

Wilf's conjecture for numerical semigroups with large second generator

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We study Wilf's conjecture for numerical semigroups S such that the second least generator a_2 of S satisfies $a_2 > \frac{c(S) + \mu(S)}{3}$, where $c(S)$ is the conductor and $\mu(S)$ the multiplicity of S . In particular, we show that for these semigroups Wilf's conjecture holds when the multiplicity is bounded by a quadratic function of the embedding dimension.

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1. Introduction and Preliminaries

A *numerical semigroup* is a subset $S \subseteq \mathbb{N}$ that contains 0, is closed under addition and such that the complement $\mathbb{N} \setminus S$ is finite. In particular, there is a largest integer not contained in S , which is called the *Frobenius number* of S and is denoted by $F(S)$. The *conductor* of S is defined as $c(S) := F(S) + 1$, and it is the minimal integer x such that $x + \mathbb{N} \subseteq S$. Calculating $F(S)$ is a classical problem (called the *Diophantine Frobenius problem*), introduced by Sylvester [11]; see [8] for a general overview.

Given coprime integers $a_1 < \dots < a_n$, the numerical semigroup *generated* by a_1, \dots, a_n is the set

$$\langle a_1, \dots, a_n \rangle := \{ \lambda_1 a_1 + \dots + \lambda_n a_n \mid \lambda_i \in \mathbb{N} \}.$$

Conversely, if S is a numerical semigroup, there are always a finite number of integers a_1, \dots, a_n such that $S = \langle a_1, \dots, a_n \rangle$; moreover, there is a unique minimal

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set of such integers, whose cardinality, called the *embedding dimension* of S , is denoted by $\nu(S)$. The integer a_1 , the smallest minimal generator of S , is called the *multiplicity* of S , and is denoted by $\mu(S)$.

In 1978, Wilf [12] suggested a relationship between the conductor and the embedding dimension of S . More precisely, set

$$L(S) := \{x \in S \mid 0 \leq x < c(S)\}.$$

Wilf hypothesized that the inequality

$$\nu(S)|L(S)| \geq c(S)$$

holds for every numerical semigroup S ; this question is known as *Wilf's conjecture*. The conjecture is still unresolved in the general case, although there have been several partial results: for example, it has been proven that Wilf's conjecture holds when $\nu(S) \leq 3$ [3, 11], when $|\mathbb{N} \setminus S| \leq 60$ [4], when $c(S) \leq 3\mu(S)$ [1, 5] and when $\nu(S) \geq \mu(S)/3$ [2, 10].

In this paper, we study Wilf's conjecture when a_2 , the second smallest generator of S , is large, in the sense that

$$a_2 > \frac{c(S) + \mu(S)}{3}.$$

This condition is favorable to the study of Wilf's condition because it implies that the conductor $c(S)$ is not too large with respect to the other parameters of S , and that the embedding dimension $\nu(S)$ is not too small with respect to the multiplicity $\mu(S)$ (see Proposition 2.4). From a technical point of view, the main advantage is that we can split the generators of S into three sets (see Sec. 2) in a way that makes it easier to estimate the cardinality of $L(S)$.

The main result of this paper is Theorem 5.2, which says that, for every $\epsilon > 0$, Wilf's conjecture holds (among the semigroups with $a_2 > \frac{c(S) + \mu(S)}{3}$) whenever

$$\mu(S) \leq \frac{8}{25}\nu(S)^2 + \frac{1}{5}\nu(S) - \frac{1}{2} - \epsilon$$

and ν is larger than a quantity $\nu_0(\epsilon)$ depending on ϵ ; this could be compared to the results in [7], where Wilf's conjecture is proved (without hypothesis on the second generator) for $\mu(S) \leq c'\nu(S)^{4/3}$, where c' is a constant (see Remark 5.4 for details).

Following the same method of proof, we also show a few variants of this result: we show that for $\nu(S) \geq 10$ we can take $\epsilon = 3/4$ (Proposition 5.3), we prove the conjecture under the additional hypothesis that $c(S) \equiv 0 \pmod{\mu(S)}$ (Proposition 5.5) and we improve the previous inequality to $\mu(S) \leq \frac{4}{9}\nu(S)^2$ provided that we allow a finite number of counterexamples for every (large) value of $\nu(S)$ (Proposition 5.6).

The structure of the paper is as follows. Section 2 focuses on the combinatorial features of semigroups satisfying the condition $a_2 > \frac{c+\mu}{3}$, and in particular on what bounds the condition imposes on the parameters of S . Section 3 estimates the cardinality of $L(S)$ in terms of the size of generators of S , first in a general way and then distinguishing six different cases according to a parameter $\theta(S)$ (see Definition 3.3).

In Sec. 4, we present two inequalities that are proved through purely analytic methods. In Sec. 5, we give a criterion summing up the previous results (Proposition 5.1) and then we prove the main Theorem 5.2 and its variants (Propositions 5.3–5.7).

For general information and results about numerical semigroups, the reader may consult [9].

2. Splitting the Generators

From now on, S will be a numerical semigroup, $\mu := \mu(S)$ its multiplicity, $\nu := \nu(S)$ its embedding dimension, and $c := c(S)$ its conductor. We denote by $\text{Ap}(S)$ the *Apéry set* of S with respect to its multiplicity, i.e.

$$\text{Ap}(S) := \{i \in S \mid i - \mu \notin S\}.$$

We recall that, for every $t \in \{0, \dots, \mu - 1\}$, there is a unique $x \in \text{Ap}(S)$ such that $x \equiv t \pmod{\mu}$; in particular, $\text{Ap}(S)$ has cardinality μ . Note also that, since $F(S) = c(S) - 1$ is the maximal integer not belonging to S , the largest element of $\text{Ap}(S)$ is $F(S) + \mu$, and thus every element of $\text{Ap}(S)$ is strictly smaller than $c + \mu$.

Let now $P := \{a_1, \dots, a_\nu\}$ be the set of minimal generators of S , with $\mu = a_1 < a_2 < \dots < a_\nu$. We shall always suppose that $a_2 > \frac{c+\mu}{3}$. Since each $x \in P \setminus \{\mu\}$ belongs to $\text{Ap}(S)$, we can subdivide $P \setminus \{\mu\}$ into the following three sets:

$$P_1 := \left\{ a \in P \setminus \{\mu\} \mid \frac{1}{3}(c + \mu) < a < \frac{1}{2}(c + \mu) \right\},$$

$$P_2 := \left\{ a \in P \setminus \{\mu\} \mid \frac{1}{2}(c + \mu) \leq a < \frac{2}{3}(c + \mu) \right\},$$

$$P_3 := \left\{ a \in P \setminus \{\mu\} \mid \frac{2}{3}(c + \mu) \leq a < c + \mu \right\}.$$

We set $q_i := |P_i|$, for $i \in \{1, 2, 3\}$.

Let $\pi : \mathbb{Z} \rightarrow \mathbb{Z}/\mu\mathbb{Z}$ be the canonical quotient map, and let $A := \pi(P)$, $A_i := \pi(P_i)$. Given two subsets $X, Y \subseteq \mathbb{Z}/\mu\mathbb{Z}$, the *sumset* of X and Y is

$$X + Y := \{x + y \mid x \in X, y \in Y\}.$$

Proposition 2.1. *Let $S = \langle a_1, a_2, \dots, a_\nu \rangle$ be a numerical semigroup with $a_2 > \frac{c(S)+\mu(S)}{3}$. Then, $\mathbb{Z}/\mu\mathbb{Z} = A \cup (A_1 + A_1) \cup (A_1 + A_2)$.*

Proof. Let $x \in \text{Ap}(S)$, $x \neq 0$. Then, $x < c + \mu$ and x is a sum of elements of $P \setminus \{\mu\}$ (since $x - n\mu \notin S$ for $n > 0$). The sum of three elements of $P \setminus \{\mu\}$ is bigger than $c + \mu$, and thus cannot be equal to x ; likewise, x cannot be the sum of two elements of $P_2 \cup P_3$, and it also cannot be the sum of an element of P_1 and an element of P_3 . Hence, the unique possibilities are $x \in P$, $x \in P_1 + P_1$, or $x \in P_1 + P_2$. The claim follows by projecting onto $\mathbb{Z}/\mu\mathbb{Z}$. \square

The following is a modification of an idea introduced by Eliahou in [2].

