Some applications of algebraic entropy to the proof of Milnor-Wolf theorem

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Some applications of Entropy to Geometric Group Theory

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Abstract

We discuss the growth function of a finitely generated cascade and its connection to the growth function of its related semi-direct product (Conjecture 1.9). The results is applied for simpler proof of well-known results in the realm of geometric group theory. We show that the finitely generated cascades on nilpotent groups obey the dichotomy rule (only polynomial and exponential growth are possible).

1 Introduction

Entropy has played an important role in different contexts since its first appearance in the middle of the 19th century in thermodynamics. During the last century, several notions of entropy were introduced, such as measure, topological and algebraic entropy. In the sequel we shall only discuss the algebraic entropy of group endomorphisms, so we adopt in the sequel the abbreviation *entropy* for *algebraic entropy*. In the abelian case it was shortly introduced in [1] and further studied thoroughly in [23, 18, 7, 20, 19] and [5] (see [6] for a complete picture in this case and see §2 for the definition of entropy). Entropy for endomorphisms of arbitrary groups was introduced in [3] and further studied in [10, 11]. The idea to combine entropy with the classical notion of growth of a finitely generated group appeared in [4] and was further developed in [3, 10, 11].

In this article we apply entropy to obtain a more transparent proof of the classical Milnor-Wolf Theorem about the growth rate of finitely generated soluble groups (Theorem 1.7). This idea of this argument was born during a semester seminar at Udine University in 2013, dedicated to Geometric Group Theory. We noticed that the massive use of entropy can give a more transparent proof than the standard proof of Milnor-Wolf Theorem [14, §5]. These results were presented by the second named author at the Conference "Gruppen und topologische Gruppen" in January 2014, as well as at the Conference "Recent advances in Commutative Ring and Module Theory" at Bressanone in 2016 (http://rings-modules-bressanone2016.blogspot.com/p/abstracts.html).

In what follows we recall basic facts on growth rate of maps (§1.1), flows in groups (§1.2), the classical notion of growth of finitely generated groups (§1.3). In §1.4 we announce our main results.

1.1 Growth rate of maps $\mathbb{N} \to \mathbb{R}$

In order to measure and classify the growth rate of maps $\mathbb{N} \to \mathbb{R}$, we need the relation $\preceq$ defined as follows. For $\gamma, \gamma' : \mathbb{N} \to \mathbb{R}$ let $\gamma \preceq \gamma'$ if there exist $n_0, C \in \mathbb{N}_+$ such that $\gamma(n) \leq \gamma'(Cn)$ for every $n \geq n_0$. Moreover $\gamma \sim \gamma'$ if $\gamma \preceq \gamma'$ and $\gamma' \preceq \gamma$ (then $\sim$ is an equivalence relation), and $\gamma \prec \gamma'$ if $\gamma \preceq \gamma'$ but $\gamma \not\sim \gamma'$.

For example, for every $\alpha, \beta \in \mathbb{R}_{\geq 0}$, $n^\alpha \sim n^\beta$ if and only if $\alpha = \beta$; if $p(t) \in \mathbb{Z}[t]$ and $p(t)$ has degree $d \in \mathbb{N}$, then $p(n) \sim n^d$. On the other hand, $a^n \sim b^n$ for every $a, b \in \mathbb{R}$ with $a, b > 1$, so in particular all exponentials are equivalent with respect to $\sim$. Finally, a map $\gamma : \mathbb{N} \to \mathbb{R}$ is called:

(a) *polynomial* if $\gamma(n) \preceq n^d$ for some $d \in \mathbb{N}_+$;

(b) *exponential* if $\gamma(n) \sim 2^n$;

(c) *subexponential* if $\gamma(n) \prec 2^n$;

(d) *intermediate* if $\gamma(n) \succ n^d$ for every $d \in \mathbb{N}_+$ and $\gamma(n) \prec 2^n$.

An easy criterion for detecting (and measuring) polynomial growth provides the *growth exponent* $\delta(\gamma)$ defined for a function $\gamma : \mathbb{N} \to \mathbb{R}$ by $\delta(\gamma) := \limsup_{n\to\infty} \frac{\log \gamma(n)}{\log n}$. Then one can easily see that γ is polynomial if and only if $\delta(\gamma) < \infty$. In such a case, $\gamma \sim n^{\delta(\gamma)}$. In other words, γ is intermediate if and only if $\delta(\gamma) = \infty$ and $\gamma(n) \prec 2^n$.

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1.2 Flows in Grp and their growth rate

In the sequel **Grp** denotes the category of groups and group homomorphisms.

Definition 1.1. A *flow* in **Grp** is a pair (G, ϕ), where G is a group and $\phi : G \to G$ is an endomorphism of G. The flow (G, ϕ) is called a *cascade*, in case ϕ is an automorphism.

If the group G is abelian (or nilpotent), then we speak of *abelian flow* (or *nilpotent flow*) and *abelian cascade* (or *nilpotent cascade*), respectively. Call a flow (G, ϕ) trivial, if $\phi = \mathrm{id}_G$.

For a flow (G, ϕ) a subgroup H of G is said to be *ϕ-invariant*, if $\phi(H) \leq H$; a ϕ-invariant subgroup is said to be *ϕ-stable*, if $\phi^{-1}(H) \leq H$. Subflow and subcascades are defined as follows.

Definition 1.2. Let (G, ϕ) be a flow (cascade) in **Grp**. A *subflow* (*subcascade*) of (G, ϕ) is a pair $(H, \phi\restriction_H)$, such that H is a ϕ-invariant (ϕ-stable) subgroup of G.

Let (G, ϕ) be a flow in **Grp**, and F be a non-empty finite subset of G. For every $n \in \mathbb{N}_+$, we call

$$T_n(\phi, F) := F \cdot \phi(F) \cdot \ldots \cdot \phi^{n-1}(F)$$

the n-trajectory of ϕ, and

$$T^+(\phi, F) = \bigcup_{n \in \mathbb{N}_+} T_n(\phi, F)$$

the trajectory of ϕ.

The subgroup $G_{\phi,F}$ of G generated by the trajectory $T^+(\phi, F)$ is ϕ-invariant, as it coincides with the subgroup generated by the orbit $\{F, \phi(F), \ldots, \phi^n(F), \ldots\}$ of F. Note that the trajectories of F and $F_1 = \{e_G\} \cup F \cup F^{-1}$ generate the same subgroup $G_{\phi,F} = G_{\phi,F_1}$ of G.

Definition 1.3. We say that the flow (G, ϕ) in **Grp** is *finitely generated* if $G = G_{\phi,F}$ for some finite subset F of G; if such an F is a singleton, we call *cyclic* the flow $(G_{\phi,F}, \phi)$.

Consider the function

$$\gamma_{\phi,F} : \mathbb{N}_+ \to \mathbb{N}_+ \text{ defined by } \gamma_{\phi,F}(n) = |T_n(\phi, F)| \text{ for every } n \in \mathbb{N}_+.$$

Since

$$|F| \leq \gamma_{\phi,F}(n) \leq |F|^n \text{ for every } n \in \mathbb{N}_+,$$

the growth of $\gamma_{\phi,F}$ is always at most exponential. Following [4] and [3], we give the following definition.

