



UNIVERSITÀ  
DEGLI STUDI  
DI UDINE

## Università degli studi di Udine

### Topology, intersections and flat modules

*Original*

*Availability:*

This version is available <http://hdl.handle.net/11390/1215710> since 2021-12-10T11:13:48Z

*Publisher:*

*Published*

DOI:10.1090/proc/13131

*Terms of use:*

The institutional repository of the University of Udine (<http://air.uniud.it>) is provided by ARIC services. The aim is to enable open access to all the world.

*Publisher copyright*

(Article begins on next page)

# TOPOLOGY, INTERSECTIONS AND FLAT MODULES

CARMELO A. FINOCCHIARO AND DARIO SPIRITO

ABSTRACT. It is well-known that, in general, multiplication by an ideal  $I$  does not commute with the intersection of a family of ideals, but that this fact holds if  $I$  is flat and the family is finite. We generalize this result by showing that finite families of ideals can be replaced by compact subspaces of a natural topological space, and that ideals can be replaced by submodules of an epimorphic extension of a base ring. As a particular case, we give a new proof of a conjecture by Glaz and Vasconcelos.

## 1. INTRODUCTION

Let  $D$  be an integral domain with quotient field  $K$ . An *overring* of  $D$  is a ring between  $D$  and  $K$ . The set of all overrings of  $D$  is denoted by  $\text{Over}(D)$ , and can be endowed with a natural topology (called the *Zariski topology*) whose basis of open sets consists of the sets of the form

$$\mathcal{B}(x_1, \dots, x_n) := \{T \in \text{Over}(D) : x_1, \dots, x_n \in T\},$$

as  $x_1, \dots, x_n$  vary in  $K$ . Under this topology,  $\text{Over}(D)$  is a compact  $T_0$  space with a unique closed point ( $D$  itself) and a generic point (the quotient field  $K$ ). One of the clues that this topology is the most natural to be put on  $\text{Over}(D)$  is that it makes the localization map

$$\begin{aligned} \lambda: \text{Spec}(D) &\longrightarrow \text{Over}(D) \\ P &\longmapsto D_P \end{aligned}$$

a topological inclusion [4, Lemma 2.4].

This topology, whose origins can be traced back to Zariski's study of the space  $\text{Zar}(D)$  of the valuation overrings of an integral domain  $D$  [25, Chapter 6, §17] (what is now called the *Zariski space* or the *Riemann-Zariski space* of  $D$ ), has recently been studied in greater detail (see for example [7, 8, 21, 20]). For example, it has been proved that  $\text{Over}(D)$  is a *spectral space*, meaning that there is a ring  $R$  such that  $\text{Spec}(R)$  is homeomorphic to  $\text{Over}(D)$  [5, Proposition 3.5]; the same can be proved of several distinguished subspaces of  $\text{Over}(D)$ , like for

---

DIPARTIMENTO DI MATEMATICA E FISICA, UNIVERSITÀ DEGLI STUDI "ROMA TRE",  
LARGO SAN LEONARDO MURIALDO, 1, 00146 ROMA, ITALY

*E-mail address:* carmelo@mat.uniroma3.it; spirito@mat.uniroma3.it.

2010 *Mathematics Subject Classification.* 13A15, 13A18, 13C11.

*Key words and phrases.* Zariski topology, overrings, flat ideals.

example local overrings [7, Corollary 2.14] or integrally closed overrings [5, Proposition 3.6].

The aim of this paper is to prove a simple and very general result (Theorem 3, in the form of Corollary 5) which intertwines the Zariski topology on  $\text{Over}(D)$  with the algebraic properties of the overrings, namely the possibility to commute intersections and products in the case of compact spaces of overrings. In this way, we generalize [13, Lemma 1.1] (which deal with locally finite intersections) and [21, Theorem 3.5] (which proves the same for Noetherian collections of integrally closed overrings). As a consequence, we obtain a new proof of the Graz-Vasconcelos conjecture [12, page 340], independent from the one obtained in [23]. Since it poses no additional challenge, we also work in a more general setting, substituting to the extension  $D \subseteq K$  any ring extension that is also an epimorphism, and using modules instead of only overrings.

