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TOPOLOGICAL ENTROPY, UPPER CARATHÉODORY CAPACITY AND FRACTAL DIMENSIONS OF SEMIGROUP ACTIONS

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Abstract. We study dynamical systems given by the action $T : G \times X \to X$ of a finitely generated semigroup G with identity 1 on a compact metric space X by continuous selfmaps and with $T(1, -) = \mathrm{id}_X$.

For any finite generating set G_1 of G containing 1, the receptive topological entropy of G_1 (in the sense of Ghys et al. (1988) and Hofmann and Stoyanov (1995)) is shown to coincide with the limit of upper capacities of dynamically defined Carathéodory structures on X depending on G_1, and a similar result holds true for the classical topological entropy when G is amenable. Moreover, the receptive topological entropy and the topological entropy of G_1 are lower bounded by respective generalizations of Katok's δ-measure entropy, for $\delta \in (0, 1)$.

In the case when $T(g, -)$ is a locally expanding selfmap of X for every $g \in G \setminus \{1\}$, we show that the receptive topological entropy of G_1 dominates the Hausdorff dimension of X modulo a factor $\log \lambda$ determined by the expanding coefficients of the elements of $\{T(g, -) : g \in G_1 \setminus \{1\}\}$.

1. Introduction. The measure entropy h_μ of a measure-preserving transformation of a probability space (X, μ) was introduced by Kolmogorov and Sinai. The topological entropy h_{top} was first defined by Adler, Konheim and McAndrew [1] for continuous selfmaps of compact spaces, while for uniformly continuous selfmaps of metric spaces a different notion of topological entropy was given by Bowen [6] and Dinaburg [10]. The two notions of topological entropy coincide for compact metric spaces, which are the spaces we are interested in.

The Krylov–Bogolyubov Theorem implies that for a continuous selfmap $f : X \to X$ of a compact metric space X, the set $M(X, f)$ of f-invariant Borel probability measures on X is not empty. The relation between the topological entropy and the measure entropy of f is stated in the famous

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Variational Principle (see [15, 16, 35]):

$$h_{\mathrm{top}}(f) = \sup\{h_\mu(f) \colon \mu \in M(X, f)\}.$$

Lind, Schmidt and Ward [23] generalized to $\mathbb{Z}^d$-actions on compact metrizable groups both the definition of topological entropy by Bowen, and the one by Adler, Konheim and McAndrew, showing that they coincide. The measure entropy for amenable group actions was introduced by Kieffer [21], while the topological entropy for amenable group actions on compact metric spaces by Stepin and Tagi-Zade [31], and Ollagnier [25] defined the topological entropy for amenable group actions on compact spaces using open covers as in [1]. A cornerstone of the theory of entropy of amenable group actions is the work by Ornstein and Weiss [26], where in particular the celebrated Ornstein–Weiss Lemma is proved. Recent relevant papers in this area are those by Chung and Thom [9] and by Li [22]. Recently, Ceccherini-Silberstein, Coornaert and Krieger [8] extended the Ornstein–Weiss Lemma to cancellative amenable semigroups. Using this result, they introduced the measure entropy and the topological entropy for actions of cancellative amenable semigroups. For further extensions of these classical entropies to the case of actions of sofic groups see the survey paper [36] by Weiss.

In a different direction, Bowen's topological entropy was generalized by Ghys, Langevin and Walczak [14] to finitely generated pseudogroups of local homeomorphisms of a compact metric space in the setting of foliations. Independently, a more general notion for locally compact semigroup actions on a compact metric space was introduced by Hofmann and Stoyanov [18] (see also [4] where the term "receptive" was coined for this kind of entropy).

The main difference between the two approaches is that in many cases when the "classical" topological entropy is zero (and this occurs very often), the receptive topological entropy is strictly positive. We follow here both lines, extending further the approach from [14] to finitely generated semigroups, by considering a particular case of the general receptive topological entropy from [18] (see also Remark 2.1).

Notation. Given a compact metric space X, we denote by $\mathrm{Con}(X)$ the semigroup of all continuous selfmaps of X, with identity $\mathrm{id}_X : X \to X$, and by

$$(G, G_1) \subseteq \mathrm{Con}(X) \tag{1.1}$$

an infinite finitely generated semigroup G contained in $\mathrm{Con}(X)$, generated by a finite set G_1 containing id_X.

For a subset Z of a compact metric space X and a finitely generated semigroup $(G, G_1) \subseteq \mathrm{Con}(X)$, we denote by $\widetilde{h}_{\mathrm{top}}(G_1, Z)$ the receptive topological entropy of (G, G_1) with respect to Z and by $h^*_{\mathrm{top}}(G_1, Z)$ the topological entropy of (G, G_1) with respect to Z. Moreover, we let $\widetilde{h}_{\mathrm{top}}(G_1) = \widetilde{h}_{\mathrm{top}}(G_1, X)$ and $h^*_{\mathrm{top}}(G_1) = h^*_{\mathrm{top}}(G_1, X)$ (see Definition 2.2).

The definitions of Hausdorff dimension and of topological entropy by Bowen share common features. Indeed, Bowen [7] defined the topological entropy of a selfmap of a compact metric space very similarly to the definition of the Hausdorff dimension. Pesin [27] presented this interrelation between dimension theory and the theory of classical dynamical systems by introducing the so-called Carathéodory structures in axiomatic way; each Carathéodory structure determines dimension characteristics called Carathéodory dimension and lower and upper Carathéodory capacities.

Following the lines of [27, §10 and §11], we introduce special Carathéodory structures τ, which we call PC-structures (see Definition 3.1), and the upper PC-capacity $\overline{\mathrm{Cap}}_\tau$ with respect to τ. Then, for a closed subset Y of a compact metric space X and a finitely generated semigroup $(G, G_1) \subseteq \mathrm{Con}(X)$ as above, we find natural PC-structures τ_γ and τ^*_γ on Y, depending on a real number $\gamma > 0$. We show that $\widetilde{h}_{\mathrm{top}}(G_1)$ and $h^*_{\mathrm{top}}(G_1)$ coincide respectively with the limits $\overline{\mathrm{CP}}_{G_1}$ and $\overline{\mathrm{CP}}^*_{G_1}$ (see Definition 3.10) of the upper PC-capacities $\overline{\mathrm{Cap}}_{\tau_\gamma}$ and $\overline{\mathrm{Cap}}^*_{\tau_\gamma}$:

Theorem 1.1. *For a compact metric space X, a finitely generated semigroup $(G, G_1) \subseteq \mathrm{Con}(X)$ and every closed subset Y of X,*

$$\widetilde{h}_{\mathrm{top}}(G_1, Y) = \overline{\mathrm{CP}}_{G_1}(Y) \quad \textit{and} \quad h^*_{\mathrm{top}}(G_1, Y) = \overline{\mathrm{CP}}^*_{G_1}(Y). \tag{1.2}$$

In particular, $\widetilde{h}_{\mathrm{top}}(G_1) = \overline{\mathrm{CP}}_{G_1}$ and $h^*_{\mathrm{top}}(G_1) = \overline{\mathrm{CP}}^*_{G_1}$.

In order to obtain this result, we appropriately modify the set of axioms from [27, §10] and develop an analogous abstract theory in §3.1. More precisely, we relax one of the axioms, otherwise the PC-structures τ_γ and τ^*_γ do not satisfy it. Since these τ_γ and τ^*_γ are those considered in [27, §11, Remark (1)] for $G = \mathbb{N}$, our work in particular fixes the gap in [27, §11, Remark (1)] (see Remark 3.5 for the details).

Katok [19] introduced the δ-measure entropy $h^\delta_\mu(f)$, with $\delta \in (0,1)$, for a continuous selfmap $f : X \to X$ of a compact metric space X with respect to an f-invariant Borel probability measure μ, and proved that $h^\delta_\mu(f) = h_\mu(f)$ for every $\delta \in (0,1)$ (see [19, Theorem 1.1]). We extend the definition of δ-measure entropy to finitely generated semigroups $(G, G_1) \subseteq \mathrm{Con}(X)$, with respect to a Borel probability measure μ on X, in two ways, analogously to what we do for the topological entropy; we denote these two new notions by $\widetilde{h}^\delta_\mu(G_1)$ and $h^{\delta,*}_\mu(G_1)$ (see Definition 2.7). Since the maps $\delta \mapsto \widetilde{h}^\delta_\mu(G_1)$ and $\delta \mapsto h^{\delta,*}_\mu(G_1)$ are decreasing, the limits

$$\overline{h}_\mu(G_1) = \lim_{\delta\to 0} \widetilde{h}^\delta_\mu(G_1) \quad \text{and} \quad \overline{h}^*_\mu(G_1) = \lim_{\delta\to 0} h^{\delta,*}_\mu(G_1)$$

exist (see §2.2). It is worth mentioning that the measure μ is not supposed to be invariant under the action of G, so $\overline{h}_\mu(G_1)$ and $\overline{h}^*_\mu(G_1)$ obtained in this way are good substitutes for the measure entropy in the absence of invariance.

