

# Expressiveness of Extended Bounded Response LTL

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Extended Bounded Response LTL with Past ( $LTL_{EBR+P}$ ) is a safety fragment of Linear Temporal Logic with Past ( $LTL+P$ ) that has been recently introduced in the context of reactive synthesis. The strength of  $LTL_{EBR+P}$  is a fully symbolic compilation of formulas into symbolic deterministic automata. Its syntax is organized in four levels. The first three levels feature (a particular combination of) future temporal modalities, the last one admits only past temporal operators. At the base of such a structuring there are algorithmic motivations: each level corresponds to a step of the algorithm for the automaton construction. The complex syntax of  $LTL_{EBR+P}$  made it difficult to precisely characterize its expressive power, and to compare it with other  $LTL+P$  safety fragments.

In this paper, we first prove that  $LTL_{EBR+P}$  is expressively complete with respect to the safety fragment of  $LTL+P$ , that is, any safety language definable in  $LTL+P$  can be formalized in  $LTL_{EBR+P}$ , and vice versa. From this, it follows that  $LTL_{EBR+P}$  and Safety-LTL are expressively equivalent. Then, we show that past modalities play an essential role in  $LTL_{EBR+P}$ : we prove that the future fragment of  $LTL_{EBR+P}$  is strictly less expressive than full  $LTL_{EBR+P}$ .

## 1 Introduction

Linear Temporal Logic (LTL) was introduced in the late seventies [13] as a modal logic for reasoning over computer programs, modeling their computations as state sequences (*i.e.*, linear orders) that represent the state a computer program is in at a given time. LTL originally used temporal modalities for moving only in the future of a time point. Later, it turned out that adding modalities for moving in the past (we refer to this logic as  $LTL+P$ ) does not add expressive power to LTL [10], but only succinctness [11]. The definition of the operators in the syntax of  $LTL+P$  was proved to be carefully designed. In fact, Kamp [8] as well as Gabbay *et al.* [6] proved that the properties that one can formalize in  $LTL+P$  are exactly those definable in the first-order fragment of the *monadic second-order theory of one successor* (S1S, for short), which is in turn decidable [1, 2].

Among the different properties that one can define in  $LTL+P$ , two notable classes are the set of *safety* and *co-safety* properties. Safety properties express the intuitive requirement that *something bad never happens*, and thus each counterexample of a safety property is finite. Co-safety properties are duals of safety properties: each state sequence that satisfies the property has a finite witness. The safety and

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co-safety classes play a crucial role in verification and synthesis, since their main feature of having *finite* witnesses makes in general the problems much simpler [9, 18].

Several safety fragments of LTL have been introduced over the years. One of the most natural examples is Safety-LTL [3, 16, 18]. The Safety-LTL logic is defined as the set of all and only those formulas of LTL (with only future modalities) such that, when in negated normal form, do *not* contain existential temporal operators (like the *until* operator). In [3], Chang *et al.* proved that all the safety properties definable in LTL are expressible in Safety-LTL as well, and *vice versa*.

Extended Bounded Response LTL with Past ( $\text{LTL}_{\text{EBR}}+\text{P}$ ) is a recently introduced safety fragment of  $\text{LTL}+\text{P}$  with an efficient reactive synthesis problem. In addition to the fact that realizability from  $\text{LTL}_{\text{EBR}}+\text{P}$  specifications is EXPTIME-complete (while LTL realizability is 2EXPTIME-complete), in practice realizability and synthesis from  $\text{LTL}_{\text{EBR}}+\text{P}$  specifications turned out to be much more efficient than other approaches [4]. The syntax of  $\text{LTL}_{\text{EBR}}+\text{P}$  is articulated over layers: the first three layers comprise a combination of future temporal modalities, while the last layer includes only past temporal operators. Each of the layers was carefully designed in order to correspond to a step of the algorithm for constructing a symbolic automaton starting from an  $\text{LTL}_{\text{EBR}}+\text{P}$  specification. This results into a great performance improvement in practice, but the syntax of  $\text{LTL}_{\text{EBR}}+\text{P}$  makes it hard to find its exact expressive power, and, consequently, makes it hard also to compare it with other safety fragments of  $\text{LTL}+\text{P}$ , like, for instance, Safety-LTL.

In this paper we prove that  $\text{LTL}_{\text{EBR}}+\text{P}$  is expressively complete with respect to the safety fragment of  $\text{LTL}+\text{P}$ . As a by-product, we obtain that  $\text{LTL}_{\text{EBR}}+\text{P}$  and Safety-LTL are expressively equivalent. The core of the proof exploits a *normal form theorem* for each safety property definable in  $\text{LTL}+\text{P}$  [3, 17], which establishes a correspondence between safety properties definable in  $\text{LTL}+\text{P}$  and properties of the form  $G\alpha$ , where  $G$  is the *globally* operator of LTL and  $\alpha$  is a pure past formula. Consequently, it is clear that the *pure past layer* of  $\text{LTL}_{\text{EBR}}+\text{P}$  plays a crucial role for the expressive equivalence of  $\text{LTL}_{\text{EBR}}+\text{P}$ . We show that this layer is really necessary. In fact, we prove that  $\text{LTL}_{\text{EBR}}$ , that is  $\text{LTL}_{\text{EBR}}+\text{P}$  devoid of the pure past layer, is *strictly less* expressive than full  $\text{LTL}_{\text{EBR}}+\text{P}$ . This is shown by proving that all the formulas of  $\text{LTL}_{\text{EBR}}$  can constrain, for any time point  $i$  in an infinite state sequence, only a *bounded* prefix before (or interval around)  $i$ . This implies that formulas that are able to constrain, for each time point  $i$ , a prefix of unbounded (although finite) length before  $i$ , like for instance  $G(p_1 \rightarrow Hp_2)$  (where  $H$  is the *historically* past operator of  $\text{LTL}+\text{P}$ ), are *not* definable in  $\text{LTL}_{\text{EBR}}$ .

The rest of the paper is organized as follows. In Section 2, we give the necessary background. The expressive power of  $\text{LTL}_{\text{EBR}}+\text{P}$  is proved in Section 3. In Section 4, we prove that the future fragment of  $\text{LTL}_{\text{EBR}}+\text{P}$  is strictly less expressive than  $\text{LTL}_{\text{EBR}}+\text{P}$ . Finally, we summarize the results of the paper in Section 5.

## 2 Preliminaries

In this section, we give the definitions that are necessary throughout the paper.

### 2.1 Linear Temporal Logic

Linear Temporal Logic (LTL) is a modal logic interpreted over infinite, discrete linear orders [5, 13]. Syntactically, LTL can be seen as an extension of propositional logic with the addition of the *next* operator ( $X\phi$ , *i.e.*, at the *next* state  $\phi$  holds) and the *until* operator ( $\phi_1 \text{ U } \phi_2$ , *i.e.*,  $\phi_2$  will eventually hold and  $\phi_1$  will hold *until* then).

LTL *with Past* (LTL+P) extends LTL with the addition of temporal operators able to talk about what happened in the *past* with respect to the current time, and it is obtained from LTL by adding the following *past* temporal operators: (i) the *yesterday* operator ( $Y\phi$ , *i.e.*, there exists a *previous* state in which  $\phi$  holds); (ii) the *weak yesterday* operator ( $Z\phi$ , *i.e.*, either a previous state does not exist or in the previous state  $\phi$  holds); (iii) and the *since* operator ( $\phi_1 S \phi_2$ , *i.e.*, there was a past state where  $\phi_2$  held, and  $\phi_1$  has held *since* then). We will now briefly recall the syntax and semantics of LTL+P, which encompasses that of LTL as well. Formally, given a set  $\Sigma$  of proposition letters, LTL+P formulas over  $\Sigma$  are generated by the following grammar:

$$\begin{array}{ll}
 \phi := p \mid \neg\phi \mid \phi_1 \vee \phi_2 \mid \phi_1 \wedge \phi_2 & \text{propositional connectives} \\
 \mid X\phi_1 \mid \phi_1 U \phi_2 \mid \phi_1 R \phi_2 \mid F\phi_1 \mid G\phi_1 & \text{future temporal operators} \\
 \mid Y\phi_1 \mid \phi_1 S \phi_2 \mid \phi_1 T \phi_2 \mid O\phi_1 \mid H\phi_1 \mid Z\phi_1 & \text{past temporal operators}
 \end{array}$$

where  $p \in \Sigma$  and  $\phi_1$  and  $\phi_2$  are LTL+P formulas. Most of the temporal operators of the language can be defined in terms of a small number of basic ones. We refer to [4] for the definition of these shortcuts. We say that an LTL+P formula is *pure past* if and only if all the temporal operators inside the formula are past operators. We call *pure past* LTL, written as  $\text{LTL}_P$ , the fragment of LTL+P containing only pure past formulas.