Definition 1.4. Let (G, ϕ) be a finitely generated flow in **Grp** and let F be a finite subset of G such that $e_G \in F = F^{-1}$ and $G = G_{\phi,F}$. We say that

- (G, ϕ) *has polynomial growth with respect to* F or (G, ϕ, F) *has polynomial growth*, if $\gamma_{\phi,F}$ is polynomial.
- (G, ϕ) *has exponential* (respectively, *subexponential*, *intermediate*) *growth with respect to* F or (G, ϕ, F) *has exponential* (respectively, *subexponential*, *intermediate*) *growth*, if $\gamma_{\phi,F}$ is exponential (respectively, subexponential, intermediate).

Before proceeding further, let us make an important point here. For a finite subset F of G, all properties considered above concern practically the ϕ-invariant subgroup $G_{\phi,F}$ of G and the restriction $\phi\restriction_{G_{\phi,F}}$. Hence, all properties listed above concern finitely generated flows in **Grp**.

1.3 The growth of a finitely generated group

The notion of growth of a finitely generated flow, applied to trivial flows, generalizes the classical notion of growth of a finitely generated group given independently by Schwarzc [21] and Milnor [15]. Indeed, if G is a finitely generated group and X is a finite symmetric set of generators of G containing e_G, then $\gamma_X = \gamma_{\mathrm{id}_G,X}$ is the classical *growth function* of G with respect to X. For a connection of the terminology coming from the theory of algebraic entropy and the classical one, note that for $n \in \mathbb{N}_+$ we have that $T_n(\mathrm{id}_G, X) = \{g \in G : \ell_X(g) \leq n\}$, where $\ell_X(g)$ is the length of the shortest word w in the alphabet X such that $w = g$. Since ℓ_X is a norm on G, the set $T_n(\mathrm{id}_G, X)$ coincides with the ball $B_n^S(e_G)$ of radius n centered at e_G and $\gamma_X(n)$ is the cardinality of this ball.

Milnor [17] proposed the following problem on the growth of finitely generated groups.

Problem 1.5 (Milnor Problem)**.** [17] *Let G be a finitely generated group, and X be a finite symmetric set of generators of G containing e_G.*

(i) Is the growth function γ_X necessarily equivalent either to a power of n or to the exponential function 2^n?

(ii) In particular, is the growth exponent $\delta_G = \limsup_{n\to\infty} \frac{\log \gamma_X(n)}{\log n}$ either a well defined integer or infinity? For which groups is δ_G finite?

Part (i) of Problem 1.5 was solved negatively by Grigorchuk in [12], where he constructed an example of finitely generated group G with intermediate growth. For part (ii) Milnor conjectured that δ_G is finite (i.e., G has polynomial growth) if and only if G is virtually nilpotent (i.e., G contains a nilpotent finite-index subgroup). The same conjecture was formulated by Wolf [24] (who proved that a nilpotent finitely generated group has polynomial growth) and Bass [2]. Gromov [13] confirmed Milnor's conjecture:

Theorem 1.6. [13] *A finitely generated group G has polynomial growth if and only if G is virtually nilpotent.*

In spite of the negative solution of Grigorchuk, the dichotomy of item (i) of Milnor problem (namely, γ_F either polynomial or exponential for a finitely generated group $G = \langle F \rangle$) turned out to hold true for some classes of groups, e.g., linear groups. This follows from the famous Tits Alternative (every finitely generated linear group either contains a free non-abelian group or a soluble subgroup of finite index, see [22]) and the following.

Theorem 1.7. [16, 24] *There exist no finitely generated soluble groups of intermediate growth.*

Our aim is to give a new proof of Theorem 1.7 (i.e., *a finitely generated soluble group of subexponential growth has polynomial growth*), by using properties of entropy in abelian groups. More precisely, we use the subtle fact that every abelian flow (G, ϕ) of entropy zero and $G \neq 0$ has non-trivial *quasi-periodic* elements (see Definition 2.6). Using this fact (via Lemma 3.10) we deduce in Theorem 4.4 that a finitely generated soluble group of subexponential growth is virtually nilpotent. Hence, it has polynomial growth by Theorem 1.6.

We split the proof of Milnor-Wolf theorem in several steps, as explained in more detailed below.

1.4 Main results

Let the pair (G, ϕ) be a cascade, i.e., $\phi \in Aut(G)$. For a finite subset F of G and $n \in \mathbb{N}$ we define

$$T_{-n}(\phi, F) := \phi^{-n}(F) \cdot \phi^{-n+1}(F) \cdot \ldots \cdot \phi^{-1}(F) = \phi^{-n}(T_n(\phi, F)),$$

$$\text{and } \mathfrak{T}_n(\phi, F) = T_{-n}(\phi, F) \cdot T_n(\phi, F).$$

We also define

$$T^-(\phi, F) = \bigcup_{n\in\mathbb{N}} T_{-n}(\phi, F) \quad \text{and} \quad T(\phi, F) = \bigcup_{n\in\mathbb{N}} \mathfrak{T}_n(\phi, F).$$

We say that the cascade (G, ϕ) is *finitely generated* (*cyclic*, respectively) if there exists a finite subset (a singleton, respectively) F of G such that $G = \langle T(\phi, F) \rangle$.

Theorem A. *Let (G, ϕ, F) be a finitely generated cascade of subexponential growth. Then the group G is finitely generated.*

The proof of Theorem A, which is showed in §3.1, follows from Remark 2.1 and Proposition 3.6.

Theorem B. *Let (N, ϕ) be a cascade and $G = N \rtimes \langle \phi \rangle$. Then G is finitely generated if and only if the cascade (N, ϕ) is finitely generated. Moreover,*

(a) if F is a finite set of generators of the cascade (N, ϕ), then $\gamma_{\phi,F} \preceq \gamma^G$;

(b) if G has subexponential growth, then (N, ϕ, F) has subexponential growth for every $F \in [N]^{<\omega}$, and N is a finitely generated group.

Theorem B is proved in §3.2. As a growth is either exponential or subexponential, combining this with item (b) in Theorem B, we obtain the following corollary.

Corollary 1.8. *If (N, ϕ, F) is a finitely generated cascade of exponential growth, then the group $G = N \rtimes \langle \phi \rangle$ has exponential growth.*

It is not clear whether one can replace here "exponential" by "subexponential" or "polynomial". The next theorem shows that this is the case when the flow is nilpotent. Its proof follows by Theorem 3.11.

Theorem C. *If (N, ϕ) is a finitely generated nilpotent cascade, then the group $G = N \rtimes \langle \phi \rangle$ has the same type of growth as the cascade (N, ϕ).*

By Theorem A, if (N, ϕ) is a finitely generated cascade of subexponential growth, then the group N is finitely generated. This suggests the following.

Conjecture 1.9. *If (N, ϕ, F) is a finitely generated cascade and $G = N \rtimes \langle \phi \rangle$, then the group G has the same type of growth as the cascade (N, ϕ, F).*

The conjecture holds true when ϕ is periodic (Remark 2.10), or when N is nilpotent (Theorem C), or when the cascade (N, ϕ, F) has exponential growth (Corollary 1.8). In fact, Giordano Bruno and Spiga have recently proved the following result.

Proposition 1.10. [11, Proposition 5.2] *If N is a finitely generated group and $\phi : N \to N$ is an automorphism, then the cascade (N, ϕ) has polynomial growth if and only if the group $G = N \rtimes \langle \phi \rangle$ has polynomial growth.*

Then the remaining part of Conjecture 1.9 can be formulated as follows (see also [11, Conjecture 5.4]):

Conjecture 1.11. *If N is a finitely generated group and (N, ϕ, F) is a cascade of subexponential growth, then the (finitely generated) group $G = N \rtimes \langle \phi \rangle$ has subexponential growth.*

1.5 Notation and terminology

We denote by $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ the sets of real, rational, integer, and natural numbers, and by $\mathbb{N}^+$ and $\mathbb{P}$ the sets of positive natural numbers and prime numbers, respectively. We denote by $[X]^{<\omega}$ the family of the finite non-empty subsets of a set X.

