# Algebraic entropies of commuting endomorphisms of torsion abelian groups

# Algebraic entropies of commuting endomorphisms of torsion abelian groups

Andzej Biś  Dikran Dikranjan  Anna Giordano Bruno
Luchezar Stoyanov

Respectfully dedicated to the 95-th birthday of Lászlo Fuchs

**Abstract**

For actions of $m$ commuting endomorphisms of a torsion abelian group we compute the algebraic entropy and the algebraic receptive entropy, showing that the latter one takes finite positive values in many cases when the former one vanishes.

2020MSC: 20M10, 20K30, 22A26, 22D05.
Keywords: algebraic entropy, receptive entropy, regular system, abelian group, torsion abelian group, $p$-group.

## 1 Introduction

A *left semigroup action* $S \overset{\alpha}{\curvearrowright} A$ of a semigroup $S$ on an abelian group $A$ (*by group endomorphisms*) is defined by $\alpha : S \times A \to A$, $(s,x) \mapsto \alpha(s)(x)$ with $\alpha(st) = \alpha(s) \circ \alpha(t)$ and $\alpha(s)(x+y) = \alpha(s)(x) + \alpha(s)(y)$ for every $s,t \in S$ and every $x,y \in A$. In case $S$ is a monoid with neutral element $e$, we impose also $\alpha(e)(x) = x$ for every every $x \in A$. For $N \subseteq S$ and $F \subseteq A$, let $T_N(\alpha,F) = \sum_{s\in N} \alpha(s)(F)$. Let $\mathcal{F}(A)$ denote the family of all finite subgroups of $A$.

We recall that a *right Følner sequence* of a semigroup $S$ is a sequence $(F_n)_{n\in\mathbb{N}}$ of finite non-empty subsets of $S$ such that $\lim_{n\to\infty} |F_n s \setminus F_n|/|F_n| = 0$ for every $s \in S$. A countable semigroup $S$ is *right-amenable* if and only if $S$ admits a right Følner sequence. Consider an action $S \overset{\alpha}{\curvearrowright} A$ of a countable cancellative right-amenable semigroup $S$ on an abelian group $A$. For $F \in \mathcal{F}(A)$ let

$$H_{alg}(\alpha,F) = \lim_{n\to\infty} \frac{\log |T_{N_n}(\alpha,F)|}{|N_n|},$$

where $(N_n)_{n\in\mathbb{N}}$ is any right Følner sequence of $S$ (the limit exists, it is finite and does not depend on the right Følner sequence). The *algebraic entropy* of $\alpha$ is $\mathrm{ent}(\alpha) = \sup\{H_{alg}(\alpha,F) : F \in \mathcal{F}(A)\}$ [3]. This concept extends in a natural way the algebraic entropy ent for $\mathbb{N}$-actions on $A$, that is, for group endomorphisms of $A$, introduced in [1, 10] and developed in [5].

The nice properties of the algebraic entropy ent stem from the fact that the action $S \overset{\alpha}{\curvearrowright} A$ provides a left $\mathbb{Z}[S]$-module structure on $A$: ent is an invariant of the category $\mathfrak{T}_S$ of left $\mathbb{Z}[S]$-modules that are torsion as abelian groups, and it is furthermore a length function of $\mathfrak{T}_S$ in the sense of Northcott and Reufel, and of Vámos (see Fact 2.5). Moreover, there is a remarkable connection of ent with the topological entropy, that is, $\mathrm{ent}(\alpha)$ coincides with the topological entropy of the dual action of $\alpha$ (i.e., the action induced by $\alpha$ of $S$ on the Pontryagin dual group $\widehat{A}$ of $A$) [3, 10].

On the other hand, ent presents some shortcomings from intuitive point of view. For example, if $S$ is finite then $\mathrm{ent}(\alpha) = \log|A|/|S|$, so in particular $\mathrm{ent}(\alpha) = \infty$ whenever $A$ is infinite. Moreover, there are many cases when $S$ is a group, $T$ is a subgroup of $S$ and the restricted action $\alpha\restriction_T$ has $\mathrm{ent}(\alpha\restriction_T) = \infty$ while $\mathrm{ent}(\alpha) = 0$; in case $T$ is a normal subgroup of $S$ always $\mathrm{ent}(\alpha) \leq \mathrm{ent}(\alpha\restriction_T)$. If again $S$ is a group and $T$ is a normal subgroup of $S$ with $T \subseteq \ker\alpha$, then $\mathrm{ent}(\alpha) \leq \mathrm{ent}(\bar{\alpha}_{S/T})$, where $\bar{\alpha}_{S/T}$ is the quotient action induced by $\alpha$; also in this case the inequality can be strict.

This suggested us to consider the new option for algebraic entropy offered in Definition 1.2 (see [2]), and inspired by [7], where counterparts of the bizzarre behavior of the topological entropy were pointed out. Following [7], a *regular system* of a finitely generated monoid $S$ is a sequence

$\Gamma = (N_n)_{n\in\mathbb{N}}$ of finite subsets of $S$ such that $N_0 = \{e\}$ and $N_i N_j \subseteq N_{i+j}$ for every $i, j \in \mathbb{N}$. In particular, $N_n \subseteq N_{n+1}$ for every $n \in \mathbb{N}$. The regular system $\Gamma = (N_n)_{n\in\mathbb{N}}$ is *exhaustive* if $S = \bigcup_{n\in\mathbb{N}} N_n$ and it is *standard* if for every $n \in \mathbb{N}$ the set $N_n = N_1 \ldots N_1$ is the $n$-fold setwise product of $N_1$, and we set $N_0 = \{e\}$, by definition.

*Example* 1.1. The following is the most natural example of an exhaustive standard regular system. If the monoid $S$ is finitely generated by $N_1$, then the standard regular system $(N_n)_{n\in\mathbb{N}}$ of $S$ is exhaustive (e.g., $S = \mathbb{N}$ with $N_n = \{0, \ldots, n\}$ for every $n \in \mathbb{N}$).

**Definition 1.2.** Let $S$ be a finitely generated monoid, $\Gamma = (N_n)_{n\in\mathbb{N}}$ a regular system of $S$, $A$ a torsion abelian group and $S \overset{\alpha}{\curvearrowright} A$. For $F \in \mathcal{F}(A)$, let

$$\widetilde{H}^{\Gamma}_{alg}(\alpha, F) = \limsup_{n\to\infty} \frac{\log |T_{N_n}(\alpha, F)|}{n}.$$

The *algebraic receptive entropy of $\alpha$ with respect to* $\Gamma$ is

$$\widetilde{\mathrm{ent}}^{\Gamma}(\alpha) = \sup\{\widetilde{H}^{\Gamma}_{alg}(\alpha, F) \colon F \in \mathcal{F}(A)\}.$$

Unlike the case of ent, it may occur that $\widetilde{H}^{\Gamma}_{alg}(\alpha, F) = \infty$ for some $F \in \mathcal{F}(A)$ (see the proof of Theorem 5.1).

Also the algebraic receptive entropy extends in a natural way the algebraic entropy ent for $\mathbb{N}$-actions, as we show in §2, where we recall the basic properties of the algebraic (receptive) entropy that we use further on.

