENDIX

TOPOLOGICAL SPACES AND GROUPS

Here, some classical results and notions used in this Thesis are presented. Moreover, we recall some basic properties of the circle group, that is one of the main object studied in this Thesis.

A.1 TOPOLOGICAL SPACES

Definition A.1.1. A topological space is said to be *Polish* if it is a complete metrizable separable space.

Theorem A.1.2 ([63, Theorem 5.3]). *If X is an Hausdorff locally compact space, then the following are equivalent:*

- (i) X is second countable;
- (ii) X is metrizable and σ -compact;
- (iii) X is Polish;
- (iv) X is homeomorphic to an open subset of a compact metrizable space.

Lemma A.1.3. *If X is a topological space and $(f_n)_{n \in \mathbb{N}}$ is a sequence of continuous real-valued functions on X that uniformly converges to 0 on a dense subset D of X , then $(f_n)_{n \in \mathbb{N}}$ uniformly converges to 0 on X .*

Proof. Pick an $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $|f_n(x)| \leq \varepsilon$ for all $n \geq n_0$ and every $x \in D$. For $n \geq n_0$ the set $f_n^{-1}([-\varepsilon, +\varepsilon])$ is closed and contains the dense subset D of X , so coincides with X ; in other words, $|f_n(x)| \leq \varepsilon$ for all $n \geq n_0$ and $x \in X$. Therefore, f_n uniformly converges to 0 on X . \square

Let $\mathcal{B}(X, \tau)$ denote the σ -algebra of Borel subsets of (X, τ) . One can define $\Sigma_\xi^0(X), \Pi_\xi^0(X)$ subfamilies of $\mathcal{B}(X, \tau)$ for $1 \leq \xi < \infty$

ω_1 , where ω_1 is the first uncountable ordinal, in the following manner.

$$\begin{aligned}\Sigma_1^0(X) &= \{U \subseteq X \mid U \text{ open in } X\}, \\ \Pi_\xi^0(X) &= \{A \subseteq X \mid X \setminus A \in \Sigma_\xi^0(X)\}, \\ \Sigma_\xi^0(X) &= \left\{ \bigcup_n A_n \mid A_n \in \Pi_{\xi_n}^0(X), \xi_n < \xi, n \in \mathbb{N} \right\}.\end{aligned}$$

Hence $\Sigma_1^0(X) = \{\text{open sets of } X\}$ e $\Pi_1^0(X) = \{\text{closed sets of } X\}$. Moreover $\Sigma_2^0(X) = \{\mathcal{F}_\sigma \text{ of } X\}$ and $\Pi_2^0(X) = \{\mathcal{G}_\delta \text{ of } X\}$. More generally,

$$\begin{aligned}\Sigma_{\xi+1}^0(X) &= (\Pi_\xi^0(X))_\sigma := \{\text{count. unions of sets in } \Pi_\xi^0(X)\}, \\ \Pi_{\xi+1}^0(X) &= (\Sigma_\xi^0(X))_\delta := \{\text{count. intersec. of sets in } \Sigma_\xi^0(X)\}.\end{aligned}$$

In particular, the following equalities hold.

$$\mathcal{B}(X) = \bigcup_{\xi < \omega_1} \Sigma_\xi^0(X) = \bigcup_{\xi < \omega_1} \Pi_\xi^0(X).$$

Definition A.1.4. Let X be a topological space.

- If (X, S) is a measurable space, then it is called a *standard Borel space* if there is a Polish topology τ on X with $S = \mathcal{B}(X, \tau)$.
- A subset of X is called *nowhere dense* if its closure has empty interior.
- A subset of X is *meager* (or *first category*) if it is the union of a countable collection of nowhere dense subsets.
- A subset A of X is said to have the *Baire property* if there is an open set U such that the symmetric difference $(U \setminus A) \cup (A \setminus U)$ is a set of first category, i. e. it is a countable union of nowhere dense sets. Some authors use the terminology *almost open*.