## 2. RESULTS

Let  $A \subseteq B$  be a ring extension; we denote by  $\mathcal{F}(B|A)$  the collection of all the  $A$ -submodules of  $B$ . The set  $\mathcal{F}(B|A)$  becomes a  $T_0$  topological space by declaring, as a basis of open sets, the family of the sets of the form  $\mathcal{B}(x_1, \dots, x_n) := \{G \in \mathcal{F}(B|A) : x_1, \dots, x_n \in G\}$ , for  $x_1, \dots, x_n$  varying in  $B$ . Note that, since  $\mathcal{B}(x_1, \dots, x_n) = \mathcal{B}(x_1) \cap \dots \cap \mathcal{B}(x_n)$ , a convenient subbasis for this topology is  $\{\mathcal{B}(x) : x \in B\}$ . We call this topology the *Zariski topology*, as it generalizes the Zariski topology on  $\text{Over}(D)$  defined in the Introduction. Note that, in particular, the set  $\mathcal{I}(A)$  of all the integral ideals of  $A$  becomes then a subspace of  $\mathcal{F}(B|A)$ . On the set  $\text{Spec}(A)$  of the prime ideals of  $A$ , this topology does *not* coincide with the classical Zariski topology, but rather with the so-called *inverse topology* (see [14] and the discussion before Example 2.2 of [22]). This should, however, not cause any confusion; the only place where we will consider  $\text{Spec}(R)$  will be Proposition 11.

If  $X$  is any topological space and  $Y \subseteq X$ , we will denote by  $\overline{Y}$  the closure of  $Y$  in  $X$ .

**Remark 1.** Let  $A \subseteq B$  be a ring extension and let  $\mathcal{F}(B|A)$  be endowed with the Zariski topology. The following properties hold.

- (1) For any  $F, G \in \mathcal{F}(B|A)$ , we have  $F \in \overline{\{G\}}$  if and only if  $F \subseteq G$ .
- (2) Any compact nonempty subspace  $C$  of  $\mathcal{F}(B|A)$  has minimal elements, with respect to the inclusion  $\subseteq$ . As a matter of fact, by Zorn's lemma it is enough to show that any chain (under inclusion)  $\Sigma \subseteq C$  has a lower bound. By (1), the collection of sets  $\mathcal{F} := \{\overline{\{F\}} \cap C : F \in \Sigma\}$  is a chain. Thus, in particular, given any finite subset  $F_1, \dots, F_n \in \Sigma$ , if  $G$  is contained in all  $F_i$ , then  $G \in \bigcap_{i=1}^n \overline{\{F_i\}} \cap C$ . This proves that  $\mathcal{F}$  is a collection of closed sets of  $C$  with the finite intersection property. By

compactness, there exists a submodule  $F^* \in \overline{\{F\}} \cap C$ , for any  $F \in \Sigma$ , and applying again (1) we see that  $F^*$  is a lower bound of  $\Sigma$  in  $C$ .

Let now  $\phi : A \rightarrow B$  be a ring homomorphism. Then,  $\phi$  is an *epimorphism* in the category of rings if, for every  $\psi_1, \psi_2 : B \rightarrow C$ , the equality  $\psi_1 \circ \phi = \psi_2 \circ \phi$  implies that  $\psi_1 = \psi_2$ . If the inclusion map  $A \hookrightarrow B$  is an epimorphism, we will call the ring extension  $A \subseteq B$  an *epimorphic extension*.

Examples of epimorphisms are surjective maps and localizations; more generally, a map  $\phi : A \rightarrow B$  such that the induced homomorphism  $\phi_{\mathfrak{p}} : A_{\mathfrak{p}} \rightarrow B_{\mathfrak{p}}$  is surjective for every  $\mathfrak{p} \in \text{Spec}(A)$  such that  $\phi(\mathfrak{p})B \neq B$  is an epimorphism (maps with this property are called *weakly surjective* [16, Chapter 1, §3]; on extensions, being an epimorphism and being weakly surjective are equivalent conditions [16, Theorem 4.4]). In particular, if  $D$  is an integral domain and  $K$  is its quotient field, the ring extension  $D \subseteq K$  is an epimorphic extension. On the other hand, if  $X$  is an indeterminate over  $A$ , then the extension  $A \subseteq A[X]$  is not epimorphic: indeed, for every  $\alpha \in A$ , we can build a ring homomorphism  $\psi_{\alpha} : A[X] \rightarrow A$  by defining  $\psi_{\alpha}(a) := a$  if  $a \in A$  and  $\psi_{\alpha}(X) = \alpha$ . In this case, we have  $\psi_{\alpha} \neq \psi_{\beta}$  if  $\alpha \neq \beta$ , but every  $i \circ \psi_{\alpha}$  is the identity on  $A$ .