Definition 2.2. Let $(a, b) \in P_1 \times P_2$. We say that (a, b) is an *Apéry pair* if $a + b \in \text{Ap}(S)$, and we denote by Σ the set of all Apéry pairs.

A subset $\{(a_i, b_i)\}_{i=1}^n \subseteq \Sigma$ is *independent* if $a_i \neq a_j$ and $b_i \neq b_j$ for every $i \neq j$; we denote by σ the maximal cardinality of an independent set of Apéry pairs.

We can relate Σ and σ through a graph-theoretic argument; see, e.g. [6] for the terminology used in the proof.

Proposition 2.3. *Let Σ and σ as above. Then,*

$$|\Sigma| \leq \sigma \cdot \max\{q_1, q_2\}.$$

Proof. Define a graph G by taking the disjoint union $P_1 \sqcup P_2$ as the set of vertices and Σ as the set of edges. Then, an independent subset of Σ is exactly an independent subset of edges of G , that is, a matching, and σ is exactly the matching number of G .

Since G is a bipartite graph, by König's theorem (see, e.g. [6, Theorem 1.1.1]) the matching number of G is equal to its point covering number, i.e. to the cardinality of the smallest set $S \subseteq V(G)$ such that every edge of G has a vertex in S .

For every $v \in V(G)$, the number of edges incident to v is at most q_1 if $v \in P_2$ and at most q_2 if $v \in P_1$; hence, the point covering number of G is at least $|E(G)| / \max\{q_1, q_2\}$. The claim follows. \square

Using this terminology, we now relate quantitatively μ , ν , q_1 and q_2 .

Proposition 2.4. *Let $S = \langle a_1, a_2, \dots, a_\nu \rangle$ be a numerical semigroup with $a_2 > \frac{c(S) + \mu(S)}{3}$. Then*

- (a) $\mu \leq \nu + \frac{q_1(q_1+1)}{2} + \sigma \cdot \max\{q_1, q_2\}$;
- (b) $\mu \leq \nu + \frac{q_1(q_1+1)}{2} + q_1 q_2$;
- (c) $\mu \leq \frac{1}{2}\nu(\nu+1)$;
- (d) $q_1 \geq \frac{2\nu-1-\sqrt{(2\nu+1)^2-8\mu}}{2}$.

Proof. (a) By Proposition 2.1, we have

$$\mu \leq |A| + |A_1 + A_1| + |A_1 + A_2|.$$

By definition, $|A| = \nu$, while $|A_1 + A_1| \leq q_1(q_1+1)/2$ by symmetry. If $x \in \text{Ap}(S) \cap (P_1 + P_2)$, then $x = a_1 + b_1$ for some Apéry pair $(a_1, b_1) \in \Sigma$; hence,

$$|\text{Ap}(S) \cap (P_1 + P_2)| \leq |\Sigma| \leq \sigma \cdot \max\{q_1, q_2\},$$

with the last inequality coming from Proposition 2.3. Since $|A_1 + A_2| \leq |\text{Ap}(S) \cap (P_1 + P_2)|$ the claim follows by summing the three bounds.

(b) is immediate from (a) and the fact that $\sigma \leq \min\{q_1, q_2\}$.

(c) Since $q_1 + q_2 \leq \nu - 1$, using (b), we have

$$\begin{aligned}\mu &\leq \nu + \frac{q_1(q_1 + 1)}{2} + q_1 q_2 \\ &\leq \nu + \frac{q_1(q_1 + 1)}{2} + q_1(\nu - 1 - q_1) = \nu - \frac{1}{2}q_1^2 + \left(\nu - \frac{1}{2}\right)q_1,\end{aligned}$$

and thus

$$q_1^2 - (2\nu - 1)q_1 + 2(\mu - \nu) \leq 0. \quad (1)$$

Therefore, the discriminant of the equation is nonnegative, that is,

$$0 \leq (2\nu - 1)^2 - 8(\mu - \nu) = (2\nu + 1)^2 - 8\mu,$$

or equivalently

$$\mu \leq \frac{1}{2}\nu^2 + \frac{1}{2}\nu + \frac{1}{8}.$$

Moreover, since μ and ν are integers, so is $\frac{1}{2}\nu^2 + \frac{1}{2}\nu = \frac{\nu(\nu+1)}{2}$, and thus we can discard the $\frac{1}{8}$.

(d) The inequality (1) holds for

$$\frac{2\nu - 1 - \sqrt{(2\nu + 1)^2 - 8\mu}}{2} \leq q_1 \leq \frac{2\nu - 1 + \sqrt{(2\nu + 1)^2 - 8\mu}}{2}.$$

The claim follows. \square

Remark 2.5. The bound in Proposition 2.4(d) may actually be negative: however, if $q_1 = 0$ then part (a) shows that $\mu \leq \nu$, and thus $\mu = \nu$. In this case, S is of maximal embedding dimension and Wilf's conjecture holds by [3, Theorem 20, Corollary 2].

3. Estimates on $|L(S)|$

The goal of this section is to estimate the cardinality of $L := L(S)$.

Lemma 3.1. *Let x, y, b, p be real numbers, with $p > 0$ and $x < y$, and let $A := b + p\mathbb{Z} := \{b + pn \mid n \in \mathbb{Z}\}$. Then*

- (a) $|A \cap [x, y)| \geq \lfloor \frac{y-x}{p} \rfloor$;
- (b) if $x \in A$ and $y \notin A$, then $|A \cap [x, y)| = \lfloor \frac{y-x}{p} \rfloor + 1$.

Proof. Let $k := \lfloor \frac{y-x}{p} \rfloor$. Then,

$$x + kp \leq x + \frac{y-x}{p} \cdot p = y;$$

hence, the k sets $[x, x+p), [x+p, x+2p), \dots, [x+(k-1)p, x+kp)$ are disjoint subintervals of $[x, y)$. In each $[x+ip, x+(i+1)p)$ there is exactly one element of A ; hence, $|A \cap [x, y)| \geq k$.

Moreover, if $x \in A$ then $x + kp \in A$; since $y \notin A$, then $x + kp \neq y$, and thus the interval $[x + kp, y]$ is nonempty and contains exactly one element of A (namely, $x + kp$). Hence, $|A \cap [x, y]| = k + 1$. \square

Proposition 3.2. *Let $S = \langle a_1, a_2, \dots, a_\nu \rangle$ be a numerical semigroup with $a_2 > \frac{c(S) + \mu(S)}{3}$. Then*

$$|L(S)| \geq \left\lfloor \frac{c}{\mu} \right\rfloor (1 + \sigma) + \left(\left\lfloor \frac{1}{2} \frac{c}{\mu} - \frac{1}{2} \right\rfloor + 1 \right) (q_1 - \sigma) + \left(\left\lfloor \frac{1}{3} \frac{c}{\mu} - \frac{2}{3} \right\rfloor + 1 \right) (q_2 - \sigma). \quad (2)$$

Proof. If x is an integer, let

$$L_x := \{a \in L(S) \mid a \equiv x \pmod{\mu}\}.$$

Clearly, L_x and L_y are disjoint if $x \not\equiv y \pmod{\mu}$. Hence,

$$|L(S)| = \sum_{x \in \text{Ap}(S)} |L_x| \geq |L_0| + \sum_{x \in P_1 \cup P_2} |L_x|.$$

If $x \in \text{Ap}(S)$, then, $L_x = (x + \mu\mathbb{Z}) \cap [x, c]$; by Lemma 3.1(a), we have $|L_x| \geq \lfloor \frac{c-x}{\mu} \rfloor$. In particular, $|L_0| \geq \lfloor \frac{c}{\mu} \rfloor$.