Remark A.1.5. In a topological space, every Borel set has the Baire property. Moreover, the sets having the Baire property form the smallest σ -algebra containing all open sets and all meager sets.

Definition A.1.6. A function $f : X_1 \rightarrow X_2$, where X_1, X_2 are two topological spaces, is said to be *Borel (Baire) measurable* if and only if $f^{-1}(U)$ is a Borel set (has the Baire property) for every open set U .

Remark A.1.7. Every Borel measurable function is a Baire measurable function.

Theorem A.1.8 (Lavrentieff Theorem). *Let X be a Polish space and $A \subseteq X$, then A is Polish with the subset topology if and only if A is a \mathcal{G}_δ subset of X .*

Recall that the *Cantor space* is $\mathcal{C} = (\{0, 1\}^{\mathbb{N}}, \tau)$ where τ is the product topology induced from the discrete one on $\{0, 1\}$.

Theorem A.1.9 (Alexandroff-Hausdorff (see [63, Theorem 13.6])). *Let X be a Polish space and A be a Borel subset of X . Then either A is countable or A contains a homeomorphic image of \mathcal{C} .*

A.2 TOPOLOGICAL GROUPS

A.2.1 The circle group

One can define the circle group as the subgroup of the complex numbers (\mathbb{C}, \cdot) of all elements with norm 1. That is the following subgroup,

$$\mathbf{T} = \{e^{2\pi i x} \mid x \in \mathbb{R}\}. \quad (\text{A.2.1})$$

Equivalently, one can define the circle group as the quotient group \mathbb{R}/\mathbb{Z} . Indeed, if $\pi : \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}$ is the canonical projection and if for every $x \in \mathbb{R}/\mathbb{Z}$ one chooses $\tilde{x} \in \pi^{-1}(x)$, then one can define the following isomorphism $\varphi : x \mapsto e^{2\pi i \tilde{x}}$ from \mathbb{R}/\mathbb{Z} to \mathbf{T} .

It is easy to prove that \mathbb{T} with the topology induced from the euclidean one of \mathbb{C} is a topological group as well \mathbb{R}/\mathbb{Z} with the quotient topology induced from the euclidean one of \mathbb{R} . Moreover, φ , defined above, is a topological isomorphism from \mathbb{R}/\mathbb{Z} to \mathbb{T} . In this Thesis, it is preferable to deal with *the additive notation* of the circle group i. e. \mathbb{R}/\mathbb{Z} , that one can denote by \mathbb{T} .

In the circle, in the usual metric the distance between two points is the length of the minimal arc connecting the two points. This metric is a compatible metric with the usual topology, i. e. the quotient topology from the euclidean one in \mathbb{R} .

Sometimes, it is preferable to deal with elements of the pre-image of an element $x \in \mathbb{T}$ with respect to the canonical projection $\pi : \mathbb{R} \rightarrow \mathbb{T}$ rather than the elements of \mathbb{T} . For example, one can consider the following restriction of π ,

$$\pi_1 = \pi \upharpoonright_{[0,1)} : [0, 1) \rightarrow \mathbb{R}/\mathbb{Z}; \quad (\text{A.2.2})$$

$$\pi_2 = \pi \upharpoonright_{[-\frac{1}{2}, \frac{1}{2})} : \left[-\frac{1}{2}, \frac{1}{2}\right) \rightarrow \mathbb{R}/\mathbb{Z}. \quad (\text{A.2.3})$$

In this way one can define the norm in \mathbb{T} in the following two equivalent ways. Let $x \in \mathbb{T}$

$$\|x\| = \begin{cases} \min\{\tilde{x}, 1 - \tilde{x}\} & \text{if } \tilde{x} = \pi_1^{-1}(x); \\ |\tilde{x}| & \text{if } \tilde{x} = \pi_2^{-1}(x). \end{cases}$$

A.2.2 Locally compact groups and compact groups

Theorem A.2.1 ([59, Theorem 5.11]). *Let $X = (G, \tau)$ be an Hausdorff group. Then $H \leq X$ is a locally compact subgroup of X (with the relative topology) if and only if H is closed. Hence every locally compact group is complete.*

Definition A.2.2. For a subgroup H of an abelian group G and a group topology τ on H , one can denote by τ^* the group topology on G , having as a local base at 0 precisely the τ -open neighbourhoods of 0 in H .

