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Algebraic entropies of commuting endomorphisms of torsion abelian groups

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Respectfully dedicated to the 95-th birthday of László Fuchs

Abstract

For actions of m commuting endomorphisms of a torsion abelian group we compute the algebraic entropy and the algebraic receptive entropy, showing that the latter one takes finite positive values in many cases when the former one vanishes.

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Keywords: algebraic entropy, receptive entropy, regular system, abelian group, torsion abelian group, p -group.

1 Introduction

A *left semigroup action* $S \overset{\alpha}{\curvearrowright} A$ of a semigroup S on an abelian group A (by group endomorphisms) is defined by $\alpha : S \times A \rightarrow A$, $(s, x) \mapsto \alpha(s)(x)$ with $\alpha(st) = \alpha(s) \circ \alpha(t)$ and $\alpha(s)(x + y) = \alpha(s)(x) + \alpha(s)(y)$ for every $s, t \in S$ and every $x, y \in A$. In case S is a monoid with neutral element e , we impose also $\alpha(e)(x) = x$ for every $x \in A$. For $N \subseteq S$ and $F \subseteq A$, let $T_N(\alpha, F) = \sum_{s \in N} \alpha(s)(F)$. Let $\mathcal{F}(A)$ denote the family of all finite subgroups of A .

We recall that a *right Følner sequence* of a semigroup S is a sequence $(F_n)_{n \in \mathbb{N}}$ of finite non-empty subsets of S such that $\lim_{n \rightarrow \infty} |F_n s \setminus F_n| / |F_n| = 0$ for every $s \in S$. A countable semigroup S is *right-amenable* if and only if S admits a right Følner sequence. Consider an action $S \overset{\alpha}{\curvearrowright} A$ of a countable cancellative right-amenable semigroup S on an abelian group A . For $F \in \mathcal{F}(A)$ let

$$H_{alg}(\alpha, F) = \lim_{n \rightarrow \infty} \frac{\log |T_{N_n}(\alpha, F)|}{|N_n|},$$

where $(N_n)_{n \in \mathbb{N}}$ is any right Følner sequence of S (the limit exists, it is finite and does not depend on the right Følner sequence). The *algebraic entropy* of α is $\text{ent}(\alpha) = \sup\{H_{alg}(\alpha, F) : F \in \mathcal{F}(A)\}$ [3]. This concept extends in a natural way the algebraic entropy ent for \mathbb{N} -actions on A , that is, for group endomorphisms of A , introduced in [1, 10] and developed in [5].

The nice properties of the algebraic entropy ent stem from the fact that the action $S \overset{\alpha}{\curvearrowright} A$ provides a left $\mathbb{Z}[S]$ -module structure on A : ent is an invariant of the category \mathfrak{T}_S of left $\mathbb{Z}[S]$ -modules that are torsion as abelian groups, and it is furthermore a length function of \mathfrak{T}_S in the sense of Northcott and Reufel, and of Vámos (see Fact 2.5). Moreover, there is a remarkable connection of ent with the topological entropy, that is, $\text{ent}(\alpha)$ coincides with the topological entropy of the dual action of α (i.e., the action induced by α of S on the Pontryagin dual group \hat{A} of A) [3, 10].

On the other hand, ent presents some shortcomings from intuitive point of view. For example, if S is finite then $\text{ent}(\alpha) = \log |A|/|S|$, so in particular $\text{ent}(\alpha) = \infty$ whenever A is infinite. Moreover, there are many cases when S is a group, T is a subgroup of S and the restricted action $\alpha \upharpoonright_T$ has $\text{ent}(\alpha \upharpoonright_T) = \infty$ while $\text{ent}(\alpha) = 0$; in case T is a normal subgroup of S always $\text{ent}(\alpha) \leq \text{ent}(\alpha \upharpoonright_T)$. If again S is a group and T is a normal subgroup of S with $T \subseteq \ker \alpha$, then $\text{ent}(\alpha) \leq \text{ent}(\bar{\alpha}_{S/T})$, where $\bar{\alpha}_{S/T}$ is the quotient action induced by α ; also in this case the inequality can be strict.

This suggested us to consider the new option for algebraic entropy offered in Definition 1.2 (see [2]), and inspired by [7], where counterparts of the bizzarre behavior of the topological entropy were pointed out. Following [7], a *regular system* of a finitely generated monoid S is a sequence

$\Gamma = (N_n)_{n \in \mathbb{N}}$ of finite subsets of S such that $N_0 = \{e\}$ and $N_i N_j \subseteq N_{i+j}$ for every $i, j \in \mathbb{N}$. In particular, $N_n \subseteq N_{n+1}$ for every $n \in \mathbb{N}$. The regular system $\Gamma = (N_n)_{n \in \mathbb{N}}$ is *exhaustive* if $S = \bigcup_{n \in \mathbb{N}} N_n$ and it is *standard* if for every $n \in \mathbb{N}$ the set $N_n = N_1 \dots N_1$ is the n -fold setwise product of N_1 , and we set $N_0 = \{e\}$, by definition.

Example 1.1. The following is the most natural example of an exhaustive standard regular system. If the monoid S is finitely generated by N_1 , then the standard regular system $(N_n)_{n \in \mathbb{N}}$ of S is exhaustive (e.g., $S = \mathbb{N}$ with $N_n = \{0, \dots, n\}$ for every $n \in \mathbb{N}$).

Definition 1.2. Let S be a finitely generated monoid, $\Gamma = (N_n)_{n \in \mathbb{N}}$ a regular system of S , A a torsion abelian group and $S \overset{\alpha}{\curvearrowright} A$. For $F \in \mathcal{F}(A)$, let

$$\widetilde{H}_{alg}^{\Gamma}(\alpha, F) = \limsup_{n \rightarrow \infty} \frac{\log |T_{N_n}(\alpha, F)|}{n}.$$

The *algebraic receptive entropy* of α with respect to Γ is

$$\widetilde{\text{ent}}^{\Gamma}(\alpha) = \sup\{\widetilde{H}_{alg}^{\Gamma}(\alpha, F) : F \in \mathcal{F}(A)\}.$$

Unlike the case of ent , it may occur that $\widetilde{H}_{alg}^{\Gamma}(\alpha, F) = \infty$ for some $F \in \mathcal{F}(A)$ (see the proof of Theorem 5.1).

Also the algebraic receptive entropy extends in a natural way the algebraic entropy ent for \mathbb{N} -actions, as we show in §2, where we recall the basic properties of the algebraic (receptive) entropy that we use further on.