In §3 we make use of the standard correspondence between $\mathbb{N}^m$-actions on abelian groups $A$ of prime exponent $p$ and $R_p$-module structures on $A$, where $R_p = \mathbb{F}_p[X_1, \ldots, X_m]$ is the ring of polynomials of $m$ variables $X_1, \ldots, X_m$ over the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. This allows us to freely pass from $\mathbb{N}^m$-actions to $R_p$-modules and viceversa, writing $\mathrm{ent}(A)$ or $\widetilde{\mathrm{ent}}(A)$ for an $R_p$-module $A$, having in mind the algebraic (receptive) entropy of the corresponding $\mathbb{N}^m$-action. As a first step we compute the algebraic (receptive) entropy of $R_p$ (see Theorem 3.3).

In §4 we compute the algebraic entropy ent. First we see that $\mathrm{ent}(A) = 0$ when $A = R_p/\mathfrak{a}$ is an infinite cyclic $R_p$-module and $\mathfrak{a}$ is a non-zero ideal of $R_p$ (see Theorem 4.1). Using this result we prove that $\mathrm{ent}(A) = \mathrm{rank}_{R_p}(A) \log p$ for an arbitrary $R_p$-module $A$ (see Theorem 4.4). This allows us to obtain a characterization of $\mathbb{N}^m$-actions on torsion abelian groups with zero algebraic entropy via Bernoulli shifts (see Proposition 4.7).

The first step in §5 is that the receptive algebraic entropy of an infinite cyclic $R_p$-module $A = R_p/\mathfrak{a}$ is infinite whenever $\mathfrak{a} \neq 0$ is a principal ideal of $R_p$ (see Theorem 5.1). Moreover, in the case $m = 2$ we compute $\widetilde{\mathrm{ent}}(A)$ even when $\mathfrak{a}$ is not necessarily principal (see Theorem 5.5 and Corollary 5.7). Remaining in the case $m = 2$, we describe when $0 < \widetilde{\mathrm{ent}}(A) < \infty$ for a finitely generated $R_p$-module $A$ (see Corollary 5.8).


## Acknowledgements

It is a pleasure to thank the referee for the careful reading.

The second and the third named author were partially supported by the "National Group for Algebraic and Geometric Structures, and Their Applications" (GNSAGA - INdAM).


## 2 Basic properties and examples

Let $S$ be a finitely generated monoid and $\Gamma = (N_n)_{n\in\mathbb{N}}$ a regular system of $S$. For $m \in \mathbb{N}_+$, the direct product $S^m$ carries two regular systems extending $\Gamma$ in a natural way:

(i) the *Cartesian extension* of $\Gamma$ is $\Gamma^m = (N_n^m)_{n\in\mathbb{N}}$;

(ii) if $N_1$ is a finite set of generators of $S$ containing $e$, the *minimal extension* $\Gamma^{(m)}$ of $\Gamma$ is the standard regular system $\Gamma^{(m)} = (N_n^{(m)})_{n\in\mathbb{N}}$ of $S^m$ with $N_1^{(m)} = \{(s, e, \ldots, e) \colon s \in N_1\} \cup \ldots \cup \{(e, \ldots, e, s) \colon s \in N_1\}$.

In $S=\mathbb{N}$ consider the exhaustive standard regular system $\Xi=(\Xi_n)_{n\in\mathbb{N}}$ with $\Xi_1=\{0,1\}$ (so $\Xi_n=\{0,\ldots,n\}$ for every $n\in\mathbb{N}$). Obviously, this is the smallest exhaustive standard regular system of $\mathbb{N}$.

For $\mathbb{N}^m$ the Cartesian extension of $\Xi$ is $\Xi^m=(\Xi^m_n)_{n\in\mathbb{N}}$ with $\Xi^m_1=\{0,1\}^m$, and the minimal extension of $\Xi$ is $\Xi^{(m)}=(\Xi^{(m)}_n)_{n\in\mathbb{N}}$ with $\Xi^{(m)}_n=\{(a_1,\ldots,a_m)\in\mathbb{N}^m\colon a_1+\ldots+a_m\leq n\}$ for every $n\in\mathbb{N}$.

*Remark* 2.1. Let $S$ be a finitely generated monoid, $A$ an abelian group and $S\overset{\alpha}{\curvearrowright}A$. If $H$ is a submonoid of $S$ and $\Gamma=(N_n)_{n\in\mathbb{N}}$ is a regular system of $H$, then $\Gamma=(N_n)_{n\in\mathbb{N}}$ is a regular system of $S$ as well, and $\widetilde{H}^{\Gamma}_{alg}(\alpha\restriction_H,F)=\widetilde{H}^{\Gamma}_{alg}(\alpha,F)$ for every $F\in\mathcal{F}(A)$, so also $\widetilde{\mathrm{ent}}^{\Gamma}(\alpha\restriction_H)=\widetilde{\mathrm{ent}}^{\Gamma}(\alpha)$.

The above notions of algebraic entropy and algebraic receptive entropy extend the usual notion of algebraic entropy of a group endomorphism:

*Remark* 2.2. (a) Let $A$ be an abelian group and $\phi:A\to A$ an endomorphism. We recall from [4, 5, 10] that, for $F\in\mathcal{F}(A)$, $T_0(\phi,F)=\{0\}$ and $T_n(\phi,F)=F+\phi(F)+\ldots+\phi^{n-1}(F)$ for every $n\in\mathbb{N}_+$; moreover

$$H_{alg}(\phi,F)=\lim_{n\to\infty}\frac{\log|T_n(\phi,F)|}{n},$$

and the limit exist by Fekete Lemma. The algebraic entropy from [10] is $\mathrm{ent}(\phi)=\sup\{H_{alg}(\phi,F)\colon F\in\mathcal{F}(A)\}$. Consider the action $\mathbb{N}\overset{\alpha_\phi}{\curvearrowright}A$ defined by $\alpha_\phi(1)=\phi$. It is easy to see that $T_{\Xi_n}(\alpha_\phi,F)=T_{n+1}(\phi,F)$ for every $n\in\mathbb{N}$ and $F\in\mathcal{F}(A)$, and so $\mathrm{ent}(\phi)=\widetilde{\mathrm{ent}}^{\Xi}(\alpha_\phi)=\mathrm{ent}(\alpha_\phi)$.

(b) If $\phi:A\to A$ is an isomorphism, $\phi$ induces also an action $\mathbb{Z}\overset{\beta_\phi}{\curvearrowright}A$ defined by $\beta_\phi(1)=\phi$. Using again the standard regular system $\Xi$ of $\mathbb{Z}$, by Remark 2.1 and item (a) we find $\mathrm{ent}(\phi)=\widetilde{\mathrm{ent}}^{\Xi}(\beta_\phi)$.

Since $\Xi$ is not exhaustive for $\mathbb{Z}$, it makes sense to consider the exhaustive standard regular system $\Xi'=(\Xi'_n)_{n\in\mathbb{N}}$ of $\mathbb{Z}$ with $\Xi'_n=\{-n,\ldots,n\}$ for every $n\in\mathbb{N}$. Then, for every $n\in\mathbb{N}$ and $F\in\mathcal{F}(A)$,

$$T_{\Xi'_n}(\beta_\phi,F)=\phi^{-n}(F)+\ldots+F+\phi(F)+\ldots+\phi^n(F)=T_{n+1}(\phi^{-1},F)+\phi(T_n(\phi,F)).$$

Since $\phi^n(T_{\Xi'_n}(\beta_\phi,F))=T_{2n+1}(\phi,F)$, and $\phi$ is bijective, we have $|T_{\Xi'_n}(\beta_\phi,F)|=|T_{2n+1}(\phi,F)|$. Therefore,

$$\begin{aligned}\widetilde{H}^{\Xi'}_{alg}(\beta_\phi,F)&=\limsup_{n\to\infty}\frac{\log|T_{\Xi'_n}(\beta_\phi,F)|}{n}=\limsup_{n\to\infty}\frac{\log|T_{2n+1}(\phi,F)|}{n}=\\&=\lim_{n\to\infty}\frac{2n+1}{n}\frac{\log|T_{2n+1}(\phi,F)|}{2n+1}=2H_{alg}(\phi,F).\end{aligned}$$

We conclude that $\widetilde{\mathrm{ent}}^{\Gamma'}(\beta_\phi)=2\mathrm{ent}(\phi)$.