Proposition A.2.3. *Let X be an abelian group and $H \leq X$.*

- (i) *If τ is a group topology on G , then $\tau \upharpoonright_H^* \supseteq \tau$ and H is a $\tau \upharpoonright_H^*$ -open subgroup of the topological group $(G, \tau \upharpoonright_H^*)$;*
- (ii) *τ^* is Hausdorff if and only if τ is Hausdorff;*
- (iii) *(G, τ^*) is locally compact if and only if (H, τ) is locally compact.*

Theorem A.2.4 (Principal Structure Theorem). *Every locally compact abelian group is topologically isomorphic to $\mathbb{R}^n \times X_0$ where X_0 contains a compact open subgroup and $n \in \mathbb{N}$.*

Theorem A.2.5 (Open Mapping Theorem). *Let X and Y be locally compact topological groups and let φ be a continuous homomorphism of X onto Y . If X is σ -compact, then φ is open.*

Lemma A.2.6 ([59, Theorem 24.25]). *Let X be compact abelian group. Then X is connected if and only if it is divisible.*

Corollary A.2.7. *Let μ be a finite-additive (translational) invariant finite measure on a compact connected abelian group X . Every proper μ -measurable subgroup of X has measure zero.*

Proof. Suppose $\mu(X) = 1$. Let H be a proper μ -measurable subgroup of X . Since X is divisible (by Lemma A.2.6), the index $[\mathbb{T} : H]$ is infinite and hence by finite-additivity $\mu(H) = 0$. Indeed, if $\mu(H) > 0$, then there exist n such that $\mu(H) > \frac{1}{n}$. Let $\alpha_i + H$ for $i = 1, \dots, n$ distinct cosets of H , then $1 \geq \mu(\sum_{i=1}^n \alpha_i + H) = n\mu(H)$, a contradiction. \square

Theorem A.2.8 ([59][41]). *Every compact connected abelian group of weight $\leq \mathfrak{c}$ non isomorphic to \mathbb{T} is a solenoid.*

A.2.3 Metrizable groups and \mathcal{G}_δ sets

Proposition A.2.9. *If X is a locally compact Hausdorff abelian group and $H \leq X$ a closed subgroup, then the following are equivalent.*

- (i) *H is a \mathcal{G}_δ subgroup of X ;*

(ii) *The quotient group X/H is a metrizable group.*

Proof. (ii) \Rightarrow (i). If X/H is a metrizable Hausdorff group (Hausdorffness is given by the fact that H is closed). By Birkhoff-Kakutani's Theorem $0_{G/H}$ has a countable base of open neighbourhood, let $\{U_n : n \in \mathbb{N}\}$ be that base. Hence

$$H = \pi^{-1}(0_{G/H}) = \pi^{-1}\left(\bigcap_{n \in \mathbb{N}} U_n\right) = \bigcap_{n \in \mathbb{N}} \pi^{-1}(U_n),$$

i. e. H is a \mathcal{G}_δ subgroup of G . (i) \Rightarrow (ii). [22, Corollary of Theorem 2].

□

Lemma A.2.10 ([22, Corollary 1 of Theorem 1]). *Let X be a locally compact group, $O \in \mathcal{G}_\delta(X)$ and $e_X \in O$. Then there exists a compact \mathcal{G}_δ subgroup N of X such that $N \subseteq O$ and X/N is a complete metrizable locally compact group.*

Theorem A.2.11 ([68][Theorem 2 §34]). *If H is a meager \mathcal{G}_δ subspace of a complete metrizable group, then H is nowhere dense.*

Theorem A.2.12 (Banach-Kuratowsky-Pettis). *If X is a topological group and $H \subseteq X$ with the Baire property and non-meager, then $\text{int}(HH^{-1}) \neq \emptyset$*

The above theorem is a generalization of the Steinhaus-Weil Theorem.