The first step of our way is the following fact, which is a generalization of [1, Theorem 2].

**Proposition 2.** *Let  $A \subseteq B$  be an epimorphic extension. Let  $I$  be a flat  $A$ -submodule of  $B$ , and let  $G_1, \dots, G_n \in \mathcal{F}(B|A)$ . Then,*

$$I(G_1 \cap \dots \cap G_n) = IG_1 \cap \dots \cap IG_n.$$

*Proof.* With a small abuse of notation, for any  $F, G \in \mathcal{F}(B|A)$ , we will denote by  $F \otimes G$  the submodule of  $B \otimes B$  generated by the elements  $f \otimes g$ , as  $f$  varies in  $F$  and  $g$  varies in  $G$ . By induction, it suffices to show the statement for  $n = 2$ . Consider the map

$$\begin{aligned} \lambda : B \otimes_A B &\rightarrow B \\ b_1 \otimes b_2 &\mapsto b_1 b_2. \end{aligned}$$

Clearly, if  $I, G \in \mathcal{F}(B|A)$ , then  $\lambda(I \otimes G) = IG$ ; therefore, by [18, Theorem 7.4]

$$I(G_1 \cap G_2) = \lambda(I \otimes (G_1 \cap G_2)) = \lambda((I \otimes G_1) \cap (I \otimes G_2)).$$

Since  $A \subseteq B$  is an epimorphic extension,  $\lambda$  is an isomorphism (indeed, this property actually characterizes epimorphisms [17, Lemma 1.0]); in particular,  $\lambda$  is a bijection, and thus

$$\lambda((I \otimes G_1) \cap (I \otimes G_2)) = \lambda(I \otimes G_1) \cap \lambda(I \otimes G_2) = IG_1 \cap IG_2.$$

This completes the proof.  $\square$

Note that this proposition does not hold if  $A \subseteq B$  is not an epimorphism: for example, if  $X$  is an indeterminate over  $A$ ,  $B = A[X] = I$ ,  $G_1 = A$ ,  $G_2 = XA[X]$ , then  $G_1 \cap G_2 = (0)$  and so  $I(G_1 \cap G_2) = (0)$ , while  $IG_1 \cap IG_2 = A[X] \cap XA[X] = XA[X]$ .

**Theorem 3.** *Let  $A \subseteq B$  be an epimorphic extension, let  $I$  be a flat  $A$ -submodule of  $B$  and let  $Y$  be a (nonempty) compact subspace of  $\mathcal{F}(B|A)$ . Then, the following equality holds:*

$$I \left( \bigcap_{J \in Y} J \right) = \bigcap_{J \in Y} IJ$$

*Proof.* The  $(\subseteq)$  containment is obvious. Take now an element  $x \in \bigcap \{IJ : J \in Y\}$ . For any  $J \in Y$ , by definition, there exist a positive integer  $n_J$  and elements  $i_1^{(J)}, \dots, i_{n_J}^{(J)} \in I$ ,  $t_1^{(J)}, \dots, t_{n_J}^{(J)} \in J$  such that

$$x = i_1^{(J)} t_1^{(J)} + \dots + i_{n_J}^{(J)} t_{n_J}^{(J)} = \sum_{h=1}^{n_J} i_h^{(J)} t_h^{(J)}.$$