*Example* 2.3. Every exhaustive standard regular system $\Gamma=(N_n)_{n\in\mathbb{N}}$ of $\mathbb{N}^m$ gives rise to a right Følner sequence of $\mathbb{N}^m$. The same occurs for every finitely generated monoid of subexponential growth.

(We have no example of a regular system of an amenable finitely generated group that fails to be a right Følner sequence.)

*Remark* 2.4. Let $S$ be a finitely generated monoid, $A$ an abelian group and $S\overset{\alpha}{\curvearrowright}A$. Suppose that $\Gamma=(N_n)_{n\in\mathbb{N}}$ is a regular system of $S$ which is also a right Følner sequence of $S$ and such that

$$|N_n|\geq cn^2 \text{ for some constant } c>0 \text{ and all } n\in\mathbb{N}_+. \tag{2.1}$$

It follows from the definitions that, if for some $F\in\mathcal{F}(A)$ we have $\widetilde{H}^{\Gamma}_{alg}(\alpha,F)<\infty$, then $H_{alg}(\alpha,F)=0$. Thus, $\widetilde{\mathrm{ent}}^{\Gamma}(\alpha)<\infty\ \Rightarrow\ \mathrm{ent}(\alpha)=0$ (equivalently, $\mathrm{ent}(\alpha)>0\ \Rightarrow\ \widetilde{\mathrm{ent}}^{\Gamma}(\alpha)=\infty$).

The condition (2.1) is available for example whenever $S$ is a finitely generated group that does not contain a cyclic subgroup of finite index.

We recall basic properties of ent that we use below. For a monoid $S$, abelian groups $A,B$ and actions $S\overset{\alpha}{\curvearrowright}A, S\overset{\beta}{\curvearrowright}B$, $\alpha$ and $\beta$ are *conjugated* by an isomorphism $\xi:A\to B$ if $\xi\circ\alpha(g)=\beta(g)\circ\xi$ for every $g\in S$.

**Fact 2.5.** *[3] Let $S$ be a cancellative right-amenable monoid, $A$ a torsion abelian group and $S \overset{\alpha}{\curvearrowright} A$. Let $B$ be an $\alpha$-invariant subgroup of $A$ and denote by $\alpha_B$ and $\alpha_{A/B}$ the induced actions of $S$ on $B$ and on $A/B$, respectively.*

*(Invariance) For $S \overset{\beta}{\curvearrowright} C$ with $C$ a torsion abelian group, if $\alpha$ and $\beta$ are conjugated, then $\mathrm{ent}(\alpha) = \mathrm{ent}(\beta)$.*

*(Monotonicity) $\mathrm{ent} \geq \max\{\mathrm{ent}(\alpha_B), \mathrm{ent}(\alpha_{A/B})\}$.*

*(Continuity) If $A$ is a direct limit of $\alpha$-invariant subgroups $\{A_i : i \in I\}$, then $\mathrm{ent}(\alpha) = \sup_{i\in I} \mathrm{ent}(\alpha_{A_i})$.*

*(weak Addition Theorem) If $A = A_1 \times A_2$, with $A_1$, $A_2$ $\alpha$-invariant subgroups of $A$, then $\mathrm{ent}(\alpha) = \mathrm{ent}(\alpha_{A_1}) + \mathrm{ent}(\alpha_{A_2})$.*

*(Addition Theorem) $\mathrm{ent}(\alpha) = \mathrm{ent}(\alpha_B) + \mathrm{ent}(\alpha_{A/B})$.*

The next theorem was inspired by a similar result for $\mathbb{N}$-actions in [5].

**Theorem 2.6.** *Let $S$ be a countable cancellative right-amenable monoid, $A$ a torsion abelian group and $S \overset{\alpha}{\curvearrowright} A$. Then $\mathrm{ent}(\alpha) > 0$ if and only if there exists a prime $p$ such that $\mathrm{ent}(\alpha_{A[p]}) > 0$.*

*Proof.* In view of the monotonicity of ent, it is enough to prove the necessity.

We consider first the case when $A$ is a bounded $p$-group for some prime $p$; let $p^k$ be the exponent of $A$. If $k = 1$ there is nothing to prove. Assume that $k > 1$ and the assertions is true for $k - 1$. If $\mathrm{ent}(\alpha_{A[p]}) > 0$, there is nothing to prove. Assume that $\mathrm{ent}(\alpha_{A[p]}) = 0$. Consider the short exact sequence $0 \to A[p] \to A \to pA \to 0$. According to the Addition Theorem and the invariance, $\mathrm{ent}(\alpha) = \mathrm{ent}(\alpha_{pA}) + \mathrm{ent}(\alpha_{A[p]}) = \mathrm{ent}(\alpha_{pA})$. Since $p^{k-1}pA = 0$, we can conclude that $\mathrm{ent}(\alpha_{(pA)[p]}) > 0$. Since $(pA)[p]$ is a subgroup of $A[p]$, monotonicity $\mathrm{ent}(\alpha_{A[p]}) > 0$, a contradiction.

If $A$ is a $p$-group for some prime $p$, $A = \bigcup_{n\in\mathbb{N}} A[p^n]$ and $\mathrm{ent}(\alpha) = \sup_{n\in\mathbb{N}} \mathrm{ent}(\alpha_{A[p^n]})$. Hence, our hypothesis $\mathrm{ent}(\alpha) > 0$ yields $\mathrm{ent}(\alpha_{A[p^n]}) > 0$ for some $n \in \mathbb{N}$. The above argument applied to $A[p^n]$, combined with the obvious equality $(A[p^n])[p] = A[p]$, entails $\mathrm{ent}(\alpha_{A[p]}) > 0$.

Let $A = \bigoplus_p A_p$, where each $A_p$ is a $p$-group. Then using the actions $S \overset{\alpha_{A_p}}{\curvearrowright} A_p$, one has $\mathrm{ent}(\alpha) = \sum_p \mathrm{ent}(\alpha_{A_p})$ by Fact 2.5. Hence, $\mathrm{ent}(\alpha) > 0$ yields the existence of a prime $p$ such that $\mathrm{ent}(\alpha_{A_p}) > 0$. Now the above argument gives $\mathrm{ent}(\alpha_{A_p[p]}) > 0$. Since $A_p[p] = A[p]$, we are done. ☐

**Fact 2.7.** *[2] Invariance, monotonicity and continuity remain valid also for the algebraic receptive entropy, while in the weak Addition Theorem only one inequality is available: if $A = A_1 \times A_2$, with $A_1$, $A_2$ $\alpha$-invariant subgroups of $A$, then $\widetilde{\mathrm{ent}}^{\Gamma}(\alpha) \leq \widetilde{\mathrm{ent}}^{\Gamma}(\alpha_{A_1}) + \widetilde{\mathrm{ent}}^{\Gamma}(\alpha_{A_2})$.*

# 3 $\mathbb{N}^m$-actions vs $\mathbb{Z}[X_1, \ldots, X_m]$-modules

**Notation 3.1.** For an action $\mathbb{N}^m \overset{\alpha}{\curvearrowright} A$ on an abelian group $A$, from now on, we simply use the notation as follows: $\widetilde{H}_{alg} = \widetilde{H}_{alg}^{\Xi^{(m)}}$ and $\widetilde{\mathrm{ent}} = \widetilde{\mathrm{ent}}^{\Xi^{(m)}}$.