As a direct consequence of the definitions (see Proposition 2.9) and as a corollary of Theorem 1.1, we find the following partial Variational Principle.

Corollary 1.2. *Let X be a compact metric space, $(G, G_1) \subseteq \mathrm{Con}(X)$ an infinite finitely generated semigroup and μ a Borel probability measure on X. Then*

$$\overline{h}_\mu(G_1) \leq \widetilde{h}_{\mathrm{top}}(G_1) = \overline{\mathrm{CP}}_{G_1} \quad \textit{and} \quad \overline{h}^*_\mu(G_1) \leq h^*_{\mathrm{top}}(G_1) = \overline{\mathrm{CP}}^*_{G_1}.$$

In the second part of the paper we restrict ourselves to the subsemigroup $\mathrm{Le}(X)$ of $\mathrm{Con}(X)$ of locally expanding selfmaps of a compact metric space X (see Definition 4.2). Locally expanding maps were studied in different contexts by Ruelle [29], Shub [30] and Mayer [24]. According to Gromov [17], the existence of a locally expanding map on a Riemannian manifold implies very strong restrictions on its geometry.

For a compact metric space X and $\lambda > 1$, let $\mathrm{Le}_\lambda(X)$ be the subsemigroup of $\mathrm{Le}(X)$ of all locally λ-expanding selfmaps of X (note that $\mathrm{id}_X \notin \mathrm{Le}_\lambda(X)$). Moreover, denote by $\dim_H(X)$ the Hausdorff dimension of X and by $\overline{\dim}_B(X)$ the upper box dimension of X (see (4.1)). We obtain the following lower bound of the receptive topological entropy by means of the upper box dimension and the Hausdorff dimension:

Theorem 1.3. *Let X be a compact metric space, μ a Borel probability measure on X, $\lambda > 1$ and $(G, G_1) \subseteq \mathrm{Con}(X)$ an infinite finitely generated semigroup such that $G_1 \setminus \{\mathrm{id}_X\} \subseteq \mathrm{Le}_\lambda(X)$. Then*

$$\widetilde{h}_{\mathrm{top}}(G_1) \geq (\log \lambda)\, \overline{\dim}_B(X) \geq (\log \lambda) \dim_H(X). \tag{1.3}$$

It is tempting to conjecture that in the hypotheses of Theorem 1.3 one can prove the sharper inequality $h^*_{\mathrm{top}}(G_1) \geq (\log \lambda)\, \overline{\dim}_B(X)$ (which would obviously imply (1.3), as $\widetilde{h}_{\mathrm{top}}(G_1) \geq h^*_{\mathrm{top}}(G_1)$, see Remark 2.3). We show below that this conjecture fails (see Example 4.5).

2. Entropies of semigroup actions. In this section we introduce the entropy functions that we consider in this paper.

As in (1.1), for a compact metric space X we denote by $\mathrm{Con}(X)$ the semigroup of all continuous selfmaps of X and we let $(G, G_1) \subseteq \mathrm{Con}(X)$ be an infinite semigroup generated by a finite set G_1 with $\mathrm{id}_X \in G_1$.

For every $n \in \mathbb{N}_+$, let

$$G_n = \underbrace{G_1 \cdot \ldots \cdot G_1}_{n};$$

moreover, let $G_0 = \{\mathrm{id}_X\}$. Then

$$G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq \bigcup_{n\in\mathbb{N}} G_n = G.$$

Remark 2.1. The setting (1.1), where X is a compact metric space and G is a subsemigroup of $\mathrm{Con}(X)$ containing id_X, defines an obvious action $T: G \times X \to X$ with $T(g,x) = g(x)$ for $g \in G$, $x \in X$.

In general, if G is a semigroup with identity 1, to every action $T: G \times X \to X$ with $T(1,-) = \mathrm{id}_X$ one can associate the subsemigroup $T(G,-) = \{T(g,-) : g \in G\}$ of $\mathrm{Con}(X)$ containing id_X. It is finitely generated whenever G is finitely generated. If the surjective semigroup homomorphism $G \to T(G,-)$ defined by $g \mapsto T(g,-)$ is injective (i.e., the action T is faithful), it is an isomorphism and one can simply replace G by $T(G,-) \subseteq \mathrm{Con}(X)$, that is, we get the setting (1.1).

In this paper we impose the setting (1.1), that is, we only consider faithful actions. This is motivated by the fact that even if the entropy $\widetilde{h}_{\mathrm{top}}$ we define later for actions (1.1) can be defined in an obvious way for arbitrary actions $T: G\times X \to X$, it coincides with the entropy of the associated faithful action of T. Hence, we can adopt the setting (1.1) without any loss of generality as far as $\widetilde{h}_{\mathrm{top}}$ is concerned.

2.1. Topological entropy and receptive topological entropy. First, we extend to finitely generated semigroups acting on compact metric spaces the receptive topological entropy introduced by Ghys, Langevin and Walczak [14] for actions of finitely generated groups (a notably more general approach covering the present one was adopted by Hofmann and Stoyanov [18]).

The action of the finitely generated semigroup $(G, G_1) \subseteq \mathrm{Con}(X)$ on the compact metric space (X,d) determines a sequence of *dynamical n-balls* (for $n \in \mathbb{N}$) centered at $x \in X$ and of radius $\gamma > 0$ defined by

$$B_n^{G_1}(x,\gamma) = \bigcap_{g\in G_n} g^{-1}(B(g(x),\gamma)),$$

where $B(x,\gamma) = \{y \in X : d(x,y) < \gamma\}$ is the standard ball in (X,d). Clearly, $B_{n+1}^{G_1}(x,\gamma) \subseteq B_n^{G_1}(x,\gamma)$ for every $n \in \mathbb{N}$, and $B_0^{G_1}(x,\gamma) = B(x,\gamma)$. For $Y \subseteq X$, $n \in \mathbb{N}$ and $\gamma > 0$, let

$$\mathcal{B}_n^{G_1}(Y,\gamma) = \{B_n^{G_1}(y,\gamma) \cap Y : y \in Y\}.$$

Note that $\mathcal{B}_0^{G_1}(Y,\gamma) = \{B(y,\gamma) \cap Y : y \in Y\}$ and that $\mathcal{B}_n^{G_1}(Y,\gamma)$ is an open cover of Y for every $n \in \mathbb{N}$.

For $n \in \mathbb{N}$ and $\gamma > 0$, a subset E of a compact metric space X is (n,γ)-*separated* if for any distinct points $a_1, a_2 \in E$ one has $a_2 \notin B_n^{G_1}(a_1,\gamma)$ (or equivalently $a_1 \notin B_n^{G_1}(a_2,\gamma)$). Since X is compact, every (n,γ)-separated subset E of X is finite. For any subset Y of X, let

$$s(n,\gamma,Y) = \max\{|E| \colon E \subseteq Y,\ E \text{ is } (n,\gamma)\text{-separated}\}.$$

DEFINITION 2.2. Let X be a compact metric space and $(G,G_1) \subseteq \mathrm{Con}(X)$ a finitely generated semigroup.

(i) The *receptive topological entropy* of (G,G_1) with respect to a subset Y of X is

$$\widetilde{h}_{\mathrm{top}}(G_1,Y) = \lim_{\gamma\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s(n,\gamma,Y).$$

If $Y = X$, let $\widetilde{h}_{\mathrm{top}}(G_1) = \widetilde{h}_{\mathrm{top}}(G_1,X)$.

(ii) The *topological entropy* of (G,G_1) with respect to a subset Y of X is

$$h^*_{\mathrm{top}}(G_1,Y) = \lim_{\gamma\to 0} \limsup_{n\to\infty} \frac{1}{|G_n|} \log s(n,\gamma,Y).$$

If $Y = X$, let $h^*_{\mathrm{top}}(G_1) = h^*_{\mathrm{top}}(G_1,X)$.