For a positive integer k, we denote by $\mathbb{Z}(k)$ the finite group with k elements, and by $\mathbb{Z}[\frac{1}{k}] \subseteq \mathbb{Q}$ the subring of $\mathbb{Q}$ generated by $\frac{1}{k}$.

Let G be a group with identity e_G. For a subset A of G, we denote by $\langle A \rangle$ the subgroup generated by A. If $x \in G$ and $\langle x \rangle$ is finite, then x is a *torsion* element of G, and $o(x) = |\langle x \rangle|$ is the *order* of x. For $n \in \mathbb{N}^+$, let $G[n] = \{g \in G : g^n = e_G\}$, and $t(G) = \bigcup_{n \in \mathbb{N}^+} G[n]$ be the torsion part of G. In particular, G is *torsion* if $t(G) = G$, while G is *torsion-free* if $t(G) = \{e_G\}$.

We denote by G' the *derived subgroup* of G, namely the subgroup of G generated by all commutators $[a, b] = aba^{-1}b^{-1}$, as $a, b \in G$. As usual $Z(G)$ denotes the *center* of G. The *n-th center* $Z_n(G)$ is defined as follows for $n \in \mathbb{N}$. Let $Z_0(G) = \{e\}$, $Z_1(G) = Z(G)$, and assume that $Z_{n-1}(G)$ is already defined for $n > 1$. Consider the canonical projection $\pi \colon G \to G/Z_{n-1}(G)$ and let $Z_n(G) = \pi^{-1} Z(G/Z_{n-1}(G))$. Note that $Z_n(G) = \{x \in G : [x, y] \in Z_{n-1}(G)$ for every $y \in G\}$.

A group N is called *nilpotent*, if there exists a finite central series

$$\{e_N\} = N_0 \lhd N_1 \lhd \ldots \lhd N_{l-1} \lhd N_l = N, \tag{1}$$

i.e., a series of normal subgroups of N such that for each $i < l$ the center of the quotient N/N_i contains N_{i+1}/N_i, i.e., the subgroup $\langle [y, g] : y \in N_{i+1}, g \in N \rangle$ is contained in N_i (or, as one briefly says, *N centralizes all the quotients N_{i+1}/N_i*). In particular, N_i is characteristic in N. The nilpotency class of a nilpotent group N is the smallest positive l for which a series as in (1) satisfies the above properties.

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2 Background

If (G, ϕ) is a flow, and $F \in [G]^{<\omega}$, the limit

$$H_{alg}(\phi, F) := \lim_n \frac{\log \gamma_{\phi,F}(n)}{n} \leq \log |F|$$

exists, due to a folklore lemma of Fekete [9] (see [3]). Throughout this paper, $\log x$ always stands for the natural logarithm of x, in base the Napier number e.

The (algebraic) entropy $h_{alg}(\phi)$ of ϕ is defined by

$$h_{alg}(\phi) := \sup\{H_{alg}(\phi, F) : F \in [G]^{<\omega}\}.$$

In the next Remark 2.1 we see that $H_{alg}(\phi, F) > 0$ if and only if $\gamma_{\phi,F}$ has exponential growth.

Remark 2.1. Let (G, ϕ) be a flow. Let us check that $h_{alg}(\phi) > 0$ if and only if $\gamma_{\phi,F}$ has exponential growth for some finite subset F of G. Indeed, $h_{alg}(\phi) > 0$ exactly when $H_{alg}(\phi, F) > 0$ for some F. If $a = H_{alg}(\phi, F)$, we have that in this case $\frac{\log \gamma_{\phi,F}(n)}{n} > \frac{a}{2}$ for all sufficiently large n, so $\gamma_{\phi,F}(n) > e^{\frac{a}{2}n}$ for all sufficiently large n. Similarly, if $b > 1$ and $\gamma_{\phi,F}(n) > b^n$ for all sufficiently large n, then $H_{alg}(\phi, F) > \log b$, so $h_{alg}(\phi) \geq \log b > 0$.

This is why, a flow with subexponential growth has zero entropy. In the abelian case, this leads to polynomial growth by the following theorem.

Theorem 2.2. [4, Dichotomy Theorem] *Every abelian algebraic flow (G, ϕ) has either exponential or polynomial growth with respect to any fixed non-empty finite subset F of G.*

If a finitely generated abelian flow (cascade) (G, ϕ) has polynomial (subexponential) growth, then so does every finitely generated subflow (subcascade) of (G, ϕ). This will be proved in Lemma 5.2 in §5.

Remark 2.3. Consider a finitely generated group G, having a subgroup G_1 which is finitely generated by some subset $F_1 \subseteq G_1$. Then the growth of G_1 is subexponential whenever the growth of G is subexponential. Indeed, if the latter holds, then by definition $\gamma_{id_G,F}$ is subexponential for every $F \in [G]^{<\infty}$, so $\gamma_{id_{G_1},F_1} = \gamma_{id_G,F_1} \restriction_{G_1}$ is subexponential.

Lemma 2.4. [4, Lemma 2.8] *Let (G, ϕ) be an abelian flow and $F \in [G]^{<\omega}$. If $H_{alg}(\phi, F) = 0$, then $h_{alg}(\phi \restriction_{G_{\phi,F}}) = 0$.*

This lemma shows that, for a finitely generated abelian flow, the property of having entropy zero does not depend on the (finite) set of generators; in other words, if an abelian flow (G, ϕ) has subexponential growth with respect to some $F \in [G]^{<\omega}$, then it has subexponential growth with respect to every $F' \in [G]^{<\omega}$. This is a particular case of Lemma 5.2.

Next we recall the notion of Pinsker subgroup $\mathbf{P}(G, \phi)$ of a flow (G, ϕ).

Theorem 2.5. [4] *Let G be an abelian group and $\phi \in \mathrm{End}(G)$. Then G has a maximum ϕ-invariant subgroup $\mathbf{P}(G, \phi)$ such that $h_{alg}(\phi \restriction_{\mathbf{P}(G,\phi)}) = 0$. It satisfies $\mathbf{P}(G/\mathbf{P}(G, \phi), \overline{\phi}) = 0$.*

2.1 Quasi-periodic elements and the QP-subgroup

Definition 2.6. An element $x \in G$ is a *quasi-periodic* point of ϕ if there exist $n > m$ in $\mathbb{N}$ such that $\phi^n(x) = \phi^m(x)$.

The quasi-periodic points of a flow (G, ϕ) form a ϕ-invariant subgroup $Q_1(G, \phi)$. For example, if G is a torsion abelian group, then $\mathbf{P}(G, \phi) = Q_1(G, \phi)$ (see [7]).

If $Q_1(G, \phi)$ is finitely generated, then there exists $n > m$ in $\mathbb{N}$, such that $\phi^n \restriction_{Q_1(G,\phi)} = \phi^m \restriction_{Q_1(G,\phi)}$. Indeed, if $Q_1(G, \phi) = \langle x_1, \ldots, x_k\rangle$, fix $1 \le i \le k$ and let $\phi^{n_i}(x_i) = \phi^{m_i}(x_i)$ for $n_i > m_i$ in $\mathbb{N}$. If m is the maximum among $m_1, \ldots, m_k$, and d is the least common multiple of the positive numbers $n_1 - m_1, \ldots, n_k - m_k$, then $\phi^m(x_i) = \phi^{m+d}(x_i)$, so $\phi^m = \phi^{m+d}$ on $Q_1(G, \phi)$.