Consider the open neighborhood  $\Omega_J := \mathcal{B}(\{t_1^{(J)}, \dots, t_{n_J}^{(J)}\})$  of  $J$ . Then the collection of sets  $\mathcal{A} := \{\Omega_J : J \in Y\}$  is an open cover of  $Y$ . By compactness,  $\mathcal{A}$  admits a finite subcover, say  $\{\Omega_{J_1}, \dots, \Omega_{J_r}\}$ , for suitable  $J_1, \dots, J_r \in Y$ . For any  $l = 1, \dots, r$ , set  $Y_l := \Omega_{J_l} \cap Y$ . By Proposition 2, we have

$$I \left( \bigcap_{J \in Y} J \right) = I \left( \bigcap_{J \in Y_1} J \cap \dots \cap \bigcap_{J \in Y_r} J \right) = I \left( \bigcap_{J \in Y_1} J \right) \cap \dots \cap I \left( \bigcap_{J \in Y_r} J \right),$$

and thus it suffices to show that  $x \in I \left( \bigcap_{J \in Y_l} J \right)$ , for each  $l = 1, \dots, r$ . However, the elements  $t_1^{(J_l)}, \dots, t_{n_{J_l}}^{(J_l)}$  belong to  $J$  for every  $J \in Y_l$ , and thus they belong to the intersection  $\bigcap \{J : J \in Y_l\}$ ; hence, the representation  $x = \sum_{h=1}^{n_{J_l}} i_h^{(J_l)} t_h^{(J_l)}$  shows that  $x \in I \left( \bigcap_{J \in Y_l} J \right)$ .  $\square$

Before giving some corollaries of independent interest, we state the following useful lemma.

**Lemma 4.** *Let  $A \subseteq B$  be a ring extension and let  $\mathcal{F}(B|A)$  be endowed with the Zariski topology. Fix a submodule  $I \in \mathcal{F}(B|A)$ . Then, the maps*

$$\begin{array}{ccc} s_I : \mathcal{F}(B|A) \longrightarrow \mathcal{F}(B|A) & & m_I : \mathcal{F}(B|A) \longrightarrow \mathcal{F}(B|A) \\ J \longmapsto I + J & \text{and} & I \longmapsto IJ \end{array}$$

*are continuous.*

*Proof.* Let  $\mathcal{B}(x)$  be a subbasic open set of  $\mathcal{F}(B|A)$ , with  $x \in B$ . If  $J_0 \in s_I^{-1}(\mathcal{B}(x))$ , then  $x = i + j$  for some  $i \in I$ ,  $j \in J_0$ ; therefore,  $\mathcal{B}(j)$  is an open neighborhood of  $J_0$  contained in  $s_I^{-1}(\mathcal{B}(x))$ , and thus  $s_I$  is

continuous. Similarly, if  $J_0 \in m_I^{-1}(\mathcal{B}(x))$ , then  $x = i_1 j_1 + \cdots + i_n j_n$  for some  $j_1, \dots, j_n \in J_0$  and  $i_1, \dots, i_n \in I$ . Then,  $J_0 \in \mathcal{B}(j_1, \dots, j_n) \subseteq m_I^{-1}(\mathcal{B}(x))$ , and this shows that  $m_I$  is continuous.  $\square$

Note that the continuity of  $m_I$  make it possible to shorten the proof of [21, Lemma 3.7].

**Corollary 5.** *Let  $D$  be an integral domain, let  $I$  and  $T$  be  $D$ -submodules of the quotient field  $K$  of  $D$ , and let  $\Delta$  be a compact subset of  $\text{Over}(D)$ , with respect to the Zariski topology. If  $T$  is flat over  $D$ , then*

$$\left( \bigcap_{U \in \Delta} IU \right) T = \bigcap_{U \in \Delta} (IUT).$$

*Proof.* By Lemma 4, the collection  $\{IU : U \in \Delta\}$  is compact, since it is the continuous image of  $\Delta$  via  $m_I$ . The conclusion is now an immediate consequence of Theorem 3.  $\square$

As a particular case of the main results, we provide now a new topological proof of the Glaz-Vasconcelos conjecture.

**Corollary 6.** [23, Theorem 1.7] *Let  $D$  be an integrally closed integral domain, and let  $I$  be a  $D$ -submodule of its quotient field  $K$ . If  $I$  is flat over  $D$ , then  $I = \bigcap \{IV : V \in \text{Zar}(D)\}$ .*

*Proof.* The space  $\text{Zar}(D)$  is compact in the Zariski topology [25, Chapter 6, Theorem 40]; moreover, since  $D$  is integrally closed,  $D = \bigcap \{V : V \in \text{Zar}(D)\}$  [3, Corollary 5.22]. Hence, by Theorem 3,

$$I = ID = I \left( \bigcap_{V \in \text{Zar}(D)} V \right) = \bigcap_{V \in \text{Zar}(D)} IV,$$

as claimed.  $\square$

Another immediate consequence of the main results deals with intersections of localizations of integral domains.