REMARK 2.3. We shall assume that the finitely generated semigroup $(G,G_1) \subseteq \mathrm{Con}(X)$ is infinite, which implies that the sequence $(a_n)_{n\in\mathbb{N}}$, where $a_n = |G_n|$ for every $n \in \mathbb{N}$, is strictly increasing, so

$$a_n \geq n \quad \text{for every } n \in \mathbb{N}. \tag{2.1}$$

One obviously has $h^*_{\mathrm{top}}(G_1,Y) \leq \widetilde{h}_{\mathrm{top}}(G_1,Y)$ for any subset Y of X, and so $h^*_{\mathrm{top}}(G_1) \leq \widetilde{h}_{\mathrm{top}}(G_1)$, due to (2.1).

(a) In case G is a group, and $G_1 = G_1^{-1}$ is a symmetric finite generating set containing id_X, $\widetilde{h}_{\mathrm{top}}(G_1)$ is the receptive topological entropy from [14].
(b) In case G is a finitely generated amenable group, and $(G_n)_{n\in\mathbb{N}}$ is a right Følner sequence for G (this is the case when G is of subexponential growth), the limit superior in the above definition of topological entropy is a limit, and it does not depend on the right Følner sequence. In particular it does not depend on the finite set of generators G_1 of G. Hence, $h^*_{\mathrm{top}}(G_1)$ does not depend on G_1 and coincides with the "classical" topological entropy $h_{\mathrm{top}}(G)$ defined for actions of amenable groups G.

REMARK 2.4. Consider the case of a single continuous selfmap $f : X \to X$ of a compact metric space X; let G be the semigroup generated by the iterations of f and $G_1 = \{\mathrm{id}_X, f\}$. Note that either the semigroup G is isomorphic to $\mathbb{N}$ (when all powers f^n are distinct), or $G = G_n = \{\mathrm{id}_X, f, \ldots, f^n\}$ for some $n \in \mathbb{N}$. In case G is infinite, $\widetilde{h}_{\mathrm{top}}(G,G_1) = h^*_{\mathrm{top}}(G,G_1) = h_{\mathrm{top}}(G) = h_{\mathrm{top}}(f)$ is Bowen's classical topological entropy.

For $n \in \mathbb{N}$ and $\gamma > 0$, a subset Z of a compact metric space X is (n,γ)-*spanning* for another subset Y of X if $Y \subseteq \bigcup_{z\in Z} B_n^{G_1}(y,\gamma)$. If Y is closed, then Y is compact, so there exist finite (n,γ)-spanning subsets Z for Y. Let

$$r(n,\gamma,Y) = \min\{|Z|\colon Z \text{ is an } (n,\gamma)\text{-spanning subset for } Y\}. \tag{2.2}$$

Using the family $\mathcal{B}_n^{G_1}(X,\gamma)$ of dynamical n-balls one can rewrite (2.2) as

$$r(n,\gamma,Y) = \min\Big\{|\mathcal{U}|\colon \mathcal{U} \subseteq \mathcal{B}_n^{G_1}(X,\gamma),\ Y \subseteq \bigcup \mathcal{U}\Big\}. \tag{2.3}$$

Slightly modifying the standard arguments for the topological entropy of a single continuous selfmap of a compact metric space (see for example [35, §7.2]), we obtain an equivalent definition of the receptive topological entropy and the topological entropy of a finitely generated semigroup with respect to a closed subset:

LEMMA 2.5. *For a compact metric space X, an infinite finitely generated semigroup $(G,G_1) \subseteq \mathrm{Con}(X)$ and a closed subset Y of X,*

$$\widetilde{h}_{\mathrm{top}}(G_1,Y) = \lim_{\gamma\to 0}\limsup_{n\to\infty}\frac{1}{n}\log r(n,\gamma,Y),$$

$$h^*_{\mathrm{top}}(G_1,Y) = \lim_{\gamma\to 0}\limsup_{n\to\infty}\frac{1}{|G_n|}\log r(n,\gamma,Y).$$

Proof. As in [35, Remarks (5)], it suffices to note that for $\gamma > 0$ and for every $n \in \mathbb{N}$,

$$r(n,\gamma,Y) \le s(n,\gamma,Y) \le r(n,\gamma/2,Y).$$

Now our assertions follow directly from the definitions. ∎

REMARK 2.6. It is known that the receptive topological entropy of a finitely generated semigroup $(G,G_1) \subseteq \mathrm{Con}(X)$, where X is a compact metric space, depends on the generating set G_1. For example, take an $\mathbb{N}$-action generated by a continuous selfmap $f : X \to X$, so G is isomorphic to $\mathbb{N}$; let $G_1 = \{\mathrm{id}_X, f\}$ and $G_1' = \{\mathrm{id}_X, f, f^2\}$. Then $\widetilde{h}_{\mathrm{top}}(G_1) = h_{\mathrm{top}}(f)$, while $\widetilde{h}_{\mathrm{top}}(G_1') = h_{\mathrm{top}}(f^2) = 2h_{\mathrm{top}}(f)$. Thus, $\widetilde{h}_{\mathrm{top}}(G_1) \neq \widetilde{h}_{\mathrm{top}}(G_1')$ whenever $0 < h_{\mathrm{top}}(f) < \infty$. On the other hand, since $\mathbb{N}$ is commutative, and so amenable, $h_{\mathrm{top}}(G_1) = h_{\mathrm{top}}(G_1') = h_{\mathrm{top}}(f)$.

Moreover, for two distinct finite generating sets G_1 and G_1' of a group $G \subseteq \mathrm{Con}(X)$, the equivalence

$$\widetilde{h}_{\mathrm{top}}(G_1) > 0 \iff \widetilde{h}_{\mathrm{top}}(G_1') > 0$$

was proved in [14]. This shows that the vanishing of the receptive topological entropy does not depend on the finite set of generators of the group G.

2.2. Measure entropies. Here, using an approach similar to that for extending Bowen's topological entropy recalled above, we extend the definition of δ-measure entropy given by Katok [19] to finitely generated semigroups $(G,G_1) \subseteq X$ for a compact metric space X.

DEFINITION 2.7. Let X be a compact metric space, $(G,G_1) \subseteq \mathrm{Con}(X)$ an infinite finitely generated semigroup, $\delta \in (0,1)$ and μ a Borel probability measure on X. For $\gamma > 0$ and $n \in \mathbb{N}$, let

$$N^{\delta}_{G_1}(n,\gamma) = \min\Big\{|\mathcal{U}| : \mathcal{U} \subseteq \mathcal{B}^{G_1}_n(X,\gamma),\ \mu\Big(\bigcup \mathcal{U}\Big) > 1-\delta\Big\}.$$

(i) The *receptive δ-measure entropy* of (G,G_1) is

$$\widetilde{h}^{\delta}_{\mu}(G_1) = \lim_{\gamma\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N^{\delta}_{G_1}(n,\gamma).$$

(ii) The *δ-measure entropy* of (G,G_1) is

$$h^{\delta,*}_{\mu}(G_1) = \lim_{\gamma\to 0} \limsup_{n\to\infty} \frac{1}{|G_n|} \log N^{\delta}_{G_1}(n,\gamma).$$

Since the map $\delta \mapsto N^{\delta}_{G_1}(n,\gamma)$ is decreasing, so too are the maps $\delta \mapsto \widetilde{h}^{\delta}_{\mu}(G_1)$ and $\delta \mapsto h^{\delta,*}_{\mu}(G_1)$, and hence the following limits exist:

$$\overline{h}_{\mu}(G_1) = \lim_{\delta\to 0} \widetilde{h}^{\delta}_{\mu}(G_1) \quad \text{and} \quad \overline{h}^{*}_{\mu}(G_1) = \lim_{\delta\to 0} h^{\delta,*}_{\mu}(G_1).$$

REMARK 2.8. Katok [19] defined the δ-measure entropy $h^{\delta}_{\mu}(f)$ of f, where $\delta \in (0,1)$, X is a compact metric space and $f : X \to X$ is a continuous selfmap, and μ is an *f-invariant* Borel probability measure on X. We retrieve Katok's measure entropy by taking in the above definition G to be the semigroup generated by f and $G_1 = \{\mathrm{id}_X, f\}$, that is, $h^{\delta}_{\mu}(f) = \widetilde{h}^{\delta}_{\mu}(G_1) = h^{\delta,*}_{\mu}(G_1)$.