In §3 we make use of the standard correspondence between \mathbb{N}^m -actions on abelian groups A of prime exponent p and R_p -module structures on A , where $R_p = \mathbb{F}_p[X_1, \dots, X_m]$ is the ring of polynomials of m variables X_1, \dots, X_m over the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. This allows us to freely pass from \mathbb{N}^m -actions to R_p -modules and viceversa, writing $\text{ent}(A)$ or $\widetilde{\text{ent}}(A)$ for an R_p -module A , having in mind the algebraic (receptive) entropy of the corresponding \mathbb{N}^m -action. As a first step we compute the algebraic (receptive) entropy of R_p (see Theorem 3.3).

In §4 we compute the algebraic entropy ent . First we see that $\text{ent}(A) = 0$ when $A = R_p/\mathfrak{a}$ is an infinite cyclic R_p -module and \mathfrak{a} is a non-zero ideal of R_p (see Theorem 4.1). Using this result we prove that $\text{ent}(A) = \text{rank}_{R_p}(A) \log p$ for an arbitrary R_p -module A (see Theorem 4.4). This allows us to obtain a characterization of \mathbb{N}^m -actions on torsion abelian groups with zero algebraic entropy via Bernoulli shifts (see Proposition 4.7).

The first step in §5 is that the receptive algebraic entropy of an infinite cyclic R_p -module $A = R_p/\mathfrak{a}$ is infinite whenever $\mathfrak{a} \neq 0$ is a principal ideal of R_p (see Theorem 5.1). Moreover, in the case $m = 2$ we compute $\widetilde{\text{ent}}(A)$ even when \mathfrak{a} is not necessarily principal (see Theorem 5.5 and Corollary 5.7). Remaining in the case $m = 2$, we describe when $0 < \widetilde{\text{ent}}(A) < \infty$ for a finitely generated R_p -module A (see Corollary 5.8).

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2 Basic properties and examples

Let S be a finitely generated monoid and $\Gamma = (N_n)_{n \in \mathbb{N}}$ a regular system of S . For $m \in \mathbb{N}_+$, the direct product S^m carries two regular systems extending Γ in a natural way:

- (i) the *Cartesian extension* of Γ is $\Gamma^m = (N_n^m)_{n \in \mathbb{N}}$;
- (ii) if N_1 is a finite set of generators of S containing e , the *minimal extension* $\Gamma^{(m)}$ of Γ is the standard regular system $\Gamma^{(m)} = (N_n^{(m)})_{n \in \mathbb{N}}$ of S^m with $N_1^{(m)} = \{(s, e, \dots, e) : s \in N_1\} \cup \dots \cup \{(e, \dots, e, s) : s \in N_1\}$.

In $S = \mathbb{N}$ consider the exhaustive standard regular system $\Xi = (\Xi_n)_{n \in \mathbb{N}}$ with $\Xi_1 = \{0, 1\}$ (so $\Xi_n = \{0, \dots, n\}$ for every $n \in \mathbb{N}$). Obviously, this is the smallest exhaustive standard regular system of \mathbb{N} .

For \mathbb{N}^m the Cartesian extension of Ξ is $\Xi^m = (\Xi_n^m)_{n \in \mathbb{N}}$ with $\Xi_1^m = \{0, 1\}^m$, and the minimal extension of Ξ is $\widetilde{\Xi}^{(m)} = (\widetilde{\Xi}_n^{(m)})_{n \in \mathbb{N}}$ with $\widetilde{\Xi}_n^{(m)} = \{(a_1, \dots, a_m) \in \mathbb{N}^m : a_1 + \dots + a_m \leq n\}$ for every $n \in \mathbb{N}$.

Remark 2.1. Let S be a finitely generated monoid, A an abelian group and $S \overset{\alpha}{\curvearrowright} A$. If H is a submonoid of S and $\Gamma = (N_n)_{n \in \mathbb{N}}$ is a regular system of H , then $\Gamma = (N_n)_{n \in \mathbb{N}}$ is a regular system of S as well, and $\widetilde{H}_{alg}^\Gamma(\alpha|_H, F) = \widetilde{H}_{alg}^\Gamma(\alpha, F)$ for every $F \in \mathcal{F}(A)$, so also $\widetilde{\text{ent}}^\Gamma(\alpha|_H) = \widetilde{\text{ent}}^\Gamma(\alpha)$.

The above notions of algebraic entropy and algebraic receptive entropy extend the usual notion of algebraic entropy of a group endomorphism:

Remark 2.2. (a) Let A be an abelian group and $\phi : A \rightarrow A$ an endomorphism. We recall from [4, 5, 10] that, for $F \in \mathcal{F}(A)$, $T_0(\phi, F) = \{0\}$ and $T_n(\phi, F) = F + \phi(F) + \dots + \phi^{n-1}(F)$ for every $n \in \mathbb{N}_+$; moreover

$$H_{alg}(\phi, F) = \lim_{n \rightarrow \infty} \frac{\log |T_n(\phi, F)|}{n},$$

and the limit exist by Fekete Lemma. The algebraic entropy from [10] is $\text{ent}(\phi) = \sup\{H_{alg}(\phi, F) : F \in \mathcal{F}(A)\}$. Consider the action $\mathbb{N} \overset{\alpha_\phi}{\curvearrowright} A$ defined by $\alpha_\phi(1) = \phi$. It is easy to see that $T_{\Xi_n}(\alpha_\phi, F) = T_{n+1}(\phi, F)$ for every $n \in \mathbb{N}$ and $F \in \mathcal{F}(A)$, and so $\text{ent}(\phi) = \widetilde{\text{ent}}^\Xi(\alpha_\phi) = \text{ent}(\alpha_\phi)$.

(b) If $\phi : A \rightarrow A$ is an isomorphism, ϕ induces also an action $\mathbb{Z} \overset{\beta_\phi}{\curvearrowright} A$ defined by $\beta_\phi(1) = \phi$. Using again the standard regular system Ξ of \mathbb{Z} , by Remark 2.1 and item (a) we find $\text{ent}(\phi) = \widetilde{\text{ent}}^\Xi(\beta_\phi)$.