Take an independent set $\{(a_t, b_t)\}_{t=1}^\sigma$ of Apéry pairs of maximal cardinality, and write $P_1 = \{a_1, \dots, a_\sigma, c_1, \dots, c_r\}$, $P_2 = \{b_1, \dots, b_\sigma, d_1, \dots, d_s\}$. Then,

$$\sum_{x \in P_1 \cup P_2} |L_x| \geq \sum_{t=1}^\sigma (|L_{a_t}| + |L_{b_t}|) + \sum_{j=1}^r |L_{c_j}| + \sum_{k=1}^s |L_{d_k}|.$$

Suppose (a, b) is an Apéry pair. By Lemma 3.1(a),

$$\begin{aligned} |L_a| + |L_b| &= \left\lfloor \frac{c-a}{\mu} \right\rfloor + \left\lfloor \frac{c-b}{\mu} \right\rfloor \\ &\geq \frac{2c - (a+b)}{\mu} - 2 > \frac{c-\mu}{\mu} - 2 = \frac{c}{\mu} - 3. \end{aligned} \quad (3)$$

Moreover, $|L_a|$ and $|L_b|$ are both integers, and thus

$$|L_a| + |L_b| \geq \left\lfloor \frac{c}{\mu} \right\rfloor - 2.$$

On the other hand, if $x = c_j$ for some j , then $x \in P_1$ and so

$$|L_x| \geq \left\lfloor \frac{c - \frac{1}{2}(c + \mu)}{\mu} \right\rfloor = \left\lfloor \frac{1}{2} \frac{c}{\mu} - \frac{1}{2} \right\rfloor,$$

while if $x = d_k$ for some k then $x \in P_2$ and thus

$$|L_x| \geq \left\lfloor \frac{c - \frac{2}{3}(c + \mu)}{\mu} \right\rfloor \geq \left\lfloor \frac{1}{3} \frac{c}{\mu} - \frac{2}{3} \right\rfloor.$$

Summing everything, we have

$$|L(S)| \geq \left\lfloor \frac{c}{\mu} \right\rfloor + \sigma \left(\left\lfloor \frac{c}{\mu} \right\rfloor - 2 \right) + \left\lfloor \frac{1}{2} \frac{c}{\mu} - \frac{1}{2} \right\rfloor (q_1 - \sigma) + \left\lfloor \frac{1}{3} \frac{c}{\mu} - \frac{2}{3} \right\rfloor (q_2 - \sigma).$$

Applying Lemma 3.1(b), for every $x \in \{0\} \cup P_1 \cup P_2$, except possibly one (namely, the x such that $c \equiv x \pmod{\mu}$), there is a further element in $L_x \cap [x, c)$; hence, we can add $q_1 + q_2$ to the quantity on the right-hand side. By distributing this quantity (adding 2σ to the second summand, $q_1 - \sigma$ to the third one and $q_2 - \sigma$ to the last one) and putting together the first two summands, we have our claim. \square

Due to the presence of the floor functions, we can get better estimates on the right-hand side of (2) if we analyze it according to the integral part of c/μ . To do so, we introduce the following parameter.

Definition 3.3. Let S be a numerical semigroup. Let k be the largest integer such that $\frac{c(S)}{\mu(S)} - (6k - 1) < 6$: then, we set

$$\theta(S) := \frac{c(S) - (6k - 1)\mu(S)}{\mu(S)}.$$

It is immediate from the definition that $\theta(S)$ is a rational number and belongs to $[0, 6)$.

Proposition 3.4. Let $S = \langle a_1, a_2, \dots, a_\nu \rangle$ be a numerical semigroup with $a_2 > \frac{c(S) + \mu(S)}{3}$, and suppose that $c(S) > 3\mu(S)$. Let

$$l := \begin{cases} 5 & \text{if } \theta(S) \in [0, 4), \\ -1 & \text{if } \theta(S) \in [4, 6), \end{cases}$$

and define

$$\begin{cases} \alpha := 1 - \frac{1}{l + \lfloor \theta \rfloor + 1} (\theta - \lfloor \theta \rfloor), \\ \beta := \min \left\{ \frac{1}{2}, \frac{1}{2} - \frac{1}{l + \lfloor \theta \rfloor + 1} \left(\frac{\theta}{2} - \left\lfloor \frac{\theta}{2} \right\rfloor - \frac{1}{2} \right) \right\}, \\ \gamma := \min \left\{ \frac{1}{3}, \frac{1}{3} - \frac{1}{l + \lfloor \theta \rfloor + 1} \left(\frac{\theta}{3} - \left\lfloor \frac{\theta}{3} \right\rfloor - \frac{1}{3} \right) \right\}. \end{cases}$$

Then,

$$\frac{\nu(S)|L(S)|}{c(S)} \geq \frac{\nu(S)}{\mu(S)} [\alpha(1 + \sigma) + \beta(q_1 - \sigma) + \gamma(q_2 - \sigma)].$$

Note that the hypothesis $c(S) > 3\mu(S)$ is not really restrictive in our context, since, by [1], Wilf's conjecture holds when $c \leq 3\mu$.

Proof. Let $\theta := \theta(S)$, $\nu := \nu(S)$, $\mu := \mu(S)$, $c := c(S)$, $L := L(S)$. By definition, $c = (6k - 1)\mu + \theta\mu$. Then,

$$\left\lfloor \frac{c}{\mu} \right\rfloor = \left\lfloor \frac{(6k - 1)\mu + \theta\mu}{\mu} \right\rfloor = 6k - 1 + \lfloor \theta \rfloor = \frac{c}{\mu} - (\theta - \lfloor \theta \rfloor);$$

analogously,

$$\left\lfloor \frac{1}{2} \frac{c}{\mu} - \frac{1}{2} \right\rfloor + 1 = \frac{c}{2\mu} + \frac{1}{2} - \left(\frac{\theta}{2} - \left\lfloor \frac{\theta}{2} \right\rfloor \right)$$

and

$$\left\lfloor \frac{1}{3} \frac{c}{\mu} - \frac{2}{3} \right\rfloor + 1 = \frac{c}{3\mu} + \frac{1}{3} - \left(\frac{\theta}{3} - \left\lfloor \frac{\theta}{3} \right\rfloor \right).$$

Multiplying (2) by $\frac{\nu}{c}$, we obtain

$$\begin{aligned} \frac{\nu|L|}{c} &\geq \frac{\nu}{\mu} \left[1 - \frac{\mu}{c}(\theta - \lfloor \theta \rfloor) \right] (1 + \sigma) \\ &\quad + \frac{\nu}{\mu} \left[\frac{1}{2} - \frac{\mu}{c} \left(\frac{\theta}{2} - \left\lfloor \frac{\theta}{2} \right\rfloor - \frac{1}{2} \right) \right] (q_1 - \sigma) \\ &\quad + \frac{\nu}{\mu} \left[\frac{1}{3} - \frac{\mu}{c} \left(\frac{\theta}{3} - \left\lfloor \frac{\theta}{3} \right\rfloor - \frac{1}{3} \right) \right] (q_2 - \sigma). \end{aligned} \quad (4)$$

Since $c > 3\mu$, we have $c \geq (l + \theta)\mu$; equivalently, $\frac{\mu}{c} \leq \frac{1}{l + \theta} \leq \frac{1}{l + \lfloor \theta \rfloor + 1}$. Since $\theta \geq \lfloor \theta \rfloor$, it follows that

$$1 - \frac{\mu}{c}(\theta - \lfloor \theta \rfloor) \geq 1 - \frac{1}{l + \lfloor \theta \rfloor + 1}(\theta - \lfloor \theta \rfloor),$$

and the right-hand side is exactly α .