Formulas from LTL+P are interpreted over *state sequences*. A *state sequence*  $\sigma = \langle \sigma_0, \sigma_1, \dots \rangle$  is an *infinite*, linearly ordered sequence of *states*, where each state  $\sigma_i$  is a set of proposition letters, that is  $\sigma_i \in 2^\Sigma$  for  $i \in \mathbb{N}$ . We will interchangeably use also the term  $\omega$ -word over the alphabet  $2^\Sigma$  for referring to a state sequence. A set of  $\omega$ -words is called  $\omega$ -language. Given two indices  $i, j \in \mathbb{Z}$ , with  $i \leq j$ , we denote as  $\sigma_{[i,j]}$  the interval of  $\sigma$  from index  $i$  to index  $j$ , that is  $\langle \sigma_i, \dots, \sigma_j \rangle$  if  $i \geq 0$ , or  $\langle \sigma_0, \dots, \sigma_j \rangle$  otherwise. With  $\sigma_{[i,\infty]}$  we denote the (infinite) suffix of  $\sigma$  starting from  $i$ .

Given a state sequence  $\sigma$ , a position  $i \geq 0$ , and an LTL+P formula  $\phi$ , we inductively define the *satisfaction* of  $\phi$  by  $\sigma$  at position  $i$ , written as  $\sigma, i \models \phi$ , as follows:

1.  $\sigma, i \models p$       iff  $p \in \sigma_i$ ;
2.  $\sigma, i \models \neg\phi$       iff  $\sigma, i \not\models \phi$ ;
3.  $\sigma, i \models \phi_1 \vee \phi_2$       iff  $\sigma, i \models \phi_1$  or  $\sigma, i \models \phi_2$ ;
4.  $\sigma, i \models X\phi$       iff  $\sigma, i+1 \models \phi$ ;
5.  $\sigma, i \models Y\phi$       iff  $i > 0$  and  $\sigma, i-1 \models \phi$ ;
6.  $\sigma, i \models \phi_1 U \phi_2$       iff there exists  $j \geq i$  such that  $\sigma, j \models \phi_2$ ,  
and  $\sigma, k \models \phi_1$  for all  $k$ , with  $i \leq k < j$ ;
7.  $\sigma, i \models \phi_1 S \phi_2$       iff there exists  $j \leq i$  such that  $\sigma, j \models \phi_2$ ,  
and  $\sigma, k \models \phi_1$  for all  $k$ , with  $j < k \leq i$ ;

We say that  $\sigma$  *satisfies*  $\phi$ , written as  $\sigma \models \phi$ , if it satisfies the formula at the first state, *i.e.*, if  $\sigma, 0 \models \phi$ : in this case, we call  $\sigma$  a *model* of  $\phi$ . We say that two formulas  $\phi$  and  $\psi$  are *equivalent* ( $\phi \equiv \psi$ ) if and only if they are satisfied by the same set of state sequences.

If  $\phi$  is a full LTL+P formula, then we define the *language* of  $\phi$ , written  $\mathcal{L}(\phi)$ , as  $\mathcal{L}(\phi) = \{\sigma \in (2^\Sigma)^\omega \mid \sigma \models \phi\}$ . If, instead,  $\phi$  contains only past operators, we change the definition of *language* as follows: for all  $\phi \in \text{LTL}_P$ , we define the language over *finite words* of  $\phi$  as  $\mathcal{L}^{<\omega}(\phi) := \{\sigma \in (2^\Sigma)^* \mid \sigma = \langle \sigma_0, \dots, \sigma_n \rangle \wedge \sigma, n \models \phi\}$ .

**Notation** From now on, given a linear temporal logic  $\mathbb{L}$ , with some abuse of notation, we will denote with  $\mathbb{L}$  also the set of formulas that *syntactically* belong to  $\mathbb{L}$ . Conversely, we denote with  $\llbracket \mathbb{L} \rrbracket$  the set of all and only those languages  $\mathcal{L}$  of infinite words for which there exists a formula  $\phi \in \mathbb{L}$  (i.e.,  $\phi$  syntactically belongs to  $\mathbb{L}$ ) such that  $\mathcal{L} = \mathcal{L}(\phi)$ . For the  $\text{LTL}_P$  logic, we write  $\llbracket \text{LTL}_P \rrbracket^{<\omega}$  for denoting the set of languages  $\mathcal{L}$  over *finite* words such that  $\mathcal{L} = \mathcal{L}^{<\omega}(\phi)$  for some  $\phi \in \text{LTL}_P$ .

It is known that past modalities do *not* add expressive power to LTL [6, 10, 11], therefore writing  $\llbracket \text{LTL} \rrbracket$  is the same as writing  $\llbracket \text{LTL}+P \rrbracket$ .

## 2.2 $\omega$ -regular expressions and (co-)Safety classes

We denote as REG the set of *regular languages* of finite words [7]. An  $\omega$ -regular language is a set of  $\omega$ -words recognized by an  $\omega$ -regular expression, that is, an expression of the form  $\bigcup_{i=1}^n U_i \cdot (V_i)^\omega$ , where  $n \in \mathbb{N}$  and  $U_i, V_i \in \text{REG}$  for  $i = 1, \dots, n$ . With  $\omega\text{-REG}$ , we denote the set of the  $\omega$ -regular languages. One of the seminal results in automata theory is the correspondence between  $\omega$ -regular languages and Büchi automata [1, 2]. An important class of  $\omega$ -regular languages comprises those languages that express the fact that something “bad” (like for instance a deadlock, or a simultaneous access into a critical section by two different processes) never happens. For this reason, they are called *safety languages* (or *safety properties*).

**Definition 1** (Safety language [9]). *Let  $\mathcal{L} \subseteq \Sigma^\omega$  be an  $\omega$ -regular language. We say that  $\mathcal{L}$  is a safety language if and only if for all the words  $\sigma \in \Sigma^\omega$  it holds that, if  $\sigma \notin \mathcal{L}$ , then  $\exists i \in \mathbb{N} \cdot \forall \sigma' \in \Sigma^\omega \cdot \sigma_{[0,i]} \cdot \sigma' \notin \mathcal{L}$ . The class of safety  $\omega$ -regular languages is denoted as SAFETY.*

Given some temporal logic  $\mathbb{L}$ , we say that  $\mathbb{L}$  is a *safety fragment* of LTL iff  $\phi \in \mathbb{L}$  implies that  $\phi \in \text{LTL}$ , and  $\mathcal{L}(\phi)$  is a safety language (Def. 1), for all formulas  $\phi$ . The class of the  $\omega$ -regular *co-safety* languages, that we call  $\text{coSAFETY}$ , is defined as the dual of SAFETY, that is the set of languages  $\mathcal{L}$  such that  $\mathcal{L} \in \text{coSAFETY}$  iff  $\overline{\mathcal{L}} \in \text{SAFETY}$ , where  $\overline{\mathcal{L}}$  is the complement language of  $\mathcal{L}$ .

The Safety-LTL logic [3, 16, 18] is defined as the set of LTL formulas such that, when in negated normal form, do *not* contain existential temporal operators (i.e., U and F). Safety-LTL is a *safety fragment* of LTL [16].

We give an alternative and equivalent definition of the SAFETY class of Def. 1, that will be useful in the following sections:  $\text{SAFETY} := \{\mathcal{L} \subseteq \Sigma^\omega \mid \overline{\mathcal{L}} = K \cdot \Sigma^\omega \wedge K \in \text{REG}\}$ .

We define the class  $\text{SAFETY}^{\text{SF}}$  ( $\text{coSAFETY}^{\text{SF}}$ ) as the set obtained from SAFETY (resp.  $\text{coSAFETY}$ ) by restricting  $K$  to be a *star-free* expression, that is, a regular expression devoid of the Kleene star [12]. In particular,  $\text{coSAFETY}^{\text{SF}} := \{\mathcal{L} \subseteq \Sigma^\omega \mid \mathcal{L} = K \cdot \Sigma^\omega \wedge K \in \text{SF}\}$ , where  $\text{SF} \subseteq \text{REG}$  is the set of star-free regular expressions. With  $\omega\text{-SF}$  we denote the set of star-free  $\omega$ -regular expressions. We now state some equivalence results that will be helpful later. Star-free expressions (SF) and pure-past LTL ( $\text{LTL}_P$ ) have the same expressive power. The same holds for the  $\omega\text{-SF}$  class and LTL.

**Proposition 1** (Thomas [17], Lichtenstein *et al.* [10]).  $\llbracket \text{LTL}_P \rrbracket^{<\omega} = \text{SF}$  and  $\llbracket \text{LTL} \rrbracket = \omega\text{-SF}$ .

Finally, we will use the following normal-form theorem, stated in [3], that proves that any LTL-definable safety (resp. co-safety) language can be expressed by a formula of the form  $G\alpha$  (resp.  $F\alpha$ ), and *vice versa*. An independent proof of this theorem can be derived also from the results by Thomas in [17].