Since Ξ is not exhaustive for \mathbb{Z} , it makes sense to consider the exhaustive standard regular system $\Xi' = (\Xi'_n)_{n \in \mathbb{N}}$ of \mathbb{Z} with $\Xi'_n = \{-n, \dots, n\}$ for every $n \in \mathbb{N}$. Then, for every $n \in \mathbb{N}$ and $F \in \mathcal{F}(A)$,

$$T_{\Xi'_n}(\beta_\phi, F) = \phi^{-n}(F) + \dots + F + \phi(F) + \dots + \phi^n(F) = T_{n+1}(\phi^{-1}, F) + \phi(T_n(\phi, F)).$$

Since $\phi^n(T_{\Xi'_n}(\beta_\phi, F)) = T_{2n+1}(\phi, F)$, and ϕ is bijective, we have $|T_{\Xi'_n}(\beta_\phi, F)| = |T_{2n+1}(\phi, F)|$. Therefore,

$$\begin{aligned} \widetilde{H}_{alg}^{\Xi'}(\beta_\phi, F) &= \limsup_{n \rightarrow \infty} \frac{\log |T_{\Xi'_n}(\beta_\phi, F)|}{n} = \limsup_{n \rightarrow \infty} \frac{\log |T_{2n+1}(\phi, F)|}{n} = \\ &= \lim_{n \rightarrow \infty} \frac{2n+1}{n} \frac{\log |T_{2n+1}(\phi, F)|}{2n+1} = 2H_{alg}(\phi, F). \end{aligned}$$

We conclude that $\widetilde{\text{ent}}^{\Gamma'}(\beta_\phi) = 2\text{ent}(\phi)$.

Example 2.3. Every exhaustive standard regular system $\Gamma = (N_n)_{n \in \mathbb{N}}$ of \mathbb{N}^m gives rise to a right Følner sequence of \mathbb{N}^m . The same occurs for every finitely generated monoid of subexponential growth.

(We have no example of a regular system of an amenable finitely generated group that fails to be a right Følner sequence.)

Remark 2.4. Let S be a finitely generated monoid, A an abelian group and $S \overset{\alpha}{\curvearrowright} A$. Suppose that $\Gamma = (N_n)_{n \in \mathbb{N}}$ is a regular system of S which is also a right Følner sequence of S and such that

$$|N_n| \geq cn^2 \text{ for some constant } c > 0 \text{ and all } n \in \mathbb{N}_+. \quad (2.1)$$

It follows from the definitions that, if for some $F \in \mathcal{F}(A)$ we have $\widetilde{H}_{alg}^\Gamma(\alpha, F) < \infty$, then $H_{alg}(\alpha, F) = 0$. Thus, $\widetilde{\text{ent}}^\Gamma(\alpha) < \infty \Rightarrow \text{ent}(\alpha) = 0$ (equivalently, $\text{ent}(\alpha) > 0 \Rightarrow \widetilde{\text{ent}}^\Gamma(\alpha) = \infty$).

The condition (2.1) is available for example whenever S is a finitely generated group that does not contain a cyclic subgroup of finite index.

We recall basic properties of ent that we use below. For a monoid S , abelian groups A, B and actions $S \overset{\alpha}{\curvearrowright} A, S \overset{\beta}{\curvearrowright} B$, α and β are *conjugated* by an isomorphism $\xi : A \rightarrow B$ if $\xi \circ \alpha(g) = \beta(g) \circ \xi$ for every $g \in S$.

Fact 2.5. [3] Let S be a cancellative right-amenable monoid, A a torsion abelian group and $S \overset{\alpha}{\curvearrowright} A$. Let B be an α -invariant subgroup of A and denote by α_B and $\alpha_{A/B}$ the induced actions of S on B and on A/B , respectively.

(Invariance) For $S \overset{\beta}{\curvearrowright} C$ with C a torsion abelian group, if α and β are conjugated, then $\text{ent}(\alpha) = \text{ent}(\beta)$.

(Monotonicity) $\text{ent} \geq \max\{\text{ent}(\alpha_B), \text{ent}(\alpha_{A/B})\}$.

(Continuity) If A is a direct limit of α -invariant subgroups $\{A_i : i \in I\}$, then $\text{ent}(\alpha) = \sup_{i \in I} \text{ent}(\alpha_{A_i})$.

(weak Addition Theorem) If $A = A_1 \times A_2$, with A_1, A_2 α -invariant subgroups of A , then $\text{ent}(\alpha) = \text{ent}(\alpha_{A_1}) + \text{ent}(\alpha_{A_2})$.

(Addition Theorem) $\text{ent}(\alpha) = \text{ent}(\alpha_B) + \text{ent}(\alpha_{A/B})$.

The next theorem was inspired by a similar result for \mathbb{N} -actions in [5].

Theorem 2.6. Let S be a countable cancellative right-amenable monoid, A a torsion abelian group and $S \overset{\alpha}{\curvearrowright} A$. Then $\text{ent}(\alpha) > 0$ if and only if there exists a prime p such that $\text{ent}(\alpha_{A[p]}) > 0$.

Proof. In view of the monotonicity of ent , it is enough to prove the necessity.

We consider first the case when A is a bounded p -group for some prime p ; let p^k be the exponent of A . If $k = 1$ there is nothing to prove. Assume that $k > 1$ and the assertions is true for $k - 1$. If $\text{ent}(\alpha_{A[p]}) > 0$, there is nothing to prove. Assume that $\text{ent}(\alpha_{A[p]}) = 0$. Consider the short exact sequence $0 \rightarrow A[p] \rightarrow A \rightarrow pA \rightarrow 0$. According to the Addition Theorem and the invariance, $\text{ent}(\alpha) = \text{ent}(\alpha_{pA}) + \text{ent}(\alpha_{A[p]}) = \text{ent}(\alpha_{pA})$. Since $p^{k-1}pA = 0$, we can conclude that $\text{ent}(\alpha_{(pA)[p]}) > 0$. Since $(pA)[p]$ is a subgroup of $A[p]$, monotonicity $\text{ent}(\alpha_{A[p]}) > 0$, a contradiction.

If A is a p -group for some prime p , $A = \bigcup_{n \in \mathbb{N}} A[p^n]$ and $\text{ent}(\alpha) = \sup_{n \in \mathbb{N}} \text{ent}(\alpha_{A[p^n]})$. Hence, our hypothesis $\text{ent}(\alpha) > 0$ yields $\text{ent}(\alpha_{A[p^n]}) > 0$ for some $n \in \mathbb{N}$. The above argument applied to $A[p^n]$, combined with the obvious equality $(A[p^n])[p] = A[p]$, entails $\text{ent}(\alpha_{A[p]}) > 0$.

Let $A = \bigoplus_p A_p$, where each A_p is a p -group. Then using the actions $S \overset{\alpha_{A_p}}{\curvearrowright} A_p$, one has $\text{ent}(\alpha) = \sum_p \text{ent}(\alpha_{A_p})$ by Fact 2.5. Hence, $\text{ent}(\alpha) > 0$ yields the existence of a prime p such that $\text{ent}(\alpha_{A_p}) > 0$. Now the above argument gives $\text{ent}(\alpha_{A_p[p]}) > 0$. Since $A_p[p] = A[p]$, we are done. \square

Fact 2.7. [2] Invariance, monotonicity and continuity remain valid also for the algebraic receptive entropy, while in the weak Addition Theorem only one inequality is available: if $A = A_1 \times A_2$, with A_1, A_2 α -invariant subgroups of A , then $\widetilde{\text{ent}}_\Gamma(\alpha) \leq \widetilde{\text{ent}}_\Gamma(\alpha_{A_1}) + \widetilde{\text{ent}}_\Gamma(\alpha_{A_2})$.