Theorem A.2.13 (Steinhaus-Weil). *If X is a locally compact group and $H \subseteq X$ a positive (left) Haar measure subset, then $\text{int}(HH^{-1}) \neq \emptyset$.*

Theorem A.2.14. *If X is a locally compact group and $H \leq X$ is a \mathcal{G}_δ -subgroup, then H is closed.*

Proof. Clearly, if H is \mathcal{G}_δ of X , then H is a \mathcal{G}_δ of its closure \bar{H} . Note that \bar{H} is also locally compact, by Theorem A.2.1. Therefore, one may suppose that H is a dense \mathcal{G}_δ subgroup of X . By Lemma A.2.10, there exists a compact subgroup of X such that $K \leq H$ and X/K is a complete metrizable locally compact group.

Therefore, H/K is a \mathcal{G}_δ dense subgroup and hence its closure has (obviously) non empty interior (hence, it is not nowhere dense). Thus, by Theorem A.2.11, H/K is non meager. Hence, by Theorem A.2.12, H/K is open and so H is open too and thus closed. \square

Corollary A.2.15. *Let X be a locally compact metrizable abelian group. $H \leq X$ is closed if and only if it is a \mathcal{G}_δ subgroup.*

Proof. If H is closed, then the quotient X/H is still metrizable. Hence by proposition A.2.9 H is \mathcal{G}_δ .

The converse is the Theorem A.2.14 \square

From Theorem A.2.12, one can deduce the following theorem.

Theorem A.2.16 ([12]). *Let $f : X_1 \rightarrow X_2$ be a Baire measurable homomorphism between Polish groups. Then f is continuous. If moreover, $f(X_1)$ is not meager, f is also open.*

Corollary A.2.17. *Any Borel measurable algebraic isomorphism between Polish groups is a topological isomorphism. In particular, if a group G admits two Polish group topologies with the same Borel sets, these topologies coincide.*

Remark A.2.18. If a group carries two different Polish group topologies they are incomparable.

A.3 PONTRYAGIN DUAL AND CHARACTERS

A.3.1 The characters group

Definition A.3.1. Let $X = (G, \tau)$ be a topological abelian group.

- An homomorphism $\chi \in \text{hom}(G, \mathbb{T})$ is said to be a *character* of X if it is continuous with respect to τ . Denote by X^* the group of all characters of X .
- Call two group topologies τ_1 and τ_2 on G *compatible* if $(G, \tau_1)^* \cong (G, \tau_2)^*$.

- Denote by \widehat{X} the character group X^* equipped with the *compact-open topology*, i.e. that one having as a base of neighbourhood of 0 the family of sets

$$W(K, U) = \{\chi \in X^* \mid \chi(K) \subseteq U\},$$

where $K \subseteq X$ is compact and U is an open neighbourhood of $0_{\mathbb{T}}$. Call \widehat{X} the *dual group* (or *Pontryagin dual*) of X .

Definition A.3.2. A subgroup H of a topological abelian group X is said to be *B-embedded* if every character $\chi \in (\widehat{G/H})_d$ is continuous in the quotient topology of G/H .

Definition A.3.3. A subgroup H of a topological group X is said to be *dually embedded* if for every $\vartheta \in H^*$, there exists $\chi \in X^*$ such that $\chi \upharpoonright_H = \vartheta$.

Definition A.3.4. A subgroup H of a topological group X is *dually closed* if for each $x \in X \setminus H$, there exists a character χ of X that separates H and x , i.e. χ is identically zero on H and $\chi(x) \neq 0$.