**Theorem 1** (Chang *et al.* [3]).  $\llbracket \text{LTL} \rrbracket \cap \text{SAFETY} = \llbracket G\alpha \rrbracket$  and  $\llbracket \text{LTL} \rrbracket \cap \text{coSAFETY} = \llbracket F\alpha \rrbracket$ .

Fig. 1 summarizes the expressive power of the various fragments and logics, included  $\text{LTL}_{\text{EBR}+P}$  and  $\text{LTL}_{\text{EBR}}$  (that are the subject of this paper).

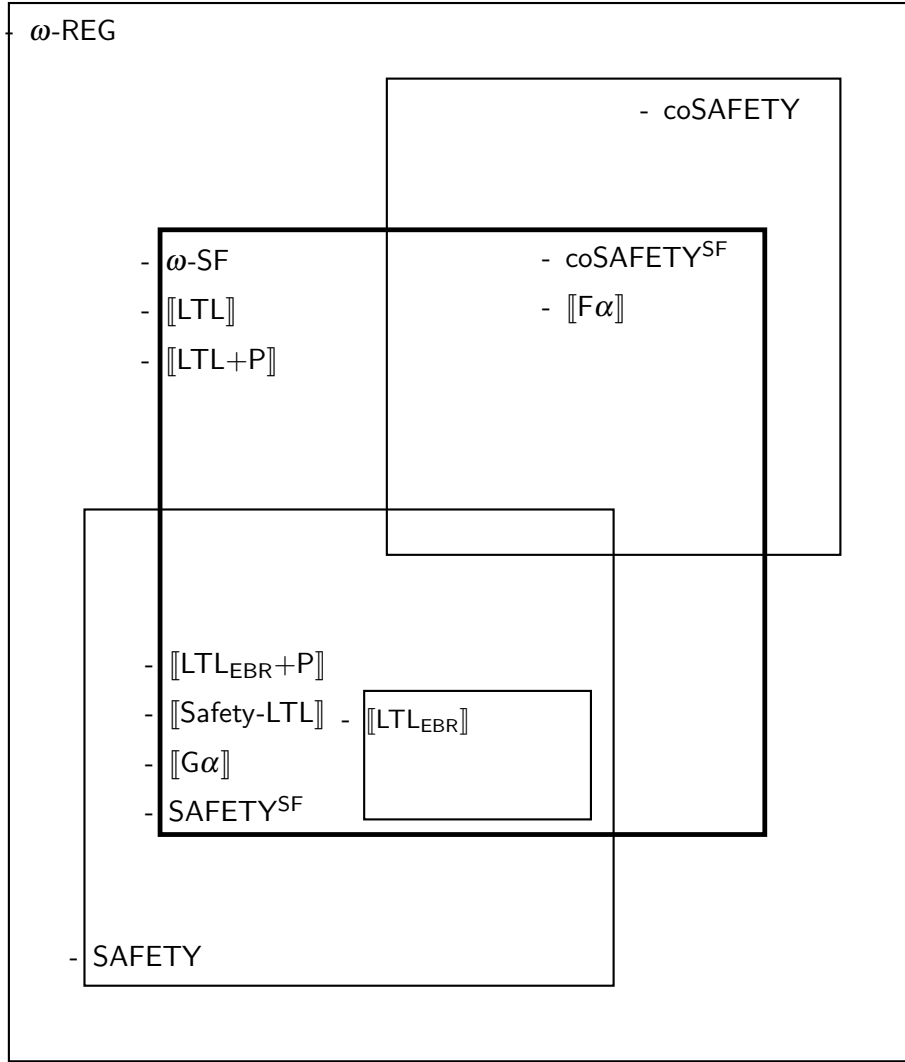


Figure 1: Comparison of expressiveness between the various formalisms. For ease of exposition, we highlighted the rectangle corresponding to LTL with thick borders.

### 2.3 Extended Bounded Response LTL

*Extended Bounded Response LTL with Past* ( $LTL_{EBR+P}$ , for short) is a fragment of  $LTL+P$ , recently introduced in the context of reactive synthesis [4]. Here below, we recall its syntax.

**Definition 2** (The logic  $LTL_{EBR+P}$  [4]). *Let  $a, b \in \mathbb{N}$ . An  $LTL_{EBR+P}$  formula  $\chi$  is inductively defined as follows:*

$\eta := p \mid \neg \eta \mid \eta_1 \vee \eta_2 \mid \Upsilon \eta \mid \eta_1 S \eta_2$	<i>Pure Past Layer</i>
$\psi := \eta \mid \neg \psi \mid \psi_1 \vee \psi_2 \mid X\psi \mid \psi_1 U^{[a,b]} \psi_2$	<i>Bounded Future Layer</i>
$\phi := \psi \mid \phi_1 \wedge \phi_2 \mid X\phi \mid G\phi \mid \psi R \phi$	<i>Future Layer</i>
$\chi := \phi \mid \chi_1 \vee \chi_2 \mid \chi_1 \wedge \chi_2$	<i>Boolean Layer</i>

We define the *bounded until* operator  $\psi_1 U^{[a,b]} \psi_2$  as a shortcut for the LTL formula  $\bigvee_{i=a}^b (X_1 \dots X_i(\psi_2) \wedge \bigwedge_{j=0}^{i-1} X_1 \dots X_j(\psi_1))$ . This means that  $\text{LTL}_{\text{EBR}} + \text{P}$  features really only universal temporal modalities (*i.e.*, X, G, and R), and thus it is a syntactical fragment of  $\text{LTL} + \text{P}$  and also a *safety* fragment (see Theorem 3.1 in [16]). We define  $\text{LTL}_{\text{EBR}}$  as the fragment of  $\text{LTL}_{\text{EBR}} + \text{P}$  devoid of the full past layer. The syntax of  $\text{LTL}_{\text{EBR}} + \text{P}$  is articulated over layers, that impose some syntactical restrictions on the formulas that can be generated from the grammar. For example,  $\text{LTL}_{\text{EBR}} + \text{P}$  forces the leftmost argument of any *release* operator to contain no universal temporal modalities (*i.e.*, R and G). Originally, the layered structure was guided by the steps of the algorithm for the construction of symbolic automata starting from  $\text{LTL}_{\text{EBR}} + \text{P}$ -formulas. We refer the reader to [4] for more details.

All formulas in  $\text{LTL}_{\text{EBR}} + \text{P}$  can be transformed into a *canonical form* (defined here below) by maintaining the equivalence.

**Definition 3** (Canonical Form of  $\text{LTL}_{\text{EBR}} + \text{P}$  [4]). *The canonical form of  $\text{LTL}_{\text{EBR}} + \text{P}$  is the set of all and only the formulas of the following type:*

$$\begin{aligned} & X^{i_1} \alpha_{i_1} \otimes \dots \otimes X^{i_j} \alpha_{i_j} \otimes \\ & X^{i_{j+1}} G \alpha_{i_{j+1}} \otimes \dots \otimes X^{i_k} G \alpha_{i_k} \otimes \\ & X^{i_{k+1}} (\alpha_{i_{k+1}} R \beta_{i_{k+1}}) \otimes \dots \otimes X^{i_h} (\alpha_{i_h} R \beta_{i_h}) \end{aligned}$$

where each  $\alpha_i, \beta_i \in \text{LTL}_{\text{P}}$ ,  $\otimes \in \{\wedge, \vee\}$ , and  $i, j, k, h \in \mathbb{N}$ .

### 3 Expressive power of $\text{LTL}_{\text{EBR}} + \text{P}$

In this section, we study the expressiveness of the  $\text{LTL}_{\text{EBR}} + \text{P}$  logic. In particular, we compare the set of languages definable in  $\text{LTL}_{\text{EBR}} + \text{P}$  with the set of safety languages expressible in LTL, and prove that the two sets are equal, that is  $\llbracket \text{LTL}_{\text{EBR}} + \text{P} \rrbracket = \llbracket \text{LTL} \rrbracket \cap \text{SAFETY}$ . Consequently,  $\text{LTL}_{\text{EBR}} + \text{P}$  and Safety-LTL are expressively equivalent (*i.e.*,  $\llbracket \text{LTL}_{\text{EBR}} + \text{P} \rrbracket = \llbracket \text{Safety-LTL} \rrbracket$ ).