Katok proved that

$$h^{\delta}_{\mu}(f) = h_{\mu}(f) \quad \text{for every } \delta \in (0,1), \tag{2.4}$$

and so in particular $\widetilde{h}^{\delta}_{\mu}(G_1) = \overline{h}_{\mu}(G_1) = \overline{h}^{*}_{\mu}(G_1) = h^{\delta,*}_{\mu}(G_1)$.

We have the following relations of the receptive δ-measure entropy to the receptive topological entropy and of the δ-measure entropy to the topological entropy, which give the inequalities in Corollary 1.2.

PROPOSITION 2.9. *Let X be a compact metric space, $(G,G_1) \subseteq \mathrm{Con}(X)$ a finitely generated semigroup, $\delta \in (0,1)$ and μ a Borel probability measure on X. Then, for all $n \in \mathbb{N}$ and $\gamma > 0$, we have $N^{\delta}_{G_1}(n,\gamma) \le r(n,\gamma,X)$, and so*

$$\widetilde{h}^{\delta}_{\mu}(G_1) \le \widetilde{h}_{\mathrm{top}}(G_1) \quad \textit{and} \quad h^{\delta,*}_{\mu}(G_1) \le h^{*}_{\mathrm{top}}(G_1). \tag{2.5}$$

Consequently,

$$\overline{h}_{\mu}(G_1) \le \widetilde{h}_{\mathrm{top}}(G_1) \quad \textit{and} \quad \overline{h}^{*}_{\mu}(G_1) \le h^{*}_{\mathrm{top}}(G_1). \tag{2.6}$$

Proof. The inequality $N^{\delta}_{G_1}(n,\gamma) \le r(n,\gamma,X)$ is a consequence of the definitions and (2.3). It immediately implies (2.5), while (2.6) follows from (2.5) by the definitions. ∎

We now discuss the situation in the presence of invariant measures. In general, a finitely generated semigroup (G, G_1) of continuous selfmaps of a compact metric space X need not admit a G-invariant Borel probability measure (e.g., see [33, Example 4.1.1]). Nevertheless, it is known (see [33, pp. 97–98]) that for two commuting homeomorphisms $f, g : X \to X$ there exists a Borel probability measure on X which is both f-invariant and g-invariant. The argument can be easily extended to the case of two, and so finitely many, pairwise commuting continuous selfmaps (see [34]). Therefore, any finitely generated commutative semigroup (G, G_1) of continuous selfmaps of X admits a G-invariant Borel probability measure. The existence of a G-invariant Borel probability measure on X is also ensured when G is an amenable (not necessarily finitely generated) group (see [11, Theorem 8.10]).

If μ is a G-invariant Borel probability measure on X, then one can also consider the receptive measure entropy $\widetilde{h}_\mu(G_1)$ from [5, 32], which is an extension of the classical measure entropy of Kolmogorov and Sinai: if $\mathcal{A} = \{A_1, \dots, A_k\}$ is a finite measurable partition of X, then the (Shannon) entropy of $\mathcal{A}$ is defined by

$$H_\mu(\mathcal{A}) = -\sum_{i=1}^{k} \mu(A_i) \log \mu(A_i),$$

with the usual agreement that $0 \log 0 = 0$ (see [35, Chapter 4]); then let

$$\widetilde{h}_\mu(G_1, \mathcal{A}) = \limsup_{n\to\infty} \frac{1}{n} H_\mu\Big(\bigvee_{g\in G_n} g^{-1}\mathcal{A}\Big), \tag{2.7}$$

and the *receptive measure entropy* of G_1 is

$$\widetilde{h}_\mu(G_1) = \sup\{\widetilde{h}_\mu(G_1, \mathcal{A})\colon \mathcal{A} \text{ a finite measurable partition of } X\}.$$

In [5] we study the relation between $\widetilde{h}_\mu(G_1)$ and $\widetilde{h}_{\mathrm{top}}(G_1)$ in the spirit of the classical Variational Principle; the problem turns out to be highly non-trivial. Moreover, Katok's above mentioned result (2.4) for a single continuous selfmap suggests the following problem, which we leave open and which could be of some help in the study of the Variational Principle in view of Proposition 2.9.

PROBLEM 2.10. *Let X be a compact metric space, $(G, G_1) \subseteq \mathrm{Con}(X)$ a finitely generated semigroup and μ a G-invariant Borel probability measure on X. What is the relation between $\overline{h}_\mu(G_1)$ and $\widetilde{h}_\mu(G_1)$? Do they coincide?*

3. Topological entropy and Carathéodory structures. In this section, first we propose a modification of the construction of Carathéodory structures introduced and elaborated by Pesin [27, §10]. Then we apply it to the case of the topological entropy (receptive or not) of a finitely generated semigroup of continuous selfmaps, following [27, §11, Remark (1)]. We compare the Carathéodory structure introduced by Pesin with its modification introduced below and called *Pesin–Carathéodory structure*. See Remark 3.5 for the details on the motivation of our construction and the comparison with that in [27, §10 and §11].

3.1. Pesin–Carathéodory structures and upper PC-capacity

Definition 3.1. Let Y and $\mathcal{S}$ be non-empty sets and $\mathcal{F} = \{U_s : s \in \mathcal{S}\}$ a cover of Y with $U_s = \emptyset$ for some $s \in \mathcal{S}$. Assume that the functions $\psi, \eta : \mathcal{S} \to \mathbb{R}_+$ are such that:

(P1) if $U_s = \emptyset$ then $\psi(s) = \eta(s) = 0$; if $\emptyset \neq U_s \in \mathcal{F}$ then $\psi(s)\eta(s) > 0$;
(P2) for every $\delta > 0$ there exists $\varepsilon > 0$ such that $\psi(s) < \varepsilon$ implies $\eta(s) < \delta$ for every $s \in \mathcal{S}$;
(P3) for every $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon)$ such that there exists $\mathcal{G} \subseteq \mathcal{S}$ finite with $\bigcup_{s \in \mathcal{G}} U_s = Y$ and $\psi(s) = \delta$ for every $s \in \mathcal{G}$.

The system $\tau = (\mathcal{S}, \mathcal{F}, \psi, \eta)$ is called a *Pesin–Carathéodory structure* (briefly, *PC-structure*) on Y.

The auxiliary set $\mathcal{S}$ is needed, since we do not require the map $\mathcal{S} \to \mathcal{F}$, $s \mapsto U_s$, to be injective; indeed, when we apply this construction in §3.2, the injectivity may not be available.

For $\mathcal{G} \subseteq \mathcal{S}$, denote

$$\mathcal{U}_{\mathcal{G}} = \{U_s : s \in \mathcal{G}\},$$

and let

$$\psi(\mathcal{G}) = \sup\{\psi(s) : s \in \mathcal{G}\} \quad \text{and} \quad \eta(\mathcal{G}) = \sup\{\eta(s) : s \in \mathcal{G}\}.$$

Adopting standard notation from set theory, for a set S let $[S]^{\leq \aleph_0}$ denote the family of all non-empty (at most) countable subsets of S. For $\varepsilon > 0$ and $\emptyset \neq Z \subseteq Y$, let

$$\mathcal{C}_\varepsilon(Z) = \Big\{\mathcal{G} \in [\mathcal{S}]^{\leq \aleph_0} : Z \subseteq \bigcup \mathcal{U}_{\mathcal{G}},\ \psi(s) = \varepsilon \text{ for every } s \in \mathcal{G}\Big\};$$

note that for every $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon)$ such that $\mathcal{C}_\delta(Z)$ is not empty by (P3); hence, there exists a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ such that $\varepsilon_n \to 0$ and $\mathcal{C}_{\varepsilon_n}(Z) \neq \emptyset$ for every $n \in \mathbb{N}$.

For $\alpha \in \mathbb{R}$ and $\mathcal{G} \subseteq \mathcal{S}$, let

$$H(\mathcal{G}, \alpha) = \sum_{s \in \mathcal{G}} \eta(s)^\alpha.$$

Remark 3.2. Let Y be a non-empty set, $\tau = (\mathcal{S}, \mathcal{F}, \psi, \eta)$ a PC-structure on Y and $Z \subseteq Y$. Let $\delta \in (0,1)$ and $\varepsilon > 0$ be such that $\eta(\mathcal{G}) \leq \delta$ if $\mathcal{G} \in \mathcal{C}_\varepsilon(Z)$ (this is possible in view of (P2)). Hence, $\mathcal{G} \in \mathcal{C}_\varepsilon(Z)$ implies $\eta(s) < 1$ for all $s \in \mathcal{G}$. Therefore, $\beta < \alpha$ implies $H(\mathcal{G}, \alpha) \leq H(\mathcal{G}, \beta)$ for $\mathcal{G} \in \mathcal{C}_\varepsilon(Z)$.