Let G be an abelian group and $\phi \in \mathrm{End}(G)$. In the general (non-torsion) case one has a chain

$$Q_0(G, \phi) \subseteq Q_1(G, \phi) \subseteq \ldots \subseteq Q_n(G, \phi) \subseteq \ldots, \qquad (2)$$

where $Q_0(G, \phi) = 0$, and $Q_{n+1}(G, \phi)/Q_n(G, \phi) = Q_1(G/Q_n(G, \phi), \bar{\phi})$ for every $n \in \mathbb{N}$, where $\bar{\phi}$ is the induced endomorphism $G/Q_n(G, \phi) \to G/Q_n(G, \phi)$. Finally, one defines the subgroup $\mathfrak{Q}(G, \phi) = \bigcup_{n\in\mathbb{N}} Q_n(G, \phi)$.

Remark 2.7. Recall that a normal subgroup H of a group G is *pure* if the quotient G/H is torsion-free. Each $Q_n(G, \phi)$, and $\mathfrak{Q}(G, \phi)$, is a ϕ-invariant subgroup of G, and these subgroups are pure whenever G is torsion-free ([4, Lemma 4.5]).

Theorem 2.8. [4] *If (G, ϕ) is an abelian flow, then $\mathfrak{Q}(G, \phi) = \mathbf{P}(G, \phi)$. Moreover, this subgroup is the maximum ϕ-invariant subgroup P of G such that $\phi\restriction_P$ has polynomial growth.*

In analogy with the series (2) one can define

$$\{0\} = \mathrm{Fix}_0(G, \phi) \subseteq \mathrm{Fix}_1(G, \phi) \subseteq \ldots \subseteq \mathrm{Fix}_n(G, \phi) \subseteq \ldots, \qquad (3)$$

where, $\mathrm{Fix}_1(G, \phi) := \{x \in G : x \text{ is a fixed point of } \phi\} \le G$ and

$$\mathrm{Fix}_{n+1}(G, \phi)/\mathrm{Fix}_n(G, \phi) = \mathrm{Fix}_1(G/\mathrm{Fix}_n(G, \phi), \bar{\phi})$$

for every $n \in \mathbb{N}$. Since ϕ acts on the quotient $G/\mathrm{Fix}_n(G, \phi)$ by means of the quotient map $\bar{\phi}$, we say that ϕ "fixes" the quotient $\mathrm{Fix}_{n+1}(G, \phi)/\mathrm{Fix}_n(G, \phi)$. Since G is an abelian group, $\psi = \phi - id_G$ is an endomorphism of G and $\mathrm{Fix}_n(G, \phi) = \ker \psi^n$.

In these terms one can characterize of the abelian cascades (G, ϕ) such that the ascending series (3) reaches G after finitely many steps as follows:

Lemma 2.9. *For an abelian group G and an automorphism ϕ of G, the extension $G^* = G \rtimes \langle\phi\rangle$ is nilpotent if and only if there exists n such that $\mathrm{Fix}_n(G, \phi) = G$.*

Proof. For every $k \in \mathbb{N}$ with $\mathrm{Fix}_k(G,\phi) \neq G$, the k-th center of G^* is $Z_k(G^*) = \mathrm{Fix}_k(G,\phi) \rtimes \langle 1 \rangle$. Then the ascending central series of G^* is

$$\mathrm{Fix}_0(G,\phi) \rtimes \langle 1 \rangle \subseteq \mathrm{Fix}_1(G,\phi) \rtimes \langle 1 \rangle \subseteq \ldots \subseteq \mathrm{Fix}_n(G,\phi) \rtimes \langle 1 \rangle \subseteq \ldots \subseteq G^*.$$

Since G^* is nilpotent if and only if there exists k such that $Z_k(G^*) = G^*$, we deduce that for the smallest such k one has $G^* \neq Z_{k-1}(G^*) = \mathrm{Fix}_{k-1}(G,\phi) \rtimes \langle 1 \rangle$, and so $\mathrm{Fix}_k(G,\phi) = G$. □

An automorphism ϕ of a group N is *periodic* if ϕ is a torsion element of the group $Aut(N)$, i.e. if $\phi^m = id_N$ for some positive integer m, so the subgroup $\langle \phi \rangle$ of $Aut(N)$ is isomorphic to some finite cyclic group $\mathbb{Z}(m)$. In this case, the cascade (N,ϕ) is called *periodic*. If (N,ϕ) is a periodic cascade, then its growth coincides with that of the group N:

Remark 2.10. Let N be a group, finitely generated by a finite symmetric subset F. Let ϕ be a periodic automorphism of N and let m be the order of ϕ in $Aut(N)$. Then the semi-direct product $G = N \rtimes \langle \phi \rangle \cong N \rtimes \mathbb{Z}(m)$ is finitely generated and its growth γ^G is of the same type as that of γ^N, as N has finite index in G.

3 Cascades

3.1 The growth of cascades – Proof of Theorem A

Example 3.1. A cascade (G,ϕ) is finitely generated whenever it is a finitely generated flow. Nevertheless, a finitely generated cascade (G,ϕ) need not be also a finitely generated flow in general. In particular, the group G need not be finitely generated, see Proposition 3.6. Take for example

(a) $G = \mathbb{Z}[1/2]$ and the automorphism $\phi : G \to G$ defined by $\phi(x) = 2x$ for any $x \in G$;

(b) the two sided right Bernoulli shift $\phi : G := \bigoplus_{\mathbb{Z}} \mathbb{Z}(2) \to G$ defined by $\phi((x_n)) = (x_{n-1})$.

Indeed, in (a) $G = \bigcup_{n \in \mathbb{Z}} C_n$, where the subgroups $C_n = \langle \frac{1}{2^n} \rangle$ form an increasing chain and $\phi(C_n) = C_{n-1}$. So, $G \neq \langle T^+(\phi,F) \rangle$ for any finite subset F of G, while $G = \langle T^-(\phi,\{1\}) \rangle = \langle T(\phi,\{1\}) \rangle$ (so the flow (G,ϕ^{-1}) is finitely generated, even cyclic).

Similarly, in (b) we have $G \neq \langle T^+(\phi,F) \rangle$ and $G \neq \langle T^-(\phi,F) \rangle$ for any finite subset F of G (so the flows (G,ϕ) and (G,ϕ^{-1}) are not finitely generated), while $G = \langle T(\phi,F_m) \rangle$, where $F_m = \{x = (x_n) : x_n = 0 \text{ for all } n \neq m\}$ and $m \in \mathbb{Z}$. So the cascade (G,ϕ) is finitely generated.

One can define the growth of a finitely generated cascade in a way similar to that of finitely generated flows:

Definition 3.2. A finitely generated cascade $(G = \langle T(\phi,F) \rangle, \phi)$ has *(sub)exponential (or polynomial) growth*, if the function $n \mapsto |\mathfrak{T}_n(\phi,F)|$ has (sub)exponential (or polynomial) growth. We say that $(G = \langle T(\phi,F) \rangle, \phi)$ has *exponential growth*, if the function $n \mapsto |\mathfrak{T}_n(\phi,F)|$ has exponential growth.