Following a well-known approach of Kaplansky, an $\mathbb{N}^m$-action on an abelian group $A$ can be viewed in a standard way as an $R_0$-module structure on $A$, where $R_0 = \mathbb{Z}[X_1, \ldots, X_m]$ is the ring of polynomials of $m$ variables $X_1, \ldots, X_m$ over $\mathbb{Z}$. Similarly, a $\mathbb{Z}^m$-action on $A$ can be viewed as a module structure on $A$ over the ring $\mathbb{Z}[X_1^{\pm 1}, \ldots, X_m^{\pm 1}]$ of Laurent polynomials of $m$ variables $X_1, \ldots, X_m$ over $\mathbb{Z}$.

In case $A$ has a prime exponent $p$, one can use also the ring $R_p = \mathbb{F}_p[X_1, \ldots, X_m]$ of polynomials of $m$ variables $X_1, \ldots, X_m$ over the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and provide as above an obvious connection between $\mathbb{N}^m$-actions and $R_p$-module structures on $A$. Moreover, the $\mathbb{Z}^m$-actions on $A$ can be viewed as module structures on $A$ over the ring $\mathbb{F}_p[X_1^{\pm 1}, \ldots, X_m^{\pm 1}]$ of Laurent polynomials of $m$ variables $X_1, \ldots, X_m$ over $\mathbb{F}_p$. In the sequel we freely pass from $\mathbb{N}^m$-actions to $R_p$-modules and viceversa, writing $\mathrm{ent}(A)$ or $\widetilde{\mathrm{ent}}(A)$ for an $R_p$-module $A$, having in mind the algebraic (receptive) entropy of the corresponding $\mathbb{N}^m$-action.

This approach is efficient in the case $m = 1$, when $R_p$ is a principal ideal domain, so $R_p$-modules have a relatively simple structure and one can easily prove that $\mathrm{ent}(A) = \widetilde{\mathrm{ent}}(A) =$

$\mathrm{rank}_{R_p}(A)$, where $\mathrm{rank}_{R_p}(A)$ denotes the maximum size of a subset of $A$ independent over $R_p$. This characterization is extended to the case $m > 1$ in Theorem 4.3. As a starting point, we consider the cyclic $R_p$-module $A = R_p$ in Theorem 3.3.

*Remark* 3.2. For the sake of simplicity, it makes sense to replace, whenever necessary, the additive monoid $(\mathbb{N}^m, +, 0)$ by the moltiplicative submonoid $M = \{X_1^{s_1} \dots X_m^{s_m} : (s_1, \dots, s_m) \in \mathbb{N}^m\}$ of the multiplicative monoid $(R_p, \cdot, 1)$ of the ring $R_p = \mathbb{F}_p[X_1, \dots, X_m]$. For $b = X_1^{s_1} \dots X_m^{s_m} \in M$, the *degree* of $b$ is $d(b) = \sum_{i=1}^m s_i$.

Substantially, for an action $\mathbb{N}^m \overset{\alpha}{\curvearrowright} A$ on an abelian group $A$ of exponent a prime $p$, the commuting endomorphisms $\phi_i := \alpha(e_i)$ of $A$, where $e_i$ is the $i$-th member of the canonical base of $\mathbb{N}^m$, make it become an $R_p$-module, as already explained above. Now the $n$-th member $\Xi_n^{(m)}$ of the standard minimal regular system $\Xi^{(m)}$ of $\mathbb{N}^m$ obviously corresponds to

$$B_n = \{b \in M : d(b) \leq n\} \subseteq M.$$

Let

$$b_n = |B_n| = |\Xi_n^{(m)}| \tag{3.1}$$

and note that $b_n$ coincides with the so called $n+1$-th simplicial $m$-polytopic number known to be equal to the binomial coefficient $C_m^{n+m}$. Hence,

$$b_n = \frac{1}{m!} n^m + \frac{m+1}{2(m-1)!} n^{m-1} + \dots \tag{3.2}$$

is a polynomial of $n$ of degree $m$, and so $\lim_{n\to\infty} b_n/n = \infty$ whenever $m > 1$.

**Theorem 3.3.** *If $m \in \mathbb{N}_+$, then $\mathrm{ent}(R_p) = \log p$. If $m = 1$, then $\widetilde{\mathrm{ent}}(R_p) = \mathrm{ent}(R_p) = \log p$, otherwise $\widetilde{\mathrm{ent}}(R_p) = \infty > \mathrm{ent}(R_p)$.*

*Proof.* If $m = 1$ the equality $\widetilde{\mathrm{ent}}(R_p) = \mathrm{ent}(R_p)$ follows from Remark 2.2(a), while the equality $\mathrm{ent}(R_p) = \log p$ is well known since the corresponding $\mathbb{N}$-action is the Bernoulli shift $(x_0, x_1, x_2, \dots) \mapsto (0, x_0, x_1, x_2, \dots)$ of $\bigoplus_{\mathbb{N}} \mathbb{Z}_p$ [5]. Assume $m > 1$ and put $F = \mathbb{F}_p \in \mathcal{F}(A)$, that is, $F$ is the set of all $p$ polynomials of degree 0 and the zero polynomial. By (3.1), $|\Xi_n^{(m)}| = |B_n| = b_n$. Since $T_{\Xi_n^{(m)}}(\alpha, F) =: V_n$ is a vector space and $B_n$ is a base of $V_n$, then $\dim V_n = |B_n| = b_n$, so $|V_n| = p^{b_n}$ and

$$\log |V_n| = b_n \log p. \tag{3.3}$$

Since $(B_n)_{n\in\mathbb{N}}$ is a Følner sequence of $R_p$, we deduce that

$$H_{alg}(\alpha, F) = \lim_{n\to\infty} \frac{\log |V_n|}{|\Xi_n^{(m)}|} = \lim_{n\to\infty} \frac{b_n \log p}{b_n} = \log p.$$

If we replace $F$ by $V_l = T_{\Xi_l^{(m)}}(\alpha, F)$ for some $l \in \mathbb{N}$, we have $T_{\Xi_n^{(m)}}(\alpha, V_l) = T_{\Xi_{n+l}^{(m)}}(\alpha, F) = V_{n+l}$, which leads to $H_{alg}(\alpha, V_l) = H_{alg}(\alpha, F) = \log p$. Since every finite subgroup of $R_p$ is contained in some $V_l$, this proves $\mathrm{ent}(R_p) = \log p$.

Since $\lim_{n\to\infty} |N_n|/n = \lim_{n\to\infty} b_n/n = \infty$ by Remark 3.2, $\widetilde{H}_{alg}(\alpha, F) = \infty$, and therefore $\widetilde{\mathrm{ent}}(\alpha) = \infty$. □

# 4 The algebraic entropy of $\mathbb{F}_p[X_1, \dots, X_m]$-modules

Here we consider $\mathbb{N}^m$-actions on an abelian group $A$ of exponent a prime $p$; so let $R_p = \mathbb{F}_p[X_1, \dots, X_m]$. Moreover, we use $\Xi^{(m)}$, which is an exhaustive standard regular system of $\mathbb{N}^m$ and also a Følner sequence. Since $\mathrm{ent}(A) = 0$ for finite abelian groups $A$, we assume in the sequel that $A$ is infinite.

