Università degli Studi di Udine

Ph.D. Course in Computer Science, Mathematics and Physics

From real-life to very strong axioms

Classification problems in Descriptive Set Theory
&
regularity properties in Generalized Descriptive Set Theory

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Ph.D. Cycle XXXV

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Acknowledgments

This thesis is the result of over three years/three years and a half of work, and I would like to thank all the people who have left their mark during this time and have influenced its writing in some way.

I can only start with my supervisor Alberto Marcone: he is a precious source of mathematical knowledge and patience without whom this project wouldn't exist. Thank you for having supported me until the last minutes of the submission of my thesis.

I would like to express my gratitude to my co-supervisor Vincenzo Dimonte, for the conversations about the academic life (often pessimistic!!!) and, of course, about rank-into-rank embeddings.

I would also like to thank my M.Sc. supervisor and collaborator Luca Motto Ros: I don't know how much he is aware of it, but all the conversations we had, even if they were few after I moved to Udine, were of great support to me from the beginning, when I decided to do a Ph.D., to the end, when we talked about the future and the possibility of doing a post-doc.

Sincere thanks to my collaborators Vadim Weinstein and Philipp Lücke, who taught me a lot and shared with me my first experiences abroad.

Thanks to you all because this Ph.D. has not only given to me the opportunity to learn and know more about the mathematical world, but also to have access to things and experiences that I would never have been able to live otherwise.

I would like to extend my gratitude to my colleagues: Manlio Valenti, whom I unfortunately met only in the last period of my Ph.D., enough to get to appreciate him as a mathematician and as a person; Sebastiano Thei, for the very long mathematical conversations (until 3 a.m.) during our trips; Andrea Volpi (and Monica) for keeping me company in the office and for motivating me in the last weeks of writing my thesis.

Special thanks to Fernando Barrera, who made me feel less alone in Udine and who is the best flatmate of ever; Vittorio Cipriani, a constant presence in these years, who has literally saved me from having to spend nights on the street many times; Davide Castelnovo, with his ways of showing affection, like filling glasses of wine.

I would like to thank my family, and in particular my parents for their effort to understand and support all my choices, my sister Francesca and my brother Leonardo, whose sensitivity and support have been fundamental in overcoming the many crises I have had during this time.

Thanks to Matteo, with whom I have shared so much of this experience, including the pandemic, and whose house has been a wonderful refuge on many occasions. He trusted in my capabilities way more than I did, and has always encouraged me to do the same.

I would also like to thank Elvira and Vittorio, very special and welcoming people in Udine.

Thanks to my friends: Giada, Laura and Letizia, who have always been present, even if at a distance; Veronica, Michele and Marisa, with whom I started my studies in mathematics, for all the conversations and their patience in waiting for my replies; Riccardo, for his lucidity and concreteness often very useful for me; Elisa, another constant and good company in Udine, for all the lasagne of the last days.

Finally thanks to Alex and Joana, Ernes, Vicent and Carol for their hospitality, dinners and drinks of this last period.

Contents

Introduction

This thesis is divided into three parts, the first and second ones focused on combinatorics and classification problems on discrete and geometrical objects in the context of descriptive set theory, and the third one on generalized descriptive set theory at singular cardinals of countable cofinality.

Descriptive Set Theory (briefly: DST) is the study of definable subsets of Polish spaces, i.e. separable completely metrizable spaces. One of the major branch of DST is invariant DST, introduced in a seemingly independent way in [FS89] and [HKL90] and successfully used in the last thirty years to solve and compare many classification problems. A very common quest in mathematics is indeed the classification of objects belonging to some set X. More formally, a classification problem consists of an equivalence relation E on X, and the goal is to find a procedure to determine whether two different elements of X are E-equivalent or not. To this aim, one want to find an assignment of complete invariants to elements of X, i.e. a pair (I, f) where I is a set whose elements are called invariants and $f \colon X \to I$ is a map assigning to each object in X an element of I so that for all $x, y \in X$, xEy if and only if $f(x) = f(y)$. Setting some suitable restrictions to the sets X and I one obtains the notion of Borel reduction. Borel reducibility measures the relative complexity of equivalence relations and it is useful, in the words of Effros, to "classify the unclassifiables" ([Eff08]). A particularly interesting dividing line in this complexity hierarchy is the so-called classification by countable structures, which divides all equivalence relations into those whose classification complexity is at most as complex as that of countable graphs and those which do not admit a "reasonable" classification by countable graphs ([Hjo00a, Gao09]). Those whose complexity is exactly the same as that of countable graphs (up to Borel bireducibility) are called Borel complete. It is known, for example, that the isomorphism relation on countable linear orders and the one on torsion free Abelian groups are Borel complete ([FS89, PS23]). Examples of other important results in this area include homeomorphism on compact Polish metric spaces and knot equivalence on wild knots in $\mathbb{R}^3$ which are both strictly more complex than isomorphism on countable graphs ([Hjo00a, Kul17]), while conformal equivalence on Riemannian surfaces and isometry on compact metric spaces are strictly below ([HK00], [Gao09, Thm 14.2.1]).

One of our goal is the classification of knots. Knots are objects very familiar and tangible in everyday life, and they also play an important role in modern mathematics. The study of knots and their properties is known as knot theory (see e.g. [BZ03]). Our plan is to gain insight into knots using discrete objects, such as linear and circular orders. This approach was already exploited in [Kul17], where it is shown that isomorphism on the Polish space of countable linear orders strictly Borel reduces to equivalence on knots.

The first part of this work is hence devoted to countable linear orders and the study of the quasi-order of convex embeddability and its induced equivalence relation. We obtain both combinatorial and descriptive set-theoretic results. We further expand our research to the case of circular orders.

Another objective of this first part is to extend the notion of convex embeddability on countable linear orders. We provide a family of quasi-orders of which embeddability is a particular case as well. We study these quasi-orders from a combinatorial point of view and analyse their complexity with respect to Borel reducibility, highlighting differences and analogies with embeddability and convex embeddability, and proving a number of additional facts about the latter. Furthermore, we extend the analysis of these quasi-orders to the set of uncountable linear orders.

The second part of the project deals with classification problems on knots and 3-manifolds. The goal here is to apply the results obtained in the first part to the study of proper arcs (which intuitively are obtained cutting a knot) and knots, establishing lower bounds (in terms of Borel reducibility) for the complexity of some natural relations between these geometrical objects. We also obtain some combinatorial results which are particularly interesting when we restrict to the set of wild proper arcs and wild knots, classes which haven't received much attention so far. These parts will be included in two forthcoming papers in collaboration with my supervisor Alberto Marcone, Luca Motto Ros (University of Torino) and Vadim Weinstein (University of Oulu). The second part of this work also includes the study of the homeomorphism between 3-manifolds and the conjugation of Cantor spaces of $\mathbb{R}^3$. Here we resort to algebraic tools. Stone duality gives a neat way to go back-and-forth between totally disconnected Polish spaces and countable Boolean algebras (see [CG01]). The main ingredient is the Stone space of all ultrafilters on a Boolean algebra. In this work we introduce a weaker concept that we call "blurry filter". Using blurry filters instead of ultrafilters enables one to extend the class of spaces under consideration beyond totally disconnected. As an application of this method, we show that both homeomorphism on 3-manifolds and conjugation of Cantor sets in $\mathbb{R}^3$ are completely classifiable by countable structures, i.e. they are Borel reducible to isomorphism on countable structures (e.g. the isomorphism on the set of countable graphs). These results are part of an upcoming paper in collaboration with Vadim Weinstein.

The last part of this thesis concerns the natural generalization of descriptive set theory that occurs when countable is replaced by uncountable, called Generalized Descriptive Set Theory (GDST). In particular, we focus on the case of GDST for a singular cardinal κ of countable cofinality. The goal here is to study when some regularity properties, as the κ^+-perfect set property and the κ^+-Baire property, hold for non-κ^+-analytic subsets of spaces defined in this context. The results obtained are included in a forthcoming paper in collaboration with my co-supervisor Vincenzo Dimonte and Philipp Lücke (University of Barcelona).

Descriptive Set Theory on discrete objects

The proof of the existence of a Borel reduction from isomorphism of linear orders to equivalence on knots in [Kul17] uses proper arcs (which intuitively are obtained cutting a knot) and the subarcs (called "components" in [Kul17]) of a proper arc, which are analogous to convex subsets of a linear order. Thus, to expand the previous results it is natural to study the following relation between linear orders.

Definition. *Given linear orders L and L', we set $L \trianglelefteq L'$ if and only if L is isomorphic to a convex subset of L'.*

We call convex embeddability the relation $\trianglelefteq$, which was already introduced and briefly studied in [BCP73]. Even if convex embeddability is a very natural relation, as far as we know it has not received much attention in the last 50 years.

We first focus on the restriction of $\trianglelefteq$ to the Polish space LO of linear orders defined on $\mathbb{N}$, denoted by $\trianglelefteq_{\mathsf{LO}}$. We begin establishing that $\trianglelefteq_{\mathsf{LO}}$ induces a structure on LO very different from that obtained using the usual embeddability relation. Indeed, as conjectured by Fraïssé in 1948 ([Fra00]) and proved by Laver in 1971 ([Lav71]), LO is a well quasi-order (briefly: a wqo) under embeddability, i.e. there are no infinite descending chains and no infinite antichains. In contrast, we show that $\trianglelefteq_{\mathsf{LO}}$ is not well-founded and has chains and antichains of size continuum (Proposition 2.2.4). We prove also other combinatorial properties of $\trianglelefteq_{\mathsf{LO}}$, showing in particular that its unbounding number $\mathfrak{b}(\bowtie_{\mathsf{LO}})$ is $\aleph_1$ and its dominating number $\mathfrak{d}(\bowtie_{\mathsf{LO}})$ equals $2^{\aleph_0}$ (Propositions 2.2.5 and 2.2.10).

We then explore the problem of classifying LO under the equivalence relation induced by $\trianglelefteq_{\mathsf{LO}}$, which we call convex biembeddability and denote by $\bowtie_{\mathsf{LO}}$. We obtain the following results (Corollaries 2.3.2 and 2.3.13):

Theorem 1. (a) *$\cong_{\mathsf{LO}}$ is Borel reducible to $\bowtie_{\mathsf{LO}}$, in symbols $\cong_{\mathsf{LO}} \leq_B \bowtie_{\mathsf{LO}}$;*

(b) *$\bowtie_{\mathsf{LO}}$ is Baire reducible to $\cong_{\mathsf{LO}}$, in symbols $\bowtie_{\mathsf{LO}} \leq_{Baire} \cong_{\mathsf{LO}}$.*

Although we are not able to show that $\bowtie_{\mathsf{LO}} \leq_B \cong_{\mathsf{LO}}$, the existence of the above Baire reduction implies that the two equivalence relations are similar in some respect, e.g. no turbulent equivalence relation Borel reduces to $\bowtie_{\mathsf{LO}}$, and $E_1 \not\leq_B \bowtie_{\mathsf{LO}}$ (Corollaries 2.3.14 and 2.3.16). In particular,

$\bowtie_{\mathsf{LO}}$ is not complete for analytic equivalence relations and thus $\trianglelefteq_{\mathsf{LO}}$ is not complete for analytic quasi-orders.

We then move to circular orders, whose notion, although not as widespread as that of linear order, is very natural and in fact has been rediscovered several times in different contexts. The oldest mention we found is in Čech's 1936 monograph (see the English version [Č69]) and a sample of more recent work is [Meg76, KM05, LM06, BR16, CMR18, PBG+18, Mat21, GM21, CMMRS23]. There is a natural notion of convex subset of a circular order, but the obvious translation of convex embeddability to circular orders fails to be transitive. However we introduce the notion of piecewise convex embeddability $\trianglelefteq_{\mathrm{c}}^{<\omega}$, which is transitive, and we study the restriction $\trianglelefteq_{\mathsf{CO}}^{<\omega}$ of $\trianglelefteq_{\mathrm{c}}^{<\omega}$ to the Polish space CO of circular orders with domain $\mathbb{N}$ and its induced equivalence relation $\bowtie_{\mathsf{CO}}^{<\omega}$. We show that $\bowtie_{\mathsf{CO}}^{<\omega}$ is strictly more complicated than $\bowtie_{\mathsf{LO}}$ in terms of Baire reducibility (Corollary 2.4.17). Indeed, while E_1 is not Borel reducible to $\bowtie_{\mathsf{LO}}$, we prove in Theorem 2.4.16 that

Theorem 2. $E_1 \leq_B \bowtie_{\mathsf{CO}}^{<\omega}$.

Let now Lin be the Polish space of linear orders defined either on $\mathbb{N}$ or on a finite subset of $\mathbb{N}$, and denote by $\preceq$ the quasi-order of embeddability on Lin.

In Section 3 we generalize $\trianglelefteq$ by defining a family of binary relations on linear orders, which depend on a nonempty class $\mathcal{L} \subseteq \mathsf{Lin}$ and is denoted by $\trianglelefteq^{\mathcal{L}}$: intuitively, given two linear orders L and L', we write $L \trianglelefteq^{\mathcal{L}} L'$ if L can be partitioned into pieces each of which is isomorphic to a convex subset of L', and these pieces are ordered both in L and L' as the same element of $\mathcal{L}$ (Definition 3.1.2).

First of all, we observe that when $\mathcal{L} = \{\mathbf{1}\}$ then $\trianglelefteq^{\mathcal{L}}$ coincides with convex embeddability. At the other extreme, if $\mathcal{L} = \mathsf{Lin}$ then the restriction $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ of $\trianglelefteq^{\mathcal{L}}$ to LO coincides with embeddability. In both cases we have a quasi-order. We thus analyse the relation $\trianglelefteq^{\mathcal{L}}$ in the other cases. We first determine when $\trianglelefteq^{\mathcal{L}}$ is a quasi-order. In Theorem 3.1.9 we prove that this is the case exactly when $\mathcal{L}$ satisfies a combinatorial property that we call *ccs* (for *closed under convex sums*, see Definition 3.1.7).

When $\mathcal{L}$ is css, we call $\mathcal{L}$-convex embeddability the quasi-order $\trianglelefteq^{\mathcal{L}}$. We then study the combinatorial properties and Borel complexity w.r.t. Borel reducibility of $\mathcal{L}$-convex embeddability. It turns out that, when $\mathcal{L} \subset \mathsf{Lin}$, $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ shares with $\trianglelefteq_{\mathsf{LO}}$ all the combinatorial properties that are established in Section 2.2. These are quite different from those of $\preceq_{\mathsf{LO}}$. As already mentioned, $\preceq_{\mathsf{LO}}$ is a wqo. Moreover LO has a maximal element under $\preceq_{\mathsf{LO}}$, the equivalence class of non-scattered linear orders, and the $\preceq_{\mathsf{LO}}$-minimal elements are ω and ω^*. In contrast, we obtain the following results.

Proposition 1. *Let $\mathcal{L}$ be ccs and different from* Lin.

(a) LO *does not have maximal elements w.r.t.* $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ *and the dominating number of* $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ *is* $2^{\aleph_0}$ *(Proposition 3.2.5).*

(b) *The unbounding number of* $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ *is* $\aleph_1$ *(Theorem 3.2.14).*

(c) $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ *has the fractal property with respect to its upper cones (Theorem 3.2.15).*

The first two results generalize those obtained for $\trianglelefteq_{\mathsf{LO}}$ in Section 2.2, while the third is new also for $\trianglelefteq_{\mathsf{LO}}$.

Recall that both $\preceq_{\mathsf{LO}}$ and $\trianglelefteq_{\mathsf{LO}}$ are proper analytic quasi-orders. While embeddability among countable graphs is complete for analytic quasi-orders, the relation $\preceq_{\mathsf{LO}}$ is far from being complete because it is combinatorially too simple. For different reasons, $\trianglelefteq_{\mathsf{LO}}$ is not complete for analytic quasi-orders as well (see Corollary 2.3.17). The descriptive set theoretic complexity of $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ depends also on the complexity of the class $\mathcal{L}$ and can fail to be analytic (Proposition 3.3.1 and Corollary 3.3.4). Nevertheless we establish some Borel reductions among different $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$'s for some ccs classes $\mathcal{L}$ (Theorems 3.4.5 and 3.4.8).

We show that for $\mathcal{L} \neq \mathsf{Lin}$ the quasi-order $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ is Borel equivalent with its natural version for coloured linear orders: this strongly contrasts with the situation for classical embeddability, where it is known that the coloured version is analytic complete ([MR04]).

We also consider the equivalence relations induced by $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ for ccs $\mathcal{L}$, which we denote by $\bowtie^{\mathcal{L}}_{\mathsf{LO}}$ and call $\mathcal{L}$-convex biembeddability, and study their Borel complexity. When $\mathcal{L} = \mathsf{Lin}$, $\bowtie^{\mathcal{L}}_{\mathsf{LO}}$ is biembeddability $\equiv_{\mathsf{LO}}$ on LO. By Laver's results ([Lav71]) it is known that $\equiv_{\mathsf{LO}}$ is an analytic equivalence relation with $\aleph_1$ equivalence classes and $\mathrm{id}(X) \not\leq_B \equiv_{\mathsf{LO}}$ for any uncountable Polish space X. In particular $\equiv_{\mathsf{LO}}$ is far from being complete for analytic equivalence relations.

When $\mathcal{L} = \{\mathbf{1}\}$, $\bowtie^{\mathcal{L}}_{\mathsf{LO}}$ is the relation of convex biembeddability $\bowtie_{\mathsf{LO}}$ on LO studied in Section 2.3: there we show that the isomorphism relation $\cong_{\mathsf{LO}}$ on LO Borel reduces to $\bowtie_{\mathsf{LO}}$ and is indeed Baire equivalent to it. Moreover, $E_1 \not\leq_{Baire} \bowtie_{\mathsf{LO}}$. In contrast we obtain:

Theorem 3. *If $\mathcal{L}$ is ccs and different from* Lin *and* $\{\mathbf{1}\}$ *then*

(a) $E_1 \leq_B \bowtie^{\mathcal{L}}_{\mathsf{LO}}$ *(Theorem 3.3.5).*

(b) $\cong_{\mathsf{LO}} <_B \bowtie^{\mathcal{L}}_{\mathsf{LO}}$*, and in fact* $\bowtie^{\mathcal{L}}_{\mathsf{LO}} \not\leq_{Baire} \cong_{\mathsf{LO}}$ *(Corollary 3.3.6).*

Most of the combinatorial techniques developed to obtain the above results actually work for uncountable linear orders as well. Working in the context of generalized descriptive set theory, we obtain the following results.

Theorem 4. (a) *It is consistent with* ZFC *that for all uncountable cardinals κ which are successors of a regular cardinal and every $\mathcal{L} \subseteq \mathsf{Lin}$ which is ccs, the relation $\bowtie^{\mathcal{L}}_{\kappa}$ of $\mathcal{L}$-convex biembeddability over linear orders of size κ is complete for all κ^+-analytic equivalence relations (Theorem 3.5.2).*

(b) *For every $\mathcal{L} \subset \mathsf{Lin}$ which is ccs there are uncountably many incomparable minimal elements w.r.t. $\mathcal{L}$-convex embeddability among uncountable linear orders (this follows from Theorem 3.5.3).*

The first result is in contrast with the situation for countable linear orders, while the second contrasts the five-elements basis theorem for embeddability on uncountable linear orders [Moo06]: there is no finite or countable basis for $\mathcal{L}$-convex embeddability on such class.

It is possible to define piecewise convex embeddability and ccs classes on CO, obtaining results which are analogous to those of Section 3. Since the ideas behind this extension to circular order are similar to the case of linear orders, we do not develop this part.

Descriptive Set Theory on geometrical objects

In Chapter 4 we deal with proper arcs and knots, proving anti-classification results in the framework of Borel reducibility and exploring the combinatorial properties of some natural relations on wild proper arcs and wild knots.

In [Kul17, Theorem 3.1] it is shown that the isomorphism $\cong_{\mathsf{LO}}$ on countable linear orders Borel reduces to equivalence $\equiv_{\mathrm{Kn}}$ on knots. Employing the same construction, we establish a similar connection between convex embeddability $\trianglelefteq_{\mathsf{LO}}$ on linear orders and the subarc relation $\precsim_{\mathrm{Ar}}$ on the standard Borel space Ar of proper arcs (Theorem 4.2.6).

Theorem 5. $\trianglelefteq_{\mathsf{LO}} \leq_B \precsim_{\mathrm{Ar}}$*, and hence* $\bowtie_{\mathsf{LO}} \leq_B \approx_{\mathrm{Ar}}$*, where* $\approx_{\mathrm{Ar}}$ *is the equivalence relation induced by* $\precsim_{\mathrm{Ar}}$.

We then show that the combinatorial structure of Ar w.r.t. $\precsim_{\mathrm{Ar}}$ is similar to that of LO w.r.t. $\trianglelefteq_{\mathsf{LO}}$.

Theorem 6. (a) $\precsim_{\mathrm{Ar}}$ *has chains of order type* $(\mathbb{R}, <)$*, as well as antichains of size* $2^{\aleph_0}$ *(Corollary 4.2.7).*

(b) *The unbounding number of* $\precsim_{\mathrm{Ar}}$ *is* $\aleph_1$ *and the dominating number of* $\precsim_{\mathrm{Ar}}$ *is* $2^{\aleph_0}$ *(Theorem 4.2.10).*

(c) *Every proper arc is the bottom of an* $\precsim_{\mathrm{Ar}}$*-unbounded chain of length* ω_1 *(Corollary 4.2.11).*

The discussion about minimal elements and basis for the relation $\precsim_{\mathrm{Ar}}$ is more delicate. If we consider only tame proper arcs, which form a $\precsim_{\mathrm{Ar}}$-downward closed subclass of the collection of all proper arcs, then the equivalence class of the trivial arc (which contains all the proper arcs equivalent to the diameter of a closed 3-ball) is $\precsim_{\mathrm{Ar}}$-minimum. Removing the (equivalence class of the) trivial arc from the collection of tame proper arcs, one can instead show that there is no finite basis; moreover, there is no infinite descending chain and every dominating family is countable. Moving to the realm of wild proper arcs WAr, and considering the restriction $\precsim_{\mathrm{WAr}}$ of $\precsim_{\mathrm{Ar}}$ to WAr, we obtain the following results which highlight the complexity of this relation from a combinatorial point of view (Theorems 4.2.13 and 4.2.14).

Theorem 7. (a) *There are infinitely many* $\precsim_{\mathrm{WAr}}$*-incomparable* $\precsim_{\mathrm{WAr}}$*-minimal elements in* WAr.

(b) *There is a strictly* $\precsim_{\mathrm{WAr}}$*-decreasing* ω*-sequence in* WAr *which is not* $\precsim_{\mathrm{WAr}}$*-bounded from below.*

(c) *No basis for* $\precsim_{\mathrm{WAr}}$ *has size smaller than* $2^{\aleph_0}$*.*

(d) *Every* $\precsim_{\mathrm{WAr}}$*-antichain is contained in a* $\precsim_{\mathrm{WAr}}$*-antichain of size* $2^{\aleph_0}$*. In particular, there are no maximal* $\precsim_{\mathrm{WAr}}$*-antichains of size smaller than* $2^{\aleph_0}$*, and every* $(\bar{B}, f) \in \mathrm{WAr}$ *belongs to a* $\precsim_{\mathrm{WAr}}$*-antichain of size* $2^{\aleph_0}$*.*

We then move to the study of knots, highlighting the natural connection they have with circular orders. We first prove the following (Theorem 4.3.2).

Theorem 8. $\cong_{\mathsf{CO}} \leq_B \equiv_{\mathrm{Kn}}$.

Trying to transfer the notion of component from arcs to knots, one encounters a number of roadblocks, as for the case of convex embeddability on circular orders. To overcome these difficulties, we introduce the (finite) piecewise subknot relation on Kn, denoted by $\precsim_{\mathrm{Kn}}^{<\omega}$. This notion is a bit technical: roughly speaking, $K \precsim_{\mathrm{Kn}}^{<\omega} K'$ means either that K is equivalent to K' or that K can be obtained as the "circularization" of a proper arc (consisting of gluing the endpoints of the proper arc) which is equivalent to the sum of finitely many subarcs of K'. We denote by $\approx_{\mathrm{Kn}}^{<\omega}$ the equivalence relation induced by $\precsim_{\mathrm{Kn}}^{<\omega}$. This relation turns out to be quite natural: it is strictly coarser than the equivalence relation $\equiv_{\mathrm{Kn}}$, but it is still able to distinguish between tame and wild knots, as shown in the next result (Proposition 4.3.8).

Proposition 2. *A knot* K *is tame if and only if* K *is* $\approx_{\mathrm{Kn}}^{<\omega}$*-equivalent to the trivial knot, that is, to a great circle of a sphere embedded in* S^3*.*

The topological notion of piecewise subknot matches well with the notion of piecewise convex embeddability $\trianglelefteq_{\mathsf{CO}}^{<\omega}$ on CO (Theorem 4.3.11).

Theorem 9. $\trianglelefteq_{\mathsf{CO}}^{<\omega} \leq_B \precsim_{\mathrm{Kn}}^{<\omega}$*, so that* $\bowtie_{\mathsf{CO}}^{<\omega} \leq_B \approx_{\mathrm{Kn}}^{<\omega}$ *and* $\cong_{\mathsf{LO}} \leq_B \approx_{\mathrm{Kn}}^{<\omega}$ *and* $E_1 \leq_B \approx_{\mathrm{Kn}}^{<\omega}$*.*

An interesting consequence of our results is that the equivalence relation associated to the piecewise subknot relation is not induced by a Borel action of a Polish group. This is in stark contrast with the relation of equivalence on knots, which is induced by a Borel action of the Polish group of homeomorphisms of S^3 onto itself (see e.g. [BZ03, Proposition 1.10]).

Let now CKn be the set of knots which are the circularization of a proper arc (intuitively, these are the knots which may be cut in at least one point). Since the relation $\precsim_{\mathrm{Kn}}$ coincide with $\equiv_{\mathrm{Kn}}$ on $\mathrm{Kn} \setminus \mathrm{CKn}$, some combinatorial properties of $\precsim_{\mathrm{Kn}}^{<\omega}$ follow easily: for example, the unbounding number of $\precsim_{\mathrm{Kn}}^{<\omega}$ is 2 and the dominating number equals $2^{\aleph_0}$. Thus it is more interesting to consider the restriction $\precsim_{\mathrm{CKn}}^{<\omega}$ of $\precsim_{\mathrm{Kn}}^{<\omega}$ to CKn. We obtain results which are similar to the case of proper arcs.

Proposition 3. (a) $\precsim_{\mathrm{CKn}}^{<\omega}$ *has chains of order type* $(\mathbb{R},<)$*, as well as antichains of size* $2^{\aleph_0}$ *(Proposition 4.3.13).*

(b) *The unbounding number of* $\precsim_{\mathrm{CKn}}^{<\omega}$ *is* $\aleph_1$ *and its dominating number of* $\precsim_{\mathrm{CKn}}^{<\omega}$ *is* $2^{\aleph_0}$ *(Theorem 4.3.15).*

(c) *Every knot* $K \in \mathrm{CKn}$ *is the bottom of an* $\precsim_{\mathrm{CKn}}^{<\omega}$*-unbounded chain of length* ω_1 *(Corollary 4.3.16).*

We finally deal with minimal elements and basis w.r.t. $\precsim_{\mathrm{CKn}}^{<\omega}$. In contrast with the case of proper arcs, it is not interesting to consider the restriction of $\precsim_{\mathrm{CKn}}^{<\omega}$ to the collection of tame knots because tame knots are all $\approx_{\mathrm{CKn}}^{<\omega}$-equivalent. We thus consider the restriction $\precsim_{\mathrm{WCKn}}^{<\omega}$ to the wild knots of CKn and show the following result, which concludes Chapter 4.

Theorem 10. (a) *There are* $2^{\aleph_0}$*-many* $\precsim_{\mathrm{WCKn}}^{<\omega}$*-incomparable* $\precsim_{\mathrm{WCKn}}^{<\omega}$*-minimal elements in the set* WCKn*. In particular, any basis for* $\precsim_{\mathrm{WCKn}}^{<\omega}$ *has size* $2^{\aleph_0}$*.*

(b) *There is a strictly* $\precsim_{\mathrm{WCKn}}^{<\omega}$*-decreasing* ω*-sequence in* WCKn *which is not* $\precsim_{\mathrm{WCKn}}^{<\omega}$*-bounded from below. In particular, all basis for* $\precsim_{\mathrm{WCKn}}^{<\omega}$ *are ill-founded.*

(c) *Every* $\precsim_{\mathrm{WCKn}}^{<\omega}$*-antichain is contained in a* $\precsim_{\mathrm{WCKn}}^{<\omega}$*-antichain of size* $2^{\aleph_0}$*. In particular, there are no maximal* $\precsim_{\mathrm{WCKn}}^{<\omega}$*-antichains of size smaller than* $2^{\aleph_0}$*, and every* $K \in \mathrm{WCKn}$ *belongs to a* $\precsim_{\mathrm{WCKn}}^{<\omega}$*-antichain of size* $2^{\aleph_0}$*.*

Chapter 5 corcerns the study of the classification of non-compact 3-manifolds up to homeomorphism w.r.t. Borel reducibility. It is already known that the isomorphism on countable structures is a lower bound for the complexity of homeomorphism on n-manifolds for $n \geqslant 2$. For the converse, it has been shown in [Gol71] that the non-compact 2-manifolds admit a complete classification by algebraic structures. At the time of [Gol71], the theory of Borel reducibility was not developed yet, so the question as to whether this classification could be realized by a Borel map was not addressed. Also, the problem of whether such classification is possible for higher dimensional manifolds was open. For non-compact 3-manifolds even less is known and in fact many have suspected that the classification of 3-manifolds is harder than the isomorphism on countable structures because of pathological examples such as the Whitehead manifolds.

In order to study manifolds in the context of Borel reducibility, we show that they can be naturally parametrized as atlases which cover subsets of the Urysohn space. We denote the standard Borel space of 3-manifolds by $\mathfrak{M}_3$. Piecewise linear manifolds can be naturally parametrized as simplicial complexes in the Urysohn space. We denote this standard Borel space by $\mathfrak{M}_3^{PL}$.

Let now $L = \{\leqslant\}$ be the first-order vocabulary with one binary relation symbol and let $\mathbb{P} \subseteq \mathrm{Mod}(L)$ be the set of partial orders. Given a non-compact piecewise linear 3-manifold M, we assign to it a countable partial order $P_M \in \mathbb{P}$ such that for any two manifolds M_1, M_2, they are homeomorphic iff P_{M_1} and P_{M_2} are isomorphic. This is proved using a weaker version of Stone duality based on a new notion that we call blurry filter. The Stone-duality states that one can move back-and-forth between totally disconnected compact Polish spaces and Boolean algebras. For one direction, given such a space X, let $\psi_0(X)$ be the Boolean algebra of the clopen sets of X ordered by set inclusion. For the other direction, given a Boolean algebra A, let $\varphi_0(A)$ be the Stone space of all ultrafilters on A. Then for all such spaces X, $\varphi_0(\psi_0(X))$ is homeomorphic to X, and for all Boolean algebras A, $\psi_0(\varphi_0(A))$ is isomorphic to A. This gives Borel reductions of homeomorphism on totally disconnected compact Polish spaces to isomorphism of Boolean algebras and vice versa ([CG01]).

We generalize this as follows. We define an object called a basis space which is defined to be a pair (X, β), where X is a set and β is a countable collection of subsets of X satisfying a number of conditions, in particular so that β is a basis for a locally compact Polish topology on X. The space of all such basis spaces is denoted by $\mathfrak{B}^C$. We say that two basis spaces (X, β) and (X', β') are equivalent, and write $(X, \beta) \equiv (X', \beta')$, if there is a bijection between their domains which takes the basis of the first one to the basis of the second one. In particular this implies that the generated topologies are homeomorphic. We then work with a weakening of Boolean algebras, called complemented algebras, and show that if (X, β) is a basis space, then we can define $\psi(X, \beta)$ to be the complemented algebra whose domain is β and the partial order is determined by strong inclusion (an open set U is strongly included in an open set V if the closure of U is contained in V). We then define a weakening of an ultrafilter, which we call blurry filter. Similarly to the Stone space of a Boolean algebra, given a complemented algebra A, one can define the basis space $\varphi(A)$ obtained by taking the set of all blurry filters of A. We then prove a partial version of Stone duality in this context, namely that $\varphi(\psi(X, \beta))$ and (X, β) are equivalent as basis spaces:

Theorem 11. *For all $(X, \beta) \in \mathfrak{B}^C$ we have that $(X, \beta) \equiv \varphi(\psi(X, \beta))$.*

Together with the fact that equivalent basis spaces give rise to isomorphic complemented algebras, we obtain the following result.

Theorem 12. *The equivalence $\equiv$ on locally compact Polish basis spaces is Borel reducible to isomorphism on countable structures.*

Using the piecewise linear structures of piecewise linear 3-manifolds, we can establish the following connection between manifolds and basis spaces.

Theorem 13. *The PL-homeomorphism relation on $\mathfrak{M}_3^{\mathrm{PL}}$ is Borel reducible to the equivalence on locally compact basis spaces.*

By Moise's theorem ([Moi52]) it is known that every 3-manifold is triangulable, i.e. it admits a unique piecewise linear structure. As far as we know, it has never been addressed whether the assignment of the PL structure to a manifold can be realized by a Borel map. We show this is indeed the case in order to establish a Borel classification of manifolds.

Theorem 14. *There is a Borel map $h\colon \mathfrak{M}_3 \to \mathfrak{M}_3^{\mathrm{PL}}$ such that for all $M \in \mathfrak{M}_3$, $M \approx h(M)$. Also, homeomorphism on $\mathfrak{M}_3$ is Borel reducible to PL-homeomorphism on $\mathfrak{M}_3^{PL}$.*

Combining all previous Borel reductions, we finally obtain the main result of Chapter 5.

Theorem 15. *Homeomorphism on 3-manifolds is Borel complete.*

A special case of the classification of 3-manifolds is that of wild Cantor sets in $\mathbb{R}^3$ [GKB13]. Two Cantor sets $C, C' \in \mathbb{R}^3$ are said to be conjugate if there is a homeomorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $h[C] = C'$. This equivalence relation arises in a natural way in the study of attractors of dynamical systems. It was shown in [GKB13] that this relation is at least as complicated as the isomorphism on countable structures and we show that this lower bound is exact answering [GKB13, Question 1.1].

Theorem 16. *The relation of conjugation of Cantor sets in $\mathbb{R}^3$ is Borel reducible to isomorphism on countable structures.*

Theorem 16 stands in contrast to [Kul17, Theorem 3], which states that if a Cantor set is replaced by the unit circle, then the corresponding equivalence relation (which is the relation $\equiv_{\mathrm{Kn}}$ mentioned in Chapter 4) is not classifiable by countable structures.

Generalized Descriptive Set Theory

Generalized Descriptive Set Theory (briefly: GDST) is a research area which has recently gained popularity. The first works in this field focus on the development of GDST on regular cardinals and the basic idea is to replace ω with an uncountable regular cardinal κ to obtain the generalized versions ${}^{\kappa}2$ and ${}^{\kappa}\kappa$ of the Cantor space ${}^{\omega}2$ and the Baire space ${}^{\omega}\omega$ (see [MV93, Vä95]). All the basic notions of classical Descriptive Set Theory related to these spaces, as the σ-algebra of Borel sets, the analytic sets, the Perfect Set Property (briefly: PSP), the Baire Property (briefly: BP), have a generalization (or more) in this settings, such as the κ^+-algebra of κ^+-Borel sets, the κ^+-analytic sets, the κ^+-Perfect Set Property (briefly: κ^+-PSP), and the κ^+-Baire Property (briefly:

κ^+-BP). Moreover, many classification problems for uncountable structures are investigated by introducing the notions of κ^+-Borel functions and κ^+-Borel reducibility, which are the analogues of Borel functions and Borel reducibility used in the classical settings. Refer to [FHK14a] for a quite comprehensive introduction to the subject.

Recently, a deeper and more general study of generalized descriptive set theory has been emerged. On one hand, GDST on regular cardinals is extended to several generalizations of Polish spaces and standard Borel spaces, which are ubiquitous in most mathematical fields (see e.g. [Gal16, CS16, LS15, ARS21]); on the other, a new theory of GDST on singular cardinals has been developed, both for those with uncountable cofinality (see [AMR22, ARon]) and those with countable cofinality (a systematic study of the latter is done in the forthcoming paper [DMRon]). Particularly interesting is the case of a singular cardinal of countable cofinality: indeed, one may recover many theorems of classical descriptive set theory that gets lost in the uncountable regular case, and in most cases the reason for this is that such a cardinal share with ω some crucial properties which are used to obtain the various results in classical DST.

Another approach to generalize classical DST resorts to large cardinals, and is based on the idea that these cardinals, especially when κ itself is a large cardinal, allow to preserve a bit more of the classical picture. For example, if κ is regular then ${}^{\kappa}\kappa$ is not homeomorphic to ${}^{\kappa}2$ (as in the classical case) if and only if κ is weakly compact. However, when κ is regular, even if it is a large cardinal, one looses some "nice" properties that hold for sets in the classical case: e.g. in ZFC one can shows that every analytic set satisfies regularity properties as the perfect set property and the Baire property, while, in contrast, the κ^+-PSP for closed/κ^+-Borel/analytic sets is independent of ZFC.

Another picture emerges when κ is singular of countable cofinality. The key large cardinal for this analysis is I0, a large cardinal which is at the very top of the hierarchy, in connection with the study of the model $L(V_{\kappa+1})$, where κ is the witness of I0 (notice that such a κ has always countable cofinality). In this context, the large cardinal version of ${}^{\omega}2$ is given by $V_{\kappa+1}$: since V_κ has size κ, $V_{\kappa+1} = \mathcal{P}(V_\kappa)$ is homeomorphic to ${}^{\kappa}2$ which is the analogue of ${}^{\omega}2$. Woodin claims that "the theory of $\mathcal{P}(V_{\kappa+1})$ in $L(V_{\kappa+1})$ under I0 is reminiscent of the theory of $\mathcal{P}(\mathbb{R})$ in $L(\mathbb{R}) = L(V_{\omega+1})$ under AD", and some results in this direction are in [Woo11].

In Chapter 6 we use this framework going through the hierarchy of large cardinals, when κ is singular of cofinality ω. We always work in the space $C(\vec{\kappa}) = \prod_{i\in\omega} \kappa_i$, where $(\kappa_i)_{i\in\omega}$ is a cofinal sequence in κ, which under our assumptions is homeomorphic to $V_{\kappa+1}$.

Our goal is to study the κ^+-PSP and the κ^+-BP for sets that belongs to the κ^+-projective and κ^+-lightface hierarchies, which are the natural generalization of the classical projective and lightface hierarchies.

Using the axiom of choice AC it is possible to build in any κ^+-Polish space of size $> \kappa$ a subset without the κ^+-PSP. The proof is the same as the classical Bernstein's proof (see [Kan09, Proposition 11.4(a)]). On the other hand, in [DMRon] it is shown that in ZFC if κ is such that $2^{<\kappa} = \kappa$, then every κ^+-analytic set A of a κ^+-Polish space X has the κ^+-PSP, i.e. either $|A| \leq \kappa$ or ${}^{\kappa}2$ embeds into A as a closed set in X. Moreover, Dimonte and Motto Ros show that if $V = L$ and κ is a limit cardinal of countable cofinality then there exists a κ^+-coanalytic subset of ${}^{\kappa}2$ without the κ^+-PSP.

Attempting to determine the exact levels in the κ^+-projective hierarchy (apart from κ^+-analytic sets) from which sets do or do not have the κ^+-PSP, one obtains no absolute answers. Recall that in the classical case, there are models with two extremes: under ZF+AD all the sets of reals have the PSP (see [Kan09, Theorem 27.9]), while in the constructible universe L there is a coanalytic set without the PSP (see [Kan09, Theorem 13.2]). In the generalized case similar results hold: in [Cra15] Cramer proves, confirming the claim by Woodin, that under ZFC+I0 all the sets of $L(V_{\kappa+1}) \cap V_{\kappa+2}$ have the κ^+-PSP in $L(V_{\kappa+1})$, while in [DMRon] it is shown that if $V = L$ and κ is a singular cardinal of cofinality ω then there is a κ^+-coanalytic subset of ${}^{\kappa}2$ without the κ^+-PSP.

This leaves wide open the answer to this question for intermediate levels.

In particular, we analyze the κ^+-PSP for sets which are definable with parameters in the effective hierarchy, which were never previously introduced. We show that under the assumption of the existence of an ω-strictly increasing sequence of measurable cardinals with limit κ there exist an inner model such that κ is a limit of measurable cardinals and a κ^+-Σ^1_2 set in it without the κ^+-PSP.

Theorem 17. *Let $\vec{\kappa} = \langle \kappa_n \mid n < \omega \rangle$ be a strictly increasing sequence of measurable cardinals with limit κ, and let $\langle U_n \mid n < \omega \rangle$ be a sequence with the property that U_n is a normal ultrafilter on κ_n for all $n < \omega$. Assume that $V = L[\mathcal{U}]$, where*

$$\mathcal{U} = \{\langle n, A\rangle \mid n < \omega,\ A \in U_n\}.$$

If $\vec{\nu} = \langle \nu_n \mid n < \omega \rangle$ is a strictly increasing sequence of cardinals of uncountable cofinality with limit κ, then there exists $x \in H(\aleph_1)$ with the property that there is a κ^+-$\Sigma^1_2(\vec{\nu}, x)$-subset of $C(\vec{\nu})$ of cardinality greater than κ that does not contain a ν^+-perfect subset.

We then prove that if κ is the witness of the large cardinal axiom I2, we obtain that every κ^+-$\Sigma^1_2(\vec{\kappa})$-subset of $C(\vec{\kappa})$ has the κ^+-PSP.

Theorem 18. *Let $\vec{\kappa}$ be a strictly increasing sequence of infinite cardinals with limit κ, let j be an I2-elementary embedding with critical sequence $\vec{\kappa}$. If A is a κ^+-$\Sigma^1_2(\vec{\kappa})$ subset of $C(\vec{\kappa})$ of cardinality greater than κ, then A contains a κ^+-perfect subset.*

Yet, the assumption of I2 on κ is not enough for the complete boldface class κ^+-$\boldsymbol{\Sigma}^1_2$.

Theorem 19. *Let $\vec{\kappa}$ be a strictly increasing sequence of infinite cardinals with limit κ, let j be an I2-elementary embedding with critical sequence $\vec{\kappa}$ and let E be a subset of κ such that V_κ is a subset of $L[E]$ and $L[E]$ contains the sequence $\vec{\kappa}$ and the restriction of j to V_κ. Then the following statements hold true in $L[E]$:*

(1) *There is an I2-elementary embedding with critical sequence $\vec{\kappa}$.*

(2) *There is a subset A of $C(\vec{\kappa})$ which is κ^+-$\boldsymbol{\Sigma}^1_2$ and does not have the κ^+-PSP.*

We now deal with the κ^+-BP. In the classical case, in ZFC one can prove that all the analytic sets have the BP, while it is not provable that $\boldsymbol{\Sigma}^1_2$ have the BP in ZFC alone (one need the $\boldsymbol{\Sigma}^1_1$-determinacy). Analogously, using techniques completely different from the classical case and a new topology on the space $C(\vec{\kappa})$, in [DMRSon] it is shown that in ZFC all the κ^+-analytic sets have the κ^+-BP (w.r.t. the so called Ellentuck-Prikry topology), and hence all the κ^+-coanalytic sets have the κ^+-BP as well. We prove here that the case of κ^+-BP for κ^+-Σ^1_2 and κ^+-$\boldsymbol{\Sigma}^1_2$ is similar to that for the κ^+-PSP.

Theorem 20. *Let j be an I2-elementary embedding with κ being the supremum of its critical sequence $\vec{\kappa} = \langle \kappa_n \mid n < \omega \rangle$. Then there exists a sequence $\mathcal{V} = \langle V_n \mid n < \omega \rangle$ such that each V_n is a normal ultrafilter on κ_n and every κ^+-$\Sigma^1_2(\mathcal{V})$ subset of $C(\vec{\kappa})$ has the κ^+-*BP *w.r.t. the Ellentuck-Prikry topology induced by $\mathcal{V}$.*

Proposition 4. *Let $\vec{\kappa} = \langle \kappa_n \mid n < \omega \rangle$ be a strictly increasing sequence of measurable cardinals with limit κ, and let $\langle U_n \mid n < \omega \rangle$ be a sequence with the property that U_n is a normal ultrafilter on κ_n for all $n < \omega$. Assume that $V = L[\mathcal{U}]$, where*

$$\mathcal{U} = \{\langle n, A\rangle \mid n < \omega,\ A \in U_n\}.$$

*Then $(C(\vec{\kappa}) \boxtimes C(\vec{\kappa})) \cap <_{L[\mathcal{U}]}$ is a κ^+-$\Sigma^1_2(\mathcal{U})$-set without the κ^+-*BP*. Thus, there is a κ^+-$\boldsymbol{\Sigma}^1_2$-set without the κ^+-*BP*.*

The previous result is a generalization of [Kan09, Corollary 13.10] in the classical setting.

As in the case of the PSP, in the classical case it is shown that under ZF+AD every sets of reals has the BP (see [Kan09, Theorem 27.9]). In the following theorem we show that under I0 all sets in $L_1(V_{\kappa+1})$ (equivalently, all κ^+-projective sets of $V_{\kappa+1}$) have the κ^+-BP.

Theorem 21. *Let $\vec{\kappa}$ be a strictly increasing sequence of infinite cardinals with limit κ, let j be an I0-embedding with critical sequence $\vec{\kappa}$. Then every subset of $C(\vec{\kappa})$ in $L_1(V_{\kappa+1})$ has the κ^+-*BP*.*

I

Descriptive set theory on discrete objects

1

Preliminaries

1.1 Borel reducibility

In this section we introduce some basic definitions and results from descriptive set theory that will be used in the sequel; the standard references are [Kec95, Gao09].

A **Polish space** is a separable and completely metrizable topological space. Examples of Polish spaces include the real line $\mathbb{R}$ and more generally all Euclidean spaces $\mathbb{R}^n$, the Cantor space $2^{\mathbb{N}}$ and the Baire space $\mathbb{N}^{\mathbb{N}}$ (both endowed with the product of the discrete topology), and all separable Banach spaces. Closed sets and G_δ sets (i.e. countable intersection of open sets) of a Polish space are Polish spaces. Also, the product and sum of a sequence of Polish spaces are Polish spaces.

A subset A of a Polish space X is **Borel** if it is an element of the smallest σ-algebra on X containing all open subsets of X.

Definition 1.1.1. A **standard Borel space** is a pair $(X, \mathcal{B})$ where X is a set, $\mathcal{B}$ is a σ-algebra on X, and there is a Polish topology on X for which $\mathcal{B}$ is precisely the collection of Borel sets. The elements of $\mathcal{B}$ are called Borel sets of X.

In particular, every Polish space is standard Borel when equipped with its σ-algebra of Borel sets. The product and sum of a sequence of standard Borel spaces are standard Borel spaces. Moreover, if $(X, \mathcal{B})$ is standard and $Y \subseteq X$ is in $\mathcal{B}$, then $(Y, \mathcal{B} \restriction Y)$ is also standard.

Let X and Y be Polish or standard Borel spaces. A function $\varphi \colon X \to Y$ is **Borel** if $\varphi^{-1}(B)$ is Borel in X for every Borel set B of Y.

We say that a subset A of a topological space X has the **Baire property** (BP for short) if $A \triangle U = (A \setminus U) \cup (U \setminus A)$ is meager for some open set U of X, i.e. $A \triangle U$ is a countable union of sets whose closure has empty interior. All Borel sets have the Baire property.

A function $\varphi \colon X \to Y$ is **Baire measurable** if $\varphi^{-1}(B)$ has the BP for every Borel set B of Y.

Definition 1.1.2. Let X be a Polish or standard Borel space. A subset $A \subseteq X$ is **analytic** (or $\mathbf{\Sigma}^1_1$) if there is a Borel subset B of $X \times \mathbb{N}^{\mathbb{N}}$ such that for all $x \in X$

$$x \in A \iff \exists y \in \mathbb{N}^{\mathbb{N}}\ (x, y) \in B,$$

i.e. A is the projection on the first coordinate of B. The set A is **coanalytic** (or $\mathbf{\Pi}^1_1$) if $X \setminus A$ is analytic, and it is **bianalytic** (or $\mathbf{\Delta}^1_1$) if it is both analytic and coanalytic. This can be further extended, but we need only the $\mathbf{\Sigma}^1_2$ sets, i.e. the projections of coanalytic subsets of $X \times \mathbb{N}^{\mathbb{N}}$.

By $D_2(\mathbf{\Pi}^1_1)$ we denote the class of sets which are the intersection of an analytic set and a coanalytic set.

Let X and Y be topological spaces and $A \subseteq X$, $B \subseteq Y$. We say that A is **Wadge reducible** to B, in symbols $A \leq_W B$, if there is a continuous map $\varphi \colon X \to Y$ such that $x \in A \iff \varphi(x) \in B$, for all $x \in X$.

Let Γ be a class of sets in Polish spaces. If Y is a Polish space, we say that the subset A of Y is **Γ-hard** if $B \leq_W A$ for any $B \in \Gamma(X)$ with X a zero-dimensional Polish space. If moreover $A \in \Gamma(Y)$, we say that A is **Γ-complete**.

An important line of research within descriptive set theory is the study of definable equivalence relations, which are typically compared using the next definition.

Definition 1.1.3. Let X and Y be sets and consider E and F equivalence relations on X and Y, respectively. A function $\varphi\colon X \to Y$ is called a **reduction** from E to F if

$$x_1 \; E \; x_2 \iff \varphi(x_1) \; F \; \varphi(x_2),$$

for all $x_1, x_2 \in X$.

We say that E is **Borel reducible** to F, and write $E \leq_B F$, if X and Y are standard Borel spaces and there exists a Borel map φ reducing E to F. The equivalence relations E and F are **Borel bireducible**, $E \sim_B F$ in symbols, if both $E \leq_B F$ and $F \leq_B E$.

Finally, we say that E is **Baire reducible** to F, and we write $E \leq_{Baire} F$, if X and Y are topological spaces and there exists a Baire measurable map $\varphi\colon X \to Y$ reducing E to F.

Definition 1.1.4. Let Γ be a collection of equivalence relations on standard Borel spaces. We say that an equivalence relation E is **complete for** Γ (or Γ**-complete**) if it belongs to Γ and any other equivalence relation in Γ Borel reduces to E. When Γ consists of all analytic equivalence relations we just say that E is **complete**.

An important class of analytic equivalence relations consists of those induced by a Borel action of a Polish group. A topological group is **Polish** if its underlying topology is Polish. Examples of Polish groups include the group of permutations of natural numbers S_∞ with the topology inherited as a subspace of the Baire space $\mathbb{N}^\mathbb{N}$, and the group of homeomorphisms of S^3 into itself with the topology induced by the uniform metric.

Definition 1.1.5. Let X and G be a standard Borel space and a Polish group, respectively. A **Borel action** of G on X is a Borel map

$$a\colon G \times X \to X$$

such that for all $x \in X$ and $g, h \in G$,

(i) $a(1_G, x) = x$, where 1_G is the identity element of G;

(ii) $a(g, a(h, x)) = a(gh, x)$.

The pair (X, a) is called a G-space. We denote by $E^X_{G,a}$ the orbit equivalence relation induced by the action, that is

$$x \; E^X_{G,a} \; y \iff \exists g \in G \; (a(g, x) = y).$$

Finally, we denote by $[x]_G$ the orbit of x, that is, the equivalence class of x with respect to $E^X_{G,a}$. When a is clear we write $g \cdot x$ in place of $a(g, x)$ and E^X_G instead of $E^X_{G,a}$.

An important class of analytic equivalence relations are those induced by a Borel action of S^∞.

Definition 1.1.6. An analytic equivalence relation is S_∞**-complete** if it is complete for the class of equivalence relations $E^Y_{S_\infty}$ arising from a Borel action of the group S_∞ on a standard Borel space Y.

Among the equivalence relations induced by an action of S_∞ we find all isomorphism relations on the countable models of a first-order theory or of an $\mathcal{L}_{\omega_1\omega}$-sentence. (The infinitary logic $\mathcal{L}_{\omega_1\omega}$ is the extension of classical first-order logic in which we allow countable conjunctions and disjunctions.)

Theorem 1.1.7 (H. Friedman-Stanley, see [FS89, Gao09])**.** *Let $\cong_{\text{C-GRAPH}}$ and $\cong_{\text{GRAPH}}$ denote the isomorphism relations on, respectively, the Polish space* C-GRAPH *of countable connected graphs and the Polish space* GRAPH *of countable graphs. Then*

$$\cong_{\text{C-GRAPH}} \sim_B \cong_{\text{GRAPH}},$$

and both equivalence relations are S_∞-complete.

Definition 1.1.8. Let E be an equivalence relation on a Polish space X. We say that E is **classifiable by countable structures** if $E \leq_B \cong_{GRAPH}$. When $E \sim_B \cong_{GRAPH}$ we say that E is **Borel complete**.

We now introduce a more general definition of a Borel reduction, looking at the restriction of a Borel map φ to sets A which are not necessarily standard Borel and such that $\varphi \restriction A$ is a reduction.

Definition 1.1.9. Let E and F be equivalence relations on standard Borel spaces X and Y respectively. Let $A \subseteq X$. We say that $E \restriction A$ **is Borel reducible to** F if there is a Borel map $\varphi \colon X \to Y$, still called a Borel reduction of $E \restriction A$ to F, such that for every $x, y \in A$,

$$x \mathrel{E} y \iff \varphi(x) \mathrel{F} \varphi(y).$$

Note that if A is a Borel set, and hence a standard Borel space, then the previous definition is equivalent to the existence of a Borel reduction $\varphi \colon A \to Y$ reducing $E \restriction A$ to F. Definition 1.1.9 is equivalent to the one given in [CMMR18, CMMR20] (where φ is required to be defined only on A) by a theorem of Kuratowski (see [Kec95, Theorem 12.2]).