If $\frac{\theta}{2} - \lfloor \frac{\theta}{2} \rfloor - \frac{1}{2} \leq 0$, then the coefficient of $q_1 - \sigma$ in (4) is at least $\frac{1}{2}$; otherwise, as above,

$$\frac{1}{2} - \frac{\mu}{c} \left(\frac{\theta}{2} - \left\lfloor \frac{\theta}{2} \right\rfloor - \frac{1}{2} \right) \geq \frac{1}{2} - \frac{1}{l + \lfloor \theta \rfloor + 1} \left(\frac{\theta}{2} - \left\lfloor \frac{\theta}{2} \right\rfloor - \frac{1}{2} \right).$$

Thus, the coefficient of $q_1 - \sigma$ is larger than β . Analogously, we obtain that the coefficient of $q_2 - \sigma$ in (4) is at least γ . Since, by definition, $1 + \sigma$, $q_1 - \sigma$ and $q_2 - \sigma$ are always nonnegative, we have

$$\frac{\nu|L|}{c} \geq \frac{\nu}{\mu} [\alpha(1 + \sigma) + \beta(q_1 - \sigma) + \gamma(q_2 - \sigma)],$$

as claimed. \square

Proposition 3.5. *Let α, β, γ, l be as in Proposition 3.4. Then*

- if $\lfloor \theta \rfloor = 0$: $\alpha \geq \frac{5}{6}$; $\beta \geq \frac{1}{2}$; $\gamma \geq \frac{1}{3}$;*
- if $\lfloor \theta \rfloor = 1$: $\alpha \geq \frac{6}{7}$; $\beta \geq \frac{3}{7}$; $\gamma \geq \frac{2}{7}$;*
- if $\lfloor \theta \rfloor = 2$: $\alpha \geq \frac{7}{8}$; $\beta \geq \frac{1}{2}$; $\gamma \geq \frac{1}{4}$;*
- if $\lfloor \theta \rfloor = 3$: $\alpha \geq \frac{8}{9}$; $\beta \geq \frac{4}{9}$; $\gamma \geq \frac{1}{3}$;*
- if $\lfloor \theta \rfloor = 4$: $\alpha \geq \frac{3}{4}$; $\beta \geq \frac{1}{2}$; $\gamma \geq \frac{1}{4}$;*
- if $\lfloor \theta \rfloor = 5$: $\alpha \geq \frac{4}{5}$; $\beta \geq \frac{2}{5}$; $\gamma \geq \frac{1}{5}$.*

Proof. The estimates for α follow immediately from Proposition 3.4 and the fact that $\theta - \lfloor \theta \rfloor \leq 1$.

If $\lfloor \theta \rfloor \equiv 0 \pmod 2$, then $\frac{\theta}{2} - \lfloor \frac{\theta}{2} \rfloor \leq \frac{1}{2}$; thus,

$$\frac{1}{2} - \left(\frac{\theta}{2} - \left\lfloor \frac{\theta}{2} \right\rfloor - \frac{1}{2} \right) \geq \frac{1}{2} - 0 = \frac{1}{2},$$

and so $\beta = \frac{1}{2}$. On the other hand, if $\lfloor \theta \rfloor \equiv 1 \pmod 2$, then $\frac{\theta}{2} - \lfloor \frac{\theta}{2} \rfloor \leq 1$ and so

$$\beta \geq \frac{1}{2} - \frac{1}{l + \lfloor \theta \rfloor + 1} \left(1 - \frac{1}{2} \right) = \frac{1}{2} \left(1 - \frac{1}{l + \lfloor \theta \rfloor + 1} \right);$$

the claim is obtained substituting l and $\lfloor \theta \rfloor$ with their actual values.

Similarly, if $\lfloor \theta \rfloor \equiv 0 \pmod 3$, then $\frac{\theta}{3} - \lfloor \frac{\theta}{3} \rfloor \leq \frac{1}{3}$ and so $\gamma = \frac{1}{3}$. On the other hand, if $\lfloor \theta \rfloor \equiv 1 \pmod 3$, then $\frac{\theta}{3} - \lfloor \frac{\theta}{3} \rfloor \leq \frac{2}{3}$ and thus

$$\gamma \geq \frac{1}{3} - \frac{1}{l + \lfloor \theta \rfloor + 1} \left(\frac{2}{3} - \frac{1}{3} \right) = \frac{1}{3} \left(1 - \frac{1}{l + \lfloor \theta \rfloor + 1} \right).$$

If $\lfloor \theta \rfloor \equiv 2 \pmod 3$, the same reasoning gives a similar estimate, but with $\frac{2}{l + \lfloor \theta \rfloor + 1}$ instead of $\frac{1}{l + \lfloor \theta \rfloor + 1}$. Substituting the actual values of l and $\lfloor \theta \rfloor$ we get the values in the statement. \square

Corollary 3.6. *Let $\alpha_i, \beta_i, \gamma_i$ be the values found in Proposition 3.5 when $\lfloor \theta \rfloor = i$, and let*

$$\ell_i(x, y, z) := \alpha_i(1 + z) + \beta_i(x - z) + \gamma_i(y - z).$$

If $i \leq 3$ and $j \geq 4$, then $\ell_i(x, y, z) \geq \ell_j(x, y, z)$ for all $x, y, z \in \mathbb{R}$ such that $x, y, z \geq 0$ and $z \leq \min\{x, y\}$.

Proof. With i, j as defined, we have $\alpha_i \geq \alpha_j$, $\beta_i \geq \beta_j$ and $\gamma_i \geq \gamma_j$; the fact that $1 + z$, $x - z$ and $y - z$ are nonnegative now guarantees that $\ell_i(x, y, z) \geq \ell_j(x, y, z)$. \square

4. Some Inequalities

We collect in this section some inequalities obtained through analytic methods, which we will use in the proof of our main result.

Definition 4.1. Let $\rho \geq 1$ be a real number. Set

$$\mathcal{I}(\rho) := \left\{ (x, y) \in \mathbb{R}^2 \mid x \geq 1, y \geq 0, \frac{x(x+1)}{2} + xy = \rho \right\}$$

and

$$\mathcal{A}(\rho) := \left\{ (x, y) \in \mathbb{R}^2 \mid x \geq 1, y \geq 0, \frac{x(x+1)}{2} + xy \geq \rho \right\}.$$

Graphically, the set $\{(x, y) \in \mathbb{R}^2 \mid x > 0, \frac{x(x+1)}{2} + xy = \rho\}$ is a branch of a hyperbola having the y -axis as a vertical asymptote and crossing the x -axis at

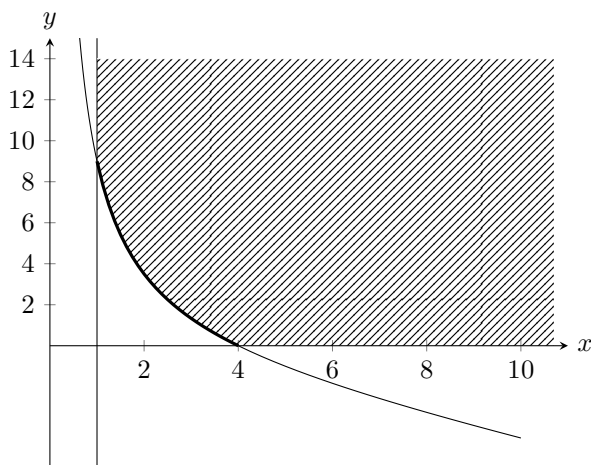


Fig. 1. The line $\mathcal{I}(10)$ (the bolded subset of the hyperbola) and the region $\mathcal{A}(10)$ (shaded).

$x_0 := \frac{-1+\sqrt{1+8\rho}}{2} \geq 1$ (since $\rho \geq 1$). In particular, $\mathcal{I}(\rho)$ is the subset of this branch defined by x varying between 1 and x_0 , while $\mathcal{A}(\rho)$ is the part of the first quadrant that is above $\mathcal{I}(\rho)$; see Fig. 1.

Lemma 4.2. *Let $f(x, y) := \alpha + \beta x + \gamma y$, where α, β, γ are positive real numbers such that $2\beta \geq \gamma$. For every $(x, y) \in \mathcal{A}(\rho)$, we have*

$$f(x, y) \geq \alpha - \frac{\gamma}{2} + \sqrt{2\gamma(2\beta - \gamma)}\sqrt{\rho}.$$

Proof. For every $(x, y) \in \mathcal{A}(\rho)$, there is an $(x', y') \in \mathcal{I}(\rho)$ such that $x' \leq x$ and $y' \leq y$. Since f is a linear function with positive coefficients, the minimum of f on $\mathcal{A}(\rho)$ must belong to $\mathcal{I}(\rho)$ (and it exists since $\mathcal{I}(\rho)$ is compact).