First we recall the normal-form theorem stated in Th. 1, establishing that  $\llbracket \text{LTL} \rrbracket \cap \text{SAFETY} = \llbracket G\alpha \rrbracket$ . Proving that  $\llbracket \text{LTL}_{\text{EBR}} + \text{P} \rrbracket = \llbracket \text{LTL} \rrbracket \cap \text{SAFETY}$  is straightforward. In [16], Sistla proved that any fragment of  $\text{LTL} + \text{P}$  with only universal (future) temporal operators (*i.e.*, X, R, and G) defines only safety properties, and thus is a safety fragment of  $\text{LTL} + \text{P}$ . Since  $\text{LTL}_{\text{EBR}} + \text{P}$ -formulas contain only universal (future) temporal operators, it follows that  $\text{LTL}_{\text{EBR}} + \text{P}$  is a safety fragment of  $\text{LTL} + \text{P}$  (this corresponds to the left-to-right direction). For the right-to-left direction it suffices to show that the normal form  $G\alpha$  is syntactically definable in  $\text{LTL}_{\text{EBR}} + \text{P}$  (*i.e.*,  $G\alpha \in \text{LTL}_{\text{EBR}} + \text{P}$  and thus also  $\mathcal{L}(G\alpha) \in \llbracket \text{LTL}_{\text{EBR}} + \text{P} \rrbracket$ , for any  $\alpha \in \text{LTL}_{\text{P}}$ ).

**Theorem 2.**  $\llbracket \text{LTL}_{\text{EBR}} + \text{P} \rrbracket = \llbracket \text{LTL} \rrbracket \cap \text{SAFETY}$ .

*Proof.* We first prove that  $\llbracket \text{LTL}_{\text{EBR}} + \text{P} \rrbracket \subseteq \llbracket \text{LTL} \rrbracket \cap \text{SAFETY}$ . Let  $\phi \in \llbracket \text{LTL}_{\text{EBR}} + \text{P} \rrbracket$ . By Def. 2,  $\phi \in \text{LTL} + \text{P}$ , and thus, since  $\llbracket \text{LTL} \rrbracket = \llbracket \text{LTL} + \text{P} \rrbracket$ , it holds that  $\mathcal{L}(\phi) \in \llbracket \text{LTL} \rrbracket$ . Moreover, since  $\text{LTL}_{\text{EBR}} + \text{P}$  contains only universal temporal operators, by Theorem 3.1 in [16], it is a *safety* fragment of LTL, and we have that  $\mathcal{L}(\phi) \in \text{SAFETY}$ . Therefore,  $\mathcal{L}(\phi) \in \llbracket \text{LTL} \rrbracket \cap \text{SAFETY}$ .

We now prove that  $\llbracket \text{LTL} \rrbracket \cap \text{SAFETY} \subseteq \llbracket \text{LTL}_{\text{EBR}} + \text{P} \rrbracket$ . Let  $\phi$  be a formula such that  $\mathcal{L}(\phi) \in \llbracket \text{LTL} \rrbracket \cap \text{SAFETY}$ . By Th. 1,  $\mathcal{L}(\phi) \in \llbracket G\alpha \rrbracket$ . Now,  $G\alpha$  (for any  $\alpha \in \text{LTL}_{\text{P}}$ ) is a formula that syntactically belongs to  $\text{LTL}_{\text{EBR}} + \text{P}$ , that is  $G\alpha \in \text{LTL}_{\text{EBR}} + \text{P}$ , and thus  $\llbracket G\alpha \rrbracket \subseteq \llbracket \text{LTL}_{\text{EBR}} + \text{P} \rrbracket$ . It follows that  $\mathcal{L}(\phi) \in \llbracket \text{LTL}_{\text{EBR}} + \text{P} \rrbracket$ .  $\square$

### 3.1 Comparison between $\text{LTL}_{\text{EBR}+P}$ , $G\alpha$ and Safety-LTL

**Comparison with  $G\alpha$**  Previously, we proved that the set of languages definable in  $\text{LTL}_{\text{EBR}+P}$  is exactly the set of safety languages definable in  $\text{LTL}+P$ . In turn, Th. 1 shows that these sets correspond to languages definable by a formula of type  $G\alpha$ , where  $\alpha \in \text{LTL}_P$ . Despite being equivalent fragments, we think that  $\text{LTL}_{\text{EBR}+P}$  offers a more natural language for safety properties than the  $G\alpha$  fragment. Consider for example the following property, expressed in natural language: either  $p_3$  holds forever, or there exists two time points  $t' \leq t$  such that (i)  $p_1$  holds in  $t$ , (ii)  $p_2$  holds in  $t'$ , and (iii)  $p_2$  holds from time point 0 to  $t$ . The property can be easily formalized in  $\text{LTL}_{\text{EBR}+P}$  by the formula  $p_1 R(p_2 R p_3)$ . The equivalent formula in the  $G\alpha$  fragment is  $G(H(p_3) \vee O(p_2 \wedge O(p_1) \wedge H(p_3)))$ , which is arguably more intricate.

**Comparison with Safety-LTL** Safety-LTL is the fragment of LTL (thus with only future temporal modalities) containing all and only the LTL-formulas that, when in negated normal form, do *not* contain any *until* or *eventually* operator. In [16], Sistla proved that this fragment expresses only safety properties, that is  $\llbracket \text{Safety-LTL} \rrbracket \subseteq \llbracket \text{LTL} \rrbracket \cap \text{SAFETY}$ . The converse direction, that is  $\llbracket \text{LTL} \rrbracket \cap \text{SAFETY} \subseteq \llbracket \text{Safety-LTL} \rrbracket$ , is reported in [3]. It immediately follows that  $\text{LTL}_{\text{EBR}+P}$  and Safety-LTL are expressively equivalent, namely  $\llbracket \text{LTL}_{\text{EBR}+P} \rrbracket = \llbracket \text{Safety-LTL} \rrbracket$ .

Differently from  $\text{LTL}_{\text{EBR}+P}$ , Safety-LTL does *not* impose any syntactic restriction on the nesting of the logical operators; as a matter of fact,  $G(p_1 \vee Gp_2)$  belongs to the syntax of Safety-LTL but not to the syntax of  $\text{LTL}_{\text{EBR}+P}$ , even though  $G(p_1 \vee Gp_2) \equiv G(\neg p_2 \rightarrow Hp_1) \in \text{LTL}_{\text{EBR}+P}$ . The restrictions on the syntax of  $\text{LTL}_{\text{EBR}+P}$  are due to algorithmic aspects: each layer of the syntax of  $\text{LTL}_{\text{EBR}+P}$  (recall Def. 2) corresponds to a step of the algorithm for the symbolic automata construction starting from  $\text{LTL}_{\text{EBR}+P}$ -formulas. As a matter of fact, in practice,  $\text{LTL}_{\text{EBR}+P}$  has shown to avoid an exponential blowup in time with respect to known algorithms for automata construction for safety specifications [4]. Last but not least, the realizability problem of  $\text{LTL}_{\text{EBR}+P}$  is EXPTIME-complete [4], as opposed to the realizability of  $\text{LTL}+P$ , which is 2EXPTIME-complete [14, 15]. Consider now  $\text{LTL}_{\text{EBR}}$ , that is the fragment of  $\text{LTL}_{\text{EBR}+P}$  devoid of past operators. Since each formula of  $\text{LTL}_{\text{EBR}}$  syntactically belongs to Safety-LTL, it immediately follows that  $\llbracket \text{LTL}_{\text{EBR}} \rrbracket \subseteq \llbracket \text{Safety-LTL} \rrbracket$ . In the next section, we will prove that the converse direction does *not* hold, that is  $\text{LTL}_{\text{EBR}}$  is *strictly* less expressive than  $\text{LTL}_{\text{EBR}+P}$ , and thus less expressive than Safety-LTL as well.

## 4 $\text{LTL}_{\text{EBR}}$ is strictly less expressive than full $\text{LTL}_{\text{EBR}+P}$

In the previous sections, we have seen that:

$$\llbracket \text{LTL}_{\text{EBR}+P} \rrbracket = \llbracket G\alpha \rrbracket = \llbracket \text{LTL} \rrbracket \cap \text{SAFETY} = \llbracket \text{Safety-LTL} \rrbracket$$

In particular, thanks to the use of the *pure past layer* (recall Def. 2),  $\text{LTL}_{\text{EBR}+P}$  can easily capture the whole class of  $\llbracket G\alpha \rrbracket$ , and thus the whole class of  $\llbracket \text{LTL} \rrbracket \cap \text{SAFETY}$ . However, one may wonder whether the pure past layer is really necessary, or whether the class  $\llbracket G\alpha \rrbracket$  can be expressed in  $\text{LTL}_{\text{EBR}+P}$  without the use of past operators.