Remark 3.3. Note that for functions $\gamma, \gamma' : \mathbb{N} \to \mathbb{R}$, with $\gamma'(n) = \gamma(n)^2$ for all but finitely many n, one has $\gamma \preceq \gamma'$, and actually, $\gamma \sim \gamma'$, if at least one of them is exponential. Anyway, such a pair of functions has always the same type of growth. On the other hand, always $\gamma \not\sim \gamma'$, when the functions are polynomial and at least one of them is of positive degree.

Lemma 3.4. *Let (G,ϕ) be a cascade which is finitely generated as a flow. Then (G,ϕ,F) has (sub)exponential (or polynomial) growth as a cascade if and only if it has (sub)exponential (or polynomial) growth as a flow.*

Proof. Since $T_{-n}(\phi,F) = \phi^{-n}(T_n(\phi,F))$, one has

$$|\mathfrak{T}_n(\phi,F)| \leq |T_{-n}(\phi,F) \cdot T_n(\phi,F)| = |\phi^{-n}(T_n(\phi,F)) \cdot T_n(\phi,F)| \leq |\phi^{-n}(T_n(\phi,F))||T_n(\phi,F)| \leq |T_n(\phi,F)|^2.$$

According to Remark 3.3, $|T_n(\phi,F)|$ and $|T_n(\phi,F)|^2$ have the same type of growth. Hence, $G = \langle T(\phi,F) \rangle$ has (sub)exponential growth if and only if $n \mapsto |\mathfrak{T}_n(\phi,F)|$ has a (sub)exponential growth, if and only if $n \mapsto |T_n(\phi,F)|$ has a (sub)exponential growth. □

Example 3.1 shows that the underlying group of a finitely generated cascade need not be a finitely generated group in general. In the following results, we show some additional assumptions under which if (G,ϕ) is a finitely generated flow, then G is finitely generated.

Fact 3.5. [8, Corollary 2.4(b)] *If (G,ϕ) is a finitely generated abelian flow with $h_{alg}(\phi) < \log 2$, then the group G is finitely generated.*

The next proposition shows that the abelian assumption in the fact above is not necessary, thus proving Theorem A.

It is proved in [10, Lemma 4.2] that if a cyclic flow $(G, \phi, \{g\})$ is not a finitely generated group, then $H_{alg}(\phi, \{e_G, g\}) > 0$.

Proposition 3.6. *Let (G, ϕ) be a flow with $h_{alg}(\phi) < \log 2$. Then the group G is finitely generated in the following two cases:*

(a) if the flow (G, ϕ) is finitely generated,

(b) if ϕ is an automorphism, and the cascade (G, ϕ) is finitely generated as a cascade.

Proof. (a) We show first that one can assume without loss of generality that (G, ϕ) is cyclic. Indeed, assume that $F = \{y_1, \dots, y_d\}$ is the finite set of generators of the flow (G, ϕ). Let $(G_i, \phi \restriction_{G_i})$ by the cyclic subflow generated by the singleton $\{y_i\}$. The subgroup H generated by the subgroups $G_1, \dots, G_d$ is ϕ-invariant and contains F, so $H = G$. Hence, it suffices to check that each subgroup G_i is finitely generated.

From now on we assume that (G, ϕ) is cyclic, and generated as a flow by $F := \{e, y\}$. As

$$\lim_n \frac{\log |T_n(\phi, F)|}{n} = H_{alg}(\phi, F) \le h_{alg}(\phi) < \log 2,$$

we obtain $|T_n(\phi, F)| < 2^n$ for all sufficiently large n, and this means the following. If $\alpha = (k_0, \dots, k_{n-1}) \in \{0, 1\}^n$ and

$$w_\alpha = y^{k_0}\phi(y^{k_1}) \dots \phi^{n-1}(y^{k_{n-1}})$$

is a generic element of $T_n(\phi, F)$, then there exists a pair $\alpha, \beta \in \{0, 1\}^n$, $\beta = (l_0, \dots, l_{n-1}) \ne \alpha$, such that $w_\alpha = w_\beta$. Choosing the minimum n with this property, we can assume $k_{n-1} = 1 \ne 0 = l_{n-1}$. From $w_\alpha = w_\beta$ we deduce that $\phi^{n-1}(y) \in L_+ := \langle T_{n-1}(\phi, F)\rangle$, so $T_n(\phi, F) \subseteq \langle T_{n-1}(\phi, F)\rangle$. Then $L_+ = \langle T_n(\phi, F)\rangle$, and L_+ is a ϕ-invariant subgroup containing F. Hence, $G = L_+$ is finitely generated.

(b) Assume now that (G, ϕ) is a finitely generated cascade. By arguing as above, one can assume again that (G, ϕ) is cyclic and find n such that the finitely generated subgroup $L_+ := \langle T_{n-1}(\phi, F)\rangle$ is ϕ-invariant. Analogously one can find n' such that $L_- := \langle T_{n'}(\phi^{-1}, F)\rangle$ is ϕ^{-1}-invariant. Then the subgroup L generated by L_-L_+ is finitely generated. Moreover, as $\phi^{-1}(L_+) \le L$ and $\phi(L_-) \le L$, we deduce that L is ϕ-stable. Since it contains F, we have $L = G$, hence G is finitely generated. □

As an application of the above result, we immediately obtain Theorem A.

Proof of Theorem A. Let (G, ϕ, F) be a finitely generated cascade of subexponential growth, and we have to prove that the group G is finitely generated.

By Remark 2.1, we have that $h_{alg}(\phi) = 0$, so Proposition 3.6 applies. □

Theorem A yields that both cascades in Example 3.1 have exponential growth.

3.2 The semi-direct product associated to a cascade – Proof of Theorem B

Every cascade (N, ϕ) gives rise to a semi-direct product $G = N \rtimes \langle\phi\rangle$. Clearly, the group G is finitely generated whenever the cascade (N, ϕ) is finitely generated. The inverse implication holds as well by the following Theorem B, that also provides a condition that ensures the stronger conclusion that N itself is a finitely generated group.

Now we are in the position to prove Theorem B, so let (N, ϕ) be a cascade and $G = N \rtimes \langle\phi\rangle$. Recall that we have to prove that G is finitely generated if and only if the cascade (N, ϕ) is finitely generated. Moreover,

(a) If F is a finite set of generators of the cascade (N, ϕ), then $\gamma_{\phi,F} \preceq \gamma^G$;

(b) If G has subexponential growth, then the cascade (N, ϕ) has subexponential growth for every $F \in [N]^{<\omega}$, and N is a finitely generated group.

Proof of Theorem B. Obviously, the group G will be finitely generated if the cascade (N, ϕ) is finitely generated. Now, suppose that the group G is finitely generated and fix a finite set F_0 of generators of G. For simplicity, denote by x the element (e_N, ϕ) of G. Then the restriction on N of the conjugation by x is simply ϕ.

Adding x to the set of generators F_0, we can arrange so that this finite set of generators has the form $F_0 = \{x\} \cup F$, where $F = \{y_1, \dots, y_d\}$ is a finite subset of N (from now on we identify N with the subgroup $N \times \{1\}$ of G, so we consider F as a subset of G as well). The subset F of N need not generate the cascade (N, ϕ). Our aim here is to see that enlarging it appropriately we can find a finite set that generates the cascade (N, ϕ). Obviously, $K = \langle T(\phi, F)\rangle$ is a ϕ-stable subgroup of N that determines a subcascade $(K, \phi \restriction_K)$ of (N, ϕ). Since $\langle x, K\rangle = \langle x, F\rangle = G$, this means that K is a normal subgroup of G and G/K is generated by the coset $xK \in G/K$. Therefore, the subgroup N/K of G/K is cyclic as well. Let yK be a generator of N/K, where $y \in N$. Then $F_1 = F \cup \{y\}$ is a finite subset of N, such that $N = \langle K \cup \{y\}\rangle = \langle T(F_1, \phi)\rangle$. Hence, the cascade (N, ϕ) is finitely generated.