Let (x_0, y_0) be the point of minimum of f on $\mathcal{I}(\rho)$. By Lagrange multipliers,

$$\begin{cases} \partial_x f(x_0, y_0) = x_0 + \frac{1}{2} + y_0 = \beta\lambda, \\ \partial_y f(x_0, y_0) = x_0 = \gamma\lambda, \end{cases}$$

for some $\lambda \in \mathbb{R}$; imposing $(x_0, y_0) \in \mathcal{I}(\rho)$, we have

$$\begin{aligned} \rho &= x_0 \left(\frac{1}{2}x_0 + \frac{1}{2} + y_0 \right) \\ &= \partial_y f(x_0, y_0) \cdot \left(\partial_x f(x_0, y_0) - \frac{1}{2}\partial_y f(x_0, y_0) \right) = \frac{\gamma(2\beta - \gamma)}{2}\lambda^2 \end{aligned}$$

and thus

$$\lambda = \sqrt{\frac{2}{\gamma(2\beta - \gamma)}}\sqrt{\rho}.$$

Substituting in f , we have

$$\begin{aligned} f(x_0, y_0) &= \alpha + \beta\gamma\lambda + \gamma \left[(\beta - \gamma)\lambda - \frac{1}{2} \right] \\ &= \alpha - \frac{\gamma}{2} + \gamma(\beta + \beta - \gamma) \sqrt{\frac{2}{\gamma(2\beta - \gamma)}} \sqrt{\rho} \\ &= \alpha - \frac{\gamma}{2} + \sqrt{2\gamma(2\beta - \gamma)} \sqrt{\rho}, \end{aligned}$$

as claimed. \square

Lemma 4.3. *Let ζ, ξ be real numbers such that $\zeta \leq 1$, and let $\lambda := \frac{4(1-\zeta)}{\xi^2}$. If x, y are positive real numbers such that*

$$\begin{cases} 2y \leq x < \xi^2 y^2 - (2\zeta - 1)y - \frac{(1 - \zeta)^2}{\xi^2} + \frac{\lambda^3 \xi^2}{32} \cdot \frac{1}{y} \left(\frac{y}{y - \lambda} \right)^{5/2}, \\ y > \max \left\{ \lambda, \frac{(\zeta - 2)^2}{\xi^2} \right\} \end{cases}, \quad (5)$$

then

$$\zeta + \xi \sqrt{x - y} \geq \frac{x}{y}. \quad (6)$$

Proof. Since $y > 0$, the inequality (6) is equivalent to

$$\xi y \sqrt{x - y} \geq x - \zeta y,$$

and squaring both sides (which makes sense since $x - \zeta y \geq x - y > 0$) this is equivalent to

$$\xi^2 y^2 (x - y) \geq x^2 - 2\zeta xy + \zeta^2 y^2,$$

that is,

$$x^2 - (2\zeta y + \xi^2 y^2)x + \zeta^2 y^2 + \xi^2 y^3 \leq 0. \quad (7)$$

The claim will follow once we prove that the roots of (7) exist and lie outside the interval defined by the first inequality in (5).

For $x = 2y$, the left-hand side of (7) becomes

$$\begin{aligned} &4y^2 - (2\zeta y + \xi^2 y^2)(2y) + \zeta^2 y^2 + \xi^2 y^3 \\ &= (4 - 4\zeta + \zeta^2)y^2 + (\xi^2 - 2\xi^2)y^3 \\ &= y^2[(\zeta - 2)^2 - \xi^2 y] \end{aligned}$$

which is negative by the second hypothesis of (5).

The larger root of Eq. (7) is equal to

$$\begin{aligned} x_+ &:= \frac{(2\zeta y + \xi^2 y^2) + \sqrt{y^2 \xi^2 [4(\zeta - 1)y + \xi^2 y^2]}}{2} \\ &= \frac{2\zeta y + \xi^2 y^2 + \xi^2 y^2 \sqrt{1 - \frac{\lambda}{y}}}{2}. \end{aligned}$$

By hypothesis, $\lambda < y$, and thus we can expand $\sqrt{1 - \frac{\lambda}{y}}$ as a Taylor series in $\frac{\lambda}{y}$ around 0. Then,

$$\begin{aligned} \xi^2 y^2 \sqrt{1 - \frac{\lambda}{y}} &= \xi^2 y^2 \left(1 - \frac{1}{2} \cdot \frac{\lambda}{y} - \frac{1}{8} \frac{\lambda^2}{y^2} + R_2(\lambda/y) \right) \\ &= \xi^2 y^2 - 2(1 - \zeta)y - \frac{2(1 - \zeta)^2}{\xi^2} + \xi^2 y^2 R_2(\lambda/y), \end{aligned}$$

where R_2 is the remainder, and thus

$$\begin{aligned} x_+ &= \frac{2\zeta y + \xi^2 y^2 + \xi^2 y^2 - 2(1 - \zeta)y - \frac{2(1 - \zeta)^2}{\xi^2} + \xi^2 y^2 R_2(\lambda/y)}{2} \\ &= \xi^2 y^2 + (2\zeta - 1)y - \frac{(1 - \zeta)^2}{\xi^2} + \frac{\xi^2 y^2 R_2(\lambda/y)}{2}. \end{aligned}$$

Since $\frac{d^3}{dz^3} \sqrt{1 - z} = -\frac{3}{8} \frac{1}{(1 - z)^{5/2}}$, by Taylor's theorem there is an $\eta \in [0, \lambda/y]$ such that

$$|\xi^2 y^2 R_2(\lambda/y)| = \xi^2 y^2 \frac{1}{6} \left(\frac{\lambda}{y} \right)^3 \frac{3}{8} \frac{1}{(1 - \eta)^{5/2}}.$$

The function $z \mapsto \frac{3}{8} \frac{1}{(1 - z)^{5/2}}$ is increasing in $[0, 1]$; hence, to bound above the remainder, we can take $\eta = \lambda/y$. We obtain

$$|\xi^2 y^2 R_2(\lambda/y)| \leq \frac{\lambda^3 \xi^2}{16} \frac{1}{(1 - (\lambda/y))^{5/2}},$$

from which the claim follows. \square

5. Wilf's Conjecture for Large Second Generator

The estimates of the previous section are connected to Wilf's conjecture in the following way.

Proposition 5.1. *Let $S = \langle a_1, a_2, \dots, a_\nu \rangle$ be a semigroup of multiplicity μ , embedding dimension ν and such that $a_2 > \frac{c+\mu}{3}$. Let α, β, γ be defined as in Proposition 3.4, and let*

$$\ell(x, y, z) := \alpha(1 + z) + \beta(x - z) + \gamma(y - z).$$

If, for every $(x, y) \in \mathcal{A}(\mu - \nu)$, we have

$$\begin{cases} \ell(x, y, x) \geq \mu/\nu, \\ \ell(x, y, y) \geq \mu/\nu, \end{cases}$$

then Wilf's conjecture holds for S .

Proof. By Proposition 3.4, we have

$$\frac{\nu|L|}{c} \geq \frac{\nu}{\mu} \ell(q_1, q_2, \sigma).$$

By Proposition 2.4(a), the triple (q_1, q_2, σ) satisfies the inequality

$$\frac{q_1(q_1 + 1)}{2} + \sigma \cdot \max\{q_1, q_2\} \geq \mu - \nu.$$

We distinguish two cases.

If $q_1 \geq q_2$, then the previous inequality becomes $\frac{q_1(q_1+1)}{2} + \sigma q_1 \geq \mu - \nu$, that is, $(q_1, \sigma) \in \mathcal{A}(\mu - \nu)$. Since the coefficients α, β, γ are positive and $q_2 \geq \sigma$, we have

$$\ell(q_1, q_2, \sigma) \geq \ell(q_1, \sigma, \sigma) \geq \frac{\mu}{\nu}$$

by hypothesis, and thus

$$\frac{\nu|L|}{c} \geq \frac{\nu}{\mu} \frac{\mu}{\nu} = 1.$$

Thus, $\nu|L| \geq c$, i.e. Wilf's conjecture holds for S .