3 \mathbb{N}^m -actions vs $\mathbb{Z}[X_1, \dots, X_m]$ -modules

Notation 3.1. For an action $\mathbb{N}^m \overset{\alpha}{\curvearrowright} A$ on an abelian group A , from now on, we simply use the notation as follows: $\widetilde{H}_{alg} = \widetilde{H}_{alg}^{\Xi(m)}$ and $\widetilde{\text{ent}} = \widetilde{\text{ent}}^{\Xi(m)}$.

Following a well-known approach of Kaplansky, an \mathbb{N}^m -action on an abelian group A can be viewed in a standard way as an R_0 -module structure on A , where $R_0 = \mathbb{Z}[X_1, \dots, X_m]$ is the ring of polynomials of m variables X_1, \dots, X_m over \mathbb{Z} . Similarly, a \mathbb{Z}^m -action on A can be viewed as a module structure on A over the ring $\mathbb{Z}[X_1^{\pm 1}, \dots, X_m^{\pm 1}]$ of Laurent polynomials of m variables X_1, \dots, X_m over \mathbb{Z} .

In case A has a prime exponent p , one can use also the ring $R_p = \mathbb{F}_p[X_1, \dots, X_m]$ of polynomials of m variables X_1, \dots, X_m over the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, and provide as above an obvious connection between \mathbb{N}^m -actions and R_p -module structures on A . Moreover, the \mathbb{Z}^m -actions on A can be viewed as module structures on A over the ring $\mathbb{F}_p[X_1^{\pm 1}, \dots, X_m^{\pm 1}]$ of Laurent polynomials of m variables X_1, \dots, X_m over \mathbb{F}_p . In the sequel we freely pass from \mathbb{N}^m -actions to R_p -modules and viceversa, writing $\text{ent}(A)$ or $\widetilde{\text{ent}}(A)$ for an R_p -module A , having in mind the algebraic (receptive) entropy of the corresponding \mathbb{N}^m -action.

This approach is efficient in the case $m = 1$, when R_p is a principal ideal domain, so R_p -modules have a relatively simple structure and one can easily prove that $\text{ent}(A) = \widetilde{\text{ent}}(A) =$

$\text{rank}_{R_p}(A)$, where $\text{rank}_{R_p}(A)$ denotes the maximum size of a subset of A independent over R_p . This characterization is extended to the case $m > 1$ in Theorem 4.3. As a starting point, we consider the cyclic R_p -module $A = R_p$ in Theorem 3.3.

Remark 3.2. For the sake of simplicity, it makes sense to replace, whenever necessary, the additive monoid $(\mathbb{N}^m, +, 0)$ by the multiplicative submonoid $M = \{X_1^{s_1} \dots X_m^{s_m} : (s_1, \dots, s_m) \in \mathbb{N}^m\}$ of the multiplicative monoid $(R_p, \cdot, 1)$ of the ring $R_p = \mathbb{F}_p[X_1, \dots, X_m]$. For $b = X_1^{s_1} \dots X_m^{s_m} \in M$, the degree of b is $d(b) = \sum_{i=1}^m s_i$.

Substantially, for an action $\mathbb{N}^m \curvearrowright^\alpha A$ on an abelian group A of exponent a prime p , the commuting endomorphisms $\phi_i := \alpha(e_i)$ of A , where e_i is the i -th member of the canonical base of \mathbb{N}^m , make it become an R_p -module, as already explained above. Now the n -th member $\Xi_n^{(m)}$ of the standard minimal regular system $\Xi^{(m)}$ of \mathbb{N}^m obviously corresponds to

$$B_n = \{b \in M : d(b) \leq n\} \subseteq M.$$

Let

$$b_n = |B_n| = |\Xi_n^{(m)}| \quad (3.1)$$

and note that b_n coincides with the so called $n+1$ -th simplicial m -polytopic number known to be equal to the binomial coefficient C_m^{n+m} . Hence,

$$b_n = \frac{1}{m!} n^m + \frac{m+1}{2(m-1)!} n^{m-1} + \dots \quad (3.2)$$

is a polynomial of n of degree m , and so $\lim_{n \rightarrow \infty} b_n/n = \infty$ whenever $m > 1$.

Theorem 3.3. *If $m \in \mathbb{N}_+$, then $\text{ent}(R_p) = \log p$. If $m = 1$, then $\widetilde{\text{ent}}(R_p) = \text{ent}(R_p) = \log p$, otherwise $\widetilde{\text{ent}}(R_p) = \infty > \text{ent}(R_p)$.*

Proof. If $m = 1$ the equality $\widetilde{\text{ent}}(R_p) = \text{ent}(R_p)$ follows from Remark 2.2(a), while the equality $\text{ent}(R_p) = \log p$ is well known since the corresponding \mathbb{N} -action is the Bernoulli shift $(x_0, x_1, x_2, \dots) \mapsto (0, x_0, x_1, x_2, \dots)$ of $\bigoplus_{\mathbb{N}} \mathbb{Z}_p$ [5]. Assume $m > 1$ and put $F = \mathbb{F}_p \in \mathcal{F}(A)$, that is, F is the set of all p polynomials of degree 0 and the zero polynomial. By (3.1), $|\Xi_n^{(m)}| = |B_n| = b_n$. Since $T_{\Xi_n^{(m)}}(\alpha, F) =: V_n$ is a vector space and B_n is a base of V_n , then $\dim V_n = |B_n| = b_n$, so $|V_n| = p^{b_n}$ and

$$\log |V_n| = b_n \log p. \quad (3.3)$$

Since $(B_n)_{n \in \mathbb{N}}$ is a Følner sequence of R_p , we deduce that

$$H_{\text{alg}}(\alpha, F) = \lim_{n \rightarrow \infty} \frac{\log |V_n|}{|\Xi_n^{(m)}|} = \lim_{n \rightarrow \infty} \frac{b_n \log p}{b_n} = \log p.$$

If we replace F by $V_l = T_{\Xi_l^{(m)}}(\alpha, F)$ for some $l \in \mathbb{N}$, we have $T_{\Xi_n^{(m)}}(\alpha, V_l) = T_{\Xi_{n+l}^{(m)}}(\alpha, F) = V_{n+l}$, which leads to $H_{\text{alg}}(\alpha, V_l) = H_{\text{alg}}(\alpha, F) = \log p$. Since every finite subgroup of R_p is contained in some V_l , this proves $\text{ent}(R_p) = \log p$.