(a) First we prove the following inclusion
$$T_n(\phi, F) \subseteq B^S_{3n-2}(e_G), \tag{1}$$
where S is any symmetric finite set of generators of G containing $\{x\} \cup F$. Indeed, a generic element of $T_n(\phi, F)$ has the form
$$z_\alpha = \prod_{k=0}^{n-1} \phi^k(y_k), \text{ where } \alpha = (y_0, \dots, y_{n-1}) \in F^n.$$
Since $\phi^k(y_k) = x^k y_k x^{-k}$, we obtain
$$z_\alpha = \prod_{k=0}^{n-1} x^k y_k x^{-k} = y_0 x y_1 x^{-1} x^2 y_2 x^{-2} \dots x^{n-1} y_{n-1} x^{-n+1} = y_0 x y_1 x y_2 x \dots x y_{n-1} x^{-n+1}.$$
Hence, $l_S(z_\alpha) \leq 3n-2$. This proves (1).

From (1) we deduce
$$|T_n(\phi, F)| \leq \gamma_S(3n-2) \leq \gamma_S(3n).$$
Since $\gamma_S(3n) \sim \gamma_S(n)$, we deduce the required relation.

(b) In case G has subexponential growth, and F generates the cascade (N, ϕ), then $\gamma_{\phi,F} \prec \gamma^G$ implies that (N, ϕ, F) has subexponential growth as well. Now Theorem A applies to conclude that N is finitely generated. □

Remark 3.7. Note that if every cyclic subcascade $\langle \phi^n(y) : n \in \mathbb{Z} \rangle$ of a finitely generated cascade (N, ϕ) has subexponential growth, one can not deduce that the whole cascade (N, ϕ) has subexponential growth as well.

This is clearly the case when N is abelian, but this property fails to be true for example in non-abelian free groups.

Corollary 3.8. *If (N, ϕ) is a finitely generated cascade and the group $G = N \rtimes \langle \phi \rangle$ has subexponential growth, then the cascade (N, ϕ) has subexponential growth. In particular, N is finitely generated.*

It is not clear whether one can replace here "subexponential growth" either by "polynomial growth", or "intermediate growth", or "exponential growth" (see Conjecture 1.9).

The following examples shows that the stronger assertion $\gamma_{\phi,F} \sim \gamma^G$ is false.

To this end, consider the Heisenberg group $\mathbb{H} = \begin{bmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{bmatrix} \rtimes \begin{bmatrix} 1 & \mathbb{Z} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ where the action of $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ on $\begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$ is given by the usual matrix conjugation in $GL_3(\mathbb{Z})$
$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & x+y \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 3.9. Let $N = \mathbb{Z}^2$, with generators $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

(a) Consider the automorphism $\phi : N \to N$ of N defined by $\phi(e_1) = e_1$, $\phi(e_2) = e_1 + e_2$, so $\phi(x, y) = (x + y, y)$. Then $G = N \rtimes \langle \phi \rangle$ is isomorphic to the Heisenberg group $\mathbb{H}$ recalled above, hence $\gamma^G \sim n^4$ (this follows from the isomorphism $G \cong \mathbb{H}$ and Bass-Guivarch Formula [2]).

On the other hand, $F = \{\pm e_1, \pm e_2\}$ is a symmetric set of generators of the cascade (N, ϕ) for which one can inductively check that both T_{-n} and T_n are contained in $[-n^2, n^2] \times [-n, n]$, so that $\mathfrak{T}_n(\phi, F) = T_{-n} + T_n \subseteq [-2n^2, 2n^2] \times [-2n, 2n]$ and $|\mathfrak{T}_n(\phi, F)| \leq cn^3$ for a positive constant number c. Hence, $\gamma_{\phi,F} \preceq n^3$.

(b) Let now be $\psi : N \to N$ the automorphism of N given by $\psi(e_1) = e_2$ and $\psi(e_2) = e_1 + e_2$. To compute $h_{alg}(\psi)$ extend ψ to the automorphism $\psi : \mathbb{Q}^2 \to \mathbb{Q}^2$ defined in the same way. Then by Yuzvinski formula, $h_{alg}(\psi) = \log \alpha > 0$, where α is the Golden Ratio, i.e., the positive root of the characteristic polynomial $x^2 - x - 1$ of ψ. By Remark 2.1, $h_{alg}(\psi) > 0$ yields the existence of a finite subset F of N such that $n \mapsto |T_n(\psi, F)|$ has exponential growth. Then $G = N \rtimes \langle \psi \rangle$ has exponential growth by Corollary 1.8.

3.3 Nilpotent cascades – Proof of Theorem C

Let us see first that Conjecture 1.9 holds true for an abelian cascade (N, ϕ) of subexponential growth, for which Theorem 2.2 implies that (N, ϕ) has indeed polynomial growth. Here $G = N \rtimes \langle\phi\rangle$ is necessarily solvable, so that if the conjecture is true, then G must be of polynomial growth, hence virtually nilpotent.

Lemma 3.10. *If (A, ψ, F) is a finitely generated abelian cascade of subexponential growth, then there exist a chain of ψ-stable subgroups $A_1 < A_2 < \ldots < A_m = A$ of A and $k \in \mathbb{N}_+$ such that ψ^k fixes all the quotients A_{i+1}/A_i.*

Proof. As (A, ψ, F) has subexponential growth, we have $h_{alg}(\psi) = 0$ by Remark 2.1. Then the group A is finitely generated by Proposition 3.6, hence its torsion subgroup $t(A)$ is finite. Since $t(A)$ is finite and ψ-stable, we deduce that $\psi \restriction_{t(A)}$ is periodic, so $t(A)$ is contained in $Q_1(A, \psi)$. The group $A/t(A)$ is torsion-free, hence all its subgroups $Q_n(A/t(A))$ are pure by Remark 2.7. Therefore, there exists a finite m such that

$$Q_1(A, \psi) \leq Q_2(A, \psi) < \ldots < Q_m(A, \psi) = \mathfrak{Q}(A, \psi).$$

As $h_{alg}(\psi) = 0$, Theorem 2.8 implies that $A = \mathbf{P}(A, \psi) = \mathfrak{Q}(A, \psi)$, and in particular $Q_m(A, \psi) = A$.

Each $A_i := Q_i(A, \psi)$ is a ψ-stable subgroup of A and each quotient A_{i+1}/A_i consists entirely of quasi-periodic elements. As the induced endomorphism $\bar{\psi}_i$ of A_{i+1}/A_i is an automorphism, these quasi-periodic elements are actually periodic. Moreover, as A_{i+1}/A_i is finitely generated, the automorphism $\bar{\psi}_i$ is periodic, say of period k_i. Now $k = k_1 \ldots k_m$ works. □

Note that, in the notation of Lemma 3.10, the subgroup $A \rtimes \langle\psi^k\rangle$ of $G = A \rtimes \langle\psi\rangle$ is nilpotent by Lemma 2.9, so G is virtually nilpotent. Now we generalize this result to the case when a finitely generated cascade is nilpotent.