Definition 1.1.10. We say that an equivalence relation E on a Polish space X is **σ-classifiable by countable structures** if there exists a countable partition $(X_i)_{i\in I}$ of X such that for all $i \in I$:

(i) X_i is closed under E (i.e. if $x \in X_i$ and $y \mathrel{E} x$ then $y \in X_i$);

(ii) X_i has the Baire property;

(iii) $E \restriction X_i$ is Borel reducible to $\cong_{GRAPH}$.

Clearly, if an equivalence relation E is classifiable by countable structures then it is σ-classifiable by countable structures.

Proposition 1.1.11. *Let E be an equivalence relation defined on a Polish space X. If E is σ-classifiable by countable structures, then $E \leq_{Baire} \cong_{\text{GRAPH}}$.*

Proof. Assume that E is σ-classifiable by countable structures and fix sets X_i witnessing this. Then by Theorem 1.1.7 for each $i \in I$ there exists a Borel reduction φ_i from $E \restriction X_i$ to $\cong_{\text{C-GRAPH}}$, so that $\varphi_i(x)$ is an infinite connected graph for every $x \in X_i$ (in particular, it is not isomorphic to the graph consisting of a single isolated vertex). Let $\tilde{\varphi}_i \colon X \to \text{GRAPH}$ be defined by

$$\tilde{\varphi}_i(x) = \varphi_i(x) \sqcup A_i,$$

where A_i is the graph consisting of i-many isolated vertices. It is easy to check that $\tilde{\varphi}_i$ is still a Borel function and it reduces $E \restriction X_i$ to $\cong_{\text{GRAPH}}$. Finally, define $\varphi \colon X \to \text{GRAPH}$ by setting $\varphi(x) = \tilde{\varphi}_i(x)$, where i is the unique index of the subset X_i of X to which x belongs.

We first show that φ is a reduction. Let x, y be two elements of X such that $x \mathrel{E} y$. Since X_i is closed under E for every $i \in I$, there exists $i_0 \in I$ such that $x, y \in X_{i_0}$. Then $\varphi_{i_0}(x) \cong_{\text{C-GRAPH}} \varphi_{i_0}(y)$, and so $\varphi(x) \cong_{\text{GRAPH}} \varphi(y)$. Conversely, suppose that $\varphi(x) = \varphi_i(x) \sqcup A_i \cong_{\text{GRAPH}} \varphi_j(y) \sqcup A_j = \varphi(y)$, for some $i, j \in I$, $x \in X_i$, and $y \in X_j$. Since isomorphism between graphs preserves connected components, we must have $i = j$ because $\varphi(x)$ contains i-many isolated vertices and $\varphi(y)$ contains j-many isolated vertices, and moreover $\varphi_i(x) \cong_{\text{C-GRAPH}} \varphi_i(y)$ because those are the only infinite connected components in $\varphi(x)$ and $\varphi(y)$, respectively. Since φ_i was a reduction we get $x \mathrel{E} y$, as desired.

Now take a Borel subset A of GRAPH. Then

$$\varphi^{-1}(A) = \bigcup_{i\in I} \left(\varphi^{-1}(A) \cap X_i\right) = \bigcup_{i\in I} \left(\tilde{\varphi}_i^{-1}(A) \cap X_i\right).$$

Since X_i has the BP and $\tilde{\varphi}_i$ is Borel for every $i \in I$, we have that $\tilde{\varphi}_i^{-1}(A) \cap X_i$ has the BP for each i. Hence also $\varphi^{-1}(A)$ has the BP and φ is a Baire measurable reduction. □

Not all orbit equivalence relations are Borel reducible, or even Baire reducible, to an S_∞-complete equivalence relation: Hjorth isolated a sufficient condition for this failure, called **turbulent**.

Theorem 1.1.12 ([Hjo00b], Corollary 3.19)**.** *There is no Baire measurable reduction of a turbulent orbit equivalence relation to any* $E^Y_{S_\infty}$.

Let E_1 be the equivalence relation defined on $\mathbb{R}^\mathbb{N}$ by

$$(x_n)_{n\in\mathbb{N}}\; E_1\; (y_n)_{n\in\mathbb{N}} \iff \exists m\, \forall n \geq m\, (x_n = y_n).$$

We also use its tail version E_1^t, defined by setting

$$(x_n)_{n\in\mathbb{N}}\; E_1^t\; (y_n)_{n\in\mathbb{N}} \iff \exists n, m\, \forall k\, (x_{n+k} = y_{m+k}).$$

Notice that E_1 and E_1^t are Borel bireducible with the analogous relations defined on $(2^\mathbb{N})^\mathbb{N}$, called $E_0(2^\mathbb{N})$ and $E_t(2^\mathbb{N})$ in [DJK94]. In the proof of [DJK94, Theorem 8.1] it is shown that $E_t(2^\mathbb{N}) \leq_B E_0(2^\mathbb{N})$, while the opposite reduction is mentioned in the observation immediately following that proof. This yields:

Proposition 1.1.13. $E_1 \sim_B E_1^t$.

The following result of Shani about E_1 generalizes a classical theorem by Kechris and Louveau [KL97]. (The additional part follows from the fact that by [Kec95, Theorem 8.38] every Baire measurable map between Polish spaces is actually continuous on a comeager G_δ set.)

Theorem 1.1.14 ([Sha21, Theorem 4.8])**.** *The restriction of E_1 to any comeager subset of $\mathbb{R}^\mathbb{N}$ is not Borel reducible to an orbit equivalence relation. Thus in particular $E_1 \not\leq_{Baire} \cong_{\mathsf{LO}}$.*

The following standard operation on equivalence relations was introduced by Friedman and Stanley in [FS89].

Definition 1.1.15. Let E be an equivalence relation on a standard Borel space X. The **Friedman-Stanley jump** of E, denoted by E^+, is the equivalence relation on the space $X^\mathbb{N} = \{(x_n)_{n\in\mathbb{N}} \mid x_n \in X\}$ defined by

$$(x_n)_{n\in\mathbb{N}}\; E^+\; (y_n)_{n\in\mathbb{N}} \iff \{[x_n]_E \mid n \in \mathbb{N}\} = \{[y_n]_E \mid n \in \mathbb{N}\}.$$

We sum up the relevant properties of the jump operator $E \mapsto E^+$ in the following proposition.

Proposition 1.1.16 (see [Gao09])**.** *Let E and F be equivalence relations on standard Borel spaces. Then:*

(a) *$E \leq_B E^+$, and if $E \leq_B F$ then $E^+ \leq_B F^+$.*

(b) *Assume E is Borel. Then E^+ is Borel, and if E has more than one equivalence class then $E <_B E^+$.*

One can transfer many of the above definitions concerning equivalence relations to the wider context of binary relations and, in particular, analytic quasi-orders. We just recall a few results in this direction.

Theorem 1.1.17 ([LR05])**.** *Every analytic quasi-order Borel reduces to the embeddability relation between countable (connected) graphs, i.e. the latter relation is complete for analytic quasi-orders.*

Every analytic quasi-order R on a standard Borel space X canonically induces the analytic equivalence relation E_R on the same space defined by

$$x\; E_R\; y \iff x\; R\; y \wedge y\; R\; x.$$

The complexities of R and E_R are linked by the following result.

Proposition 1.1.18 ([LR05])**.** *If a quasi-order R on a standard Borel space X is complete for analytic quasi-orders, then E_R is complete for analytic equivalence relations.*

1.2 Countable linear orders

Any $L \in 2^{\mathbb{N}\times\mathbb{N}}$ can be seen as a code of a binary relation on $\mathbb{N}$, namely, the one relating n and m if and only if $L(n,m) = 1$. Denote by LO the set of codes for linear orders on $\mathbb{N}$, i.e.

$$\mathsf{LO} = \{L \in 2^{\mathbb{N}\times\mathbb{N}} \mid L \text{ codes a reflexive linear order on } \mathbb{N}\}.$$

When $L \in \mathsf{LO}$ we denote by $\leq_L$ the order on $\mathbb{N}$ coded by L, and by $<_L$ its strict part.

It is easy to see that LO is a closed subset of the Polish space $2^{\mathbb{N}\times\mathbb{N}}$, thus it is a Polish space as well. Given $L \in \mathsf{LO}$, a neighbourhood base of L in LO is determined by the sets

$$\{L' \in \mathsf{LO} \mid L' \restriction n = L \restriction n\}$$

where n varies over $\mathbb{N}$ and $L \restriction n = L' \restriction n$ means that $m \leq_L m' \iff m \leq_{L'} m'$ for every $m, m' < n$. We also denote by WO the set of all well-orders on $\mathbb{N}$, and recall that it is a proper coanalytic subset of LO.

We denote by $\preccurlyeq$ the quasi-order of embeddability on linear orders, that is: $L \preccurlyeq L'$ if there exists an injection f from L to L', called embedding, such that $n \leq_L m \Rightarrow f(n) \leq_{L'} f(m)$ for every $n, m \in L$. (By linearity and antisymmetry of the orders, such an f also satisfies $f(n) \leq_{L'} f(m) \Rightarrow n \leq_L m$.) The restriction $\preccurlyeq_{\mathsf{LO}}$ of $\preccurlyeq$ to LO is clearly an analytic quasi-order. In contrast with embeddability among countable graphs and Theorem 1.1.17, the relation $\preccurlyeq_{\mathsf{LO}}$ is far from being complete because it is combinatorially simple. Recall that a quasi-order $\leq$ on a set X is a well quasi-order (briefly: a wqo) if for each sequence $(x_n)_{n\in\omega}$ of elements of X, there exist $n < m$ such that $x_n \leq x_m$, or equivalently, if $\leq$ has no infinite descending chain and no infinite antichain ([Ros82]). In 1948 Fraïssé conjectured that the set of countable linear orders is well quasi-ordered under the quasi-order of embeddability and Laver in 1971 showed that this is indeed the case [Lav71]. Moreover LO has a maximal element under $\preccurlyeq_{\mathsf{LO}}$, the equivalence class of non-scattered linear orders (recall that a linear order is scattered if the rationals do not embed into it).

The isomorphism relation on LO is denoted by $\cong_{\mathsf{LO}}$, and it is an analytic equivalence relation.

Theorem 1.2.1 (H. Friedman-Stanley,[FS89])**.** $\cong_{\mathsf{LO}}$ *is* S_∞*-complete.*

Recall that $E \leq_B E^+$ for any analytic equivalence relation E (Proposition 1.1.16). In the case of $\cong_{\mathsf{LO}}$, we also have the converse.

Proposition 1.2.2 (Folklore)**.** $(\cong_{\mathsf{LO}})^+ \sim_B \cong_{\mathsf{LO}}$.

Proof. By Theorem 1.1.7 and Theorem 1.2.1, we have that $\cong_{\text{C-GRAPH}} \sim_B \cong_{\text{GRAPH}} \sim_B \cong_{\mathsf{LO}}$, so it is enough to prove that $(\cong_{\text{C-GRAPH}})^+ \leq_B \cong_{\text{GRAPH}}$ because $(\cong_{\text{C-GRAPH}})^+ \sim_B (\cong_{\mathsf{LO}})^+$ by Proposition 1.1.16. Given a sequence of countable connected graphs $(A_n)_{n\in\mathbb{N}}$, let $G_A = \bigsqcup_{n,i\in\mathbb{N}} A_{n,i}$ be the disjoint union of the graphs $A_{n,i}$, where $A_{n,i} \cong A_n$ for every $n, i \in \mathbb{N}$. Then the Borel map from $(\text{C-GRAPH})^{\mathbb{N}}$ to GRAPH which sends $(A_n)_{n\in\mathbb{N}}$ to G_A is a reduction of $(\cong_{\text{C-GRAPH}})^+$ to $\cong_{\text{GRAPH}}$. □

We need to deal also with finite linear orders, which are missing in LO. For this reason, we let Lin be the subset of $2^{\mathbb{N}\times\mathbb{N}}$ consisting of all (codes for) linear orders defined either on a finite subset of $\mathbb{N}$ or on the whole $\mathbb{N}$. Thus Lin is the union of LO and Fin, where $\mathsf{Fin} \subset \mathsf{Lin}$ is the set of (codes for) finite linear orders. It is easy to see that Lin is a F_σ subset of $2^{\mathbb{N}\times\mathbb{N}}$, and hence it is a standard Borel space, and that isomorphism on Lin is induced by a Borel action of S_∞.

When $L \in \mathsf{Lin}$ we denote by $\leq_L$ the order on the domain of L coded by L, and by $<_L$ its strict part. We denote by L also its domain. For convenience, sometimes we use the notation n_L to emphasize that n is an element of the domain of L.

We recall some isomorphism invariant operations on the class of linear orders that are useful to build Borel reductions. They can all be construed as Borel maps from Lin, $(\mathsf{Lin})^n$, or $\mathsf{Lin}^{\mathbb{N}}$ to Lin, and their restriction to LO has range contained in LO.

- The **reverse** L^* of a linear order L is the linear order on the domain of L defined by setting $x \leq_{L^*} y \iff y \leq_L x$.

- If L and K are linear orders, their **sum** $L+K$ is the linear order defined on the disjoint union of L and K by setting $x \leq_{L+K} y$ if and only if either $x \in L$ and $y \in K$, or $x, y \in L$ and $x \leq_L y$, or $x, y \in K$ and $x \leq_K y$.
- In a similar way, given a linear order K and a sequence of linear orders $(L_k)_{k \in K}$ we can define the K-**sum** $\sum_{k \in K} L_k$ on the disjoint union of the L_k's by setting $x \leq_{\sum_{k \in K} L_k} y$ if and only if there are $k <_K k'$ such that $x \in L_k$ and $y \in L_{k'}$, or $x, y \in L_k$ for the same $k \in K$ and $x \leq_{L_k} y$. Formally, $\sum_{k \in K} L_k$ is thus defined on the set $\{(x, k) \mid k \in K, x \in L_k\}$ by stipulating that $(x, k) \leq_{\sum_{k \in K} L_k} (x', k')$ if and only if $k <_K k'$ or else $k = k'$ and $x \leq_{L_k} x'$.
- The **product** LK of two linear orders L and K is the cartesian product $L \times K$ ordered antilexicographically. Equivalently, $LK = \sum_{k \in K} L_k$, where $L_k = L$ for every $k \in K$.

For every $n \in \mathbb{N}$, we denote by $\boldsymbol{n}$ the element of Fin with domain $\{0, ..., n-1\}$ ordered as usual. Similarly, for every infinite ordinal $\alpha < \omega_1$ we fix a well-order $\boldsymbol{\alpha} \in \mathsf{LO}$ with order type α. We also fix computable copies of $(\mathbb{N}, \leq)$, $(\mathbb{Z}, \leq)$ and $(\mathbb{Q}, \leq)$ in LO, and denote them by ω, ζ and η, respectively. We denote by $\min L$ and $\max L$ the minimum and maximum of L, if they exist. Finally, we let Scat and WO be the subsets of Lin consisting of scattered linear orders and well-orders, respectively (recall that a linear order is scattered if the rationals do not embed into it).

Definition 1.2.3. A subset I of the domain of a linear order L is **(L-)convex** if $x \leq_L y \leq_L z$ with $x, z \in I$ implies $y \in I$. An L-convex set is **proper** if it is neither empty nor the entire L.

The **L-convex closure** of a set $A \subseteq L$ is the smallest L-convex subset of L containing A, that is, the set of all $\ell \in L$ such that $a_0 \leq_L \ell \leq_L a_1$ for some (possibly equal) $a_0, a_1 \in A$.

Remark 1.2.4. If a linear order$L$ has order type η and $L_0 \trianglelefteq L$, then the order type of L_0 is one of $\mathbf{1}$, η, $\mathbf{1} + \eta$, $\eta + \mathbf{1}$, or $\mathbf{1} + \eta + \mathbf{1}$.

An **initial segment** of a linear order L is a subset I of its domain which is $\leq_L$-downward closed, i.e. $x \in I$ whenever $x \leq_L y$ for some $y \in I$. Dually, $I \subseteq L$ is a **final segment** of L if it is $\leq_L$-upward closed, i.e. if $y \in I$ and $y \leq_L x$ imply $x \in I$. Clearly, initial and final segments are always convex subsets of L.

If $m, n \in L$, we adopt the notations $[m, n]_L$, $(m, n)_L$, $(-\infty, n]_L$, $(-\infty, n)_L$, $[n, +\infty)_L$, and $(n, +\infty)_L$ to indicate the obvious L-convex sets. Notice however that not all L-convex sets are of one of these forms.

Given $L \in \mathsf{LO}$, we write $L_0 \subseteq L$ (resp. $L_0 \subset L$) if L_0 is a (resp. proper) sub-order of L, and $L_0 \sqsubseteq L$ (resp. $L_0 \sqsubset L$) if L_0 is a (resp. proper) convex subset of L. If $L_0, L_1 \subseteq L$, we write $L_0 \leq_L L_1$ (resp. $L_0 <_L L_1$) iff $n \leq_L m$ (resp. $n <_L m$) for every $n \in L_0$ and $m \in L_1$. Notice that if $L_0 \leq_L L_1$ then either L_0 and L_1 are disjoint, in which case $L_0 <_L L_1$, or the only element in their intersection is the maximum of L_0 and the minimum of L_1.

When studying combinatorial properties of our quasi-orders we use the following standard terminology.

Definition 1.2.5. Let $\leq$ be a quasi-order on a set X. We say that $\mathcal{F} \subseteq X$ is a **dominating family** if for every L there exists $L' \in \mathcal{F}$ such that $L \leq L'$. Let $\mathfrak{d}(\leq)$ be the **dominating number** of $\leq$, i.e. the least size of a dominating family with respect to $\leq$.

We say that $\mathcal{B} \subseteq X$ is a **basis** with respect to $\leq$ if for every L there is $L' \in \mathcal{B}$ such that $L' \leq L$. In other words, a basis for $\leq$ is a dominating family for $\geq$.

The **unbounding number** $\mathfrak{b}(\leq)$ is the smallest size of a subset of X which is unbounded with respect to $\leq$.

We need to recall some other basic notions about linear orders (see [Ros82]). Let L be a linear order. The **(finite) condensation** of L is determined by the map $c_F^L \colon L \to \mathcal{P}(L)$ defined by

$$c_F^L(n) = \{m \mid [n, m]_L \cup [m, n]_L \text{ is finite}\}$$

for every $n \in L$. It is immediate that if $m \in c_F^L(n)$ then $c_F^L(m) = c_F^L(n)$. We call a set $c_F^L(n)$ a **condensation class**. A condensation class may be finite or infinite, and in the latter case its order type is one of ω, ω^* and ζ. We denote by L_F the set of condensation classes of L.

In the sequel we use the basic properties of condensation classes which are collected in the following proposition.

Proposition 1.2.6. *Let L be any linear order.*

(a) *For every $\ell \in L$, $c_F^L(\ell)$ is convex.*

(b) $\bigcup_{\ell \in L} c_F^L(\ell) = L$, *and* $c_F^L(\ell) \cap c_F^L(\ell') = \emptyset$ *if* $c_F^L(\ell) \neq c_F^L(\ell')$; *hence L_F is a partition of L.*

(c) *If $c_F^L(\ell)$ and $c_F^L(\ell')$ are two different condensation classes, then $\ell <_L \ell'$ if and only if $c_F^L(\ell) <_L c_F^L(\ell')$; hence L_F is linearly ordered.*

(d) *Let L, L' be linear orders. If f is an isomorphism from L to L' then the restriction of f to each $c_F^L(\ell)$ is an isomorphism between $c_F^L(\ell)$ and $c_F^{L'}(f(\ell))$ and hence $|c_F^L(\ell)| = |c_F^{L'}(f(\ell))|$. Moreover, $L_F \cong L'_F$ via the well-defined map $c_F^L(\ell) \mapsto c_F^{L'}(f(\ell))$.*

Clearly, if L and L' are arbitrary linear orders such that $L \cong L'$, then $\zeta L \cong \zeta L'$. The converse is true as well.

Lemma 1.2.7. *Given two linear orders L, L', $\zeta L \cong \zeta L'$ if and only if $L \cong L'$.*

Proof. For the nontrivial direction, notice that $c_F^{\zeta L}(i, n) = \zeta \times \{n\}$, and similarly for the condensation classes of $\zeta L'$. It follows that $(\zeta L)_F \cong L$ and $(\zeta L')_F \cong L'$. By Proposition 1.2.6, if $\zeta L \cong \zeta L'$ then $(\zeta L)_F \cong (\zeta L')_F$, hence $L \cong L'$. □

We conclude this section recalling the definition of the powers of $\mathbb{Z}$ and some of their properties. When α is an ordinal we can define $\mathbb{Z}^\alpha$ in two equivalent ways: by induction on α ([Ros82, Definition 5.34]) and by explicitly defining a linear order on a set ([Ros82, Definition 5.35]); the latter can actually be used to define $\mathbb{Z}^L$ for any linear order L.

Definition 1.2.8. (1) $\mathbb{Z}^0 = \mathbf{1}$,

(2) $\mathbb{Z}^{\alpha+1} = (\mathbb{Z}^\alpha\omega)^* + \mathbb{Z}^\alpha + \mathbb{Z}^\alpha\omega$,

(3) $\mathbb{Z}^\alpha = \left(\sum_{\beta<\alpha} \mathbb{Z}^\beta\omega\right)^* + \mathbf{1} + \sum_{\beta<\alpha} \mathbb{Z}^\beta\omega$ if α is limit.

Definition 1.2.9. Let L be a linear order. For any map $f : L \to \mathbb{Z}$, we define the support of f as the set $\operatorname{Supp}(f) = \{n \in L \mid f(n) \neq 0\}$. The L-power of $\mathbb{Z}$, denoted by $\mathbb{Z}^L$, is the linear order on $\{f : L \to \mathbb{Z} \mid \operatorname{Supp}(f) \text{ is finite}\}$ defined by the following: if $f, g : L \to \mathbb{Z}$ are maps with finite support let $f \leq_{\mathbb{Z}^L} g$ if and only if $f = g$ or $f(n_0) <_{\mathbb{Z}} g(n_0)$ where $n_0 = \max\{n \in \operatorname{Supp}(f) \cup \operatorname{Supp}(g) \mid f(n) \neq g(n)\}$.

Sometimes we need the following properties (see [CCM19, Section 3.2]).

Proposition 1.2.10. *For all ordinals $\beta < \alpha$, we have*

$$\mathbb{Z}^\alpha \cong \left(\sum_{\beta\leq\gamma<\alpha} \mathbb{Z}^\gamma\omega\right)^* + \left(\sum_{\beta\leq\gamma<\alpha} \mathbb{Z}^\gamma\omega\right).$$

Proposition 1.2.11. *For any linear orders L and L' we have*

(a) $(\mathbb{Z}^L)^* = \mathbb{Z}^L$,

(b) $\mathbb{Z}^{L+L'} \cong \mathbb{Z}^L\mathbb{Z}^{L'}$,

(c) *if L is countable and not a well-order then there is a countable ordinal α such that $\mathbb{Z}^L \cong \mathbb{Z}^\alpha\eta$.*

1.3 Countable circular orders

We now describe the basic notation and notions regarding circular orders. The prototype of a circular order is the unit circle S^1 traversed counterclockwise, which we denote by C_{S^1}.

Definition 1.3.1. ([KM05, Definition 2.1]) A ternary relation $C \subset X^3$ on a set X is said to be a **circular order** if the following conditions are satisfied for every $x, y, z, w \in X$:

(i) Cyclicity: $(x, y, z) \in C \Rightarrow (y, z, x) \in C$;

(ii) Antisymmetry and reflexivity: $(x, y, z) \in C \wedge (y, x, z) \in C \iff x = y \vee y = z \vee z = x$;

(iii) Transitivity: $(x, y, z) \in C \Rightarrow \forall t((x, y, t) \in C \vee (t, y, z) \in C)$;

(iv) Totality: $(x, y, z) \in C \vee (y, x, z) \in C$.

Notice that, assuming the other conditions, (iii) is equivalent to asserting that $(x, y, z) \in C$ and $(x, z, w) \in C$ imply $(x, y, w) \in C$ whenever $x \neq z$. In the sequel we often make use of this reformulation. Definition 1.3.1 is different from [Č69, 5.1]: indeed, the latter characterizes the strict relation associated to C, i.e. the set of all triples (x, y, z) such that $(x, y, z) \in C$ and x, y, z are all distinct.

By abuse of notation, when C is a circular order on X we write $C(x, y, z)$ instead of $(x, y, z) \in C$, for $x, y, z \in X$. The **reverse** C^* of a circular order C on X is the circular order on X defined by $C^*(x, y, z) \iff C(z, y, x)$ for all $x, y, z \in X$.

Definition 1.3.2. Let C and C' be circular orders on sets X and X', respectively. We say that C is **embeddable** into C', and write $C \preccurlyeq_c C'$, if there exists an injective function $f \colon X \to X'$, called embedding, such that for every $x, y, z \in X$,

$$C(x, y, z) \Rightarrow C'(f(x), f(y), f(z)).$$

We say that C and C' are **isomorphic**, and write $C \cong_c C'$, if there exists f as above which is a bijection (in which case f is called isomorphism).

Notice that by totality and antisymmetry, an f as in Definition 1.3.2 satisfies also

$$C'(f(x), f(y), f(z)) \Rightarrow C(x, y, z).$$

For a circular order, the notions of successor and predecessor of an element are meaningless. However, we can still define a notion of immediate successor or immediate predecessor.

Definition 1.3.3. Given a circular order C on the set X and $x, y \in X$, we say that x is the **immediate predecessor** (resp. **immediate successor**) of y in C if $x \neq y$ and $C(x, y, z)$ (resp. $C(y, x, z)$) for every $z \in X$.

Definition 1.3.4. Given a linear order L, we define a circular order $C[L]$ by setting $C[L](x, y, z)$ if and only if one of the following conditions is satisfied:

$$x \leq_L y \leq_L z, \quad y \leq_L z \leq_L x, \quad z \leq_L x \leq_L y.$$

Notice that every circular order C is of the form $C[L]$ for some (in general non unique) linear order L. Clearly, for two linear orders L and L' such that $L \preccurlyeq L'$ we have $C[L] \preccurlyeq_c C[L']$.

Denote by CO the set of codes for circular orders on $\mathbb{N}$, i.e.

$$\mathsf{CO} = \{C \in 2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}} \mid C \text{ codes a circular order on } \mathbb{N}\}.$$

Since CO is a closed subset of the Polish space $2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$, we have that it is a Polish space as well. Denote by $\preccurlyeq_{\mathsf{CO}}$ and $\cong_{\mathsf{CO}}$ the restriction of the relations of embeddability $\preccurlyeq_c$ and isomorphism $\cong_c$ to CO, respectively. It is immediate that both $\preccurlyeq_{\mathsf{CO}}$ and $\cong_{\mathsf{CO}}$ are analytic.

Proposition 1.3.5. $\preccurlyeq_{\mathsf{CO}}$ *is a wqo.*

Proof. Recall that a quasi-order $(X, \leq_X)$ is a wqo if for every sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X, there exist $n < m$ such that $x_n \leq_X x_m$. Suppose that $(C_n)_{n\in\mathbb{N}}$ is a sequence of elements of CO. For every $n \in \mathbb{N}$ consider the linear order L_n defined by

$$x \leq_{L_n} y \iff C_n(0, x, y) \wedge (y = 0 \Rightarrow x = 0).$$

Notice that $C[L_n] = C_n$.

Since the embeddability relation $\preccurlyeq_{\mathsf{LO}}$ on LO is wqo, there are $n < m$ such that $L_n \preccurlyeq_{\mathsf{LO}} L_m$ and hence $C_n \preccurlyeq_{\mathsf{CO}} C_m$. □

The isomorphism $\cong_{\mathsf{CO}}$ is an equivalence relation on CO. Clearly, for $L, L' \in \mathsf{LO}$, we have that $L \cong_{\mathsf{LO}} L'$ implies $C[L] \cong_{\mathsf{CO}} C[L']$. The converse implication is not true, as showed by $C[\omega + \mathbf{1}]$ and $C[\omega]$, for which we have $C[\omega + \mathbf{1}] \cong_{\mathsf{CO}} C[\omega]$, but $\omega + \mathbf{1} \ncong_{\mathsf{LO}} \omega$.

Theorem 1.3.6. $\cong_{\mathsf{CO}} \sim_B \cong_{\mathsf{LO}}$.

Proof. For the Borel reduction from $\cong_{\mathsf{CO}}$ to $\cong_{\mathsf{LO}}$, it is enough to note that $\cong_{\mathsf{CO}}$ is an equivalence relation arising from a Borel action of the group S_∞. Then $\cong_{\mathsf{CO}} \leq_B \cong_{\mathsf{LO}}$ by Theorem 1.2.1.

For the converse, consider the Borel map $\varphi \colon \mathsf{LO} \to \mathsf{CO}$ defined by

$$\varphi(L) = C[\mathbf{1} + \zeta L].$$

If $L \cong_{\mathsf{LO}} L'$ we have immediately that $\varphi(L) \cong_{\mathsf{CO}} \varphi(L')$. Suppose now that $\varphi(L) \cong_{\mathsf{CO}} \varphi(L')$ via the map f. Since $\mathbf{1}$ is the only element which has no immediate successor in both $\varphi(L)$ and $\varphi(L')$, we have that $f(\mathbf{1}) = \mathbf{1}$. Thus $\zeta L \cong_{\mathsf{LO}} \zeta L'$ and by Lemma 1.2.7 we obtain $L \cong_{\mathsf{LO}} L'$. □

2

Convex embeddability on linear/circular orders

2.1 Definition and basic facts

This is the main definition of Chapter 2.

Definition 2.1.1 ([BCP73])**.** Let L and L' be linear orders. We say that an embedding f from L to L' is a **convex embedding** if $f(L)$ is an L'-convex set. We write $L \trianglelefteq L'$ when such f exists, and call **convex embeddability** the resulting binary relation.

Remark 2.1.2. Notice that $L \trianglelefteq L'$ if and only if

$$L' \cong L_l + L + L_r,$$

for some (possibly empty) L_l and L_r, if and only if L is isomorphic to an L'-convex set.

While $L \preccurlyeq \eta$, for every countable linear order L, we have $L \trianglelefteq \eta$ if and only if L has order type $\mathbf{1}$, η, $\mathbf{1} + \eta$, $\eta + \mathbf{1}$ or $\mathbf{1} + \eta + \mathbf{1}$.

One easily sees that the restriction of convex embeddability to the Polish space LO is an analytic quasi-order, which we denote by $\trianglelefteq_{\mathsf{LO}}$. The strict part of $\trianglelefteq_{\mathsf{LO}}$ is denoted by $\triangleleft_{\mathsf{LO}}$, that is, $L \triangleleft_{\mathsf{LO}} L'$ if and only if $L \trianglelefteq_{\mathsf{LO}} L'$ but $L' \not\trianglelefteq_{\mathsf{LO}} L$. We call **convex biembeddability**, and denote it by $\bowtie_{\mathsf{LO}}$, the equivalence relation on LO induced by $\trianglelefteq_{\mathsf{LO}}$, that is

$$L \bowtie_{\mathsf{LO}} L' \iff L \trianglelefteq_{\mathsf{LO}} L' \text{ and } L' \trianglelefteq_{\mathsf{LO}} L.$$

Clearly, if $L \cong_{\mathsf{LO}} L'$ then $L \bowtie_{\mathsf{LO}} L'$. The converse implication does not hold, as witnessed by $\zeta\omega$ and $\omega + \zeta\omega$.

Finally, notice that if $L \bowtie_{\mathsf{LO}} L'$ then $L \equiv_{\mathsf{LO}} L'$, where $\equiv_{\mathsf{LO}}$ is the equivalence relation of biembeddability on LO induced by $\preccurlyeq_{\mathsf{LO}}$. The converse is not true: the linear orders of the form $\boldsymbol{k}\eta$, for $k > 0$, belong to the same $\equiv_{\mathsf{LO}}$-equivalence class, but they are pairwise $\trianglelefteq_{\mathsf{LO}}$ incomparable.

2.2 Combinatorial properties of $\trianglelefteq_{\mathsf{LO}}$

In this section we explore the combinatorial properties of convex embeddability, pointing out several differences between $\trianglelefteq_{\mathsf{LO}}$ and the embeddability relation $\preccurlyeq_{\mathsf{LO}}$ on LO. For example, we show that $\trianglelefteq_{\mathsf{LO}}$ has antichains of size the continuum and chains of order type $(\mathbb{R}, \leq)$ (hence descending and ascending chains of arbitrary countable length), that well-orders are unbounded with respect to $\trianglelefteq_{\mathsf{LO}}$ (hence the unbounding number of $\trianglelefteq_{\mathsf{LO}}$ is ω_1), that $\trianglelefteq_{\mathsf{LO}}$ has dominating number $2^{\aleph_0}$ (thus in particular there is no $\trianglelefteq_{\mathsf{LO}}$-maximal element), and that all bases for $\trianglelefteq_{\mathsf{LO}}$ have maximal size $2^{\aleph_0}$. This is in stark contrast with the fact that $\preccurlyeq_{\mathsf{LO}}$ is a wqo (and hence has neither infinite antichains nor infinite descending chains), that η is the maximum with respect to $\preccurlyeq_{\mathsf{LO}}$ (hence there are no $\preccurlyeq_{\mathsf{LO}}$-unbounded sets and the dominating number of $\preccurlyeq_{\mathsf{LO}}$ is 1), and that $\{\omega, \omega^*\}$ is a two-elements basis for $\preccurlyeq_{\mathsf{LO}}$.

Applying Proposition 1.2.6 and recalling that a convex embedding $f\colon L \to L'$ is just an isomorphism between L and a convex subset of L', we easily obtain the following useful fact.

Proposition 2.2.1. *Let L, L' be arbitrary linear orders. If $L \trianglelefteq L'$ via some convex embedding $f\colon L \to L'$, then the restriction of f witnesses $c_F^L(\ell) \cong c_F^{L'}(f(\ell)) \cap f(L)$ and hence $|c_F^L(\ell)| = |c_F^{L'}(f(\ell)) \cap f(L)| \leq |c_F^{L'}(f(\ell))|$ for every $\ell \in L$. Moreover, $f(c_F^L(\ell)) = c_F^{L'}(f(\ell))$ for every $\ell \in L$, except for the first and last condensation classes of L (if they exist). Finally, $L_F \trianglelefteq L'_F$ via the well-defined map $c_F^L(\ell) \mapsto c_F^{L'}(f(\ell))$.*

Arguing as in the case of $\cong$, we obtain a result for $\trianglelefteq$ which is analogous to Lemma 1.2.7.

Proposition 2.2.2. *Given two linear orders L, L', $\zeta L \trianglelefteq \zeta L'$ if and only if $L \trianglelefteq L'$.*

Proof. For the nontrivial direction, recall that by the proof of Lemma 1.2.7 we have $(\zeta L)_F \cong L$ and $(\zeta L')_F \cong L'$. By Proposition 2.2.1, if $\zeta L \trianglelefteq \zeta L'$ then $(\zeta L)_F \trianglelefteq (\zeta L')_F$, hence $L \trianglelefteq L'$. □

Given a map $f\colon \mathbb{Q} \to \mathsf{Scat}$, let $\eta_f \in \mathsf{LO}$ be (an isomorphic copy on $\mathbb{N}$ of) the linear order $\eta_f = \sum_{q\in\mathbb{Q}} f(q)$.

Lemma 2.2.3. *There is an embedding from the partial order $(\mathrm{Int}(\mathbb{R}), \subseteq)$ into $(\mathsf{LO}, \trianglelefteq_{\mathsf{LO}})$, where $\mathrm{Int}(\mathbb{R})$ is the set of the open intervals of $\mathbb{R}$.*

Proof. Consider an injective map $f\colon \mathbb{Q} \to \{\boldsymbol{n} \mid n \in \mathbb{N} \setminus \{0\}\}$ and consider the resulting linear order η_f. Notice that for each $(\ell, q) \in \eta_f$, $|c_F^{\eta_f}(\ell, q)| = |f(q)|$ is finite. Moreover if q and q' are distinct rational numbers then $|c_F^{\eta_f}(\ell, q)| \neq |c_F^{\eta_f}(\ell', q')|$ for every $\ell \in f(q)$ and $\ell' \in f(q')$ by injectivity of f.

An element of $\mathrm{Int}(\mathbb{R})$ is of the form (x, y) where $x \in \{-\infty\} \cup \mathbb{R}$ and $y \in \mathbb{R} \cup \{+\infty\}$ with $x < y$. For such (x, y) we define the linear order $L_{(x,y)} \cong \sum_{q\in\mathbb{Q}\cap(x,y)} f(q)$ as the restriction of η_f to $\{(\ell, q) \in \eta_f \mid q \in \mathbb{Q} \cap (x, y)\}$, which is a convex subset of η_f with no first and last condensation class.

We show that, after canonically coding each $L_{(x,y)}$ as an element of LO, the map $(x, y) \mapsto L_{(x,y)}$ is an embedding of the partial order $(\mathrm{Int}(\mathbb{R}), \subseteq)$ into $(\mathsf{LO}, \trianglelefteq_{\mathsf{LO}})$. Clearly, if $(x, y) \subseteq (x', y')$, then $L_{(x,y)} \sqsubseteq L_{(x',y')}$ and in particular we have $L_{(x,y)} \trianglelefteq_{\mathsf{LO}} L_{(x',y')}$. Vice versa, take $(x, y), (x', y') \in \mathrm{Int}(\mathbb{R})$, with $(x, y) \not\subseteq (x', y')$ and fix $q \in (x, y) \setminus (x', y')$. The condensation class of $(0, q)$ in $L_{(x,y)}$ has cardinality $f(q)$ and, by injectivity of f, no condensation class in $L_{(x',y')}$ has the same cardinality. Since there is no first and last condensation class in $L_{(x,y)}$, we get $L_{(x,y)} \not\trianglelefteq_{\mathsf{LO}} L_{(x',y')}$ by Proposition 2.2.1. □

Proposition 2.2.4. *$\trianglelefteq_{\mathsf{LO}}$ has chains of order type $(\mathbb{R}, <)$, as well as antichains of size $2^{\aleph_0}$.*

Proof. This is immediate from Lemma 2.2.3 and the fact that $(\mathrm{Int}(\mathbb{R}), \subseteq)$ has the same properties: consider e.g. the families $\{(x, +\infty) \mid x \in \mathbb{R}\}$ and $\{(x, x + 1) \mid x \in \mathbb{R}\}$, respectively. □

Let $\mathfrak{b}(\trianglelefteq_{\mathsf{LO}})$ be the **unbounding number** of $\trianglelefteq_{\mathsf{LO}}$, i.e. the smallest size of a family $\mathcal{F} \subseteq \mathsf{LO}$ which is unbounded with respect to $\trianglelefteq_{\mathsf{LO}}$. Using infinite (countable) sums of linear orders, one can easily prove that $\mathfrak{b}(\trianglelefteq_{\mathsf{LO}}) > \aleph_0$. The next result thus shows that $\mathfrak{b}(\trianglelefteq_{\mathsf{LO}})$ attains the smallest possible value.

Proposition 2.2.5. *WO is a maximal ω_1-chain without an upper bound in LO with respect to $\trianglelefteq_{\mathsf{LO}}$. Hence $\mathfrak{b}(\trianglelefteq_{\mathsf{LO}}) = \aleph_1$.*

Proof. Fix $L \in \mathsf{LO}$ and for every $n \in L$ define

$$\alpha_{n,L} = \sup\{\mathrm{ot}(L') \mid L' \text{ is a well-order, } L' \sqsubseteq L, \text{ and } n = \min L'\}.$$

Notice that $\alpha_{n,L}$ is actually attained by definition of $\sqsubseteq$. Therefore, $\alpha_{n,L} < \omega_1$ because L is countable. Let $\alpha_L = \sup_{n\in L} \alpha_{n,L} < \omega_1$. By construction, if $L' \sqsubseteq L$ and L is well-ordered, then $\mathrm{ot}(L') \leq \alpha_L$, thus $\boldsymbol{\alpha}_L + \mathbf{1} \not\trianglelefteq L$. Since L was arbitrary, we showed that for every $L \in \mathsf{LO}$ there exists $L' \in \mathsf{WO}$ such that $L' \not\trianglelefteq L$, i.e. that WO is $\trianglelefteq_{\mathsf{LO}}$-unbounded in LO.

Clearly, WO is a $\trianglelefteq_{\mathsf{LO}}$-chain: maximality then follows from unboundedness of WO, together with the observation that for $\omega \leq \alpha < \omega_1$ and $L \in \mathsf{LO}$, if $L \trianglelefteq_{\mathsf{LO}} \boldsymbol{\alpha}$ and $\boldsymbol{\beta} \triangleleft_{\mathsf{LO}} L$ for every $\beta < \alpha$, then $L \cong \boldsymbol{\alpha}$. □

An easy consequence of Proposition 2.2.5 is that no $L \in \mathsf{LO}$ is a node with respect to $\trianglelefteq_{\mathsf{LO}}$. This will be subsumed by Proposition 2.2.9.

Corollary 2.2.6. *For every $L \in \mathsf{LO}$ there is $M \in \mathsf{LO}$ which is $\trianglelefteq_{\mathsf{LO}}$-incomparable with L, i.e. $L \ntrianglelefteq_{\mathsf{LO}} M$ and $M \ntrianglelefteq_{LO} L$.*

Proof. If L is not a well-order, then it is enough to let $M \in \mathsf{WO}$ be such that $M \ntrianglelefteq_{\mathsf{LO}} L$ (the existence of such an M is granted by Proposition 2.2.5). If instead L is a well-order, then it is enough to set $M = \omega^*$. □

Another easy consequence of Proposition 2.2.5 is that $\trianglelefteq_{\mathsf{LO}}$ has no maximal element. In fact, much more is true.

Corollary 2.2.7. *Every $L \in \mathsf{LO}$ is the bottom element of a $\trianglelefteq_{\mathsf{LO}}$-unbounded chain of length ω_1.*

Proof. For every $\beta < \omega_1$ set $L_\beta = L + \boldsymbol{\beta}$ (in particular, $L_0 = L$), and consider the (not necessarily strictly) $\trianglelefteq_{\mathsf{LO}}$-increasing sequence $\langle L_\beta \mid \beta < \omega_1 \rangle$. Since $\boldsymbol{\beta} \trianglelefteq_{\mathsf{LO}} L_\beta$ for every $\beta < \omega_1$, the above sequence is $\trianglelefteq_{\mathsf{LO}}$-unbounded by Proposition 2.2.5. Moreover, for every $\beta < \omega_1$ there is $\beta' < \omega_1$ such that $L_\beta \triangleleft_{\mathsf{LO}} L_{\beta'}$. Indeed, it is enough to set $\beta' = \alpha_{L_\beta} + 1$, where α_{L_β} is as in the proof of Proposition 2.2.5: then $\boldsymbol{\beta'} \ntrianglelefteq_{\mathsf{LO}} L_\beta$, and thus also $L_{\beta'} \ntrianglelefteq_{\mathsf{LO}} L_\beta$. This easily implies that $\langle L_\beta \mid \beta < \omega_1 \rangle$ contains a strictly $\trianglelefteq_{\mathsf{LO}}$-increasing cofinal (hence $\trianglelefteq_{\mathsf{LO}}$-unbounded in LO) chain of length ω_1 beginning with L_0, as desired. □

We say that a collection $\mathcal{B}$ of (infinite) linear orders on $\mathbb{N}$ is a **basis** for $\trianglelefteq_{\mathsf{LO}}$ if for every $L \in \mathsf{LO}$ there is $L' \in \mathcal{B}$ such that $L' \trianglelefteq_{\mathsf{LO}} L$. The next result shows that each basis with respect to $\trianglelefteq_{\mathsf{LO}}$ is as large as possible.

Proposition 2.2.8. (a) *There are $2^{\aleph_0}$-many $\trianglelefteq_{\mathsf{LO}}$-incomparable $\trianglelefteq_{\mathsf{LO}}$-minimal elements in LO. In particular, if $\mathcal{B}$ is a basis for $\trianglelefteq_{\mathsf{LO}}$ then $|\mathcal{B}| = 2^{\aleph_0}$.*

(b) *There is a $\trianglelefteq_{\mathsf{LO}}$-decreasing ω-sequence in LO which is not $\trianglelefteq_{\mathsf{LO}}$-bounded from below.*

Proof. (a) Consider an infinite subset $S \subseteq \mathbb{N}$. Let $f_S \colon \mathbb{Q} \to \{\boldsymbol{n} \mid n \in S\}$ be a map such that

$$\forall q, q' (q < q' \to \forall n \in S \, \exists q'' (q < q'' < q' \land f_S(q'') = \boldsymbol{n})),$$

so that in particular f_S is surjective, and consider the linear order η_{f_S}. Let $q < q'$ be arbitrary rational numbers. By a back-and-forth argument on the condensation classes, it is easy to see that by choice of f_S the linear order η_{f_S} is isomorphic to the restriction $\eta_{f_S} \restriction (q, q') \cong \sum_{q'' \in \mathbb{Q} \cap (q,q')} f_S(q'')$ of η_{f_S} to its convex subset $\{(\ell, q'') \in \eta_{f_S} \mid q < q'' < q'\}$. This implies that each η_{f_S} is $\trianglelefteq_{\mathsf{LO}}$-minimal, because by density of η and finiteness of the condensation classes of η_{f_S}, any infinite convex subset of η_{f_S} contains some $\eta_{f_S} \restriction (q, q')$. Finally, by the choice of f_S for every $n \in S$ there are densely many condensation classes in $(\eta_{f_S})_F$ of size exactly n. Thus if $S \neq S'$ we have $\eta_{f_S} \ntrianglelefteq_{\mathsf{LO}} \eta_{f_{S'}}$ and $\eta_{f_{S'}} \ntrianglelefteq_{\mathsf{LO}} \eta_{f_S}$ by Proposition 2.2.1, as desired.

(b) Consider the family $\{L_{(n,+\infty)} \mid n \in \mathbb{N}\}$, where $L_{(n,+\infty)}$ is as in the proof Lemma 2.2.3. It is a strictly $\trianglelefteq_{\mathsf{LO}}$-decreasing chain, and we claim that it is $\trianglelefteq_{\mathsf{LO}}$-unbounded from below. To this aim, it is enough to consider any $L \in \mathsf{LO}$ with $L \trianglelefteq_{\mathsf{LO}} L_{(0,+\infty)}$, and show that $L \ntrianglelefteq_{\mathsf{LO}} L_{(m,+\infty)}$ for some $m \in \mathbb{N}$. Since $L \trianglelefteq_{\mathsf{LO}} L_{(0,+\infty)}$, all the condensation classes of L are finite by Proposition 2.2.1. Let $\ell \in L$ be such that $c^L_F(\ell)$ is not the minimum or the maximum of L_F, and let $q \in \mathbb{Q}$ be such that $f(q) = |c^L_F(\ell)|$, where $f \colon \mathbb{Q} \to \{\boldsymbol{n} \mid n \in \mathbb{N} \setminus \{0\}\}$ is the function used to defined the linear orders $L_{(n,+\infty)}$. Let $m \in \mathbb{N}$ be such that $q < m$. Then $L \ntrianglelefteq_{\mathsf{LO}} L_{(m,+\infty)}$ because otherwise by choice of ℓ the latter would have a condensation class of size $f(q)$ by Proposition 2.2.1, which is impossible by choice of m and the fact that f is an injection. □

Proposition 2.2.8 allows us to considerably improve Corollary 2.2.6 as follows.

Proposition 2.2.9. *Every $\trianglelefteq_{\mathsf{LO}}$-antichain is contained in a $\trianglelefteq_{\mathsf{LO}}$-antichain of size $2^{\aleph_0}$. In particular, there are no maximal $\trianglelefteq_{\mathsf{LO}}$-antichains of size smaller than $2^{\aleph_0}$, and every $L \in \mathsf{LO}$ belongs to a $\trianglelefteq_{\mathsf{LO}}$-antichain of size $2^{\aleph_0}$.*

Proof. Let $\mathcal{B}$ be a $\trianglelefteq_{\mathsf{LO}}$-antichain and assume that $|\mathcal{B}| < 2^{\aleph_0}$ (otherwise the statement is trivial). Consider the antichain $\mathcal{A} = \{\eta_{f_S} \mid S \subseteq \mathbb{N} \wedge S \text{ is infinite}\}$ of size $2^{\aleph_0}$ from Proposition 2.2.8. From $\trianglelefteq_{\mathsf{LO}}$-minimality of η_{f_S} it follows that $\mathcal{B} \cup (\mathcal{A} \setminus \bigcup_{L\in\mathcal{B}}\{K \in \mathcal{A} \mid K \trianglelefteq_{\mathsf{LO}} L\})$ is a $\trianglelefteq_{\mathsf{LO}}$-antichain. To show that this antichain has size $2^{\aleph_0}$ it suffices to show that

Claim 2.2.9.1. $\{K \in \mathcal{A} \mid K \trianglelefteq_{\mathsf{LO}} L\}$ is countable for every $L \in \mathsf{LO}$,

so that $|\bigcup_{L\in\mathcal{B}}\{K \in \mathcal{A} \mid K \trianglelefteq_{\mathsf{LO}} L\}| \leq \aleph_0 \cdot |\mathcal{B}| = \max\{\aleph_0, |\mathcal{B}|\} < 2^{\aleph_0}$.

To prove the claim, suppose that $S \subseteq \mathbb{N}$ is such that $\eta_{f_S} \trianglelefteq_{\mathsf{LO}} L$, so that without loss of generality we can write $L = L_l + \eta_{f_S} + L_r$. If f were a convex embedding of $\eta_{f_{S'}}$ into L with $f(\eta_{f_{S'}}) \cap \eta_{f_S} \neq \emptyset$, then by density of η and finiteness of the condensation classes of $\eta_{f_{S'}}$ there would be rationals $q < q'$ such that $f(\eta_{f_{S'}} \restriction (q, q')) \subseteq \eta_{f_S}$, and since $\eta_{f_{S'}} \cong \eta_{f_{S'}} \restriction (q, q')$ we would get $\eta_{f_{S'}} \trianglelefteq_{\mathsf{LO}} \eta_{f_S}$. Thus if $S \neq S'$, then $f(\eta_{f_{S'}}) \cap \eta_{f_S} = \emptyset$. Since L is countable, this means that there are only countably many distinct $S \subseteq \mathbb{N}$ for which $\eta_{f_S} \trianglelefteq_{\mathsf{LO}} L$ can hold.

Finally, the additional part of the statement follows by viewing $L \in \mathsf{LO}$ as the element of an antichain of size 1. □

We say that a collection $\mathcal{F} \subseteq \mathsf{LO}$ is a dominating family with respect to $\trianglelefteq_{\mathsf{LO}}$ if and only if for every $L \in \mathsf{LO}$ there exists $L' \in \mathcal{F}$ such that $L \trianglelefteq_{\mathsf{LO}} L'$. Let $\mathfrak{d}(\trianglelefteq_{\mathsf{LO}})$ be the **dominating number** of $\trianglelefteq_{\mathsf{LO}}$, i.e. the least size of a dominating family with respect to $\trianglelefteq_{\mathsf{LO}}$. The next proposition shows that $\mathfrak{d}(\trianglelefteq_{\mathsf{LO}})$ is as large as it can be.

Proposition 2.2.10. $\mathfrak{d}(\trianglelefteq_{\mathsf{LO}}) = 2^{\aleph_0}$.

Proof. Consider again the antichain $\mathcal{A} = \{\eta_{f_S} \mid S \subseteq \mathbb{N}\}$ from the proof of Proposition 2.2.8. If $\mathcal{F}$ were a dominating family with respect to $\trianglelefteq_{\mathsf{LO}}$ of size $\kappa < 2^{\aleph_0}$, then by $|\mathcal{A}| = 2^{\aleph_0}$ there would be $M \in \mathcal{F}$ such that $\{K \in \mathcal{A} \mid K \trianglelefteq_{\mathsf{LO}} M\}$ is uncountable, contradicting Claim 2.2.9.1. □

2.3 Borel complexity of $\trianglelefteq_{\mathsf{LO}}$ and $\bowtie_{\mathsf{LO}}$

At the beginning of Section 2.1 we introduced the equivalence relation $\bowtie_{\mathsf{LO}}$ of convex biembeddability on LO, observing that it is different from both isomorphism and biembeddability. We now focus on determining the complexity of $\bowtie_{\mathsf{LO}}$ with respect to Borel reducibility.

Theorem 2.3.1. *The map φ sending a linear order L to $\varphi(L) = \mathbf{1} + \zeta L + \mathbf{1}$ is such that*

(a) $L \cong L' \iff \varphi(L) \cong \varphi(L') \iff \varphi(L) \bowtie \varphi(L') \iff \varphi(L) \trianglelefteq \varphi(L')$;

(b) $|\varphi(L)| = \max\{\aleph_0, |L|\}$.

Proof. We claim that φ reduces $\cong_{\mathsf{LO}}$ to $\bowtie_{\mathsf{LO}}$. The second part is obvious, so let us concentrate on the first one. It is immediate that if $L \cong L'$ then $\varphi(L) \cong \varphi(L')$ and hence $\varphi(L) \bowtie \varphi(L')$, while $\varphi(L) \bowtie \varphi(L')$ clearly implies $\varphi(L) \trianglelefteq \varphi(L')$.

Let now f witness $\varphi(L) \trianglelefteq \varphi(L')$. The only elements of $\varphi(L)$ and $\varphi(L')$ without immediate successor and immediate predecessor are their minimum and maximum, respectively. Therefore, we must have $f(\min\varphi(L)) = \min\varphi(L')$ and $f(\max\varphi(L)) = \max\varphi(L')$. Hence f is also surjective (hence an isomorphism), and $f \restriction (\zeta L)$ witnesses $\zeta L \cong \zeta L'$. Thus $L \cong L'$ by Lemma 1.2.7. □

Noticing that when restricted to LO the map from Theorem 2.3.1 is Borel, we immediately get

Corollary 2.3.2. $\cong_{\mathsf{LO}} \leq_B \bowtie_{\mathsf{LO}}$.