$\text{LTL}_{\text{EBR}}$  is defined as the fragment of  $\text{LTL}_{\text{EBR}+P}$  devoid of the pure past layer (recall Section 2.3). In this section, we investigate the problem of establishing whether  $\text{LTL}_{\text{EBR}}$  has the same expressive power of  $\text{LTL}_{\text{EBR}+P}$ , or equivalently, whether  $\text{LTL}_{\text{EBR}}$  can express every language in  $\llbracket \text{Safety-LTL} \rrbracket$ . We will

prove that this is *not* the case, that is

$$\llbracket \text{LTL}_{\text{EBR}} \rrbracket \subsetneq \llbracket \text{LTL}_{\text{EBR}} + \text{P} \rrbracket \quad (1)$$

This result proves that *past modalities*, although being not important for the expressiveness of full LTL (since  $\llbracket \text{LTL} \rrbracket = \llbracket \text{LTL} + \text{P} \rrbracket$  [6, 10, 11]), can play a crucial role for the expressive power of *fragments* of LTL, like, for instance,  $\text{LTL}_{\text{EBR}}$ .

#### 4.1 The general idea

We will prove Eq. (1) by showing that  $\llbracket \text{LTL}_{\text{EBR}} \rrbracket \subsetneq \llbracket \text{Safety-LTL} \rrbracket$ . The result in Eq. (1) follows from the fact that  $\llbracket \text{Safety-LTL} \rrbracket = \llbracket \text{LTL}_{\text{EBR}} + \text{P} \rrbracket$ . We will prove that the language of the Safety-LTL-formula  $\varphi_G := G(p_1 \vee G(p_2))$  cannot be expressed by any  $\text{LTL}_{\text{EBR}}$ -formula. The formula  $\varphi_G$  belongs *syntactically* to Safety-LTL, and thus  $\mathcal{L}(\varphi_G) \in \llbracket \text{Safety-LTL} \rrbracket$ . We also note that  $\varphi_G$  can be expressed in  $\text{LTL}_{\text{EBR}} + \text{P}$ . In fact, it holds that:

$$G(p_1 \vee G(p_2)) \equiv G(\neg p_2 \rightarrow H(p_1)) \quad (2)$$

Since  $G(\neg p_2 \rightarrow H(p_1)) \in \text{LTL}_{\text{EBR}} + \text{P}$ , it holds that  $\mathcal{L}(\varphi_G) \in \llbracket \text{LTL}_{\text{EBR}} + \text{P} \rrbracket$ . It is worth noticing the following points: (i)  $G(\neg p_2 \rightarrow H(p_1))$  is of the form  $G\alpha$ , where  $\alpha \in \text{LTL}_{\text{P}}$  ( $\alpha$  is a pure past formula); (ii) the formula  $\varphi_G$  is equivalent to  $G(p_2) \vee ((XGp_2)Rp_1)$ , but the latter formula does not syntactically belong to  $\text{LTL}_{\text{EBR}}$ , due to the restriction that forces the leftmost argument of any *release* operator to contain no universal temporal operators (*i.e.*,  $R$  and  $G$ ). In fact, in the following, we will prove that  $\mathcal{L}(\varphi_G) \notin \llbracket \text{LTL}_{\text{EBR}} \rrbracket$ .

The proof of the undefinability of  $\varphi_G$  is based on the fact that each formula of  $\text{LTL}_{\text{EBR}}$  cannot constrain an arbitrarily long prefix of a state sequence, but only a finite prefix whose maximum length depends on the maximum number of nested *next* operators.

Consider again the formula  $\varphi_G := G(p_1 \vee G(p_2))$ . The language  $\mathcal{L}(\varphi_G)$  is expressed by the  $\omega$ -regular expression  $(\{p_1\})^\omega + (\{p_1\})^* \cdot (\{p_2\})^\omega$ . Written in natural language, each model of  $\varphi_G$  cannot contain a position in which  $\neg p_2$  holds preceded by a position in which  $\neg p_1$  holds.

**Remark 1.** Let  $\sigma \subseteq (2^\Sigma)^\omega$  be a state sequence. It holds that:

$$\sigma \models \varphi_G \Rightarrow \neg \exists i, j (j \leq i \wedge \sigma_j \models \neg p_1 \wedge \sigma_i \models \neg p_2)$$

We define  $^{i,k}\sigma^j$  as the state sequence such that at the time points  $i$  and  $k$  it holds  $p_1 \wedge \neg p_2$ , at time point  $j$  it holds  $\neg p_1 \wedge p_2$ , and for all the other time points  $p_1 \wedge p_2$  holds. The membership of  $^{i,k}\sigma^j$  to  $\mathcal{L}(\varphi_G)$  depends on the value of the three indices  $i$ ,  $j$  and  $k$ , as follows.

**Remark 2.** If  $i < j$  and  $k < j$ , then  $^{i,k}\sigma^j \models \varphi_G$ . Conversely, if  $i \geq j$  or  $k \geq j$ , then  $^{i,k}\sigma^j \not\models \varphi_G$ .

As we will see, given a generic formula  $\psi \in \text{LTL}_{\text{EBR}}$ , one can always find some values for the indices  $i$ ,  $j$  and  $k$  such that (a)  $j$  is chosen sufficiently greater than  $i$ ; (b)  $k$  is chosen sufficiently greater than  $j$ ; (c)  $\psi$  is not able to distinguish the state sequence  $^{i,i}\sigma^j$  from  $^{i,k}\sigma^j$ . Since, by Remark 2,  $^{i,i}\sigma^j \in \mathcal{L}(\varphi_G)$  but  $^{i,k}\sigma^j \notin \mathcal{L}(\varphi_G)$ , this proves the undefinability of  $\varphi_G$  in  $\text{LTL}_{\text{EBR}}$ . The rationale is that the  $\text{LTL}_{\text{EBR}}$  logic combines bounded future formulas (*i.e.*, formulas obtained by a Boolean combination of propositional atoms and  $X$  operators) and universal temporal operators (*i.e.*,  $G$  and  $R$ ). This implies the fact that, for a generic model  $\sigma$  of an  $\text{LTL}_{\text{EBR}}$ -formula  $\psi$ , at each time point  $i \geq 0$  of  $\sigma$  (this corresponds to the universal temporal operators) only a *finite and bounded suffix* after  $i$  (this corresponds to the  $\text{LTL}_{\text{B}}$ -formulas) can be constrained by  $\psi$  (this can be thought of as a sort of bounded memory property of this logic). Equivalently, this means that each  $\text{LTL}_{\text{EBR}}$ -formula is *not* able to constrain any finite but arbitrarily long (unbounded) prefix of a state sequence, contrary, for instance, to the case of the formula  $G(\neg p_2 \rightarrow H(p_1))$  (that is equivalent to  $\varphi_G$ , see Eq. (2)).



## 4.2 The Canonical Form

The limitation of  $\text{LTL}_{\text{EBR}}$ -formulas mentioned before is more evident in the *canonical form* for the  $\text{LTL}_{\text{EBR}}$  logic, that we will define in this part. We first give some preliminaries definitions. We define *Bounded Past*  $\text{LTL}_P$  ( $\text{LTL}_{\text{BP}}$ , for short) as the set of all and only the  $\text{LTL}_{\text{EBR}}+P$  formulas that are a Boolean combination of propositional atoms and *yesterday* operators ( $Y$ ). We use the shortcut  $\psi_1 S^{[a,b]} \psi_2$  for denoting the formula  $\bigvee_{i=a}^b (Y_1 \dots Y_i(\psi_2) \wedge \bigwedge_{j=0}^{i-1} Y_1 \dots Y_j(\psi_1))$ . Given a formula  $\alpha \in \text{LTL}_{\text{BP}}$ , we define its *temporal depth*, denoted as  $D(\alpha)$ , as follows:

- $D(p) = 0$ , for all  $p \in \Sigma$
- $D(\neg\alpha_1) = D(\alpha_1)$
- $D(\alpha_1 \wedge \alpha_2) = \max\{D(\alpha_1), D(\alpha_2)\}$
- $D(Y\alpha_1) = 1 + D(\alpha_1)$
- $D(\alpha_1 S^{[a,b]} \alpha_2) = b + \max\{D(\alpha_1), D(\alpha_2)\}$

For each  $\alpha \in \text{LTL}_{\text{BP}}$ , the language  $\mathcal{L}^{<\omega}(\alpha)$  consists only of words of length at most  $D(\alpha) + 1$ . Recall from Section 2 that, given a infinite state sequence  $\sigma = \langle \sigma_0, \sigma_1, \dots \rangle$  and some  $n \geq 0$ ,  $\sigma_{[n-d, n]}$  is the interval of  $\sigma$  of length *at most*  $d$  ending at index  $n$ . The crucial property of  $\text{LTL}_{\text{BP}}$ -formulas, that can be shown with a simple induction, is that their truth over a state sequence  $\sigma$  can be checked by considering only a finite and *bounded* interval of  $\sigma$ , whose length depends on the *temporal depth* of the formula.

**Remark 3.** For any  $\alpha \in \text{LTL}_{\text{BP}}$ , with temporal depth  $d = D(\alpha)$ , and for any  $n \geq 0$ , it holds that  $\sigma, n \models \alpha$  if and only if  $\sigma_{[n-d, n]} \models \alpha$ .