**Corollary 7.** *Let  $D$  be an integral domain, let  $Y$  be a compact nonempty subspace of  $\text{Over}(D)$  such that  $D = \bigcap \{R : R \in Y\}$ , and let  $S$  be a multiplicative subset of  $D$ . Then,  $S^{-1}D = \bigcap \{S^{-1}R : R \in Y\}$ .*

*Proof.* It suffices to use Theorem 3, keeping in mind that  $S^{-1}D$  is a flat  $D$ -module.  $\square$

**Corollary 8.** [11, Proposition 43.5] *Let  $D$  be an integral domain, let  $Y$  be a locally finite subspace of  $\text{Over}(D)$  (i.e., any nonzero element of  $D$  is noninvertible only in finitely many members of  $Y$ ) such that  $D = \bigcap \{R : R \in Y\}$ , and let  $S$  be a multiplicative subset of  $D$ . Then,  $S^{-1}D = \bigcap \{S^{-1}R : R \in Y\}$*

*Proof.* By Corollary 7, it is enough to show that a locally finite collection of overrings of  $D$  is compact, with respect to the Zariski topology of  $\text{Over}(D)$ .

Let  $\mathcal{A}$  be an open cover of  $Y$ . By Alexander's Subbasis Theorem (see e.g. [15, Chapter 5, Theorem 6, page 139]), we can assume, without loss of generality, that  $\mathcal{A}$  consists of subbasic open sets of  $\text{Over}(D)$ , say  $\mathcal{A} = \left\{ \mathcal{B} \left( \frac{a_i}{b_i} \right) : i \in I \right\}$ , where  $a_i, b_i \in D, b_i \neq 0$ , for any  $i \in I$ . Fix now an index  $i' \in I$  and note that, by assumption, the set  $Y' := \{R \in Y : b_{i'}^{-1} \notin R\}$  is finite, say  $Y' = \{R_1, \dots, R_n\}$ . Thus, any member of  $Y - Y'$  belongs to  $\mathcal{B} \left( \frac{a_{i'}}{b_{i'}} \right)$  and any  $R_j \in Y'$  belongs to some  $\mathcal{B} \left( \frac{a_{i_j}}{b_{i_j}} \right)$ . The proof is now complete.  $\square$

Note that the main part of the proof of the previous corollary is also a consequence of [8, Proposition 2.9], where it was proved in the more general context of semistar operations; we inserted the proof here for the reader's convenience. Moreover, the proof of the previous corollary also extends [9, Remark 4.7], where the authors proved that any locally finite family of localizations is compact.

**Corollary 9.** *Let  $D$  be a Prüfer domain with quotient field  $K$ , let  $\mathfrak{a}$  be an ideal of  $D$ , and let  $Y \subseteq \mathcal{I}(D)$  be compact. Then,*

$$\mathfrak{a} + \bigcap_{\mathfrak{b} \in Y} \mathfrak{b} = \bigcap_{\mathfrak{b} \in Y} (\mathfrak{a} + \mathfrak{b}).$$

*Proof.* It suffices to prove that, for every prime ideal  $\mathfrak{p}$ , the equality

$$\mathfrak{a}D_{\mathfrak{p}} + \left( \bigcap_{\mathfrak{b} \in Y} \mathfrak{b} \right) D_{\mathfrak{p}} = \left( \bigcap_{\mathfrak{b} \in Y} (\mathfrak{a} + \mathfrak{b}) \right) D_{\mathfrak{p}}$$

holds. Fix thus a prime ideal  $\mathfrak{p}$ , and let  $V := D_{\mathfrak{p}}$ ; since  $D$  is a Prüfer domain,  $V$  is a valuation domain.