Theorem 3.11. *If (N, ϕ, F) is a finitely generated nilpotent cascade, then the following conditions are equivalent:*

(a) there exists $k \in \mathbb{N}$ such that $G_1 = N \rtimes \langle\phi^k\rangle$ is nilpotent;

(b) the group $G = N \rtimes \langle\phi\rangle$ is virtually nilpotent;

(c) G has polynomial growth;

(d) G has subexponential growth;

(e) the cascade (N, ϕ, F) has subexponential growth.

In particular, the group $G = N \rtimes \langle\phi\rangle$ has the same type of growth as the cascade (N, ϕ, F).

Proof. First, note that the group G is finitely generated by Theorem B.

If (N, ϕ, F) has exponential growth, then so does G according to Corollary 1.8.

Obviously (a) → (b), while (b) → (c) by Theorem 1.6. The implication (c) → (d) is trivial, and (d) → (e) follows by item (b) of Theorem B.

It remains to prove that (e) → (a). Assume that (N, ϕ, F) has subexponential growth. By Theorem B, N is finitely generated. As N is nilpotent, we can find a central series (1), i.e., a normal series such that N centralizes all the quotients $A_i := N_{i+1}/N_i$, and ϕ stabilizes each subgroup N_i. For the finitely generated abelian group A_i we find a power k_i, such that ϕ^{k_i} centralizes A_i by applying Lemma 3.10. Take $k = k_0 k_1 \cdots k_{l-1}$, so the automorphism ϕ^k centralizes all the quotients N_{i+1}/N_i. As G_1 centralizes all the quotients N_{i+1}/N_i with $i < l$ and the quotient G_1/N_{l-1} is abelian as ϕ^k (so G_1 as well) stabilizes N_l/N_{l-1}, we deduce that the subgroup $G_1 = N \rtimes \langle\phi^k\rangle$ is nilpotent. □

The conclusion of Theorem 3.11 above covers the proof of Theorem C.

Now we show that there exists no nilpotent cascade of intermediate growth. This generalizes Theorem 2.2 in the case of cascades.

Corollary 3.12. *A finitely generated nilpotent cascade of subexponential growth is of polynomial growth.*

By Theorem 3.11, Conjecture 1.9 has positive answer for nilpotent groups. In the next proposition we prove that it also has positive answer for solvable cascades of polynomial growth.

Proposition 3.13. *Let (N, ϕ, F) be a finitely generated solvable cascade. If the group $G = N \rtimes \langle\phi\rangle$ has subexponential growth, then the group N is finitely generated, and the cascade (N, ϕ, F_1) has polynomial growth for some finite subset $F_1 \supseteq F$ of N.*

Proof. The finitely generated solvable group G has polynomial growth by Theorem 1.7, so it is virtually nilpotent by Theorem 1.6. Let G^* be a finite index nilpotent subgroup of G. Inside G^*, we can find a finite index characteristic subgroup of G, so we can just assume G^* to be a finite index nilpotent characteristic subgroup of G.

Theorem B implies that the cascade (N, ϕ, F) has subexponential growth, and that the group N is finitely generated. Consider the subgroup $N^* = G^* \cap N$ of G^*. Then N^* is nilpotent, and it has finite index in N, so N^* is a finitely generated nilpotent group.

Moreover, N^* is ϕ-stable, being the intersection of two ϕ-stable groups, so we can consider the group $G_0 = N^* \rtimes \langle \phi \rangle$, and note that G_0 has finite index in G, so also G_0 has polynomial growth.

Now Theorem 3.11 ensures that the finitely generated cascade (N^*, ϕ) has polynomial growth, say with respect to some finite $F_0 \subseteq N^*$. Finally, let $F_1 = F \cup F_0$. $\square$

4 Applications towards Milnor-Wolf Theorem

A group is called *polycyclic*, if there exists a normal series

$$\{e_G\} = G_0 \lhd G_1 \lhd \ldots \lhd G_{l-1} \lhd G_l = G, \tag{4}$$

with cyclic quotients G_{i+1}/G_i. The number $h(G)$ of infinite quotients, called *Hirsch length*, does not depend on the choice of the series (4).

Lemma 4.1 (Milnor)**.** *Let G be a finitely generated group of subexponential growth. Then the derived subgroup G' of G is also finitely generated.*

Proof. The quotient G/G' is a finitely generated abelian group. So $G/G' \cong \mathbb{Z}^d \times F$, where $d \in \mathbb{N}$ and F is a finite group. Hence one can build a series $G' = N_0 \leq N_1 \leq N_2 \leq \ldots \leq N_d \leq G$, such that $N_{i+1}/N_i \cong \mathbb{Z}$ for all $0 \leq i < d$ and $G/N_d \cong F$ is finite. Then N_d is a finitely generated group of subexponential growth and all N_i are normal subgroups of G. We prove that each N_i is finitely generated by induction on d. The case $d = 0$ follows from the fact that a finite index subgroup of a finitely generated group is itself finitely generated. Clearly, it suffices to check the assertion for $d = 1$. Now $N_1/N_0 \cong \mathbb{Z}$, so there exists $x \in N_1$ such that $N_1 = \langle x \rangle N_0$ and $\langle x \rangle \cap N_0$ is trivial. Then $N_1 \cong N_0 \rtimes \langle x \rangle$. Let ϕ be the internal automorphism of N_1 induced by the conjugation by x. According to Theorem B, the cascade $(N_0, \phi \restriction_{N_0})$ is finitely generated. Hence, Proposition 3.6 applies to conclude that $N_0 = G'$ is finitely generated. $\square$

Corollary 4.2 (Milnor)**.** *A finitely generated soluble group of subexponential growth is polycyclic.*

Proof. Applying Lemma 4.1, we can prove that all members of the derived series of G are finitely generated. Since this is an abelian series, this implies that one can find a refinement that has cyclic factors, i.e., G is polycyclic. $\square$

Now we show that the conclusion of Lemma 4.1 does not hold for finitely generated groups of exponential growth.

Example 4.3. Let $N = \mathbb{Z}[\frac{1}{2}]$ and let $\psi : N \to N$ be the automorphism given by $\psi(x) = 2x$ for all $x \in N$. Then $G = N \rtimes \langle \psi \rangle$ is a finitely generated metaabelian group that is not polycyclic, as $G' = N$ is not finitely generated. By Lemma 4.1, G has exponential growth. This follows also from Corollary 1.8 as one can verify that $h_{alg}(\psi) > 0$.

Taking G as in Example 4.3, and $\phi = id_G$, we see that one may have a cascade (G, ϕ) of exponential growth and a subcascade (N, ϕ_N) of polynomial growth such that the quotient $(G/N, \bar{\phi})$ has polynomial growth as well.

As the Example 3.9(2) shows, a polycyclic group may have exponential growth.

From Theorem 3.11 and Corollary 4.2 we deduce the next theorem providing a proof of Gromov's theorem in the case of soluble groups.

Theorem 4.4. *A finitely generated soluble group of subexponential growth is virtually nilpotent.*

Proof. According to Corollary 4.2, a finitely generated soluble group G of subexponential growth is polycyclic. We can carry out an inductive argument on $h(G)$. In case $h(G) = 0$ the groups in question are finite so there is nothing to prove.