The main question now becomes whether $\bowtie_{\mathsf{LO}} \leq_B \cong_{\mathsf{LO}}$. This is still open and the answer is not obvious because e.g. it is not even clear if $\bowtie_{\mathsf{LO}}$ is induced by a Borel action of S_∞. We now embark in a deeper analysis of $\bowtie_{\mathsf{LO}}$, leading at least to $\bowtie_{\mathsf{LO}} \leq_{Baire} \cong_{\mathsf{LO}}$.

In the spirit of the definition of convex embeddability and recalling Remark 2.1.2, we introduce the following notions.

Definition 2.3.3. Let L be a linear order. We say that

(1) L is **right compressible** if $L = L' + L_r$, with $L' \cong L$ and $L_r \neq \emptyset$;

(2) L is **left compressible** if $L = L_l + L'$, with $L' \cong L$ and $L_l \neq \emptyset$;

(3) L is **bicompressible** if it is both left compressible and right compressible,

(4) L is **incompressible** if it is neither left nor right compressible.

Notice that the set of right compressible linear orders is invariant with respect to isomorphism. The same holds for the set of left compressible linear orders, the set of bicompressible linear orders, and that of incompressible linear orders.

It is clear that ω^* is right compressible but not left compressible, ω is left compressible but not right compressible, $\omega + \omega^*$ and η are bicompressible, and ζ is incompressible.

The following characterizations of the above notions turn out to be useful.

Lemma 2.3.4. *Let L be a linear order. Then*

(a) *L is right compressible if and only if $L = L_l + \widetilde{L} + L_r$, with $\widetilde{L} \cong L$ and $L_r \neq \emptyset$.*

(b) *L is left compressible if and only if $L = L_l + \widetilde{L} + L_r$, with $\widetilde{L} \cong L$ and $L_l \neq \emptyset$.*

(c) *L is bicompressible if and only if $L = L_l + \widetilde{L} + L_r$, with $\widetilde{L} \cong L$ and $L_l, L_r \neq \emptyset$.*

Proof. (a) For the non trivial direction, suppose that $L = L_l + \widetilde{L} + L_r$, with $L \cong \widetilde{L}$ via some $f \colon L \to \widetilde{L}$ and $L_r \neq \emptyset$. Let $M_0 = f(L_r) \subseteq \widetilde{L}$ and for every $n \in \mathbb{N}$ define $M_{n+1} = f(M_n) \subseteq \widetilde{L}$. Let $M = \bigcup_{n\in\mathbb{N}} M_n \subseteq \widetilde{L}$ and note that $f \restriction (M + L_r) \colon M + L_r \to M$ is an isomorphism. Then the map $g \colon L \to L_l + \widetilde{L}$ defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \in M + L_r, \\ x, & \text{otherwise} \end{cases}$$

is an isomorphism witnessing $L \cong L_l + \widetilde{L}$. Thus, we can write $L = L' + L_r$, with $L' = L_l + \widetilde{L}$.

(b) is similar to (a).

(c) If $L = L_l + L' + L_r$ with $L' \cong L$ and $L_l, L_r \neq \emptyset$, then by (a) and (b) we immediately obtain that L is bicompressible. Conversely, suppose that L is bicompressible. Since L is left compressible, then $L = L_l + L'$, with $L' \cong L$ and $L_l \neq \emptyset$. Since $L' \cong L$ is right compressible, we can write $L' = \widetilde{L} + L_r$, with $\widetilde{L} \cong L'$ and $L_r \neq \emptyset$. Hence, $L = L_l + \widetilde{L} + L_r$, with $\widetilde{L} \cong L' \cong L$ and $L_l, L_r \neq \emptyset$. □

We denote by $\mathsf{LO}_r \subseteq \mathsf{LO}$ the set of (codes for) right compressible linear orders on $\mathbb{N}$, and by $\mathsf{LO}_l \subseteq \mathsf{LO}$ the set of (codes for) left compressible linear orders on $\mathbb{N}$. Note that $\mathsf{LO}_r = \{L \in \mathsf{LO} \mid L^* \in \mathsf{LO}_l\}$, and vice versa. Moreover, each of the four sets

$$\mathsf{LO} \setminus (\mathsf{LO}_r \cup \mathsf{LO}_l) \qquad \mathsf{LO}_l \setminus \mathsf{LO}_r \qquad \mathsf{LO}_r \setminus \mathsf{LO}_l \qquad \mathsf{LO}_r \cap \mathsf{LO}_l \tag{2.3.1}$$

is closed under isomorphism. The next proposition shows that they are also closed under $\bowtie_{\mathsf{LO}}$.

Proposition 2.3.5. *If L is a right compressible linear order and $L' \bowtie L$ (which implies $|L'| = |L|$), then L' is right compressible as well. Similarly, if $L' \bowtie L$ and L is left compressible (respectively: bicompressible, incompressible), then so is L'.*

In particular, the four subsets $\mathsf{LO} \setminus (\mathsf{LO}_r \cup \mathsf{LO}_l)$, $\mathsf{LO}_l \setminus \mathsf{LO}_r$, $\mathsf{LO}_r \setminus \mathsf{LO}_l$ and $\mathsf{LO}_r \cap \mathsf{LO}_l$ are invariant with respect to $\bowtie_{\mathsf{LO}}$.

Proof. It is clearly enough to consider the case of right compressible linear orders. Since L is right compressible, then $L = \widetilde{L} + L_r$ for some $\widetilde{L} \cong L$ and $L_r \neq \emptyset$. Let $f \colon L' \to \widetilde{L}$ and $g \colon L \to L'$ be convex embeddings witnessing $L' \trianglelefteq \widetilde{L}$ and $L \trianglelefteq L'$, respectively, so that $\widetilde{L} = \widetilde{L}_l + f(L') + \widetilde{L}_r$ and $L' = L'_l + g(L) + L'_r$. Then

$$\begin{aligned} L' &= L'_l + g(L) + L'_r \\ &= L'_l + g(\widetilde{L}) + g(L_r) + L'_r \\ &= L'_l + g(\widetilde{L}_l) + g(f(L')) + g(\widetilde{L}_r) + g(L_r) + L'_r. \end{aligned}$$

Since $g(f(L')) \cong L'$ and $g(\widetilde{L}_r) + g(L_r) + L'_r \supseteq g(L_r) \neq \emptyset$, by Lemma 2.3.4 we have $L' \in \mathsf{LO}_r$, as desired. □

We are now ready to go back to the study of the complexity of convex biembeddability. We can prove that the restrictions of $\bowtie_{\mathsf{LO}}$ to each of the four sets in (2.3.1), which we denote by $\bowtie_{\mathsf{LO}\setminus(\mathsf{LO}_r\cup\mathsf{LO}_l)}$, $\bowtie_{\mathsf{LO}_l\setminus\mathsf{LO}_r}$, $\bowtie_{\mathsf{LO}_r\setminus\mathsf{LO}_l}$ and $\bowtie_{\mathsf{LO}_r\cap\mathsf{LO}_l}$, respectively, are Borel bireducible with $\cong_{\mathsf{LO}}$. To this aim, we first observe that the map $\varphi_0 = \varphi$ from Theorem 2.3.1 reduces isomorphism to convex biembeddability restricted to incompressible linear orders, and that suitable variations of it do the same job but with left compressible (respectively, right compressible, bicompressible) linear orders.

Proposition 2.3.6. *Given a linear order L, set*

$$\begin{aligned} \varphi_0(L) &= \mathbf{1} + \zeta L + \mathbf{1} \\ \varphi_1(L) &= \eta + \zeta L + \mathbf{1} \\ \varphi_2(L) &= \mathbf{1} + \zeta L + \eta \\ \varphi_3(L) &= \eta + \zeta L + \eta. \end{aligned}$$

Then $\varphi_0(L)$ is incompressible, $\varphi_1(L)$ is left compressible but not right compressible, $\varphi_2(L)$ is right compressible but not left compressible, and $\varphi_3(L)$ is bicompressible.

Moreover, Theorem 2.3.1 is still true when φ is replaced by any of the above φ_i's.

Proof. As the minimum and the maximum of $\varphi_0(L)$ are the only elements without immediate predecessor and successor, respectively, we have that $\varphi_0(L)$ is not isomorphic to any of its proper convex subsets, i.e. it is incompressible. Hence we are done with φ_0 by Theorem 2.3.1.

Using a similar argument, one easily sees that $\varphi_1(L)$ is not right compressible. Indeed, any convex embedding f of $\varphi_1(L)$ into itself cannot send $\max \varphi_1(L)$ into ζL (by the argument in the previous paragraph) and cannot send it into η either (because otherwise $f(\zeta L) \subseteq \eta$, which is clearly impossible). On the other hand, $\varphi_1(L)$ is trivially left compressible because one can map η onto any of its (proper) final segments. Obviously $|\varphi_1(L)| = \max\{\aleph_0, |L|\}$, $L \cong L' \Rightarrow \varphi_1(L) \cong \varphi_1(L')$, $\varphi_1(L) \cong \varphi_1(L') \Rightarrow \varphi_1(L) \bowtie \varphi_1(L')$, and $\varphi_1(L) \bowtie \varphi_1(L') \Rightarrow \varphi_1(L) \trianglelefteq \varphi_1(L')$, so it remains to prove that if $\varphi_1(L) \trianglelefteq \varphi_1(L')$ then $L \cong L'$. Let $f \colon \varphi_1(L) \to \varphi_1(L')$ be a convex embedding. Since the elements of η are the unique non-maximal points without immediate predecessor and immediate successor (both in $\varphi_1(L)$ and $\varphi_1(L')$), then $f(\eta) \subseteq \eta$. Similarly, since the elements of ζL and $\zeta L'$ are the only elements having both an immediate predecessor and an immediate successor, then $f(\zeta L) \subseteq \zeta L'$. Moreover, the maximal element $\mathbf{1}$ has no immediate predecessor, which forbids $f(\mathbf{1}) \in \zeta L$, and we cannot have $f(\mathbf{1}) \in \eta$ because otherwise $f(\zeta L) \subseteq \eta$: thus $f(\mathbf{1}) = \mathbf{1}$. Since the range of f is convex, it then follows that $f(\zeta L) = \zeta L'$, hence $\zeta L \cong \zeta L'$ and thus $L \cong L'$ by Lemma 1.2.7.

The cases of $\varphi_2(L)$ and $\varphi_3(L)$ are similar. □

When restricted to LO, the functions φ_i are clearly Borel, thus we obtain:

Corollary 2.3.7. *The isomorphism relation $\cong_{\mathsf{LO}}$ is Borel reducible to any of $\bowtie_{\mathsf{LO}\setminus(\mathsf{LO}_r\cup\mathsf{LO}_l)}$, $\bowtie_{\mathsf{LO}_l\setminus\mathsf{LO}_r}$, $\bowtie_{\mathsf{LO}_r\setminus\mathsf{LO}_l}$, and $\bowtie_{\mathsf{LO}_r\cap\mathsf{LO}_l}$.*

Notice that the ranges of the four reductions used in the proof of Corollary 2.3.7 are all Borel, and that on such ranges isomorphism and convex biembeddability coincide.

Theorem 2.3.8. (a) *On the set* $\mathsf{LO} \setminus (\mathsf{LO}_r \cup \mathsf{LO}_l)$ *the relations* $\bowtie_{\mathsf{LO}}$ *and* $\cong_{\mathsf{LO}}$ *coincide, so that* $\bowtie_{\mathsf{LO} \setminus (\mathsf{LO}_r \cup \mathsf{LO}_l)}$ *is Borel reducible to* $\cong_{\mathsf{LO}}$ *via the identity map.*

(b) *Each of* $\bowtie_{\mathsf{LO}_r \cap \mathsf{LO}_l}$, $\bowtie_{\mathsf{LO}_l \setminus \mathsf{LO}_r}$, *and* $\bowtie_{\mathsf{LO}_r \setminus \mathsf{LO}_l}$ *is Borel reducible to* $(\cong_{\mathsf{LO}})^+$, *and thus to* $\cong_{\mathsf{LO}}$.

Proof. (a) Let $L, L' \in \mathsf{LO} \setminus (\mathsf{LO}_r \cup \mathsf{LO}_l)$. It is obvious that if $L \cong_{\mathsf{LO}} L'$ then $L \bowtie_{\mathsf{LO}} L'$. For the other direction, assume $L \bowtie_{\mathsf{LO}} L'$ and let f and g be convex embeddings witnessing $L \trianglelefteq_{\mathsf{LO}} L'$ and $L' \trianglelefteq_{\mathsf{LO}} L$, respectively. Then $L' = L'_l + f(L) + L'_r$ and $L = L_l + g(L'_l) + g(f(L)) + g(L'_r) + L_r$. Since L is incompressible and $g(f(L)) \cong L$ we have $L_l + g(L'_l) = g(L'_r) + L_r = \emptyset$ and hence $L'_l = L'_r = \emptyset$, showing that f is an isomorphism.

(b) We start by considering the case of $\bowtie_{\mathsf{LO}_r \cap \mathsf{LO}_l}$. Let $\varphi_{r+l} \colon \mathsf{LO} \setminus [\zeta, \omega, \omega^*]_{\cong} \to \mathsf{LO}^{\mathbb{N}}$ be a Borel map such that $\varphi_{r+l}(L)$ is an enumeration (possibly with repetitions) of all the *infinite* subsets of L of the form $[n, m]_L$. Since we are omitting the isomorphism types of ζ, ω, and ω^* the map is well-defined, i.e. for each L in its domain there is at least one infinite interval $[n, m]_L$, and clearly $\mathsf{LO}_l \cap \mathsf{LO}_r \subseteq \mathrm{dom}(\varphi_{r+l})$. By the same reason, its domain is Borel because we are omitting finitely many $\cong_{\mathsf{LO}}$-classes, which are Borel themselves. We claim that for all $L, L' \in \mathsf{LO}_l \cap \mathsf{LO}_r$

$$L \bowtie_{\mathsf{LO}} L' \iff \varphi_{r+l}(L) \ (\cong_{\mathsf{LO}})^+ \ \varphi_{r+l}(L'),$$

so that any Borel extension of φ_{r+l} to LO witnesses $\bowtie_{\mathsf{LO}_r \cap \mathsf{LO}_l} \leq_B (\cong_{\mathsf{LO}})^+$, and hence $\bowtie_{\mathsf{LO}_r \cap \mathsf{LO}_l} \leq_B \cong_{\mathsf{LO}}$ by Theorem 1.2.2.

Assume first that $L \bowtie_{\mathsf{LO}} L'$, and let f be a convex embedding witnessing $L \trianglelefteq_{\mathsf{LO}} L'$. Given any infinite $[n, m]_L$, we have $[n, m]_L \cong_{\mathsf{LO}} [f(n), f(m)]_{L'}$, so that in particular the latter is infinite and appears among the linear orders in $\varphi_{r+l}(L')$. Symmetrically, if g is a convex embedding witnessing $L' \trianglelefteq_{\mathsf{LO}} L$, then for every infinite $[n, m]_{L'}$ we have $[n, m]_{L'} \cong [g(n), g(m)]_L$. It follows that $\varphi_{r+l}(L) \ (\cong_{\mathsf{LO}})^+ \ \varphi_{r+l}(L')$.

Conversely, observe that since $L \in \mathsf{LO}_l \cap \mathsf{LO}_r$ then by Lemma 2.3.4 we have $L = L_l + \widetilde{L} + L_r$, with $\widetilde{L} \cong L$ and both L_l and L_r nonempty. Fix $k \in L_l$ and $m \in L_r$. Then $\widetilde{L} \sqsubseteq [k, m]_L$, and hence $L \trianglelefteq [k, m]_L$ and $[k, m]_L$ is infinite. Thus if $\varphi_{r+l}(L) \ (\cong_{\mathsf{LO}})^+ \ \varphi_{r+l}(L')$, there are $k', m' \in L'$ such that $[k, m]_L \cong [k', m']_{L'}$. But then $L \trianglelefteq_{\mathsf{LO}} L'$ because $L \trianglelefteq [k, m]_L \cong [k', m']_{L'} \trianglelefteq L'$. The argument to show that if $\varphi_{r+l}(L) \ (\cong_{\mathsf{LO}})^+ \ \varphi_{r+l}(L')$ then $L' \trianglelefteq_{\mathsf{LO}} L$ is symmetric.

We now move to the case of $\bowtie_{\mathsf{LO}_l \setminus \mathsf{LO}_r}$. Let $\varphi_l \colon \mathsf{LO} \setminus [\omega^*]_{\cong} \to \mathsf{LO}^{\mathbb{N}}$ be a Borel map such that $\varphi_l(L)$ is an enumeration of all the *infinite* subsets of L of the form $[n, +\infty)_L$, which is well-defined on all $L \not\cong \omega^*$ and such that $\mathsf{LO}_l \setminus \mathsf{LO}_r \subseteq \mathrm{dom}(\varphi_l)$. Arguing as above, it is enough to show that for all $L, L' \in \mathsf{LO}_l \setminus \mathsf{LO}_r$

$$L \bowtie_{\mathsf{LO}} L' \iff \varphi_l(L) \ (\cong_{\mathsf{LO}})^+ \ \varphi_l(L').$$

For the forward direction, let f and g be convex embeddings witnessing $L \trianglelefteq_{\mathsf{LO}} L'$ and $L' \trianglelefteq_{\mathsf{LO}} L$, respectively. We first show that $f(L)$ is a final segment of L'. Since f is a convex embedding, $L' = L'_l + f(L) + L'_r$ with L'_l and L'_r possibly empty. Then $L = L_l + g(f(L)) + L_r$ with $L_r \supseteq g(L'_r)$. Since $g(f(L)) \cong L$ and $L \notin \mathsf{LO}_r$, we have $L_r = \emptyset$ and hence $L'_r = \emptyset$, i.e. $L' = L'_l + f(L)$. Thus if $[n, \infty)_L$ is infinite, then $f([n, \infty)_L) = [f(n), \infty)_{L'}$, so that, being infinite, the latter appears in $\varphi_l(L')$ and $[n, \infty)_L \cong [f(n), \infty)_{L'}$. Analogously, $g(L')$ is a final segment of L because $L' \notin \mathsf{LO}_r$, hence for every infinite $[n, +\infty)_{L'}$, we have $[n, +\infty)_{L'} \cong [g(n), +\infty)_L$. It follows that $\varphi_l(L) \ (\cong_{\mathsf{LO}})^+ \ \varphi_l(L')$.

Conversely, assume that $\varphi_l(L) \ (\cong_{\mathsf{LO}})^+ \ \varphi_l(L')$. Using $L \in \mathsf{LO}_l$, let $L = L_l + \widetilde{L}$ with $L_l \neq \emptyset$ and $\widetilde{L} \cong L$, and fix any $m \in L_l$. Then $\widetilde{L} \sqsubseteq [m, +\infty)_L$, and thus the latter, being infinite, appears in $\varphi_l(L)$ and $L \trianglelefteq [m, +\infty)_L$. Let $m' \in L'$ be such that $[m, +\infty)_L \cong_{\mathsf{LO}} [m', +\infty)_{L'}$: then $L \trianglelefteq_{\mathsf{LO}} [m, +\infty)_L \cong_{\mathsf{LO}} [m', +\infty)_{L'} \trianglelefteq_{\mathsf{LO}} L'$. Reversing the role of L and L' we get $L' \trianglelefteq_{\mathsf{LO}} L$ and we are done.

The case of $\bowtie_{\mathsf{LO}_r \setminus \mathsf{LO}_l}$ is symmetric, with the desired Borel reduction be given by any Borel map $\varphi_r \colon \mathsf{LO} \setminus [\omega]_{\cong} \to \mathsf{LO}^{\mathbb{N}}$ such that $\varphi_r(L)$ is an enumeration of all the *infinite* subsets of L of the form $(-\infty, n]_L$. □

Remark 2.3.9. The statement and proof of Theorem 2.3.8 can easily be adapted to deal with uncountable linear orders of a given cardinality κ. However, since we have no use for this in the present project, for the sake of simplicity we decided to stick to the countable case.

If LO_r and LO_l were Borel subsets of LO, then we could glue the reductions from the proof of Theorem 2.3.8 and obtain a Borel reduction from the whole $\bowtie_{\mathsf{LO}}$ to $\cong_{\mathsf{LO}}$. Unfortunately, this is not the case: none of the subclasses of LO involved in Theorem 2.3.8 is Borel. To prove this, we need the following lemmas.

Lemma 2.3.10. *Let $\alpha > 0$. For any $z \in \mathbb{Z}^\alpha$ and $\beta < \alpha$ there exists γ such that $\beta \leq \gamma < \alpha$ and $\mathbb{Z}^\gamma\omega \trianglelefteq_{\mathsf{LO}} [z, +\infty)_{\mathbb{Z}^\alpha}$.*

Proof. We consider the isomorphic copy of $\mathbb{Z}^\alpha$ given by Proposition 1.2.10:

$$\Big(\sum_{\beta\leq\gamma<\alpha} \mathbb{Z}^\gamma\omega\Big)^* + \Big(\sum_{\beta\leq\gamma<\alpha} \mathbb{Z}^\gamma\omega\Big).$$

Without loss of generality we can assume $z \in \sum_{\beta\leq\gamma<\alpha} \mathbb{Z}^\gamma\omega$, so that there exists γ with $\beta \leq \gamma < \alpha$ such that $z \in \mathbb{Z}^\gamma\omega$. Since $\mathbb{Z}^\gamma\omega \trianglelefteq_{\mathsf{LO}} [z, +\infty)_{\mathbb{Z}^\gamma\omega} \trianglelefteq_{\mathsf{LO}} [z, +\infty)_{\mathbb{Z}^\alpha}$, this γ works. □

Lemma 2.3.11. *For every ordinal $\alpha > 0$, $\mathbb{Z}^\alpha$ is incompressible.*

Proof. By induction on $\alpha > 0$. We have already noticed that $\mathbb{Z}^1 \cong_{\mathsf{LO}} \zeta$ is incompressible. Fix $\alpha > 1$ and assume that $\mathbb{Z}^\beta$ is incompressible for every $\beta < \alpha$.

We consider the isomorphic copy of $\mathbb{Z}^\alpha$ given by Proposition 1.2.10 with $\beta = 0$:

$$\Big(\sum_{\gamma<\alpha} \mathbb{Z}^\gamma\omega\Big)^* + \Big(\sum_{\gamma<\alpha} \mathbb{Z}^\gamma\omega\Big).$$

We just prove that $\mathbb{Z}^\alpha \notin \mathsf{LO}_r$, as $\mathbb{Z}^\alpha \notin \mathsf{LO}_l$ can be proved in a symmetric way. Suppose, towards a contradiction, that $\mathbb{Z}^\alpha \in \mathsf{LO}_r$ and let f be a convex embedding of $\mathbb{Z}^\alpha$ into a proper initial segment of $\mathbb{Z}^\alpha$. Assume first that $f(\mathbb{Z}^\alpha) \cap \big(\sum_{\gamma<\alpha} \mathbb{Z}^\gamma\omega\big) \neq \emptyset$. Let $\beta < \alpha$ be least such that $\mathbb{Z}^\gamma\omega \nsubseteq f(\mathbb{Z}^\alpha)$ for every $\gamma \geq \beta$. (Such a β exists by the choice of f.)

Claim 2.3.11.1. $f(\mathbb{Z}^\alpha) \cap \mathbb{Z}^\gamma\omega = \emptyset$ for every $\gamma \geq \beta$, so that $f(\mathbb{Z}^\alpha)$ is a final segment of

$$\Big(\sum_{\gamma<\alpha} \mathbb{Z}^\gamma\omega\Big)^* + \Big(\sum_{\gamma<\beta} \mathbb{Z}^\gamma\omega\Big).$$

Proof of the Claim. If $\gamma > \beta$ the convexity of $f(\mathbb{Z}^\alpha)$ implies immediately $f(\mathbb{Z}^\alpha) \cap \mathbb{Z}^\gamma\omega = \emptyset$, so we only need to consider the case $\gamma = \beta$. Towards a contradiction, assume that $f(\mathbb{Z}^\alpha)$ intersects $\mathbb{Z}^\beta\omega$, and using $f(\mathbb{Z}^\alpha) \nsupseteq \mathbb{Z}^\beta\omega$ let n be maximum such that $f(\mathbb{Z}^\alpha) \cap (\mathbb{Z}^\beta \times \{n\}) \neq \emptyset$. Pick $z \in \mathbb{Z}^\alpha$ such that $f(z) \in \mathbb{Z}^\beta \times \{n\}$. By Lemma 2.3.10 there exists $\beta \leq \gamma < \alpha$ such that $\mathbb{Z}^\gamma\omega \trianglelefteq_{\mathsf{LO}} [z, +\infty)_{\mathbb{Z}^\alpha}$ and hence $\mathbb{Z}^\gamma\omega \trianglelefteq_{\mathsf{LO}} [f(z), +\infty)_{\mathbb{Z}^\beta\times\{n\}}$. But then $\mathbb{Z}^\gamma\omega \trianglelefteq_{\mathsf{LO}} \mathbb{Z}^\beta$, and since $\mathbb{Z}^\beta \trianglelefteq_{\mathsf{LO}} \mathbb{Z}^\gamma \cong \mathbb{Z}^\gamma \times \{0\}$ by $\beta \leq \gamma$ (see Definition 1.2.8) this shows that $\mathbb{Z}^\beta$ is right compressible, against the induction hypothesis. □

Using Proposition 1.2.10 again, we have

$$\begin{aligned}\Big(\sum_{\gamma<\alpha} \mathbb{Z}^\gamma\omega\Big)^* + \Big(\sum_{\gamma<\beta} \mathbb{Z}^\gamma\omega\Big) &= \Big(\sum_{\beta\leq\gamma<\alpha} \mathbb{Z}^\gamma\omega\Big)^* + \Big(\sum_{\gamma<\beta} \mathbb{Z}^\gamma\omega\Big)^* + \Big(\sum_{\gamma<\beta} \mathbb{Z}^\gamma\omega\Big)\\ &\cong \Big(\sum_{\beta\leq\gamma<\alpha} \mathbb{Z}^\gamma\omega\Big)^* + \mathbb{Z}^\beta\end{aligned}$$

Let g be the isomorphism between the first and last element of this chain. Choose $z \in \mathbb{Z}^\alpha$ such that $g(f(z)) \in \mathbb{Z}^\beta$ — such a z exists because $f(\mathbb{Z}^\alpha)$ is cofinal in $\big(\sum_{\gamma<\alpha} \mathbb{Z}^\gamma\omega\big)^* + \big(\sum_{\gamma<\beta} \mathbb{Z}^\gamma\omega\big)$ by Claim 2.3.11.1. Arguing as before, $\mathbb{Z}^\gamma\omega \trianglelefteq_{\mathsf{LO}} [g(f(z)), +\infty)_{\mathbb{Z}^\beta}$ for some $\beta \leq \gamma < \alpha$, contradicting again the incompressibility of $\mathbb{Z}^\beta$.

Finally, assume that $f(\mathbb{Z}^\alpha) \cap \big(\sum_{\gamma<\alpha} \mathbb{Z}^\gamma\omega\big) = \emptyset$, i.e. $f(\mathbb{Z}^\alpha) \subseteq \big(\sum_{\gamma<\alpha} \mathbb{Z}^\gamma\omega\big)^*$. Let $\beta < \alpha$ be smallest such that $f(\mathbb{Z}^\alpha) \cap (\mathbb{Z}^\beta\omega)^* \neq \emptyset$, and let n be smallest such that $f(\mathbb{Z}^\alpha) \cap (\mathbb{Z}^\beta \times \{n\})^* \neq \emptyset$. Pick $z \in \mathbb{Z}^\alpha$ such that $f(z) \in (\mathbb{Z}^\beta \times \{n\})^*$. Arguing as before, there is $\beta \leq \gamma < \alpha$ such that $\mathbb{Z}^\gamma\omega \trianglelefteq_{\mathsf{LO}} (\mathbb{Z}^\beta \times \{n\})^*$. Since $(\mathbb{Z}^\beta \times \{n\})^* \cong (\mathbb{Z}^\beta)^* \cong \mathbb{Z}^\beta$ by Proposition 1.2.11, this would mean that $\mathbb{Z}^\gamma\omega \trianglelefteq_{\mathsf{LO}} \mathbb{Z}^\beta$, contradicting again the incompressibility of the latter. □

Theorem 2.3.12. (a) LO_l, LO_r *and* $\mathsf{LO}_r \cap \mathsf{LO}_l$ *are* $\mathbf{\Sigma}^1_1$*-complete.*

(b) $\mathsf{LO} \setminus (\mathsf{LO}_r \cup \mathsf{LO}_l)$ *is* $\mathbf{\Pi}^1_1$*-complete.*

(c) $\mathsf{LO}_r \setminus \mathsf{LO}_l$ *and* $\mathsf{LO}_l \setminus \mathsf{LO}_r$ *are* $D_2(\mathbf{\Pi}^1_1)$*-complete.*

Proof. (a) First, we check that LO_r is $\mathbf{\Sigma}^1_1$. Indeed, $L \in \mathsf{LO}_r$ if and only if

$$\begin{aligned}\exists f\colon \mathbb{N} \to \mathbb{N}\big[&\forall n, m\ (n <_L m \to f(n) <_L f(m)) \wedge \\ &\forall n, m, k\ (f(n) \leq_L k \leq_L f(m) \to \exists k'(f(k') = k)) \wedge \\ &\exists n\ \forall m\ (f(m) <_L n)\big].\end{aligned}$$

In a similar way, one can prove that LO_l (and hence also $\mathsf{LO}_l \cap \mathsf{LO}_r$) is $\mathbf{\Sigma}^1_1$.

We now show that LO_l, LO_r and $\mathsf{LO}_r \cap \mathsf{LO}_l$ are $\mathbf{\Sigma}^1_1$-hard by continuously reducing the $\mathbf{\Sigma}^1_1$-complete set $\mathsf{LO} \setminus \mathsf{WO}$ to each of them. We can actually use the continuous function $L \mapsto \mathbb{Z}^L$ for all three sets. Indeed, if $L \notin \mathsf{WO}$, by Proposition 1.2.11 we have $\mathbb{Z}^L \cong \mathbb{Z}^\alpha \eta$ for some ordinal α, and hence $\mathbb{Z}^L$ is obviously bicompressible. If $L \in \mathsf{WO}$, then $\mathbb{Z}^L$ is incompressible by Lemma 2.3.11.

(b) is immediate from the proof of (a).

(c) By (a) it follows that $\mathsf{LO}_r \setminus \mathsf{LO}_l$ and $\mathsf{LO}_l \setminus \mathsf{LO}_r$ are $D_2(\mathbf{\Pi}^1_1)$. Consider now the set $A = \{(L, L') \in \mathsf{LO} \times \mathsf{LO} \mid L \notin \mathsf{WO} \text{ and } L' \in \mathsf{WO}\}$ and recall that it is $D_2(\mathbf{\Pi}^1_1)$-complete. Define the continuous map $\psi\colon \mathsf{LO} \times \mathsf{LO} \to \mathsf{LO}$ by $\psi(L, L') = \mathbb{Z}^{1+L'} + \eta + \mathbb{Z}^{1+L}$.

We claim that $\psi(L, L')$ is left compressible if and only if $L' \notin \mathsf{WO}$. One direction is obvious: if $L' \notin \mathsf{WO}$, then $\mathbb{Z}^{1+L'} \cong \mathbb{Z}^\alpha \eta$ for some $\alpha \geq 1$, and thus it has a convex self-embedding onto a proper final segment of it, which can then be naturally extended to a witness of $\psi(L, L') \in \mathsf{LO}_l$. For the other direction, we use the fact that every convex subset of η consists of points which have neither an immediate predecessor nor an immediate successors, while convex subsets of $\mathbb{Z}^{1+L}$ and $\mathbb{Z}^{1+L'}$ with at least two points always contain elements with both an immediate predecessor and an immediate successor in the given linear order. (Here we use again the fact that $\mathbb{Z}^{1+L}$ and $\mathbb{Z}^{1+L'}$ are either of the form $\mathbb{Z}^\alpha$ or $\mathbb{Z}^\alpha \eta$ for some $\alpha \geq 1$, depending on whether L and L' are well-orders or not.) Thus if $f\colon \psi(L, L') \to \psi(L, L')$ is a convex embedding we must have $f(\eta) = \eta$, and hence $f(\mathbb{Z}^{1+L'}) \subseteq \mathbb{Z}^{1+L'}$. Thus if $L' \in \mathsf{WO}$ then $\mathbb{Z}^{1+L'} \notin \mathsf{LO}_l$ by Lemma 2.3.11, which implies $f(\mathbb{Z}^{1+L'}) = \mathbb{Z}^{1+L'}$: since f was arbitrary, this shows that $\psi(L, L') \notin \mathsf{LO}_l$.

Analogously, one can check that $\psi(L, L')$ is right compressible if and only if $L \notin \mathsf{WO}$. Using these facts, it is then easy to prove that $(L, L') \in A$ if and only if $\psi(L, L') \in \mathsf{LO}_r \setminus \mathsf{LO}_l$, hence ψ witnesses that $\mathsf{LO}_r \setminus \mathsf{LO}_l$ is $D_2(\mathbf{\Pi}^1_1)$-hard.

For $\mathsf{LO}_l \setminus \mathsf{LO}_r$ it suffices to switch the positions of $\mathbb{Z}^{1+L}$ and $\mathbb{Z}^{1+L'}$ in the definition of ψ. $\square$

Even if they are not Borel, the sets $\mathsf{LO} \setminus (\mathsf{LO}_r \cup \mathsf{LO}_l)$, $\mathsf{LO}_l \setminus \mathsf{LO}_r$, $\mathsf{LO}_r \setminus \mathsf{LO}_l$ and $\mathsf{LO}_r \cap \mathsf{LO}_l$ belong to the σ-algebra generated by the analytic sets, and hence have the Baire property and are universally measurable. By Theorem 2.3.8 and Proposition 1.1.11 we thus obtain the following result.

Corollary 2.3.13. *The equivalence relation* $\bowtie_{\mathsf{LO}}$ *is* σ*-classifiable by countable structures, and therefore* $\bowtie_{\mathsf{LO}} \leq_{Baire} \cong_{\mathsf{LO}}$.

Notice that, since the partition of LO given by (2.3.1) is finite, we actually have that the preimages of Borel sets via the reduction of $\bowtie_{\mathsf{LO}}$ to $\cong_{\mathsf{LO}}$ are Boolean combinations of analytic sets. It remains open the problem whether $\bowtie_{\mathsf{LO}}$ is Borel reducible to $\cong_{\mathsf{LO}}$. However, from the reductions above we can derive some more information about the complexity of $\bowtie_{\mathsf{LO}}$, showing that it shares some important properties with $\cong_{\mathsf{LO}}$.

Corollary 2.3.14. *If* X *is a Polish space on which the action of a Polish group* G *is turbulent, then* $E^X_G \not\leq_B \bowtie_{\mathsf{LO}}$.

Proof. If $E^X_G \leq_B \bowtie_{\mathsf{LO}}$, then by Corollary 2.3.13 we would have that $E^X_G \leq_{Baire} \cong_{\mathsf{LO}}$, against Theorem 1.1.12. $\square$

In Proposition 1.2.2 we observed that $(\cong_{\mathsf{LO}})^+ \leq_B \cong_{\mathsf{LO}}$. Replacing Borel reducibility with Baire reducibility, we get an analogous result for $\bowtie_{\mathsf{LO}}$.

Corollary 2.3.15. $(\bowtie_{\mathsf{LO}})^+ \leq_{Baire} \bowtie_{\mathsf{LO}}$.

Proof. Since $\bowtie_{\mathsf{LO}} \leq_{Baire} \cong_{\mathsf{LO}}$, we have that $(\bowtie_{\mathsf{LO}})^+ \leq_{Baire} (\cong_{\mathsf{LO}})^+$, but $(\cong_{\mathsf{LO}})^+ \leq_B \cong_{\mathsf{LO}} \leq_B \bowtie_{\mathsf{LO}}$, so $(\bowtie_{\mathsf{LO}})^+ \leq_{Baire} \bowtie_{\mathsf{LO}}$. □

Corollary 2.3.16. $E_1 \not\leq_{Baire} \bowtie_{\mathsf{LO}}$.

Proof. If $E_1 \leq_{Baire} \bowtie_{\mathsf{LO}}$, by Corollary 2.3.13 we would have $E_1 \leq_{Baire} \cong_{\mathsf{LO}}$, contradicting Theorem 1.1.14. □

Each one of Corollaries 2.3.14 and 2.3.16 implies that $\bowtie_{\mathsf{LO}}$ is not complete for analytic equivalence relations, thus by Proposition 1.1.18 we obtain:

Corollary 2.3.17. $\trianglelefteq_{\mathsf{LO}}$ *is not complete for analytic quasi-orders.*

Recall that by $\mathrm{Int}(\mathbb{R})$ we denote the set of the open intervals of $\mathbb{R}$. We can naturally equip $\mathrm{Int}(\mathbb{R})$ with a Polish topology: indeed, if we extend the usual order on $\mathbb{R}$ to $\mathbb{R} \cup \{\pm\infty\}$ in the obvious way, then $\mathrm{Int}(\mathbb{R})$ is the open subset $\{(x, y) \mid x < y\}$ of the Polish space $(\mathbb{R} \cup \{\pm\infty\})^2$. The inclusion relation on $\mathrm{Int}(\mathbb{R})$ is then closed. Notice now that the embedding from $(\mathrm{Int}(\mathbb{R}), \subseteq)$ to $(\mathsf{LO}, \trianglelefteq_{\mathsf{LO}})$ defined in the proof of Lemma 2.2.3 is actually a Borel reduction. Thus we have the following corollary.

Corollary 2.3.18. $(\mathrm{Int}(\mathbb{R}), \subseteq) \leq_B (\mathsf{LO}, \trianglelefteq_{\mathsf{LO}})$.

2.4 Convex embeddability between countable circular orders

Our goal in this section is to define a relation of convex embeddability among circular orders. We first recall the definition of convex subset of a circular order as given by Kulpeshov and Macpherson ([KM05]).

Definition 2.4.1. Let C be a circular order. The set $A \subseteq C$ is said to be **convex** in C, in symbols $A \sqsubseteq C$, if for any distinct $x, y \in A$ one of the following holds:

(i) for every $c \in C$ with $C(x, c, y)$ we have $c \in A$;

(ii) for every $c \in C$ with $C(y, c, x)$ we have $c \in A$.

If A is a proper subset of C we write $A \sqsubset C$.

Note that if $A \sqsubset C$ then exactly one of (i) and (ii) holds for each pair of distinct $x, y \in A$.

The following propositions collect some basic properties of convex subsets of circular orders.

Proposition 2.4.2. *If C is a circular order and $A \sqsubseteq C$ then $C \setminus A$ is a convex subset of A as well.*

Proof. If $C \setminus A$ is empty or a singleton the result is trivial, so we can assume that $C \setminus A$ contains at least two points. Toward a contradiction, suppose $x, y \in C \setminus A$ are distinct and such that:

(1) there exists $c_0 \in A$ with $C(x, c_0, y)$, and

(2) there exists $c_1 \in A$ with $C(y, c_1, x)$.

By cyclicity and transitivity we obtain $C(c_0, y, c_1)$ and $C(c_1, x, c_0)$, and since A is convex we would have that at least one of x and y belongs to A, a contradiction. □

The previous proposition highlights a major difference between convex subsets of circular and linear orders: the complement of a convex subset of a linear order is not in general convex. On the other hand, convex subsets of linear orders are closed under intersections, while this is not the case for circular orders: consider the circular order $C[\mathbf{4}]$ and its convex subsets $\{0,1,2\}$ and $\{2,3,0\}$. However the intersection of two convex subsets of a circular order is not convex only in some circumstances.

Proposition 2.4.3. *Let C be a circular order. If $A, B \trianglelefteq C$ then $A \cap B$ is the union of two convex subsets of C. Moreover, if $A \cap B$ is not convex then $A \cup B = C$.*

Proof. If $B \subseteq A$ or $A \subseteq B$, the result is trivial. So, suppose there exist $w \in A \setminus B$ and $z \in B \setminus A$, and consider the partition of $A \cap B$ given by the sets

$$A_1 = \{x \in A \cap B \mid C(w,x,z)\} \quad \text{and} \quad A_2 = \{x \in A \cap B \mid C(z,x,w)\}.$$

We claim that $A_1 \trianglelefteq C$. Let $x, y \in A_1$ be distinct: since $x, y \in A$, without loss of generality we can assume that $u \in A$ for every $u \in C$ such that $C(x,u,y)$. Since $z \notin A$ we have that $C(x,z,y)$ fails and, by totality and cyclicity, we have $C(x,y,z)$. Using cyclicity, transitivity and $C(w,x,z)$ we obtain $C(y,w,x)$. Since $w \notin B$ and B is convex this implies that $u \in B$ for every $u \in C$ such that $C(x,u,y)$. If now u is such that $C(x,u,y)$ we already showed that $u \in A \cap B$. From $C(y,w,x)$ and $C(x,u,y)$ it follows that $C(y,w,u)$ which, combined with $C(w,y,z)$, yields $C(w,u,z)$ and hence $u \in A_1$. The proof that A_2 is convex is symmetric.

Now assume that $A \cap B$ is not convex, and hence both A_1 and A_2 are non empty. Fix $x \in A_1$ and $y \in A_2$. From $C(w,x,z)$ and $C(z,y,w)$ it follows that we have $C(x,z,y)$ and $C(y,w,x)$. Since $x, y \in A$ but $z \notin A$ we must have that $C(y,u,x)$ implies $u \in A$. Similarly we obtain that $C(x,u,y)$ implies $u \in B$. By totality it follows that $A \cup B = C$. □

Proposition 2.4.4. *Let C be a circular order. Let $\{A_i \mid i \in I\}$ and $\{B_j \mid j \in J\}$ be two collections of pairwise disjoint convex subsets of C. Then there exists at most one pair $(i,j) \in I \times J$ such that $A_i \cap B_j$ is not convex.*

Proof. Suppose that $A_i \cap B_j$ is not convex. By the second part of Proposition 2.4.3 we have $A_i \cup B_j = C$. Hence for every $i' \neq i$ and $j' \neq j$ we have that $A_{i'} \subseteq B_j$ and $B_{j'} \subseteq A_i$. Therefore $A_{i'} \cap B_j = A_{i'}$, $A_i \cap B_{j'} = B_{j'}$ and $A_{i'} \cap B_{j'} \subseteq A_{i'} \cap A_i = \emptyset$ are all convex. □

If f is an embedding between linear orders L and L' and $f(L) \trianglelefteq L'$, then $f(B) \trianglelefteq L'$ for every $B \trianglelefteq L$. This ceases to be true for circular orders, as shown by the following example. The identity map between $C = C[\zeta]$ and $C' = C[\zeta + \mathbf{1}]$ has convex range, but the image of the convex set $B = C \setminus \{0\} \trianglelefteq C$ is no longer convex in C'. The following proposition gives a weakening of the above property which is however sufficient for the ensuing proofs.

Proposition 2.4.5. *Let f be an embedding between the circular orders C into C'. If $A' \trianglelefteq C'$, then $f^{-1}(A') \trianglelefteq C$. Conversely, if $A \triangleleft C$ is such that $f(A) \trianglelefteq C'$, then $f(B) \trianglelefteq C'$ for all $B \trianglelefteq C$ with $B \subseteq A$.*

Proof. The first part is obvious, so let us consider $A \triangleleft C$ with $f(A) \trianglelefteq C'$, and fix any $B \trianglelefteq C$ contained in A. Pick distinct points $f(x), f(y) \in f(B) \subseteq f(A)$, so that $x, y \in B$ and $x \neq y$ because f is injective. Since $A \triangleleft C$, without loss of generality we might assume that $c \in A$ for all $c \in C$ with $C(x,c,y)$ and that there is $d \in C$ with $C(y,d,x)$ and $d \notin A$, so that the same is true with A replaced by B because $B \subseteq A$ is convex. Since f is an embedding, $f(d)$ is such that $C'(f(y), f(d), f(x))$ but $f(d) \notin f(A)$. Since $f(A) \trianglelefteq C'$ by hypothesis, this means that $c' \in f(A)$ for all $c' \in C$ such that $C'(f(x), c', f(y))$. So for such a $c' \in C'$ there is $c \in A$ such that $c' = f(c)$: then $C(x,c,y)$ because f is an embedding, and so $c \in B$ and $f(c) = c' \in f(B)$. This shows that $f(B)$ satisfies (i) of Definition 2.4.1 with respect to x and y. Hence $f(B) \trianglelefteq C'$. □

The first natural attempt to define convex embeddability between circular orders is the following.

Definition 2.4.6. Let C and C' be circular orders. We say that C is **convex embeddable** into C', and write $C \trianglelefteq_{\mathrm{c}} C'$, if there exists an embedding f from C to C' such that $f(C) \sqsubseteq C'$.

However, $\trianglelefteq_{\mathrm{c}}$ is **not** transitive, as witnessed by $C[\zeta] \trianglelefteq_{\mathrm{c}} C[\zeta+\mathbf{1}]$, $C[\zeta+\mathbf{1}] \trianglelefteq_{\mathrm{c}} C[\omega+\mathbf{1}+\omega^*+\eta]$ (because $C[\zeta+\mathbf{1}] \cong_c C[\omega+\mathbf{1}+\omega^*]$), and $C[\zeta] \ntrianglelefteq_{\mathrm{c}} C[\omega+\mathbf{1}+\omega^*+\eta]$. Nevertheless, notice that if we partition $C[\zeta]$ into the *two* convex subsets ω^* and ω then they are isomorphic to the two convex subsets ω^* and ω of $C[\omega+\mathbf{1}+\omega^*+\eta]$.

By taking the transitive closure of $\trianglelefteq_{\mathrm{c}}$ (i.e. the smallest binary relation containing $\trianglelefteq_{\mathrm{c}}$) we are naturally led to the following definition. We call **finite convex partition** of the circular order C any finite partition $\{C_i \mid i < n\}$ of C such that

- $C_i \sqsubseteq C$ for all $i < n$, and
- for all $x, y, z \in C$, if $C(x,y,z)$ then $C[\boldsymbol{n}](i,j,k)$ for the unique $i, j, k < n$ such that $x \in C_i$, $y \in C_j$, and $z \in C_k$.

Notice that this implies that the C_i's are ordered as $C[\boldsymbol{n}]$, that is: if $i, j, k < n$ are distinct and $C[\boldsymbol{n}](i,j,k)$ then $C(x,y,z)$ for every $x \in C_i$, $y \in C_j$, and $z \in C_k$. Also, the convexity of the C_i's follows from the second condition if $n \geq 3$.

Definition 2.4.7. Let C and C' be circular orders. We say that C is **piecewise convex embeddable** into C', and write $C \trianglelefteq_{\mathrm{c}}^{<\omega} C'$, if there are a finite convex partition $\{C_i \mid i < n\}$ of C and an embedding f of C into C' such that $f(C_i) \sqsubseteq C'$ for all $i < n$.

We denote by $\trianglelefteq_{\mathsf{CO}}^{<\omega}$ the restriction of $\trianglelefteq_{\mathrm{c}}^{<\omega}$ to the set CO of (codes for) circular linear orders on $\mathbb{N}$.

Clearly, $C \trianglelefteq_{\mathrm{c}} C'$ implies $C \trianglelefteq_{\mathrm{c}}^{<\omega} C'$. Notice also that when C has at least two elements and $C \trianglelefteq_{\mathsf{CO}}^{<\omega} C'$ as witnessed by $\{C_i \mid i < n\}$ and f, without loss of generality we can assume that $n > 1$ and hence $C_i \sqsubset C$. (If not, split $f(C_0) \sqsubseteq C'$ into two nonempty convex subsets A, B of C', and consider the finite convex partition $\{f^{-1}(A), f^{-1}(B)\}$ of C together with the same embedding f.)

Proposition 2.4.8. *$\trianglelefteq_{\mathrm{c}}^{<\omega}$ is transitive.*

Proof. Suppose that $C \trianglelefteq_{\mathrm{c}}^{<\omega} C'$, as witnessed by the embedding f and the finite convex partition $\{C_i \mid i < n\}$ of C, and that $C' \trianglelefteq_{\mathrm{c}}^{<\omega} C''$ via the embedding g and the finite convex partition $\{C'_j \mid j < m\}$ of C'. If C' has only one element than so does C and $C \trianglelefteq_{\mathrm{c}}^{<\omega} C''$ is immediate. Thus, without loss of generality, we can assume that $m > 1$, so that $C'_j \sqsubset C'$ for all $j < m$. Notice that $\{f(C_i) \mid i < n\}$ and $\{C'_j \mid j < m\}$ are two collections of pairwise disjoint convex subsets of C'. We distinguish two cases.

If $C'_{i,j} = f(C_i) \cap C'_j$ is a convex subset of C' for every $i < n$ and $j < m$, then we can order the family of pairwise disjoint convex sets

$$\{C'_{i,j} \mid (i,j) \in n \times m \wedge C'_{i,j} \neq \emptyset\}$$

following the circular order of C'. In this way we obtain a family $\{D'_k \mid k < \ell\}$, for the suitable $\ell \leq n \cdot m$, such that if $x_0 \in D'_{k_0}$, $x_1 \in D'_{k_1}$, and $x_2 \in D'_{k_2}$ satisfy $C'(x_0,x_1,x_2)$ then $C[\boldsymbol{\ell}](k_0,k_1,k_2)$. Then $\{f^{-1}(D'_k) \mid k < \ell\}$ is a finite convex partition of C and $g \circ f$ is an embedding of C into C''. Moreover, for every $k < \ell$ we have $(g \circ f)(f^{-1}(D'_k)) = g(D'_k) \sqsubseteq C''$ because $D'_k \subseteq C'_j \sqsubset C'$ for some $j < m$ (Proposition 2.4.5). Thus $C \trianglelefteq_{\mathrm{c}}^{<\omega} C''$.

Suppose now that $C'_{i,j} = f(C_i) \cap C'_j$ is not convex for some (i,j). By Proposition 2.4.4 there is at most one such pair $(\bar{\imath}, \bar{\jmath})$. By Proposition 2.4.3, $C'_{\bar{\imath},\bar{\jmath}}$ is the union of two disjoint convex subsets A_0 and A_1 of C'. Then we can argue as in the previous paragraph but starting with the family

$$\{C'_{i,j} \mid (i,j) \in n \times m \wedge (i,j) \neq (\bar{\imath},\bar{\jmath}) \wedge C'_{i,j} \neq \emptyset\} \cup \{A_0, A_1\}. \qquad \square$$

Thus $\trianglelefteq_{\mathrm{c}}^{<\omega}$ is a quasi-order, and it is easy to see that its restriction $\trianglelefteq_{\mathsf{CO}}^{<\omega}$ to the Polish space CO is analytic.

We first show that $\trianglelefteq_{\mathsf{CO}}^{<\omega}$ satisfies combinatorial properties similar to those proved for $\trianglelefteq_{\mathsf{LO}}$ in Section 2.2. A key point is that it still makes sense to talk about (finite) condensation in the realm of circular orders. Indeed, given a circular order C the **condensation class** $c_F^C(\ell)$ of ℓ is the collection of those m such that either $\{k \mid C(\ell,k,m)\}$ or $\{k \mid C(m,k,\ell)\}$ is finite. Each $c_F^C(\ell)$ is convex in C, and it again holds that the condensation classes form a partition of C. This allows us to define the (**finite**) **condensation** C_F of C in the obvious way. The crucial observation is that we can substitute Proposition 2.2.1 with the following lemma.

Lemma 2.4.9. *Let f be an embedding between the circular orders C and C'. Fix any $\ell \in C$, and let $A \sqsubset C$ and $a,b \in A \setminus c_F^C(\ell)$ be such that $f(A) \sqsubseteq C'$, $c_F^C(\ell) \subseteq A$ and $C(a,\ell',b)$ for all $\ell' \in c_F^C(\ell)$. Then the restriction of f to $c_F^C(\ell)$ is an isomorphism between $c_F^C(\ell)$ and $c_F^{C'}(f(\ell))$, and thus $|c_F^C(\ell)| = |c_F^{C'}(f(\ell))|$.*

Proof. By Proposition 2.4.5, we have $f(c_F^C(\ell)) \sqsubseteq C'$, which easily implies $f(c_F^C(\ell)) \subseteq c_F^{C'}(f(\ell))$. Conversely, pick any $d' \in c_F^{C'}(f(\ell))$ distinct from $f(\ell)$, and first assume that $\{k \mid C'(d',k,f(\ell))\}$ is finite. Consider the set $B = \{k \in C \mid C(a,k,\ell)\} \subseteq A$. Since $a \notin c_F^C(\ell)$ and $B \sqsubseteq C$, the set $f(B)$ is infinite and by Proposition 2.4.5 $f(B) \sqsubseteq C'$. We cannot have $C'(d',f(a),f(\ell))$, otherwise $\{k \mid C'(d',k,f(\ell))\} \supseteq f(B)$ and the former would be infinite. Thus $C'(f(a),d',f(\ell))$, and so $\{k \mid C'(d',k,f(\ell))\} \subseteq f(B)$ because $f(B) \sqsubseteq C'$. This easily implies that $d' = f(d)$ for some $d \in c_F^C(\ell)$, and we are done. (When $\{k \mid C'(f(\ell),k,d')\}$ is finite, we work symmetrically on the other side of ℓ and use b instead of a.) $\square$

Proposition 2.4.10. (a) *There is an embedding from the partial order $(\mathrm{Int}(\mathbb{R}),\subseteq)$ into $\trianglelefteq_{\mathsf{CO}}^{<\omega}$, and indeed $(\mathrm{Int}(\mathbb{R}),\subseteq) \leq_B \trianglelefteq_{\mathsf{CO}}^{<\omega}$.*

(b) *$\trianglelefteq_{\mathsf{CO}}^{<\omega}$ has chains of order type $(\mathbb{R},<)$, as well as antichains of size $2^{\aleph_0}$.*

Proof. Given an interval $(x,y) \in \mathrm{Int}(\mathbb{R})$, consider the circular order $C_{(x,y)} \in \mathsf{CO}$ defined by $C_{(x,y)} = C[L_{(x,y)}]$, where $L_{(x,y)}$ is as in the proof of Lemma 2.2.3. We claim that the map $(x,y) \mapsto C_{(x,y)}$ witnesses (a). Pick two intervals $(x,y),(x',y') \in \mathrm{Int}(\mathbb{R})$. If $(x,y) \subseteq (x',y')$ then the identity map witnesses $C_{(x,y)} \trianglelefteq_{\mathrm{c}} C_{(x',y')}$, hence $C_{(x,y)} \trianglelefteq_{\mathsf{CO}}^{<\omega} C_{(x',y')}$. Suppose now that $(x,y) \nsubseteq (x',y')$, and for the sake of definiteness assume that $x < x'$. Towards a contradiction suppose that there are a finite convex partition $\{C_i \mid i < n\}$ of $C_{(x,y)}$ and an embedding f witnessing $C_{(x,y)} \trianglelefteq_{\mathsf{CO}}^{<\omega} C_{(x',y')}$. As usual, we can assume $n > 1$, so that $C_i \sqsubset C_{(x,y)}$ for all $i < n$. Since there are infinitely many rationals between x and x' and all condensation classes of $C_{(x,y)}$ are finite, we can find $i < n$ and $q,q',r \in \mathbb{Q}$ such that $x < q < r < q' < x'$ and the hypothesis of Lemma 2.4.9 are satisfied with $A = C_i$, $a = (0,q)$, $b = (0,q')$ and $\ell = (0,r)$. Thus the condensation class of $f(0,r)$ has the same size of the condensation class of $(0,r)$, which by construction can happen only if $r \in (x',y')$, a contradiction.