Since  $V$  is flat over  $D$  and  $\{\mathfrak{a} + \mathfrak{b} : \mathfrak{b} \in Y\}$  is compact (Lemma 4), we have, by Theorem 3,

$$(1) \quad \left( \bigcap_{\mathfrak{b} \in Y} (\mathfrak{a} + \mathfrak{b}) \right) V = \bigcap_{\mathfrak{b} \in Y} ((\mathfrak{a} + \mathfrak{b})V) = \bigcap_{\mathfrak{b} \in Y} (\mathfrak{a}V + \mathfrak{b}V)$$

Observe now that, since  $V$  is a valuation domain, the collection of ideals  $Y' := \{\mathfrak{b}V : \mathfrak{b} \in Y\}$  is totally ordered and compact, by Lemma 4. Thus, since by compactness  $Y'$  has minimal elements under inclusion (Remark 1), it follows that  $Y'$  has a minimum. Then, there is an ideal  $\mathfrak{b}_0 \in Y$  such that  $\mathfrak{b}_0V \subseteq \mathfrak{b}V$ , for any  $\mathfrak{b} \in Y$ . It follows that the last member of the equality (1) becomes

$$\bigcap_{\mathfrak{b} \in Y} (\mathfrak{a}V + \mathfrak{b}V) = \mathfrak{a}V + \mathfrak{b}_0V = \mathfrak{a}V + \left( \bigcap_{\mathfrak{b} \in Y} \mathfrak{b}V \right) = \mathfrak{a}V + \left( \bigcap_{\mathfrak{b} \in Y} \mathfrak{b} \right) V,$$

where the last equality is again a consequence of Theorem 3. The proof is now complete.  $\square$

**Remark 10.** The previous corollary is closely related to the dual AB-5\* of Grothendieck AB-5 (see, for example, [2]). Precisely, if  $D$  is a Prüfer domain and any filter base of ideals of  $D$  is compact, with respect to the Zariski topology of  $\mathcal{I}(D)$ , then  $D$  is AB-5\* (as  $D$ -module).

In the case of Prüfer domains, we can also prove a partial converse of Theorem 3. Recall that a prime ideal  $P$  of a Prüfer domain  $D$  is *branched* if the set of prime ideals of  $D$  properly contained in  $P$  has a maximum (see e.g. [11, Theorem 17.3]). If the dimension of  $D$  is finite, every prime ideal is branched.

**Proposition 11.** *Let  $D$  be a Prüfer domain with quotient field  $K$ , and let  $\Delta \subseteq \text{Spec}(D)$  be a nonempty set.*

- (a)  *$\Delta$  is compact (in the “classical” Zariski topology of  $\text{Spec}(D)$ ) if and only if, for every flat  $D$ -submodule  $I$  of  $K$ ,  $\bigcap_{\mathfrak{p} \in \Delta} ID_{\mathfrak{p}} = I \left( \bigcap_{\mathfrak{p} \in \Delta} D_{\mathfrak{p}} \right)$ .*
- (b) *Suppose that every prime ideal of  $D$  is branched. Then,  $\Delta$  is compact (in the “classical” Zariski topology of  $\text{Spec}(R)$ ), if and only if  $\bigcap_{\mathfrak{p} \in \Delta} D_{\mathfrak{q}} D_{\mathfrak{p}} = D_{\mathfrak{q}} \left( \bigcap_{\mathfrak{p} \in \Delta} D_{\mathfrak{p}} \right)$  for every  $\mathfrak{q} \in \text{Spec}(D)$ .*

*Proof.* In both points, one implication follows from Corollary 7 and the fact that the map  $\lambda : \text{Spec}(D) \rightarrow \text{Over}(D)$ ,  $P \mapsto D_P$ , is a topological inclusion. Suppose  $\Delta$  is not compact, and let  $T := \bigcap_{\mathfrak{p} \in \Delta} D_{\mathfrak{p}}$ ; note that, without loss of generality, we can suppose that  $\Delta = \Delta^{\downarrow} = \{\mathfrak{q} \in \text{Spec}(D) : \mathfrak{q} \subseteq \mathfrak{p} \text{ for some } \mathfrak{p} \in \Delta\}$ , since  $\Delta$  is compact if and only if  $\Delta^{\downarrow}$  is compact.