Since $\lim_{n \rightarrow \infty} |N_n|/n = \lim_{n \rightarrow \infty} b_n/n = \infty$ by Remark 3.2, $\widetilde{H}_{\text{alg}}(\alpha, F) = \infty$, and therefore $\widetilde{\text{ent}}(\alpha) = \infty$. \square

4 The algebraic entropy of $\mathbb{F}_p[X_1, \dots, X_m]$ -modules

Here we consider \mathbb{N}^m -actions on an abelian group A of exponent a prime p ; so let $R_p = \mathbb{F}_p[X_1, \dots, X_m]$. Moreover, we use $\Xi^{(m)}$, which is an exhaustive standard regular system of \mathbb{N}^m and also a Følner sequence. Since $\text{ent}(A) = 0$ for finite abelian groups A , we assume in the sequel that A is infinite.

Theorem 4.1. *If $A = R_p/\mathfrak{a}$ is an infinite cyclic R_p -module, where $\mathfrak{a} \neq 0$ is an ideal of R_p , then $\text{ent}(A) = 0$.*

Proof. Denote by $\bar{\alpha}$ the \mathbb{N}^m -action corresponding to the R_p -module structure of $A = R_p/\mathfrak{a}$, and let $q : R_p \rightarrow A = R_p/\mathfrak{a}$ be the quotient map. We keep the notation from the proof of Theorem 3.3. In particular, $V_n = T_{\Xi_n^{(m)}}(\alpha, F)$ is a linear subspace of R_p for $n \in \mathbb{N}$, so $q(V_n) = T_{\Xi_n^{(m)}}(\bar{\alpha}, q(\mathbb{F}_p))$

is a linear subspace of A . Since $q(V_n) \cong V_n/V_n \cap \ker q = V_n/V_n \cap \mathfrak{a}$, so $|q(V_n)| = |V_n/V_n \cap \mathfrak{a}| = |V_n|/|V_n \cap \mathfrak{a}|$, and hence

$$\log |q(V_n)| = \log |V_n| - \log |V_n \cap \mathfrak{a}|. \quad (4.1)$$

Consider the principal ideal $\mathfrak{a} = (a(X_1, \dots, X_m))$ and let d be the degree of a . Then $p(X_1, \dots, X_m) \in V_n \cap \mathfrak{a}$ precisely when $p(X_1, \dots, X_m) = r(X_1, \dots, X_m)a(X_1, \dots, X_m)$ for some $r(X_1, \dots, X_m) \in R_p$ of degree at most $n - d$, i.e., $r(X_1, \dots, X_m) \in V_{n-d}$. Since R_p is a domain, the map

$$V_{n-d} \ni r(X_1, \dots, X_m) \mapsto r(X_1, \dots, X_m)a(X_1, \dots, X_m) \in V_n \cap \mathfrak{a}$$

provides a bijection, so $|V_n \cap \mathfrak{a}| = |V_{n-d}|$. Hence, (3.2), (3.3) and (4.1) give

$$\log |q(V_n)| = (b_n - b_{n-d}) \log p = \left(\frac{d}{(m-1)!} n^{m-1} + \dots \right) \log p; \quad (4.2)$$

in particular $\log |q(V_n)|$ is a polynomial of n of degree $m-1$. This implies

$$H_{alg}(\bar{\alpha}, q(\mathbb{F}_p)) = 0. \quad (4.3)$$

For any $F' \in \mathcal{F}(R_p)$ one can find $n_0 \in \mathbb{N}$ such that $F' \subseteq T_{\Xi_{n_0}}^{(m)}(\alpha, \mathbb{F}_p)$, so

$$T_{\Xi_{n+n_0}}^{(m)}(\alpha, \mathbb{F}_p) \supseteq T_{\Xi_n}^{(m)}(\alpha, F') \text{ for every } n \in \mathbb{N}. \quad (4.4)$$

For $F^* \in \mathcal{F}(A)$ there is $F' \in \mathcal{F}(R_p)$ with $q(F') = F^*$. By (4.4), this gives

$$T_{\Xi_{n+n_0}}^{(m)}(\bar{\alpha}, q(\mathbb{F}_p)) \supseteq T_{\Xi_n}^{(m)}(\bar{\alpha}, F^*). \quad (4.5)$$

Dividing by $|\Xi_n^{(m)}|$ and using (4.3) we deduce that $H_{alg}(\bar{\alpha}, F^*) = 0$ as well. Therefore $\text{ent}(A) = 0$.

In case when \mathfrak{a} is not necessarily principal, find a principal ideal $0 \neq \mathfrak{b} \subseteq \mathfrak{a}$. Then $\text{ent}(R_p/\mathfrak{b}) = 0$ by the above argument. Since $A = R_p/\mathfrak{a}$ is a quotient of R_p/\mathfrak{b} by Fact 2.5 we deduce that $\text{ent}(A) = 0$ as well. \square

The computation of ent for non-cyclic R_p -modules can be somehow reduced to the case of cyclic ones.

Definition 4.2. Let R be a domain and A be an R -module. Call $a \in A$ *R -torsion* if $\text{ann}(a) = \{r \in R : ra = 0\} \neq 0$. Let $t_R(A)$ denote the R -submodule of A consisting of all R -torsion elements of A . Call A *R -torsion free* (resp., *R -torsion*), if $t_R(A) = 0$ (resp., $t_R(A) = A$).

Clearly, $A/t_R(A)$ is R -torsion free. The next result shows that we can study ent in R_p -torsion free modules A .

Lemma 4.3. *Let A be an R_p -module. Then $\text{ent}(t_{R_p}(A)) = 0$ and $\text{ent}(A) = \text{ent}(A/t_{R_p}(A))$. In particular $\text{ent}(A) = 0$ if A is R_p -torsion.*

Proof. If $a \in A$ is R_p -torsion, then $\text{ann}(a) \neq 0$ and $aR_p \cong R_p/\text{ann}(a)$. Hence $\text{ent}(aR_p) = 0$ either because it is finite or by Theorem 4.1. By Fact 2.5, $\text{ent}(t_{R_p}(A)) = 0$ and so $\text{ent}(A) = \text{ent}(A/t_{R_p}(A))$. \square

Theorem 4.4. *For every R_p -module A , $\text{ent}(A) = \text{rank}_{R_p}(A) \log p$.*

Proof. Every R_p -independent subset of A is contained in some maximal R_p -independent subset X' of A . The submodule A_0 of A generated by X' is free and $\text{rank}_{R_p}(A) = \text{rank}_{R_p}(A_0) = |X'|$. Then A/A_0 is R_p -torsion, so $\text{ent}(A/A_0) = 0$ by Lemma 4.3, and hence $\text{ent}(A) = \text{ent}(A_0)$ by Fact 2.5.