Suppose that $q_1 \leq q_2$, and consider the following subset of the three-dimensional space:

$$\Omega := \left\{ (x, y, z) \in \mathbb{R}^3 \mid x, y, z \geq 0, z \leq x \leq y, \frac{x(x+1)}{2} + yz \geq \mu - \nu \right\}.$$

Then, $(q_1, q_2, \sigma) \in \Omega$. By definition, ℓ is a linear function with positive coefficients. Thus, as in the proof of Lemma 4.2, the point of minimum of ℓ on Ω belongs to the boundary

$$\Delta := \left\{ (x, y, z) \mid x, y, z \geq 0, z \leq x \leq y, \frac{x(x+1)}{2} + yz = \mu - \nu \right\}.$$

Let also

$$\Delta' := \left\{ (x, y, z) \mid x, y, z \geq 0, \frac{x(x+1)}{2} + yz = \mu - \nu \right\}.$$

The point of minimum (x_0, y_0, z_0) of ℓ on Δ' , by Lagrange multipliers, must satisfy

$$\begin{cases} \partial_x \ell(x_0, y_0, z_0) = x_0 + \frac{1}{2} = \beta\lambda, \\ \partial_y \ell(x_0, y_0, z_0) = z_0 = \gamma\lambda, \\ \partial_z \ell(x_0, y_0, z_0) = y_0 = (\alpha - \beta - \gamma)\lambda, \end{cases}$$

for some λ . For every choice of $[\theta]$, we have $\gamma > \alpha - \beta - \gamma$; hence, we have $z_0 > y_0$. Since $z \leq y$ for all points of Ω , the point of minimum of ℓ on Δ must satisfy another boundary condition, that is, either $x_0 = z_0$ or $x_0 = y_0$. In the former case, $(x_0, y_0) \in \mathcal{I}(\mu - \nu)$, and thus as in the previous case $\ell(x_0, y_0, x_0) \geq \mu/\nu$ by hypothesis.

On the other hand, if $x_0 = y_0$ then $(x_0, z_0) \in \mathcal{I}(\mu - \nu)$, and thus $\ell(x_0, y_0, z_0) = \ell(x_0, x_0, z_0)$. However,

$$\ell(x_0, x_0, z_0) = \alpha(1 + z_0) + (\beta + \gamma)(x_0 - z_0) \geq \alpha(1 + z_0) + \beta(x_0 - z_0) = \ell(x_0, z_0, z_0)$$

since $x_0 \geq z_0$ and $\gamma > 0$. By hypothesis, $\ell(x_0, z_0, z_0) \geq \mu/\nu$, and thus also $\ell(x_0, x_0, z_0) \geq \mu/\nu$. In both cases, we have

$$\frac{\nu|L|}{c} \geq \frac{\nu}{\mu} \ell(q_1, q_2, \sigma) \geq \frac{\nu}{\mu} \frac{\mu}{\nu} = 1,$$

and thus Wilf's conjecture holds for S . \square

Theorem 5.2. *For every $\epsilon > 0$, there is a $\nu_0(\epsilon)$ such that, if $S = \langle a_1, a_2, \dots, a_\nu \rangle$ is a numerical semigroup such that*

- $a_2 > \frac{c(S) + \mu(S)}{3}$,
- $\nu(S) = \nu \geq \nu_0(\epsilon)$, and
- $\mu(S) \leq \frac{8}{25}\nu^2 + \frac{1}{5}\nu - \frac{1}{2} - \epsilon$,

then S satisfies Wilf's conjecture.

Proof. If $c \leq 3\mu$, then Wilf's conjecture holds by [1]. Thus, we can suppose $c > 3\mu$.

Let $\ell(x, y, z) := \alpha(1 + z) + \beta(x - z) + \gamma(y - z)$, where α, β, γ are defined as in Proposition 3.4. By Proposition 5.1, we need to show that $\ell(x, y, x) \geq \mu/\nu$ and $\ell(x, y, y) \geq \mu/\nu$ for all $(x, y) \in \mathcal{A}(\mu - \nu)$.

Let $f(x, y) := A + Bx + Cy$ be either $\ell(x, y, x)$ or $\ell(x, y, y)$. By Lemma 4.2, we have

$$f(x, y) \geq \left(A - \frac{C}{2}\right) + \sqrt{2C(2B - C)}\sqrt{\mu - \nu}.$$

Set $\zeta := A - \frac{C}{2}$ and $\xi := \sqrt{2C(2B - C)}$. By Lemma 4.3, for every $\epsilon > 0$ and for large enough ν the inequality $f(x, y) \geq \mu/\nu$ holds as long as

$$\mu < \xi^2\nu^2 + (2\zeta - 1)\nu - \frac{(1 - \zeta)^2}{\xi^2} - \epsilon \quad (8)$$

since the last term of the first equation of (5) goes to 0 as ν goes to infinity.

Since we are interested in the asymptotic worst case for the right-hand side of (8), we need to take the smallest ξ^2 from the possible ones. If $f(x, y) = \ell(x, y, x)$, then we have $f(x, y) = \alpha + (\alpha - \gamma)x + \gamma y$ and thus

$$\xi^2 = 2\gamma(2\alpha - 2\gamma - \gamma) = 2\gamma(2\alpha - 3\gamma);$$

Table 1. Values of ξ^2 for $f(x, y) = \ell(x, y, x)$ and $f(x, y) = \ell(x, y, y)$.

	$\ell(x, y, x)$	$\ell(x, y, y)$
$\lfloor \theta \rfloor = 0$	$4/9$	$4/9$
$\lfloor \theta \rfloor = 1$	$24/49$	$18/49$
$\lfloor \theta \rfloor = 2$	$1/2$	$15/32$
$\lfloor \theta \rfloor = 3$	$14/27$	$32/81$
$\lfloor \theta \rfloor = 4$	$3/8$	$3/8$
$\lfloor \theta \rfloor = 5$	$2/5$	$8/25$

on the other hand, if $f(x, y) = \ell(x, y, y)$ then $f(x, y) = \alpha + \beta x + (\alpha - \beta)y$ and thus

$$\xi^2 = 2(\alpha - \beta)(2\beta - (\alpha - \beta)) = 2(\alpha - \beta)(3\beta - \alpha).$$

Substituting the α , β and γ with the estimates found in Proposition 3.5 (and note that, by Corollary 3.6, it would be enough to consider the cases $\lfloor \theta \rfloor = 4$ and $\lfloor \theta \rfloor = 5$), the minimum among these values is $8/25$, which happens for $\lfloor \theta \rfloor = 5$ and for the case $\ell(x, y, y)$ (see Table 1). Therefore, for large ν , if μ is smaller than the right-hand side of (8) when ξ and ζ are calculated in this case then they are smaller in every case. By making the substitution, we have our claim. \square

A more explicit version is the following.

Proposition 5.3. *Let $S = \langle a_1, a_2, \dots, a_\nu \rangle$ be a numerical semigroup with $\nu(S) = \nu \geq 10$. If $a_2 > \frac{c(S) + \mu(S)}{3}$ and*

$$\mu(S) \leq \frac{8}{25}\nu(S)^2 + \frac{1}{5}\nu(S) - \frac{5}{4},$$

then S satisfies Wilf's conjecture.

Proof. As in the proof of Theorem 5.2, we can suppose $c > 3\mu$.

Following the proof of Theorem 5.2, let $\ell(x, y, z) := \alpha(1+z) + \beta(x-z) + \gamma(y-z)$, where α, β, γ are as in Proposition 3.4. By Corollary 3.6, it is enough to consider the cases $(\alpha, \beta, \gamma) = (\frac{3}{4}, \frac{1}{2}, \frac{1}{4})$ and $(\alpha, \beta, \gamma) = (\frac{4}{5}, \frac{2}{5}, \frac{1}{5})$.

We need to consider four different $f(x, y) := A + Bx + Cy$:

- $\lfloor \theta \rfloor = 4$ and $f(x, y) = \ell(x, y, x)$: then, $A = \frac{3}{4}$, $B = \frac{1}{2}$, $C = \frac{1}{4}$;
- $\lfloor \theta \rfloor = 4$ and $f(x, y) = \ell(x, y, y)$: then, $A = \frac{3}{4}$, $B = \frac{1}{2}$, $C = \frac{1}{4}$;
- $\lfloor \theta \rfloor = 5$ and $f(x, y) = \ell(x, y, x)$: then, $A = \frac{4}{5}$, $B = \frac{3}{5}$, $C = \frac{1}{5}$;
- $\lfloor \theta \rfloor = 5$ and $f(x, y) = \ell(x, y, y)$: then, $A = \frac{4}{5}$, $B = \frac{2}{5}$, $C = \frac{2}{5}$.