Take $\alpha \in \mathbb{R}$ and $\varepsilon > 0$. For $\emptyset \neq Z \subseteq Y$, let $R_\alpha(Z, \varepsilon) = 0$ if $\mathcal{C}_\varepsilon(Z) = \emptyset$, otherwise, when $\mathcal{C}_\varepsilon(Z) \neq \emptyset$, let

$$R_\alpha(Z, \varepsilon) = \inf \{H(\mathcal{G}, \alpha) \colon \mathcal{G} \in \mathcal{C}_\varepsilon(Z)\}. \tag{3.1}$$

Now let

$$\overline{r}_\alpha(Z) = \limsup_{\varepsilon \to 0} R_\alpha(Z, \varepsilon).$$

Proposition 3.3. *Let Y be a non-empty set, $\tau = (\mathcal{S}, \mathcal{F}, \psi, \eta)$ a PC-structure on Y and $\emptyset \neq Z \subseteq Y$. There exists $\beta_C \in [-\infty, \infty]$ such that:*

(1) $\overline{r}_\beta(Z) = \infty$ *for every* $\beta < \beta_C$*;*
(2) $\overline{r}_\beta(Z) = 0$ *for every* $\beta > \beta_C$*.*

Proof. (1) Assume that $\overline{r}_\delta(Z) = \infty$ for some $\delta \in \mathbb{R}$ and that $\beta < \delta$. Fix $\varepsilon > 0$ and $\mathcal{G} \in \mathcal{C}_\varepsilon(Z)$. Then $H(\mathcal{G}, \delta) < H(\mathcal{G}, \beta)$ by Remark 3.2, assuming without loss of generality that $\eta(\mathcal{G}) < 1$. By definition it follows that $R_\delta(Z, \varepsilon) \leq R_\beta(Z, \varepsilon)$ and so also $\overline{r}_\delta(Z) \leq \overline{r}_\beta(Z)$. Therefore, $\overline{r}_\beta(Z) = \infty$.

(2) Assume that $\overline{r}_\delta(Z) \leq B$ for some $\delta \in \mathbb{R}$ and some constant $B \in \mathbb{R}_+$. Let $\beta > \delta$ and fix $\gamma > 0$. By (P2) and (P3) there exists $\varepsilon > 0$ such that $\mathcal{C}_\varepsilon(Z) \neq \emptyset$, and if $\mathcal{G} \in \mathcal{C}_\varepsilon(Z)$, then $\eta(\mathcal{G})^{\beta - \delta} < \gamma$; therefore,

$$H(\mathcal{G}, \beta) = \sum_{U \in \mathcal{G}} \eta(U)^\delta \eta(U)^{\beta - \delta} < \gamma \sum_{U \in \mathcal{G}} \eta(U)^\delta = \gamma H(\mathcal{G}, \alpha).$$

Hence, $R_\beta(Z, \varepsilon) < \gamma R_\delta(Z, \varepsilon)$, and so

$$\overline{r}_\beta(Z) \leq \gamma \overline{r}_\delta(Z) \leq \gamma B.$$

We conclude that $\overline{r}_\beta(Z) = 0$. ∎

Definition 3.4. Let Y be a non-empty set, $\tau = (\mathcal{S}, \mathcal{F}, \psi, \eta)$ a PC-structure on Y and $\emptyset \neq Z \subseteq Y$. The *upper PC-capacity* of Z with respect to τ is the critical value

$$\overline{\mathrm{Cap}}_\tau(Z) = \beta_C \in [-\infty, \infty]$$

of the map $\alpha \mapsto \overline{r}_\alpha(Z)$ from Proposition 3.3.

Remark 3.5. In [27, §10] one can find an abstract general construction that permits one to define the upper PC-capacity as above. It is based on the axioms called there (A1), (A2) and (A3′). While (A1) is our (P1) and (A2) is our (P2), the axiom (A3′) differs from our (P3):

(A3′) there exists $\varepsilon > 0$ such that for every $\delta \in (0, \varepsilon)$ there exists $\mathcal{G} \subseteq \mathcal{S}$ finite with $\bigcup_{s \in \mathcal{G}} U_s = Y$ and $\psi(s) = \delta$ for every $s \in \mathcal{G}$.

Clearly, (A3′) is strictly stronger than (P3), although the difference is quite subtle. Anyway (P3) is sufficient to get $\overline{r}_\alpha(Z)$ and so the upper PC-capacity, since the value 0 that we assign to $R_\alpha(Z,\varepsilon)$ corresponding to empty $\mathcal{C}_\varepsilon(Z)$ does not play any role in the lim sup defining $\overline{r}_\alpha(Z)$. So, the quantity $\overline{r}_\alpha(Z)$ coincides with that defined in [27, §10] under the stronger axiom (A3′), and for this reason we keep the same notation.

In §3.2 we apply the above abstract construction to the specific cases of the receptive topological entropy and the topological entropy of finitely generated semigroup actions on compact metric spaces, following what is done in [27, §11, Remark (1)] for the topological pressure in the case of continuous selfmaps of compact metric spaces, that is, the case of $\mathbb{N}$-actions.

First of all, since the topological entropy is a particular case of the topological pressure, we need only two auxiliary functions η and ψ, while for the topological pressure a third function ξ was needed (in our modification of PC-structure we can set $\xi \equiv 1$).

Moreover, the function ψ defined in (3.8) for our applications to the receptive topological entropy and to the topological entropy does not satisfy (A3′) while it satisfies (P3); so τ_γ and τ_γ^*, defined in (3.7) and (3.9) respectively, are PC-structures in our sense (see Lemma 3.8) but not in the sense of [27, §10]. On the other hand, τ_γ and τ_γ^* are practically the same as the structure considered in [27, §11, Remark (1)], where it is wrongly claimed that it also satisfies (A3′). Starting from this problem, we introduced (P3) in place of (A3′) and verified that all still works properly in the abstract construction. In particular, this also fixes the gap in [27, §11, Remark (1)].

Definition 3.6. A PC-structure $\tau = (\mathcal{S},\mathcal{F},\psi,\eta)$ on a non-empty set Y is a *strong PC-structure* if τ also satisfies

(P4) if $s,t \in \mathcal{S}$ and $\psi(s)=\psi(t)$ then $\eta(s)=\eta(t)$.

For a strong PC-structure $\tau = (\mathcal{S},\mathcal{F},\psi,\eta)$ on a non-empty set Y, and for $\emptyset \neq Z \subseteq Y$ and $\varepsilon > 0$, let

$$\Lambda_\tau(Z,\varepsilon) = \begin{cases} \inf\{|\mathcal{G}| : \mathcal{G} \in \mathcal{C}_\varepsilon(Z)\} & \text{if } \mathcal{C}_\varepsilon(Z) \neq \emptyset, \\ 1 & \text{if } \mathcal{C}_\varepsilon(Z) = \emptyset. \end{cases} \tag{3.2}$$

Observe that (P4) implies that, for every $\mathcal{G} \in \mathcal{C}_\varepsilon(Z)$, the restriction $\eta\restriction_{\mathcal{G}}$ is constant as $\psi\restriction_{\mathcal{G}}$ is constant, that is, denoting this constant value by η_ε we have

$$\eta(s) = \eta_\varepsilon \quad \text{for every } \mathcal{G} \in \mathcal{C}_\varepsilon(Z) \text{ and every } s \in \mathcal{G}.$$

In particular, this implies that, for $\alpha \in \mathbb{R}_+$,

$$H(\mathcal{G},\alpha) = |\mathcal{G}|\eta_\varepsilon^\alpha. \tag{3.3}$$

The following is the main result of this section on upper PC-capacity that we apply in the proof of Theorem 1.1; we give the proof for the reader's convenience although it is similar to that of [27, Theorem 2.2].