We give now the *canonical form* for  $\text{LTL}_{\text{EBR}}$ , and we refer to it as Canonical- $\text{LTL}_{\text{EBR}}$ . The canonical form of  $\text{LTL}_{\text{EBR}}$  forces any universal unbounded operator, like *globally* or *release*, to contain only  $\text{LTL}_{\text{BP}}$ -formulas. Formally, we define Canonical- $\text{LTL}_{\text{EBR}}$  as the canonical form described in Def. 3 but such that each  $\alpha_i, \beta_i$  is a *bounded past* LTL formula. By applying the same transformation from  $\text{LTL}_{\text{EBR}}+P$  to its canonical form given in [4], one obtain the following lemma.

**Lemma 1.**  $\llbracket \text{LTL}_{\text{EBR}} \rrbracket = \llbracket \text{Canonical-LTL}_{\text{EBR}} \rrbracket$ .

*Proof.* Obviously  $\llbracket \text{Canonical-LTL}_{\text{EBR}} \rrbracket \subseteq \llbracket \text{LTL}_{\text{EBR}} \rrbracket$ , since each formula  $\psi$  that belongs to Canonical- $\text{LTL}_{\text{EBR}}$  can be turned into an equivalent one  $\psi' \in \text{LTL}_{\text{EBR}}$  by expanding each bounded past operators into conjunctions/disjunctions of *yesterday* operators.

For proving  $\llbracket \text{LTL}_{\text{EBR}} \rrbracket \subseteq \llbracket \text{Canonical-LTL}_{\text{EBR}} \rrbracket$ , it is sufficient to apply the transformations described in [4] for the translation of  $\text{LTL}_{\text{EBR}}+P$  into canonical form. In particular, since by definition  $\psi$  has no past temporal operators, the only past operators in  $\psi'$  are the ones introduced by the *pastification* step described in [4], which are all *bounded*, that is either  $Y$  or  $S^{[a,b]}$ .  $\square$

The canonical form of  $\text{LTL}_{\text{EBR}}$  makes it easier to prove Eq. (1). Take for example the formula  $\text{XXG}(p \vee Yp \vee YYp)$ , that belongs to Canonical- $\text{LTL}_{\text{EBR}}$ . It is clear that, at each time point, this formula can constrain only the interval consisting of the current state and its two previous states (in fact its temporal depth is 3).

## 4.3 The main proof

In this part, we show the undefinability of the formula  $\phi_G$  in the Canonical- $\text{LTL}_{\text{EBR}}$  logic. The undefinability in  $\text{LTL}_{\text{EBR}}$  follows from Lemma 1.

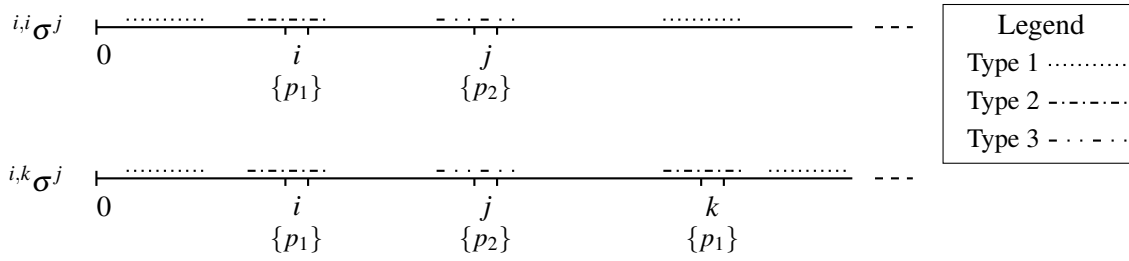


Figure 2

Given three indices  $i, j, k \in \mathbb{N}$  such that  $i \neq j$  and  $k \neq j$ , we formally define the state sequence  $^{i,k}\sigma^j = \langle ^{i,k}\sigma_0^j, ^{i,k}\sigma_1^j, \dots \rangle$  as follows:

$$^{i,k}\sigma_h^j = \begin{cases} \{p_1\} & \text{if } h \in \{i, k\} \\ \{p_2\} & \text{if } h = j \\ \{p_1, p_2\} & \text{otherwise} \end{cases}$$

The core of the main theorem is based on the fact that any formula of type  $G\alpha$  or  $\alpha R\beta$ , where  $\alpha$  and  $\beta$  are *bounded past*  $LTL_P$  formulas, is not able to distinguish the state sequence  $^{i,i}\sigma^j$  with  $i < j$  (which is a model of  $\varphi_G$ ) from  $^{i,k}\sigma^j$  with  $k > j$  (which is *not* a model of  $\varphi_G$ ), for sufficiently large values of  $i, j$  and  $k$ . The choice for the values of the three indices is based on the values of the *temporal depth* of  $\alpha$  and  $\beta$ . Since the *globally* operator is a special case of the *release* operator, that is  $G\alpha \equiv \perp R\alpha$ , it suffices to prove the property for formulas of type  $\alpha R\beta$ . We first prove the two fundamental properties that show that, for any interval of  $^{i,i}\sigma^j$  of length at most  $d$  (for any  $d \in \mathbb{N}$ ), we can find the exact same interval in  $^{i,k}\sigma^j$ , and *vice versa*. Fig. 2 shows the idea of this correspondence.

**Lemma 2.** *Let  $d \in \mathbb{N}$ . For all  $i \geq d$ , for all  $j \geq i + d$ , and for all  $k \geq j + d$ , it holds that:*

$$\text{Property 1: } \forall n' \geq 0. \exists n \geq 0. ^{i,k}\sigma_{[n'-d, n']}^j = ^{i,i}\sigma_{[n-d, n]}^j$$

$$\text{Property 2: } \forall n \geq 0. \exists n' \geq 0. ^{i,i}\sigma_{[n-d, n]}^j = ^{i,k}\sigma_{[n'-d, n']}^j$$

*Proof.* Take any value for  $i, j$ , and  $k$  such that: (i)  $i \geq d$ , (ii)  $j \geq i + d$ , (iii)  $k \geq j + d$ . Given any interval of length  $d$  of the state sequence  $^{i,i}\sigma^j$ , we show how to find an exact same one in  $^{i,k}\sigma^j$ , and viceversa.

The constraints above on the three indices ensure that both the state sequences  $^{i,i}\sigma^j$  and  $^{i,k}\sigma^j$  contain *only three* types of intervals of length at most  $d$ . Consider  $^{i,k}\sigma^j$  (the case for  $^{i,i}\sigma^j$  is specular). The three types are the following:

Type 1:  $(\{p_1, p_2\})^n$  for some  $0 \leq n \leq d$ ;

Type 2:  $(\{p_1, p_2\})^n \cdot (\{p_1\}) \cdot (\{p_1, p_2\})^{d-n-1}$ , for some  $0 \leq n < d$ ;

Type 3:  $(\{p_1, p_2\})^n \cdot (\{p_2\}) \cdot (\{p_1, p_2\})^{d-n-1}$ , for some  $0 \leq n < d$ ;

The situation is depicted in Fig. 2. Given any interval of any of the three types above, we show below how to find the very same interval in  $^{i,i}\sigma^j$  (Fig. 2 tries to show visually this correspondence):

- each interval of  $^{i,k}\sigma^j$  of type  $(\{p_1, p_2\})^n$  is equal to  $^{i,i}\sigma_{[0, n]}^j$ ;
- each interval of  $^{i,k}\sigma^j$  of type  $(\{p_1, p_2\})^n \cdot (\{p_1\}) \cdot (\{p_1, p_2\})^{d-n-1}$  is equal to  $^{i,i}\sigma_{[i-n, i+d-n-1]}^j$ .

- each interval of  $^{i,k}\sigma^j$  of type  $(\{p_1, p_2\})^n \cdot (\{p_2\}) \cdot (\{p_1, p_2\})^{d-n-1}$  is equal to  $^{i,i}\sigma_{[j-n, j+d-n-1]}^j$ ;

This proves *Property 1*.

Similarly, the correspondence between intervals of  $^{i,i}\sigma^j$  and intervals of  $^{i,k}\sigma^j$  is the following:

- each interval of  $^{i,i}\sigma^j$  of type  $(\{p_1, p_2\})^n$  is equal to  $^{i,k}\sigma_{[0,n]}^j$ ;
- each interval of  $^{i,i}\sigma^j$  of type  $(\{p_1, p_2\})^n \cdot (\{p_1\}) \cdot (\{p_1, p_2\})^{d-n-1}$  is equal to  $^{i,k}\sigma_{[i-n, i+d-n-1]}^j$ ;
- each interval of  $^{i,i}\sigma^j$  of type  $(\{p_1, p_2\})^n \cdot (\{p_2\}) \cdot (\{p_1, p_2\})^{d-n-1}$  is equal to  $^{i,k}\sigma_{[j-n, j+d-n-1]}^j$ ;

This proves *Property 2*.  $\square$

We can now prove that the state sequences  $^{i,i}\sigma^j$  and  $^{i,k}\sigma^j$  are indistinguishable for each formula of type  $\alpha R \beta$  (and, consequently, of type  $G\alpha$ ), with  $\alpha, \beta \in \text{LTL}_{\text{BP}}$ .