**Theorem 4.1.** *If $A = R_p/\mathfrak{a}$ is an infinite cyclic $R_p$-module, where $\mathfrak{a} \neq 0$ is an ideal of $R_p$, then* $\mathrm{ent}(A) = 0$.

*Proof.* Denote by $\overline{\alpha}$ the $\mathbb{N}^m$-action corresponding to the $R_p$-module structure of $A = R_p/\mathfrak{a}$, and let $q : R_p \to A = R_p/\mathfrak{a}$ be the quotient map. We keep the notation from the proof of Theorem 3.3. In particular, $V_n = T_{\Xi_n^{(m)}}(\alpha, F)$ is a linear subspace of $R_p$ for $n \in \mathbb{N}$, so $q(V_n) = T_{\Xi_n^{(m)}}(\overline{\alpha}, q(\mathbb{F}_p))$

is a linear subspace of $A$. Since $q(V_n) \cong V_n/V_n \cap \ker q = V_n/V_n \cap \mathfrak{a}$, so $|q(V_n)| = |V_n/V_n \cap \mathfrak{a}| = |V_n|/|V_n \cap \mathfrak{a}|$, and hence

$$\log |q(V_n)| = \log |V_n| - \log |V_n \cap \mathfrak{a}|. \tag{4.1}$$

Consider the principal ideal $\mathfrak{a} = (a(X_1, \dots, X_m))$ and let $d$ be the degree of $a$. Then $p(X_1, \dots, X_m) \in V_n \cap \mathfrak{a}$ precisely when $p(X_1, \dots, X_m) = r(X_1, \dots, X_m)a(X_1, \dots, X_m)$ for some $r(X_1, \dots, X_m) \in R_p$ of degree at most $n - d$, i.e., $r(X_1, \dots, X_m) \in V_{n-d}$. Since $R_p$ is a domain, the map

$$V_{n-d} \ni r(X_1, \dots, X_m) \mapsto r(X_1, \dots, X_m)a(X_1, \dots, X_m) \in V_n \cap \mathfrak{a}$$

provides a bijection, so $|V_n \cap \mathfrak{a}| = |V_{n-d}|$. Hence, (3.2), (3.3) and (4.1) give

$$\log |q(V_n)| = (b_n - b_{n-d}) \log p = \left( \frac{d}{(m-1)!} n^{m-1} + \dots \right) \log p; \tag{4.2}$$

in particular $\log |q(V_n)|$ is a polynomial of $n$ of degree $m - 1$. This implies

$$H_{alg}(\overline{\alpha}, q(\mathbb{F}_p)) = 0. \tag{4.3}$$

For any $F' \in \mathcal{F}(R_p)$ one can find $n_0 \in \mathbb{N}$ such that $F' \subseteq T_{\Xi_{n_0}^{(m)}}(\alpha, \mathbb{F}_p)$, so

$$T_{\Xi_{n+n_0}^{(m)}}(\alpha, \mathbb{F}_p) \supseteq T_{\Xi_n^{(m)}}(\alpha, F') \quad \text{for every} \quad n \in \mathbb{N}. \tag{4.4}$$

For $F^* \in \mathcal{F}(A)$ there is $F' \in \mathcal{F}(R_p)$ with $q(F') = F^*$. By (4.4), this gives

$$T_{\Xi_{n+n_0}^{(m)}}(\overline{\alpha}, q(\mathbb{F}_p)) \supseteq T_{\Xi_n^{(m)}}(\overline{\alpha}, F^*). \tag{4.5}$$

Dividing by $|\Xi_n^{(m)}|$ and using (4.3) we deduce that $H_{alg}(\overline{\alpha}, F^*) = 0$ as well. Therefore $\mathrm{ent}(A) = 0$.

In case when $\mathfrak{a}$ is not necessarily principal, find a principal ideal $0 \neq \mathfrak{b} \subseteq \mathfrak{a}$. Then $\mathrm{ent}(R_p/\mathfrak{b}) = 0$ by the above argument. Since $A = R_p/\mathfrak{a}$ is a quotient of $R_p/\mathfrak{b}$ by Fact 2.5 we deduce that $\mathrm{ent}(A) = 0$ as well. □

The computation of ent for non-cyclic $R_p$-modules can be somehow reduced to the case of cyclic ones.

**Definition 4.2.** Let $R$ be a domain and $A$ be an $R$-module. Call $a \in A$ *$R$-torsion* if $\mathrm{ann}(a) = \{r \in R \colon ra = 0\} \neq 0$. Let $t_R(A)$ denote the $R$-submodule of $A$ consisting of all $R$-torsion elements of $A$. Call $A$ *$R$-torsion free* (resp., *$R$-torsion*), if $t_R(A) = 0$ (resp., $t_R(A) = A$).

Clearly, $A/t_R(A)$ is $R$-torsion free. The next result shows that we can study ent in $R_p$-torsion free modules $A$.

**Lemma 4.3.** *Let $A$ be an $R_p$-module. Then* $\mathrm{ent}(t_{R_p}(A)) = 0$ *and* $\mathrm{ent}(A) = \mathrm{ent}(A/t_{R_p}(A))$. *In particular* $\mathrm{ent}(A) = 0$ *if $A$ is $R_p$-torsion.*

*Proof.* If $a \in A$ is $R_p$-torsion, then $\mathrm{ann}(a) \neq 0$ and $aR_p \cong R_p/\mathrm{ann}(a)$. Hence $\mathrm{ent}(aR_p) = 0$ either because it is finite or by Theorem 4.1. By Fact 2.5, $\mathrm{ent}(t_{R_p}(A)) = 0$ and so $\mathrm{ent}(A) = \mathrm{ent}(A/t_{R_p}(A))$. □

**Theorem 4.4.** *For every $R_p$-module $A$,* $\mathrm{ent}(A) = \mathrm{rank}_{R_p}(A) \log p$.

*Proof.* Every $R_p$-independent subset of $A$ is contained in some maximal $R_p$-independent subset $X'$ of $A$. The submodule $A_0$ of $A$ generated by $X'$ is free and $\mathrm{rank}_{R_p}(A) = \mathrm{rank}_{R_p}(A_0) = |X'|$. Then $A/A_0$ is $R_p$-torsion, so $\mathrm{ent}(A/A_0) = 0$ by Lemma 4.3, and hence $\mathrm{ent}(A) = \mathrm{ent}(A_0)$ by Fact 2.5.

If $\mathrm{rank}_{R_p}(A)$ is infinite, then $A_0$ contains a submodule $M \cong \bigoplus_{\mathbb{N}} R_p$, and so $\mathrm{ent}(A_0) \geq \mathrm{ent}(M) = \infty$ by Fact 2.5. If $\mathrm{rank}_{R_p}(A) = t$ is finite, then $A_0 = R_p^t$, and so $\mathrm{ent}(A_0) = t\,\mathrm{ent}(R_p) = t \log p$ by Theorem 3.3. □

Combining Theorem 4.4 with Theorem 4.1 one obtains the following.