THEOREM 3.7. *For a strong PC-structure $\tau = (\mathcal{S}, \mathcal{F}, \psi, \eta)$ on a non-empty set Y and for $\emptyset \neq Z \subseteq Y$,*

$$\overline{\mathrm{Cap}}_\tau(Z) = \limsup_{\varepsilon \to 0} \frac{\log \Lambda_\tau(Z, \varepsilon)}{\log(1/\eta_\varepsilon)}.$$

Proof. Let

$$\alpha = \overline{\mathrm{Cap}}_\tau(Z) \quad \text{and} \quad \beta = \limsup_{\varepsilon \to 0} \frac{\log \Lambda_\tau(Z, \varepsilon)}{\log(1/\eta_\varepsilon)}.$$

Since τ is a strong PC-structure, it satisfies (P4). Hence, for $\varepsilon > 0$ such that $\mathcal{C}_\varepsilon(Z) \neq \emptyset$, the function η is constant on each $\mathcal{G} \in \mathcal{C}_\varepsilon(Z)$ with value η_ε. Therefore, for $\theta > 0$,

$$R_\theta(Z, \varepsilon) = \Lambda_\tau(Z, \varepsilon)(\eta_\varepsilon)^\theta, \tag{3.4}$$

in view of (3.1)–(3.3). Fix an arbitrary $\gamma > 0$. As $\alpha - \gamma < \alpha$, there exists a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ with $\varepsilon_n \to 0$, and with $\eta_{\varepsilon_n} < 1$ for all $n \in \mathbb{N}$ by (P2), such that $\mathcal{C}_{\varepsilon_n}(Z) \neq \emptyset$ for all $n \in \mathbb{N}$ by (P3), and

$$\infty = \overline{r}_{\alpha - \gamma}(Z) = \lim_{n \to \infty} R_{\alpha - \gamma}(Z, \varepsilon_n).$$

In particular, there exists $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, we have $R_{\alpha - \gamma}(Z, \varepsilon_n) \geq 1$, and (3.4) gives

$$\Lambda_\tau(Z, \varepsilon_n)(\eta_{\varepsilon_n})^{\alpha - \gamma} \geq 1.$$

Taking log and using the inequality $\eta_{\varepsilon_n} < 1$, we obtain

$$\alpha - \gamma \leq \frac{\log \Lambda_\tau(Z, \varepsilon_n)}{\log(1/\eta_{\varepsilon_n})}.$$

Therefore,

$$\alpha - \gamma \leq \limsup_{n \to \infty} \frac{\log \Lambda_\tau(Z, \varepsilon_n)}{\log(1/\eta_{\varepsilon_n})} \leq \beta. \tag{3.5}$$

There exists a sequence $(\varepsilon'_n)_{n \in \mathbb{N}}$ with $\varepsilon'_n \to 0$, and with $\eta_{\varepsilon'_n} < 1$ for all $n \in \mathbb{N}$ by (P2), such that $\mathcal{C}_{\varepsilon'_n}(Z) \neq \emptyset$ for all $n \in \mathbb{N}$ by (P3), and

$$\beta = \lim_{n \to \infty} \frac{\log \Lambda_\tau(Z, \varepsilon'_n)}{\log(1/\eta_{\varepsilon'_n})}.$$

As $\alpha + \gamma > \alpha$, we have $\overline{r}_{\alpha + \gamma}(Z) = 0$, so

$$\limsup_{n \to \infty} R_{\alpha + \gamma}(Z, \varepsilon'_n) \leq \overline{r}_{\alpha + \gamma}(Z) = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that, for every $n \geq n_1$, $R_{\alpha+\gamma}(Z, \varepsilon'_n) \leq 1$, and so (3.4) gives

$$\Lambda_\tau(Z, \varepsilon'_n)(\eta_{\varepsilon'_n})^{\alpha+\gamma} \leq 1.$$

Taking log and using the inequality $\eta_{\varepsilon'_n} < 1$, we obtain

$$\alpha + \gamma \geq \frac{\log \Lambda_\tau(Z, \varepsilon'_n)}{\log(1/\eta_{\varepsilon'_n})}.$$

Therefore,

$$\alpha + \gamma \geq \limsup_{n\to\infty} \frac{\log \Lambda_\tau(Z, \varepsilon'_n)}{\log(1/\eta_{\varepsilon'_n})} = \beta. \tag{3.6}$$

From (3.5) and (3.6), we deduce that

$$\beta - \gamma \leq \alpha \leq \beta + \gamma.$$

Since $\gamma > 0$ is arbitrary, we conclude that $\alpha = \beta$, as required. ∎

3.2. Topological entropy and receptive topological entropy as limits of upper PC-capacities. We now define an appropriate strong PC-structure determined by the action of a finitely generated semigroup (G, G_1) on a compact metric space X with respect to some closed subset Y of X and $\gamma > 0$. To this end, consider

$$\mathcal{F}_\gamma(Y) = \{\emptyset\} \cup \bigcup_{n=0}^{\infty} \mathcal{B}_n(Y, \gamma) = \{\emptyset\} \cup \{B_n^{G_1}(y, \gamma) \cap Y : y \in Y,\, n \in \mathbb{N}\}.$$

Let

$$\mathcal{S}_\gamma(Y) = (Y \times \mathbb{N}) \cup \{s_\emptyset\} \quad \text{and} \quad \mathcal{F}_\gamma(Y) = \{U_s : s \in \mathcal{S}_\gamma(Y)\},$$

with $U_{(y,n)} = B_n^{G_1}(y, \gamma) \cap Y$ for every $s = (y, n) \in Y \times \mathbb{N}$ and $U_{s_\emptyset} = \emptyset$.

The utility of the auxiliary set $\mathcal{S}_\gamma(Y)$ of indices for the definition of the following functions, which would not be well posed if considered directly on the set $\mathcal{F}_\gamma(Y)$, is explained by the following observation. Indeed, the map $\mathcal{S}_\gamma(Y) \to \mathcal{F}_\gamma(Y)$ defined by $s \mapsto U_s$ need not be injective. For example, for $X = Y$, it may happen that $B_m^{G_1}(x, \gamma) = B_n^{G_1}(x, \gamma)$ whenever $G_m = G_n$; this may occur with $m \neq n$, in such a case G is finite. On the other hand, injectivity may also fail when $x \neq y$ while $B_n^{G_1}(x, \gamma) = B_n^{G_1}(y, \gamma)$ for some $n \in \mathbb{N}$ (e.g., in the Cantor set C take two distinct points x, y with $B(x, \gamma) = B(y, \gamma)$).

Let

$$\tau_\gamma(Y) = (\mathcal{S}_\gamma(Y), \mathcal{F}_\gamma(Y), \psi, \eta) \tag{3.7}$$

with $\psi, \eta : \mathcal{S}_\gamma(Y) \to \mathbb{R}_+$ defined by

$$\psi(y, n) = 1/n \quad \text{and} \quad \eta(y, n) = e^{-n} \quad \text{for } (y, n) \in Y \times \mathbb{N}_+, \tag{3.8}$$

$\psi(y, 0) = \eta(y, 0) = 1$ for every $y \in Y$ and $\psi(s_\emptyset) = \eta(s_\emptyset) = 0$.

An alternative choice is

$$\tau_\gamma^*(Y) = (\mathcal{S}_\gamma(Y), \mathcal{F}_\gamma(Y), \psi, \eta^*) \tag{3.9}$$

with

$$\eta^*(y,n) = e^{-|G_n|} \quad \text{for } (y,n) \in Y \times \mathbb{N}_+,$$

$\eta^*(y,0) = 1$ for every $y \in Y$ and $\eta^*(s_\emptyset) = 0$.

Note that $\tau_\gamma(Y)$ and $\tau_\gamma^*(Y)$ share the same $\mathcal{S}_\gamma(Y)$ and $\mathcal{F}_\gamma(Y)$, as well as the same ψ. They differ only in the fourth component $\eta^* \neq \eta$. Consequently, they also share the same families $\mathcal{C}_\varepsilon(-)$ and so the same $\Lambda_{\tau_\gamma^*} = \Lambda_{\tau_\gamma}$.

Now, we are able to show that any infinite finitely generated semigroup acting on a compact metric space by continuous selfmaps dynamically determines two special PC-structures, which reflects the complexity of this action.

LEMMA 3.8. *For a compact metric space X, an infinite finitely generated semigroup $(G, G_1) \subseteq \mathrm{Con}(X)$, a closed subset Y of X and $\gamma > 0$, the systems*

$$\tau_\gamma(Y) = (\mathcal{S}_\gamma(Y), \mathcal{F}_\gamma(Y), \psi, \eta) \quad \text{and} \quad \tau_\gamma^*(Y) = (\mathcal{S}_\gamma(Y), \mathcal{F}_\gamma(Y), \psi, \eta^*)$$

are strong PC-structures on Y.