**Lemma 3.** *Let  $\alpha, \beta \in \text{LTL}_{\text{BP}}$ , and let  $d = \max\{D(\alpha), D(\beta)\}$  be the maximum between the temporal depths of  $\alpha$  and  $\beta$ . It holds that  $^{i,i}\sigma^j \models \alpha R \beta$  iff  $^{i,k}\sigma^j \models \alpha R \beta$ , for all  $i \geq d$ , for all  $j \geq i + d$ , and for all  $k \geq j + d$ .*

*Proof.* Take any value for  $i, j$ , and  $k$  such that: (i)  $i \geq d$ , (ii)  $j \geq i + d$ , (iii)  $k \geq j + d$ .

We first prove the left-to-right direction. Suppose that  $^{i,i}\sigma^j \models \alpha R \beta$ . We divide in cases:

1. Suppose that  $^{i,i}\sigma^j, n \models \beta$  for all  $n \geq 0$ . Since  $\beta \in \text{LTL}_{\text{BP}}$  and  $D(\beta) \leq d$ , it holds that  $^{i,i}\sigma_{[n-d, n]}^j \models \beta$ , for all  $n \geq 0$ . Suppose by contradiction that there exists some  $n' \geq 0$  such that  $^{i,k}\sigma_{[n'-d, n']}^j \models \neg\beta$ . By *Property 1* of Lemma 2, this means that there exists some  $n'' \geq 0$  such that  $^{i,i}\sigma_{[n''-d, n'']}^j \models \neg\beta$ . But this is a contradiction. Thus, it holds that  $^{i,k}\sigma_{[n'-d, n']}^j \models \beta$  for all  $n' \geq 0$ , that is, for all  $n' \geq 0$ , and thus  $^{i,k}\sigma^j \models \alpha R \beta$ .
2. Suppose that  $\exists n \geq 0 . (^{i,i}\sigma^j, n \models \alpha \wedge \forall 0 \leq m \leq n . ^{i,i}\sigma^j, m \models \beta)$ . We divide again in cases:
  - (a) Suppose that  $n < k$ . Then  $^{i,i}\sigma_{[0, n]}^j = ^{i,k}\sigma_{[0, n]}^j$ . Clearly, it holds that  $^{i,k}\sigma^j, n \models \alpha$  and  $^{i,k}\sigma^j, m \models \beta$  for all  $0 \leq m \leq n$ . Therefore  $^{i,k}\sigma^j \models \alpha R \beta$ .
  - (b) Suppose that  $n \geq k$ . In particular, it holds that  $^{i,i}\sigma_{[n-d, n]}^j \models \alpha \wedge \beta$ . We use a *contraction argument* for proving that in this case there exists a smaller index at which the *release* satisfies its existential part (i.e., the formula  $\alpha$ ). Consider the time point  $i - 1$ . It holds that  $^{i,i}\sigma_{[i-1-d, i-1]}^j = ^{i,i}\sigma_{[n-d, n]}^j$  and thus, since  $^{i,i}\sigma_{[n-d, n]}^j \models \alpha \wedge \beta$  and  $\alpha, \beta \in \text{LTL}_{\text{BP}}$ , we have that  $^{i,i}\sigma_{[i-1-d, i-1]}^j \models \alpha \wedge \beta$ . Moreover,  $^{i,i}\sigma_{[0, i-1]}^j$  is a prefix of  $^{i,i}\sigma_{[0, n]}^j$ , and thus, given that  $^{i,i}\sigma_{[p-d, p]}^j \models \beta$  for all  $0 \leq p \leq n$ , it holds that  $^{i,i}\sigma_{[p-d, p]}^j \models \beta$  for all  $0 \leq p \leq i - 1$ . From this, it follows that  $^{i,i}\sigma^j, i - 1 \models \alpha$  and  $^{i,i}\sigma^j, m \models \beta$  for all  $0 \leq m \leq i - 1$ . Since  $i - 1 < k$ , by Item 2a, it holds that  $^{i,k}\sigma^j \models \alpha R \beta$ .

We now prove the right-to-left direction. Suppose that  $^{i,k}\sigma^j \models \alpha R \beta$ . We divide in cases:

1. Suppose that  $^{i,k}\sigma^j, n \models \beta$ . This case is specular to Item 1.
2. Suppose that  $\exists n \geq 0 . (^{i,k}\sigma^j, n \models \alpha \wedge \forall 0 \leq m \leq n . ^{i,k}\sigma^j, m \models \beta)$ . Since  $\alpha, \beta \in \text{LTL}_{\text{BP}}$  and  $D(\alpha), D(\beta) \leq d$ , it holds that  $\exists n \geq 0 . (^{i,k}\sigma_{[n-d, n]}^j \models \alpha \wedge \forall 0 \leq m \leq n . ^{i,k}\sigma_{[m-d, m]}^j \models \beta)$ . We divide again in cases:
  - (a) If  $n < k$ , then  $^{i,k}\sigma_{[0, n]}^j = ^{i,i}\sigma_{[0, n]}^j$  and thus  $^{i,i}\sigma^j, n \models \alpha$  and  $^{i,i}\sigma^j, m \models \beta$  for all  $0 \leq m \leq n$ , that is  $^{i,i}\sigma^j \models \alpha R \beta$ .

- (b) If  $k \leq n \leq k + d$ , then  ${}^{i,k}\sigma_{[n-d,n]}^j = {}^{i,k}\sigma_{[n-k-i-d,n-k-i]}^j$  (we used again a contraction argument). Since by hypothesis  ${}^{i,k}\sigma_{[n-d,n]}^j \models \alpha$ , it holds also that  ${}^{i,k}\sigma_{[n-k-i-d,n-k-i]}^j \models \alpha$ . Moreover,  ${}^{i,k}\sigma_{[0,n-k-i]}^j$  is a prefix of  ${}^{i,k}\sigma_{[0,n]}^j$ , and thus, since by hypothesis  ${}^{i,k}\sigma_{[p-d,p]}^j \models \beta$  for all  $0 \leq p \leq n$ , it also holds that  ${}^{i,k}\sigma_{[p-d,p]}^j \models \beta$  for all  $0 \leq p \leq n - k - i$ . Therefore  ${}^{i,k}\sigma_{[n-k-i-d,n-k-i]}^j \models \alpha$  and  ${}^{i,k}\sigma_{[m-d,m]}^j \models \beta$  for all  $0 \leq m \leq n - k - i$ . Since  $l + n - i < k$ , by Item 2a, it holds that  ${}^{i,i}\sigma^j \models \alpha R \beta$ .
- (c) Otherwise  $n > k + d$ . We have that  ${}^{i,k}\sigma_{[n-d,n]}^j = {}^{i,k}\sigma_{[i-1,i-1-d]}^j$  (also in this case we used a contraction argument). Since by hypothesis  ${}^{i,k}\sigma_{[n-d,n]}^j \models \alpha$ , it also holds that  ${}^{i,k}\sigma_{[i-1,i-1-d]}^j \models \alpha$ . Moreover,  ${}^{i,k}\sigma_{[0,i-1]}^j$  is a prefix of  ${}^{i,k}\sigma_{[0,n]}^j$  and thus, since by hypothesis  ${}^{i,k}\sigma_{[p-d,p]}^j \models \beta$  for all  $0 \leq p \leq n$ , it also holds that  ${}^{i,k}\sigma_{[p-d,p]}^j \models \beta$  for all  $0 \leq p \leq i - 1$ . Therefore  ${}^{i,k}\sigma_{[i-1,i-1-d]}^j \models \alpha$  and  ${}^{i,k}\sigma_{[m-d,m]}^j \models \beta$  for all  $0 \leq m \leq i - 1$ . Since  $i - 1 < k$ , by Item 2a, it holds that  ${}^{i,i}\sigma^j \models \alpha R \beta$ .  $\square$

By using Lemma 3 as the proof for the base case, we prove by induction on the structure of the formula that any formula in Canonical-LTL<sub>EBR</sub> is not able to distinguish the state sequences  ${}^{i,i}\sigma^j$  and  ${}^{i,k}\sigma^j$  for sufficiently large values of  $i, j, k$ . In the following, given a formula  $\psi \in \text{Canonical-LTL}_{\text{EBR}}$ , we will denote with  $m_\psi$  the maximum number of nested *next* operators in  $\psi$ , and with  $d_\psi$  the maximum temporal depth between all its LTL<sub>BP</sub>-subformulas.