Lemma A.3.5 ([59, Lemma 24.4][75, Theorem 27]). *Let X be a locally compact abelian group. If H is a closed subgroup of X and ψ is a continuous character of H , then there exists a continuous character χ of X such that $\chi(x) = \psi(x)$ for all $x \in H$, i.e. $\chi \upharpoonright_H = \psi$ i.e. H is dually embedded in X .*

Proposition A.3.6 ([59, 23.19]). *Let X be a compact abelian group and $\chi \in \widehat{X}$. Then $\int_X e^{2\pi i \chi(x)} dx = 0$ unless $\chi \equiv 0_{\mathbb{T}}$, in which case the integral is 1.*

Definition A.3.7. A subset A of a topological abelian group X is called *quasi-convex* if for every $x \in X \setminus A$ there exists $\chi \in \widehat{X}$ such that

$$\chi(A) \subset \mathbb{T}_+ \text{ and } \chi(x) \notin \mathbb{T}_+$$

where \mathbb{T}_+ is the image of the segment $[-\frac{1}{4}, \frac{1}{4}]$ with respect to the natural quotient map $\pi : \mathbb{R} \rightarrow \mathbb{T}$.

Definition A.3.8. A topological group (G, τ) (as well as its topology τ) is called *locally quasi-convex* if (G, τ) has a basis of neighborhoods of 0 consisting of quasi-convex subsets of (G, τ) .

Remark A.3.9. Vilenkin in [85] proved that for every topological abelian group X , its dual \widehat{X} is locally quasi-convex. Indeed, if one consider the set

$$W(K, \mathbb{T}_+) = \left\{ \chi \in \widehat{X} \mid \chi(K) \subseteq \mathbb{T}_+ \right\},$$

where K is a compact subset of X , i. e. $K \in \mathcal{K}(X)$, then

$$\{W(K, \mathbb{T}_+) \mid K \in \mathcal{K}(X)\}$$

is a neighbourhood basis of $e_{\widehat{X}}$.

Theorem A.3.10 (Peter-Weyl the abelian case). *Let X be a discrete abelian group. A subgroup $H \leq \widehat{X}$ separates the points of X if and only if H is dense in \widehat{X} .*

A.3.2 Pontryagin duality

Theorem A.3.11 (Pontryagin-van Kampen). *If X is a locally compact abelian group, then*

$$\alpha_X : X \rightarrow \widehat{\widehat{X}} \text{ where } \alpha_X(x) : \chi \mapsto \chi(x);$$

is a topological isomorphism.

Definition A.3.12. Call a topological abelian group $X = (G, \tau)$ (as well its topology) *reflexive* if α_X is a topological isomorphism.

Proposition A.3.13. *Let X be a topological abelian group.*

- (i) *If X is locally compact, then \widehat{X} is locally compact;*
- (ii) *if X is compact, then \widehat{X} is discrete;*
- (iii) *if X is discrete, then \widehat{X} is compact.*

Theorem A.3.14. *If X is a locally compact abelian group and H is a closed subgroup, then*

- (i) ([59, 23.25]) $\widehat{X/H} \cong H^\perp$ and the adjoint of the canonical projection is the isomorphism;
- (ii) ([59, 24.5]) if H is open or compact $\widehat{H} \cong \widehat{X}/H^\perp$
- (iii) $w(X) = w(\widehat{X})$.

Corollary A.3.15. *Let H be a subgroup of locally compact abelian group. Then $(H^\perp)^\perp = H$.*

Proposition A.3.16. *If X is a compact abelian group, then*

- (i) X is connected if and only if \widehat{X} is torsion-free;
- (ii) X is totally disconnected if and only if \widehat{X} is torsion;
- (iii) $c(X) = (t(\widehat{X}))^\perp$.
- (iv) $w(X) = |\widehat{X}|$
- (v) X is metrizable if and only if \widehat{X} is countable.

A.3.3 Topologies induced by characters

Definition A.3.17. A subset A of an abelian semigroup G is said to be a *large set* if there exist a finite subset F of G such that $F + A = G$.

Definition A.3.18. A topological abelian group is said to be *totally bounded* (as well its topology) if every open non-empty subset is a large subset. A Hausdorff totally bounded group is called *precompact*. Analogously, a subset E of a topological abelian group is totally bounded if for every open neighbourhood U of 0 , there exists a finite subset F of E such that $F + U \supseteq E$.

Definition A.3.19. A topological abelian group is called *locally precompact* if it is Hausdorff and there exists an open precompact neighbourhood of 0 .