Proof. Clearly, (P1), (P2) and (P4) are satisfied by $\tau_\gamma(Y)$ and by $\tau_\gamma^*(Y)$. To see that also (P3) is satisfied for both $\tau_\gamma(Y)$ and $\tau_\gamma^*(Y)$, note that for every $\gamma > 0$ there exists $n \in \mathbb{N}_+$ with $1/n < \gamma$, and since Y is compact, there exists a finite subcover $\{B_n(y_1,\gamma), \dots, B_n(y_k,\gamma)\} \subseteq \mathcal{B}_n(Y,\gamma)$ of Y, i.e., a finite $\mathcal{G} = \{(y_1,n), \dots, (y_k,n)\} \subseteq \mathcal{S}$ such that $\mathcal{U}_\mathcal{G} = \{B_n(y_1,\gamma), \dots, B_n(y_k,\gamma)\}$ is a cover of Y. Therefore, $\mathcal{G} \in \mathcal{C}_{1/n}(Y)$, as $\psi(s) = 1/n$ for every $s \in \mathcal{G}$. ∎

To simplify notation, in the rest of the section we put

$$\Lambda_\gamma = \Lambda_{\tau_\gamma} = \Lambda_{\tau_\gamma^*}, \qquad \overline{\mathrm{Cap}}_\gamma = \overline{\mathrm{Cap}}_{\tau_\gamma} \quad \text{and} \quad \overline{\mathrm{Cap}}_\gamma^* = \overline{\mathrm{Cap}}_{\tau_{\gamma^*}}.$$

LEMMA 3.9. *For a compact metric space X, an infinite finitely generated semigroup $(G, G_1) \subseteq \mathrm{Con}(X)$, a closed subset Y of X and $\gamma > 0$, we have $\Lambda_\gamma(Y, 1/n) = r(n,\gamma,Y)$ for every $n \in \mathbb{N}$. Therefore,*

$$\overline{\mathrm{Cap}}_\gamma(Y) = \limsup_{n\to\infty} \frac{1}{n} \log \Lambda_\gamma\left(Y, \frac{1}{n}\right),$$
$$\overline{\mathrm{Cap}}_\gamma^*(Y) = \limsup_{n\to\infty} \frac{1}{|G_n|} \log \Lambda_\gamma\left(Y, \frac{1}{n}\right)$$

and hence

$$\begin{aligned} \overline{\mathrm{Cap}}_\gamma(Y) &= \limsup_{n\to\infty} \frac{1}{n} \log r(n,\gamma,Y), \\ \overline{\mathrm{Cap}}_\gamma^*(Y) &= \limsup_{n\to\infty} \frac{1}{|G_n|} \log r(n,\gamma,Y). \end{aligned} \tag{3.10}$$

In particular, the maps $\gamma \mapsto \overline{\mathrm{Cap}}_\gamma(Y)$ and $\gamma \mapsto \overline{\mathrm{Cap}}_\gamma^*(Y)$ are monotone.

Proof. By the definition of $\varLambda_\gamma(Y,1/n)$, we have, for every $n\in\mathbb{N}_+$,

$$\begin{aligned}\varLambda_\gamma(Y,1/n)&=\inf\{|\mathcal{G}|\colon \mathcal{G}\in\mathcal{C}_{1/n}(Y)\}\\&=\min\Big\{|\mathcal{G}|\colon \mathcal{G}\subseteq Y\times\{n\},\ \mathcal{G}\text{ finite},\ Y\subseteq\bigcup\mathcal{U}_{\mathcal{G}}\Big\}\\&=\min\Big\{|\mathcal{U}|\colon \mathcal{U}\subseteq\mathcal{B}_n^{G_1}(Y,\gamma),\ \mathcal{U}\text{ finite},\ Y\subseteq\bigcup\mathcal{U}\Big\}\\&=r(n,\gamma,Y),\end{aligned}$$

where we apply (2.3) in the last equality.

By Theorem 3.7 and the definition of $\varLambda_\gamma(Y,\varepsilon)$, we get

$$\begin{aligned}\overline{\mathrm{Cap}}_\gamma(Y)&=\limsup_{\varepsilon\to0}\frac{\log\varLambda_\gamma(Y,\varepsilon)}{\log(1/\eta_\varepsilon)}=\limsup_{n\to\infty}\frac{1}{n}\log\varLambda_\gamma(Y,1/n),\\\overline{\mathrm{Cap}}^*_\gamma(Y)&=\limsup_{\varepsilon\to0}\frac{\log\varLambda_\gamma(Y,\varepsilon)}{\log(1/\eta^*_\varepsilon)}=\limsup_{n\to\infty}\frac{1}{|G_n|}\log\varLambda_\gamma(Y,1/n).\end{aligned}$$

This proves (3.10).

To prove the last assertion note that for every $n\in\mathbb{N}$ the map $\gamma\mapsto r(n,\gamma,Y)$ is monotone. Hence, the maps $\gamma\mapsto\frac{1}{n}\log r(n,\gamma,Y)$ and $\gamma\mapsto\frac{1}{|G_n|}\log r(n,\gamma,Y)$ are monotone as well (see also Remark 2.3). Now (3.10) applies. ∎

By the conclusion of Lemma 3.9 and following the idea in [27, §11], we get the existence of a limit upper PC-capacity:

Definition 3.10. For a compact metric space X, an infinite finitely generated semigroup $(G,G_1)\subseteq\mathrm{Con}(X)$ and a closed subset Y of X, let

$$\overline{\mathrm{CP}}_{G_1}(Y)=\lim_{\gamma\to0}\overline{\mathrm{Cap}}_\gamma(Y)\quad\text{and}\quad\overline{\mathrm{CP}}^*_{G_1}(Y)=\lim_{\gamma\to0}\overline{\mathrm{Cap}}^*_\gamma(Y).$$

Let $\overline{\mathrm{CP}}_{G_1}=\overline{\mathrm{CP}}_{G_1}(X)$ and $\overline{\mathrm{CP}}^*_{G_1}=\overline{\mathrm{CP}}^*_{G_1}(X)$.

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. By (3.10), we have

$$\overline{\mathrm{Cap}}_\gamma(Y)=\limsup_{n\to\infty}\frac{1}{n}\log r(n,\gamma,Y).$$

Hence, by Lemma 2.5, we conclude that

$$\overline{\mathrm{CP}}_{G_1}(Y)=\lim_{\gamma\to0}\overline{\mathrm{Cap}}_\gamma(Y)=\lim_{\gamma\to0}\limsup_{n\to\infty}\frac{1}{n}\log r(n,\gamma,Y)=\widetilde{h}_{\mathrm{top}}(G_1,Y).$$

Analogously, by Lemma 2.5,

$$\overline{\mathrm{CP}}^*_{G_1}(Y)=\lim_{\gamma\to0}\overline{\mathrm{Cap}}^*_\gamma(Y)=\lim_{\gamma\to0}\limsup_{n\to\infty}\frac{1}{|G_n|}\log r(n,\gamma,Y)=\widetilde{h}^*_{\mathrm{top}}(G_1,Y).$$

The second statement of the theorem is an immediate consequence of the first. ∎

4. Receptive topological entropy of locally expanding maps and Hausdorff dimension. In this section, we study a particular class of semigroups called locally λ-expanding. The properties of a locally λ-expanding semigroup (G, G_1) acting on a compact metric space X show the interrelations between dynamical systems and fractal dimension theory. In particular, the receptive topological entropy of (G, G_1) is related to the upper box dimension of X.

First, following [28] (see also [12, Chapter 2]), recall that the *upper box dimension* of a closed subset Z of a compact metric space X is

$$\overline{\dim}_B(Z) = \limsup_{\gamma\to 0} \frac{\log N(Z,\gamma)}{-\log\gamma} \tag{4.1}$$

where $N(Z,\gamma)$ denotes the least number of balls $B(x,\gamma)$ of radius $\gamma > 0$ needed to cover Z (hence, $N(Z,\gamma) = r(0,\gamma,Z)$).

We apply the following inequality (see [12, p. 24]), where $\dim_H(X)$ denotes the Hausdorff dimension of a compact metric space X.

FACT 4.1. *Let X be a compact metric space. Then* $\dim_H(X) \leq \overline{\dim}_B(X)$.

The following notion of locally expanding map is inspired by Ruelle [29].

DEFINITION 4.2. Let (X, d) be a compact metric space and $f : X \to X$ a continuous selfmap. For $\lambda > 1$, f is *locally λ-expanding* if there exists $\varepsilon > 0$ such that for every pair of distinct points $x, y \in X$,

$$d(x,y) < \varepsilon \implies d(f(x), f(y)) \geq \lambda \cdot d(x,y).$$

In that case we say that λ is an *expanding coefficient* of f.