If $\text{rank}_{R_p}(A)$ is infinite, then A_0 contains a submodule $M \cong \bigoplus_{\mathbb{N}} R_p$, and so $\text{ent}(A_0) \geq \text{ent}(M) = \infty$ by Fact 2.5. If $\text{rank}_{R_p}(A) = t$ is finite, then $A_0 = R_p^t$, and so $\text{ent}(A_0) = t \text{ent}(R_p) = t \log p$ by Theorem 3.3. \square

Combining Theorem 4.4 with Theorem 4.1 one obtains the following.

Corollary 4.5. *For an infinite R_p -module A , $\text{ent}(A) = 0$ if and only if $\text{rank}_{R_p}(A) = 0$.*

We aim to obtain a counterpart of Corollary 4.5 for an arbitrary torsion abelian group A , but the condition $\text{rank}_{R_p}(A) = 0$ becomes meaningless, so we give an alternative characterization of the property $\text{ent}(A) = 0$.

Remark 4.6. Let A be an R_p -module, corresponding to the action $\mathbb{N}^m \overset{\alpha}{\curvearrowright} A$. According to Corollary 4.5, $\text{rank}_{R_p}(A) > 0$ if and only if A contains a submodule B isomorphic to $R_p \cong \bigoplus_{\mathbb{N}^m} \mathbb{Z}_p$. Since the action of $\mathbb{N}^m \overset{\alpha_B}{\curvearrowright} B$ coincides with the m -dimensional Bernoulli shift over \mathbb{Z}_p (i.e., the m -th Cartesian power of the usual one-dimensional Bernoulli shift – see the proof of Theorem 3.3), we shall refer to this circumstance by simply saying that $\mathbb{N}^m \overset{\alpha}{\curvearrowright} A$ “contains an m -dimensional Bernoulli shift over” \mathbb{Z}_p . In these terms, Corollary 4.5 says that *an action $\mathbb{N}^m \overset{\alpha}{\curvearrowright} A$ on an abelian group A of exponent a prime p has $\text{ent}(A) = 0$ if and only if $\mathbb{N}^m \overset{\alpha}{\curvearrowright} A$ does not contain any m -dimensional Bernoulli shift over \mathbb{Z}_p .*

Obviously, this terminology can be used also when the torsion abelian group A is not necessarily an R_p -module; in such a case, for a prime p , by saying that $\mathbb{N}^m \overset{\alpha}{\curvearrowright} A$ contains an m -dimensional Bernoulli shift over \mathbb{Z}_p we mean that A contains an α -invariant subgroup $B \cong \bigoplus_{\mathbb{N}^m} \mathbb{Z}_p$ such that $\mathbb{N}^m \overset{\alpha_B}{\curvearrowright} B$ is conjugated to the m -dimensional Bernoulli shift over \mathbb{Z}_p .

By means of Theorem 2.6, we obtain the following extension.

Proposition 4.7. *Let $m \in \mathbb{N}_+$, let A be a torsion abelian group and $\mathbb{N}^m \overset{\alpha}{\curvearrowright} A$. Then $\text{ent}(\alpha) > 0$ if and only if there exists a prime p such that $\mathbb{N}^m \overset{\alpha_{A[p]}}{\curvearrowright} A$ contains an m -dimensional Bernoulli shift over \mathbb{Z}_p .*

Proof. According to Theorem 2.6, $\text{ent}(\alpha) > 0$ if and only if $\text{ent}(\alpha_{A[p]}) > 0$ for a prime p . Now apply Corollary 4.5 and Remark 4.6 to $\alpha_{A[p]}$. \square

The algebraic entropy of m commuting endomorphisms ϕ_1, \dots, ϕ_m of an abelian p -group A was already studied in [5]. Since A is a module over the ring \mathbb{J}_p of p -adic integers, one obtains also a natural structure of a $\mathbb{J}_p[X_1, \dots, X_m]$ -module on A . If $\text{ent}(\phi_1) = \dots = \text{ent}(\phi_m) = 0$, then $\text{ent}(\psi) = 0$ for every $\psi \in \mathbb{J}_p[\phi_1, \dots, \phi_m]$ by [5, Lemma 2.5]. Let us see that $\text{ent}(A) = 0$ as well. Indeed, if $\text{ent}(A) > 0$, then Proposition 4.7 provides an m -dimensional Bernoulli shift over \mathbb{Z}_p in A , i.e., a submodule $B \cong \mathbb{F}_p[X_1, \dots, X_m]$. Since B is ϕ_1 -invariant and $\phi_1 \upharpoonright_B$ is conjugated to the multiplication by X_1 , by Fact 2.5 $\text{ent}(\phi_1) \geq \text{ent}(\phi_1 \upharpoonright_B) \geq \log p > 0$, a contradiction.

With a more careful housekeeping, the above argument proves that, if $m > 1$, $\text{ent}(\phi_1) = \infty$ under the assumption that $\text{ent}(A) > 0$ and even more. Taking for simplicity $A = B = \mathbb{J}_p[\phi_1, \dots, \phi_m]$, then $\text{ent}(\psi) = \infty$ for every endomorphism of A induced by the multiplication by any polynomial $\psi \in \mathbb{J}_p[X_1, \dots, X_m]$ of positive degree.

5 The algebraic receptive entropy of $\mathbb{F}_p[X_1, \dots, X_m]$ -modules

We start with the computation of $\widetilde{\text{ent}}$ for cyclic R_p -modules, where $R_p = \mathbb{F}_p[X_1, \dots, X_m]$ and $m > 2$.

Theorem 5.1. *If $m > 2$ and $A = R_p/\mathfrak{a}$ is an infinite cyclic R_p -module, where $\mathfrak{a} \neq 0$ is a principal ideal of R_p , then $\widetilde{\text{ent}}(A) = \infty$.*

Proof. We keep the notation from the proofs of Theorems 3.3 and 4.1. In particular, α is the \mathbb{N}^m -action on R_p and $\bar{\alpha}$ is the \mathbb{N}^m -action on A determined by the R_p -module structure of A , $\mathfrak{a} = (a)$ with $d(a) = d$; for $n \in \mathbb{N}$, let $V_n = T_{\Xi_n^{(m)}}(\alpha, F)$ and so $q(V_n) = T_{\Xi_n^{(m)}}(\bar{\alpha}, q(\mathbb{F}_p))$, where $q : R_p \rightarrow A = R_p/\mathfrak{a}$ is the quotient map. By (4.2), since $m - 1 > 1$ by hypothesis, we conclude that $\widetilde{H}_{\text{alg}}(\bar{\alpha}, q(\mathbb{F}_p)) = \infty$, and so $\widetilde{\text{ent}}(\bar{\alpha}) = \infty$. \square

The next example shows that the conclusion of the above theorem need not be true if \mathfrak{a} is not principal.