Part (b) is derived from (a) as in Proposition 2.2.4. $\square$

Proposition 2.4.11. $\mathfrak{b}(\trianglelefteq_{\mathsf{CO}}^{<\omega}) = \aleph_1$*, and indeed every $C \in \mathsf{CO}$ is the bottom of a strictly increasing $\trianglelefteq_{\mathsf{CO}}^{<\omega}$-unbounded chain of length ω_1.*

Proof. For $C \in \mathsf{CO}$ and $\ell \in C$, let $\alpha_{\ell,C}$ be the sup of those $\omega \leq \alpha < \omega_1$ such that $C[\boldsymbol{\alpha}] \trianglelefteq_{\mathrm{c}} C$ via some f satisfying $f(0) = \ell$. Since $\alpha_{\ell,C}$ is attained by definition of convexity, the ordinal $\alpha_C = \sup_{\ell \in C} \alpha_{\ell,C}$ is countable, and by construction $C[\boldsymbol{\alpha_C + 1}] \ntrianglelefteq_c C$. Let α be an additively indecomposable[1] countable ordinal above $\alpha_C + 1$: we claim that $C[\boldsymbol{\alpha}] \ntrianglelefteq_{\mathsf{CO}}^{<\omega} C$. Suppose towards a contradiction that $\{C_i \mid i < n\}$ is a finite convex partition and f an embedding witnessing $C[\boldsymbol{\alpha}] \trianglelefteq_{\mathsf{CO}}^{<\omega} C$. As usual we can assume $n > 1$. Then there are $i < n$ and $\gamma < \alpha$ such that $A'_\gamma = \{\beta \in C[\boldsymbol{\alpha}] \mid \beta \geq \gamma\}$ is contained in C_i. Since α is additively indecomposable, the linear order determined by A_γ has order type $\alpha \geq \alpha_C + 1$, thus we can consider the set $A_\gamma = \{\beta \in A'_\gamma \mid \beta < \gamma + \alpha_C + 1\}$, which has order type $\alpha_C + 1$. Since $A_\gamma \sqsubseteq C[\boldsymbol{\alpha}]$ and $A_\gamma \subseteq C_i \sqsubset C[\boldsymbol{\alpha}]$, by Proposition 2.4.5 the restriction of f to A_γ witnesses $C[\boldsymbol{\alpha_C + 1}] \trianglelefteq_{\mathrm{c}} C$, a contradiction.

[1]An ordinal α is additively idecomposable if $\beta + \gamma < \alpha$ for all $\beta, \gamma < \alpha$. Additively indecomposable ordinals are precisely those of the form ω^δ for some ordinal δ.

This shows that the family $\{C[\boldsymbol{\alpha}] \mid \omega \leq \alpha < \omega_1\}$ is $\trianglelefteq_{\mathsf{CO}}^{<\omega}$-unbounded in CO. Since $C[\boldsymbol{\alpha}] \trianglelefteq_{\mathrm{c}} C[\boldsymbol{\beta}]$ when $\alpha \leq \beta$, we can extract from it a strictly increasing chain witnessing $\mathfrak{b}(\trianglelefteq_{\mathsf{CO}}^{<\omega}) \leq \aleph_1$. To show that $\mathfrak{b}(\trianglelefteq_{\mathsf{CO}}^{<\omega}) > \aleph_0$, consider a countable family $\{C_i \mid i \in \mathbb{N}\} \subseteq \mathsf{CO}$. For each $i \in \mathbb{N}$ pick an arbitrary $\ell_i \in C_i$ and define $L_i \in \mathsf{LO}$ by setting $x \leq_{L_i} y$ iff $C_i(\ell_i, x, y)$. Then the circular order $C = C\left[\sum_{i\in\mathbb{N}} L_i\right] \in \mathsf{CO}$ is such that $C_i \trianglelefteq_{\mathrm{c}} C$ for all $i \in \mathbb{N}$, and thus the given family is $\trianglelefteq_{\mathsf{CO}}^{<\omega}$-bounded.

For the second part, pick $\ell \in C$ and let $L \in \mathsf{LO}$ be defined by $x \leq_L y$ iff $C(\ell, x, y)$. Consider the $\trianglelefteq_{\mathsf{CO}}^{<\omega}$-nondecreasing sequence $(C_\alpha)_{\alpha<\omega_1}$ of circular orders defined by $C_0 = C$ and $C_\alpha = C[L+\boldsymbol{\omega}+\boldsymbol{\alpha}]$ when $\alpha > 0$. Since $C[\boldsymbol{\omega}+\boldsymbol{\alpha}] \trianglelefteq_{\mathrm{c}} C_\alpha$ for all $\alpha \neq 0$, such a sequence is $\trianglelefteq_{\mathsf{CO}}^{<\omega}$-unbounded. Thus we can extract from it a strictly $\trianglelefteq_{\mathsf{CO}}^{<\omega}$-increasing subsequence of length ω_1 with C_0 as first element: being cofinal in the original sequence, it will be $\trianglelefteq_{\mathsf{CO}}^{<\omega}$-unbounded too, as required. $\square$

Proposition 2.4.12. (a) *There are $2^{\aleph_0}$-many $\trianglelefteq_{\mathsf{CO}}^{<\omega}$-incomparable $\trianglelefteq_{\mathsf{CO}}^{<\omega}$-minimal elements in CO. In particular, all bases for $\trianglelefteq_{\mathsf{CO}}^{<\omega}$ are of maximal size.*

(b) *There is a $\trianglelefteq_{\mathsf{CO}}^{<\omega}$-decreasing ω-sequence in CO which is not $\trianglelefteq_{\mathsf{CO}}^{<\omega}$-bounded from below.*

Proof. (a) Given an infinite $S \subseteq \mathbb{N}$, let $C_S = C[\eta_{f_S}]$ where η_{f_S} is as in the proof of Proposition 2.2.8(a). If $A \sqsubseteq C_S$ is infinite, then there exist $q, q' \in \mathbb{Q}$ with $q < q'$ such that $\{(\ell, q'') \in C_S \mid q \leq q'' \leq q'\} \subseteq A$, and thus C_S is convex embeddable into (the circular order determined by) A.

Let $C \trianglelefteq_{\mathsf{CO}}^{<\omega} C_S$, as witnessed by the finite convex partition $\{C_i \mid i < n\}$ and the embedding f. Then there is $i < n$ such that C_i, and hence also $f(C_i)$ is infinite. Setting $A = f(C_i)$ in the previous paragraph, we get that $C_S \trianglelefteq_{\mathrm{c}} f(C_i) \cong C_i \sqsubseteq C$, hence $C_S \trianglelefteq_{\mathsf{CO}}^{<\omega} C$. This shows that C_S is $\trianglelefteq_{\mathsf{CO}}^{<\omega}$-minimal.

Assume now that $C_S \trianglelefteq_{\mathsf{CO}}^{<\omega} C_{S'}$ for some infinite $S, S' \subseteq \mathbb{N}$, and let $\{C_i \mid i < n\}$ be a finite convex partition of C_S and $f \colon C_S \to C_{S'}$ be an embedding witnessing this. As usual, we may assume $n > 1$, so that $C_i \sqsubset C_S$ and $f(C_i) \sqsubset C_{S'}$. Fix any $i < n$ such that C_i is infinite. By the first paragraph, there are $q < q'$ such that $\{(\ell, q'') \in C_S \mid q \leq q'' \leq q'\} \subseteq C_i$. Given an arbitrary $m \in S$, pick $q'' \in \mathbb{Q}$ such that $q < q'' < q'$ and $f_S(q'') = m$. Then the hypotheses of Lemma 2.4.9 are satisfied when we set $A = C_i$, $a = (0, q)$, $b = (0, q')$, and $\ell = (0, q'')$. Thus $C_{S'}$ must contain a condensation class of size m, which is possible only if $m \in S'$. This shows that $S \subseteq S'$. Conversely, given $m \in S'$ we work with the infinite set $f(C_i) \sqsubset C_{S'}$ and pick $q, q' \in \mathbb{Q}$ such that $\{(\ell, q'') \in C_{S'} \mid q \leq q'' \leq q'\} \subseteq f(C_i)$. Then we pick $q'' \in \mathbb{Q}$ such that $q < q'' < q'$ and $f_{S'}(q'') = m$. Applying (the proof of) Lemma 2.4.9 we get that the condensation class of $f^{-1}(0, q'')$ has size m, hence $m \in S$. Since $m \in S'$ was arbitrary, $S' \subseteq S$, and thus $S = S'$. This shows that $\{C_S \mid S \subseteq \mathbb{N} \wedge S \text{ is infinite}\}$ is a $\trianglelefteq_{\mathsf{CO}}^{<\omega}$-antichain and we are done.

(b) Consider the family $\{C_{(m,+\infty)} \mid m \in \mathbb{N}\}$, where $C_{(m,+\infty)}$ is as in the proof of Proposition 2.4.10(a). It is a strictly $\trianglelefteq_{\mathsf{CO}}^{<\omega}$-decreasing chain, so we only need to show that it is $\trianglelefteq_{\mathsf{CO}}^{<\omega}$-unbounded from below. Let $C \in \mathsf{CO}$ be such that $C \trianglelefteq_{\mathsf{CO}}^{<\omega} C_{(0,+\infty)}$, as witnessed by the finite convex partition $\{C_i \mid i < n\}$ (for some $n > 1$) and the embedding f. Then there is $i < n$ such that C_i is infinite, which means that $\{(\ell, q'') \in C_{(0,+\infty)} \mid q < q'' < q'\} \subseteq f(C_i) \sqsubset C_{(0,+\infty)}$ for some rational numbers $0 \leq q < q'$. Thus C contains a convex subset isomorphic to $C_{(q,q')}$ by Lemma 2.4.5. Pick $m \in \mathbb{N}$ with $m > q'$. Then $C \ntrianglelefteq_{\mathsf{CO}}^{<\omega} C_{(m,+\infty)}$ because otherwise $C_{(q,q')} \trianglelefteq_{\mathrm{c}} C \trianglelefteq_{\mathsf{CO}}^{<\omega} C_{(m,+\infty)}$, contradicting (the proof of) Proposition 2.4.10(a). $\square$

Proposition 2.4.13. *Every $\trianglelefteq_{\mathsf{CO}}^{<\omega}$-antichain is contained in a $\trianglelefteq_{\mathsf{CO}}^{<\omega}$-antichain of size $2^{\aleph_0}$. In particular, there are no maximal $\trianglelefteq_{\mathsf{CO}}^{<\omega}$-antichains of size smaller than $2^{\aleph_0}$, and every $C \in \mathsf{CO}$ belongs to a $\trianglelefteq_{\mathsf{CO}}^{<\omega}$-antichain of size $2^{\aleph_0}$.*

Proof. Following the proof of Proposition 2.2.9, we only need to verify that for every $C \in \mathsf{CO}$ the set $\{S \subseteq \mathbb{N} \mid S \text{ is infinite} \wedge C_S \trianglelefteq_{\mathsf{CO}}^{<\omega} C\}$ is countable, where the C_S's are defined in the proof Proposition 2.4.12(a).

First observe that arguing as at the beginning of that proof and using Proposition 2.4.5 one can prove that if $C_S \trianglelefteq_{\mathsf{CO}}^{<\omega} C$ then $C_S \trianglelefteq_{\mathrm{c}} C$. Indeed, let $C_S \trianglelefteq_{\mathsf{CO}}^{<\omega} C$ be witnessed by the finite convex partition $\{C_i \mid i < n\}$ (for some $n > 1$) of C_S and the embedding $f \colon C_S \to C$. Then some C_i

must be infinite, so there is an embedding $g\colon C_S \to C_S$ such that $\operatorname{Im} g \sqsupseteq C_S$ and $\operatorname{Im} g \subseteq C_i \sqsubset C_S$. Hence $f(\operatorname{Im} g) \sqsupseteq C$, and so $f \circ g$ witnesses $C_S \trianglelefteq_{\mathrm{c}} C$.

Suppose that $S, S' \subseteq \mathbb{N}$ are distinct infinite sets such that $C_S \trianglelefteq^{<\omega}_{\mathsf{CO}} C$ and $C_{S'} \trianglelefteq^{<\omega}_{\mathsf{CO}} C$ via corresponding embeddings f and g, respectively. Without loss of generality, we may assume that $\operatorname{Im} f \neq C$ and $\operatorname{Im} g \neq C$, as otherwise $C_{S'} \trianglelefteq^{<\omega}_{\mathsf{CO}} C_S$ or $C_S \trianglelefteq^{<\omega}_{\mathsf{CO}} C_{S'}$, contradicting (the proof of) Proposition 2.4.12(a). If $\operatorname{Im} f \cap \operatorname{Im} g \neq \emptyset$, then by Proposition 2.4.3 such intersection is the union of (at most) two proper convex subsets A_0, A_1 of C, each of which must be infinite by definition of C_S and $C_{S'}$. Thus $f^{-1}(A_0)$ is an infinite convex proper subset of C_S, and so $C_S \trianglelefteq_{\mathrm{c}} f^{-1}(A_0)$, which in turn implies $C_S \trianglelefteq_{\mathrm{c}} A_0$ and $C_S \trianglelefteq_{\mathrm{c}} C_{S'}$, a contradiction. Thus $\operatorname{Im} f \cap \operatorname{Im} g = \emptyset$. Since C is countable, there can be at most countably many infinite $S \subseteq \mathbb{N}$ such that $C_S \trianglelefteq^{<\omega}_{\mathsf{CO}} C$ and the claim follows. □

Once we know that for every $C \in \mathsf{CO}$ there are at most countably many infinite sets $S \in \mathbb{N}$ such that $C_S \trianglelefteq^{<\omega}_{\mathsf{CO}} C$, arguing as in Proposition 2.2.10 we easily get

Proposition 2.4.14. $\mathfrak{d}(\trianglelefteq^{<\omega}_{\mathsf{CO}}) = 2^{\aleph_0}$.

We now move to the study of the (analytic) equivalence relation $\bowtie^{<\omega}_{\mathsf{CO}}$ induced by $\trianglelefteq^{<\omega}_{\mathsf{CO}}$. Obviously if $C \cong_{\mathsf{CO}} C'$ then we also have $C \bowtie^{<\omega}_{\mathsf{CO}} C'$.

Theorem 2.4.15. $\cong_{\mathsf{LO}} \leq_B \bowtie^{<\omega}_{\mathsf{CO}}$.

Proof. Consider the Borel map $\varphi\colon \mathsf{LO} \to \mathsf{CO}$ defined by

$$\varphi(L) = C[(\mathbf{1} + \zeta L)\omega].$$

We claim that φ is a reduction. Clearly, if $L \cong_{\mathsf{LO}} L'$ then $\varphi(L) \cong_{\mathsf{CO}} \varphi(L')$ and hence $\varphi(L) \bowtie^{<\omega}_{\mathsf{CO}} \varphi(L')$. For the converse, let the finite convex partition $\{C_i \mid i < n\}$ of $\varphi(L)$ and the embedding f of $\varphi(L)$ into $\varphi(L')$ witness $\varphi(L) \trianglelefteq^{<\omega}_{\mathsf{CO}} \varphi(L')$. Without loss of generality $n > 1$, so that $C_i \sqsubset \varphi(L)$ for all $i < n$. Since n is finite, there exists some $j < n$ such that C_j contains at least two copies of $\mathbf{1} + \zeta L$, so we can consider a convex set of the form $\mathbf{1} + \zeta L + \mathbf{1} \subseteq C_j$, so that $f(\mathbf{1} + \zeta L + \mathbf{1}) \sqsupseteq \varphi(L')$ by Proposition 2.4.5. Since the $\mathbf{1}$'s are the only elements which do not have immediate predecessor and successor both in $\varphi(L)$ and in $\varphi(L')$, and since $f(\mathbf{1} + \zeta L + \mathbf{1})$ is convex, we have that the images via f of the two $\mathbf{1}$'s in $\mathbf{1} + \zeta L + \mathbf{1} \subseteq C_j$ are two necessarily "consecutive" $\mathbf{1}$'s in $\varphi(L')$. It follows that $\mathbf{1} + \zeta L + \mathbf{1} \subseteq C_j$ is isomorphic to a copy of $\mathbf{1} + \zeta L' + \mathbf{1}$ in $\varphi(L')$. We thus obtain $\zeta L \cong_{\mathsf{LO}} \zeta L'$, hence $L \cong_{\mathsf{LO}} L'$ by Lemma 1.2.7. □

The next results contrasts with Corollary 2.3.16. To simplify the notation, we let $\vec{x}$ and $\vec{y}$ denote the sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, respectively.

Theorem 2.4.16. $E_1 \leq_B \bowtie^{<\omega}_{\mathsf{CO}}$.

Proof. By Proposition 1.1.13 it suffices to define a Borel reduction from E_1^t to $\bowtie^{<\omega}_{\mathsf{CO}}$. To this end, fix an injective map $f\colon \mathbb{Q} \to \{\boldsymbol{n} \mid n \in \mathbb{N} \setminus \{0, 1\}\}$ and, as in the proofs of Lemma 2.2.3 and Proposition 2.2.4, consider the linear orders η_f and $L_{(x,x+1)}$, with $x \in \mathbb{R}$. By Lemma 2.2.3, $L_{(x,x+1)}$ and $L_{(x',x'+1)}$ are isomorphic if and only if $x = x'$. Consider the Borel map that sends $\vec{x} = (x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ to the linear order

$$L(\vec{x}) = \sum_{n\in\mathbb{Z}} \overline{L}_n,$$

where $\overline{L}_n = \eta_f + \eta$ if $n < 0$ and $\overline{L}_n = L_{(x_n,x_n+1)} + \eta$ if $n \geq 0$. We claim that the Borel map $\varphi\colon \mathbb{R}^{\mathbb{N}} \to \mathsf{CO}$ defined by $\varphi(\vec{x}) = C[L(\vec{x})]$ is a reduction from E_1^t to $\bowtie^{<\omega}_{\mathsf{CO}}$.

First suppose that $\vec{x} \ E_1^t \ \vec{y}$, i.e. that there are $\bar{n}, \bar{m} \in \mathbb{N}$ such that $x_{\bar{n}+k} = y_{\bar{m}+k}$ for all $k \in \mathbb{N}$. Consider the finite convex partition $\{C_i \mid i < 2\bar{n}+2\}$ of $\varphi(\vec{x})$ given by setting for $0 \leq j < \bar{n}$

$$
\begin{aligned}
C_0 &= \sum_{n \in \mathbb{Z} \setminus \mathbb{N}} \overline{L}_n = \{(\ell, n) \in L(\vec{x}) \mid n < 0\} \\
C_{2j+1} &= L_{(x_j, x_j+1)} \times \{j\} \\
C_{2j+2} &= \eta \times \{j\} \\
C_{2\bar{n}+1} &= \sum_{n \geq \bar{n}} \overline{L}_n = \{(\ell, n) \in L(\vec{x}) \mid n \geq \bar{n}\}.
\end{aligned}
$$

Consider the embedding f of $\varphi(\vec{x})$ into $\varphi(\vec{y})$ defined by

$$
f(\ell, n) = \begin{cases} (\ell, n - \bar{n}) & \text{if } n < \bar{n} \\ (\ell, \bar{m} + (n - \bar{n})) & \text{if } n \geq \bar{n}. \end{cases}
$$

By choice of $\bar{n}, \bar{m} \in \mathbb{N}$ and since $L_{(x,x+1)} \sqsupseteq \eta_f$ for all $x \in \mathbb{R}$, it is easy to verify that f is well-defined and that $f(C_i) \sqsupseteq \varphi(\vec{y})$ for all $i < 2\bar{n}+2$. This witnesses $\varphi(\vec{x}) \trianglelefteq_{\mathsf{CO}}^{<\omega} \varphi(\vec{y})$, and since $\varphi(\vec{y}) \trianglelefteq_{\mathsf{CO}}^{<\omega} \varphi(\vec{x})$ can be proved symmetrically, we obtain $\varphi(\vec{x}) \bowtie_{\mathsf{CO}}^{<\omega} \varphi(\vec{y})$.

Suppose now that[2] $\varphi(\vec{x}) \bowtie_{\mathsf{CO}}^{<\omega} \varphi(\vec{y})$. Let $\{C_i \mid i < b\}$ with $b \in \mathbb{N} \setminus \{0\}$ be a finite convex partition of $\varphi(\vec{x})$ and f be an embedding of $\varphi(\vec{x})$ into $\varphi(\vec{y})$ witnessing $\varphi(\vec{x}) \trianglelefteq_{\mathsf{CO}}^{<\omega} \varphi(\vec{y})$. (As usual, we can assume $b > 1$, so that Proposition 2.4.5 can be applied when necessary.) Since b is finite, for some $i < b$ and $\bar{n} \in \mathbb{N} \setminus \{0\}$ we must have $\sum_{n \geq \bar{n}-1} \overline{L}_n \subseteq C_i$. Notice that for every $n \geq \bar{n}-1$ and $q \in \eta$, the point $(q, n) \in \eta \times \{n\} \sqsubset \varphi(\vec{x})$ has no immediate predecessor and immediate successor, while points of the form (ℓ, m) for $\ell \in L_{(y_m, y_m+1)}$ and $m \in \mathbb{N}$ or $\ell \in \eta_f$ and $m \in \mathbb{Z} \setminus \mathbb{N}$ have an immediate predecessor or an immediate successor (or both): thus $f(q, n) \in \eta \times m$ for some $m \in \mathbb{Z}$. By a similar argument, $f(L_{(x_n, x_n+1)} \times \{n\}) \subseteq L_{(y_m, y_m+1)} \times \{m\}$ or $f(L_{(x_n, x_n+1)} \times \{n\}) \subseteq \eta_f \times \{m\}$ for a suitable $m \in \mathbb{Z}$. This two facts together with the convexity of $f(C_i)$ and the fact that, by the proof of Lemma 2.2.3, the only convex subset of η_f isomorphic to $L_{(x,x+1)}$ is $L_{(x,x+1)}$ itself, imply that $f(L_{(x_{\bar{n}}, x_{\bar{n}}+1)} \times \{\bar{n}\}) = L_{(x_{\bar{m}}, x_{\bar{m}}+1)} \times \{\bar{m}\}$ for some $\bar{m} \in \mathbb{N}$, and in turn $f(L_{(x_{\bar{n}+k}, x_{\bar{n}+k}+1)} \times \{\bar{n}+k\}) = L_{(x_{\bar{m}+k}, x_{\bar{m}+k}+1)} \times \{\bar{m}+k\}$ for all $k \in \mathbb{N}$. But by Lemma 2.2.3 again, this means that $x_{\bar{n}+k} = y_{\bar{m}+k}$ for all $k \in \mathbb{N}$, hence $\vec{x} \ E_1^t \ \vec{y}$. □

Corollary 2.4.17. *$\cong_{\mathsf{LO}} <_B \bowtie_{\mathsf{CO}}^{<\omega}$ and $\bowtie_{\mathsf{LO}} <_{Baire} \bowtie_{\mathsf{CO}}^{<\omega}$. Moreover $\bowtie_{\mathsf{CO}}^{<\omega}$ is not Baire reducible to an orbit equivalence relation..*

Proof. All the statements follow from Theorem 2.4.16 and some of the previous results. The first two statements need Theorem 2.4.15 and Corollaries 2.3.16 and 2.3.13; the last one follows from Theorem 1.1.14. □

[2] Our proof actually shows that $\varphi(\vec{x}) \trianglelefteq_{\mathsf{CO}}^{<\omega} \varphi(\vec{y})$ already suffices to obtain $\vec{x} \ E_1^t \ \vec{y}$, so that in particular we get $\varphi(\vec{x}) \trianglelefteq_{\mathsf{CO}}^{<\omega} \varphi(\vec{y}) \iff \varphi(\vec{x}) \bowtie_{\mathsf{CO}}^{<\omega} \varphi(\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^{\mathbb{N}}$.

3

Piecewise convex embeddability on linear orders

3.1 The ccs property

In this section we introduce a binary relation among linear orders which captures the idea of "piecewise" convex embeddability, where the pieces are orderly indexed by an element of a fixed class $\mathcal{L} \subseteq \mathsf{Lin}$. Unless otherwise stated, *from now on we let $\mathcal{L}$ be a nonempty downward $\preceq$-closed subset of* Lin. Among such classes we find those of the form $\mathcal{L}_{\preceq L_0} = \{L \in \mathsf{Lin} \mid L \preceq L_0\}$ and $\mathcal{L}_{\prec L_0} = \{L \in \mathsf{Lin} \mid L \prec L_0\}$, for some $L_0 \in \mathsf{Lin}$.

Definition 3.1.1. Given $K \in \mathsf{Lin}$ and a linear order L, a **K-convex partition of** L is a partition $(L_k)_{k \in K}$ of L such that $k <_K k'$ if and only if $L_k <_L L_{k'}$ for every $k, k' \in K$.

Notice that if $(L_k)_{k \in K}$ is a K-convex partition of L, each L_k is a convex subset of L. Let us stress that in the following definition, our index class $\mathcal{L}$ is contained in Lin, but $\trianglelefteq^{\mathcal{L}}$ is defined on the class of *all* linear orders.

Definition 3.1.2. Given $\mathcal{L} \subseteq \mathsf{Lin}$ as above and linear orders L, L', we write $L \trianglelefteq^{\mathcal{L}} L'$ if and only if there exist $K \in \mathcal{L}$, a K-convex partition $(L_k)_{k \in K}$ of L, and an embedding f of L into L' such that $f(L_k) \sqsubseteq L'$ for all $k \in K$. The binary relation $\trianglelefteq^{\mathcal{L}}$ is called **$\mathcal{L}$-convex embeddability**.

Equivalently, $L \trianglelefteq^{\mathcal{L}} L'$ if and only if there is $K \in \mathcal{L}$ and a family $(L_k)_{k \in K}$ of nonempty linear orders such that, up to isomorphism, $L = \sum_{k \in K} L_k$ and there is an embedding $f \colon L \to L'$ such that $f(L_k) \sqsubseteq L'$ for all $k \in K$. Yet another equivalent reformulation of $L \trianglelefteq^{\mathcal{L}} L'$ is the following: there are $K \in \mathcal{L}$, $K' \in \mathsf{Lin}$, an embedding $f \colon K \to K'$, a K-convex partition $(L_k)_{k \in K}$ of L, and a K'-convex partition $(L'_k)_{k \in K'}$ of L' such that $L_k \cong L'_{f(k)}$ for all $k \in K$.

Although in general $\trianglelefteq^{\mathcal{L}}$ needs not to be a quasi-order, we also consider its "strict part" $\triangleleft^{\mathcal{L}}$ defined by $L \triangleleft^{\mathcal{L}} L'$ if $L \trianglelefteq^{\mathcal{L}} L'$ but $L' \ntrianglelefteq^{\mathcal{L}} L$, and write $L \bowtie^{\mathcal{L}} L'$ if both $L \trianglelefteq^{\mathcal{L}} L'$ and $L' \trianglelefteq^{\mathcal{L}} L$. As usual, we denote by $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ the restriction of $\trianglelefteq^{\mathcal{L}}$ to the set LO of (codes for) linear orders on the whole $\mathbb{N}$, and similarly for $\triangleleft^{\mathcal{L}}_{\mathsf{LO}}$ and $\bowtie^{\mathcal{L}}_{\mathsf{LO}}$.

If $\mathcal{L} = \{\mathbf{1}\} = \mathcal{L}_{\preceq \mathbf{1}}$, then $\trianglelefteq^{\mathcal{L}}$ is simply convex embeddability $\trianglelefteq$. Moreover, if $\mathcal{L} \subseteq \mathcal{L}'$ then $L \trianglelefteq^{\mathcal{L}} L' \Rightarrow L \trianglelefteq^{\mathcal{L}'} L'$ for all linear orders L, L'. Since each $\mathcal{L}$ is tacitly assumed to be nonempty and downward $\preceq$-closed, it follows that $\mathcal{L}_{\preceq \mathbf{1}} \subseteq \mathcal{L}$ and hence $L \trianglelefteq L' \Rightarrow L \trianglelefteq^{\mathcal{L}} L'$.

At the other extreme, we have the case where $\mathcal{L} = \mathsf{Lin} = \mathcal{L}_{\preceq \eta}$ (equivalently: $\mathcal{L} \nsubseteq \mathsf{Scat}$). In this case, if L is countable and L' is an arbitrary linear order, then $L \preceq L' \Rightarrow L \trianglelefteq^{\mathcal{L}} L'$, as we can always partition L in singletons. More generally, by the same reasoning we have the following useful fact.

Fact 3.1.3. *If $L \in \mathcal{L}$ and L' is arbitrary, then $L \trianglelefteq^{\mathcal{L}} L'$ if and only if $L \preceq L'$.*

Another useful fact is the following:

Proposition 3.1.4. *For every $\mathcal{L} \subseteq \mathsf{Lin}$ and $L \in \mathsf{Scat}$, we have $L \trianglelefteq^{\mathcal{L}} \eta$ if and only if $L \in \mathcal{L}$.*

Proof. Assume that $K \in \mathcal{L}$, $(L_k)_{k\in K}$, and $f\colon L \to \eta$ witness $L \trianglelefteq^{\mathcal{L}} \eta$. By Remark 1.2.4 and $L \in \mathsf{Scat}$, each L_k has order type $\mathbf{1}$, hence $L \cong_{\mathsf{LO}} K \in \mathcal{L}$. □

Combining the above observations, one can determine the mutual relationships among the relations $\trianglelefteq^{\mathcal{L}}$. More precisely, say that $\trianglelefteq^{\mathcal{L}}$ **refines** $\trianglelefteq^{\mathcal{L}'}$ if $\trianglelefteq^{\mathcal{L}} \subseteq \trianglelefteq^{\mathcal{L}'}$, i.e. $L \trianglelefteq^{\mathcal{L}} L' \Rightarrow L \trianglelefteq^{\mathcal{L}'} L'$ for all linear orders L and L'.[1]

Proposition 3.1.5. *The relation $\trianglelefteq^{\mathcal{L}}$ refines $\trianglelefteq^{\mathcal{L}'}$ if and only if $\mathcal{L} \subseteq \mathcal{L}'$.*

Proof. As observed, one direction is obvious, so let us assume that $\trianglelefteq^{\mathcal{L}}$ refines $\trianglelefteq^{\mathcal{L}'}$.

Proposition 3.1.4 implies that $\mathcal{L}\cap\mathsf{Scat} \subseteq \mathcal{L}'\cap\mathsf{Scat}$, so we only need to show that if $\mathcal{L} = \mathsf{Lin}$ then $\mathcal{L}' = \mathsf{Lin}$ too. But if $\eta \in \mathcal{L}$ then $\eta \trianglelefteq^{\mathcal{L}} \mathbf{2}\eta$ by Fact 3.1.3, which by our initial assumption implies $\eta \trianglelefteq^{\mathcal{L}'} \mathbf{2}\eta$. Assume towards a contradiction that $\mathcal{L}' \neq \mathsf{Lin}$, i.e. $\mathcal{L}' \subseteq \mathsf{Scat}$. Let $K \in \mathcal{L}'$, $(L'_k)_{k\in K}$ and $f\colon \eta \to \mathbf{2}\eta$ witness $\eta \trianglelefteq^{\mathcal{L}'} \mathbf{2}\eta$. Since $K \in \mathsf{Scat}$, at least one of the convex sets L'_k contains a copy of η by Remark 1.2.4, hence $\eta \trianglelefteq \mathbf{2}\eta$, which is not the case. □

Since $\trianglelefteq$ refines $\trianglelefteq^{\mathcal{L}}$ for all the families $\mathcal{L}$ under consideration, it easily follows that the relation $\trianglelefteq^{\mathcal{L}}$ is always reflexive. However, the next example shows that $\trianglelefteq^{\mathcal{L}}$ might lack transitivity.

Example 3.1.6. Consider $\mathcal{L} = \mathcal{L}_{\preceq\mathbf{2}}$. It is immediate that $\zeta\mathbf{3} \trianglelefteq^{\mathcal{L}} \zeta + \mathbf{1} + \zeta\mathbf{2} \trianglelefteq^{\mathcal{L}} (\zeta + \mathbf{1})\mathbf{3}$, but $\zeta\mathbf{3} \ntrianglelefteq^{\mathcal{L}} (\zeta+\mathbf{1})\mathbf{3}$ because to find an embedding as in Definition 3.1.2 we need to have a linear order $K \in \mathcal{L}$ with three elements, which is not the case. More generally, if $\mathcal{L} = \mathcal{L}_{\preceq\mathbf{n}}$ with $n > 1$, we have that $\zeta(\mathbf{2n}-\mathbf{1}) \trianglelefteq^{\mathcal{L}} (\zeta + \mathbf{1})(\mathbf{n} - \mathbf{1}) + \zeta\mathbf{n} \trianglelefteq^{\mathcal{L}} (\zeta + \mathbf{1})(\mathbf{2n} - \mathbf{1})$, but $\zeta(\mathbf{2n} - \mathbf{1}) \ntrianglelefteq^{\mathcal{L}} (\zeta + \mathbf{1})(\mathbf{2n} - \mathbf{1})$. Hence transitivity fails for all binary relations $\trianglelefteq^{\mathcal{L}_{\preceq\mathbf{n}}}$ with $n > 1$.

Since we want to work with quasi-orders, we thus have to first determine when $\trianglelefteq^{\mathcal{L}}$ is transitive. Consider linear orders L, L', L'' such that $L \trianglelefteq^{\mathcal{L}} L'$ with witnesses $K \in \mathcal{L}$, $(L_k)_{k\in K}$ and $f\colon L \to L'$, and $L' \trianglelefteq^{\mathcal{L}} L''$ with witnesses $K' \in \mathcal{L}$, $(L'_{k'})_{k'\in K'}$ and $f'\colon L' \to L''$. We would like to have that $L \trianglelefteq^{\mathcal{L}} L''$. To this aim, for every $k \in K$ define the set

$$K'_k = \{k' \in K' \mid f(L_k) \cap L'_{k'} \neq \emptyset\}.$$

Notice that each K'_k is a nonempty convex subset of K', and that $\forall k_0, k_1 \in K\ (k_0 <_K k_1 \Rightarrow K'_{k_0} \leq_{K'} K'_{k_1})$ because $f(L_k) \sqsubseteq L'$ for each $k \in K$ by choice of f. Now, consider the linear order

$$M = \sum_{k\in K} K'_k,$$

i.e. M is the set $\{(k', k) \mid k \in K \text{ and } k' \in K'_k\}$ ordered antilexicographically. For every $(k', k) \in M$, let

$$L_{(k',k)} = \{n \in L \mid n \in L_k \text{ and } f(n) \in L'_{k'}\}.$$

Notice that $L_{(k',k)}$ is a nonempty convex subset of L_k, and hence of L, and that $f(L_{(k,k')}) \sqsubseteq L'_{k'}$, hence $(f' \circ f)(L_{(k',k)}) \sqsubseteq L''$. Thus, *if M were a member of $\mathcal{L}$*, then M, $\big(L_{(k',k)}\big)_{(k',k)\in M}$ and $f' \circ f$ would witness $L \trianglelefteq^{\mathcal{L}} L''$. This motivates the following technical definition.

Definition 3.1.7. Let $\mathcal{L} \subseteq \mathsf{Lin}$ be downward closed under embeddability. We say that $\mathcal{L}$ is **closed under convex sums**, **ccs** for short, if for every $K, K' \in \mathcal{L}$ and for every $(K'_k)_{k\in K}$ such that each K'_k is a nonempty convex subset of K' and

$$\forall k_0, k_1 \in K\ (k_0 <_K k_1 \Rightarrow K'_{k_0} \leq_{K'} K'_{k_1}),$$

we have that $\sum_{k\in K} K'_k \in \mathcal{L}$.

[1] The result would not change if one restricts this definition to *countable* linear orders.

Many natural classes are ccs, for example: $\mathcal{L}_{\preceq \mathbf{1}}$, Fin, WO, Scat, and Lin. (Further examples of ccs classes are given later in Section 3.4.) Moreover, it is immediate to see that if $\mathcal{L}$ is ccs then so is $\mathcal{L}^* = \{L^* \mid L \in \mathcal{L}\}$. Since $\sum_{k \in K} K'_k$ is a suborder of $K'K$ it is immediate that any downward $\preceq$-closed $\mathcal{L}$ which is closed under products is ccs. In Section 3.4 we however exhibit examples of ccs classes that are not closed under products. On the other hand, notice that the ccs property does not hold for all $\mathcal{L}$ which are downward $\preceq$-closed. Indeed, a crucial property of the convex sums involved in Definition 3.1.7 is that if $K'_{k_0} \cap K'_{k_1} = \{k'\}$ for some distinct $k_0, k_1 \in K$, then k' "appears" at least twice in $\sum_{k \in K} K'_k$ and the latter is not necessarily isomorphic to a suborder of K'. This observation allows us to show that the classes considered in Example 3.1.6 are not ccs, and hence there is no ccs class between $\mathcal{L}_{\preceq \mathbf{1}}$ and Fin.

Example 3.1.8. Every class $\mathcal{L}_{\preceq \mathbf{n}}$ with $n > 1$ is not ccs. Indeed, it is enough to consider $K = \mathbf{2}$ and $K' = \mathbf{n}$ and define $K'_0 = \mathbf{n}$ and $K'_1 = \{n-1\}$ to obtain that $\sum_{k \in K} K'_k = K'_0 + K'_1 \cong \mathbf{n} + \mathbf{1}$ does not belong to $\mathcal{L}_{\preceq \mathbf{n}}$.

We now show that the ccs property is not only sufficient to obtain the transitivity of $\trianglelefteq^{\mathcal{L}}$, but it is also necessary, and thus characterizes those $\mathcal{L} \subseteq \mathsf{Lin}$ for which $\trianglelefteq^{\mathcal{L}}$ is a quasi-order.

Theorem 3.1.9. *Let $\mathcal{L} \subseteq \mathsf{Lin}$ be nonempty and downward $\preceq$-closed. Then the following are equivalent:*

(i) *$\mathcal{L}$ is ccs;*

(ii) *$\trianglelefteq^{\mathcal{L}}$ is transitive;*

(iii) *$\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ is transitive.*

Proof. We already showed that (i) $\Rightarrow$ (ii) in the discussion preceding Definition 3.1.7, while (ii) $\Rightarrow$ (iii) is obvious, so let us prove (iii) $\Rightarrow$ (i). If $\mathcal{L} = \mathsf{Lin}$ or $\mathcal{L} = \mathcal{L}_{\preceq \mathbf{1}}$ then $\mathcal{L}$ is trivially ccs, while if $\mathcal{L} \neq \mathcal{L}_{\preceq \mathbf{1}}$ but $\mathcal{L}$ does not contain all finite linear orders, then $\mathcal{L} = \mathcal{L}_{\preceq \mathbf{n}}$ for some $n > 1$, and hence $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ is not transitive by Example 3.1.6. We can thus assume without loss of generality that $\mathcal{L}$ is such that $\mathsf{Fin} \subseteq \mathcal{L} \subseteq \mathsf{Scat}$.

Suppose that $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ is transitive: given $K, K' \in \mathcal{L}$ and $(K'_k)_{k \in K}$ such that $\emptyset \neq K'_k \sqsubseteq K'$ and $\forall k_0, k_1 \in K$ $(k_0 <_K k_1 \Rightarrow K'_{k_0} \leq_{K'} K'_{k_1})$, we want to show that $\sum_{k \in K} K'_k \in \mathcal{L}$. If $\sum_{k \in K} K'_k$ is finite then it belongs to $\mathcal{L}$ by $\mathsf{Fin} \subseteq \mathcal{L}$, hence we can further assume that $L = \sum_{k \in K} K'_k$ is infinite, i.e. $L \in \mathsf{LO}$. Let $L' = \sum_{k \in K}(K'_k + Q_k)$, where

$$Q_k = \begin{cases} \emptyset & \text{if } K'_k \cap K'_j = \emptyset \text{ for all } k <_K j \\ \eta & \text{otherwise.} \end{cases}$$

Then $L' \in \mathsf{LO}$ as well, and we claim that $L' \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} \eta$. To see this, let $K'' = \bigcup_{k \in K} K'_k \subseteq K'$ (notice that in general this is **not** a disjoint union), so that $K'' \in \mathcal{L}$ because the latter is downward $\preceq$-closed. For each $k' \in K''$ let $A_{k'} = \{k \in K \mid k' \in K'_k\}$ and let $L'_{k'}$ be the L'-convex closure of $\{(k', k) \mid k \in A_{k'}\}$. Then $L'_{k'} \sqsubseteq L'$ is of the form $\sum_{k \in A_{k'}}(\mathbf{1} + Q_k)$, where the singleton $\mathbf{1}$ in the k-th summand is the point $\{(k', k)\}$, and thus it has order type $\mathbf{1}$ (if $A_{k'}$ is a singleton), or one of η, $\mathbf{1} + \eta$, $\eta + \mathbf{1}$, $\mathbf{1} + \eta + \mathbf{1}$ (if $A_{k'}$ is not a singleton, the four cases depending on whether $A_{k'}$ has a minimum or a maximum). It is easy to verify that $(L'_{k'})_{k' \in K''}$ is a K''-convex partition of L', and since $L'_{k'} \trianglelefteq \eta$ because of its order type (Remark 1.2.4), it is easy to recursively construct an embedding $f \colon L' \to \eta$ which, together with $K'' \in \mathcal{L}$ and $(L'_{k'})_{k' \in K''}$, witnesses $L' \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} \eta$.

Clearly $L \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} L'$, as witnessed by $K \in \mathcal{L}$ and $(K'_k)_{k \in K}$ themselves, hence by transitivity of $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ we get $L \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} \eta$. But $L \in \mathsf{Scat}$ because it is a scattered sum of scattered linear orders (see [Ros82, Proposition 2.17]), thus $L \in \mathcal{L}$ by Claim 3.1.4. □

We conclude this section with a couple of technical results that will be useful later on. Although we will apply them only when $\mathcal{L}$ is ccs, we prove them in full generality. A subset $A \subseteq M$ of a linear order M is **inherently cofinal** if for every embedding $f \colon A \to M$ the image of $f(A)$ is cofinal in M. Notice that if M is either ζ or an infinite cardinal κ, then every tail $[m_0, +\infty)_M$ of M is inherently cofinal. The following proposition was already proved in Proposition 2.2.2 for the special case $M = \zeta$ and $\mathcal{L} = \mathcal{L}_{\preceq \mathbf{1}}$.

Proposition 3.1.10. *Suppose that the linear order M has an inherently cofinal tail $[m_0, +\infty)_M$. Then for every downward $\preceq$-closed $\mathcal{L} \subseteq \mathsf{Lin}$ and all linear orders L and L' we have $ML \trianglelefteq^{\mathcal{L}} ML'$ if and only if $L \trianglelefteq^{\mathcal{L}} L'$.*

Proof. For the nontrivial direction, suppose that $ML \trianglelefteq^{\mathcal{L}} ML'$ as witnessed by $K \in \mathcal{L}$, the K-convex partition $(L_k)_{k \in K}$ of ML and $f \colon ML \to ML'$. For every $k \in K$, let $\tilde{L}_k = \{\ell \in L \mid (m_0, \ell) \in L_k\}$. Let also $\tilde{K} = \{k \in K \mid \tilde{L}_k \neq \emptyset\}$, so that $\tilde{K} \in \mathcal{L}$ because the latter is downward $\preceq$-closed. Define the map $g \colon L \to L'$ by setting $g(\ell) = \ell'$ if and only if $\ell' \in L'$ is such that $f(m_0, \ell) \in M \times \{\ell'\}$. We claim that $\tilde{K}$, $(\tilde{L}_k)_{k \in \tilde{K}}$ and $g \colon L \to L'$ witness $L \trianglelefteq^{\mathcal{L}} L'$.

It is easy to see that $(\tilde{L}_k)_{k \in \tilde{K}}$ is a $\tilde{K}$-convex partition of L, and that g is order-preserving since f was. To see that g is also injective, consider any $\ell_0, \ell_1 \in L$ with $\ell_0 <_L \ell_1$. If $g(\ell_0) = g(\ell_1)$, then $f([m_0, +\infty)_M \times \{\ell_0\})$ would be a non-cofinal subset of $M \times \{g(\ell_0)\}$ (as witnessed by $f(m_0, \ell_1)$), contradicting the fact that $[m_0, +\infty)_M$ was inherently cofinal in M. This shows that g is an embedding. It remains to show that $g(\tilde{L}_k) \sqsubseteq L'$ for all $k \in \tilde{K}$. Fix $\ell_0, \ell_1 \in \tilde{L}_k$ and $\ell' \in L'$ such that $g(\ell_0) <_{L'} \ell' <_{L'} g(\ell_1)$: our goal is to show that $\ell' = g(\ell)$ for some $\ell \in \tilde{L}_k$. Since $f(L_k) \sqsubseteq ML'$, there is $\ell \in [\ell_0, \ell_1]_L \sqsubseteq \tilde{L}_k$ such that $f^{-1}(m_0, \ell') \in M \times \{\ell\}$: we claim that $g(\ell) = \ell'$. Suppose towards a contradiction that $\ell' <_{L'} g(\ell)$, which together with $\ell_0 \leq_L \ell \leq_L \ell_1$ implies $(m_0, \ell_0) \leq_L f^{-1}(m_0, \ell') <_L (m_0, \ell) \leq_L (m_0, \ell_1)$. Since $[(m_0, \ell_0), (m_0, \ell_1)]_L \sqsubseteq L_k$ and $f(L_k) \sqsubseteq ML'$, we get that $f^{-1} \restriction ([m_0, +\infty)_M \times \{\ell'\})$ is a well-defined embedding of $[m_0, +\infty)_M \times \{\ell'\}$ into $M \times \{\ell\}$ with a non-cofinal range (as witnessed by (m_0, ℓ)), against the fact that $[m_0, +\infty)_M$ was inherently cofinal. The case $g(\ell) <_{L'} \ell'$ is symmetric: in this case the range of the embedding obtained by restricting f^{-1} to $[m_0, +\infty)_M \times \{g(\ell)\}$ would not be cofinal in $M \times \{\ell\}$ (as witnessed by $f^{-1}(m_0, \ell')$), a contradiction. Therefore we must conclude that $g(\ell) = \ell'$, as desired. □

The next result plays a crucial role in transferring some of the properties of $\trianglelefteq_{\mathsf{LO}}$ uncovered in Chapter 2 to the more general context of an arbitrary $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$.

Proposition 3.1.11. *Let $\mathcal{L} \subseteq \mathsf{Lin}$ be downward $\preceq$-closed, and let L, L' and M be linear orders with $M \notin \mathcal{L}$. If $LM \trianglelefteq^{\mathcal{L}} L'$ then $L \trianglelefteq L'$.*

Proof. Suppose that $K \in \mathcal{L}$, $(L_k)_{k \in K}$ and $f \colon LM \to L'$ witness $LM \trianglelefteq^{\mathcal{L}} L'$. For each $m \in M$ set $K_m = \{k \in K \mid L_k \cap (L \times \{m\}) \neq \emptyset\}$. If one of the sets K_m is a singleton $\{k\}$, then $L \times \{m\} \subseteq L_k$, hence $L \cong L \times \{m\} \sqsubseteq L_k \trianglelefteq L'$ and we are done. Otherwise each K_m has at least two elements. In particular, this entails that $K_{m_0} \leq_K K_{m_1}$ if and only if $m_0 <_M m_1$. Now define $g \colon M \to K$ by letting $g(m)$ be an element of K_m distinct from its maximum (if the latter exists). It is easy to see that g is an embedding, which is against the hypothesis $M \notin \mathcal{L}$ because $K \in \mathcal{L}$. □

Corollary 3.1.12. *Let $\mathcal{L} \subseteq \mathsf{Scat}$ be downward $\preceq$-closed. For all linear orders L and L', we have that $L\eta \trianglelefteq^{\mathcal{L}} L'$ if and only if $L\eta \trianglelefteq L'$.*

Proof. Since $\eta\eta \cong \eta$, if $L\eta \trianglelefteq^{\mathcal{L}} L'$ then also $(L\eta)\eta \trianglelefteq^{\mathcal{L}} L'$, hence $L\eta \trianglelefteq L'$ by Proposition 3.1.11. The other direction is trivial. □

3.2 Combinatorial properties of $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$

In this section, we explore the combinatorial properties of $\mathcal{L}$-convex embeddability on countable linear orders. We always assume that $\mathcal{L}$ is downward $\preceq$-closed and ccs. Actually, the ccs hypothesis is never used in our proofs but, since we employ the usual terminology for the combinatorial properties of quasi-orders, it is natural to assume that $(\mathsf{LO}, \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$ is indeed a quasi-order (if $\mathcal{L}$ is not ccs we could view $(\mathsf{LO}, \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$ as an oriented graph and speak e.g. of independent sets instead of antichains).

We exclude from our analysis the case $\mathcal{L} = \mathsf{Lin}$ because $\trianglelefteq_{\mathsf{LO}}^{\mathsf{Lin}}$ coincides with embeddability on LO, whose combinatorial properties are well known. We thus usually assume $\eta \notin \mathcal{L}$, that is, $\mathcal{L} \subseteq \mathsf{Scat}$.

As in Section 2.2, the following standard construction of linear orders η^f in which one replaces each $q \in \mathbb{Q}$ with the linear order $f(q)$ plays a central role. Notice that each η^f is not scattered and contains a copy of $\mathbb{Q}$ which is both coinitial and cofinal. Actually, it follows from a classic result of Hausdorff (see e.g. [Ros82, Theorem 4.9]) that every countable linear order which has no scattered initial and final sets is of the form η^f for some $f \colon \mathbb{Q} \to \mathsf{Scat}$.

Definition 3.2.1. Given a map $f \colon \mathbb{Q} \to \mathsf{Lin}$, let η^f the linear order $\sum_{q\in\mathbb{Q}} f(q)$, i.e. the set $\{(\ell, q) \mid q \in \mathbb{Q} \text{ and } \ell \in f(q)\}$ ordered antilexicographically. When $A \subseteq \mathbb{Q}$ we let $\eta^f_A = \sum_{q\in A} f(q)$ be the restriction of η^f to $\{(\ell, q) \in \eta^f \mid q \in A\}$; when $I \subseteq \mathbb{R}$, with a minor abuse of notation we write η^f_I in place of $\eta^f_{I\cap\mathbb{Q}}$.

A crucial property of linear orders of the form η^f is that if $L \sqsubseteq \eta^f$ is such that its projection on the second coordinate has more than one element, then $L \notin \mathsf{Scat}$.

Lemma 3.2.2. (a) *Let $f \colon \mathbb{Q} \to \mathsf{Lin}$, $K \in \mathsf{Scat}$, and let $(L_k)_{k\in K}$ be a K-convex partition of η^f. Then there exist $k \in K$ and $q_0, q_1 \in \mathbb{Q}$ with $q_0 < q_1$ such that $\eta^f_{(q_0,q_1)} \sqsubseteq L_k$. The same applies if we start from a partition of $\eta^f_{(r_0,r_1)}$ for any $r_0, r_1 \in \mathbb{R}$ with $r_0 < r_1$.*

(b) *Let $f \colon \mathbb{Q} \to \mathsf{Lin}$ be such that $f^{-1}(L)$ is dense in $\mathbb{Q}$ for every $L \in f(\mathbb{Q})$. Then $\eta^f_{(r_0,r_1)} \cong_{\mathsf{LO}} \eta^f$ for every $r_0, r_1 \in \mathbb{R}$ with $r_0 < r_1$. Moreover, for every $\mathcal{L} \subseteq \mathsf{Scat}$ and every $L \in \mathsf{Lin}$, we have $\eta^f \trianglelefteq_{\mathsf{LO}}^{\mathcal{L}} L$ if and only if $\eta^f \trianglelefteq_{\mathsf{LO}} L$.*

(c) *Let $f_0, f_1 \colon \mathbb{Q} \to \mathsf{Scat}$, and let $h \colon \eta^{f_0} \to \eta^{f_1}$ witness $\eta^{f_0} \trianglelefteq_{\mathsf{LO}} \eta^{f_1}$. Then there are $r_0, r_1 \in \mathbb{R} \cup \{-\infty, +\infty\}$ with $r_0 < r_1$ and an order-preserving bijection $g \colon \mathbb{Q} \to (r_0, r_1) \cap \mathbb{Q}$ such that $h(f_0(q) \times \{q\}) = f_1(g(q)) \times \{g(q)\}$ for all $q \in \mathbb{Q}$. In particular, for every $q \in \mathbb{Q}$ there is $q' \in (r_0, r_1) \cap \mathbb{Q}$ such that $f_0(q) \cong f_1(q')$.*

(d) *Let $\mathcal{L} \subseteq \mathsf{Scat}$ and $f_0, f_1 \colon \mathbb{Q} \to \mathsf{Scat}$ be as in part (b). Then $\eta^{f_0} \trianglelefteq_{\mathsf{LO}}^{\mathcal{L}} \eta^{f_1} \iff \eta^{f_0} \trianglelefteq_{\mathsf{LO}} \eta^{f_1} \iff \eta^{f_0} \cong_{\mathsf{LO}} \eta^{f_1} \iff f_0(\mathbb{Q})$ and $f_1(\mathbb{Q})$ are the same up to isomorphism.*

Proof. (a) Fix $m_q \in f(q)$ for each $q \in \mathbb{Q}$, and for every $k \in K$ let L'_k be the projection of L_k on its second coordinate. Then each L'_k is convex (in $\mathbb{Q}$) because $L_k \sqsubseteq \eta^f$. If every L'_k were a singleton, then $\mathbb{Q} \preceq K$ via the map sending $q \in \mathbb{Q}$ to the unique $k \in K$ such that $(m_q, q) \in L_k$. This is impossible because $K \in \mathsf{Scat}$, hence by convexity there are $k \in K$ and $q_0, q_1 \in \mathbb{Q}$ such that $q_0 < q_1$ and $[q_0, q_1]_{\mathbb{Q}} \sqsubseteq L'_k$. Thus $\eta^f_{(q_0,q_1)} \sqsubseteq L_k$, as required. The additional part follows by the simple observation that $\eta^f_{(r_0,r_1)} \cong \eta^{f'}$ for $f' = f \circ h$ and $h \colon \mathbb{Q} \to (r_0, r_1) \cap \mathbb{Q}$ an order-preserving bijection.