**Corollary 4.5.** *For an infinite $R_p$-module $A$,* $\mathrm{ent}(A) = 0$ *if and only if* $\mathrm{rank}_{R_p}(A) = 0$.

We aim to obtain a counterpart of Corollary 4.5 for an arbitrary torsion abelian group $A$, but the condition $\mathrm{rank}_{R_p}(A) = 0$ becomes meaningless, so we give an alternative characterization of the property $\mathrm{ent}(A) = 0$.

*Remark* 4.6. Let $A$ be an $R_p$-module, corresponding to the action $\mathbb{N}^m \overset{\alpha}{\curvearrowright} A$. According to Corollary 4.5, $\mathrm{rank}_{R_p}(A) > 0$ if and only if $A$ contains a submodule $B$ isomorphic to $R_p \cong \bigoplus_{\mathbb{N}^m} \mathbb{Z}_p$. Since the action of $\mathbb{N}^m \overset{\alpha_B}{\curvearrowright} B$ coincides with the $m$-dimensional Bernoulli shift over $\mathbb{Z}_p$ (i.e., the $m$-th Cartesian power of the usual one-dimensional Bernoulli shift – see the proof of Theorem 3.3), we shall refer to this circumstance by simply saying that $\mathbb{N}^m \overset{\alpha}{\curvearrowright} A$ "contains an $m$-dimensional Bernoulli shift over" $\mathbb{Z}_p$. In these terms, Corollary 4.5 says that *an action $\mathbb{N}^m \overset{\alpha}{\curvearrowright} A$ on an abelian group $A$ of exponent a prime $p$ has* $\mathrm{ent}(A) = 0$ *if and only if* $\mathbb{N}^m \overset{\alpha}{\curvearrowright} A$ *does not contain any $m$-dimensional Bernoulli shift over $\mathbb{Z}_p$.*

Obviously, this terminolgy can be used also when the torsion abelian group $A$ is not necessarily an $R_p$-module; in such a case, for a prime $p$, by saying that $\mathbb{N}^m \overset{\alpha}{\curvearrowright} A$ contains an $m$-dimensional Bernoulli shift over $\mathbb{Z}_p$ we mean that $A$ contains an $\alpha$-invariant subgroup $B \cong \bigoplus_{\mathbb{N}^m} \mathbb{Z}_p$ such that $\mathbb{N}^m \overset{\alpha_B}{\curvearrowright} B$ is conjugated to the $m$-dimensional Bernoulli shift over $\mathbb{Z}_p$.

By means of Theorem 2.6, we obtain the following extension.

**Proposition 4.7.** *Let $m \in \mathbb{N}_+$, let $A$ be a torsion abelian group and $\mathbb{N}^m \overset{\alpha}{\curvearrowright} A$. Then* $\mathrm{ent}(\alpha) > 0$ *if and only if there exists a prime $p$ such that $\mathbb{N}^m \overset{\alpha_{A[p]}}{\curvearrowright} A$ contains an $m$-dimensional Bernoulli shift over $\mathbb{Z}_p$.*

*Proof.* According to Theorem 2.6, $\mathrm{ent}(\alpha) > 0$ if and only if $\mathrm{ent}(\alpha_{A[p]}) > 0$ for a prime $p$. Now apply Corollary 4.5 and Remark 4.6 to $\alpha_{A[p]}$. □

The algebraic entropy of $m$ commuting endomorphisms $\phi_1, \ldots, \phi_m$ of an abelian $p$-group $A$ was already studied in [5]. Since $A$ is a module over the ring $\mathbb{J}_p$ of $p$-adic integers, one obtains also a natural structure of a $\mathbb{J}_p[X_1, \ldots, X_m]$-module on $A$. If $\mathrm{ent}(\phi_1) = \ldots = \mathrm{ent}(\phi_m) = 0$, then $\mathrm{ent}(\psi) = 0$ for every $\psi \in \mathbb{J}_p[\phi_1, \ldots, \phi_m]$ by [5, Lemma 2.5]. Let us see that $\mathrm{ent}(A) = 0$ as well. Indeed, if $\mathrm{ent}(A) > 0$, then Proposition 4.7 provides an $m$-dimensional Bernoulli shift over $\mathbb{Z}_p$ in $A$, i.e., a submodule $B \cong \mathbb{F}_p[X_1, \ldots, X_m]$. Since $B$ is $\phi_1$-invariant and $\phi_1 \restriction_B$ is conjugated to the multiplication by $X_1$, by Fact 2.5 $\mathrm{ent}(\phi_1) \geq \mathrm{ent}(\phi_1 \restriction_B) \geq \log p > 0$, a contradiction.

With a more careful housekeeping, the above argument proves that, if $m > 1$, $\mathrm{ent}(\phi_1) = \infty$ under the assumption that $\mathrm{ent}(A) > 0$ and even more. Taking for simplicity $A = B = \mathbb{J}_p[\phi_1, \ldots, \phi_m]$, then $\mathrm{ent}(\psi) = \infty$ for every endomorphism of $A$ induced by the multiplication by any polynomial $\psi \in \mathbb{J}_p[X_1, \ldots, X_m]$ of positive degree.

## 5 The algebraic receptive entropy of $\mathbb{F}_p[X_1, \ldots, X_m]$-modules

We start with the computation of $\widetilde{\mathrm{ent}}$ for cyclic $R_p$-modules, where $R_p = \mathbb{F}_p[X_1, \ldots, X_m]$ and $m > 2$.

**Theorem 5.1.** *If $m > 2$ and $A = R_p/\mathfrak{a}$ is an infinite cyclic $R_p$-module, where $\mathfrak{a} \neq 0$ is a principal ideal of $R_p$, then $\widetilde{\mathrm{ent}}(A) = \infty$.*

*Proof.* We keep the notation from the proofs of Theorems 3.3 and 4.1. In particular, $\alpha$ is the $\mathbb{N}^m$-action on $R_p$ and $\overline{\alpha}$ is the $\mathbb{N}^m$-action on $A$ determined by the $R_p$-module structure of $A$, $\mathfrak{a} = (a)$ with $d(a) = d$; for $n \in \mathbb{N}$, let $V_n = T_{\Xi_n^{(m)}}(\alpha, F)$ and so $q(V_n) = T_{\Xi_n^{(m)}}(\overline{\alpha}, q(\mathbb{F}_p))$, where $q : R_p \to A = R_p/\mathfrak{a}$ is the quotient map. By (4.2), since $m - 1 > 1$ by hypothesis, we conclude that $\widetilde{H}_{alg}(\overline{\alpha}, q(\mathbb{F}_p)) = \infty$, and so $\widetilde{\mathrm{ent}}(\overline{\alpha}) = \infty$. □

The next example shows that the conclusion of the above theorem need not be true if $\mathfrak{a}$ is not principal.

*Example* 5.2. (a) Let $m = 3$, that is, $R_p = \mathbb{F}_p[X, Y, Z]$, and let $A = R_p/\mathfrak{a} \cong \mathbb{F}_p[X]$, with $\mathfrak{a} = (Y, Z) = (Y) + (Z)$. So $A \cong \mathbb{F}_p[X]$. Denote by $\alpha$ the $\mathbb{N}^3$-action on $A$ induced by the $R_p$-module structure. This $R_p$-module induces an $A$-module structure on $A$, and we denote by $\bar{\alpha}$ the associated $\mathbb{N}$-action on $A$. Since $A$ is a quotient of $R_p$, $\bar{\alpha}$ is a quotient action of $\alpha$. Therefore,

$\widetilde{\mathrm{ent}}^{\Xi^{(3)}}(\alpha) = \widetilde{\mathrm{ent}}^{\Xi}(\bar{\alpha})$, and $\widetilde{\mathrm{ent}}^{\Xi}(\bar{\alpha}) = \log p < \infty$ since $X$ acts on $A \cong \mathbb{F}_p[X]$ as the Bernoulli shift (see [5]).