Example 5.2. (a) Let $m = 3$, that is, $R_p = \mathbb{F}_p[X, Y, Z]$, and let $A = R_p/\mathfrak{a} \cong \mathbb{F}_p[X]$, with $\mathfrak{a} = (Y, Z) = (Y) + (Z)$. So $A \cong \mathbb{F}_p[X]$. Denote by α the \mathbb{N}^3 -action on A induced by the R_p -module structure. This R_p -module induces an A -module structure on A , and we denote by $\bar{\alpha}$ the associated \mathbb{N} -action on A . Since A is a quotient of R_p , $\bar{\alpha}$ is a quotient action of α . Therefore,

$\widetilde{\text{ent}}^{\Xi^{(3)}}(\alpha) = \widetilde{\text{ent}}^{\Xi}(\bar{\alpha})$, and $\widetilde{\text{ent}}^{\Xi}(\bar{\alpha}) = \log p < \infty$ since X acts on $A \cong \mathbb{F}_p[X]$ as the Bernoulli shift (see [5]).

(b) Let $m > 2$ and $R_p = \mathbb{F}_p[X_1, \dots, X_m]$. Fix a positive $d < m$ and put $\mathfrak{a}_d = (X_{d+1}, \dots, X_m) = (X_{d+1}) + \dots + (X_m)$, and $A_d = R_p/\mathfrak{a}_d \cong \mathbb{F}_p[X_1, \dots, X_d]$. Denote by α the \mathbb{N}^m -action on A_d induced by the R_p -module structure. This R_p -module induces an A_d -module structure on A_d , and we denote by $\bar{\alpha}$ the associated \mathbb{N}^d -action on A_d . Since A_d is a quotient of R_p , $\bar{\alpha}$ is a quotient action of α . Therefore $\widetilde{\text{ent}}(\alpha) = \widetilde{\text{ent}}(\bar{\alpha})$ (see [3]), and $\widetilde{\text{ent}}(\bar{\alpha}) = \log p < \infty$ for $d = 1$ while $\widetilde{\text{ent}}(\bar{\alpha}) = \infty$ for $d > 1$ by Theorem 3.3.

We recall some well-known facts regarding $\mathbb{F}_p[X_1, X_2]$ necessary for the proof of the sharper Theorem 5.5.

Fact 5.3. *If R is a principal ideal domain, then an ideal $\mathfrak{a} \neq 0$ of $R[X]$ is prime if and only if one of the following two cases occur:*

- (a) $\mathfrak{a} = \langle f(X) \rangle$ for some irreducible element $f(X) \in R[X]$ (two cases are possible here: either $\deg f > 0$ or $f(X) = p$ for some prime $p \in R$);
- (b) $\mathfrak{a} = \langle p, f(X) \rangle$ for some prime $p \in R$ and $f(X) \in R[X]$ such that $\deg f > 0$ and its projection $\bar{f}(X) \in R/pR[X]$ is irreducible; in this case \mathfrak{a} is a maximal ideal of $R[X]$.

For $R = k[Y]$ with k a finite field, the maximal ideals of $R[X]$ have finite index.

The next theorem is focused on \mathbb{N}^2 -actions, so now $R_p = \mathbb{F}_p[X_1, X_2]$. We recall that we always consider the regular system $\Xi^{(2)}$, so we omit to write it every time. Our aim is to compute the algebraic receptive entropy of finitely generated $\mathbb{F}_p[X_1, X_2]$ -modules. To this end we start with cyclic R_p -modules, recalling that $\widetilde{\text{ent}}(R_p) = \infty$ according to Theorem 3.3.

Lemma 5.4. *Let \mathfrak{a} be a non-trivial ideal of R_p such that R_p/\mathfrak{a} is infinite and cyclic. Then there exists a principal prime ideal \mathfrak{p} of R_p containing \mathfrak{a} .*

Proof. We can apply Lasker-Noether Theorem to deduce that

$$\mathfrak{a} = \bigcap_{i=1}^s \mathfrak{q}_i, \quad (5.1)$$

where \mathfrak{q}_i are primary ideals of R_p for $i \in \{1, \dots, s\}$. Let $\mathfrak{p}_i = \text{rad}(\mathfrak{q}_i)$, then \mathfrak{p}_i is prime and clearly $\mathfrak{a} \subseteq \mathfrak{q}_i \subseteq \mathfrak{p}_i$. Since R_p is Noetherian, \mathfrak{q}_i is finitely generated, so there exists $k_i \in \mathbb{N}_+$ such that $\mathfrak{p}_i^{k_i} \subseteq \mathfrak{q}_i$.

Suppose that all \mathfrak{p}_i are maximal. Then, using Fact 5.3, we deduce that all R_p/\mathfrak{p}_i are finite. Since \mathfrak{p}_i is finitely generated, $\mathfrak{p}_i/\mathfrak{p}_i^2$ is finitely generated as an R_p/\mathfrak{p}_i -module, hence finite. So, R_p/\mathfrak{p}_i^2 is finite as well. Arguing by induction, one can see that $R_p/\mathfrak{p}_i^{k_i}$ is finite. Therefore, R_p/\mathfrak{q}_i is finite for all $i \in \{1, \dots, s\}$. From (5.1) we deduce that R_p/\mathfrak{a} embeds into the direct product $\prod_{i=1}^s R_p/\mathfrak{q}_i$, that is finite. Hence $A = R_p/\mathfrak{a}$ is finite as well, a contradiction. Then at least one \mathfrak{p}_i is not maximal. Being \mathfrak{p}_i prime and non-maximal, according to Fact 5.3 it is principal. \square

Theorem 5.5. *If $A = R_p/\mathfrak{a}$ is an infinite cyclic R_p -module, with $\mathfrak{a} \neq 0$ ideal of R_p , then $0 < \widetilde{\text{ent}}(A) < \infty$. Moreover, when $\mathfrak{a} = (a(X_1, X_2))$ is a principal ideal, then $\widetilde{\text{ent}}(A) = \deg a \cdot \log p$.*