(b) Use a back-and-forth argument to find an order-preserving bijection $g \colon (r_0, r_1) \cap \mathbb{Q} \to \mathbb{Q}$ such that $f(q) = f(g(q))$ for all $q \in (r_0, r_1) \cap \mathbb{Q}$ — this can be done by the hypothesis on f. Then the map sending (ℓ, q) to $(\ell, g(q))$ is the desired isomorphism. For the non trivial implication of the additional part, assume that $\eta^f \trianglelefteq_{\mathsf{LO}}^{\mathcal{L}} L$ as witnessed by $K \in \mathcal{L}$ and the K-convex partition of η^f. By part (a) there are $k \in K$ and $q_0 < q_1$ such that $\eta^f_{(q_0,q_1)} \sqsubseteq L_k$, hence $\eta^f \cong_{\mathsf{LO}} \eta^f_{(q_0,q_1)} \trianglelefteq_{\mathsf{LO}} L$ and we are done.

(c) Since $h(\eta^{f_0}) \sqsubseteq \eta^{f_1}$, its projection I on its second coordinate is $\mathbb{Q}$-convex: set $r_0 = \inf I$ and $r_1 = \sup I$. Fix an arbitrary $q \in \mathbb{Q}$. If the projection on the second coordinate of $h(f_0(q) \times \{q\})$ was not a singleton, then $h(f_0(q) \times \{q\})$ would be non-scattered, which is impossible because $h(f_0(q) \times \{q\}) \cong f_0(q) \times \{q\} \cong f_0(q)$ and the latter belongs to Scat. Therefore the map $g \colon \mathbb{Q} \to (r_0, r_1) \cap \mathbb{Q}$ sending $q \in \mathbb{Q}$ to the unique q' such that $h(f_0(q) \times \{q\}) \subseteq f_1(q') \times \{q'\}$ is a well-defined surjection, and it is order-preserving since h was. Moreover, it is also injective: if $q_0 < q_1$ were such that $g(q_0) = g(q_1)$, then $h \restriction \eta^{f_0}_{(q_0,q_1)}$ would be an embedding sending the non-scattered linear order $\eta^{f_0}_{(q_0,q_1)}$ into $f_1(g(q_0)) \times \{g(q_0)\} \in \mathsf{Scat}$, a contradiction. Thus g is an order-preserving bijection such that $h(f_0(q) \times \{q\}) \subseteq f_1(g(q)) \times \{g(q)\}$ for every $q \in \mathbb{Q}$, so we only need to show that

$h(f_0(q) \times \{q\}) = f_1(g(q)) \times \{g(q)\}$. If not, since $f_1(g(q)) \times \{g(q)\} \sqsupseteq h(\eta^{f_0})$ there would be $q' \neq q$ such that $h(f_0(q') \times \{q'\}) \cap (f_1(g(q)) \times \{g(q)\}) \neq \emptyset$, hence $h(f_0(q') \times \{q'\}) \subseteq f_1(g(q)) \times \{g(q)\}$ and by definition $g(q') = g(q)$, against injectivity of g.

(d) If $f_0(\mathbb{Q})$ and $f_1(\mathbb{Q})$ contain the same linear orders up to isomorphism, then using a back-and-forth argument as in part (b) one can easily show that $\eta^{f_0} \cong_{\mathsf{LO}} \eta^{f_1}$; this implies $\eta^{f_0} \trianglelefteq_{\mathsf{LO}} \eta^{f_1}$, which in turn implies $\eta^{f_0} \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} \eta^{f_1}$. So we only need to show that if $\eta^{f_0} \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} \eta^{f_1}$, then for every $q \in \mathbb{Q}$ there is $q' \in \mathbb{Q}$ such that $f_0(q) \cong f_1(q')$, and vice versa. Fix $K \in \mathcal{L}$, a K-convex partition $(L_k)_{k \in K}$ of η^{f_0} and an embedding $h \colon \eta^{f_0} \to \eta^{f_1}$ witnessing $\eta^{f_0} \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} \eta^{f_1}$. By part (a) there are $k \in K$ and $q_0 < q_1$ such that $\eta^{f_0}_{(q_0,q_1)} \sqsubseteq L_k$, and $\eta^{f_0}_{(q_0,q_1)} \cong_{\mathsf{LO}} \eta^{f_0}$ by part (b). Thus $\eta^{f_0} \trianglelefteq_{\mathsf{LO}} \eta^{f_1}$ and we can find $r_0, r_1 \in \mathbb{R}$ and g as in part (c). Then for every $q \in \mathbb{Q}$ there is $q' \in \mathbb{Q}$ such that $f_0(q) \cong f_1(q')$. Conversely, given any $q' \in \mathbb{Q}$ there is $q'' \in (r_0, r_1) \cap \mathbb{Q}$ such that $f_1(q'') = f_1(q')$ (because by hypothesis $f_1^{-1}(f_1(q')))$ is dense in $\mathbb{Q}$), and hence $q = g^{-1}(q')$ is such that $f_0(q) \cong f_1(q'') = f_1(q')$, as desired. □

The following lemma generalizes Lemma 2.2.3.

Lemma 3.2.3. *For every ccs $\mathcal{L} \subseteq \mathsf{Scat}$, there is an embedding of $(\mathrm{Int}(\mathbb{R}), \subseteq)$ into $(\mathsf{LO}, \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$.*

Proof. Let $f \colon \mathbb{Q} \to \{\mathbf{n} \mid n \in \mathbb{N} \setminus \{0\}\}$ be injective: we claim that the map which sends the interval $(x, y) \in \mathrm{Int}(\mathbb{R})$ to the linear order $\eta^f_{(x,y)} \in \mathsf{LO}$ from Definition 3.2.1 is the desired embedding.

If $(x, y) \subseteq (x', y')$, then $\eta^f_{(x,y)} \sqsubseteq \eta^f_{(x',y')}$, and thus $\eta^f_{(x,y)} \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} \eta^f_{(x',y')}$. Vice versa, let (x, y) and (x', y') be elements of $\mathrm{Int}(\mathbb{R})$ and such that $(x, y) \nsubseteq (x', y')$. Towards a contradiction, suppose that $\eta^f_{(x,y)} \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} \eta^f_{(x',y')}$. Consider the restriction $\eta^f_{(r_0,r_1)}$ of $\eta^f_{(x,y)}$, where (r_0, r_1) is a nonempty open interval contained in $(x, y) \setminus (x', y')$, so that $\eta^f_{(r_0,r_1)} \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} \eta^f_{(x',y')}$ because $\eta^f_{(r_0,r_1)} \trianglelefteq_{\mathsf{LO}} \eta^f_{(x,y)}$. Fix a K-convex partition $(L_k)_{k \in K}$ of $\eta^f_{(r_0,r_1)}$ witnessing $\eta^f_{(r_0,r_1)} \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} \eta^f_{(x',y')}$, for some $K \in \mathcal{L} \subseteq \mathsf{Scat}$. By Lemma 3.2.2(a) there exist $k \in K$ and $q_0, q_1 \in \mathbb{Q}$ with $r_0 \leq q_0 < q_1 \leq r_1$ such that $\eta^f_{(q_0,q_1)} \sqsubseteq L_k$. Hence $\eta^f_{(q_0,q_1)} \trianglelefteq_{\mathsf{LO}} \eta^f_{(x',y')}$, and using the fact that $\eta^f_{(q_0,q_1)} \cong \eta^{f'}$ for a suitable $f' \colon \mathbb{Q} \to \mathsf{Scat}$, we can apply Lemma 3.2.2(c) and get that for any $q_0 < q < q_1$ there is $x' < q' < y'$ such that $f(q) \cong f(q')$. But this contradicts the injectivity of f, as $q \neq q'$ because $(q_0, q_1) \cap (x', y') = \emptyset$. □

Theorem 3.2.4. *For every ccs $\mathcal{L} \subseteq \mathsf{Scat}$, there are chains of order type $(\mathbb{R}, <)$ and antichains of size $2^{\aleph_0}$ in $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$.*

Proof. By Lemma 3.2.3 the family $\left\{\eta^f_{(0,x)} \mid x > 0\right\}$ is a chain of order type $(\mathbb{R}, <)$, while $\left\{\eta^f_{(x,x+1)} \mid x \in \mathbb{R}\right\}$ is an antichain of size the continuum. Alternatively, to build a large antichain we can fix a family $(L_\alpha)_{\alpha < 2^{\aleph_0}}$ of pairwise non-isomorphic scattered linear orders and notice that if $f_\alpha \colon \mathbb{Q} \to \mathsf{Scat}$ is the constant function with value L_α, then by Lemma 3.2.2(d) the family $\mathcal{A} = \{\eta^{f_\alpha} \mid \alpha < 2^{\aleph_0}\}$ is a $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-antichain. □

We now show that the dominating number $\mathfrak{d}(\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$ of $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ (Definition 1.2.5) is as large as possible.

Theorem 3.2.5. *For every ccs $\mathcal{L} \subseteq \mathsf{Scat}$, the quasi-order $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ does not have maximal elements, and every dominating family with respect to $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ has size $2^{\aleph_0}$. Thus $\mathfrak{d}(\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}) = 2^{\aleph_0}$.*

Proof. Let $L \in \mathsf{LO}$. Corollary 2.2.7 there exists L' such that $L \triangleleft_{\mathsf{LO}} L'$. Thus using Proposition 3.1.11 we have $L \triangleleft^{\mathcal{L}}_{\mathsf{LO}} L'\eta$ and L is not $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-maximal.

Let now $\mathcal{F}$ be a dominating family with respect to $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$: we claim that $\mathcal{F}$ is also a dominating family with respect to $\trianglelefteq_{\mathsf{LO}}$, so that $|\mathcal{F}| = 2^{\aleph_0}$ by Proposition 2.2.10. Fix an arbitrary $L \in \mathsf{LO}$. Since $\mathcal{F}$ is $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-dominating, there is $L' \in \mathcal{F}$ such that $L\eta \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} L'$. But then $L \trianglelefteq_{\mathsf{LO}} L'$ by Proposition 3.1.11, hence we are done. □

We now look at bases and minimal elements in LO with respect to $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$. Recall that by Proposition 2.2.8, if $\mathcal{L} = \{\mathbf{1}\}$ then there are $2^{\aleph_0}$-many $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-incomparable $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-minimal elements. In contrast, the following result extends to most ccs classes $\mathcal{L}$ a basic fact about $\preceq_{\mathsf{LO}}$.

Theorem 3.2.6. *For every ccs $\mathcal{L} \subseteq \mathsf{Lin}$, if either $\boldsymbol{\omega}^* \in \mathcal{L}$ or $\boldsymbol{\omega} \in \mathcal{L}$ then $\{\boldsymbol{\omega}, \boldsymbol{\omega}^*\}$ is a basis for $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$.*

Proof. Assume that $\boldsymbol{\omega}^* \in \mathcal{L}$. By Fact 3.1.3 we have that $\boldsymbol{\omega}^* \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} L$ for every ill-founded $L \in \mathsf{LO}$. On the other hand, if $L \in \mathsf{WO}$ then trivially $\boldsymbol{\omega} \trianglelefteq_{\mathsf{LO}} L$, and hence $\boldsymbol{\omega} \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} L$. The case when $\boldsymbol{\omega} \in \mathcal{L}$ is symmetric. □

Since $\mathcal{L}$ is downward $\preceq$-closed, if $\mathcal{L}$ contains at least an infinite linear order then Theorem 3.2.6 applies and $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ has a basis of size 2. It thus remains to consider families $\mathcal{L}$ such that $\mathcal{L} \subseteq \mathsf{Fin}$, which by the ccs property amounts to $\mathcal{L} = \mathcal{L}_{\preceq \mathbf{1}}$ or $\mathcal{L} = \mathsf{Fin}$. In this case, we can reproduce the result obtained for $\trianglelefteq_{\mathsf{LO}}$ in Proposition 2.2.8 and show that there are $2^{\aleph_0}$-many $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-incomparable $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-minimal elements. To motivate the next technical result, notice that by Fact 3.1.3 the relation $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ coincides with embeddability on $\mathcal{L}$, so that all $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-antichains have finite intersection with $\mathcal{L}$. Therefore, in order to find infinite antichains (of minimal elements) we have to search in $\mathsf{LO} \setminus \mathcal{L}$.

For every infinite $S \subseteq \mathbb{N} \setminus \{0\}$, fix a surjective map $f_S \colon \mathbb{Q} \to \{\mathbf{n} \mid n \in S\}$ such that $f_S^{-1}(\mathbf{n})$ is dense for every $n \in S$.

Proposition 3.2.7. *Let $S, S' \subseteq \mathbb{N} \setminus \{0\}$ be infinite, and consider any ccs $\mathcal{L} \subseteq \mathsf{Scat}$.*

(a) *If $S \neq S'$, then $\eta^{f_S} \ntrianglelefteq^{\mathcal{L}}_{\mathsf{LO}} \eta^{f_{S'}}$.*

(b) *η^{f_S} is $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-minimal in $\mathsf{LO} \setminus \mathcal{L}$ if and only if the following condition holds:*

$$\text{If } K \in \mathcal{L} \cap \mathsf{LO} \text{ and } L_k \in \mathsf{Fin} \text{ for all } k \in K, \text{ then } \sum_{k \in K} L_k \in \mathcal{L}. \qquad (\star)$$

Proof. (a) This is just an application of Lemma 3.2.2(d).

(b) If $L = \sum_{k \in K} L_k$ witnesses the failure of ($\star$), then $L \in \mathsf{LO} \setminus \mathcal{L}$ is such that $L \triangleleft^{\mathcal{L}}_{\mathsf{LO}} \eta^{f_S}$ and hence η^{f_S} is not $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-minimal over $\mathsf{LO} \setminus \mathcal{L}$. To see this, find an embedding $g \colon K \to \mathbb{Q}$ such that $f_S(g(k)) \geq |L_k|$ for all $k \in K$ (this is possible because each $f_S^{-1}(\mathbf{n})$ is dense in $\mathbb{Q}$), and then lift it to an embedding $h \colon L \to \eta^{f_S}$ sending $L_k \times \{k\}$ into $f_S(g(k)) \times \{g(k)\}$ in the obvious way. Then $K \in \mathcal{L}$, $(L_k)_{k \in K}$ and h witness that $L \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} \eta^{f_S}$. On the other hand, $\eta^{f_S} \ntrianglelefteq^{\mathcal{L}}_{\mathsf{LO}} L$ because $L \in \mathsf{Scat}$ while $\eta^{f_S} \in \mathsf{LO} \setminus \mathsf{Scat}$, hence there is no embedding at all from η^{f_S} to L.

Assume now that condition ($\star$) holds and that $L \in \mathsf{LO} \setminus \mathcal{L}$ is such that $L \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} \eta^{f_S}$, as witnessed by $K \in \mathcal{L}$, $(L_k)_{k \in K}$ and $h \colon L \to \eta^{f_S}$. By ($\star$) and $L \notin \mathcal{L}$ there is some $k \in K$ for which L_k is infinite. But then $h(L_k)$ is an infinite convex subset of η^{f_S}, which means that $\eta^{f_S}_{(q_0,q_1)} \sqsubseteq h(L_k)$ for some $q_0 < q_1$, and hence $\eta^{f_S}_{(q_0,q_1)} \trianglelefteq_{\mathsf{LO}} L$ via h^{-1}. Since $\eta^{f_S}_{(q_0,q_1)} \cong \eta^{f_S}$ by Lemma 3.2.2(b), it follows that $\eta^{f_S} \trianglelefteq_{\mathsf{LO}} L$, and thus also $\eta^{f_S} \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} L$. This proves that there is no $L \in \mathsf{LO} \setminus \mathcal{L}$ such that $L \triangleleft^{\mathcal{L}}_{\mathsf{LO}} \eta^{f_S}$, as desired. □

Albeit artificial, condition ($\star$) is satisfied by $\{\mathbf{1}\}$, Fin, WO, Scat, and all other examples of ccs families from Section 3.4. Indeed, we do not know if ($\star$) is actually satisfied by *all* ccs families $\mathcal{L} \subseteq \mathsf{Scat}$.

Theorem 3.2.8. *For any ccs $\mathcal{L} \subseteq \mathsf{Fin}$ there are $2^{\aleph_0}$-many $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-incomparable $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-minimal elements in LO. Thus every basis for $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ has cardinality $2^{\aleph_0}$.*

Proof. Since $\mathsf{LO} \cap \mathsf{Fin} = \emptyset$, condition ($\star$) of Proposition 3.2.7(b) is trivially satisfied and $\mathsf{LO} \setminus \mathcal{L} = \mathsf{LO}$. Therefore by Proposition 3.2.7 the family $\mathcal{B} = \left\{\eta^{f_S} \mid S \subseteq \mathbb{N} \setminus \{0\} \text{ is infinite}\right\}$ is the desired $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-antichain of $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-minimal elements. □

Proposition 3.2.9. *Consider any ccs $\mathcal{L} \subseteq \mathsf{Scat}$ satisfying ($\star$) of Proposition 3.2.7(b). Then any $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-antichain of size less than $2^{\aleph_0}$ contained in $\mathsf{LO} \setminus \mathcal{L}$ can be extended to a $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-antichain of size $2^{\aleph_0}$ also contained in $\mathsf{LO} \setminus \mathcal{L}$. In particular for every $L \in \mathsf{LO} \setminus \mathcal{L}$ there is $M \in \mathsf{LO} \setminus \mathcal{L}$ which is $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-incomparable with L, and indeed L belongs to a $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-antichain of size $2^{\aleph_0}$.*

Proof. Let $\mathcal{A} = \{L_\alpha \mid \alpha < \kappa\}$ be a $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-antichain of size $\kappa < 2^{\aleph_0}$ with $L_\alpha \notin \mathcal{L}$ for all $\alpha < \kappa$, and let $\mathcal{B}$ be the $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-antichain of size $2^{\aleph_0}$ from the proof of Theorem 3.2.8. Given $\alpha < \kappa$, consider the set $\mathcal{B}_\alpha = \{\eta^{fs} \in \mathcal{B} \mid \eta^{fs} \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} L_\alpha\}$. By Lemma 3.2.2(b), if $\eta^{fs} \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} L_\alpha$ then also $\eta^{fs} \trianglelefteq_{\mathsf{LO}} L_\alpha$, thus $\mathcal{B}_\alpha = \{L \in \mathcal{B} \mid L \trianglelefteq_{\mathsf{LO}} L_\alpha\}$ and so $\mathcal{B}_\alpha$ is countable by Claim 2.2.9.1. Therefore $\bigcup_{\alpha<\kappa} \mathcal{B}_\alpha$ has size at most $\max\{\kappa, \aleph_0\}$. From this and $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-minimality over $\mathsf{LO} \setminus \mathcal{L}$ of the linear orders η^{fs} (Proposition 3.2.7(b)), it then follows that $\mathcal{A} \cup (\mathcal{B} \setminus \bigcup_{\alpha<\kappa} \mathcal{B}_\alpha)$ is the desired $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-antichain of size $2^{\aleph_0}$ extending $\mathcal{A}$. □

Corollary 3.2.10. *For every ccs $\mathcal{L} \subseteq \mathsf{Fin}$ there are no maximal $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-antichains of size smaller than $2^{\aleph_0}$.*

Corollary 3.2.11. *All maximal $\trianglelefteq^{\mathsf{Scat}}_{\mathsf{LO}}$-antichains $\mathcal{A}$ are either finite or of size $2^{\aleph_0}$. More precisely:*

(a) *If $\mathcal{A} \cap \mathsf{Scat} \neq \emptyset$, then $\mathcal{A} \subseteq \mathsf{Scat}$ and $\mathcal{A}$ is also an antichain with respect to $\preceq$, hence it is finite.*

(b) *If $\mathcal{A} \cap \mathsf{Scat} = \emptyset$, then $|\mathcal{A}| = 2^{\aleph_0}$.*

Thus there is no countably infinite maximal $\trianglelefteq^{\mathsf{Scat}}_{\mathsf{LO}}$-antichain.

Proof. (a) Let $L \in \mathcal{A} \cap \mathsf{Scat}$. If $L' \notin \mathsf{Scat}$, then $L \trianglelefteq^{\mathsf{Scat}}_{\mathsf{LO}} L'$ by Fact 3.1.3, hence $\mathcal{A} \subseteq \mathsf{Scat}$. Moreover, on Scat the relations $\trianglelefteq^{\mathsf{Scat}}_{\mathsf{LO}}$ and $\preceq_{\mathsf{LO}}$ coincide by Fact 3.1.3, hence we are done.
(b) Apply Proposition 3.2.9. □

Remark 3.2.12. For an arbitrary $\mathcal{L}$, if an antichain $\mathcal{A}$ intersects $\mathcal{L}$ then it is included in Scat because $L \preceq_{\mathsf{LO}} L'$ whenever $L \in \mathsf{LO}$ and $L' \notin \mathsf{Scat}$. However, in contrast with Corollary 3.2.11.(a), this does not rule out the existence of large $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-antichains of scattered linear orders when $\mathcal{L} \subsetneq \mathsf{Scat}$. For example consider for every $f \in \mathbb{N}^{\mathbb{N}}$ the linear order $L_f = \zeta\omega^* + \sum_{n\in\mathbb{N}}(\zeta + f(n))$; then $L_f \trianglelefteq^{\mathsf{Fin}}_{\mathsf{LO}} L_{f'}$ if and only if $L_f \bowtie^{\mathsf{Fin}}_{\mathsf{LO}} L_{f'}$ if and only if $\exists n, n' \forall i\, f(n+i) = f'(n'+i)$; we thus have a $\trianglelefteq^{\mathsf{Fin}}$-antichain of size $2^{\aleph_0}$ contained in Scat.

Other configurations of maximal antichains are possible as well. For example, $\mathcal{L}_{\preceq\boldsymbol{\omega}}$ is ccs by Proposition 3.4.2, and it is easy to check using Proposition 3.2.9 that every maximal $\trianglelefteq^{\mathcal{L}_{\preceq\omega}}_{\mathsf{LO}}$-antichain either is of the form $\{\boldsymbol{\omega}, \boldsymbol{\alpha}^*\}$ for some infinite $\alpha < \omega_1$, or else has size $2^{\aleph_0}$.

Motivated by Proposition 2.2.5, we now analyse the (un)boundedness of WO in LO with respect to $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$. We have to distinguish two cases.

Proposition 3.2.13. *Consider any ccs $\mathcal{L} \subseteq \mathsf{Lin}$.*

(a) *If $\mathsf{WO} \subseteq \mathcal{L}$, then WO is bounded with respect to $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ in LO.*

(b) *If $\mathsf{WO} \nsubseteq \mathcal{L}$, then WO is unbounded with respect to $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ in LO.*

Proof. (a) By Fact 3.1.3, any upper $\preceq_{\mathsf{LO}}$-bound for WO is also an upper bound with respect to $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$. Thus every non-scattered linear order $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-bounds WO from above.
(b) Let $\beta < \omega_1$ be such that $\boldsymbol{\beta} \notin \mathcal{L}$, and consider any $L \in \mathsf{LO}$. By Proposition 2.2.5 there is $\alpha < \omega_1$ such that $\boldsymbol{\alpha} \ntrianglelefteq_{\mathsf{LO}} L$, hence $\boldsymbol{\alpha\beta} \ntrianglelefteq^{\mathcal{L}}_{\mathsf{LO}} L$ by Proposition 3.1.11. Since $\boldsymbol{\alpha\beta} \in \mathsf{WO}$ and L was arbitrary, this shows that WO is $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-unbounded. □

Using infinite (countable) sums of linear orders, it is immediate to prove that $\mathfrak{b}(\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}) > \aleph_0$. Taking this into account, we show that $\mathfrak{b}(\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$ is as small as possible.

Theorem 3.2.14. *For every ccs $\mathcal{L} \subseteq \mathsf{Scat}$ there exists a family $\mathcal{F}$ of size $\aleph_1$ which is unbounded with respect to $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$. Thus, $\mathfrak{b}(\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}) = \aleph_1$.*

Proof. Let $\mathcal{F} = \{\boldsymbol{\alpha}\eta \mid \alpha < \omega_1\}$. Since $\boldsymbol{\alpha}\eta = \eta^{f_\alpha}$ where $f_\alpha \colon \mathbb{Q} \to \mathsf{Scat}$ is the constant function with value $\boldsymbol{\alpha}$, by Lemma 3.2.2(d) the family $\mathcal{F}$ is a $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-antichain of size $\aleph_1$: we claim that it is $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-unbounded in LO. Indeed, suppose towards a contradiction that $\mathcal{F}$ is $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-bounded from above by some $L \in \mathsf{LO}$. Then $\boldsymbol{\alpha}\eta \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} L$ for every $\alpha < \omega_1$, hence by Proposition 3.1.11 we would have $\boldsymbol{\alpha} \trianglelefteq_{\mathsf{LO}} L$ for every $\alpha < \omega_1$, against Proposition 2.2.5. □

The next result shows that $(\mathsf{LO}, \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$ exhibits a high degree of self-similarity when $\mathcal{L} \neq \mathsf{Lin}$ (the statement obviously fails for $\preceq_{\mathsf{LO}}$). Given $L_0 \in \mathsf{LO}$, we let $L_0{\uparrow}^{\mathcal{L}} = \{L \in \mathsf{LO} \mid L_0 \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} L\}$ be the $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-upper cone above L_0.

Theorem 3.2.15. *For every ccs $\mathcal{L} \subseteq \mathsf{Scat}$, the partial order $(\mathsf{LO}, \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$ has the fractal property with respect to its upper cones, that is, $(\mathsf{LO}, \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$ embeds into $(L_0{\uparrow}^{\mathcal{L}}, \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$ for every $L_0 \in \mathsf{LO}$.*

Proof. Fix $L_0 \in \mathsf{LO}$ and, using Proposition 2.2.5, fix $\alpha < \omega_1$ such that $\boldsymbol{\alpha} \ntrianglelefteq_{\mathsf{LO}} L_0$ (in particular, $\alpha \geq \omega$). Consider the map $\varphi \colon \mathsf{LO} \to L_0{\uparrow}^{\mathcal{L}}$ defined by

$$\varphi(L) = (\boldsymbol{\alpha}\eta_0 + \eta_1 + L_0 + \eta_2)L,$$

where to help the reader we denote by η_j distinct copies of η: we show that φ is an embedding from $(\mathsf{LO}, \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$ to $(L_0{\uparrow}^{\mathcal{L}}, \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$.

Clearly, if $L \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} L'$ via $K \in \mathcal{L}$, the K-convex partition $(L_k)_{k\in K}$ of L and the embedding $g \colon L \to L'$, then K itself, the K-convex partition $(L'_k)_{k\in K}$ of $\varphi(L)$ given by $L'_k = (\boldsymbol{\alpha}\eta_0 + \eta_1 + L_0 + \eta_2)L_k$ and the embedding $h \colon \varphi(L) \to \varphi(L')$ defined by $h(x, \ell) = (x, g(\ell))$ witness that $\varphi(L) \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} \varphi(L')$.

For the other direction, suppose that $\varphi(L) \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} \varphi(L')$ as witnessed by $K \in \mathcal{L}$, the K-convex partition $(M_k)_{k\in K}$ of $\varphi(L)$ and the embedding $h \colon \varphi(L) \to \varphi(L')$. For each $\ell \in L$, consider the partition of $\boldsymbol{\alpha}\eta_0 \times \{\ell\} \sqsubseteq \varphi(L)$ given by the nonempty sets of the form $M_k \cap (\boldsymbol{\alpha}\eta_0 \times \{\ell\})$, which is a K'-convex partition for some $K' \subseteq K \in \mathcal{L} \subseteq \mathsf{Scat}$: since $\boldsymbol{\alpha}\eta_0 \cong \eta^f$ where $f \colon \mathbb{Q} \to \mathsf{Lin}$ is the constant function with value $\boldsymbol{\alpha}$, by Lemma 3.2.2(a) we can choose[2] $N_\ell = \eta^f_{(q_0^{(\ell)}, q_1^{(\ell)})} \times \{\ell\} \cong \boldsymbol{\alpha}\eta_0$ and $k_\ell \in K$ such that $N_\ell \sqsubseteq (\boldsymbol{\alpha}\eta_0 \times \{\ell\}) \cap M_{k_\ell}$, so that $h \restriction N_\ell$ witnesses $N_\ell \trianglelefteq_{\mathsf{LO}} \varphi(L')$. If $h(N_\ell) \cap (\eta_j \times \{\ell'\}) \neq \emptyset$ for some $j \in \{1, 2\}$ and $\ell' \in L'$, then N_ℓ would contain a convex subset with order type η, which is not the case. Therefore either $h(N_\ell) \sqsubseteq \boldsymbol{\alpha}\eta_0 \times \{\ell'\}$ or $h(N_\ell) \sqsubseteq L_0 \times \{\ell'\}$ for some (necessarily unique) $\ell' \in L'$. But $\boldsymbol{\alpha} \trianglelefteq_{\mathsf{LO}} N_\ell$ and $\boldsymbol{\alpha} \ntrianglelefteq_{\mathsf{LO}} L_0$, hence the second possibility cannot hold. This shows that there is a well-defined map $g \colon L \to L'$ such that $h(N_\ell) \sqsubseteq \boldsymbol{\alpha}\eta_0 \times \{g(\ell)\}$ for all $\ell \in L$: we claim that g is an embedding. Indeed, for every $\ell_0, \ell_1 \in L$ we have

$$\ell_0 <_L \ell_1 \iff N_{\ell_0} <_{\varphi(L)} N_{\ell_1} \iff h(N_{\ell_0}) <_{\varphi(L')} h(N_{\ell_1})$$

because h is an embedding. If there were $\ell_0 <_L \ell_1$ such that $g(\ell_0) = g(\ell_1)$, then $h(\eta_1 \times \{\ell_0\}) \subseteq \boldsymbol{\alpha}\eta_0 \times \{g(\ell_0)\}$ because $N_{\ell_0} <_{\varphi(L)} \eta_1 \times \{\ell_0\} <_{\varphi(L)} N_{\ell_1}$. Let $k \in K$ be such that $M_k \cap (\eta_1 \times \{\ell_0\})$ contains an interval $(q_0, q_1) \times \{\ell_0\}$ of $\eta_1 \times \{\ell_0\}$, for some $q_0 < q_1$. (Such a k exists by Lemma 3.2.2(a) applied to $\eta_1 \times \{\ell_0\}$, which is isomorphic to η^f where f the constant function with value $\mathbf{1}$.) Then $h((q_0, q_1) \times \{\ell_0\})$ would be a convex subset of $\boldsymbol{\alpha}\eta_0 \times \{g(\ell_0)\}$ homeomorphic to η, which is clearly impossible because $\alpha > 1$. Thus g is injective, and hence for all $\ell_0, \ell_1 \in L$

$$\ell_0 <_L \ell_1 \iff h(N_{\ell_0}) <_{\varphi(L')} h(N_{\ell_1}) \iff \boldsymbol{\alpha}\eta_0 \times \{g(\ell_0)\} <_{\varphi(L')} \boldsymbol{\alpha}\eta_0 \times \{g(\ell_1)\} \iff g(\ell_0) <_{L'} g(\ell_1).$$

Now set $L_k = \{\ell \in L \mid k_\ell = k\}$ for each $k \in K$, and let $K' = \{k \in K \mid L_k \neq \emptyset\} \subseteq K$, so that $K' \in \mathcal{L}$ by downward $\preceq$-closure of $\mathcal{L}$. Clearly, $\bigcup_{k\in K'} L_k = L$. Moreover, for every $k, k' \in K'$ we have

$$k <_{K'} k' \iff M_k <_{\varphi(L)} M_{k'} \iff \forall \ell_0 \in L_k \forall \ell_1 \in L_{k'} \, (N_{\ell_0} <_{\varphi(L)} N_{\ell_1}) \iff L_k <_L L_{k'},$$

and thus $(L_k)_{k\in K'}$ is a K'-convex partition of L. In particular, every L_k is L-convex.

[2] In general, the choice of N_ℓ and k_ℓ is not unique.

We also claim that $g(L_k) \sqsubseteq L'$ for all $k \in K'$. Pick arbitrary $\ell_0, \ell_1 \in L_k$ such that $g(\ell_0) <_{L'} g(\ell_1)$, and consider any $m' \in L'$ such that $g(\ell_0) <_{L'} m' <_{L'} g(\ell_1)$ (if there is any), so that in particular $h(N_{\ell_0}) <_{\varphi(L')} \boldsymbol{\alpha}\eta_0 \times \{m'\} <_{\varphi(L')} h(N_{\ell_1})$ and $\boldsymbol{\alpha}\eta_0 \times \{m'\} \sqsubseteq h(M_k)$. Since $h \restriction M_k$ is an isomorphism between M_k and the $\varphi(L')$-convex set $h(M_k)$, and since $\boldsymbol{\alpha}\eta_0 \times \{m'\}$ does not contain any $\varphi(L')$-convex subset isomorphic to η, then $h^{-1}(\boldsymbol{\alpha}\eta_0 \times \{m'\}) \cap (\eta_j \times \{m\}) = \emptyset$ for every $j \in \{1, 2\}$ and $m \in L$. Since $h^{-1}(\boldsymbol{\alpha}\eta_0 \times \{m'\} \subseteq L_0 \times \{m\}$ is impossible by choice of α, we conclude that there is $m \in L$ such that $h^{-1}(\boldsymbol{\alpha}\eta_0 \times \{m'\}) \sqsubseteq (\boldsymbol{\alpha}\eta_0 \times \{m\})$. Notice that $\ell_0 \leq_L m \leq_L \ell_1$ because $N_{\ell_0} <_{\varphi(L)} h^{-1}(\boldsymbol{\alpha}\eta_0 \times \{m'\}) <_{\varphi(L)} N_{\ell_1}$, hence $m \in L_k$ because the latter is L-convex, and so $k_m = k$. Suppose towards a contradiction that $m = \ell_0$. Then the h-preimage of $\eta_1 \times \{g(\ell_0)\}$, which is a $\varphi(L')$-convex subset of $\varphi(L')$ between $h(N_{\ell_0})$ and $\boldsymbol{\alpha}\eta_0 \times \{m'\}$, would be a $\varphi(L)$-convex subset of $\boldsymbol{\alpha}\eta_0 \times \{\ell_0\}$, which is impossible. A similar argument excludes $m = \ell_1$: hence $\ell_0 <_L m <_L \ell_1$ and $\boldsymbol{\alpha}\eta_0 \times \{m\} \sqsubseteq M_k$. By the usual argument, this entails that $h(\boldsymbol{\alpha}\eta_0 \times \{m\}) \subseteq \boldsymbol{\alpha}\eta_0 \times \{\ell'\}$ for some $\ell' \in L'$, and necessarily $\ell' = m'$ by choice of m. Thus $m' = g(m)$, so $m' \in g(L_k)$. Since m' was arbitrary, $g(L_k)$ is L'-convex.

This concludes the proof because we have shown that $K' \in \mathcal{L}$, the K'-convex partition $(L_k)_{k \in K}$ and g witness $L \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} L'$, as desired. □

In contrast, it is often not possible to embed $(\mathsf{LO}, \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$ into a lower cone $(L_0{\downarrow}^{\mathcal{L}}, \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$, where $L_0{\downarrow}^{\mathcal{L}} = \{L \in \mathsf{LO} \mid L \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} L_0\}$. This is trivial if we consider a $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$-minimal element in LO, such as ω or ω^* when $\mathcal{L} \nsubseteq \mathsf{Fin}$ or the non-scattered minimal elements from Theorem 3.2.8 if $\mathcal{L} \subseteq \mathsf{Fin}$.

Besides the ones determined by minimal elements, there are many other lower cones in which $(\mathsf{LO}, \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$ cannot be embedded. For example, if $\mathcal{L} \subseteq \mathsf{Fin}$ and $L_0 \in \mathsf{Scat}$, then $L_0{\downarrow}^{\mathcal{L}}$ contains countably many equivalence classes under $\bowtie^{\mathcal{L}}_{\mathsf{LO}}$ (this follows from the fact that a countable scattered linear order has countably many convex subsets, [Bon75]), and thus by Theorem 3.2.4 there is again no embedding from $(\mathsf{LO}, \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$ into $(L_0{\downarrow}^{\mathcal{L}}, \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$. If instead $\mathsf{Fin} \subsetneq \mathcal{L} \subseteq \mathsf{Scat}$, we can notice that if $L_0 \in \mathsf{LO} \cap \mathcal{L}$ then $(\mathsf{LO}, \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$ is not embeddable in $(L_0{\downarrow}^{\mathcal{L}}, \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$ because the latter coincides with $(L_0{\downarrow}^{\mathcal{L}}, \preceq_{\mathsf{LO}})$ by Fact 3.1.3, and hence it is a wqo.

In fact, we have no examples of $L_0 \in \mathsf{LO}$ and ccs $\mathcal{L} \subseteq \mathsf{Scat}$ such that $(\mathsf{LO}, \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$ embeds into $(L_0{\downarrow}^{\mathcal{L}}, \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$. If instead $\mathcal{L} = \mathsf{Lin}$ the situation is clearer: since $\trianglelefteq^{\mathsf{Lin}}_{\mathsf{LO}}$ is $\preceq_{\mathsf{LO}}$, then[3] $(\mathsf{LO}, \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$ embeds into $(L_0{\downarrow}^{\mathcal{L}}, \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$ if and only if L_0 is not scattered (in which case $L_0{\downarrow}^{\mathcal{L}} = \mathsf{LO}$).

3.3 Borel complexity of $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ and $\bowtie^{\mathcal{L}}_{\mathsf{LO}}$

In this section we analyze the descriptive set-theoretic complexity of the quasi-order $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ and of its associated equivalence relation $\bowtie^{\mathcal{L}}_{\mathsf{LO}}$. We again mostly work with ccs families $\mathcal{L} \subsetneq \mathsf{Lin}$, as $\bowtie^{\mathsf{Lin}}_{\mathsf{LO}}$ is just the well-studied relation $\equiv_{\mathsf{LO}}$ of biembeddability (also called equimorphism) on LO.

We first determine bounds on the complexity of $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ and $\bowtie^{\mathcal{L}}_{\mathsf{LO}}$ as subsets of $\mathsf{LO} \times \mathsf{LO}$. Since their definition includes an existential quantification over $\mathcal{L}$, it is not surprising that their complexity depends on that of $\mathcal{L}$.

Proposition 3.3.1. *Let $\mathcal{L} \subseteq \mathsf{Lin}$ be downward $\preceq$-closed.*

(a) *$\mathcal{L}$ is a coanalytic subset of* Lin, *and thus it cannot be proper analytic.*

(b) *The relations $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ and $\bowtie^{\mathcal{L}}_{\mathsf{LO}}$ are both $\boldsymbol{\Sigma}^1_2$.*

(c) *If $\mathcal{L}$ is Borel, then $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ and $\bowtie^{\mathcal{L}}_{\mathsf{LO}}$ are analytic.*

(d) *If $\mathcal{L}$ is closed under doublings, i.e. $\mathbf{2}L \in \mathcal{L}$ for all $L \in \mathcal{L} \cap \mathsf{LO}$, then $\mathcal{L} \leq_W \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ and $\mathcal{L} \leq_W \bowtie^{\mathcal{L}}_{\mathsf{LO}}$. Thus if $\mathcal{L}$ is also proper coanalytic (which in particular implies $\mathcal{L} \neq \mathsf{Lin}$ and hence $\mathcal{L} \subseteq \mathsf{Scat}$) then $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ and $\bowtie^{\mathcal{L}}_{\mathsf{LO}}$ are not analytic, while if $\mathcal{L}$ is even $\boldsymbol{\Pi}^1_1$-complete then $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ and $\bowtie^{\mathcal{L}}_{\mathsf{LO}}$ are $\boldsymbol{\Pi}^1_1$-hard.*

[3]For the nontrivial direction, notice that if there were an embedding f of $(\mathsf{LO}, \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$ into $(L_0{\downarrow}^{\mathcal{L}}, \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$ then $f(L_0) \prec f(\eta) \preceq L_0$. Thus also $f^{(2)}(L_0) = (f \circ f)(L_0) \prec f(L_0)$, and iterating the process $f^{(n+1)}(L_0) \prec f^{(n)}(L_0)$ for every $n \in \omega$. But then $(f^{(n)}(L_0))_{n \in \mathbb{N}}$ would be an infinite descending chain, contradicting the fact that $\prec$ is wqo.

Part (c) applies e.g. to the families $\mathcal{L}_{\preceq \mathbf{1}}$, Fin, Lin, and all the ccs classes considered in Section 3.4; instead the hypothesis of part (d) follows from condition ($\star$) of Proposition 3.2.7 and applies also to WO and Scat.

Proof. (a) Since $\preceq_{\mathsf{LO}}$ is a wqo, $\mathsf{Lin} \setminus \mathcal{L}$ is a finite union of upward $\preceq$-closed cones, each of which is analytic because $\preceq_{\mathsf{LO}}$ is an analytic relation. Then $\mathsf{Lin} \setminus \mathcal{L}$ is analytic and $\mathcal{L}$ is coanalytic.

(b) The upper bound directly comes from Definition 3.1.2, taking into account part (a).

(c) Similar to (b).

(d) Consider the continuous map $\varphi\colon \mathsf{LO} \to \mathsf{LO}$ defined by $\varphi(L) = (\eta + \mathbf{2})L$. We claim that $L \in \mathcal{L} \iff \varphi(L) \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} \eta \iff \varphi(L) \bowtie^{\mathcal{L}}_{\mathsf{LO}} \eta$, which amounts to just showing the first equivalence because $\eta \trianglelefteq_{\mathsf{LO}} \varphi(L)$ for every $L \in \mathsf{LO}$. If $L \in \mathcal{L}$, then $K = \mathbf{2}L \in \mathcal{L}$ by hypothesis, and the K-convex partition $(L_k)_{k \in K}$ of $\varphi(L)$ whose first element of each pair is $\eta + \mathbf{1}$ and the second element is $\mathbf{1}$ can be used to witness $\varphi(L) \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} \eta$ in the obvious way. Conversely, assume that $\varphi(L) \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} \eta$ via some $K \in \mathcal{L}$ and some K-convex partition $(L_k)_{k \in K}$ of $\varphi(L)$. Notice that whenever $\ell, \ell' \in L$ are distinct then no convex subset of $\varphi(L)$ isomorphic to a convex subset of η contains both $(0, \ell)$ and $(0, \ell')$. Therefore the map associating to each $\ell \in L$ the unique $k \in K$ such that $(0, \ell) \in L_k$ is order-preserving and injective, so that $L \preceq_{\mathsf{LO}} K$ and $L \in \mathcal{L}$. □

We now move to the classification of $\bowtie^{\mathcal{L}}_{\mathsf{LO}}$ with respect to Borel reducibility. When $\mathcal{L} = \{\mathbf{1}\}$, the relation $\bowtie^{\mathcal{L}}_{\mathsf{LO}}$ coincides with convex biembeddability $\bowtie_{\mathsf{LO}}$ on LO: its classification has already been studied in Section 2.3. We generalize Corollary 2.3.2 to all ccs classes $\mathcal{L} \subsetneq \mathsf{Lin}$ through the following more general result.

Theorem 3.3.2. *Let $\mathcal{L} \subseteq \mathsf{Lin}$ be downward $\preceq$-closed, and let $M \notin \mathcal{L}$. The map φ sending each linear order L to $\varphi(L) = (\mathbf{1} + \zeta L + \mathbf{1})M$ is such that $L \cong L' \iff \varphi(L) \trianglelefteq^{\mathcal{L}} \varphi(L') \iff \varphi(L) \bowtie^{\mathcal{L}} \varphi(L')$.*

Proof. Obviously, if $L \cong L'$ then $\varphi(L) \bowtie^{\mathcal{L}} \varphi(L')$, and the latter implies $\varphi(L) \trianglelefteq^{\mathcal{L}} \varphi(L')$. So it remains to show that if $\varphi(L) \trianglelefteq^{\mathcal{L}} \varphi(L')$, then $L \cong L'$. Since $M \notin \mathcal{L}$, by Proposition 3.1.11 we obtain from $\varphi(L) \trianglelefteq^{\mathcal{L}} \varphi(L')$ that $\mathbf{1} + \zeta L + \mathbf{1} \trianglelefteq \varphi(L')$ via some embedding g with convex range. Since the $\mathbf{1}$'s are the only elements that do not have immediate predecessor and successor both in $\mathbf{1} + \zeta L + \mathbf{1}$ and in $\varphi(L')$, we have that the two $\mathbf{1}$'s in $\mathbf{1} + \zeta L + \mathbf{1}$ are mapped by g into the two $\mathbf{1}$'s of $(\mathbf{1} + \zeta L' + \mathbf{1}) \times \{m\}$ for some $m \in M$, hence $g(\mathbf{1} + \zeta L + \mathbf{1}) = (\mathbf{1} + \zeta L' + \mathbf{1}) \times \{m\}$ and $\mathbf{1} + \zeta L + \mathbf{1} \cong \mathbf{1} + \zeta L' + \mathbf{1}$. But then $\zeta L \cong \zeta L'$, and so $L \cong L'$ by Lemma 1.2.7. □

Noticing that if $M \in \mathsf{Lin}$ the restriction to LO of the map φ from Theorem 3.3.2 is Borel, we get:

Corollary 3.3.3. *For every ccs $\mathcal{L} \subseteq \mathsf{Scat}$, we have $\cong_{\mathsf{LO}} \leq_B \bowtie^{\mathcal{L}}_{\mathsf{LO}}$.*

Proof. Apply Theorem 3.3.2 with $M = \eta$. □

Corollary 3.3.3 also provides lower bounds for the complexity of $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ and $\bowtie^{\mathcal{L}}_{\mathsf{LO}}$ as subsets of $\mathsf{LO} \times \mathsf{LO}$.

Corollary 3.3.4. *For every downward $\preceq$-closed $\mathcal{L} \subseteq \mathsf{Lin}$, the relations $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ and $\bowtie^{\mathcal{L}}_{\mathsf{LO}}$ are $\mathbf{\Sigma}^1_1$-hard. Therefore if $\mathcal{L}$ is Borel then $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$ and $\bowtie^{\mathcal{L}}_{\mathsf{LO}}$ are complete analytic (as subsets of $\mathsf{LO} \times \mathsf{LO}$); if instead $\mathcal{L}$ is proper coanalytic and satisfies the closure property from Proposition 3.3.1(d), then they are neither analytic nor coanalytic, hence they are at least $\mathbf{\Delta}^1_2$.*

Proof. If $\mathcal{L} = \mathsf{Lin}$, then the map $L \mapsto (\eta, L)$ simultaneously reduces the $\mathbf{\Sigma}^1_1$-complete set $\mathsf{Lin} \setminus \mathsf{Scat}$ to $\trianglelefteq^{\mathsf{Lin}}_{\mathsf{LO}}$ and $\bowtie^{\mathsf{Lin}}_{\mathsf{LO}}$ because they coincide with $\preceq_{\mathsf{LO}}$ and $\equiv_{\mathsf{LO}}$, respectively. If instead $\mathcal{L} \subseteq \mathsf{Scat}$, use Corollary 3.3.3 and the well-known fact that $\cong_{\mathsf{LO}}$ is a $\mathbf{\Sigma}^1_1$-complete subset of $(\mathsf{LO})^2$. □

Corollary 2.3.16 does not generalize to an arbitrary $\bowtie^{\mathcal{L}}_{\mathsf{LO}}$, and actually we have the opposite situation for every $\mathcal{L}$ different from both $\mathcal{L}_{\preceq \mathbf{1}}$ and Lin.

Theorem 3.3.5. *For every ccs $\mathcal{L}$ such that $\mathsf{Fin} \subseteq \mathcal{L} \subseteq \mathsf{Scat}$ we have $E_1 \leq_B \bowtie^{\mathcal{L}}_{\mathsf{LO}}$.*

Proof. Let $(\mathbb{R}^+)^{\mathbb{N}}$ be the set of sequences of positive real numbers, whose elements will be denoted by $(x_n)_{n\in\omega}$ or, for the sake of brevity, by $\vec{x}$. Consider the restriction $E_1 \restriction (\mathbb{R}^+)^{\mathbb{N}}$ of E_1 to $(\mathbb{R}^+)^{\mathbb{N}}$: applying the exponential function pointwise, one immediately sees that $E_1 \restriction (\mathbb{R}^+)^{\mathbb{N}} \sim_B E_1$. Fix an injective $f\colon \mathbb{Q} \to \{\mathbf{n} \mid n \in \mathbb{N} \setminus \{0\}\}$, and consider once again the linear order η^f. To simplify the notation, given any $r \in \mathbb{R}$ we write η^f_r in place of $\eta^f_{(r,r+1)}$. Let $\varphi\colon (\mathbb{R}^+)^{\mathbb{N}} \to \mathsf{LO}$ be the Borel map given by

$$\varphi(\vec{x}) = \eta^f \boldsymbol{\omega}^* + \sum_{n\in\mathbb{N}} (\eta^f_{-(n+1)} + \eta^f_{x_n}).$$

We claim that φ reduces $E_1 \restriction (\mathbb{R}^+)^{\mathbb{N}}$ to $\bowtie^{\mathcal{L}}_{\mathsf{LO}}$.

Suppose that $\vec{x}, \vec{y} \in (\mathbb{R}^+)^{\mathbb{N}}$ are such that $\vec{x}\ E_1\ \vec{y}$, and let $n_0 \in \mathbb{N}$ be such that $x_n = y_n$ for all $n \geq n_0$. Let $m = 2n_0 + 2$, so that $\mathbf{m} \in \mathsf{Fin} \subseteq \mathcal{L}$. Consider the $\mathbf{m}$-convex partition $(L_k)_{k<m}$ of $\varphi(\vec{x})$ given by

$$L_k = \begin{cases} \eta^f \boldsymbol{\omega}^* & \text{if } k = 0 \\ \eta^f_{-(i+1)} & \text{if } k = 2i+1 \text{ for some } i < n_0 \\ \eta^f_{x_i} & \text{if } k = 2i+2 \text{ for some } 0 \leq i < n_0 \\ \sum_{n\geq n_0} (\eta^f_{-(n+1)} + \eta^f_{x_n}) & \text{if } k = 2n_0 + 1. \end{cases}$$

We now define an embedding $g\colon \varphi(\vec{x}) \to \varphi(\vec{y})$ as follows. First send L_0 into the $\varphi(L')$-convex set $\{(\ell, j) \in \eta^f \boldsymbol{\omega}^* \mid j \leq_{\boldsymbol{\omega}^*} 2n_0\} \trianglelefteq \eta^f \boldsymbol{\omega}^*$ of $\varphi(\vec{y})$ by traslating each summand of L_0 to the left by $2n_0$-many places. Then send each L_k with $0 < k \leq 2n_0$ into the summand $\eta^f \times \{2n_0 - k\} \trianglelefteq \eta^f \boldsymbol{\omega}^* \trianglelefteq \varphi(\vec{y})$ in the obvious way, using the fact that L_k is the restriction of η^f to an open interval. Finally, map L_{2n_0+1} identically to itself (viewed as a tail of $\varphi(\vec{y})$), which is possible because $\eta^f_{x_n} = \eta^f_{y_n}$ for all $n \geq n_0$ by choice of n_0. Then $g(L_k) \trianglelefteq \varphi(\vec{y})$ for every $k < m$, hence $\varphi(\vec{x}) \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} \varphi(\vec{y})$ as witnessed by $\mathbf{m}$, $(L_k)_{k<m}$ and g.