Assume that $h(G) > 0$ and the statement holds true for all groups of Hirsch length $< h(G)$. Let (4) be a normal series with cyclic quotients witnessing that. By choosing the greatest k such that G/G_k is infinite, we can replace G by G_{k+1} and prove that G_{k+1} is virtually nilpotent (this will imply that G is virtually nilpotent as well, as G_{k+1} is a finite-index subgroup of G). So we can assume without loss of generality that $k + 1 = l$, i.e. that $G/G_{l-1} \cong \mathbb{Z}$.

Let $x \in G$ be such that $G/G_{l-1} = \langle x, G_{l-1} \rangle / G_{l-1}$. As $\langle x \rangle \cap G_{l-1}$ is trivial, if ϕ be the automorphism of G_{l-1} induced by the conjugation by x in G, then $G \cong G_{l-1} \rtimes \langle \phi \rangle$.

As $h(G) = h(G_{l-1}) + 1$, we have $h(G_{l-1}) < h(G)$. Since the group G_{l-1} is a finitely generated soluble group of subexponential growth (by Remark 2.3) and $h(G_{l-1}) < h(G)$, we deduce that the subgroup G_{l-1} is virtually nilpotent. Let N^* be a finite index nilpotent subgroup of G_{l-1}. One can prove that N^* contains a finite index subgroup N that is a characteristic subgroup of G_{l-1}. In particular, N will be a ϕ-stable nilpotent subgroup of G_{l-1} of finite index. Therefore, $G^* = N \rtimes \langle\phi\rangle$ is a finite index subgroup of G. As the subgroup G^* of G has subexponential growth by Remark 2.3, we can apply Theorem 3.11 to deduce that G^* is virtually nilpotent. $\square$

5 Final remarks and open questions

The next example shows that in general $\gamma_{\phi,F}$ and $\gamma_{\phi,F'}$ can be quite different for a flow (G,ϕ), in particular $H_{alg}(\phi,F) \neq H_{alg}(\phi,F')$ (recall, that when $H_{alg}(\phi,F) = \log b > 0$, then $\gamma_{\phi,F}(n)$ has the same asymptotic behavior as b^n by Remark 2.1).

Example 5.1. (a) Let $k > 2$ be an integer. Consider the group $G = \mathbb{Z}[\frac{1}{k}]$, and its automorphism $\phi : G \to G$ defined by $\phi(x) = \frac{1}{k}x$. Finally, let $F = \{0,1\}$ and $F' = \{0,1,\dots,k-1\}$ and note that both F and F' generate the flow (G,ϕ). Nevertheless, $\gamma_{\phi,F}(n) = 2^n$, while $\gamma_{\phi,F'} = k^n$. So $\log 2 = H_{alg}(\phi,F) < H_{alg}(\phi,F') = \log k$.

(b) Take a cyclic flow or cascade (N,ϕ), generated by the singleton $F = \{y\} \subseteq N$, for example the one in Example 3.9(2), with $F = \{e_1\}$. One can see that $\gamma_{\phi,\{y\}}(n) = 1$ for all n, yet $N = N_{\phi,F}$ contains a finite subset $F' = \{0,e_1\}$ such that $\gamma_{\phi,F'}(n)$ has exponential growth.

Lemma 5.2. *Let (G,ϕ) be a finitely generated abelian flow. If (G,ϕ,F) has polynomial (resp., subexponential) growth for some $F \in [G]^{<\infty}$ generating (G,ϕ), then (G,ϕ,F_0) has polynomial (resp., subexponential) growth for every $F_0 \in [G]^{<\infty}$ generating (G,ϕ).*

Proof. We use additive notation for the abelian group G, and let $F \in [G]^{<\infty}$ be a finite symmetric set containing the identity element of G and generating (G,ϕ). By assumption, the function $n \mapsto |T_n(\phi,F)|$ has polynomial (resp., subexponential) growth.

Let us introduce the notation
$$(m)A := A + A + ... + A$$
for m copies of a finite subset A of G.

Let F_0 be any finite subset of G. We have to prove that the growth of the function
$$n \mapsto |T_n(\phi,F_0)| \tag{5}$$
is polynomial (resp., subexponential). Since $G = \bigcup_k \langle T_k(\phi,F)\rangle$, there exists k such that $F_0 \subseteq \langle T_k(\phi,F)\rangle$. Then there is also m such that $F_0 \subseteq (m)T_k(\phi,F)$, so that
$$T_n(\phi,F_0) \subseteq T_n(\phi,(m)T_k(\phi,F)) = (m)T_n(\phi,T_k(\phi,F)). \tag{6}$$

Now we check that
$$T_n(\phi,T_k(\phi,F)) \subseteq (k)T_{n+k}(\phi,F). \tag{7}$$
As
$$T_n(\phi,T_k(\phi,F)) = \sum_{i=0}^{n-1}\phi^i\left(\sum_{j=0}^{k-1}\phi^j(F)\right) = \sum_{i=0}^{n-1}\sum_{j=0}^{k-1}\phi^{i+j}(F),$$
looking at the range of i and j it is not hard to realize that for all possible values of s, between 0 and $n+k-2$, there are at most k pairs i,j such that $i+j=s$ (assuming that, asymptotically, k is smaller than n). Therefore, the above sum is contained in the sum $\sum_{s=0}^{n+k-1}(k)\phi^s(F) = (k)\sum_{s=0}^{n+k-1}\phi^s(F) = (k)T_{n+k}(\phi,F)$. This proves (7).

From (6) and (7) we obtain
$$T_n(\phi,F_0) \subseteq (m)(k)T_{n+k}(\phi,F) = (mk)T_{n+k}(\phi,F),$$
so the map (5) is polynomial (resp., subexponential) whenever the map $n \mapsto |(mk)T_{n+k}(\phi,F)|$ is polynomial (resp., subexponential). Finally, the latter holds true as by assumption there exists a polynomial (resp., subexponential) function $P(t)$ such that $|T_{n+k}(\phi,F)| \leq P(n+k)$, therefore
$$|(mk)T_{n+k}(\phi,F)| \leq |T_{n+k}(\phi,F)|^{mk} \leq P(n+k)^{mk}.$$
Since $P(n+k)^{mk}$ is a polynomial function of n of degree mkd, if d is the degree of P (resp., subexponential function), we are done. $\square$

Another important issue is pointed out in the next question, but no answer is available so far.

Question 5.3. If (G,ϕ) is a finitely generated cascade, and if $G = G_{\phi,F}$ and $G = G_{\phi,F'}$ for some finite subsets F and F' of G both containing e_G, are then $\gamma_{\phi,F}$ and $\gamma_{\phi,F'}$ of the same type (polynomial, exponential or intermediate)?

A positive answer to Question 5.3 will guarantee that the type of growth of a finitely generated flow $G_{\phi,F}$ does not depend on the specific finite set of generators F (so, for example, F can always be taken symmetric, etc.). In particular, one may speak of growth of a finitely generated flow without any reference to a specific finite set of generators.

Without the limitation $e_G \in F$ on the finite set F in Question 5.3 one has an immediate negative answer to it as we saw in Example 5.1(b) (note that both finite sets in (a) of that example give equivalent growth functions).

One more comment on Question 5.3:

Remark 5.4. Let (G, ϕ) be a finitely generated abelian flow in **Grp** and let F be a finite subset of G such that $G = G_{\phi,F}$. According to Lemma 5.2 (see also [4]), if $G_{\phi,F}$ has a polynomial (respectively, exponential, intermediate) growth, this is independent on the finite set of generators F (i.e., Question 5.3 has positive answer in the abelian case).

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