Proof. First assume that $\mathfrak{a} = (a)$ is principal and let $d = d(a)$. Denote by α the \mathbb{N}^2 -action on R_p and $\bar{\alpha}$ the \mathbb{N}^2 -action corresponding to the R_p -module structure of $A = R_p/\mathfrak{a}$, moreover let $q : R_p \rightarrow A = R_p/\mathfrak{a}$ be the quotient map. Let $F \in \mathcal{F}(A)$ and $F' \in \mathcal{F}(R_p)$ with $q(F') = F$. There exists $n_0 \in \mathbb{N}$ such that $F' \subseteq T_{\Xi_{n_0}^{(2)}}(\alpha, \mathbb{F}_p)$, so as in (4.5), $T_{\Xi_{n+n_0}^{(2)}}(\bar{\alpha}, q(\mathbb{F}_p)) \supseteq T_{\Xi_n^{(2)}}(\bar{\alpha}, F)$ for every $n \in \mathbb{N}$. Since, in the notation of Theorem 4.1, for $V_n = T_{\Xi_n^{(2)}}(\alpha, \mathbb{F}_p)$ one has $q(V_n) = T_{\Xi_n^{(2)}}(\bar{\alpha}, q(\mathbb{F}_p))$, using (4.2) and (4.5), we deduce that for every $n \in \mathbb{N}$,

$$\log |T_{\Xi_n^{(2)}}(\bar{\alpha}, F)| \leq d(n + n_0) \log p;$$

so $\widetilde{H}_{\text{alg}}(\bar{\alpha}, F) \leq d \log p$ for every $F \in \mathcal{F}(A)$, and we conclude that $\widetilde{\text{ent}}(\alpha) \leq d \log p$. On the other hand, from (4.2) we get

$$\widetilde{\text{ent}}(\bar{\alpha}) \geq \widetilde{H}_{\text{alg}}(\bar{\alpha}, q(\mathbb{F}_p)) = d \log p > 0.$$

If \mathfrak{a} is not principal, pick a principal ideal $0 \neq \mathfrak{b} \leq \mathfrak{a}$. The above argument applied to $A' = R_p/\mathfrak{b}$ gives $\widetilde{\text{ent}}(A') < \infty$. Since A is isomorphic to a quotient of A' , by Fact 2.7 we conclude that $\widetilde{\text{ent}}(A) \leq \widetilde{\text{ent}}(A') < \infty$. On the other hand, by Lemma 5.4 there exists a principal ideal \mathfrak{p} of R_p containing \mathfrak{a} . So $\widetilde{\text{ent}}(R_p/\mathfrak{p}) > 0$ by the previous part of the proof, and hence $\widetilde{\text{ent}}(R_p/\mathfrak{a}) > \widetilde{\text{ent}}(R_p/\mathfrak{p}) > 0$ by Fact 2.7. \square

Example 5.6. Consider the ideal $\mathfrak{a} = (X_1 - X_2)$ of $R_p = \mathbb{F}_p[X_1, X_2]$. Then the actions of X_1 and X_2 on the R_p -module $A = R_p/\mathfrak{a}$ are the same, say α . This means that, calling ϕ the multiplication by X_1 (or X_2) in A and of α_ϕ the relative \mathbb{N} -action on A , α coincides with the co-diagonal action $\mathbb{N}^2 \xrightarrow{\alpha_\phi^{(2)}} A$ of α_ϕ , defined by $\alpha_\phi^{(2)}(n, m) = \alpha_\phi(n + m) = \phi^{n+m}$ for every $n, m \in \mathbb{N}$. By Theorem 5.5, $\widetilde{\text{ent}}(\alpha_\phi^{(2)}) = \widetilde{\text{ent}}(A) = \log p$.

Now we describe when $\widetilde{\text{ent}}(R_p/\mathfrak{a}) < \infty$ for a non-zero ideal \mathfrak{a} of $R_p = \mathbb{F}_p[X_1, \dots, X_m]$.

Corollary 5.7. *Let $R_p = \mathbb{F}_p[X_1, \dots, X_m]$ for some $m > 1$. Then the following conditions are equivalent:*

- (a) $m = 2$;
- (b) *there exists a principal ideal $\mathfrak{a} \neq 0$ of R_p such that $\widetilde{\text{ent}}(R_p/\mathfrak{a}) < \infty$;*
- (c) $0 < \widetilde{\text{ent}}(R_p/\mathfrak{a}) < \infty$ *for every ideal $\mathfrak{a} \neq 0$ of R_p .*

Proof. (a) \Rightarrow (c) follows from Theorem 5.5, (c) \Rightarrow (b) is trivial and (b) \Rightarrow (a) follows from Theorem 5.1. \square

To conclude, we obtain a complete description when $0 < \widetilde{\text{ent}}(A) < \infty$ for a finitely generated R_p -module A for $R_p = \mathbb{F}_p[X_1, X_2]$.

Corollary 5.8. *For an infinite finitely generated R_p -module A the following conditions are equivalent:*

- (a) $\text{ent}(A) = 0$;
- (b) $0 < \widetilde{\text{ent}}(A) < \infty$;
- (c) $\text{rank}_{R_p}(A) = 0$.

Proof. (a) \Leftrightarrow (c) was proved in Corollary 4.5, and (b) \Rightarrow (a) follows from Remark 2.4. To prove (c) \Rightarrow (b) write $A = C_1 + \dots + C_n$, where C_i are cyclic submodules of A . By hypothesis, for $i \in \{1, \dots, n\}$ we can write $C_i \cong R_p/\mathfrak{a}_i$ for some ideal $\mathfrak{a}_i \neq 0$ of R_p . By Theorem 5.5, $\widetilde{\text{ent}}(C_i) < \infty$ for $i \in \{1, \dots, n\}$. For $A' = C_1 \times \dots \times C_n$ we have $\widetilde{\text{ent}}(A') < \infty$ by Fact 2.7. As A is a quotient of A' , we conclude that $\widetilde{\text{ent}}(A) < \infty$ by Fact 2.7. At least one of the cyclic submodules C_i is infinite, so $\widetilde{\text{ent}}(A) \geq \widetilde{\text{ent}}(C_i) > 0$ by Fact 2.7. \square

Remark 5.9. If $S = \mathbb{F}_p[X_1, \dots, X_m]$ for some $m > 1$ and $A = S/\mathfrak{a}$ for an ideal $\mathfrak{a} \neq 0$ of S contained in the maximal ideal $\mathfrak{m} = (X_1, \dots, X_m)$, we conjecture that $0 < \widetilde{\text{ent}}(A) < \infty$ if and only if $\dim A = 1$, where \dim denotes the Krull dimension of the quotient ring A .

If \mathfrak{a} is principal, then $\dim S/\mathfrak{a} = m - 1$ by Krull's Principal Ideal Theorem, so this conjecture is consistent with Corollary 5.7. On the other hand, this conjecture covers also Example 5.2(a), where $\dim S/\mathfrak{a} = 1$.

We conclude with the following open problem.

Question 5.10. Let p be a prime and $m > 1$ and integer. Is $\widetilde{\text{ent}}$ a length function in the category of $\mathbb{F}_p[X_1, \dots, X_m]$ -modules?

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