Conversely, suppose that $\varphi(\vec{x}) \bowtie^{\mathcal{L}}_{\mathsf{LO}} \varphi(\vec{y})$, and fix some $K \in \mathcal{L}$, a K-convex partition $(L_k)_{k\in K}$ of $\varphi(\vec{x})$, and an embedding $g\colon \varphi(\vec{x}) \to \varphi(\vec{y})$ witnessing $\varphi(\vec{x}) \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} \varphi(\vec{y})$. By Lemma 3.2.2(a) for each $n \in \mathbb{N}$ there are $-(n+1) \leq q^{(n)}_0 < q^{(n)}_1 \leq -n$ such that $M_n = \eta^f_{(q^{(n)}_0, q^{(n)}_1)} \trianglelefteq \eta^f_{-(n+1)} \cap L_k$ for some $k \in K$, so that g itself witnesses $M_n \trianglelefteq_{\mathsf{LO}} \varphi(\vec{y})$. Notice also that all linear orders M_n and $\varphi(\vec{y})$ are of the form $\eta^{f'}$ for suitable functions $f'\colon \mathbb{Q} \to \{\mathbf{n} \mid n \in \mathbb{N} \setminus \{0\}\}$. By Lemma 3.2.2(c) and injectivity of f (and using also $\vec{x}, \vec{y} \in (\mathbb{R}^+)^{\mathbb{N}}$), it easily follows that either $g(M_n) = M_n \times \{j_n\} \trianglelefteq \eta^f \boldsymbol{\omega}^* \trianglelefteq \varphi(\vec{y})$ for some $j_n \in \boldsymbol{\omega}^*$, or else $g(M_n) = M_n \trianglelefteq \eta^f_{-(n+1)} \trianglelefteq \varphi(\vec{y})$. But since $M_n <_{\varphi(\vec{x})} M_m$ for all $n, m \in \mathbb{N}$ such that $n < m$, if the first case occur for both M_n and M_m then $j_n <_{\boldsymbol{\omega}^*} j_m$ (equivalently: $j_n > j_m$) because otherwise $g(M_m) = M_m \times \{j_m\} <_{\varphi(\vec{y})} M_n \times \{j_n\} = g(M_n)$. (Here we use that if $m > n$ then $(-(m+1), -m) <_{\mathbb{R}} (-(n+1), -n)$.) On the other hand, if the second case occurs for some M_n, then it also occurs for all M_m with $m \geq n$ because g is order-preserving. Combining these two facts, we obtain that there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the second case, namely $g(M_n) = M_n \trianglelefteq \eta^f_{-(n+1)} \trianglelefteq \varphi(\vec{y})$, occurs: we claim that $x_n = y_n$ for every $n \geq n_0$, so that $\vec{x}\ E_1\ \vec{y}$. Suppose towards a contradiction that $x_n \neq y_n$ for some $n \geq n_0$. Since $M_n <_{\varphi(\vec{x})} \eta^f_{x_n} <_{\varphi(\vec{x})} M_{n+1}$, by choice of n_0 we have that $\eta^f_{x_n} \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} \eta^f_{-(n+1)} + \eta^f_{y_n} + \eta^f_{-(n+2)} \trianglelefteq \varphi(\vec{y})$. Fix $r_0 < r_1$ such that $(r_0, r_1) \subseteq (x_n, x_n + 1) \setminus (y_n, y_n + 1)$, so that also $\eta^f_{(r_0,r_1)} \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} \eta^f_{-(n+1)} + \eta^f_{y_n} + \eta^f_{-(n+2)}$. By Lemma 3.2.2(a) again there are $r_0 \leq q_0 < q_1 \leq r_1$ such that $\eta^f_{(q_0,q_1)} \trianglelefteq_{\mathsf{LO}} \eta^f_{-(n+1)} + \eta^f_{y_n} + \eta^f_{-(n+2)}$. Since both $\eta^f_{(q_0,q_1)}$ and $\eta^f_{-(n+1)} + \eta^f_{y_n} + \eta^f_{-(n+2)}$ are of the form $\eta^{f'}$ for a suitable f', Lemma 3.2.2(c) applies, yielding the desired contradiction because f is injective and $(q_0, q_1) \cap [(-(n+1), -n) \cup (y_n, y_n + 1) \cup (-(n+2), -(n+1))] = \emptyset$. ☐

Corollary 3.3.6. *Let the ccs class $\mathcal{L} \subseteq \mathsf{Scat}$ be different from $\mathcal{L}_{\preceq \mathbf{1}}$. Then $\cong_{\mathsf{LO}} <_B \bowtie^{\mathcal{L}}_{\mathsf{LO}}$, and moreover $\bowtie^{\mathcal{L}}_{\mathsf{LO}} \not\leq_{Baire} \bowtie_{\mathsf{LO}}$ and $\bowtie^{\mathcal{L}}_{\mathsf{LO}} \not\leq_{Baire} E$ for every orbit equivalence relation E.*

Proof. Corollary 3.3.3 gives $\cong_{\mathsf{LO}} \leq_B \bowtie^{\mathcal{L}}_{\mathsf{LO}}$, while the non-reducibility results follow at once from Theorem 1.1.14 and Corollary 2.3.16. ☐

Along the same lines, we have:

Proposition 3.3.7. *For every ccs $\mathcal{L} \subseteq \mathsf{Scat}$ we have $\bowtie^{\mathcal{L}}_{\mathsf{LO}} \not\leq_B \equiv_{\mathsf{LO}}$, and thus $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} \not\leq_B \preceq_{\mathsf{LO}}$.*

Proof. By Corollary 3.3.3 the identity on $\mathbb{R}$ Borel reduces to $\bowtie^{\mathcal{L}}_{\mathsf{LO}}$, while by Laver's classic result it does not Borel reduce to $\equiv_{\mathsf{LO}}$. □

We do not know whether $\bowtie^{\mathcal{L}}_{\mathsf{LO}}$ is complete for analytic equivalence relations when $\mathcal{L}_{\preceq \mathbf{1}} \subsetneq \mathcal{L} \subsetneq \mathsf{Lin}$ is Borel. It remains also open whether $\bowtie^{\mathcal{L}}_{\mathsf{LO}}$ is proper $\mathbf{\Sigma}^1_2$ in the case of a $\mathbf{\Pi}^1_1$-complete class $\mathcal{L}$.

Notice now that the embedding from $(\mathrm{Int}(\mathbb{R}), \subseteq)$ to $(\mathsf{LO}, \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$ defined in the proof of Lemma 3.2.3 is actually a Borel reduction. Thus we have the following proposition.

Proposition 3.3.8. *For every ccs class $\mathcal{L} \subseteq \mathsf{Scat}$, $(\mathrm{Int}(\mathbb{R}), \subseteq) \leq_B (\mathsf{LO}, \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}})$.*

We conclude this section by studying what happens if we move to *coloured* linear orders. Consider the Polish space $\mathsf{LO}_{\mathbb{N}} = \mathsf{LO} \times \mathbb{N}^{\mathbb{N}}$. Each element $(L, c) \in \mathsf{LO}_{\mathbb{N}}$ can be interpreted as the linear order L on $\mathbb{N}$ where each of its elements $\ell \in L$ is coloured with $c(\ell)$.

Definition 3.3.9. Let $\mathcal{L} \subseteq \mathsf{Lin}$ and $(L, c), (L', c') \in \mathsf{LO}_{\mathbb{N}}$. We say that (L, c) is **$\mathcal{L}$-convex embeddable** in (L', c'), in symbols $(L, c) \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}_{\mathbb{N}}} (L', c')$, if and only if for some embedding $f\colon L \to L'$ witnessing $L \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} L'$ we have $c'(f(n)) = c(n)$ for every $n \in \mathbb{N}$. When $\mathcal{L} = \mathcal{L}_{\preceq \mathbf{1}}$ we just write $\trianglelefteq_{\mathsf{LO}_{\mathbb{N}}}$ instead of $\trianglelefteq^{\mathcal{L}_{\preceq \mathbf{1}}}_{\mathsf{LO}_{\mathbb{N}}}$, while if $\mathcal{L} = \mathsf{Lin}$ we write $\preceq_{\mathsf{LO}_{\mathbb{N}}}$ instead of $\trianglelefteq^{\mathsf{Lin}}_{\mathsf{LO}_{\mathbb{N}}}$.

Notice that $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}_{\mathbb{N}}}$ is always reflexive, and it is transitive (i.e. a quasi-order) if and only is so is $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$, i.e. if and only if $\mathcal{L}$ is ccs.

Marcone and Rosendal [MR04] showed that the quasi-order $\preceq_{\mathsf{LO}_{\mathbb{N}}}$ of embeddability between coloured linear orders is complete for analytic quasi-orders, and thus $\trianglelefteq^{\mathsf{Lin}} <_B \trianglelefteq^{\mathsf{Lin}}_{\mathsf{LO}_{\mathbb{N}}}$. In contrast, when considering ccs families $\mathcal{L} \neq \mathsf{Lin}$, we have the opposite situation.

Theorem 3.3.10. *If $\mathcal{L} \subseteq \mathsf{Scat}$ is a ccs class then $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}_{\mathbb{N}}} \sim_B \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$.*

Proof. Clearly, $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} \leq_B \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}_{\mathbb{N}}}$ via the reduction $L \mapsto (L, c)$ with c the constant map with value 0. For the converse, let $\mathsf{LO}'_{\mathbb{N}}$ be the collection of those linear orders (L, c) such that $c(\ell) > 0$ for all $\ell \in L$ and c is not constant on any closed interval $[\ell_0, \ell_1]_L$ with $\ell_0 <_L \ell_1$. Notice that the Borel map $(L, c) \mapsto (\mathbf{2}L, c')$ with $c'(0, \ell) = c(\ell) + 2$ and $c'(1, \ell) = 1$ for all $\ell \in L$ reduces $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}_{\mathbb{N}}}$ to $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}_{\mathbb{N}}} \restriction \mathsf{LO}'_{\mathbb{N}}$, so it is enough to show that $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}_{\mathbb{N}}} \restriction \mathsf{LO}'_{\mathbb{N}} \leq_B \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$.

Consider the Borel map $\varphi\colon \mathsf{LO}'_{\mathbb{N}} \to \mathsf{LO}$ defined by $\varphi(L, c) = \sum_{\ell \in L} \eta^{f_\ell}$, where f_ℓ is the constant map with value $c(\ell)$ (viewed as a finite linear order). We claim that φ reduces $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}_{\mathbb{N}}} \restriction \mathsf{LO}'_{\mathbb{N}}$ to $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$.

One direction is obvious, so let us assume that $\varphi(L, c) \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}} \varphi(L', c')$, as witnessed by $K \in \mathcal{L}$, the K-convex partition $(M_k)_{k \in K}$ of $\varphi(L, c)$ and the embedding $g\colon \varphi(L) \to \varphi(L')$. We follow the strategy used in the proof of Theorem 3.2.15, although in a simplified situation. By Lemma 3.2.2(a), for every $\ell \in L$ we can fix $k_\ell \in K$ and $N_\ell = \eta^{f_\ell}_{(q^{(n)}_0, q^{(n)}_1)} \sqsubseteq \eta^{f_\ell} \cap M_{k_\ell}$, so that $N_\ell \cong_{\mathsf{LO}} \eta^{f_\ell}$ and $N_\ell \trianglelefteq_{\mathsf{LO}} \varphi(L', c')$ as witnessed by g itself. Since $(L', c') \in \mathsf{LO}'_{\mathbb{N}}$ and $\varphi(L', c')$ is of the form $\eta^{f'}$ for a suitable $f'\colon \mathbb{Q} \to \{\mathbf{n} \mid n \in \mathbb{N} \setminus \{0\}\}$, by Lemma 3.2.2(a) there is a (necessarily unique) $\ell' \in L'$ such that $g(N_\ell) \subseteq \eta^{f_{\ell'}} \sqsubseteq \varphi(L', c')$ and $f_{\ell'}$ has the same value of f_ℓ: we claim that the map $h\colon L \to L'$ defined by $h(\ell) = \ell'$ is a colour-preserving embedding. It is clearly order-preserving because so is g. If there were $\ell_0, \ell_1 \in L$ with $\ell_0 <_L \ell_1$ and $h(\ell_0) = h(\ell_1)$, then f_ℓ would have the same value as $f_{h(\ell_0)}$ for all $\ell \in [\ell_0, \ell_1]_L$, contradicting $(L, c) \in \mathsf{LO}'_\ell$. Therefore h is also injective, and it is order preserving because f_ℓ and $f_{h(\ell)}$ have the same value for all $\ell \in L$.

For each $k \in K$ set $L_k = \{\ell \in L \mid k_\ell = k\}$ and $K' = \{k \in K \mid L_k \neq \emptyset\} \in \mathcal{L}$. Observe that $(L_k)_{k \in K'}$ is a K'-convex partition of L: indeed, since $\ell_0 <_L \ell_1 \iff N_{\ell_0} <_{\varphi(L,c)} N_{\ell_1}$, for all $k, k' \in K'$ we have

$$k <_{K'} k' \iff M_k <_{\varphi(L,c)} M_{k'} \iff \forall \ell_0 \in L_k \forall \ell_1 \in L_{k'} \, (N_{\ell_0} <_{\varphi(L)} N_{\ell_1}) \iff L_k <_L L_{k'}.$$

We also claim that each $h(L_k)$ is L'-convex. Fix $\ell_0, \ell_1 \in L_k$ such that $h(\ell_0) <_{L'} h(\ell_1)$. Since $g \restriction M_k$ is an isomorphism between M_k and $g(M_k)$, then the corresponding restriction of g^{-1}

witnesses $\sum_{h(\ell_0)<_{L'}\ell'<_{L'}h(\ell_1)} \eta^{f_{\ell'}} \trianglelefteq_{\mathsf{LO}} \sum_{l_0 \leq_L \ell \leq_L \ell_1} \eta^{f_\ell}$. Both these linear orders are of the form $\eta^{f'}$, so by Lemma 3.2.2(c) for each $\ell' \in (h(\ell_0), h(\ell_1))_{L'}$ there is $\ell \in [\ell_0, \ell_1]_L$ such that $g^{-1}(\eta^{f_{\ell'}}) \subseteq \eta^{f_\ell}$. We cannot have $\ell = \ell_0$ because otherwise c' would be constant on $[h(\ell_0), \ell']_{L'}$, and $\ell \neq \ell_1$ as well because otherwise c' would be constant on $[\ell', h(\ell_1)]_{L'}$. Hence $\ell_0 <_L \ell <_L \ell_1$, which implies $\eta^{f_\ell} \sqsubseteq M_k$, so that necessarily $\ell \in L_k$ and $\eta^{f_\ell} \trianglelefteq_{\mathsf{LO}} \varphi(L', c')$ via $g \restriction \eta^{f_\ell}$. By Lemma 3.2.2(c) again, we have that $g(\eta^{f_\ell}) \sqsubseteq \eta^{f_{\ell''}}$ for some unique $\ell'' \in L'$, so that in particular $h(\ell) = \ell''$. Since $g^{-1}(\eta^{f_{\ell'}}) \subseteq \eta^{f_\ell}$ we must have $\ell'' = \ell'$, so that $\ell' \in h(L_k)$, as desired.

We have shown that K', the K'-convex partition $(L_k)_{k \in K'}$ of L, and the embedding h witness $(L, c) \trianglelefteq^{\mathcal{L}}_{\mathsf{LO}_\mathbb{N}} (L', c')$, hence we are done. □

Recalling that $\trianglelefteq_{\mathsf{LO}}$ is not complete for analytic quasi-orders (Corollary 2.3.17), we obtain the following result, which is in contrast with the situation for $\preceq_{\mathsf{LO}_\mathbb{N}}$ ([MR04]).

Corollary 3.3.11. *The relation $\trianglelefteq_{\mathsf{LO}_\mathbb{N}}$ of convex embeddability between coloured linear orders is not complete for analytic quasi-orders.*

3.4 Examples of ccs families

In this section we provide more examples of classes $\mathcal{L}$ of countable linear orders which are ccs. We also compare $\mathcal{L}$-convex embeddability for some of these $\mathcal{L}$'s according to Borel reducibility.

We first notice that the collection of ccs classes is not closed under union. Indeed, let $\mathcal{L} = \mathsf{WO} \cup \mathsf{WO}^*$. It is easy to check that WO and WO^* are ccs classes. Let now $K = \omega$, $K' = \omega^*$, and set $K'_0 = \omega^*$ and $K'_k = \{\max \omega^*\}$ for every $k > 0$. Then $\sum_{k \in K} K'_k \cong \omega^* + \omega = \zeta \notin \mathcal{L}$, and hence $\mathcal{L}$ is not ccs.

On the other hand, it is immediate to see that the collection of ccs classes is closed under intersection.

Remark 3.4.1. Notice that in the previous discussion and examples of Section 3.1 of classes which are not ccs, we are using the following fact: if $\mathcal{L}$ is ccs and $L + \mathbf{1}, \mathbf{1} + L' \in \mathcal{L}$ then $L + \mathbf{1} + L'$ belongs to $\mathcal{L}$ as well.

However the latter condition is not equivalent to being ccs, as witnessed by the following example. Let $\mathcal{L} = \{L \in \mathsf{LO} \mid \zeta\omega \not\trianglelefteq L \wedge \zeta\omega^* \not\trianglelefteq L\}$. It is immediate that if $L + \mathbf{1}$, $\mathbf{1} + L'$ are elements of $\mathcal{L}$ then $L + \mathbf{1} + L'$ is in $\mathcal{L}$ as well. On the other hand $\omega^2, \omega^*\omega \in \mathcal{L}$ and there is a convex sum using $K = \omega^2$ and $K' = \omega^*\omega$ which is isomorphic to $\zeta\omega \notin \mathcal{L}$.

Recall that an **(additively) indecomposable ordinal** γ is any nonzero ordinal number such that for any $\alpha, \beta < \gamma$, we have $\alpha + \beta < \gamma$. The indecomposable ordinals are precisely those of the form ω^δ for some ordinal δ. From the normality of addition in its right argument, it follows that γ is indecomposable if and only if $\alpha + \gamma = \gamma$ for every $\alpha < \gamma$. We use these properties to show the following proposition.

Proposition 3.4.2. *Let γ be an infinite countable ordinal and consider the class $\mathcal{L}_{\prec\gamma}$. Then $\trianglelefteq^{\mathcal{L}_{\prec\gamma}}$ is ccs if and only if γ is either the successor of an indecomposable ordinal or an indecomposable ordinal.*

Proof. $\Rightarrow$) We show the contrapositive. First we consider the case $\gamma = \alpha + 1$ assuming that α is not indecomposable. Fix $\beta < \alpha$ such that $\alpha < \beta + \alpha$. To show that $\mathcal{L}_{\prec\gamma}$ is not ccs (so that by Theorem 3.1.9 $\trianglelefteq^{\mathcal{L}_{\prec\gamma}}$ is not transitive) we consider $\boldsymbol{\beta} + \mathbf{1}, \boldsymbol{\alpha} \in \mathcal{L}_{\prec\gamma}$. Define the following convex subsets of $\boldsymbol{\alpha}$:

$$\boldsymbol{\alpha}_k = \{\min \boldsymbol{\alpha}\} \text{ for all } k <_{\boldsymbol{\beta}+\mathbf{1}} \max\{\boldsymbol{\beta} + \mathbf{1}\}, \qquad \text{and} \qquad \boldsymbol{\alpha}_{\max\{\boldsymbol{\beta}+\mathbf{1}\}} = \boldsymbol{\alpha}.$$

Then $\sum_{k \in \boldsymbol{\beta}+\mathbf{1}} \boldsymbol{\alpha}_k = \boldsymbol{\beta} + \boldsymbol{\alpha} \notin \mathcal{L}_{\prec\gamma}$.

We now prove that $\mathcal{L}_{\prec\gamma}$ is not ccs when γ is limit but not indecomposable. Fix $\alpha, \beta < \gamma$ such that $\gamma = \alpha + \beta$. Since γ is limit then β is limit as well and in particular $\beta > 1$, so that $\alpha + 1 < \gamma$. Consider $\boldsymbol{\beta}, \boldsymbol{\alpha} + \mathbf{1} \in \mathcal{L}_{\prec\gamma}$, and define the following convex subsets of $\boldsymbol{\alpha} + \mathbf{1}$:

$$\boldsymbol{\alpha}_{\min \boldsymbol{\beta}} = \boldsymbol{\alpha} + \mathbf{1}, \qquad \text{and} \qquad \boldsymbol{\alpha}_k = \{\max\{\boldsymbol{\alpha} + \mathbf{1}\}\} \text{ for all } k >_{\boldsymbol{\beta}} \min \boldsymbol{\beta}.$$

Then $\sum_{k\in\boldsymbol{\beta}}\boldsymbol{\alpha}_k=\boldsymbol{\alpha}+\boldsymbol{\beta}=\boldsymbol{\gamma}$ does not belong to $\mathcal{L}_{\prec\boldsymbol{\gamma}}$.

$\Leftarrow$) By Theorem 3.1.9 it suffices to show that if γ satisfies the hypothesis then $\mathcal{L}_{\prec\boldsymbol{\gamma}}$ is ccs. We prove this by induction on γ. If $\gamma=\omega$ then $\mathcal{L}_{\prec\boldsymbol{\gamma}}=\mathsf{Fin}$ which is ccs as noticed in Example 3.1.8. Fix now $\gamma>\omega$ indecomposable or successor of an indecomposable and assume that $L_{\prec\boldsymbol{\gamma}'}$ is ccs for every $\gamma'<\gamma$ which is indecomposable or successor of an indecomposable.

First suppose $\gamma=\omega^\delta$ is indecomposable with $\delta>1$. Let $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{L}_{\prec\boldsymbol{\omega}^\delta}$ and consider nonempty convex subsets $(\boldsymbol{\beta}_k)_{k\in\boldsymbol{\alpha}}$ of $\boldsymbol{\beta}$ such that $\forall k,k'\in\boldsymbol{\alpha}\ (k<_{\boldsymbol{\alpha}}k'\rightarrow\boldsymbol{\beta}_k\leq_{\boldsymbol{\beta}}\boldsymbol{\beta}_{k'})$. We want to show that $\sum_{k\in\alpha}\beta_k<\omega^\delta$. Since $\alpha,\beta<\omega^\delta$ there exist $\xi,\xi'<\delta$ and $n,n'\in\mathbb{N}$ minimal such that $\alpha\leq\omega^\xi n$ and $\beta\leq\omega^{\xi'}n'$. We can decompose

$$\boldsymbol{\alpha}=\boldsymbol{\alpha}_0\cup\dots\cup\boldsymbol{\alpha}_{n-1}$$

so that $\boldsymbol{\alpha}_0<_{\boldsymbol{\alpha}}\dots<_{\boldsymbol{\alpha}}\boldsymbol{\alpha}_{n-1}$ and $0<\alpha_i\leq\omega^\xi$ for every $i<n$. For every $i<n$ decompose, for some $l_i\leq n'$,

$$\bigcup_{k\in\boldsymbol{\alpha}_i}\boldsymbol{\beta}_k=\boldsymbol{\beta}_{i,0}\cup\dots\cup\boldsymbol{\beta}_{i,l_i-1},$$

so that $\boldsymbol{\beta}_{i,0}<_{\boldsymbol{\beta}}\dots<_{\boldsymbol{\beta}}\boldsymbol{\beta}_{i,l_i-1}$ and $0<\beta_{i,j}\leq\omega^{\xi'}$ for every $j<l_i$. For every $i<n$ and $j<l_i$ let

$$\boldsymbol{\alpha}_{i,j}=\{k\in\boldsymbol{\alpha}_i\mid\boldsymbol{\beta}_{i,j}\cap\boldsymbol{\beta}_k\neq\emptyset\},$$

and for each $k\in\boldsymbol{\alpha}_{i,j}$ set $\boldsymbol{\beta}_{i,j,k}=\boldsymbol{\beta}_{i,j}\cap\boldsymbol{\beta}_k$. We order the indices (i,j,k) such that $i<n$, $j<l_i$ and $k\in\boldsymbol{\alpha}_{i,j}$ lexicographically. It is easy to see that if $(i,j,k)<_{\text{lex}}(i',j',k')$ then $\boldsymbol{\beta}_{i,j,k}\leq_{\boldsymbol{\beta}}\boldsymbol{\beta}_{i',j',k'}$. Since every element of a $\boldsymbol{\beta}_k$ belongs to some $\boldsymbol{\beta}_{i,j,k}$ such that $k\in\boldsymbol{\alpha}_{i,j}$ we can write

$$\sum_{k\in\boldsymbol{\alpha}}\boldsymbol{\beta}_k=\sum_{i<n}\Big(\sum_{j<l_i}\Big(\sum_{k\in\boldsymbol{\alpha}_{i,j}}\boldsymbol{\beta}_{i,j,k}\Big)\Big).\tag{$\star$}$$

Consider now $\sum_{k\in\boldsymbol{\alpha}_{i,j}}\boldsymbol{\beta}_{i,j,k}$ for some $i<n$ and $j<l_i$. Let $\xi_0=\max(\xi,\xi')<\delta$. Notice that $\alpha_{i,j}\leq\omega^\xi\leq\omega^{\xi_0}$ as $\boldsymbol{\alpha}_{i,j}\subseteq\boldsymbol{\alpha}_i$ and $\bigcup_{k\in\boldsymbol{\alpha}_{i,j}}\boldsymbol{\beta}_{(i,j,k)}\leq\omega^{\xi'}\leq\omega^{\xi_0}$ as $\bigcup_{k\in\boldsymbol{\alpha}_{(i,j)}}\boldsymbol{\beta}_{(i,j,k)}\subseteq\boldsymbol{\beta}_{i,j}$. We can apply the induction hypothesis that $\mathcal{L}_{\prec\boldsymbol{\omega}^{\boldsymbol{\xi_0}}+\mathbf{1}}$ is css to $\boldsymbol{\alpha}_{i,j}$ and $\bigcup_{k\in\boldsymbol{\alpha}_{i,j}}\boldsymbol{\beta}_{i,j,k}$ to obtain $\sum_{k\in\alpha_{i,j}}\beta_{i,j,k}\leq\omega^{\xi_0}$. Then ($\star$) yields $\sum_{k\in\alpha}\beta_k\leq\omega^{\xi_0}nn'<\omega^\delta$.

Consider now γ to be the successor of an indecomposable ordinal: let $\gamma=\omega^\delta+1$ with $\delta>0$. Let $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathcal{L}_{\prec\boldsymbol{\omega}^\delta+\mathbf{1}}$ and consider nonempty convex subsets $(\boldsymbol{\beta}_k)_{k\in\boldsymbol{\alpha}}$ of $\boldsymbol{\beta}$ such that $\forall k,k'\in\boldsymbol{\alpha}\ (k<_{\boldsymbol{\alpha}}k'\rightarrow\boldsymbol{\beta}_k\leq_{\boldsymbol{\beta}}\boldsymbol{\beta}_{k'})$. We want to show that $\sum_{k\in\alpha}\beta_k\leq\omega^\delta$. We distinguish two cases:

- If $\alpha=\xi+1$ is a successor ordinal then let $k_0=\max\alpha$. Notice that $\alpha<\omega^\delta$, and a fortiori $\xi<\omega^\delta$. Let $\bar{\boldsymbol{\beta}}=(\boldsymbol{\beta}\setminus\boldsymbol{\beta}_{k_0})\cup\{\min\boldsymbol{\beta}_{k_0}\}$ and notice that $\boldsymbol{\beta}_k\subseteq\bar{\boldsymbol{\beta}}$ for every $k<_{\boldsymbol{\alpha}}k_0$. Moreover $\bar{\beta}<\omega^\delta$ because $\bar{\boldsymbol{\beta}}$ is a subset of $\boldsymbol{\beta}$ with a maximum. Since by induction hypothesis $\mathcal{L}_{\prec\boldsymbol{\omega}^\delta}$ is ccs we have $\sum_{k<_\alpha k_0}\beta_k<\omega^\delta$. Hence, as $\beta_{k_0}\leq\omega^\delta$, we have
$$\sum_{k\in\alpha}\beta_k=\sum_{k<_\alpha k_0}\beta_k+\beta_{k_0}\leq\omega^\delta.$$
- If α is limit, for every $k'\in\alpha$ let $\bar{\boldsymbol{\beta}}_{k'}=\bigcup_{k<_\alpha k'}\boldsymbol{\beta}_k$. Notice that $\bar{\beta}_{k'}<\omega^\delta$ as $\bar{\boldsymbol{\beta}}_{k'}$ is a subset of $\boldsymbol{\beta}$ bounded by $\min\boldsymbol{\beta}_{k'}$. Moreover $\{k\in\boldsymbol{\alpha}\mid k<_{\boldsymbol{\alpha}}k'\}$ is bounded in $\boldsymbol{\alpha}$ and hence has order type $<\omega^\delta$. Since by induction hypothesis $\mathcal{L}_{\prec\boldsymbol{\omega}^\delta}$ is ccs we have $\sum_{k<_\alpha k'}\beta_k<\omega^\delta$ for every $k'\in\boldsymbol{\alpha}$. Therefore
$$\sum_{k\in\alpha}\beta_k=\sup\Big\{\sum_{k<_\alpha k'}\beta_k\mid k'\in\boldsymbol{\alpha}\Big\}\leq\omega^\delta,$$
where equality holds because α is limit. □

Given $L \in \mathsf{LO}$, we now consider the classes $\mathcal{L}_{\prec \mathbb{Z}^L}$ and $\mathcal{L}_{\preceq \mathbb{Z}^L}$. First of all, if L is not a well order then by Proposition 1.2.10 $\mathbb{Z}^L$ is not scattered and so $\mathcal{L}_{\prec \mathbb{Z}^L} = \mathsf{Scat}$ and $\mathcal{L}_{\preceq \mathbb{Z}^L} = \mathsf{Lin}$ are both ccs as already noticed.

We thus restrict our analysis to the classes $\mathcal{L}_{\prec \mathbb{Z}^L}$ and $\mathcal{L}_{\preceq \mathbb{Z}^L}$, with $L \in \mathsf{WO}$. Let γ be the order type of L.

Using Proposition 1.2.10, if we let $L = \sum_{\beta<\gamma} \mathbb{Z}^\beta \omega$, we have $\mathbb{Z}^\gamma \cong_{\mathsf{LO}} L^* + \mathbf{1} + L$. Then Remark 3.4.1 implies that $\mathcal{L}_{\prec \mathbb{Z}^\gamma}$ is not ccs, because $L^* + \mathbf{1}, \mathbf{1} + L \in \mathcal{L}_{\prec \mathbb{Z}^\gamma}$, yet $L^* + \mathbf{1} + L \notin \mathcal{L}_{\prec \mathbb{Z}^\gamma}$.

Our next goal is to prove that $\mathcal{L}_{\preceq \mathbb{Z}^\gamma}$ is ccs for every ordinal γ and hence that $\mathcal{L}_{\preceq \mathbb{Z}^L}$ is ccs for every countable linear order L. First we need a technical lemma.

Lemma 3.4.3. *Let $\gamma > 0$. If $A \subset \mathbb{Z}^\gamma$ is bounded below then $A \preceq \sum_{\beta<\gamma} \mathbb{Z}^\beta \omega$. Symmetrically, if A is bounded above then $A \preceq \left(\sum_{\beta<\gamma} \mathbb{Z}^\beta \omega\right)^*$.*

Proof. We distinguish two cases. If $\gamma = \beta + 1$ we have $\mathbb{Z}^\gamma \cong_{\mathsf{LO}} \mathbb{Z}^\beta \zeta$. Then it is easy to see that $A \preceq \mathbb{Z}^\beta \omega$.

Let γ be limit. By the boundedness of A from below there exists $\alpha < \gamma$ such that

$$A \preceq \left(\sum_{\beta<\alpha} \mathbb{Z}^\beta \omega\right)^* + \mathbf{1} + \sum_{\beta<\gamma} \mathbb{Z}^\beta \omega = \left(\sum_{\beta<\alpha} \mathbb{Z}^\beta \omega\right)^* + \mathbf{1} + \sum_{\beta<\alpha} \mathbb{Z}^\beta \omega + \sum_{\alpha\le\beta<\gamma} \mathbb{Z}^\beta \omega$$
$$\cong \mathbb{Z}^\alpha + \sum_{\alpha\le\beta<\gamma} \mathbb{Z}^\beta \omega \cong \sum_{\alpha\le\beta<\gamma} \mathbb{Z}^\beta \omega \preceq \sum_{\beta<\gamma} \mathbb{Z}^\beta \omega,$$

where we are using Proposition 1.2.10. □

Proposition 3.4.4. *$\mathcal{L}_{\preceq \mathbb{Z}^\gamma}$ is ccs for every ordinal γ.*

Proof. We argue by induction on γ. If $\gamma = 0$ we have $\mathbb{Z}^\gamma \cong \mathbf{1}$ and $\mathcal{L}_{\preceq \mathbb{Z}^\gamma} = \{\mathbf{1}\}$ is ccs.

Now fix $\gamma \ge 1$ and assume that $\mathcal{L}_{\preceq \mathbb{Z}^\beta}$ is ccs for every $\beta < \gamma$. Consider $K, K' \in \mathcal{L}_{\preceq \mathbb{Z}^\gamma}$ and nonempty convex subsets $(K'_k)_{k\in K}$ of K' such that $\forall k, k' \in K$ $(k <_K k' \to K'_k \le_{K'} K'_{k'})$. We want to show that $\sum_{k\in K} K'_k \preceq \mathbb{Z}^\gamma$. It is convenient to think of K and K' as subsets of $\mathbb{Z}^\gamma$.

We assume that K has a minimum but no maximum: the other cases (no extrema, maximum only and both extrema) can be treated similarly. Pick a sequence $\{k_i \mid i \in \mathbb{N}\}$ cofinal in K with $k_0 = \min K$.

For every i let $B_i = \{k \in K \mid k_i <_K k \le_K k_{i+1}\}$. Then

$$\sum_{k\in K} K'_k = K'_{k_0} + \sum_{i\in\mathbb{N}} \left(\sum_{k\in B_i} K'_k\right)$$

Since K'_{k_0} is bounded above in $\mathbb{Z}^\gamma$, by Lemma 3.4.3 $K'_{k_0} \preceq \left(\sum_{\beta<\gamma} \mathbb{Z}^\beta \omega\right)^*$.

Fix i. Since B_i and $\bigcup_{k\in B_i} K'_k$ are bounded in $\mathbb{Z}^\gamma$ we have $B_i \preceq \mathbb{Z}^{\beta_i} n_i$ and $\bigcup_{k\in B_i} K'_k \preceq \mathbb{Z}^{\beta_i} n'_i$ for some $\beta_i < \gamma$ and $n_i, n'_i \in \mathbb{N}$ (actually, if γ is limit we can choose $n_i = n'_i = 1$, while if $\gamma = \beta + 1$ we can choose $\beta_i = \beta$). We decompose $B_i = \bigcup_{j<n_i} B_{i,j}$ so that $B_{i,j} <_K B_{i,j+1}$ and $\emptyset \ne B_{i,j} \preceq \mathbb{Z}^{\beta_i}$ for every $j < n_i$. For every $j < n_i$ we can write $\bigcup_{k\in B_{i,j}} K'_k = \bigcup_{h<\ell_{i,j}} K'_{i,j,h}$ for some $\ell_{i,j} \le n'_i$, so that $K'_{i,j,h} <_{K'} K'_{i,j,h+1}$ and $\emptyset \ne K'_{i,j,h} \preceq \mathbb{Z}^{\beta_i}$ for every $h < \ell_{i,j}$. For every $j < n_i$ and $h < \ell_{i,j}$ let $K_{i,j,h} = \{k \in B_{i,j} \mid K'_{i,j,h} \cap K'_k \ne \emptyset\}$, and for each $k \in K_{i,j,h}$ set $K'_{i,j,h,k} = K'_{i,j,h} \cap K'_k$. It is clear that $K'_{i,j,h,k} \le_{K'} K'_{i,j,h,k'}$ whenever $k, k' \in K_{i,j,h}$ are such that $k <_K k'$. We can therefore apply the induction hypothesis to $\beta_i < \gamma$ and obtain

$$\sum_{k\in K_{i,j,h}} K'_{i,j,h,k} \preceq \mathbb{Z}^{\beta_i}. \qquad (\star)$$

Since if $k \in B_i$ every element of K'_k belongs to some $K'_{i,j,h,k}$ such that $k \in K_{i,j,h}$ we can write

$$\sum_{k\in B_i} K'_k = \sum_{j<n_i} \left(\sum_{h<\ell_{i,j}} \left(\sum_{k\in K_{i,j,h}} K'_{i,j,h,k}\right)\right).$$

Then by ($\star$) and $\ell_{i,j} \leq n'_i$ we have $\sum_{k\in B_i} K'_k \preceq \mathbb{Z}^{\beta_i}(n'_i n_i)$.

It is now easy to obtain $\sum_{i\in\mathbb{N}}(\sum_{k\in B_i} K'_k) \preceq \sum_{\beta<\gamma} \mathbb{Z}^\beta\omega$. Therefore

$$\sum_{k\in K} K'_k = K'_{k_0} + \sum_{i\in\mathbb{N}}\Big(\sum_{k\in B_i} K'_k\Big) \preceq \Big(\sum_{\beta<\gamma}\mathbb{Z}^\beta\omega\Big)^* + \sum_{\beta<\gamma}\mathbb{Z}^\beta\omega \cong \mathbb{Z}^\gamma,$$

where in the last step we use Proposition 1.2.10. □

We now establish some connections in the context of Borel reducibility among the $\trianglelefteq^{\mathcal{L}}_{\mathsf{LO}}$'s for some of the ccs classes $\mathcal{L}$ we discussed above. Recall that by Proposition 3.4.2 if γ is an indecomposable ordinal both $\mathcal{L}_{\prec\gamma}$ and $\mathcal{L}_{\prec\boldsymbol{\gamma+1}}$ are ccs.

Theorem 3.4.5. *For every additively indecomposable γ we have $\trianglelefteq^{\mathcal{L}_{\prec\gamma}} \leqslant_B \trianglelefteq^{\mathcal{L}_{\prec\boldsymbol{\gamma+1}}}$.*

Proof. We prove that the Borel map $\varphi\colon \mathsf{LO} \to \mathsf{LO}$ defined by $\varphi(L) = L + \mathbf{1}$ is a reduction from $\trianglelefteq^{\mathcal{L}_{\prec\gamma}}$ to $\trianglelefteq^{\mathcal{L}_{\prec\boldsymbol{\gamma+1}}}$. For the not obvious direction, suppose that $L + \mathbf{1} \trianglelefteq^{\mathcal{L}_{\prec\boldsymbol{\gamma+1}}} L' + \mathbf{1}$ with witness $\boldsymbol{\alpha} \in \mathcal{L}$ and the $\boldsymbol{\alpha}$-convex partition $(L_k)_{k\in\boldsymbol{\alpha}}$ of $L+\mathbf{1}$. Since the L_k's partition $L+\mathbf{1}$, there exists $k_0 \in \boldsymbol{\alpha}$ such that $\max(L+\mathbf{1}) \in L_{k_0}$. Then $k_0 = \max\boldsymbol{\alpha}$. Since γ is limit we have $\alpha < \gamma$. Thus $L + \mathbf{1} \trianglelefteq^{\mathcal{L}_{\prec\gamma}} L' + \mathbf{1}$, and in particular we get $L \trianglelefteq^{\mathcal{L}_{\prec\gamma}} L'$. □

Attempting to compare $\trianglelefteq^{\mathcal{L}_{\prec\beta}}$ and $\trianglelefteq^{\mathcal{L}_{\prec\gamma}}$ for β and γ which are far apart seems to be more difficult. We are able to show the existence of a Borel reduction only in certain cases. Recall that an ordinal $\alpha > 1$ is **multiplicatively indecomposable** if $\beta\gamma < \alpha$ for every $\beta, \gamma < \alpha$. It is well known that the infinite multiplicatively indecomposable ordinals are exactly those of the form ω^{ω^ξ} for some ordinal ξ.

Remark 3.4.6. Let $\gamma = \omega^\delta$ be an infinite additively indecomposable ordinal. Writing δ in Cantor normal form, it is easy to see that there is a largest multiplicatively indecomposable ordinal $\beta \leq \gamma$. We call β the **threshold** of γ. This terminology is justified because, writing $\beta = \omega^{\omega^\xi}$ and hence $\gamma = \omega^{\omega^\xi+\theta}$ for some $\theta < \delta$, it is easy to check that:

(a) $\alpha\gamma = \gamma$, for every $0 < \alpha < \beta$;

(b) $\alpha\gamma > \gamma$, for every $\alpha \geq \beta$.

Theorem 3.4.7. *Let γ be infinite additively indecomposable and let β be its threshold. Then $\trianglelefteq^{\mathcal{L}_{\prec\beta}} \leqslant_B \trianglelefteq^{\mathcal{L}_{\prec\boldsymbol{\gamma+1}}}$.*

Proof. Let h be an embedding of γ into $\mathbb{Q}$, and for each $\alpha < \gamma$ consider the linear order $\eta^f(\alpha) = \eta^f_{(h(\alpha),h(\alpha+1))}$ with $f\colon \mathbb{Q} \to \{\mathbf{n} \mid n \in \mathbb{N} \setminus \{0\}\}$ injective. Define the Borel map $\varphi\colon \mathsf{LO} \to \mathsf{LO}$ by

$$\varphi(L) = \sum_{\alpha<\gamma}(\zeta L + \eta^f(\alpha)).$$

We prove that φ is a reduction from $\trianglelefteq^{\mathcal{L}_{\prec\beta}}$ to $\trianglelefteq^{\mathcal{L}_{\prec\boldsymbol{\gamma+1}}}$. Suppose that $L \trianglelefteq^{\mathcal{L}_{\prec\beta}} L'$ with witnesses $\boldsymbol{\xi} \in \mathcal{L}_{\prec\boldsymbol{\beta}}$, $\boldsymbol{\xi}$-convex partition $(L_k)_{k\in\boldsymbol{\xi}}$ of L and embedding f'. Fix $\alpha < \gamma$. Then $(\zeta L)_{\varphi(L)} \trianglelefteq^{\mathcal{L}_{\prec\beta}} (\zeta L')_{\varphi(L')}$ with witnesses $\boldsymbol{\xi}$, $\boldsymbol{\xi}$-convex partition $(\zeta L_k)_{k\in\boldsymbol{\xi}}$ of ζL and embedding $g\colon \zeta L \to \zeta L'$ defined by $g(z,\ell) = (z, f'(\ell))$. Moreover, $(\eta^f(\alpha))_{\varphi(L)} \trianglelefteq_{\mathsf{LO}} (\eta^f(\alpha))_{\varphi(L')}$ via the identity. Notice that $\boldsymbol{\xi}+\mathbf{1} \in \mathcal{L}_{\prec\boldsymbol{\beta}}$ since β is limit. We thus obtain that $\boldsymbol{\xi}+\mathbf{1}$, the $(\boldsymbol{\xi}+\mathbf{1})$-convex partition $\big((\zeta L_k)_{k\in\boldsymbol{\xi}}, \eta^f(\alpha)\big)$ of $\zeta L + \eta^f(\alpha)$ and map $g \cup \mathrm{id}_{\eta^f(\alpha)}$ witness that $(\zeta L + \eta^f(\alpha))_{\varphi(L)} \trianglelefteq^{\mathcal{L}_{\prec\beta}} (\zeta L' + \eta^f(\alpha))_{\varphi(L')}$. By Remark 3.4.6.(a) we have $(\xi+1)\gamma = \gamma$. It is now easy to see that $\varphi(L) \trianglelefteq^{\mathcal{L}_{\prec\boldsymbol{\gamma+1}}} \varphi(L')$ with witnesses $(\boldsymbol{\xi}+\mathbf{1})\boldsymbol{\gamma}$, $((\boldsymbol{\xi}+\mathbf{1})\boldsymbol{\gamma})$-convex partition $\big((\zeta L_k)_{k\in\boldsymbol{\xi}}, \eta^f(\alpha)\big)_{\alpha<\gamma}$ of $\varphi(L)$ and map $\bigcup_{\alpha<\gamma} g \cup \mathrm{id}_{\eta^f(\alpha)}$.

Vice versa, suppose $\varphi(L) \trianglelefteq^{\mathcal{L}_{\prec\boldsymbol{\gamma+1}}} \varphi(L')$ with witnesses $\boldsymbol{\xi} \in \mathcal{L}_{\prec\boldsymbol{\gamma+1}}$, $\boldsymbol{\xi}$-convex partition $(L_k)_{k\in\boldsymbol{\xi}}$ of $\varphi(L)$ and map g. We first claim that $g((\eta^f(\alpha)\times\{\alpha\})_{\varphi(L)}) = (\eta^f(\alpha)\times\{\alpha\})_{\varphi(L')}$ for each $\alpha < \gamma$.

Fix α. Toward a contradiction, suppose that $g((\eta^f(\alpha)\times\{\alpha\})_{\varphi(L)}) \cap \zeta L' \neq \emptyset$. By Lemma 3.2.2(a) there exists a convex subset of $(\eta^f(\alpha)\times\{\alpha\})_{\varphi(L)}$ of the form $\eta^{f''}$ such that $g(\eta^{f''}) \sqsubseteq \zeta L'$.

If $L' \in \mathsf{Scat}$ then $\zeta L'$ is scattered as well and hence $g(\eta^{f''}) \not\trianglelefteq \zeta L'$, reaching a contradiction. If instead $L' \notin \mathsf{Scat}$ then L' is of the form $L' = L'_0 + \eta^{f'} + L'_1$ for some (possibly empty) $L'_0, L'_1 \in \mathsf{Scat}$ and map $f' \colon \mathbb{Q} \to \mathsf{Scat}$. Notice that $\zeta L' \cong_{\mathsf{LO}} \zeta L'_0 + \zeta\eta^{f'} + \zeta L'_1$. Since $\zeta L'_0$ and $\zeta L'_1$ are scattered, by applying the previous argument to $\eta^{f''}$ and $\zeta L'_0$ (respectively $\zeta L'_1$) instead of $(\eta^f(\alpha) \times \{\alpha\})_{\varphi(L)})$ and $\zeta L'$, we obtain that $g(\eta^{f''}) \cap \zeta L'_0 = \emptyset$ and $g(\eta^{f''}) \cap \zeta L'_1 = \emptyset$. Thus, $g(\eta^{f''}) \trianglelefteq \zeta\eta^{f'}$ and, since $\zeta\eta^{f'} \cong_{\mathsf{LO}} \eta^{f'''}$ for some suitable $f''' \colon \mathbb{Q} \to \mathsf{Scat}$, by Lemma 3.2.2(c) we reach a contradiction.

Hence $g((\eta^f(\alpha) \times \{\alpha\})_{\varphi(L)}) \subseteq (\sum_{\alpha<\gamma} \eta^f(\alpha) \times \{\alpha\})_{\varphi(L')}$, and by applying Lemma 3.2.2(c) once more, it follows that $g((\eta^f(\alpha) \times \{\alpha\})_{\varphi(L)}) = (\eta^f(\alpha) \times \{\alpha\})_{\varphi(L')}$.

Therefore for each α, we have that $g(\zeta L \times \{\alpha\}) \subseteq \zeta L' \times \{\alpha\}$. For every $\alpha < \gamma$ let

$$K_\alpha = \{k \in \boldsymbol{\xi} \mid (\zeta L \times \{\alpha\})_{\varphi(L)} \cap L_k \neq \emptyset\}.$$

Since $K_\alpha \trianglelefteq \boldsymbol{\xi}$ we have $K_\alpha \in \mathcal{L}_{\prec\boldsymbol{\gamma+1}}$. Moreover, if $\alpha < \alpha'$ then $K_\alpha \leq_{\boldsymbol{\xi}} K_{\alpha'}$ and, since $\mathcal{L}_{\prec\boldsymbol{\gamma+1}}$ is ccs, we have $\sum_{\alpha<\gamma} K_\alpha \in \mathcal{L}_{\prec\boldsymbol{\gamma+1}}$. Let $\delta = \min\{\xi_\alpha \mid \alpha < \gamma\}$, where ξ_α is the order type of K_α for every α. Then $\delta\gamma \leq \sum_{\alpha<\gamma} \xi_\alpha \leq \gamma$, and hence by Remark 3.4.6(b) we obtain that $\delta < \beta$. Thus there exists α such that $\xi_\alpha < \beta$: then K_α witnesses $\zeta L \trianglelefteq^{\mathcal{L}_{\prec\beta}} \zeta L'$, and by Proposition 3.1.10 we have $L \trianglelefteq^{\mathcal{L}_{\prec\beta}} L'$ as desired. □

The idea of the proof of Theorem 3.4.7 allows us to Borel reduce also $\trianglelefteq^{\mathsf{Fin}}$ to $\trianglelefteq^{\mathcal{L}_{\preceq\zeta}}$.

Theorem 3.4.8. $\trianglelefteq^{\mathsf{Fin}} \leqslant_B \trianglelefteq^{\mathcal{L}_{\preceq\zeta}}$.

Proof. Let h be an embedding of ζ into $\mathbb{Q}$, and for each $z \in \zeta$ consider the linear order $\eta^f(z) = \eta^f_{(h(z),h(z+1))}$, where $f \colon \mathbb{Q} \to \{\mathbf{n} \mid n \in \mathbb{N} \setminus \{0\}\}$ is injective. We imitate the proof of Theorem 3.4.7 and define the Borel function

$$\varphi(L) = \sum_{z\in\zeta} (\zeta L + \eta^f(z)),$$

and show that it is a reduction from $\trianglelefteq^{\mathsf{Fin}}$ to $\trianglelefteq^{\mathcal{L}_{\preceq\zeta}}$. Suppose that $L \trianglelefteq^{\mathsf{Fin}} L'$ with witness $\mathbf{n} \in \mathsf{Fin}$. As in the proof of Theorem 3.4.7, one can show that $\varphi(L) \trianglelefteq^{\mathcal{L}_{\preceq\zeta}} \varphi(L')$ can be witnessed by the linear order $(\mathbf{n}+\mathbf{1})\zeta \cong_{\mathsf{LO}} \zeta$.

Conversely, suppose that $\varphi(L) \trianglelefteq^{\mathcal{L}_{\preceq\zeta}} \varphi(L')$ with witnesses $K \preceq \zeta$, K-convex partition $(L_k)_{k\in K}$ of $\varphi(L)$ and map g. For every $z \in \zeta$ define the set $K_z = \{k \in K \mid (\zeta L \times \{z\})_{\varphi(L)} \cap L_k \neq \emptyset\}$. Arguing as in the proof of Theorem 3.4.7, we obtain that $\sum_{z\in\zeta} K_z \preceq \zeta$. Then by Lemma 3.4.3 it follows that $K_z \preceq \omega^*$ and $K_z \preceq \omega$, and hence K_z is finite for each z. Thus $\zeta L \trianglelefteq^{\mathsf{Fin}} \zeta L'$ and by Proposition 3.1.10 we obtain $L \trianglelefteq^{\mathsf{Fin}} L'$. □

3.5 Uncountable linear orders

Roughly speaking, generalized descriptive set theory is obtained by replacing ω with an uncountable cardinal κ in all basic definitions and notions from classical descriptive set theory. For example, one considers the **generalized Cantor space** 2^κ of all binary κ-sequences equipped with the topology generated by the sets of the form $\{x \in 2^\kappa \mid s \subseteq x\}$, where s varies among all binary sequences of length $< \kappa$. Borel sets are then replaced by **κ^+-Borel sets**, i.e. sets in the κ^+-algebra generated by the open sets of the given topological space. The notions of **κ^+-Borel function** and **κ^+-Borel reducibility** $\leq^\kappa_B$ are defined accordingly. (See [AMR22] for a quite comprehensive introduction to the subject.)

The usefulness of this approach is that it allows us to tackle classification problems for uncountable structures with tools which resemble, to some extent, those used in the classical setting—one can look at [FHK14b, MMR21, HKM17] for some of the most significant results in this direction connecting classification/complexity in terms of generalized descriptive set theory with Shelah's stability theory.

In the present setting, we can form the space

$$\mathsf{LO}_\kappa = \{L \in 2^{\kappa\times\kappa} \mid L \text{ codes a reflexive linear order on } \kappa\}$$

of (codes for) linear orders on κ. This is endowed with the relative topology inherited from $2^{\kappa\times\kappa}$, once the latter is identified in the obvious way with the generalized Cantor space 2^κ. This is the same as the topology generated by the neighborhood base of $L \in \mathsf{LO}_\kappa$ determined by the sets $\{L' \in \mathsf{LO}_\kappa \mid L' \upharpoonright \alpha = L \upharpoonright \alpha\}$ for $\alpha < \kappa$.

Working in this setup, we consider $\mathcal{L}$-convex embeddability only when $\mathcal{L}$ is a set of countable linear orders, i.e. $\mathcal{L} \subseteq \mathsf{Lin}$.

We denote by $\trianglelefteq^{\mathcal{L}}_\kappa$ the restriction of $\trianglelefteq^{\mathcal{L}}$ to LO_κ. In Theorem 3.1.9 we showed that $\trianglelefteq^{\mathcal{L}}_\kappa$ is a quasi-order if and only if $\mathcal{L}$ is ccs.

Denote now by $\bowtie^{\mathcal{L}}_\kappa$ the equivalence relation induced by $\trianglelefteq^{\mathcal{L}}_\kappa$ on LO_κ. Notice that in this generalized context every class $\mathcal{L}$ is κ^+-Borel, and hence $\bowtie^{\mathcal{L}}_\kappa$ is κ^+-analytic for any $\mathcal{L}$.

It is easy to check that the map φ from the proof of Theorem 3.3.3 is a κ^+-Borel map from LO_κ to itself that witnesses the following theorem.

Theorem 3.5.1. *Let κ be any uncountable cardinal and $\mathcal{L} \subseteq \mathsf{Lin}$ be ccs. Then the isomorphism relation $\cong_\kappa$ on LO_κ is κ^+-Borel reducible to $\bowtie^{\mathcal{L}}_\kappa$.*

Combining this with [HK15, Theorem 1.13] we immediately get the following completeness result, which in the case of $\mathcal{L} = \{\mathbf{1}\}$ is in stark contrast with the countable setting (Corollaries 2.3.14 and 2.3.16).

Theorem 3.5.2. *Assume $\mathsf{V} = \mathsf{L}$, and let $\kappa = \lambda^+$ with λ regular and $\mathcal{L} \subseteq \mathsf{Lin}$ be ccs. Then the relation $\bowtie^{\mathcal{L}}_\kappa$ is complete for κ^+-analytic equivalence relations (with respect to κ^+-Borel reducibility).*

The construction from Proposition 3.2.7 can be used to uncover a significant difference between embeddability and $\mathcal{L}$-convex embeddability among uncountable linear orders when $\mathcal{L} \neq \mathsf{Lin}$. By the celebrated five-element basis theorem of J. Moore [Moo06], assuming PFA there is a finite basis (of size 5) for the embeddability relation on uncountable linear orders. If we move to $\mathcal{L}$-convex embeddability, working in ZFC alone we instead obtain the following result which, when $\kappa = \aleph_1$, implies that there is no finite or even countable basis for the class of uncountable linear orders.