The set of prime ideals  $\mathfrak{p}$  such that  $\mathfrak{p}T \neq T$  is the image of  $\text{Spec}(T)$  under the canonical map  $\text{Spec}(T) \rightarrow \text{Spec}(D)$ ; since it contains  $\Delta$ , and  $\Delta$  is not compact, it must also contain a prime ideal  $\mathfrak{q} \notin \Delta$ . Since  $D$  is a Prüfer domain,  $\mathfrak{q}$  is a flat  $D$ -module; however,

$$\bigcap_{\mathfrak{p} \in \Delta} \mathfrak{q} D_{\mathfrak{p}} = \bigcap_{\mathfrak{p} \in \Delta} D_{\mathfrak{p}} = T \neq \mathfrak{q} T = \mathfrak{q} \left( \bigcap_{\mathfrak{p} \in \Delta} D_{\mathfrak{p}} \right),$$

against the hypothesis. Therefore, part (a) is proved.

If every prime ideal of  $D$  is branched, so is  $\mathfrak{q}$ ; therefore, there is a prime ideal  $\mathfrak{q}_0$  directly below  $\mathfrak{q}$ . No ideal  $\mathfrak{p} \in \Delta$  contains  $\mathfrak{q}$ ; therefore,  $D_{\mathfrak{p}} D_{\mathfrak{q}} \supsetneq D_{\mathfrak{q}}$ , and in particular  $D_{\mathfrak{q}_0} \subseteq D_{\mathfrak{p}} D_{\mathfrak{q}}$ . Hence,

$$\bigcap_{\mathfrak{p} \in \Delta} D_{\mathfrak{q}} D_{\mathfrak{p}} \supsetneq D_{\mathfrak{q}_0} \supsetneq D_{\mathfrak{q}} = D_{\mathfrak{q}} T = D_{\mathfrak{q}} \left( \bigcap_{\mathfrak{p} \in \Delta} D_{\mathfrak{p}} \right),$$

against the hypothesis. Part (b) is proved.  $\square$



Note that part (b) of the previous proposition does not hold without the hypothesis that the prime ideals are branched: indeed, if  $V$  is a valuation domain with maximal ideal  $\mathfrak{m}$  unbranched, and  $\Delta := \operatorname{Spec}(V) \setminus \{\mathfrak{m}\}$ , then

$$V_{\mathfrak{m}}V = V = \bigcap_{\mathfrak{p} \in \Delta} V_{\mathfrak{p}} = \bigcap_{\mathfrak{p} \in \Delta} V_{\mathfrak{p}}V_{\mathfrak{m}},$$

despite  $\Delta$  not being compact.

Another question arising from Theorem 3 is if the equality  $I(\bigcap_{J \in Y} J) = \bigcap_{J \in Y} IJ$ , for all compact families  $Y$  of submodules of an epimorphic extension  $A \subseteq B$ , implies that  $I$  is flat. This is true if the base ring  $A$  is a domain, but fails in general (see [1, Theorem 2] and the subsequent discussion).

**Remark 12.** While Theorem 3 is quite general, it may be in general hard, or at least not easy, to find examples of compact subspaces to which it can be applied, or to prove that a given family is actually compact.

Some examples can be constructed using the fact that, under the Zariski topology,  $\mathcal{F}(B|A)$  is a spectral space, i.e., it is homeomorphic to the prime spectrum of a ring [22, Example 2.2(2)]. For example, it follows from Remark 1(1) and either [24, Proposition 2.3] or [19, Proposition 2.2] that a subset  $Y$  of  $\mathcal{F}(B|A)$  is compact if and only if every element of the closure of  $Y$ , with respect to the constructible topology, contains a point of  $Y$ , where the *constructible topology* on  $\mathcal{F}(B|A)$  is the coarsest topology on  $\mathcal{F}(B|A)$  for which any open and compact subspace of  $\mathcal{F}(B|A)$  is both open and closed.

Another class of examples comes from the domination map  $d : \operatorname{Zar}(A) \longrightarrow \operatorname{Spec}(A)$  of the Zariski space of a domain  $A$  (i.e., the set of valuation overrings of  $A$ ). For example, if  $S$  is a compact subspace of  $\operatorname{Spec}(A)$ , then  $d^{-1}(S)$  is compact, by [19, Proposition 2.2 and Lemma 2.7(3)].

#### ACKNOWLEDGMENTS

The authors would like to thank the referee for her/his helpful comments and suggestions.