Moreover, f is *locally expanding* if it is λ-locally expanding for some $\lambda > 1$.

For $\lambda > 1$, compositions of locally λ-expanding selfmaps are locally λ-expanding as well, so compositions of locally expanding selfmaps are locally expanding. Consider the semigroup $\mathrm{Le}(X) \subseteq \mathrm{Con}(X)$ of all locally expanding selfmaps of X and note that $\mathrm{id}_X \notin \mathrm{Le}(X)$. Moreover, for $\lambda > 1$, let $\mathrm{Le}_\lambda(X)$ be the subsemigroup of $\mathrm{Le}(X)$ of all locally λ-expanding selfmaps of X; therefore, for $\gamma \geq \lambda > 1$,

$$\mathrm{Le}_\gamma(X) \subseteq \mathrm{Le}_\lambda(X) \subseteq \mathrm{Le}(X).$$

LEMMA 4.3. *Let X be a compact metric space and $(G, G_1) \subseteq \mathrm{Con}(X)$ an infinite finitely generated semigroup. Let $G_1 = \{\mathrm{id}_X, f_1, \dots, f_k\}$ with $f_i \in \mathrm{Le}_{\lambda_i}(X)$ for $\lambda_i > 1$ and every $i \in \{1, \dots, k\}$, and let $\lambda = \min\{\lambda_i \colon i \in \{1, \dots, k\}\}$. Let $\varepsilon > 0$ be such that, for every pair of distinct points $x, y \in X$,*

$$d(x,y) < \varepsilon \implies d(f_i(x), f_i(y)) \geq \lambda_i \cdot d(x,y)$$

for every $i \in \{1,\dots,k\}$. *Then* $(G, G_1) \subseteq \mathrm{Le}_\lambda(X) \cup \{\mathrm{id}_X\}$, *and for every* $x \in X$, $n \in \mathbb{N}$ *and* $\gamma \in (0,\varepsilon)$,

$$B_n^{G_1}(x,\gamma) \subseteq B(x,\gamma/\lambda^n). \tag{4.2}$$

Consequently,

$$N(X,\gamma/\lambda^n) \le r(n,\gamma,X). \tag{4.3}$$

Proof. Let $y \in B_n^{G_1}(x,\gamma)$; then $d(h(x),h(y)) < \gamma$ for every $h \in G_n$, so in particular also $d(x,y) < \gamma < \varepsilon$. If $y = x$, then clearly $x \in B(x,\gamma/\lambda^n)$. So assume that $y \ne x$ and take $h \in G_n$. Then there exist $f_{i_1},\dots,f_{i_n}$ in $G_1 \setminus \{\mathrm{id}_X\}$ such that $h = f_{i_1}\dots f_{i_n}$. Since $d(x,y) < r < \varepsilon$, we have

$$\begin{aligned}\gamma > d(h(x),h(y)) &= d(f_{i_1}\dots f_{i_n}(x), f_{i_1}\dots f_{i_n}(y))\\ &\ge \lambda_{i_1} d(f_{i_2}\dots f_{i_n}(x), f_{i_2}\dots f_{i_n}(y)) \ge \dots \ge\\ &\ge \lambda_{i_1}\dots\lambda_{i_n} d(x,y) \ge \lambda^n d(x,y).\end{aligned}$$

Therefore, $d(x,y) < \gamma/\lambda^n$. This gives (4.2).

Now (4.3) follows from (4.2) and (2.3). ∎

In the proof of Theorem 1.3 we need the following result.

LEMMA 4.4 ([13, Lemma 6.2]). *Let* $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ *be an unbounded decreasing function,* $\delta \in (0,1)$ *and* $\gamma > 0$. *Then*

$$\limsup_{r\to 0} \frac{\log \phi(r)}{\log r} = \limsup_{n\to\infty} \frac{\log \phi(\delta^n\gamma)}{\log(\delta^n\gamma)}. \tag{4.4}$$

Obviously, unboundedness of ϕ is equivalent to $\lim_{r\to 0}\phi(r) = \infty$. Clearly, (4.4) remains true also when ϕ is bounded, since in that case both limits superior are limits with common value 0.

Proof of Theorem 1.3. We have to prove that if X is a compact metric space, $\lambda > 1$ and $(G,G_1) \subseteq \mathrm{Con}(X)$ an infinite finitely generated semigroup with $G_1 \setminus \{\mathrm{id}_X\} \subseteq \mathrm{Le}_\lambda(X)$, then

$$\widetilde{h}_{\mathrm{top}}(G_1) \ge (\log\lambda)\overline{\dim}_B(X) \ge (\log\lambda)\dim_H(X). \tag{4.5}$$

Let $G_1 = \{\mathrm{id}_X, f_1,\dots,f_k\}$ and let $\varepsilon \in (0,1)$ be such that, for every pair of distinct points $x,y \in X$ and for every $i \in \{1,\dots,k\}$,

$$d(x,y) < \varepsilon \implies d(f_i(x), f_i(y)) \ge \lambda\cdot d(x,y).$$

Applying Lemma 4.4 for the first equality and (4.3) for the subsequent inequality, we get

$$\begin{aligned}\overline{\dim}_B(Z) &= \limsup_{n\to\infty} \frac{\log N(X,(\gamma/\lambda^n))}{-\log(\gamma/\lambda^n)}\\ &\le \limsup_{n\to\infty} \frac{\log r(n,\gamma,X)}{-\log(\gamma/\lambda^n)} = \frac{1}{\log\lambda}\cdot \limsup_{n\to\infty}\frac{\log r(n,\gamma,X)}{n}.\end{aligned}$$

Therefore,

$$\widetilde{h}_{\mathrm{top}}(G_1) = \lim_{\gamma\to 0} \limsup_{n\to\infty} \frac{\log r(n,\gamma,X)}{n} \geq (\log\lambda)\,\overline{\dim}_B(X).$$

This proves the first inequality in (4.5); the second is given by Fact 4.1. ∎

The next example shows that in Theorem 1.3 one cannot replace $\widetilde{h}_{\mathrm{top}}(G_1)$ by $h^*_{\mathrm{top}}(G_1)$.

EXAMPLE 4.5. Let $X = \mathbb{T}$ be the unit circle in the complex plane. It is known that $\dim_H(\mathbb{T}) = 1$. It is also known that the topological entropy of the map $f : \mathbb{T} \to \mathbb{T}$ given by $f(z) = z^2$ is $h_{\mathrm{top}}(f) = \log 2$ (e.g., see [35]). It is easy to see that f is locally λ-expanding for every $\lambda \in (1, 2)$.

Consider the additive semigroup $G = \mathbb{N}^2_+ = \{(m,k)\colon m,k \geq 0\}$ with $G_1 = \{(1,0),(0,1)\}$ as a generating set. Then, for every $n \in \mathbb{N}$, $G_n = \{(m,k)\colon m,k \geq 0,\ m+k \leq n\}$, and so

$$|G_n| \geq n + (n-1) + \cdots + 1 = n(n+1)/2 > n^2/2. \tag{4.6}$$

Consider the action $T : G \times \mathbb{T} \to \mathbb{T}$ of G on $\mathbb{T}$ defined by $T((m,k),z) = z^{2(m+k)} = f^{m+k}(z)$ for all $(m,k) \in G$ and $z \in \mathbb{T}$ (that is, $T((0,1),-) = f = T((1,0),-)$). Clearly, for every $\lambda \in (1,2)$, this action is locally λ-expanding, that is, $T((m,k),-)$ is locally λ-expanding for every $(m,k) \in G$.

Given $\gamma > 0$ and $n \in \mathbb{N}$, it is easy to see that every (n,γ)-spanning set for $\mathbb{T}$ with respect to f is an (n,γ)-spanning set for $\mathbb{T}$ with respect to the action T of G and its set of generators G_1, and vice versa. This implies that $h_{\mathrm{top}}(G) = h_{\mathrm{top}}(G_1) = \log 2$. A similar argument using (4.6) shows that $h^*_{\mathrm{top}}(G_1) = 0$. Thus, for any $\lambda \in (1,2)$,

$$h^*_{\mathrm{top}}(G_1) = 0 < (\log\lambda)\dim_H(\mathbb{T}) = \log\lambda \leq \log 2 = h_{\mathrm{top}}(G_1).$$

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