Theorem 3.5.3. *For every cardinal κ of uncountable cofinality and $\mathcal{L} \subseteq \mathsf{Scat}$ ccs, there are at least $2^{\aleph_0}$-many $\trianglelefteq^{\mathcal{L}}_\kappa$-incomparable $\trianglelefteq^{\mathcal{L}}_\kappa$-minimal elements in LO_κ.*

Proof. First observe that for every countable L, the linear order $L\kappa = \sum_{\alpha<\kappa} L$ is $\trianglelefteq^{\mathcal{L}}_\kappa$-minimal in LO_κ. This is basically because $L\kappa$ is κ-like, i.e. for every $n \in L\kappa$ the initial segment $(-\infty, n]_{L\kappa}$ has size $< \kappa$. Thus if $K \in \mathcal{L}$, the K-convex partition $(L_n)_{n\in K}$ and $f\colon L' \to L\kappa$ witness $L' \trianglelefteq^{\mathcal{L}}_\kappa L\kappa$ for some $L' \in \mathsf{LO}_\kappa$, then there is $n \in K$ such that L_n has size κ and hence is a final segment of L' (here we are using that κ has uncountable cofinality) and $f(L_n) \sqsupseteq \sum_{\beta\leq\alpha<\kappa} L$ for some $\beta < \kappa$, which we can assume to be the least with this property. But since clearly $L\kappa \cong \sum_{\beta<\alpha<\kappa} L \sqsubseteq f(L_n)$, we then get that $L\kappa \trianglelefteq^{\mathcal{L}}_\kappa L'$ and thus $L' \bowtie^{\mathcal{L}}_\kappa L\kappa$.

Let now η^{f_S} be defined as in Proposition 3.2.7 for every infinite $S \subseteq \mathbb{N}$, and consider $\mathcal{A} = \{\eta^{f_S}\kappa \mid S \subseteq \mathbb{N}\}$. Then by the previous argument each $\eta^{f_S}\kappa$ is $\trianglelefteq^{\mathcal{L}}_\kappa$-minimal. We now claim that the elements of $\mathcal{A}$ are pairwise $\trianglelefteq^{\mathcal{L}}_\kappa$-incomparable. Suppose that $\eta^{f_S}\kappa \trianglelefteq^{\mathcal{L}}_\kappa \eta^{f_{S'}}\kappa$ with witnesses $K \in \mathcal{L}$, $(L_k)_{k\in K}$ and $g\colon \eta^{f_S}\kappa \to \eta^{f_{S'}}\kappa$. Since K is countable while $\eta^{f_S}\kappa$ is uncountable, there is $k \in K$ such that $\eta^{f_S} \sqsubseteq L_k$. Thus $g(\eta^{f_S}) \sqsubseteq \eta^{f_{S'}}\kappa$, and since $g(\eta^{f_S})$ is countable there is a countable $\alpha < \kappa$ such that $g(\eta^{f_S}) \sqsubseteq \eta^{f_{S'}}\boldsymbol{\alpha}$. Then by Proposition 3.2.7(a) $S = S'$. By the fact that $|\mathcal{A}| = 2^{\aleph_0}$ we finally get the desired result. □

Remark 3.5.4. Theorem 3.5.3 is optimal in the context of Moore's theorem because under PFA we have $2^{\aleph_0} = 2^{\aleph_1}$, and thus the family of $\trianglelefteq^{\mathcal{L}}_{\aleph_1}$-minimal elements that we constructed is as large as possible. In particular, all bases for $\trianglelefteq^{\mathcal{L}}_{\aleph_1}$ on $\mathsf{LO}_{\aleph_1}$ have maximal size.

II

Descriptive set theory on geometrical objects

4

Anti-classification results in knot theory

4.1 Knots and proper arcs: definitions and basic facts

In mathematics, there are essentially two ways to formalize the intuitive concept of a knot: a (mathematical) *knot* is obtained from a real-life knot by joining its ends so that it cannot be undone, while a *proper arc* is obtained by embedding the real-life knot in a closed 3-ball and sticking its ends to the border of the ball, so that again it cannot be undone. The two concepts are strictly related, although not equivalent as there exist knots that cannot be "cut" to obtain a proper arc ([Bin56]). Let us recall the main definitions and related concepts.

Depending on the situation, we think of S^1 as either the unit circle in $\mathbb{R}^2$ or the one-point compactification of $\mathbb{R}$ obtained by adding ∞ to the space. Similarly, S^3 can be viewed as the one-point compactification of $\mathbb{R}^3$.

Definition 4.1.1. A **knot** K is a homeomorphic image of S^1 in S^3, that is, a subspace of S^3 of the form $K = \operatorname{Im} f$ for some topological embedding $f\colon S^1 \to S^3$.

The collection of all knots is denoted by Kn; as shown in [Kul17], it can be construed as a standard Borel space. Obviously, if $K \subseteq S^3$ is a knot and $\varphi\colon S^3 \to S^3$ is an embedding, then $\varphi(K)$ is a knot as well.

Remark 4.1.2. Knots can be naturally endowed with a circular order induced by the standard circular order C_{S^1} defined on S^1 (see Section 1.3). More precisely, let $f\colon S^1 \to S^3$ be an embedding and $K = \operatorname{Im} f$ be the knot induced by f. Then for every $x, y, z \in K$ we can set

$$C_f(x, y, z) \iff C_{S^1}(f^{-1}(x), f^{-1}(y), f^{-1}(z)).$$

If $f, f'\colon S^1 \to S^3$ are two embeddings giving rise to the same knot $K = \operatorname{Im} f = \operatorname{Im} f'$, then $f^{-1} \circ f'\colon S^1 \to S^1$ is a homeomorphism, and thus it is either order-preserving or order-reversing with respect to C_{S^1}. It follows that either $C_f = C_{f'}$ or $C_f = C^*_{f'}$. Thus a knot K can be endowed with exactly two circular orders, corresponding to the two possible orientations of K sometimes used in knot theory, which are one the reverse of the other one and depend on the specific embedding used to witness $K \in \mathrm{Kn}$. We speak of **oriented** knot K when we single out one specific orientation between the two possibilities.

Two knots $K, K' \in \mathrm{Kn}$ are **equivalent**, in symbols $K \equiv_{\mathrm{Kn}} K'$, if there exists a homeomorphism $\varphi\colon S^3 \to S^3$ such that $\varphi(K) = K'$. The relation $\equiv_{\mathrm{Kn}}$ is an analytic equivalence relation on Kn. A knot is **trivial** if it is equivalent to the unit circle $I_{\mathrm{Kn}} = \{(x, y, z) \in S^3 \mid x^2 + y^2 = 1 \wedge z = 0\}$.

Remark 4.1.3. In knot theory it is more common two consider the oriented version of $\equiv_{\mathrm{Kn}}$, according to which two knots K and K' are equivalent if there is an *orientation-preserving* homeomorphism $\varphi\colon S^3 \to S^3$ such that $\varphi(K) = K'$ or, equivalently, an ambient isotopy sending K to K'. Nevertheless, we are mostly going to prove anti-classification results, and thus they become even stronger if we consider the coarser equivalence $\equiv_{\mathrm{Kn}}$. For the interested reader, however, we point out that all our results remain true if we stick to common practice and replace all the relevant equivalence relations and quasi-orders with their oriented versions. Similar considerations apply to the ensuing definitions and results concerning proper arcs.

We now move to proper arcs. Given $x \in \mathbb{R}^3$ and a positive $r \in \mathbb{R}$, the closed ball with center x and radius r is denoted by $\bar{B}(x,r)$. The origin $(0,0,0)$ of $\mathbb{R}^3$ is sometimes denoted by $\bar{0}$. To avoid repetitions, we convene that from now $\bar{B}$, possibly with subscripts and/or superscripts, is always a closed topological 3-ball, i.e. a homeomorphic copy of a closed ball in $\mathbb{R}^3$. Recall that by compactness of $\bar{B}$ and the invariance of domain theorem, the notion of boundary of $\bar{B}$ as a topological subspace of $\mathbb{R}^3$ and the notion of boundary of $\bar{B}$ as a topological 3-manifold coincide. Thus we can unambiguously denote by $\partial\bar{B}$ the **boundary** of $\bar{B}$, and set $\operatorname{Int}\bar{B} = \bar{B} \setminus \partial\bar{B}$. Notice also that by the same reasons, if $\varphi\colon \bar{B} \to \mathbb{R}^3$ is an embedding, then $\varphi(\partial\bar{B}) = \partial\,\varphi(\bar{B})$ and $\varphi(\operatorname{Int}\bar{B}) = \operatorname{Int}\varphi(\bar{B})$.

Definition 4.1.4. Given a topological embedding $f\colon [0,1] \to \bar{B}$ we say that the pair $(\bar{B}, \operatorname{Im} f)$ is an **proper arc** if $f(x) \in \partial\bar{B} \iff x = 0 \vee x = 1$. With an abuse of notation which is standard in knot theory, when there is no danger of confusion *we identify f with its image* $\operatorname{Im} f$ and write e.g. $(\bar{B}, f)$ in place of $(\bar{B}, \operatorname{Im} f)$.

Any proper arc $(\bar{B}, f)$ can be canonically turned (up to knot equivalence) into a knot $K_{(\bar{B},f)}$ by joining its ends $f(0)$ and $f(1)$ with a simple curve running on the boundary $\partial\bar{B}$ of its ambient space. The collection of proper arcs is denoted by Ar, and can be construed as a standard Borel subspace of the product $K(\mathbb{R}^3) \times K(\mathbb{R}^3)$ of the Vietoris space $K(\mathbb{R}^3)$ over $\mathbb{R}^3$ (see [Kul17] for the analogous construction of the coding space Kn of knots). Notice that if $(\bar{B}, f)$ is a proper arc and $\varphi\colon \bar{B} \to \mathbb{R}^3$ is an embedding, then $(\varphi(\bar{B}), \varphi(f)) = (\varphi(\bar{B}), \varphi(\operatorname{Im} f))$ is a proper arc, as witnessed by the embedding $\varphi \circ f\colon [0,1] \to \varphi(\bar{B})$.

Remark 4.1.5. (1) Every specific embedding f giving rise to an arc $(\bar{B}, \operatorname{Im} f)$ induces an orientation on it, namely, the linear order $\leq_f$ on $\operatorname{Im} f$ defined by

$$b_0 \leq_f b_1 \iff f^{-1}(b_0) \leq f^{-1}(b_1).$$

If $f, f'\colon [0,1] \to \bar{B}$ are two topological embeddings inducing the same proper arc (that is, $\operatorname{Im} f = \operatorname{Im} f'$), then $f^{-1} \circ f'\colon [0,1] \to [0,1]$ is a homeomorphism, and thus it is either order-preserving or order-reversing. It follows that every proper arc has exactly two orientations. Moreover, the minimum and the maximum of $\leq_f$ always exists and they can be identified, independently of f, as the only points of $\operatorname{Im} f$ belonging to $\partial\bar{B}$. We speak of **oriented** proper arc $(\bar{B}, f)$ when we equip it with the specific orientation given by the displayed f.

(2) If $(\bar{B}, f)$ and $(\bar{B}', g)$ are proper arcs and $\varphi\colon \bar{B} \to \bar{B}'$ is a topological embedding such that $\varphi(\operatorname{Im} f) \subseteq \operatorname{Im} g$, then $h = g^{-1} \circ \varphi \circ f\colon [0,1] \to [0,1]$ is a topological embedding. It follows that when $(\bar{B}, f)$ and $(\bar{B}', g)$ are construed as oriented proper arcs, then φ is either order-preserving (that is, $\varphi(b_0) \leq_g \varphi(b_1)$ for all $b_0, b_1 \in \bar{B}$ with $b_0 \leq_f b_1$) or order-reversing (that is, $\varphi(b_1) \leq_g \varphi(b_0)$ for all $b_0, b_1 \in \bar{B}$ with $b_0 \leq_f b_1$).

Two proper arcs $(\bar{B}, f)$ and $(\bar{B}', g)$ are **equivalent**, in symbols $(\bar{B}, f) \equiv_{\mathrm{Ar}} (\bar{B}', g)$, if there exists a homeomorphism $\varphi\colon \bar{B} \to \bar{B}'$ such that $\varphi(\operatorname{Im} f) = \operatorname{Im} g$. The relation $\equiv_{\mathrm{Ar}}$ is an analytic equivalence relation on the standard Borel space Ar. A proper arc $(\bar{B}, f)$ is **trivial** if it is equivalent to $I_{\mathrm{Ar}} = (\bar{B}(\bar{0}, 1), [-1,1] \times \{(0,0)\})$.

An important dividing line among knots (respectively, proper arcs) is given by tameness, i.e. the absence of singular points. Given a knot $K \in \mathrm{Kn}$, a **subarc** of K is any proper arc of the form $(\bar{B}, K \cap \bar{B})$. A point $x \in K$ is called **singular**, or a **singularity**, of K if there is no $\bar{B}$ such that $x \in \operatorname{Int}\bar{B}$ and $(\bar{B}, K \cap \bar{B})$ is a trivial proper subarc of K. The space of singularities of K is denoted by Σ_K. An **isolated** singular point of K is an isolated point of the topological space Σ_K, and the (sub)space of isolated singular points of K is denoted by $I\Sigma_K$. Finally, a knot K is **tame**[1]

[1] Our definition of tame knot is equivalent to the classical one, according to which a knot is tame if it is equivalent to a finite polygon (see [BZ03, Definition 1.3]).

if it has no singular point, and **wild** otherwise. Notice also that if $x \in K$ is not a singularity of K, then there are arbitrarily small closed topological 3-balls $\bar{B}$ witnessing this.

The previous definitions can be naturally adapted to proper arcs. Let $(\bar{B}, f) \in \mathrm{Ar}$. A point $x \in \operatorname{Im} f$ is called **singular**, or a **singularity**, of $(\bar{B}, f)$ if it belongs to $\Sigma_{K_{(\bar{B},f)}}$, while an **isolated** singular point of $(\bar{B}, f)$ is an an element of $I\Sigma_{K_{(\bar{B},f)}}$. Accordingly, the space of singularities of $(\bar{B}, f)$ is denoted by $\Sigma_{(\bar{B},f)}$, while the space of isolated singular points is denoted by $I\Sigma_{(\bar{B},f)}$. An arc $(\bar{B}, f)$ is **tame** if $\Sigma_{(\bar{B},f)} = \emptyset$ (equivalently, if $K_{(\bar{B},f)}$ is tame), and **wild** otherwise. Notice that if $x \in \operatorname{Im} f \cap \operatorname{Int} \bar{B}$, then $x \notin \Sigma_{(\bar{B},f)}$ if and only if there is $\bar{B}' \subseteq \bar{B}$ such that $x \in \operatorname{Int} \bar{B}'$ and $(\bar{B}', f \cap \bar{B}')$ is a trivial proper arc. For points on the boundary $\partial\bar{B}$, instead, it is not enough to consider closed topological 3-balls $\bar{B}' \subseteq \bar{B}$, as we necessarily need to consider a "trivial prolungation" of the curve $\operatorname{Im} f$ beyond its extreme points in order to determine whether they are singular or not.

We also introduce a notion of a circularization of a proper arc, which generates a knot and gives a characterization of tame knots.

Definition 4.1.6. Let $(\bar{B}, f) \in \mathrm{Ar}$. Up to equivalence, we can assume that $\bar{B} = [-1, 1]^3$, $f(0) = (-1, 0, 0)$ and $f(1) = (1, 0, 0)$. Consider the equivalence relation obtained setting $(-1, y, z) \sim (1, y, z)$ for all $(y, z) \in [-1, 1]^2$, so that in the quotient space $T = [-1, 1]^3/\sim$ the two lateral faces of the cube $\bar{B}$ are glued and we have a solid torus. Given a topological embedding h of T into S^3, we call **circularitazion of** $(\bar{B}, f)$, denoted by $C^h[(\bar{B}, f)]$, the knot which is obtained as the image of $\operatorname{Im} f/\sim$ via h.

Notice that the circularization of a proper arc depends on the topological embedding of the solid torus into S^3, hence it is not unique. Moreover, we have that $K \in \mathrm{Kn}$ is tame if and only if $K = C^h[I_{\mathrm{Ar}}]$ for some topological embedding $h \colon T \to S^3$.

A substantial part of the analysis of tame knots relies on their prime factorization, which is in turn based on the classical notion of sum (see [BZ03, Definition 2.7], where the sum is actually called product). We introduce a corresponding sum for proper arcs which is even more natural than the sum of knots, and in fact it applies to arbitrary proper arcs. As in the case of knot sums, in order to have a well-defined operation (up to equivalence) we need to consider oriented proper arcs.

Definition 4.1.7. Let $(\bar{B}_0, f_0)$ and $(\bar{B}_1, f_1)$ be *oriented* proper arcs. Up to equivalence, we may assume that $\bar{B}_0 = [-1, 0] \times [-1, 1]^2$, $\bar{B}_1 = [0, 1] \times [-1, 1]^2$, $f_0(0) = (-1, 0, 0)$, $f_0(1) = f_1(0) = (0, 0, 0)$, and $f_1(1) = (1, 0, 0)$. The **sum** $(\bar{B}_0, f_0) \oplus (\bar{B}_1, f_1)$ is the proper arc $(\bar{B}, f)$ where $\bar{B} = [-1, 1]^3$ and $f \colon [0, 1] \to \bar{B}$ is defined by $f(x) = f_0(2x)$ if $x \leq \frac{1}{2}$ and $f(x) = f_1(2x - 1)$ if $x \geq \frac{1}{2}$.

By induction, one can then define finite sums of proper arcs $(\bar{B}_0, f_0) \oplus \cdots \oplus (\bar{B}_n, f_n)$, abbreviated by $\bigoplus_{i \leq n} (\bar{B}_i, f_i)$, for every $n \in \mathbb{N}$.

Remark 4.1.8. Although the sum of two oriented proper arcs is again oriented, in this thesis we will tacitly consider it as an unoriented proper arc. Also, we will often sum unoriented proper arcs: what we mean in this case is that the arcs are summed using the natural orientation coming from the way we present them.

Finally, we notice that the sum # of tame knots (see [BZ03, Definition 7.1]) can be defined using $\oplus$. Indeed, every (oriented) tame knot can be turned into a(n oriented) proper arc $(\bar{B}_K, f_K)$ as follows: choose $\bar{B}_K$ so that $K \subseteq \operatorname{Int} \bar{B}_K$, cut K at an "external" point of any of its planar projections, and attach the two ends to distinct points on the boundary of $\bar{B}_K$ by means of trivial arcs running outside the knot. Given two oriented tame knots we then have that, up to equivalence, $K_0 \# K_1 = K_{(\bar{B}_{K_0}, f_{K_0}) \oplus (\bar{B}_{K_1}, f_{K_1})}$. Notice also that if $(\bar{B}_0, f_0)$ and $(\bar{B}_1, f_1)$ are (oriented) tame proper arcs, then $K_{(\bar{B}_0,f_0) \oplus (\bar{B}_1,f_1)} \equiv_{\mathrm{Kn}} K_{(\bar{B}_0,f_0)} \# K_{(\bar{B}_1,f_1)}$. Recall that a nontrivial tame knot is **prime** if it cannot be written as a sum of nontrivial knots. Every tame knot can be uniquely written as a finite sum of prime knots (see [BZ03, Theorem 7.12]).

4.2 Proper arcs and their classification

The following notion was introduced, with a different terminology, in [Kul17, Definition 2.10].

Definition 4.2.1. Let $(\bar{B}, f), (\bar{B}', g) \in \mathrm{Ar}$. We say that $(\bar{B}, f)$ is a **subarc** of $(\bar{B}', g)$, or that $(\bar{B}', g)$ has $(\bar{B}, f)$ as subarc, if there is $(\bar{B}_0, h) \in \mathrm{Ar}$ and $\bar{B}_1 \subseteq \bar{B}_0$ such that $(\bar{B}', g) \equiv_{\mathrm{Ar}} (\bar{B}_0, h)$, $(\bar{B}_1, h \cap \bar{B}_1) \in \mathrm{Ar}$, and $(\bar{B}, f) \equiv_{\mathrm{Ar}} (\bar{B}_1, h \cap \bar{B}_1)$.

It is convenient to reformulate the notion of subarc as follows.

Definition 4.2.2. Let $(\bar{B}, f), (\bar{B}', g) \in \mathrm{Ar}$. We set

$$(\bar{B}, f) \precsim_{\mathrm{Ar}} (\bar{B}', g)$$

if there exists a topological embedding $\varphi\colon \bar{B} \to \bar{B}'$ such that $\varphi(f) = g \cap \mathrm{Im}\ \varphi$. (Notice that we automatically have that $(\varphi(\bar{B}), g \cap \mathrm{Im}\,\varphi))$ is a proper arc.)

Proposition 4.2.3. *Let $(\bar{B}, f), (\bar{B}', g) \in \mathrm{Ar}$. Then $(\bar{B}, f)$ is a subarc of $(\bar{B}', g)$ if and only if $(\bar{B}, f) \precsim_{\mathrm{Ar}} (\bar{B}', g)$.*

Proof. We only prove the forward direction, as the other implication is obvious. Let $(\bar{B}_0, h)$ and $\bar{B}_1 \subseteq \bar{B}_0$ witness that $(\bar{B}, f)$ is a subarc of $(\bar{B}', g)$, and let $\varphi_0\colon \bar{B}' \to \bar{B}_0$ and $\varphi_1\colon \bar{B} \to \bar{B}_1$ be homeomorphisms witnessing $(\bar{B}', g) \equiv_{\mathrm{Ar}} (\bar{B}_0, h)$ and $(\bar{B}, f) \equiv_{\mathrm{Ar}} (\bar{B}_1, h \cap \bar{B}_1)$, respectively. Then the map $\varphi_0^{-1} \circ \varphi_1\colon \bar{B} \to \bar{B}'$ is a topological embedding such that

$$(\varphi_0^{-1} \circ \varphi_1)(f) = \varphi_0^{-1}(h \cap \bar{B}_1) = \varphi_0^{-1}(h) \cap \varphi_0^{-1}(\bar{B}_1) = g \cap \mathrm{Im}(\varphi_0^{-1} \circ \varphi_1).$$

Therefore $(\bar{B}, f) \precsim_{\mathrm{Ar}} (\bar{B}', g)$. □

We can thus identify the relation $\precsim_{\mathrm{Ar}}$ and the **subarc relation** on Ar of Definition 4.2.1. Clearly, the relation $\precsim_{\mathrm{Ar}}$ is an analytic quasi-order on the standard Borel space Ar. We denote by $\prec_{\mathrm{Ar}}$ the strict part of $\precsim_{\mathrm{Ar}}$, i.e.

$$(\bar{B}, f) \prec_{\mathrm{Ar}} (\bar{B}', g) \iff (\bar{B}, f) \precsim_{\mathrm{Ar}} (\bar{B}', g) \wedge (\bar{B}', g) \not\precsim_{\mathrm{Ar}} (\bar{B}, f).$$

The analytic equivalence relation associated to $\precsim_{\mathrm{Ar}}$ is denoted by $\approx_{\mathrm{Ar}}$, and we say that two proper arcs $(\bar{B}, f)$ and $(\bar{B}', g)$ are **mutual subarcs** if

$$(\bar{B}, f) \approx_{\mathrm{Ar}} (\bar{B}', g).$$

This may be interpreted as asserting that the two arcs have the "same complexity" because each of them is a subarc of the other one. Notice also that $(\bar{B}, f) \equiv_{\mathrm{Ar}} (\bar{B}', g)$ trivially implies $(\bar{B}, f) \approx_{\mathrm{Ar}} (\bar{B}', g)$.

If $(\bar{B}, f), (\bar{B}', g) \in \mathrm{Ar}$ and φ witnesses $(\bar{B}, f) \equiv_{\mathrm{Ar}} (\bar{B}', g)$, then φ induces a homeomorphism between the spaces $\Sigma_{(\bar{B}, f)}$ and $\Sigma_{(\bar{B}', g)}$, and hence also a homeomorphism between $I\Sigma_{(\bar{B}, f)}$ and $I\Sigma_{(\bar{B}', g)}$. If instead $\varphi\colon \bar{B} \to \bar{B}'$ is just an embedding witnessing $(\bar{B}, f) \precsim_{\mathrm{Ar}} (\bar{B}', g)$, then we still have that φ induces an embedding of $\Sigma_{(\bar{B}, f)}$into $\Sigma_{(\bar{B}', g)}$, but needs not send isolated singular points into isolated singular points: if $x \in I\Sigma_{(\bar{B}, f)} \cap \partial \bar{B}$, then it might happen that $\varphi(x) \in \Sigma_{(\bar{B}', g)} \setminus I\Sigma_{(\bar{B}', g)}$. However, this is the only exception.

Lemma 4.2.4. (a) *Let $(\bar{B}, f) \in \mathrm{Ar}$ and $\bar{B}' \subseteq \bar{B}$ be such that $(\bar{B}', f \cap \bar{B}') \in \mathrm{Ar}$. Then we have $\Sigma_{(\bar{B}', f \cap \bar{B}')} \subseteq \Sigma_{(\bar{B}, f)}$, and $\Sigma_{(\bar{B}', f \cap \bar{B}')} \cap \mathrm{Int}\, \bar{B}' = \Sigma_{(\bar{B}, f)} \cap \mathrm{Int}\, \bar{B}'$.*

(b) *Let $(\bar{B}, f), (\bar{B}', g) \in \mathrm{Ar}$, and let $\varphi\colon \bar{B} \to \bar{B}'$ witness $(\bar{B}, f) \precsim_{\mathrm{Ar}} (\bar{B}', g)$. If $x \in I\Sigma_{(\bar{B}, f)} \cap \mathrm{Int}\, \bar{B}$, then $\varphi(x) \in I\Sigma_{(\bar{B}', g)}$.*

Proof. (a) The first part is easy and is left to the reader. For the nontrivial inclusion of the second part, assume that $x \in \mathrm{Int}\, \bar{B}'$ (so that $x \in \mathrm{Int}\, \bar{B}$ as well because $\bar{B}' \subseteq \bar{B}$) and $x \notin \Sigma_{(\bar{B}', f \cap \bar{B}')}$. Let $\bar{B}'' \subseteq \bar{B}'$ be a witness of this: then $\bar{B}''$ also witnesses $x \notin \Sigma_{(\bar{B}, f)}$.

(b) By hypothesis and the fact that $\varphi\colon \bar{B} \to \varphi(\bar{B})$ is a homeomorphism, $\varphi(x) \in I\Sigma_{(\varphi(\bar{B}), g \cap \mathrm{Im}\, \varphi)} \cap \mathrm{Int}\, \varphi(\bar{B})$. By part (a), this implies that $\varphi(x) \in \Sigma_{(\bar{B}', g)}$. Using $\varphi(x) \in \mathrm{Int}\, \varphi(\bar{B})$, pick a small enough open set $U \subseteq \mathrm{Int}\, \varphi(\bar{B})$ such that $U \cap \Sigma_{(\varphi(\bar{B}), g \cap \mathrm{Im}\, \varphi)} = \{\varphi(x)\}$: then by part (a) again $U \cap \Sigma_{(\bar{B}', g)} = U \cap \Sigma_{(\varphi(\bar{B}), g \cap \mathrm{Im}\, \varphi)}$, and thus $U \cap \Sigma_{(\bar{B}', g)}$ witnesses $\varphi(x) \in I\Sigma_{(\bar{B}', g)}$. □

We now define an infinitary version of the sum operation for (tame) proper arcs introduced in Definition 4.1.7. Since the ambient space $\bar{B}$ in the definition of a proper arc is a compact space, in order to define such infinitary sums we need the summands to accumulate towards a point $b \in \bar{B}$, which thus becomes a singularity when infinitely many summands are not trivial.

Definition 4.2.5. Let $(\bar{B}_i, f_i)$ be oriented proper arcs, for $i \in \mathbb{N}$.[2] The **(infinite) sum with limit** $b \in \bar{B}$, denoted by $\bigoplus_{i\in\mathbb{N}}^{b} (\bar{B}_i, f_i)$, is defined up to equivalence as follows. Without loss of generality, we may assume that b is of the form $(b', 0, 0)$ for some b' with $0 < b' \leq 1$. Up to equivalence, we may also assume that $\bar{B}_i = [b' - 2^{-i}, b' - 2^{-(i+1)}] \times [-2^{-i}, 2^{-i}]^2$, and that $f_i(0) = (b' - 2^{-i}, 0, 0)$ and $f_i(1) = (b' - 2^{-(i+1)}, 0, 0)$ for all $i \in \mathbb{N}$. Then $\bigoplus_{i\in\mathbb{N}}^{b} (\bar{B}_i, f_i)$ is the arc $(\bar{B}, f)$ where $\bar{B} = [-1, 1]^3$ and $\operatorname{Im} f$ is the union of $\bigcup_{i\in\mathbb{N}} \operatorname{Im} f_i$ together with $[-1, b' - 1] \times \{(0, 0)\}$ and $[b', 1] \times \{(0, 0)\}$ (the latter might reduce to the point $(1, 0, 0)$ if $b' = 1$ or, equivalently, if $b \in \partial\bar{B}$).

Trivially, $(\bar{B}_j, f_j) \precsim_{\mathrm{Ar}} \bigoplus_{i\leq n} (\bar{B}_i, f_i)$ for all $n \geq j$ and $(\bar{B}_j, f_j) \precsim_{\mathrm{Ar}} \bigoplus_{i\in\mathbb{N}}^{b}(\bar{B}_i, f_i)$ for all $b \in \bar{B}$. Notice that, up to $\equiv_{\mathrm{Ar}}$, Definition 4.2.5 gives rise to precisely two non-equivalent proper arcs, depending on whether $b \in \partial\bar{B}$ or not—besides this dividing line the actual choice of the limit point $b \in \bar{B}$ is completely irrelevant. Therefore we can simplify the notation by denoting with $\bigoplus_{i\in\mathbb{N}} (\bar{B}_i, f_i)$ the infinite sum $\bigoplus_{i\in\mathbb{N}}^{b} (\bar{B}_i, f_i)$ for some/any $b \in \operatorname{Int}\bar{B}$, and with $\bigoplus_{i\in\mathbb{N}}^{\partial} (\bar{B}_i, f_i)$ the infinite sum $\bigoplus_{i\in\mathbb{N}}^{b} (\bar{B}_i, f_i)$ for some/any $b \in \partial\bar{B}$. It is not hard to see that $\bigoplus_{i\in\mathbb{N}}^{\partial} (\bar{B}_i, f_i) \prec_{\mathrm{Ar}} \bigoplus_{i\in\mathbb{N}} (\bar{B}_i, f_i)$. Finally, if all the proper arcs $(\bar{B}_i, f_i)$ are equivalent to the same arc $(\bar{B}', g)$, the two possible infinite sums will be denoted by $\bigoplus_{\mathbb{N}} (\bar{B}', g)$ and $\bigoplus_{\mathbb{N}}^{\partial} (\bar{B}', g)$, respectively. Obviously, we can also replace $\mathbb{N}$ with any infinite $A \subseteq \mathbb{N}$ and write $\bigoplus_{j\in A}^{(\partial)} (\bar{B}_j, f_j)$ to denote $\bigoplus_{i\in\mathbb{N}}^{(\partial)} (\bar{B}_{r(j)}, f_{r(j)})$, where $r \colon \mathbb{N} \to A$ is the increasing enumeration of A; similarly for $\bigoplus_{A}^{(\partial)} (\bar{B}', g)$.

Figure 4.1 presents the arc $\bigoplus_{\mathbb{N}} (\bar{B}', g)$ where $(\bar{B}', g)$ is the trefoil; its variant $\bigoplus_{\mathbb{N}}^{\partial}(\bar{B}', g)$ would be obtained my moving the current limit point $(0, 0, 0)$ to the point $(1, 0, 0)$ on $\partial\bar{B}$.

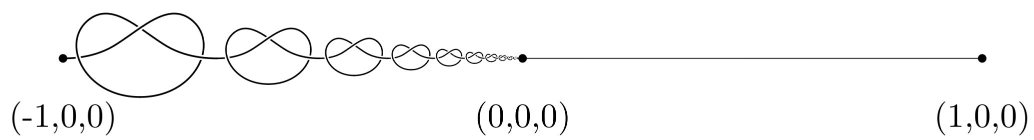

Figure 4.1: Infinite sum of trefoils, with limit point internal to the ambient space $\bar{B} = [-1, 1]^3$.

In [Kul17, Theorem 3.1] it is shown that the isomorphism $\cong_{\mathsf{LO}}$ on countable linear orders Borel reduces to equivalence $\equiv_{\mathrm{Kn}}$ on knots. Employing the same construction, we establish a similar connection between convex embeddability $\trianglelefteq_{\mathsf{LO}}$ on linear orders and the subarc relation $\precsim_{\mathrm{Ar}}$ on proper arcs.

Fix a proper arc $(\bar{B}^*, f^*)$ of the form $\bigoplus_{\mathbb{N}} (\bar{B}_i, f_i)$ with all the proper arcs $(\bar{B}_i, f_i)$ tame and not trivial. (For the sake of definiteness, one can e.g. assume that $(\bar{B}^*, f^*)$ is the sum of infinitely many trefoils depicted in Figure 4.1.) An important feature of such a $(\bar{B}^*, f^*)$ is that

> Any embedding $\varphi \colon \bar{B}^* \to \bar{B}^*$ with $\varphi(f^*) = f^* \cap \operatorname{Im}\varphi$ preserves the (natural) orientation of the arc. (†)

Notice also that the only singularity of $(\bar{B}^*, f^*)$, which is trivially isolated, belongs to $\operatorname{Int}\bar{B}^*$.

We first define a Borel map that given $L \in \mathsf{LO}$ produces an order-embedding h_L of L into $(\mathbb{Q}, \leq)$ and a function $r_L \colon L \to \mathbb{Q}$ such that:

(a) the open intervals $V_n^L = (h_L(n) - 2r_L(n), h_L(n) + 2r_L(n))$ are included in $[-1, 1]$ and pairwise disjoint;

(b) $\bigcup_{n\in\mathbb{N}} V_n^L$ is dense in $[-1, 1]$.

[2] When summing unoriented proper arcs, if not specified otherwise we use the natural orientation coming from their presentation.

To this end, we first establish in a Borel way whether L has extrema, what are they, and when one element of the linear order is the immediate successor of another.

Notice that $\lim_{n\to\infty} r_L(n) = 0$ and that we can assume that $r_L(n+1) < r_L(n)$ for every $n \in \mathbb{N}$. Let $U_n^L = [h_L(n)-r_L(n), h_L(n)+r_L(n)]$. Thinking of $[-1,1]$ as lying on the x-axis, we replace U_n^L with the cube $\bar{B}_n^L = U_n^L \times [-r_L(n), r_L(n)]^2$. Let $(\bar{B}_n^L, f_n^L)$ be equivalent to $(\bar{B}^*, f^*)$ and such that $f_n^L(0)$ and $f_n^L(1)$ both belong to the x-axis, and set $f_r^L = ([-1,1] \setminus \bigcup_{n\in\mathbb{N}} U_n^L) \times \{(0,0)\}$. Then we define the map

$$F\colon \mathsf{LO} \to \mathrm{Ar}, \qquad L \mapsto (\bar{B}, f_L) \tag{4.2.1}$$

by letting $\bar{B} = [-1,1]^3$ and $f_L = f_r^L \cup \bigcup_{n\in\mathbb{N}} f_n^L$.

By construction, every $(h_L(n), 0, 0)$ is singular and isolated in $\Sigma_{F(L)}$ by (the trace of) $\bar{B}_n^L$, and every other member of $\Sigma_{F(L)}$ is a limit of these singular points. Thus $\Sigma_{F(L)}$ is contained in the x-axis and $I\Sigma_{F(L)} = \{(h_L(n), 0, 0) \mid n \in \mathbb{N}\}$. The latter is naturally ordered by considering the restriction of $\leq_{f_L}$ to $I\Sigma_{F(L)}$, or equivalently, by considering first coordinates ordered as elements of $\mathbb{R}$. Then the map $n \mapsto (h_L(n), 0, 0)$ is an isomorphism between the linear orders L and $I\Sigma_{F(L)}$.

Since the entire construction really depends on the proper arc $(\bar{B}^*, f^*)$, when relevant we will add this information to the notation and write e.g. $F_{(\bar{B}^*, f^*)}(L)$. For future reference, we also notice that by construction $I\Sigma_{F(L)} \subseteq \operatorname{Int} \bar{B}$.

Theorem 4.2.6. *The map F from equation (4.2.1) simultaneously witnesses $\trianglelefteq_{\mathsf{LO}} \leq_B \precsim_{\mathrm{Ar}}$ (hence also $\bowtie_{\mathsf{LO}} \leq_B \approx_{\mathrm{Ar}}$) and $\cong_{\mathsf{LO}} \leq_B \equiv_{\mathrm{Ar}}$.*

The lower bound $\cong_{\mathsf{LO}} \leq_B \equiv_{\mathrm{Ar}}$ for the relation $\equiv_{\mathrm{Ar}}$ is implicit in (the proof of) [Kul17, Theorem 3.1]. Notice however that our proof is more natural, as it avoids reducing first $\cong_{\mathsf{LO}}$ to its restriction to linear orders with minimum and without maximum, and then the latter to the relations on arcs and knots, as it is done instead in [Kul17].

Proof. In order to check that F is a Borel function between the Polish space LO and the standard Borel space Ar one can argue as in [Kul17], so we only need to prove that F is a reduction.

Assume first that $L, L' \in \mathsf{LO}$ are such that $L \trianglelefteq_{\mathsf{LO}} L'$, and let $g\colon L \to L'$ witness this. For every $n \in L$ the proper arcs $(\bar{B}_n^L, f_n^L)$ and $(\bar{B}_{g(n)}^{L'}, f_{g(n)}^{L'})$ are both equivalent to $(\bar{B}^*, f^*)$, and hence we can consider a homeomorphism $\varphi_1^n\colon \bar{B}_n^L \to \bar{B}_{g(n)}^{L'}$ witnessing this. Notice that φ_1^n is necessarily order-preserving by (†). Let $\varphi_1 = \bigcup_{n\in\mathbb{N}} \varphi_1^n$ and notice that $\varphi_1(h_L(n), 0, 0) = (h_{L'}(g(n)), 0, 0)$ for every $n \in L$, so that the restriction of φ_1 to $I\Sigma_{F(L)}$ is order-preserving into $I\Sigma_{F(L')}$.

For every $n \in L$ let $M_n = \max\{2r_L(n), 2r_{L'}(g(n))\}$ and let $\varphi_2^n\colon \overline{V_n^L} \times [-M_n, M_n]^2 \to \overline{V_{g(n)}^{L'}} \times [-M_n, M_n]^2$ be a homeomorphism which extends φ_1^n and has the following properties:

(i) for all $(y,z) \in [-M_n, M_n]^2$, $\varphi_2^n(h_L(n) \pm 2r_L(n), y, z) = (h_{L'}(g(n)) \pm 2r_{L'}(g(n))y, z)$;

(ii) for all $(y,z) \in [-M_n, M_n]^2$ with $\max\{|y|, |z|\} = M_n$ and for all $t \in [-1,1]$ we have

$$\varphi_2^n(h_L(n) + 2r_L(n)t, y, z) = (h_{L'}(g(n)) + 2r_{L'}(g(n))t, y, z)$$

(this condition is missing in [Kul17]).

Let $W_n^L = \overline{V_n^L} \times [-1,1]^2$ and $W_{g(n)}^{L'} = \overline{V_{g(n)}^{L'}} \times [-1,1]^2$. We can then define a homeomorphism $\varphi_3^n\colon W_n^L \to W_{g(n)}^{L'}$ which extends φ_2^n and is such that:

(iii) for every $(y,z) \in [-1,1]^2$ such that $\max\{|y|, |z|\} \geq M_n$ we have

$$\varphi_3^n(h_L(n) + 2r_L(n)t, y, z) = (h_{L'}(g(n)) + 2r_{L'}(g(n))t, y, z),$$

so that outside $\overline{V_n^L} \times [-M_n, M_n]^2$ the lines parallel to the x-axis are mapped into themselves.

Then $\varphi_3 = \bigcup_{n\in\mathbb{N}} \varphi_3^n$ is a homeomorphism between $\bigcup_{n\in\mathbb{N}} W_n^L$ and $\bigcup_{n\in\mathbb{N}} W_{g(n)}^{L'}$.

We finally extend φ_3 to $\varphi\colon \bar{B} \to \bar{B}$ by looking at each $x_0 \in [-1,1] \setminus \bigcup_{n\in\mathbb{N}} \overline{V_n^L}$ (which is a cluster point of $\operatorname{Im} h_L$) and setting $\varphi(x_0,y,z) = (x_0',y,z)$ for every $(y,z) \in [-1,1]^2$, where $x_0 = \lim_{i\to\infty} h_L(n_i)$ and $x_0' = \lim_{i\to\infty} h_{L'}(g(n_i))$. Condition (iii) ensures that φ is continuous and indeed a homeomorphism. It is immediate that φ witnesses $F(L) \precsim_{\mathrm{Ar}} F(L')$, and that if $g\colon L \to L'$ was actually an isomorphism, then φ witnesses $F(L) \equiv_{\mathrm{Ar}} F(L')$.

Conversely, suppose that $\varphi\colon \bar{B} \to \bar{B}$ is an embedding witnessing $F(L) \precsim_{\mathrm{Ar}} F(L')$. Since all isolated points of $F(L)$ belong to $\operatorname{Int} \bar{B}$, by Lemma 4.2.4 the map $\varphi \restriction I\Sigma_L$ embeds $I\Sigma_{F(L)}$ into $I\Sigma_{F(L')}$. Furthermore, as explained in [Kul17], the embedding φ preserves the betweenness relation. By (†), for any $n \in L$ the restriction of φ to the arc $(\bar{B}_n^L, f_L \cap \bar{B}_n^L)$, which maps it to $(\varphi(\bar{B}_n^L), f_{L'} \cap \varphi(\bar{B}_n^L))$, is order-preserving and hence $\varphi \restriction I\Sigma_{F(L)}$ is order-preserving too. Moreover, since φ is continuous and f_L is connected we get that also $\varphi(f_L)$ is connected: it follows that $\varphi(I\Sigma_{F(L)})$ is a convex subset of $I\Sigma_{F(L')}$. Summing up, $\varphi \restriction I\Sigma_{F(L)}$ witnesses that $I\Sigma_{F(L)} \trianglelefteq I\Sigma_{F(L')}$, hence $L \trianglelefteq_{\mathsf{LO}} L'$ because $L \cong I\Sigma_{F(L)} \trianglelefteq I\Sigma_{F(L')} \cong L'$. Obviously, if $\varphi\colon \bar{B} \to \bar{B}$ was actually a homeomorphism, then $\varphi \restriction I\Sigma_{F(l)}$ would be onto $I\Sigma_{F(L')}$, and thus it would witness $I\Sigma_{F(L)} \cong I\Sigma_{F(L')}$, which in turn implies $L \cong_{\mathsf{LO}} L'$. □

By Theorem 4.2.6, the reduction $F\colon \mathsf{LO} \to \mathrm{Ar}$ allows us to transfer some combinatorial properties of $\trianglelefteq_{\mathsf{LO}}$ discussed in Section 2.2 to the quasi-order $\precsim_{\mathrm{Ar}}$ (cfr. Lemma 2.2.3, Proposition 2.2.4, and Corollary 2.3.18).

Corollary 4.2.7. (a) *There is an embedding from the partial order* $(\operatorname{Int}(\mathbb{R}),\subseteq)$ *into* $\precsim_{\mathrm{Ar}}$, *and indeed* $(\operatorname{Int}(\mathbb{R}),\subseteq) \leq_B \precsim_{\mathrm{Ar}}$.

(b) $\precsim_{\mathrm{Ar}}$ *has chains of order type* $(\mathbb{R},<)$, *as well as antichains of size* $2^{\aleph_0}$.

In contrast, the combinatorial properties uncovered in Propositions 2.2.5, 2.2.8, and 2.2.10, being universal statements, do not transfer through the reduction F. We overcome some of these difficulties by using the following construction.

Using the orientation induced by f, when $(\bar{B},f)$ is a proper arc the set $I\Sigma_{(\bar{B},f)}$ can naturally be viewed as a linear order $L_{(\bar{B},f)} = (I\Sigma_{(\bar{B},f)}, \leq_f)$. Since $\Sigma_{(\bar{B},f)}$, being a subspace of the Polish space $\bar{B}$, is second-countable, the set $I\Sigma_{(\bar{B},f)}$ is (at most) countable and thus up to isomorphism $L_{(\bar{B},f)}$ is an element of Lin. We remark that the linear order $L_{(\bar{B},f)}$ really depends on the topological embedding f (or, more precisely, on the orientation it induces) rather than its image. However, if f and f' are two topological embeddings giving rise to the same arc, then either $L_{(\bar{B},f)} = L_{(\bar{B},f')}$ or $L_{(\bar{B},f)} = (L_{(\bar{B},f')})^*$ — indeed the two linear orders correspond to the two possible orientations of the arc $(\bar{B},\operatorname{Im} f)$. Recall that by construction, for proper arcs of the form[3] $F(L) = (\bar{B},f_L)$ we have $I\Sigma_{F(L)} \cong L$.

Lemma 4.2.8. *Let* $(\bar{B},f),(\bar{B}',g) \in \mathrm{Ar}$ *be such that* $(\bar{B},f) \precsim_{\mathrm{Ar}} (\bar{B}',g)$, *and let* $K = (I\Sigma_{(\bar{B},f)} \cap \operatorname{Int} \bar{B}, \leq_f)$. *Then either* $K \trianglelefteq L_{(\bar{B}',g)}$ *or* $K \trianglelefteq (L_{(\bar{B}',g)})^*$.

Proof. Let $\varphi\colon \bar{B} \to \bar{B}'$ be an embedding witnessing $(\bar{B},f) \precsim_{\mathrm{Ar}} (\bar{B}',g)$. By Lemma 4.2.4(b), $\varphi \restriction (I\Sigma_{(\bar{B},f)} \cap \operatorname{Int} \bar{B})$ is an embedding of $I\Sigma_{(\bar{B},f)} \cap \operatorname{Int} \bar{B}$ into $I\Sigma_{(\bar{B}',g)}$, and arguing as in the proof of Theorem 4.2.6 we can observe that $\varphi(I\Sigma_{(\bar{B},f)} \cap \operatorname{Int} \bar{B})$ is a convex subset of $I\Sigma_{(\bar{B}',g)}$ with respect to $\leq_g$ because $\operatorname{Im} f \cap \operatorname{Int} \bar{B}$, which is homeomorphic to $(0,1)$, is connected. As already noticed, the embedding φ is either order-preserving or order-reversing (Remark 4.1.5(2)): thus $\varphi \restriction (I\Sigma_{(\bar{B},f)} \cap \operatorname{Int} \bar{B})$ witnesses $K \trianglelefteq L_{(\bar{B}',g)}$ in the former case, and $K \trianglelefteq (L_{(\bar{B}',g)})^*$ in the latter. □

Remark 4.2.9. In the special case where $(\bar{B},f)$ is of the form $F(L)$ for some $L \in \mathsf{LO}$, then all its isolated singular points belong to $\operatorname{Int} \bar{B}$ and $I\Sigma_{F(L)} \cong L$. Thus in this case Lemma 4.2.8 reads as follows: For every $L \in \mathsf{LO}$ and $(\bar{B}',g) \in \mathrm{Ar}$ with $F(L) \precsim_{\mathrm{Ar}} (\bar{B}',g)$, either $L \trianglelefteq L_{(\bar{B}',g)}$ or $L \trianglelefteq (L_{(\bar{B}',g)})^*$.

[3] If not specified otherwise, we always choose the natural orientation of $F(L)$.

Lemma 4.2.8 allows us to prove analogues of Propositions 2.2.5 and 2.2.10.

Theorem 4.2.10. $\mathfrak{b}(\precsim_{\mathrm{Ar}}) = \aleph_1$ *and* $\mathfrak{d}(\precsim_{\mathrm{Ar}}) = 2^{\aleph_0}$.

Proof. We begin with the unbounding number. First notice that $\mathfrak{b}(\precsim_{\mathrm{Ar}}) > \aleph_0$ because given a countable family of proper arcs $\{(\bar{B}_i, f_i) \mid i \in \mathbb{N}\}$, their infinite sum $\bigoplus_{\mathbb{N}} (\bar{B}_i, f_i)$ is a $\precsim_{\mathrm{Ar}}$-upper bound for them. To show the existence of an $\precsim_{\mathrm{Ar}}$-unbounded family of arcs of size $\aleph_1$ we use Proposition 2.2.5 as follows. Let $F \colon \mathsf{LO} \to \mathrm{Ar}$ be the reduction introduced in (4.2.1), and consider the family $\{F(\boldsymbol{\alpha}) \mid \omega \leq \alpha < \omega_1\}$. It is strictly $\precsim_{\mathrm{Ar}}$-increasing by Theorem 4.2.6. Suppose towards a contradiction that there is $(\bar{B}, f) \in \mathrm{Ar}$ such that $F(\boldsymbol{\alpha}) \precsim_{\mathrm{Ar}} (\bar{B}, f)$ for all $\alpha < \omega_1$. Then $I\Sigma_{(\bar{B},f)}$ would be infinite and thus the linear order $L = L_{(\bar{B},f)}$ would be, up to isomorphism, an element of LO. By Lemma 4.2.8 and Remark 4.2.9, this would lead to the fact that $L + L^*$ is a $\trianglelefteq_{\mathsf{LO}}$-upper bound for WO, contradicting Proposition 2.2.5.

We now deal with the dominating number. Consider once again the $\trianglelefteq_{\mathsf{LO}}$-antichain $\mathcal{A} = \{L_S \mid S \subseteq \mathbb{N}\}$, where $L_S = \eta_{f_S}$ is as in the proof of Proposition 2.2.8(a), and notice that by the usual back-and-forth argument $L_S \cong (L_S)^*$. We first prove the analogue of Claim 2.2.9.1.

Claim 4.2.10.1. For every proper arc $(\bar{B}, f) \in \mathrm{Ar}$, the collection

$$\{F(L_S) \mid F(L_S) \precsim_{\mathrm{Ar}} (\bar{B}, f)\}$$

is countable.

Proof of the Claim. By Lemma 4.2.8, Remark 4.2.9, and $L_S \cong (L_S)^*$ we have that

$$\{L_S \in \mathcal{A} \mid F(L_S) \precsim_{\mathrm{Ar}} (\bar{B}, f)\} \subseteq \{L_S \in \mathcal{A} \mid L_S \trianglelefteq L_{(\bar{B},f)}\}.$$

If $L_{(\bar{B},f)}$ is finite, then the latter set is empty and so is the set in the claim; if instead $L_{(\bar{B},f)}$ is infinite then, up to isomorphism, it is a member of LO, and thus the result easily follows from Claim 2.2.9.1. □

The proof of the theorem can now be completed using the same argument of Proposition 2.2.10: every element of a $\precsim_{\mathrm{Ar}}$-dominating family has only countably many proper arcs of the form $F(L_S)$ below it, and since by Theorem 4.2.6 there are $2^{\aleph_0}$-many such arcs the dominating family must have size $2^{\aleph_0}$ too. □

As in the case of linear orders, one can then derive the following analogue of Corollary 2.2.7. However, the proof is slightly more delicate.

Corollary 4.2.11. *Every proper arc* $(\bar{B}, f)$ *is the bottom of an* $\precsim_{\mathrm{Ar}}$*-unbounded chain of length* ω_1.

Proof. Consider the sequence of proper arcs $((\bar{B}_\alpha, f_\alpha))_{\alpha<\omega_1}$ where $(\bar{B}_0, f_0) = (\bar{B}, f)$ and, for $\alpha \geq 1$, $(\bar{B}_\alpha, f_\alpha) = (\bar{B}, f) \oplus F(\boldsymbol{\omega} + \boldsymbol{\alpha})$. For every $\alpha \leq \beta < \omega_1$ we have $(\bar{B}_\alpha, f_\alpha) \precsim_{\mathrm{Ar}} (\bar{B}_\beta, f_\beta)$, and if $\alpha > 0$ we also have $F(\boldsymbol{\omega} + \boldsymbol{\alpha}) \precsim_{\mathrm{Ar}} (\bar{B}_\alpha, f_\alpha)$. By (the proof of) Theorem 4.2.10, this implies that the sequence is $\precsim_{\mathrm{Ar}}$-unbounded. Moreover, for every $\alpha < \omega_1$ there is $\beta > \alpha$ such that $(\bar{B}_\beta, f_\beta) \not\precsim_{\mathrm{Ar}} (\bar{B}_\alpha, f_\alpha)$ (and hence $(\bar{B}_\alpha, f_\alpha) \prec_{\mathrm{Ar}} (\bar{B}_\beta, f_\beta)$), as otherwise $(\bar{B}_\alpha, f_\alpha)$ would be an upper bound for the sequence $((\bar{B}_\alpha, f_\alpha))_{\alpha<\omega_1}$. It follows that we can extract from the latter a strictly $\precsim_{\mathrm{Ar}}$-increasing subsequence of length ω_1 with $(\bar{B}, f)$ as a first element: since such a subsequence is $\precsim_{\mathrm{Ar}}$-cofinal in $((\bar{B}_\alpha, f_\alpha))_{\alpha<\omega_1}$, it is $\precsim_{\mathrm{Ar}}$-unbounded as well and the proof is complete. □

We now move to the possible generalizations of Proposition 2.2.8, i.e. we discuss minimal elements and bases for the relation $\precsim_{\mathrm{Ar}}$.