#### REFERENCES

- [1] David D. Anderson, On the ideal equation  $I(B \cap C) = IB \cap IC$ , *Canad. Math. Bull.* **26** (1983), no. 3, 331–332.
- [2] Pham Ngoc Anh, Dolors Herbera, Claudia Menini. AB-5\* and linear compactness. *J. Algebra* **200** (1998), no. 1, 99–117.
- [3] Michael F. Atiyah and Ian G. Macdonald, *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.
- [4] David E. Dobbs, Richard Fedder, and Marco Fontana. Abstract Riemann surfaces of integral domains and spectral spaces. *Ann. Mat. Pura Appl.* (4), 148:101–115, 1987.

- [5] Carmelo Antonio Finocchiaro. Spectral spaces and ultrafilters. *Comm. Algebra*, 42(4):1496–1508, 2014.
- [6] Carmelo Antonio Finocchiaro, Marco Fontana, K. Alan Loper, The constructible topology on spaces of valuation domains, *Trans. Am. Math. Soc.* **365** (2013), 6199–6216.
- [7] Carmelo Antonio Finocchiaro, Marco Fontana, and Dario Spirito. New distinguished classes of spectral spaces: a survey, *Proceedings of the Graz conference*, to appear.
- [8] Carmelo Antonio Finocchiaro, Dario Spirito, Some topological considerations on semistar operations, *J. Algebra* **409**, 199–218, 2014.
- [9] Marco Fontana, James Huckaba, Localizing systems and semistar operations. Non-Noetherian commutative ring theory, 169–197, *Math. Appl.*, 520, Kluwer Acad. Publ., Dordrecht, 2000.
- [10] Marco Fontana, James Huckaba, Ira Papick, Prüfer domains, New York, Marcel Dekker Inc., 1997.
- [11] Robert Gilmer, Multiplicative ideal theory, *Queen’s Papers in Pure and Applied Mathematics*, Volume 90, 1992.
- [12] Sarah Glaz and Wolmer V. Vasconcelos. Flat ideals. II. *Manuscripta Math.*, 22(4):325–341, 1977.
- [13] William Heinzer and Jack Ohm. Noetherian intersections of integral domains. *Trans. Amer. Math. Soc.*, 167:291–308, 1972.
- [14] Melvin Hochster, Prime ideal structure in commutative rings, *Trans. Amer. Math. Soc.* **142** (1969), 43–60.
- [15] John Kelley, *General topology*. Graduate Texts in Mathematics, Springer, 1975.
- [16] Manfred Knebusch and Digen Zhang. *Manis valuations and Prüfer extensions. I*. Springer-Verlag, Berlin, 2002. Lecture Notes in Mathematics, Vol. 1791.
- [17] Daniel Lazard. Autour de la platitude. *Bull. Soc. Math. France*, 97:81–128, 1969.
- [18] Hideyuki Matsumura. *Commutative ring theory*. Cambridge University Press, Cambridge, 1986 Cambridge Studies in Advanced Mathematics, Vol. 8.
- [19] Bruce Olberding, Affine schemes and topological closures in the Zariski-Riemann space of valuation rings. *J. Pure Appl. Algebra* **219** (2015), no. 5, 1720–1741.
- [20] Bruce Olberding. Overrings of two-dimensional Noetherian domains representable by Noetherian spaces of valuation rings. *J. Pure Appl. Algebra*, 212(7):1797–1821, 2010.
- [21] Bruce Olberding. Noetherian spaces of integrally closed rings with an application to intersections of valuation rings. *Comm. Algebra*, 38(9):3318–3332, 2010.
- [22] Bruce Olberding, Topological aspects of irredundant intersections of ideals and valuation rings, arXiv:1510.02000.
- [23] Giampaolo Picozza and Francesca Tartarone. Flat ideals and stability in integral domains. *J. Algebra*, 324(8):1790–1802, 2010.
- [24] Niels Schwartz, Marcus Tressl, Elementary properties of minimal and maximal points in Zariski spectra. *J. Algebra* **323** (2010), no. 3, 698–728.
- [25] Oscar Zariski and Pierre Samuel. *Commutative algebra. Vol. II*. Springer-Verlag, New York, 1975. Reprint of the 1960 edition, Graduate Texts in Mathematics, Vol. 29.