Set $\xi := \sqrt{2C(2B - C)}$, $\zeta := A - \frac{C}{2}$ and $\lambda := \frac{4(1-\zeta)}{\xi^2}$. By Lemmas 4.2 and 4.3, we have $f(x, y) \geq \mu/\nu$ on $\mathcal{A}(\mu - \nu)$ if

$$\mu < \xi^2 \nu^2 + (2\zeta - 1)\nu - \frac{(1 - \zeta)^2}{\xi^2} - \frac{\lambda^3 \xi^2}{32} \cdot \frac{1}{\nu} \left(\frac{\nu}{\nu - \lambda} \right)^{5/2}.$$

Substituting the values of ξ, ζ, λ of each case and using $\nu \geq 10$, we have that $f(x, y) \geq \mu/\nu$ under the following conditions:

- if $\lfloor \theta \rfloor = 4$, when

$$\mu < \frac{3}{8}\nu^2 + \frac{1}{4}\nu - \frac{3}{8} - \frac{3}{4} \cdot \frac{1}{10} \left(\frac{10}{6} \right)^{5/2};$$

- if $\lfloor \theta \rfloor = 5$ and $f(x, y) = \ell(x, y, x)$, when

$$\mu < \frac{2}{5}\nu^2 + \frac{2}{5}\nu - \frac{9}{40} - \frac{27}{80} \cdot \frac{1}{10} \left(\frac{10}{7} \right)^{5/2};$$

- if $\lfloor \theta \rfloor = 5$ and $f(x, y) = \ell(x, y, y)$, when

$$\mu < \frac{8}{25}\nu^2 + \frac{1}{5}\nu - \frac{1}{2} - \frac{5}{4} \cdot \frac{1}{10} \left(\frac{10}{5} \right)^{5/2}.$$

Among these, the worst case (for all coefficients) is the last one, and thus it is enough to consider that one. We have

$$\frac{5}{4} \cdot \frac{1}{10} \left(\frac{10}{5} \right)^{5/2} = \frac{5 \cdot 4\sqrt{2}}{40} = \frac{\sqrt{2}}{2} < \frac{3}{4},$$

and thus $f(x, y) \geq \mu/\nu$ if

$$\mu < \frac{8}{25}\nu^2 + \frac{1}{5}\nu - \frac{1}{2} - \frac{3}{4}.$$

By Proposition 5.1, Wilf's conjecture holds in this case, as claimed. \square

Remark 5.4. Let $\rho := \lceil \frac{\mu}{\nu} \rceil$. In [7], it is shown that Wilf's conjecture holds when $\mu \geq \frac{f(\rho)}{g(\rho)}$, where f is a polynomial of degree 5 and g a polynomial of degree 1, and if all prime factors of μ are at least equal to ρ . Asymptotically, the first condition can be rephrased as $\mu \geq c\rho^4$, where c is a constant. Since $\rho \in [\mu/\nu, \mu/\nu + 1)$, this is equivalent to

$$\mu \geq c' \left(\frac{\mu}{\nu} \right)^4 \Rightarrow \mu \leq c' \nu^{4/3}$$

for some constant c' .

Written in this way, this result can be compared with Theorem 5.2: in the latter, we are able to increase the exponent of ν from $4/3$ to 2 , at the price of adding the hypothesis $a_2 > \frac{c+\mu}{3}$.

To conclude the paper, we give three variants of Theorem 5.2 that can be proved with a similar argument. The first one looks at case $c \equiv 0 \pmod{\mu}$, the second one strengthens the coefficient $\frac{8}{25}$ and the third one weakens Wilf's conjecture.

Proposition 5.5. *If $S = \langle a_1, a_2, \dots, a_\nu \rangle$ is a numerical semigroup such that*

- $a_2 > \frac{c(S)+\mu(S)}{3}$,
- $\nu(S) \geq 10$, and
- $c \equiv 0 \pmod{\mu}$,

then S satisfies Wilf's conjecture.

Proof. Let α, β, γ, l be as in Proposition 3.4. Since $c \equiv 0 \pmod{\mu}$, the number θ is an integer. Thus, α is always equal to 1; likewise, β is always equal to $\frac{1}{2}$, since $\frac{\theta}{2} - \lfloor \frac{\theta}{2} \rfloor$ is either 0 or $\frac{1}{2}$. If θ is equivalent to 0 or 1 modulo 3, then in the same way $\gamma = \frac{1}{3}$; on the other hand, if $\theta = 2$ we have

$$\gamma = \frac{1}{3} - \frac{1}{8} \left(\frac{2}{3} - \frac{1}{3} \right) = \frac{7}{24},$$

while if $\theta = 5$ then

$$\gamma = \frac{1}{3} - \frac{1}{5} \left(\frac{2}{3} - 1 - \frac{1}{3} \right) = \frac{4}{15}.$$

Since the minimal γ is $\frac{4}{15}$, we have always $\ell(x, y, z) \geq (1+z) + \frac{1}{2}(x-z) + \frac{4}{15}(y-z)$.

Let $f(x, y) := \ell(x, y, y)$. Then, $f(x, y) = 1 + \frac{1}{2}x + \frac{1}{2}y$. Let $A := 1$, $B := \frac{1}{2} =: C$. As in the proof of Theorem 5.2, define

$$\begin{cases} \xi := \sqrt{2C(2B-C)} = \sqrt{\frac{1}{2}}, \\ \zeta := A - \frac{C}{2} = \frac{3}{4}, \\ \lambda := \frac{4(1-\zeta)}{\xi^2} = 2. \end{cases}$$

By Lemma 4.3, the inequality $f(x, y) \geq \mu/\nu$ holds as long as

$$\begin{aligned} \mu &< \xi^2 \nu^2 + (2\zeta - 1)\nu - \frac{(1-\zeta)^2}{\xi^2} - \frac{\lambda^3 \xi^2}{32} \cdot \frac{1}{\nu} \left(\frac{\nu}{\nu - \lambda} \right)^{5/2} \\ &= \frac{1}{2} \nu^2 + \frac{1}{2} \nu - \frac{1}{4} - \frac{1}{2\nu} \left(\frac{\nu}{\nu - \lambda} \right)^{5/2}. \end{aligned}$$

Since $\nu \geq 10$, we have

$$\frac{1}{2\nu} \left(\frac{\nu}{\nu - \lambda} \right)^{5/2} \leq \frac{1}{20} \left(\frac{5}{4} \right)^{5/2} < \frac{1}{10}$$

and thus the claim holds for

$$\mu < \frac{1}{2} \nu^2 + \frac{1}{2} \nu - \frac{1}{4} - \frac{1}{10}.$$

Since $\frac{1}{2} \nu^2 + \frac{1}{2} \nu$ is always an integer and $\frac{1}{4} + \frac{1}{10} < \frac{1}{2}$, we have $f(x, y) \geq \mu/\nu$ (and thus that Wilf's conjecture holds) whenever $\mu \leq \frac{1}{2} \nu^2 + \frac{1}{2} \nu - 1$.

By Proposition 2.4(c), we always have $\mu \leq \frac{1}{2} \nu^2 + \frac{1}{2} \nu$, so the only case left to consider is $\mu = \frac{1}{2} \nu^2 + \frac{1}{2} \nu = \frac{\nu(\nu+1)}{2}$. Under this condition, we have, by Proposition 2.4(d),

$$q_1 \geq \frac{2\nu - 1 - 1}{2} = \nu - 1.$$

Since also $q_1 \leq \nu - 1$, we must have $q_1 = \nu - 1$ and $q_2 = 0$. In this case,

$$\frac{\nu|L|}{c} \geq \frac{\nu}{\mu} \left[1 + \frac{1}{2}(\nu - 1) \right] = \frac{\nu}{\mu} \cdot \frac{\nu + 1}{2} = \frac{\nu(\nu + 1)}{2\mu} = 1$$

and thus S satisfies Wilf's conjecture.