**Lemma 4.** *Let  $\psi \in \text{Canonical-LTL}_{\text{EBR}}$ . It holds that  ${}^{i,i}\sigma^j \models \psi$  iff  ${}^{i,k}\sigma^j \models \psi$ , for all  $i \geq m_\psi + d_\psi$ , for all  $j \geq i + d_\psi$ , and for all  $k \geq j + d_\psi$ .*

*Proof.* Take any value for  $i, j$ , and  $k$  such that: (i)  $i \geq m_\psi + d_\psi$ , (ii)  $j \geq i + d_\psi$ , (iii)  $k \geq j + d_\psi$ . We proceed by induction on the structure of the formula  $\psi$ .

For the base case, we consider three cases: (i) formulas in LTL<sub>BP</sub>, that is such that all its temporal operators refer to the past and are bounded; (ii) formulas of type  $G\alpha$ , where  $\alpha \in \text{LTL}_{\text{BP}}$ ; (iii) formulas of type  $\alpha R \beta$ , where  $\alpha, \beta \in \text{LTL}_{\text{BP}}$ ;

We consider the case of a formula  $\alpha \in \text{LTL}_{\text{BP}}$ , and suppose that  ${}^{i,i}\sigma^j \models \alpha$ . By definition of  ${}^{i,i}\sigma^j$  and  ${}^{i,k}\sigma^j$ , it always holds that  ${}^{i,i}\sigma_0^j = {}^{i,k}\sigma_0^j$ . Since  $\alpha \in \text{LTL}_{\text{BP}}$  refers only to the current state or to the past, it follows that  ${}^{i,i}\sigma^j \models \alpha$  if and only if  ${}^{i,k}\sigma^j \models \alpha$ .

Consider now the case for  $\alpha R \beta$ , where  $\alpha, \beta \in \text{LTL}_{\text{BP}}$ . Since  $m_{\alpha R \beta} = 0$  (i.e., there are no *next* operators in this formula), we can apply Lemma 3, having that  ${}^{i,i}\sigma^j \models \alpha R \beta$  if and only if  ${}^{i,k}\sigma^j \models \alpha R \beta$ . Since  $G\alpha = \perp R \alpha$ , this proves also the case for the *globally* operator.

For the inductive step, since by hypothesis  $\psi$  belongs to the canonical form of LTL<sub>EBR</sub>, it suffices to consider only the case for the *next* operator, *conjunctions* and *disjunctions*.

Consider first the case for the *next* operator, and suppose that  ${}^{i,i}\sigma^j \models X\psi'$ . For any indices  $k, i$  and  $j$  such that  $i \geq m_{X\psi'} + d_{X\psi'}$ ,  $j \geq i + d_{X\psi'}$  and  $k \geq j + d_{X\psi'}$ , we want to prove that  ${}^{i,k}\sigma^j \models X\psi'$ . By definition of the next operator, it holds that  ${}^{i,i}\sigma^j, 1 \models \psi'$ . Now, let  $\tau$  be the state sequence obtained from  ${}^{i,i}\sigma^j$  by discarding its initial state, that is  $\tau := {}^{i,i}\sigma_{[1,\infty]}^j$ . Obviously,  $\tau \models \psi'$ . We observe that  $\tau$  is equal to the state sequence  ${}^{i-1,i-1}\sigma^{j-1}$ . Since the maximum number  $m_{\psi'}$  of nested next operators in  $\psi'$  is  $m_{X\psi'} - 1$  (while  $\alpha_{\psi'}$  remains the same), we can apply the inductive hypothesis on  $\psi'$ , having that  ${}^{i-1,k-1}\sigma^{j-1} \models \psi'$ . By definition of  $\tau$ , it follows that  ${}^{i,k}\sigma^j \models X\psi'$ .

We consider now the case for conjunctions, and suppose that  ${}^{i,i}\sigma^j \models \psi_1 \wedge \psi_2$ , for generic indices  $k, i$  and  $j$  such that  $i \geq m_{\psi_1 \wedge \psi_2} + d_{\psi_1 \wedge \psi_2}$ ,  $j \geq i + d_{\psi_1 \wedge \psi_2}$ , and  $k \geq j + d_{\psi_1 \wedge \psi_2}$ . It holds that  ${}^{i,i}\sigma^j \models \psi_1$  and

$^{i,i}\sigma^j \models \psi_2$ . Moreover,  $m_{\psi_1} \leq m_{\psi_1 \wedge \psi_2}$  and  $m_{\psi_2} \leq m_{\psi_1 \wedge \psi_2}$ . Similarly,  $d_{\psi_1} \leq d_{\psi_1 \wedge \psi_2}$  and  $d_{\psi_2} \leq d_{\psi_1 \wedge \psi_2}$ . This means that we can apply the inductive hypothesis both on  $\psi_1$  and  $\psi_2$  on the *current* indices  $k$ ,  $i$  and  $j$ . By inductive hypothesis, we have that  $^{i,k}\sigma^j \models \psi_1$  and  $^{i,k}\sigma^j \models \psi_2$ . It follows that  $^{i,k}\sigma^j \models \psi_1 \wedge \psi_2$ . The case for  $\psi_1 \vee \psi_2$  is specular.  $\square$

Thanks to Lemma 4, it is simple to prove the undefinability of  $G(p_1 \vee G(p_2))$  in  $\text{LTL}_{\text{EBR}}$ , that proves that  $\text{LTL}_{\text{EBR}}$  is *strictly less expressive* than Safety-LTL.

**Theorem 3.**  $\llbracket \text{LTL}_{\text{EBR}} \rrbracket \subsetneq \llbracket \text{Safety-LTL} \rrbracket$ .

*Proof.* Consider the formula  $\varphi_G := G(p_1 \vee G(p_2))$ . We prove that there does *not* exist a formula  $\psi \in \text{LTL}_{\text{EBR}}$  such that  $\mathcal{L}(\psi) = \mathcal{L}(\varphi_G)$ . We proceed by contradiction. Suppose that there exists a formula  $\psi \in \text{LTL}_{\text{EBR}}$  such that  $\mathcal{L}(\psi) = \mathcal{L}(\varphi_G)$ . By Lemma 1, there exists a formula  $\psi' \in \text{Canonical-LTL}_{\text{EBR}}$  such that  $\mathcal{L}(\psi) = \mathcal{L}(\psi')$ . Let  $m_{\psi'}$  be the maximum number of *nested next* operators in  $\psi'$ , and let  $d_{\psi'}$  be the maximum temporal depth between all the  $\text{LTL}_{\text{BP}}$ -subformulas in  $\psi'$ . Let  $k$ ,  $i$  and  $j$  be three indices such that: (i)  $i \geq m_{\psi'} + d_{\psi'}$ ; (ii)  $j \geq i + d_{\psi'}$ ; (iii) and  $k \geq j + d_{\psi'}$ . Consider the two state sequences  $^{i,i}\sigma^j$  and  $^{i,k}\sigma^j$ . By Lemma 4,  $^{i,i}\sigma^j \in \mathcal{L}(\psi')$  if and only if  $^{i,k}\sigma^j \in \mathcal{L}(\psi')$ , that is  $^{i,i}\sigma^j \in \mathcal{L}(\varphi_G)$  if and only if  $^{i,k}\sigma^j \in \mathcal{L}(\varphi_G)$ . Since it holds that  $^{i,i}\sigma^j \in \mathcal{L}(\varphi_G)$  but  $^{i,k}\sigma^j \notin \mathcal{L}(\varphi_G)$ , this is clearly a contradiction.  $\square$

**Corollary 1.**  $\llbracket \text{LTL}_{\text{EBR}} \rrbracket \subsetneq \llbracket \text{LTL}_{\text{EBR}} + \text{P} \rrbracket$ .

## 5 Conclusions

We considered the logic  $\text{LTL}_{\text{EBR}} + \text{P}$ , a recently introduced safety fragment of LTL with an efficient realizability problem. The syntax of  $\text{LTL}_{\text{EBR}} + \text{P}$  made it difficult to exactly characterize its expressive power. We studied the expressive power of  $\text{LTL}_{\text{EBR}} + \text{P}$  and of its pure future fragment,  $\text{LTL}_{\text{EBR}}$ , and compare it with other safety fragments of LTL. It turned out that  $\text{LTL}_{\text{EBR}} + \text{P}$  is expressively complete with respect to the safety fragment of LTL, and, consequently, it is expressively equivalent to Safety-LTL. We found out that past modalities are crucial for the expressive power of  $\text{LTL}_{\text{EBR}} + \text{P}$ . In fact,  $\text{LTL}_{\text{EBR}}$  is strictly less expressive than full  $\text{LTL}_{\text{EBR}} + \text{P}$ . This was somehow surprising, since it proves that, despite not being fundamental for the expressiveness of full LTL, past modalities are crucial for fragments of LTL, like, for instance,  $\text{LTL}_{\text{EBR}} + \text{P}$ .

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