Remark A.3.20. One can prove that precompact groups are precisely topological isomorphic images of dense subgroup of compact groups. Therefore, precompact groups are precisely the groups with compact completion. Analogously, the locally precompact groups are precisely the groups with locally compact completion.

Definition A.3.21. Let G be an abelian group and $H \subseteq \text{hom}(G, \mathbb{T})$. Let τ_H be the weakest group topology on G that makes each character of H continuous. That is the topology generated by the following pre-base

$$\{\chi^{-1}(B) \mid B \in \mathcal{B}, \chi \in H\},$$

where \mathcal{B} is a base of the usual topology in \mathbb{T} . Call such a topology the *initial topology* of H .

Proposition A.3.22 ([25]). *The map*

$$H \mapsto \tau_H$$

is monotonically increasing with respect to the inclusion.

Proposition A.3.23 ([25]). *If $H \subseteq \text{hom}(G, \mathbb{T})$, then $\tau_H = \tau_{\langle H \rangle}$.*

Theorem A.3.24 (Comfort-Ross [25][Theorem 1.2]). *Let (G, τ) be a topological abelian group. Its topology τ is totally bounded if and only if there exist $H \leq G^*$ such that $\tau = \tau_H$.*

Moreover the following hold

- (i) $w(G, \tau_H) = |H|$ and $H_1 < H_2$ if and only if $\tau_{H_1} \subset \tau_{H_2}$;
- (ii) (G, τ_H) is Hausdorff if and only if H separates the points of G ;
- (iii) (G, τ_H) is linear (there exists a local base at 0 consisting of open subgroups) if and only if H is torsion.

A linear topology in \mathbb{Z} generated by a torsion subgroup of \mathbb{T} is the p -adic topology.

Example A.3.25 (p-adic topology). Let $H = \mathbb{Z}(p^\infty) \leq \mathbb{T} = \mathbb{Z}^*$ be the Prüfer group for a prime p . Then $\tau_H = \tau_p$ where τ_p is the p-adic topology, i. e. the topology such that $\mathcal{V}_{\tau_p}(0)$ is generated by $p^n\mathbb{Z}$ for $n \in \mathbb{N}$.

Indeed, if $\chi_{\frac{1}{p^n}} : 1 \mapsto \frac{1}{p^n} + \mathbb{Z}$, then $\mathcal{V}_{\tau_H}(0)$ is generated by

$$U_{n,m} = \chi_{\frac{1}{p^n}}^{-1} \left(\left(-\frac{1}{p^m}, \frac{1}{p^m} \right) \right)$$

One has $U_{n,m} = p^n\mathbb{Z}$ for every $m \geq n$.

Note that the p-adic topology is Hausdorff, since $\mathbb{Z}(p^\infty)$ is infinite and hence dense in $\mathbb{T} = \widehat{\mathbb{Z}}$.

Example A.3.26 (Fürstenberg topology [49]). Let $\tau_{\mathcal{F}}$ be the Fürstenberg topology, i. e. the topology such that $\{n\mathbb{Z} \mid n \in \mathbb{N}_+\}$ is a base of $\mathcal{V}_{\tau_{\mathcal{F}}}(0)$ and $\{p^n\mathbb{Z} \mid p \in \mathbb{P}, n \in \mathbb{N}\}$ is a pre-base. Let $H = \mathbb{Q}/\mathbb{Z} = \bigoplus_{p \in \mathbb{P}} \mathbb{Z}(p^\infty)$, that is the torsion subgroup of \mathbb{T} . Then $\tau_H = \tau_{\mathcal{F}}$ and hence it is linear. Indeed, $\tau_{\mathcal{F}} = \sup\{\tau_p \mid p \in \mathbb{P}\} = \sup\{\tau_{\mathbb{Z}(p^\infty)} \mid p \in \mathbb{P}\} = \tau_{\mathbb{Q}/\mathbb{Z}}$.

Also the Fürstenberg topology is Hausdorff and hence pre-compact.

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