Suppose now $f(x, y) = \ell(x, y, x) = \alpha + (\alpha - \gamma)x + \gamma y$. Since $\gamma \geq \frac{4}{15}$, it is enough to consider the case $\alpha = 1$, $\gamma = \frac{4}{15}$. Define $A, B, C, \xi, \zeta, \lambda$ as above; then, we have $A = 1$, $B = \frac{11}{15}$, $C = \frac{4}{15}$ and so

$$\begin{cases} \xi := \sqrt{2C(2B - C)} = \sqrt{\frac{48}{75}}, \\ \zeta := A - \frac{C}{2} = \frac{13}{15}, \\ \lambda := \frac{4(1 - \zeta)}{\xi^2} = \frac{55}{12}. \end{cases}$$

Then,

$$\frac{\lambda^3 \xi^2}{32} \cdot \frac{1}{\nu} \left(\frac{\nu}{\nu - \lambda} \right)^{5/2} \leq \frac{\left(\frac{55}{12} \right)^3 \cdot \frac{48}{75}}{32} \cdot \frac{1}{10} \left(\frac{10}{10 - \frac{55}{12}} \right)^{5/2} \leq \frac{9}{10}$$

and so Wilf's conjecture holds for

$$\mu < \frac{48}{75}\nu^2 + \frac{11}{15}\nu - \frac{1}{18} - \frac{9}{10}.$$

Since both $\frac{48}{75}$ and $\frac{11}{15}$ are larger than $1/2$ and $\frac{1}{18} + \frac{9}{10} < 1$, the quantity on the right-hand side is strictly larger than $\frac{1}{2}\nu^2 + \frac{1}{2}\nu$. Hence, the previous inequality holds whenever $a_2 > \frac{c+\mu}{3}$, and thus Wilf's conjecture holds also in this case. \square

Proposition 5.6. *There is an integer N such that, for every $\nu \geq N$, there are only finitely many numerical semigroups $S = \langle a_1, a_2, \dots, a_\nu \rangle$ with*

- $a_2 > \frac{c(S) + \mu(S)}{3}$,
- $\nu = \nu(S)$, and
- $\mu(S) \leq \frac{4}{9}\nu^2$,

and that do not satisfy Wilf's conjecture.

Proof. Fix any $\chi \in (0, 1/3)$, and consider the function

$$f(q_1, q_2) := 1 - \chi + \frac{1}{2}q_1 + \frac{1}{3}q_2.$$

By Lemmas 4.2 and 4.3, for every $\epsilon > 0$, there is an $N_1(\chi, \epsilon)$ such that, for every point $(q_1, q_2) \in \mathcal{A}(\mu, \nu)$, with $\nu \geq N_1(\chi, \epsilon)$, we have $f(q_1, q_2) \geq \mu/\nu$ whenever

$$\mu \leq \frac{4}{9}\nu^2 + \left(\frac{2}{3} - 2\chi \right) \nu - \frac{1}{16} - \epsilon.$$

Let $N_2(\chi, \epsilon) := (\epsilon + \frac{1}{16})(\frac{2}{3} - 2\chi)^{-1}$: then, for $\nu \geq N_2(\chi, \epsilon)$, we have

$$\left(\frac{2}{3} - 2\chi\right)\nu - \frac{1}{16} - \epsilon \geq 0.$$

Therefore, for every $\nu \geq N := N(\chi, \epsilon) := \max\{N_1(\chi, \epsilon), N_2(\chi, \epsilon)\}$, we have $f(q_1, q_2) \geq \mu/\nu$ whenever $\mu \leq \frac{4}{9}\nu^2$. Equivalently, we have

$$1 + \frac{1}{2}q_1 + \frac{1}{3}q_2 \geq \frac{\mu}{\nu} + \chi.$$

Using the inequality $\lfloor x \rfloor > x - 1$ on Proposition 3.2, we have

$$\frac{\nu|L|}{c} \geq \frac{\nu}{\mu} \left(1 + \frac{1}{2}q_1 + \frac{1}{3}q_2\right) - \frac{\nu}{c} \left(1 + \frac{1}{2}q_1 + \frac{2}{3}q_2\right)$$

which for $\nu \geq N$ is bigger than

$$\frac{\nu}{\mu} \left(\frac{\mu}{\nu} + \chi\right) - \frac{\nu}{c} \left(\frac{\mu}{\nu} + \chi + \frac{1}{3}q_2\right) \geq 1 + \frac{\nu}{\mu}\chi - \frac{1}{c} \left(\mu + \chi\nu + \frac{\nu(\nu-1)}{3}\right),$$

using also the fact that $q_2 \leq \nu - 1$. The quantity on the right-hand side is bigger than 1 when

$$\frac{\nu}{\mu}\chi - \frac{1}{c} \left(\mu + \chi\nu + \frac{\nu(\nu-1)}{3}\right) \geq 0;$$

since c, ν, μ and χ are positive, this is equivalent to

$$c \geq \frac{\mu}{\chi\nu} \left(\mu + \chi\nu + \frac{\nu(\nu-1)}{3}\right), \quad (9)$$

and all semigroups satisfying this inequality satisfy Wilf's conjecture.

In particular, for any value of ν, μ and χ , there are only a finite number of semigroups that do not satisfy this condition. For any ν , there are also a finite number of multiplicities μ satisfying $\mu \leq \frac{4}{9}\nu^2$; hence, for any fixed $\nu \geq N$ there are only finitely many numerical semigroups that verify the hypothesis of the theorem and that do not satisfy Wilf's conjecture. \square

We note that the right-hand side of (9) is very large: for example, if $\nu = 10$, $\mu = 50$ and $\chi = \frac{1}{6}$, then it is equal to 26,050. The strategy used in the proof of Theorem 5.2 (i.e. writing $c = (6k-1)\mu + \theta\mu$ and using different estimates for different $\lfloor \theta \rfloor$) can be employed to obtain numerically better bounds (but still with the hypothesis $\mu \leq \frac{4}{9}\nu^2$).

Proposition 5.7. *For every $\lambda < \frac{4}{5}$, there is a $\nu_0(\lambda)$ such that, if $S = \langle a_1, a_2, \dots, a_\nu \rangle$ is a numerical semigroup such that $a_2 > \frac{c(S)+\mu(S)}{3}$ and $\nu \geq \nu_0(\lambda)$, then*

$$\nu(S)|L(S)| \geq \lambda \cdot c(S). \quad (10)$$

Proof. Fix a $\lambda < \frac{4}{5}$. Let $\ell(x, y, z) := \alpha(1+z) + \beta(x-z) + \gamma(y-z)$, where α, β, γ are defined as in Proposition 3.4. Let $f(x, y) := A + Bx + Cy$ be either $\ell(x, y, x)$ or

$\ell(x, y, y)$; as in the proof of Theorem 5.2, we need to prove that $f(x, y) \geq \lambda(\mu/\nu)$ for both choices of f and all $(x, y) \in \mathcal{I}(\mu - \nu)$, that is, we have to show that $\lambda^{-1}f(x, y) \geq \mu/\nu$.

By Lemmas 4.2 and 4.3, for every $\epsilon > 0$, there is a $\nu_1(\epsilon)$ such that, for every $\nu \geq \nu_1(\epsilon)$ this inequality holds for

$$\mu \leq (2(C\lambda^{-1})(2B\lambda^{-1} - C\lambda^{-1}) - \epsilon)\nu^2 = \left(\frac{2C(2B - C)}{\lambda^2} - \epsilon\right)\nu^2.$$

By the proof of Theorem 5.2, $2C(2B - C)$ is at least $\frac{8}{25}$; if $\lambda < \frac{4}{5}$, then

$$\frac{2C(2B - C)}{\lambda^2} > \frac{8}{25} \cdot \frac{25}{16} = \frac{1}{2}.$$

Therefore, we can choose an ϵ satisfying

$$0 < \epsilon < \frac{2C(2B - C)}{\lambda^2} - \frac{1}{2}$$

and for such an ϵ there is a $\nu_2(\epsilon, \lambda)$ such that

$$\left(\frac{2C(2B - C)}{\lambda^2} - \epsilon\right)\nu^2 > \frac{1}{2}\nu^2 + \frac{1}{2}\nu$$

for all $\nu \geq \nu_1(\epsilon, \lambda)$. Setting $\nu_0(\lambda) := \max\{\nu_1(\epsilon), \nu_2(\epsilon, \lambda)\}$, we have that the inequality (10) holds for $\nu \geq \nu_0(\lambda)$ and $\mu \leq \frac{1}{2}\nu^2 + \frac{1}{2}\nu$. Since every semigroup with $a_2 > \frac{c(S)+\mu(S)}{3}$ satisfies the latter condition (by Proposition 2.4(c)), the claim holds. \square

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