(b) Let $m > 2$ and $R_p = \mathbb{F}_p[X_1, \dots, X_m]$. Fix a positive $d < m$ and put $\mathfrak{a}_d = (X_{d+1}, \dots, X_m) = (X_{d+1}) + \dots + (X_m)$, and $A_d = R_p/\mathfrak{a}_d \cong \mathbb{F}_p[X_1, \dots, X_d]$. Denote by $\alpha$ the $\mathbb{N}^m$-action on $A_d$ induced by the $R_p$-module structure. This $R_p$-module induces an $A_d$-module structure on $A_d$, and we denote by $\bar{\alpha}$ the associated $\mathbb{N}^d$-action on $A_d$. Since $A_d$ is a quotient of $R_p$, $\bar{\alpha}$ is a quotient action of $\alpha$. Therefore $\widetilde{\mathrm{ent}}(\alpha) = \widetilde{\mathrm{ent}}(\bar{\alpha})$ (see [3]), and $\widetilde{\mathrm{ent}}(\bar{\alpha}) = \log p < \infty$ for $d = 1$ while $\widetilde{\mathrm{ent}}(\bar{\alpha}) = \infty$ for $d > 1$ by Theorem 3.3.

We recall some well-known facts regarding $\mathbb{F}_p[X_1, X_2]$ necessary for the proof of the sharper Theorem 5.5.

**Fact 5.3.** *If $R$ is a principal ideal domain, then an ideal $\mathfrak{a} \neq 0$ of $R[X]$ is prime if and only if one of the following two cases occur:*

*(a) $\mathfrak{a} = \langle f(X) \rangle$ for some irreducible element $f(X) \in R[X]$ (two cases are possible here: either $\deg f > 0$ or $f(X) = p$ for some prime $p \in R$);*

*(b) $\mathfrak{a} = \langle p, f(X) \rangle$ for some some prime $p \in R$ and $f(X) \in R[X]$ such that $\deg f > 0$ and its projection $\bar{f}(X) \in R/pR[X]$ is irreducible; in this case $\mathfrak{a}$ is a maximal ideal of $R[X]$.*

*For $R = k[Y]$ with $k$ a finite filed, the maximal ideals of $R[X]$ have finite index.*

The next theorem is focused on $\mathbb{N}^2$-actions, so now $R_p = \mathbb{F}_p[X_1, X_2]$. We recall that we always consider the regular system $\Xi^{(2)}$, so we omit to write it every time. Our aim is to compute the algebraic receptive entropy of finitely generated $\mathbb{F}_p[X_1, X_2]$-modules. To this end we start with cyclic $R_p$-modules, recalling that $\widetilde{\mathrm{ent}}(R_p) = \infty$ according to Theorem 3.3.

**Lemma 5.4.** *Let $\mathfrak{a}$ be a non-trivial ideal of $R_p$ such that $R_p/\mathfrak{a}$ is infinite and cyclic. Then there exists a principal prime ideal $\mathfrak{p}$ of $R_p$ containing $\mathfrak{a}$.*

*Proof.* We can apply Lasker-Noether Theorem to deduce that

$$\mathfrak{a} = \bigcap_{i=1}^{s} \mathfrak{q}_i, \tag{5.1}$$

where $\mathfrak{q}_i$ are primary ideals of $R_p$ for $i \in \{1, \dots, s\}$. Let $\mathfrak{p}_i = \mathrm{rad}(\mathfrak{q}_i)$, then $\mathfrak{p}_i$ is prime and clearly $\mathfrak{a} \subseteq \mathfrak{q}_i \subseteq \mathfrak{p}_i$. Since $R_p$ is Noetherian, $\mathfrak{q}_i$ is finitely generated, so there exists $k_i \in \mathbb{N}_+$ such that $\mathfrak{p}_i^{k_i} \subseteq \mathfrak{q}_i$.

Suppose that all $\mathfrak{p}_i$ are maximal. Then, using Fact 5.3, we deduce that all $R_p/\mathfrak{p}_i$ are finite. Since $\mathfrak{p}_i$ is finitely generated, $\mathfrak{p}_i/\mathfrak{p}_i^2$ is finitely generated as an $R_p/\mathfrak{p}_i$-module, hence finite. So, $R_p/\mathfrak{p}_i^2$ is finite as well. Arguing by induction, one can see that $R_p/\mathfrak{p}_i^{k_i}$ is finite. Therefore, $R_p/\mathfrak{q}_i$ is finite for all $i \in \{1, \dots, s\}$. From (5.1) we deduce that $R_p/\mathfrak{a}$ embeds into the direct product $\prod_{i=1}^{s} R_p/\mathfrak{q}_i$, that is finite. Hence $A = R_p/\mathfrak{a}$ is finite as well, a contradiction. Then at least one $\mathfrak{p}_i$ is not maximal. Being $\mathfrak{p}_i$ prime and non-maximal, according to Fact 5.3 it is principal. □

**Theorem 5.5.** *If $A = R_p/\mathfrak{a}$ is an infinite cyclic $R_p$-module, with $\mathfrak{a} \neq 0$ ideal of $R_p$, then $0 < \widetilde{\mathrm{ent}}(A) < \infty$. Moreover, when $\mathfrak{a} = (a(X_1, X_2))$ is a principal ideal, then $\widetilde{\mathrm{ent}}(A) = \deg a \cdot \log p$.*

*Proof.* First assume that $\mathfrak{a} = (a)$ is principal and let $d = d(a)$. Denote by $\alpha$ the $\mathbb{N}^2$-action on $R_p$ and $\overline{\alpha}$ the $\mathbb{N}^2$-action corresponding to the $R_p$-module structure of $A = R_p/\mathfrak{a}$, moreover let $q : R_p \to A = R_p/\mathfrak{a}$ be the quotient map. Let $F \in \mathcal{F}(A)$ and $F' \in \mathcal{F}(R_p)$ with $q(F') = F$. There exists $n_0 \in \mathbb{N}$ such that $F' \subseteq T_{\Xi^{(2)}_{n_0}}(\alpha, \mathbb{F}_p)$, so as in (4.5), $T_{\Xi^{(2)}_{n+n_0}}(\overline{\alpha}, q(\mathbb{F}_p)) \supseteq T_{\Xi^{(2)}_n}(\overline{\alpha}, F)$ for every $n \in \mathbb{N}$. Since, in the notation of Theorem 4.1, for $V_n = T_{\Xi^{(2)}_n}(\alpha, \mathbb{F}_p)$ one has $q(V_n) = T_{\Xi^{(2)}_n}(\overline{\alpha}, q(\mathbb{F}_p))$, using (4.2) and (4.5), we deduce that for every $n \in \mathbb{N}$,

$$\log |T_{\Xi^{(2)}_n}(\overline{\alpha}, F)| \leq d(n + n_0) \log p;$$

so $\widetilde{H}_{alg}(\overline{\alpha}, F) \leq d \log p$ for every $F \in \mathcal{F}(A)$, and we conclude that $\widetilde{\mathrm{ent}}(\alpha) \leq d \log p$. On the other hand, from (4.2) we get

$$\widetilde{\mathrm{ent}}(\overline{\alpha}) \geq \widetilde{H}_{alg}(\overline{\alpha}, q(\mathbb{F}_p)) = d \log p > 0.$$

If $\mathfrak{a}$ is not principal, pick a principal ideal $0 \neq \mathfrak{b} \leq \mathfrak{a}$. The above argument applied to $A' = R_p/\mathfrak{b}$ gives $\widetilde{\mathrm{ent}}(A') < \infty$. Since $A$ is isomorphic to a quotient of $A'$, by Fact 2.7 we conclude that $\widetilde{\mathrm{ent}}(A) \leq \widetilde{\mathrm{ent}}(A') < \infty$. On the other hand, by Lemma 5.4 there exists a principal ideal $\mathfrak{p}$ of $R_p$ containing $\mathfrak{a}$. So $\widetilde{\mathrm{ent}}(R_p/\mathfrak{p}) > 0$ by the previous part of the proof, and hence $\widetilde{\mathrm{ent}}(R_p/\mathfrak{a}) > \widetilde{\mathrm{ent}}(R_p/\mathfrak{p}) > 0$ by Fact 2.7. □

*Example* 5.6. Consider the ideal $\mathfrak{a} = (X_1 - X_2)$ of $R_p = \mathbb{F}_p[X_1, X_2]$. Then the actions of $X_1$ and $X_2$ on the $R_p$-module $A = R_p/\mathfrak{a}$ are the same, say $\alpha$. This means that, calling $\phi$ the multiplication by $X_1$ (or $X_2$) in $A$ and of $\alpha_\phi$ the relative $\mathbb{N}$-action on $A$, $\alpha$ coincides with the co-diagonal action action $\mathbb{N}^2 \overset{\alpha_\phi^{(2)}}{\curvearrowright} A$ of $\alpha_\phi$, defined by $\alpha_\phi^{(2)}(n,m) = \alpha_\phi(n+m) = \phi^{n+m}$ for every $n, m \in \mathbb{N}$. By Theorem 5.5, $\widetilde{\mathrm{ent}}(\alpha_\phi^{(2)}) = \widetilde{\mathrm{ent}}(A) = \log p$.

Now we describe when $\widetilde{\mathrm{ent}}(R_p/\mathfrak{a}) < \infty$ for a non-zero ideal $\mathfrak{a}$ of $R_p = \mathbb{F}_p[X_1, \dots, X_m]$.

**Corollary 5.7.** *Let $R_p = \mathbb{F}_p[X_1, \dots, X_m]$ for some $m > 1$. Then the following conditions are equivalent:*

*(a)* $m = 2$*;*

*(b) there exists a principal ideal $\mathfrak{a} \neq 0$ of $R_p$ such that $\widetilde{\mathrm{ent}}(R_p/\mathfrak{a}) < \infty$;*

*(c)* $0 < \widetilde{\mathrm{ent}}(R_p/\mathfrak{a}) < \infty$ *for every ideal $\mathfrak{a} \neq 0$ of $R_p$.*

*Proof.* (a)⇒(c) follows from Theorem 5.5, (c)⇒(b) is trivial and (b)⇒(a) follows from Theorem 5.1. □

To conclude, we obtain a complete description when $0 < \widetilde{\mathrm{ent}}(A) < \infty$ for a finitely generated $R_p$-module $A$ for $R_p = \mathbb{F}_p[X_1, X_2]$.

**Corollary 5.8.** *For an infinite finitely generated $R_p$-module $A$ the following conditions are equivalent:*

*(a)* $\mathrm{ent}(A) = 0$*;*

*(b)* $0 < \widetilde{\mathrm{ent}}(A) < \infty$*;*

*(c)* $\mathrm{rank}_{R_p}(A) = 0$.

*Proof.* (a)⇔(c) was proved in Corollary 4.5, and (b)⇒(a) follows from Remark 2.4. To prove (c)⇒(b) write $A = C_1 + \ldots + C_n$, where $C_i$ are cyclic submodules of $A$. By hypothesis, for $i \in \{1, \dots, n\}$ we can write $C_i \cong R_p/\mathfrak{a}_i$ for some ideal $\mathfrak{a}_i \neq 0$ of $R_p$. By Theorem 5.5, $\widetilde{\mathrm{ent}}(C_i) < \infty$ for $i \in \{1, \dots, n\}$. For $A' = C_1 \times \ldots \times C_n$ we have $\widetilde{\mathrm{ent}}(A') < \infty$ by Fact 2.7. As $A$ is a quotient of $A'$, we conclude that $\widetilde{\mathrm{ent}}(A) < \infty$ by Fact 2.7. At least one of the cyclic submodules $C_i$ is infinite, so $\widetilde{\mathrm{ent}}(A) \geq \widetilde{\mathrm{ent}}(C_i) > 0$ by Fact 2.7. □

*Remark* 5.9. If $S = \mathbb{F}_p[X_1, \dots, X_m]$ for some $m > 1$ and $A = S/\mathfrak{a}$ for an ideal $\mathfrak{a} \neq 0$ of $S$ contained in the maximal ideal $\mathfrak{m} = (X_1, \dots, X_m)$, we conjecture that $0 < \widetilde{\mathrm{ent}}(A) < \infty$ if and only if $\dim A = 1$, where dim denotes the Krull dimension of the quotient ring $A$.

If $\mathfrak{a}$ is principal, then $\dim S/\mathfrak{a} = m - 1$ by Krull's Principal Ideal Theorem, so this conjecture is consistent with Corollary 5.7. On the other hand, this conjecture covers also Example 5.2(a), where $\dim S/\mathfrak{a} = 1$.

We conclude with the following open problem.

**Question 5.10.** Let $p$ be a prime and $m > 1$ and integer. Is $\widetilde{\mathrm{ent}}$ a length function in the category of $\mathbb{F}_p[X_1, \dots, X_m]$-modules?

## References


[1] R. L. Adler, A. G. Konheim, M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. 114 (1965) 309–319.

[2] A. Biś, D. Dikranjan, A. Giordano Bruno, L. Stoyanov, *Topological and algebraic receptive entropy*, preprint.

[3] D. Dikranjan, A. Fornasiero, A. Giordano Bruno, *Algebraic entropy for amenable semigroup actions*, J. Algebra 556 (2020) 467–546.

[4] D. Dikranjan, A. Giordano Bruno, *Entropy on abelian groups*, Adv. Math. 298 (2016) 612–653.

[5] D. Dikranjan, B. Goldsmith, L. Salce, P. Zanardo, *Algebraic entropy for abelian groups*, Trans. Amer. Math. Soc. 361 (2009) 3401–3434.

[6] É. Ghys, R. Langevin, P. Walczak, *Entropie géométrique des feuilletages*, Acta Math. 160 (1988) 105–142.

[7] K.-H. Hofmann, L. N. Stoyanov, *Topological entropy of group and semigroup actions*, Adv. Math. 115 (1995) 54–98.

[8] D. Ornstein, B. Weiss, *Entropy and isomorphism theorems for actions of amenable groups*, J. Analyse Math. 48 (1987) 1–141.

[9] S. Virili, *Algebraic entropy of amenable group actions*, Math. Z. 291 (2019) 1389–1417.

[10] M. D. Weiss, *Algebraic and other entropies of group endomorphisms*, Math. Systems Theory 8 (3) (1974/75) 243–248.