If we consider only tame proper arcs, which form a $\precsim_{\mathrm{Ar}}$-downward closed subclass of the collection of all proper arcs, then the situation is pretty clear: the trivial arc I_{Ar} is the $\precsim_{\mathrm{Ar}}$-minimum within this class. Call **prime arc** any proper arc of the form $(\bar{B}_K, f_K)$ for K a prime knot: then one can observe that prime arcs are $\precsim_{\mathrm{Ar}}$-minimal above I_{Ar} (and are the unique such). Indeed, assume that $(\bar{B}, f)$ is a prime arc and that $(\bar{B}', g) \precsim_{\mathrm{Ar}} (\bar{B}, f)$ for some $(\bar{B}', g) \in \mathrm{Ar}$. Since $(\bar{B}, f)$ is tame, without loss of generality we can assume that $\bar{B}' \subseteq \mathrm{Int}\, \bar{B}$. Let $K_1 = K_{(\bar{B}',g)}$, and let

K_2 be the knot obtained from the remainder $f \setminus \operatorname{Int} \bar{B}'$ by connecting $g(0)$ and $g(1)$ with a simple curve lying on $\partial \bar{B}'$ and the extrema $f(0)$ and $f(1)$ with a simple curve on $\partial \bar{B}$. By construction, the prime knot used to construct $(\bar{B}, f)$ is the sum of K_1 and K_2, thus one of K_1 and K_2 is trivial. In the former case $(\bar{B}', g) \equiv_{\mathrm{Ar}} I_{\mathrm{Ar}}$, while in the latter $(\bar{B}', g) \equiv_{\mathrm{Ar}} (\bar{B}, f)$.

Prime arcs play the same role in the realm of tame proper arcs as prime knots do in the realm of tame knots: every tame proper arc is of the form $\bigoplus_{i \leq n} (\bar{B}^p_i, f^p_i)$ for some (unique, up to permutations) sequence of prime arcs $(\bar{B}^p_i, f^p_i)$. This has a number of consequences on the structure of *nontrivial* tame proper arcs under $\precsim_{\mathrm{Ar}}$:

(1) There are no infinite descending chains.

(2) Since up to equivalence there are only countably many tame proper arcs, and since there are infinitely many prime arcs (consider e.g. the prime arcs obtained from the (p, q) torus knots, where $p, q > 1$), then the collection of prime arcs constitute a countably infinite antichain basis. In particular, there are no finite bases.

(3) If $(\bar{B}^p_i, f^p_i)$, for $i \in \mathbb{N}$, is an enumeration of the prime arcs, then $(\bigoplus_{i \leq n} (\bar{B}^p_i, f^p_i))_{n \in \mathbb{N}}$ is an unbounded ω-chain. In particular, there is no $\precsim_{\mathrm{Ar}}$-maximal tame proper arc, and the unbounding number of $\precsim_{\mathrm{Ar}}$ restricted to tame proper arcs is $\aleph_0$.

(4) Every dominating family is infinite: below every tame proper arc there are only finitely many of the infinitely many pairwise $\precsim_{\mathrm{Ar}}$-incomparable prime arcs, thus no finite family can be dominating with respect to $\precsim_{\mathrm{Ar}}$. Hence the dominating number of $\precsim_{\mathrm{Ar}}$ restricted to tame proper arcs is $\aleph_0$.

Having obtained the desired information in the realm of tame proper arcs, it is now natural to move to the wild side and consider the restriction $\precsim_{\mathrm{WAr}}$ of $\precsim_{\mathrm{Ar}}$ to the collection WAr of wild arcs. By $\trianglelefteq_{\mathsf{LO}}$-minimality of η_{f_S}, Lemma 4.2.8 and Remark 4.2.9, one may be tempted to conjecture that the proper arcs $F(\eta_{f_S})$ used in the proof of Theorem 4.2.10 are $\precsim_{\mathrm{WAr}}$-minimal. That is not quite true, as the arc $(\bar{B}^*, f^*) = \bigoplus_{\mathbb{N}} (\bar{B}_i, f_i)$ used to define the reduction $F = F_{(\bar{B}^*, f^*)} \colon \mathsf{LO} \to \mathrm{Ar}$ from (4.2.1) is such that $(\bar{B}^*, f^*) \prec_{\mathrm{WAr}} F(\eta_{f_S})$, and moreover the proper arc $(\bar{B}^\partial, f^\partial) = \bigoplus^\partial_{\mathbb{N}} (\bar{B}_i, f_i)$ is such that $(\bar{B}^\partial, f^\partial) \prec_{\mathrm{WAr}} (\bar{B}^*, f^*) \prec_{\mathrm{WAr}} F(\eta_{f_S})$. However, the following lemma allows us to obtain useful information on the $\precsim_{\mathrm{WAr}}$-predecessors of $F(\eta_{f_S})$.

Lemma 4.2.12. *Let $\{(\bar{B}^p_i, f^p_i) \mid i \in \mathbb{N}\}$ be a family of (oriented) prime arcs, and let $(\bar{B}^*, f^*) = \bigoplus_{i \in \mathbb{N}} (\bar{B}^p_i, f^p_i)$ and $(\bar{B}^\partial, f^\partial) = \bigoplus^\partial_{i \in \mathbb{N}} (\bar{B}^p_i, f^p_i)$. We consider the proper arc $F_{(\bar{B}^*, f^*)}(L_S)$ for some $S \subseteq \mathbb{N}$, where $L_S = \eta_{f_S}$ is as in the proof of Proposition 2.2.8(a).*

(a) *If $(\bar{B}', g)$ is a prime arc, then $(\bar{B}', g) \precsim_{\mathrm{Ar}} (\bar{B}^\partial, f^\partial)$ if and only if there is $\bar{\imath} \in \mathbb{N}$ such that $(\bar{B}', g) \equiv_{\mathrm{Ar}} (\bar{B}^p_{\bar{\imath}}, f^p_{\bar{\imath}})$. The same is true if $(\bar{B}^\partial, f^\partial)$ is replaced by $F_{(\bar{B}^*, f^*)}(L_S)$.*

Let now $(\bar{B}, f)$ be an arbitrary wild proper arc. Then:

(b) *$(\bar{B}, f) \precsim_{\mathrm{WAr}} (\bar{B}^\partial, f^\partial)$ if and only if $(\bar{B}, f) \equiv_{\mathrm{Ar}} \bigoplus^\partial_{j \in A} (\bar{B}^p_j, f^p_j)$ for some infinite set $A \subseteq \mathbb{N}$.*

(c) *If $(\bar{B}, f) \precsim_{\mathrm{WAr}} F_{(\bar{B}^*, f^*)}(L_S)$, then there is $\bar{B}' \subseteq \bar{B}$ such that $(\bar{B}', f \cap \bar{B}') \in \mathrm{WAr}$ and $(\bar{B}', f \cap \bar{B}') \precsim_{\mathrm{WAr}} (\bar{B}^\partial, f^\partial)$.*

Proof. (a) One direction is obvious. For the other direction, assume that $(\bar{B}', g) \precsim_{\mathrm{Ar}} (\bar{B}^\partial, f^\partial)$. Recall that the ambient space $\bar{B}^\partial$ of $\bigoplus^\partial_{i \in \mathbb{N}} (\bar{B}^p_i, f^p_i)$ is the cube $[-1, 1]^3$, and that its only singularity is the point $(1, 0, 0)$. By the way we defined infinite sums, without loss of generality we may assume that $\bar{B}^p_i = [1 - 2^{-i}, 1 - 2^{-(i+1)}] \times [-1, 1]^2$ and that $f^p_i(0) = (1 - 2^{-i}, 0, 0)$, and $f^p_i(1) = (1 - 2^{-(i+1)}, 0, 0)$ for all $i \in \mathbb{N}$. Let $\varphi \colon \bar{B}' \to [-1, 1]^3$ witness $(\bar{B}', g) \precsim_{\mathrm{Ar}} (\bar{B}^\partial, f^\partial)$, and notice that since $(\bar{B}^p_0, f^p_0)$ is tame we may assume $\operatorname{Im} \varphi \subseteq [0, 1] \times [-1, 1]^2$. Let $I = \{i \in \mathbb{N} \mid \varphi(\bar{B}') \cap f^p_i \cap \operatorname{Int} \bar{B}^p_i \neq \emptyset\}$: it is convex (with respect to the usual order $\leq$ on $\mathbb{N}$) because g is connected. Without loss of generality, we may assume that for all $i \in I$ the space $\bar{B}'_i = \varphi(\bar{B}') \cap \bar{B}^p_i$ is a closed topological 3-ball. Consider the (tame) proper arcs $(\bar{B}'_i, f_i \cap \bar{B}'_i)$, which by construction are such that either

$(\bar{B}', g) \equiv_{\mathrm{Ar}} \bigoplus_{i \in I} (\bar{B}'_i, f_i \cap \bar{B}'_i)$ if I is finite, or $(\bar{B}', g) \equiv_{\mathrm{Ar}} \bigoplus^{\partial}_{i \in I} (\bar{B}'_i, f_i \cap \bar{B}'_i)$ if I is infinite. Moreover each $(\bar{B}'_i, f_i \cap \bar{B}'_i)$ is either trivial or equivalent to the corresponding $(\bar{B}^p_i, f^p_i)$ because $(\bar{B}'_i, f_i \cap \bar{B}'_i) \precsim (\bar{B}^p_i, f^p_i)$ and the latter is prime. If all the $(\bar{B}'_i, f_i \cap \bar{B}'_i)$'s were trivial, then $(\bar{B}', g)$ would be trivial too, a contradiction. Let $\bar{\imath} \in I$ be such that $(\bar{B}'_{\bar{\imath}}, f_{\bar{\imath}} \cap \bar{B}'_{\bar{\imath}})$ is not trivial: then $(\bar{B}'_{\bar{\imath}}, f_{\bar{\imath}} \cap \bar{B}'_{\bar{\imath}}) \equiv_{\mathrm{Ar}} (\bar{B}^p_{\bar{\imath}}, f^p_{\bar{\imath}})$, and since $(\bar{B}', g)$ is prime and $\varphi^{-1} \restriction \bar{B}'_{\bar{\imath}}$ witnesses $(\bar{B}'_{\bar{\imath}}, f_{\bar{\imath}} \cap \bar{B}'_{\bar{\imath}}) \precsim_{\mathrm{Ar}} (\bar{B}', g)$ it follows that $(\bar{B}^p_{\bar{\imath}}, f^p_{\bar{\imath}}) \equiv_{\mathrm{Ar}} (\bar{B}', g)$, as desired.

Suppose now that $(\bar{B}', g) \precsim_{Ar} F_{(\bar{B}^*, f^*)}(L_S)$ via some φ. Recall the notation used in the proof of Theorem 4.2.6 and in the discussion preceding it. Since $(\bar{B}', g)$ is tame and not trivial, $\varphi(g)$ is tame and cannot be contained in $[h_{L_S}(m), h_{L_S}(m) + 2r_{L_S}(m)] \times [-1,1]^2$ for any $m \in L_S$; therefore $\varphi(g)$ must be contained either in $[h_{L_S}(n) - 2r_{L_S}(n), h_{L_S}(n) - \varepsilon] \times [-1,1]^2$ or in $[h_{L_S}(m), h_{L_S}(n) - \varepsilon] \times [-1,1]^2$ for some consecutive $m, n \in L_S$ and small enough $\varepsilon > 0$. However, since the part on the right of the singularity $h_{L_S}(m)$ is trivial and $(\bar{B}^p_0, f^p_0)$ is tame, we can actually assume that we are always in the first case and that $\operatorname{Im} \varphi \subseteq [h_{L_S}(n) - 2r_{L_S}(n), h_{L_S}(n)] \times [-1,1]^2$. Since the subarc of $F_{(\bar{B}^*, f^*)}(L_S)$ determined by the latter set is equivalent to $\bigoplus^{\partial}_{i \in \mathbb{N}} (\bar{B}^p_i, f^p_i)$, we are done by the first part.

(b) Let $\varphi \colon \bar{B} \to [-1,1]^3$ witness $(\bar{B}, f) \precsim_{\mathrm{WAr}} (\bar{B}^\partial, f^\partial)$. Being wild and $\precsim_{\mathrm{WAr}}$-below an arc with only one singularity, the proper arc $(\bar{B}, f)$ has a unique singularity $x \in \bar{B}$: clearly, $\varphi(x) = (1,0,0)$ by Lemma 4.2.4(a) and thus, necessarily, $x \in \partial \bar{B}$. As before, set $I = \{ i \in \mathbb{N} \mid \varphi(\bar{B}) \cap f^p_i \cap \operatorname{Int} \bar{B}^p_i \neq \emptyset \}$: now, we know that I is a final segment of $(\mathbb{N}, \leq)$ because $\varphi(x) = (1,0,0)$. We can assume that $\operatorname{Im} \varphi \setminus (\{1\} \times [-1,1]^2) \subseteq \bigcup_{i \in I} \bar{B}^p_i$ and that for all $i \in I$ the space $\bar{B}'_i = \varphi(\bar{B}) \cap \bar{B}^p_i$ is a closed topological 3-ball. Then φ witnesses $(\bar{B}, f) \equiv_{\mathrm{Ar}} \bigoplus^{\partial}_{i \in I} (\bar{B}'_i, f^p_i \cap \bar{B}'_i)$. Each of the proper arcs $(\bar{B}'_i, f^p_i \cap \bar{B}'_i)$, being a subarc of the prime arc $(\bar{B}^p_i, f^p_i)$, is either trivial or equivalent to it: set

$$A = \{ i \in I \mid (\bar{B}'_i, f^p_i \cap \bar{B}'_i) \equiv_{\mathrm{Ar}} (\bar{B}^p_i, f^p_i) \}.$$

The set A is infinite because otherwise $\bigoplus^{\partial}_{i \in I} (\bar{B}'_i, f^p_i \cap \bar{B}'_i)$, and hence also $(\bar{B}, f)$, would be tame. Moreover, each of the $(\bar{B}'_i, f^p_i \cap \bar{B}'_i)$ is tame, so the trivial arcs $(\bar{B}'_i, f^p_i \cap \bar{B}'_i)$ occurring in the sequence can be "absorbed" by the next $(\bar{B}'_j, f^p_j \cap \bar{B}'_j)$ with $j \in A$. Therefore

$$(\bar{B}, f) \equiv_{\mathrm{Ar}} \bigoplus^{\partial}_{i \in I} (\bar{B}'_i, f^p_i \cap \bar{B}'_i) \equiv_{\mathrm{Ar}} \bigoplus^{\partial}_{j \in A} (\bar{B}'_j, f^p_j \cap \bar{B}'_j) \equiv_{\mathrm{Ar}} \bigoplus^{\partial}_{j \in A} (\bar{B}^p_j, f^p_j).$$

Conversely, assume that $(\bar{B}, f) \equiv_{\mathrm{Ar}} \bigoplus^{\partial}_{j \in A} (\bar{B}^p_j, f^p_j)$ for some infinite $A \subseteq \mathbb{N}$. For each $i \in \mathbb{N}$, set $(\bar{B}'_i, f'_i) = (\bar{B}^p_i, f^p_i)$ if $i \in A$ and $(\bar{B}'_i, f'_i) = I_{\mathrm{Ar}}$ if $i \notin A$: since all the proper arcs $(\bar{B}^p_i, f^p_i)$ are tame, we get $\bigoplus^{\partial}_{j \in A} (\bar{B}^p_j, f^p_j) \equiv_{\mathrm{Ar}} \bigoplus^{\partial}_{i \in \mathbb{N}} (\bar{B}'_i, f'_i)$, so it is enough to show that the latter is a subarc of $\bigoplus^{\partial}_{i \in \mathbb{N}} (\bar{B}^p_i, f^p_i)$. Without loss of generality, $\bar{B}_i = \bar{B}'_i = [1 - 2^{-i}, 1 - 2^{-(i+1)}] \times [-2^{-i}, 2^{-i}]^2$, $f_i(0) = f'_i(0) = (1 - 2^{-i}, 0, 0)$, and $f_i(1) = f'_i(1) = (1 - 2^{-(i+1)}, 0, 0)$. We can further assume that the ambient space of $\bigoplus^{\partial}_{i \in \mathbb{N}} (\bar{B}'_i, f'_i)$ is the "step pyramid" $([-1,0] \times [-1,1]^2) \cup \bigcup_{i \in \mathbb{N}} \bar{B}'_i \cup \{(1,0,0)\}$. For each $i \notin A$, fix a tubular neighborhood $\bar{B}''_i \subseteq \bar{B}^p_i$ of f^p_i, i.e. a "cylinder" of radius ε_i with rotation axis given by f^p_i itself — this is possible because $(\bar{B}^p_i, f^p_i)$ is tame. Moreover, since each block of consecutive $i \in \mathbb{N} \setminus A$ is finite, we can assume that $\varepsilon_i = \varepsilon_{i+1}$ if $i, i+1 \notin A$. For $i \in A$ pick instead $\bar{B}''_i \subseteq \bar{B}^p_i$ so that: $(\bar{B}''_i, f^p_i)$ is a proper arc; $\bar{B}''_i$ intersects the left face $\{1 - 2^{-i}\} \times [-2^{-i}, 2^{-i}]^2$ of $\bar{B}^p_i$ in a disc of radius ε_i centered in $(1 - 2^{-i}, 0, 0)$, where $\varepsilon_i = \varepsilon_{i-1}$ if $i > 0$ and $i - 1 \notin A$ and $\varepsilon_i = 2^{-i}$ otherwise; similarly, $\bar{B}''_i$ intersects the right face $\{1 - 2^{-(i+1)}\} \times [-2^{-i}, 2^{-i}]^2$ of $\bar{B}_i$ in a disc of radius ε_i centered in $(1 - 2^{-(i+1)}, 0, 0)$, where $\varepsilon_i = \varepsilon_{i+1}$ if $i + 1 \notin A$ and $\varepsilon_i = 2^{-(i+1)}$ otherwise. Finally, let $\bar{B}''_{-1} \subseteq [-1,0] \times [-1,1]^2$ be such that $(\bar{B}''_{-1}, [-1,0] \times \{(0,0)\})$ is a proper arc and $\bar{B}''_{-1}$ intersects the left face $\{0\} \times [-1,1]^2$ of $\bar{B}^p_0$ in a disc of radius 1 centered in the origin $(0,0,0)$. By construction, $\bar{B}''_{-1} \cup \bigcup_{i \in \mathbb{N}} \bar{B}''_i \cup \{(1,0,0)\}$ is homeomorphic to a (closed) cone, and thus it is a closed topological 3-ball. Moreover, every $(\bar{B}'_i, f'_i)$ is equivalent to $(\bar{B}''_i, f^p_i)$ via some $\varphi_i \colon \bar{B}'_i \to \bar{B}''_i$. Fix also a homeomorphism $\varphi_{-1} \colon [-1,0] \times [-1,1]^2 \to \bar{B}''_{-1}$ fixing the interval $[-1,0] \times \{(0,0)\}$, and let φ_∞ be the identity on the singleton $(1,0,0)$. Then $\varphi = \varphi_{-1} \cup \bigcup_{i \in \mathbb{N}} \varphi_i \cup \varphi_\infty$ is an embedding witnessing $\bigoplus^{\partial}_{i \in \mathbb{N}} (\bar{B}'_i, f'_i) \precsim_{\mathrm{Ar}} \bigoplus^{\partial}_{i \in \mathbb{N}} (\bar{B}^p_i, f^p_i)$, as desired.

(c) Let $\varphi\colon \bar{B} \to [-1,1]^3$ witness $(\bar{B}, f) \precsim_{\mathrm{Ar}} F_{(\bar{B}^*,f^*)}(L_S)$. We claim that there is $x \in \Sigma_{(\bar{B},f)}$ such that $\varphi(x) \in I\Sigma_{F_{(\bar{B}^*,f^*)}(L_S)}$. Pick any $x \in \Sigma_{(\bar{B},f)}$, so that $\varphi(x) \in \Sigma_{F_{(\bar{B}^*,f^*)}(L_S)}$ as well by Lemma 4.2.4(a). If $\varphi(x) \notin I\Sigma_{F_{(\bar{B}^*,f^*)}(L_S)}$, we use the fact that by construction every singularity $y \in \Sigma_{F_{(\bar{B}^*,f^*)}(L_S)} \setminus I\Sigma_{F_{(\bar{B}^*,f^*)}(L_S)}$ is a limit of isolated singularities *from both sides* (unless $y \in \partial\bar{B}^*$, in which case there is only one side available), and hence for all $\bar{B}' \subseteq [-1,1]^3$ with $y \in \bar{B}'$ and $(\bar{B}', f_{L_S} \cap \bar{B}') \in \mathrm{Ar}$ the set $I\Sigma_{F_{(\bar{B}^*,f^*)}(L_S)} \cap \bar{B}'$ is infinite. Applying this to $\bar{B}' = \varphi(\bar{B})$ and $y = \varphi(x)$, we get that there is some (in fact, infinitely many) $y' \in I\Sigma_{F_{(\bar{B}^*,f^*)}(L_S)} \cap \operatorname{Int} \bar{B}'$: by Lemma 4.2.4, replacing x with $\varphi^{-1}(y')$ we are done. Using the same notation as in the proof of Theorem 4.2.6, let $n \in L_S$ be such that $\varphi(x) = (h_{L_S}(n), 0, 0)$. Without loss of generality, we may assume that $\bar{B}'' = \operatorname{Im} \varphi \cap [h_{L_S}(n) - 2r_{L_S}(n), h_{L_S}(n)] \times [-1,1]^2$ is a closed topological 3-ball. Moreover, $\varphi(x) = (h_{L_S}(n), 0, 0) \in \Sigma_{(\bar{B}'', f_{L_S} \cap \bar{B}'')}$ because otherwise x would not be a singularity of $(\bar{B}, f)$ (here we use the fact that on the right of $(h_{L_S}(n), 0, 0)$ there is a trivial arc), thus $(\bar{B}'', f_{L_S} \cap \bar{B}'') \in \mathrm{WAr}$. Since the subarc of $F_{(\bar{B}^*,f^*)}(L_S)$ determined by $[h_{L_S}(n) - 2r_{L_S}(n), h_{L_S}(n)] \times [-1,1]^2$ is equivalent to $(\bar{B}^\partial, f^\partial)$, setting $B' = \varphi^{-1}(B'')$ we are done. □

We are not able to get a full analogue of Proposition 2.2.8, but Lemma 4.2.12 allows us to get a similar, although slightly weaker, result.

Theorem 4.2.13. (a) *There are infinitely many $\precsim_{\mathrm{WAr}}$-incomparable $\precsim_{\mathrm{WAr}}$-minimal elements in* WAr.

(b) *There is a strictly $\precsim_{\mathrm{WAr}}$-decreasing ω-sequence in* WAr *which is not $\precsim_{\mathrm{WAr}}$-bounded from below.*

(c) *No basis for $\precsim_{\mathrm{WAr}}$ has size smaller than $2^{\aleph_0}$.*

Proof. Fix an enumeration without repetitions $\{(\bar{B}^p_i, f^p_i) \mid i \in \mathbb{N}\}$ of all prime arcs.

(a) For each $k \in \mathbb{N}$ set $(\bar{B}'_k, g_k) = \bigoplus^\partial_{\mathbb{N}} (\bar{B}^p_k, f^p_k)$. Every $(\bar{B}'_k, g_k)$ is $\precsim_{\mathrm{WAr}}$-minimal as a consequence of Lemma 4.2.12(b) (and the fact that all arcs in the infinitary sum are the same), and if $k \neq k'$ then $(\bar{B}'_k, g_k) \not\precsim_{\mathrm{WAr}} (\bar{B}'_{k'}, g_{k'})$ because $(\bar{B}^p_k, f^p_k) \precsim_{\mathrm{Ar}} (\bar{B}'_k, g_k)$ but $(\bar{B}^p_k, f^p_k) \not\precsim_{\mathrm{Ar}} (\bar{B}'_{k'}, g_{k'})$ by Lemma 4.2.12(a).

(b) Now let $(\bar{B}'_k, g_k) = \bigoplus^\partial_{i \geq k} (\bar{B}^p_i, f^p_i)$. By parts (a) and (b) of Lemma 4.2.12, if $k < k'$ then $(\bar{B}'_{k'}, g_{k'}) \prec_{\mathrm{Ar}} (\bar{B}'_k, g_k)$. Moreover, by Lemma 4.2.12(b) if $(\bar{B}, f) \in \mathrm{WAr}$ is such that $(\bar{B}, f) \precsim_{\mathrm{WAr}} (\bar{B}'_0, g_0)$ then $(\bar{B}, f) \equiv_{\mathrm{Ar}} \bigoplus^\partial_{j \in A} (\bar{B}^p_j, f^p_j)$, for some infinite $A \subseteq \mathbb{N}$. Let $k = \min A$: since $(\bar{B}^p_k, f^p_k) \precsim_{\mathrm{WAr}} \bigoplus^\partial_{j \in A} (\bar{B}^p_j, f^p_j)$ but by Lemma 4.2.12(a) $(\bar{B}^p_k, f^p_k) \not\precsim_{\mathrm{WAr}} (\bar{B}'_{k+1}, g_{k+1})$, we have $(\bar{B}, f) \not\precsim_{\mathrm{WAr}} (\bar{B}'_{k+1}, g_{k+1})$. Thus the chain formed by the proper arcs $(\bar{B}'_k, g_k)$ is as required.

(c) Let $\{A_x \mid x \in 2^{\mathbb{N}}\}$ be a family of infinite sets $A_x \subseteq \mathbb{N}$ such that $A_x \cap A_y$ is finite for all distinct $x, y \in 2^{\mathbb{N}}$. (For the sake of definiteness, set $A_x = \{h(x \restriction n) \mid n \in \mathbb{N}\}$, where h is a bijection from all finite binary sequences to the natural numbers.) Fix a basis $\mathcal{B}$ for $\precsim_{\mathrm{WAr}}$. Then for every $\bigoplus^\partial_{j \in A_x} (\bar{B}^p_j, f^p_j)$ there is some $(\bar{B}_x, f_x) \in \mathcal{B}$ such that $(\bar{B}_x, f_x) \precsim_{\mathrm{WAr}} \bigoplus^\partial_{j \in A_x} (\bar{B}^p_j, f^p_j)$. By Lemma 4.2.12(b), $(\bar{B}_x, f_x) \equiv_{\mathrm{Ar}} \bigoplus^\partial_{j \in A'_x} (\bar{B}^p_j, f^p_j)$ for some infinite $A'_x \subseteq A_x$. If there were distinct $x, y \in 2^{\mathbb{N}}$ such that $(\bar{B}_x, f_x) \equiv_{\mathrm{Ar}} (\bar{B}_y, f_y)$, then we would get $A'_x = A'_y$ by Lemma 4.2.12(a), and thus $A'_x \subseteq A_x \cap A_y$, which is impossible because A'_x is infinite. Thus all the proper arcs $(\bar{B}_x, f_x) \in \mathcal{B}$ are distinct, and thus $|\mathcal{B}| \geq 2^{\aleph_0}$, as desired. □

Lemma 4.2.12 is also sufficient to recover an analogue of Proposition 2.2.9 for proper arcs.

Theorem 4.2.14. *Every $\precsim_{\mathrm{WAr}}$-antichain is contained in a $\precsim_{\mathrm{WAr}}$-antichain of size $2^{\aleph_0}$. In particular, there are no maximal $\precsim_{\mathrm{WAr}}$-antichains of size smaller than $2^{\aleph_0}$, and every $(\bar{B}, f) \in \mathrm{WAr}$ belongs to a $\precsim_{\mathrm{WAr}}$-antichain of size $2^{\aleph_0}$.*

Proof. Let $\mathcal{A} = \{(\bar{B}'_m, g_m) \mid m < \kappa\}$ be a $\precsim_{\mathrm{WAr}}$-antichain, where $\kappa < 2^{\aleph_0}$. Let $(\bar{B}^p_i, f^p_i)$, for $i \in \mathbb{N}$, be an enumeration without repetitions of all prime arcs, and for $S \subseteq \mathbb{N}$ let $L_S = \eta_{f_S}$ be the linear order from the proof of Proposition 2.2.8(a). Let $\{A_S \mid S \subseteq \mathbb{N}\}$ be a family of sets $A_S \subseteq \mathbb{N}$ such that $A_S \cap A_{S'}$ is finite for all distinct $S, S' \subseteq \mathbb{N}$. (Such a family can be constructed as in the proof of Theorem 4.2.13(c).) For each $S \subseteq \mathbb{N}$, set $(\bar{B}^*_S, f^*_S) = \bigoplus_{j \in A_S} (\bar{B}^p_j, f^p_j)$ and

$$(\bar{B}_S, f_S) = F_{(\bar{B}^*_S, f^*_S)}(L_S).$$

Let $\mathcal{B}$ be the collection of all proper arcs of the form $(\bar{B}_S, f_S)$ which are $\precsim_{\mathrm{WAr}}$-incomparable with every $(\bar{B}'_m, g_m) \in \mathcal{A}$.

Claim 4.2.14.1. $|\mathcal{B}| = 2^{\aleph_0}$.

Proof of the Claim. By the proof of Claim 4.2.10.1 and $\kappa < 2^{\aleph_0}$ there are $2^{\aleph_0}$-many proper arcs $(\bar{B}_S, f_S)$ such that $(\bar{B}_S, f_S) \not\precsim_{\mathrm{WAr}} (\bar{B}'_m, g_m)$ for all $m < \kappa$. On the other hand, we claim that there are at most κ-many proper arcs $(\bar{B}_S, f_S)$ such that $(\bar{B}'_m, g_m) \precsim_{\mathrm{WAr}} (\bar{B}_S, f_S)$ for some $m < \kappa$, which suffices to prove the claim. Indeed, suppose that $m < \kappa$ and $S \subseteq \mathbb{N}$ are such that $(\bar{B}'_m, g_m) \precsim_{\mathrm{WAr}} (\bar{B}_S, f_S)$. Then by parts (c) and (b) of Lemma 4.2.12 there is $\bar{B}'' \subseteq \bar{B}'_m$ such that $(\bar{B}'', g_m \cap \bar{B}'') \equiv_{\mathrm{Ar}} \bigoplus^{\partial}_{j \in A} (\bar{B}^p_j, f^p_j)$ for some infinite $A \subseteq A_S$. Since if $S' \subseteq \mathbb{N}$ is different from S then $A_S \cap A_{S'}$ is finite, there is $\bar{\jmath} \in A$ such that $\bar{\jmath} \notin A_{S'}$. If $(\bar{B}'_m, g_m) \precsim_{\mathrm{WAr}} (\bar{B}_{S'}, f_{S'})$ then $(\bar{B}'', g_m \cap \bar{B}'') \precsim_{\mathrm{WAr}} (\bar{B}_{S'}, f_{S'})$, and thus $(\bar{B}^p_{\bar{\jmath}}, f^p_{\bar{\jmath}}) \precsim_{\mathrm{WAr}} (\bar{B}_{S'}, f_{S'})$, which is impossible by Lemma 4.2.12(a) and the choice of $\bar{\jmath}$. Thus for every $m < \kappa$ there is at most one $S \subseteq \mathbb{N}$ such that $(\bar{B}'_m, g_m) \precsim_{\mathrm{WAr}} (\bar{B}_S, f_S)$ and we are done. □

By (the proof of) Proposition 2.2.8, Lemma 4.2.8 and Remark 4.2.9 (together with $L_S \cong (L_S)^*$), if $S, S' \subseteq \mathbb{N}$ are distinct then $(\bar{B}_S, f_S)$ and $(\bar{B}_{S'}, f_{S'})$ are $\precsim_{\mathrm{WAr}}$-incomparable. Thus $\mathcal{A} \cup \mathcal{B}$ is a $\precsim_{\mathrm{WAr}}$-antichain of size $2^{\aleph_0}$ containing $\mathcal{A}$, as desired. □

4.3 Knots and their classification

In the proof of [Kul17, Theorem 3.1], it is defined a function from LO to Kn, that we here call G, by setting $G(L) = K_{F(\mathbf{1}+L+\mathbf{2}+\eta)}$, where F is the reduction from (4.2.1). It was claimed that G was a reduction of $\cong_{\mathsf{LO}}$ to $\equiv_{\mathrm{Kn}}$, but this is not the case. Indeed, notice that if M is a linear order, then we have $G(\eta+\mathbf{1}+M) \equiv_{\mathrm{Kn}} G(M)$, essentially because $C[\mathbf{1}+\eta+\mathbf{1}+M+\mathbf{2}+\eta] \cong_{\mathsf{CO}} C[\mathbf{1}+M+\mathbf{2}+\eta]$; however if M is scattered (and in many other cases) $\eta + \mathbf{1} + M \not\cong_{\mathsf{LO}} M$.

One can easily fix this problem by replacing $K_{F(\mathbf{1}+L+\mathbf{2}+\eta)}$ with $K_{F(\mathbf{1}+L+\mathbf{2}+\eta)+\bigoplus_{\mathbb{N}}(\bar{B}^*, f^*)}$, where $(\bar{B}^*, f^*)$ is a figure-eight arc. More precisely, we can derive [Kul17, Theorem 3.1] from (the proof of) Theorem 4.2.6 connecting the endpoints of each arc $F(L)$ with $\bigoplus_{\mathbb{N}}(\bar{B}^*, f^*)$ and get:

Corollary 4.3.1. $\cong_{\mathsf{LO}} \leq_B \equiv_{\mathrm{Kn}}$.

The next result follows from Theorem 1.3.6 and Corollary 4.3.1. However, exploiting the obvious analogy between circular orders and knots one obtains a direct and more natural proof. (The reduction F_{Kn} will be used also in the proof of Theorem 4.3.11).

Theorem 4.3.2. $\cong_{\mathsf{CO}} \leq_B \equiv_{\mathrm{Kn}}$.

Proof. We define a Borel reduction $F_{\mathrm{Kn}} \colon \mathsf{CO} \to \mathrm{Kn}$ similar to the reduction of the proof of Theorem 4.2.6. Instead of embedding a linear order $L \in \mathsf{LO}$ into $[-1, 1] \subseteq \mathbb{R}$, we embed $C \in \mathsf{CO}$ into $S^1 = \mathbb{R} \cup \{\infty\}$ by defining a sequence of intervals $(h_C(n) - 2r_C(n), h_C(n) + 2r_C(n))_{n \in \mathbb{N}}$ of $\mathbb{R}$ denoted by V^C_n, satisfying conditions analogous to (a)-(b) of the proof of Theorem 4.2.6.

As before, for every $n \in \mathbb{N}$ let $U^C_n = [h_C(n) - r_C(n), h_C(n) + r_C(n)]$, consider $\bar{B}^C_n = U^C_n \times [-r_C(n), r_C(n)]^2$ and define a proper arc $(\bar{B}^C_n, f^C_n)$ as in Figure 4.1. Set $f^C = \{(x, 0, 0) \mid (x, 0, 0) \notin \bigcup_{n \in \mathbb{N}} \bar{B}^C_n\} \cup \{\infty\}$. Finally we consider the knot $F_{\mathrm{Kn}}(C)$ given by $\bigcup_{n \in \mathbb{N}} f^C_n \cup f^C$. The rest of the proof is an adaptation of the proof of Theorem 4.2.6 to this case. □

Remark 4.3.3. Theorem 4.1 of [Kul17] shows that a certain equivalence relation induced by a turbulent action is Borel reducible to $\equiv_{\mathrm{Kn}}$. Therefore, since $\cong_{\mathsf{LO}}$ and $\cong_{\mathsf{CO}}$ are induced by actions of S_∞ the reductions in Corollary 4.3.1 and Theorem 4.3.2 are actually strict by Theorem 1.1.12.

In order to extend to knots the analysis of $\precsim_A$ previously developed, one may be tempted to transfer the subarc relation from proper arcs to knots through the transformation $(\bar{B}, f) \mapsto K_{(\bar{B},f)}$ previously exploited. The resulting definition would be the following:

> Given two knots $K, K' \in \mathrm{Kn}$, we say that K is a **subknot** of K' if $K \equiv_{\mathrm{Kn}} K_{(\bar{B}', K' \cap \bar{B}')}$ for some subarc $(\bar{B}', K' \cap \bar{B}')$ of K'.

However, as in the case of convex embeddability for circular orders, this relation is not transitive and we need to define a piecewise version of the subarc relation, which is the analogue for knots of $\trianglelefteq^{<\omega}_{\mathsf{CO}}$ (recall Definition 2.4.7). To this aim, we first introduce the following notion.

Definition 4.3.4. Let $\{(\bar{B}_i, f_i) \mid i \leq n\}$ be a collection of oriented proper arcs, $h \colon T \to S^3$ a topological embedding, and $K \in \mathrm{Kn}$. We say that K is the **circular sum of** the $(\bar{B}_i, f_i)$'s via h, if $K = C^h[\bigoplus_{i \leq n} (\bar{B}_i, f_i)]$ (recall Definitions 4.1.6 and 4.1.7).

Remark 4.3.5. A topological embedding h of T in S^3 is canonical if the closure of $S^3 \setminus h(T)$ is a solid torus as well (recall the solid torus theorem, see e.g. [Rol90, p. 107]). For any $(\bar{B}, f) \in \mathrm{Ar}$, the knot $K_{(\bar{B},f)}$ previously defined is equivalent to $C^h[(\bar{B}, f) \bigoplus I_{\mathrm{Ar}}]$ for any canonical h. Moreover, when $(\bar{B}, f)$ is tame we have $K_{(\bar{B},f)} \equiv_{\mathrm{Kn}} C^h[(\bar{B}, f)]$ for every such h, i.e. the two operations of joining the endpoints of $(\bar{B}, f)$ with a trivial arc and of circularization of $(\bar{B}, f)$ yield the same knot (up to equivalence).

Definition 4.3.6. Let $K, K' \in \mathrm{Kn}$. Then K is a **(finite) piecewise subknot** of K', in symbols

$$K \precsim^{<\omega}_{\mathrm{Kn}} K',$$

if and only if either $K \equiv_{\mathrm{Kn}} K'$ or there exist oriented proper arcs $\{(\bar{B}_i, f_i) \mid i \leq k\}$ and $\{(\bar{B}'_j, f'_j) \mid j \leq k'\}$, topological embeddings $h, h' \colon T \to S^3$ and an embedding of circular orders $c \colon C[\mathbf{k+1}] \to C[\mathbf{k'+1}]$ (so that we must have $k' \geq k$) such that:

(i) $K = C^h[\bigoplus_{i \leq k} (\bar{B}_i, f_i)]$ and $K' = C^{h'}[\bigoplus_{i \leq k'} (\bar{B}'_j, f'_j)]$;

(ii) for every $i \leq k$, $(\bar{B}_i, f_i)$ is equivalent to $(\bar{B}'_{c(i)}, f'_{c(i)})$ (as oriented proper arcs).

The **(finite) piecewise mutual subknot relation** is the relation defined by $K \approx^{<\omega}_{\mathrm{Kn}} K'$ if and only if $K \precsim^{<\omega}_{\mathrm{Kn}} K'$ and $K' \precsim^{<\omega}_{\mathrm{Kn}} K$.

Proposition 4.3.7. *$\precsim^{<\omega}_{\mathrm{Kn}}$ and $\approx^{<\omega}_{\mathrm{Kn}}$ are an analytic quasi-order and an analytic equivalence relation on* Kn*, respectively.*

Proof. It is easy to see that $\precsim^{<\omega}_{\mathrm{Kn}}$ is reflexive and analytic. To prove transitivity we can mostly mimic the proof of Proposition 2.4.8. □

The quasi-order $\precsim^{<\omega}_{\mathrm{Kn}}$ is fine enough to distinguish between tame and wild knots, as shown in the next proposition.

Proposition 4.3.8. *Let $K \in \mathrm{Kn}$, and recall that we denote by I_{Kn} the trivial knot. Then the following are equivalent:*

(1) *K is tame;*

(2) *$K \approx^{<\omega}_{\mathrm{Kn}} I_{\mathrm{Kn}}$;*

(3) *$K \precsim^{<\omega}_{\mathrm{Kn}} I_{\mathrm{Kn}}$.*

In particular, the $\approx^{<\omega}_{\mathrm{Kn}}$-class of the tame knots is minimal with respect to (the quotient order of) $\precsim^{<\omega}_{\mathrm{Kn}}$.

Proof. The proof is immediate using the facts that a knot is tame if and only if it is the circularization of the trivial arc and that the trivial knot can be written as a circular sum only if all summands are trivial arcs and the embedding of the solid torus is canonical. □

Remark 4.3.9. If $(\bar{B}, f)$ is a proper arc with a tame subarc and $(\bar{B}', g)$ is a tame arc then it is easy to check that $C[(\bar{B}, f)] \approx^{<\omega}_{\mathrm{Kn}} C[(\bar{B}, f) \oplus (\bar{B}', g)]$.

Notice that the relations $\precsim^{<\omega}_{\mathrm{Kn}}$ and $\approx^{<\omega}_{\mathrm{Kn}}$ differ from $\equiv_{\mathrm{Kn}}$ only on the set of knots which are circularizations of proper arcs. For this reason we focus on the following subset of Kn.

Definition 4.3.10. We denote by CKn and WCKn, respectively, the set of knots which are a circularization of a proper arc (that is, up to knot equivalence, those of the form $C^h[(\bar{B}, f)]$ for some $(\bar{B}, f) \in \mathrm{Ar}$ and some embedding $h\colon T \to S^3$), and its subset consisting of wild knots. Let $\precsim^{<\omega}_{\mathrm{CKn}}$ and $\precsim^{<\omega}_{\mathrm{WCKn}}$ be the restrictions of $\precsim^{<\omega}_{\mathrm{Kn}}$ to these sets.

Notice that CKn is a proper subset of Kn: for example, the knot constructed by Bing in [Bin56] cannot be "cut" at any point and thus it does not belong to CKn. However CKn is quite rich, as it includes any wild knot K satisfying any of the following equivalent conditions: K has at least one isolated singularity (i.e. $I\Sigma_K \neq \emptyset$), the set Σ_K of singularities of K is not dense in K, there exists a point of K which is not a singularity (i.e. $\Sigma_K \neq K$). Moreover, the wild knots built by Artin and Fox in [FA48] do not satisfy the previous conditions, yet they belong to CKn. Further evidence of the complexity and richness of $\precsim^{<\omega}_{\mathrm{CKn}}$ is provided in the results below (see Proposition 4.3.13 and Theorems 4.3.15–4.3.19).

Since $C^h[(\bar{B}, f)] \approx^{<\omega}_{\mathrm{Kn}} C^{h'}[(\bar{B}, f)]$ for any topological embeddings h and h', every $K \in \mathrm{CKn}$ can be assumed to be, up to $\approx^{<\omega}_{\mathrm{CKn}}$, of the form $C^h[(\bar{B}, f)]$ for some canonical embedding $h\colon T \to S^3$. To simplify the notation we write $C[(\bar{B}, f)]$ in place of $C^h[(\bar{B}, f)]$ when h is canonical and we do not mention h and h' witnessing $K \precsim^{<\omega}_{\mathrm{Kn}} K'$ when they are canonical.

The next theorem establishes a lower bound for the complexity of $\precsim^{<\omega}_{\mathrm{CKn}}$ w.r.t. Borel reducibility.

Theorem 4.3.11. $\trianglelefteq^{<\omega}_{\mathsf{CO}} \leq_B \precsim^{<\omega}_{\mathrm{CKn}}$.

Proof. We claim that the Borel map $F_{\mathrm{Kn}}\colon \mathsf{CO} \to \mathrm{Kn}$ from the proof of Theorem 4.3.2 is the desired reduction. First of all, notice that $\mathrm{Im}(F_{\mathrm{Kn}})$ is contained in CKn by construction. Fix now $C, C' \in \mathsf{CO}$.

Assume first that $C \trianglelefteq^{<\omega}_{\mathsf{CO}} C'$, and let the finite convex partition $(C_i)_{i\leq k}$ of C and the embedding g witness this. For every $i \leq k$, let $\bar{B}_i = [a_i, b_i] \times [-1, 1]^2$, where $a_i = \inf \bigcup_{n\in C_i} V^C_n$ and $b_i = \sup \bigcup_{n\in C_i} V^C_n$, and $f_i = F_{\mathrm{Kn}}(C) \cap \bar{B}_i$, so that $(\bar{B}_i, f_i)$ is a proper arc. The proper arcs $(\bar{B}'_i, f'_i)$ are defined similarly using $g(C_i)$ and $F_{\mathrm{Kn}}(C')$. Since C_i and $g(C_i)$ are isomorphic as linear orders we have $(\bar{B}_i, f_i) \equiv_{\mathrm{Ar}} (\bar{B}'_i, f'_i)$. Moreover, $F_{\mathrm{Kn}}(C) = C[\bigoplus_{i\leq k}(\bar{B}_i, f_i)]$. Since each $(\bar{B}'_i, f'_i)$ is a subarc of $F_{\mathrm{Kn}}(C')$, adding the subarcs which cover $F_{\mathrm{Kn}}(C') \setminus \mathrm{Int}\left(\bigcup_{i\leq k} \bar{B}'_i\right)$, the conditions of Definition 4.3.6 are satisfied. Hence, $F_{\mathrm{Kn}}(C) \precsim^{<\omega}_{\mathrm{Kn}} F_{\mathrm{Kn}}(C')$.

Conversely, suppose that $F_{\mathrm{Kn}}(C)$ and $F_{\mathrm{Kn}}(C')$ (which are elements of CKn) are such that $F_{\mathrm{Kn}}(C) \precsim^{<\omega}_{\mathrm{Kn}} F_{\mathrm{Kn}}(C')$, and let $\{(\bar{B}_i, f_i) \mid i \leq k\}$, $\{(\bar{B}'_j, f'_j) \mid j \leq k'\}$ and $c\colon C[\mathbf{k}+\mathbf{1}] \to C[\mathbf{k}'+\mathbf{1}]$ witness this. By definition of $F_{\mathrm{Kn}}(C)$ when $B_i \cap B_m$ contains a point $x \in I\Sigma_{F_{\mathrm{Kn}}(C)}$ then x is a singular point of only one of $(\bar{B}_i, f_i)$ and $(\bar{B}_m, f_m)$; by reindexing the sequence $\{(\bar{B}_i, f_i) \mid i \leq k\}$ we can assume this occurs always for the index which is the immediate predecessor of the other in $C[\mathbf{k}+\mathbf{1}]$. The same can be done for the sequence $\{(\bar{B}'_j, f'_j) \mid j \leq k'\}$ and, by an analogue of (†), c is still an embedding of $C[\mathbf{k}+\mathbf{1}]$ into $C[\mathbf{k}'+\mathbf{1}]$.

Recall that h_C is an isomorphism of circular orders between C and $I\Sigma_{F_{\mathrm{Kn}}(C)}$ and let $C_i = h_C^{-1}(I\Sigma_{F_{\mathrm{Kn}}(C)} \cap \bar{B}_i \setminus \bar{B}_m)$ where m is the immediate predecessor of i in $C[\mathbf{k}+\mathbf{1}]$. Notice that $(C_i)_{i\leq k}$ is a finite convex partition of C. Moreover, since $(\bar{B}_i, f_i) \equiv_{\mathrm{Ar}} (\bar{B}'_{c(i)}, f'_{c(i)})$ for every $i \leq k$, we have that each $I\Sigma_{F_{\mathrm{Kn}}(C')} \cap \bar{B}'_{c(i)} \setminus \bar{B}'_j$ (for j the immediate predecessor of $c(i)$ in $C[\mathbf{k}'+\mathbf{1}]$) is convex in $I\Sigma_{F_{\mathrm{Kn}}(C')}$ and isomorphic to $I\Sigma_{F_{\mathrm{Kn}}(C)} \cap \bar{B}_i \setminus \bar{B}_m$ (we are again using the analogue of (†)). Finally, since $I\Sigma_{F_{\mathrm{Kn}}(C')} \cong C'$ via $h_{C'}^{-1}$, then $C \trianglelefteq^{<\omega}_{\mathsf{CO}} C'$, as desired. □

Corollary 4.3.12. $\bowtie^{<\omega}_{\mathsf{CO}} \leq_B \approx^{<\omega}_{\mathrm{CKn}}$, *whence also* $\cong_{\mathsf{LO}} \leq_B \approx^{<\omega}_{\mathrm{CKn}}$ *and* $E_1 \leq_B \approx^{<\omega}_{\mathrm{CKn}}$.

The fact that the isomorphism on linear orders is Borel reducible to $\approx^{<\omega}_{\mathrm{CKn}}$ implies that $\approx^{<\omega}_{\mathrm{CKn}}$ is proper analytic. Moreover, $\approx^{<\omega}_{\mathrm{CKn}}$ is not Baire reducible to an orbit equivalence relation because it Borel reduces E_1, in stark contrast with knot equivalence $\equiv_{\mathrm{Kn}}$; in particular we have that $\approx^{<\omega}_{\mathrm{CKn}}$ is not Borel, or even Baire, reducible to $\equiv_{\mathrm{Kn}}$.

Using Theorem 4.3.11, we can transfer the combinatorial properties of $\trianglelefteq^{<\omega}_{\mathsf{CO}}$ proved in Proposition 2.4.10 to $\precsim^{<\omega}_{\mathrm{CKn}}$.

Proposition 4.3.13. (a) *There is an embedding from the partial order* $(\mathrm{Int}(\mathbb{R}), \subseteq)$ *into* $\precsim^{<\omega}_{\mathrm{CKn}}$*, and indeed* $(\mathrm{Int}(\mathbb{R}), \subseteq) \leq_B \precsim^{<\omega}_{\mathrm{CKn}}$.

(b) $\precsim^{<\omega}_{\mathrm{CKn}}$ *has chains of order type* $(\mathbb{R}, <)$*, as well as antichains of size* $2^{\aleph_0}$.

To extend the other combinatorial properties of $\trianglelefteq^{<\omega}_{\mathsf{CO}}$ to $\precsim^{<\omega}_{\mathrm{CKn}}$ we need an analogous of Lemma 4.2.8. When K is a knot and f is such that $\mathrm{Im}\, f = K$, the set $I\Sigma_K$ can naturally be viewed as a circular order $C^K_f = (I\Sigma_K, C_f)$. As it was the case for proper arcs, the set $I\Sigma_K$ is (at most) countable and thus C^K_f is either a finite or a countable circular order. If $f, f' \colon S^1 \to S^3$ are topological embeddings giving rise to the same knot, then either $C^K_f = C^K_{f'}$ or $C^K_f = (C^K_{f'})^*$. Recall that by construction, for knots of the form[4] $F_{\mathrm{Kn}}(C)$ we have $C^{F_{\mathrm{Kn}}(C)}_f \cong_{\mathsf{CO}} C$.

Lemma 4.3.14. *Let* $K, K' \in \mathrm{CKn}$ *be such that* $K \precsim^{<\omega}_{\mathrm{Kn}} K'$ *and let* f *and* f' *be such that* $\mathrm{Im}\, f = K$ *and* $\mathrm{Im}\, f' = K'$. *Then there exists a finite set* $A \subseteq I\Sigma_K$ *such that either* $C^K_f \setminus A \trianglelefteq^{<\omega}_{\mathrm{c}} C^{K'}_{f'}$ *or* $C^K_f \setminus A \trianglelefteq^{<\omega}_{\mathrm{c}} (C^{K'}_{f'})^*$.

Proof. Let $\{(\bar{B}_i, f_i) \mid i \leq k\}$, $\{(\bar{B}'_j, f'_j) \mid j \leq k'\}$ and the embedding $c \colon C[\mathbf{k+1}] \to C[\mathbf{k'+1}]$ witness $K \precsim^{<\omega}_{\mathrm{Kn}} K'$. Let $A = \{x \in I\Sigma_K \mid \exists i \leq k\, (x \in \partial \bar{B}_i)\}$, and notice that A contains at most $k+1$ points. We can assume that c agrees with the orientations induced on K and K' by f and f', in which case we show that $C^K_f \setminus A \trianglelefteq^{<\omega}_{\mathrm{c}} C^{K'}_{f'}$ (if c agrees with only one of the orientations we obtain $C^K_f \setminus A \trianglelefteq^{<\omega}_{\mathrm{c}} (C^{K'}_{f'})^*$, and if it disagrees with both it suffices to reverse both orientations).

For every $i \leq k$ let $C_i = I\Sigma_K \cap \mathrm{Int}\, \bar{B}_i$. Then each C_i is convex, and $\{C_i \mid i \leq k\}$ is a finite convex partition of $C^K_f \setminus A$ (some of the C_i's might actually be empty, in which case we would obtain a convex partition with less than $k+1$ sets, but for notational ease we avoid keeping track of this). We now define an embedding h of $C^K_f \setminus A$ into $C^{K'}_{f'}$ such that $h(C_i) \sqsubseteq C^{K'}_{f'}$ for all $i \leq k$. For $x \in I\Sigma_K \setminus A$ there exists a unique $i \leq k$ such that $x \in C_i$, and thus we can define $h(x) = \varphi_i(x)$ where φ_i witnesses $(\bar{B}_i, f_i) \equiv_{\mathrm{Ar}} (\bar{B}'_{c(i)}, f'_{c(i)})$. Since $x \in I\Sigma_K \cap \mathrm{Int}\, \bar{B}_i$ we have $x \in I\Sigma_{(\bar{B}_i, f_i)}$; therefore $\varphi_i(x) \in I\Sigma_{(\bar{B}'_{c(i)}, f'_{c(i)})} \cap \mathrm{Int}\, B'_{c(i)}$ and hence $\varphi_i(x) \in I\Sigma_{K'}$. It is easy to check that h is an embedding of circular orders and that $h(C_i) \sqsubseteq C^{K'}_{f'}$ for all $i \leq k$. □

Since in $\mathrm{Kn} \setminus \mathrm{CKn}$ the relation $\precsim^{<\omega}_{\mathrm{Kn}}$ is $\equiv_{\mathrm{Kn}}$, it is easy to show that $\mathfrak{b}(\precsim^{<\omega}_{\mathrm{Kn}}) = 2$ and $\mathfrak{d}(\precsim^{<\omega}_{\mathrm{Kn}}) = 2^{\aleph_0}$. It is therefore more interesting to compute the unbounding and dominating number of $\precsim^{<\omega}_{\mathrm{CKn}}$.

Theorem 4.3.15. $\mathfrak{b}(\precsim^{<\omega}_{\mathrm{CKn}}) = \aleph_1$ *and* $\mathfrak{d}(\precsim^{<\omega}_{\mathrm{CKn}}) = 2^{\aleph_0}$.

Proof. We first show the existence of an $\precsim^{<\omega}_{\mathrm{CKn}}$-unbounded family of knots of size $\aleph_1$. Consider the map $F_{\mathrm{Kn}} \colon \mathsf{CO} \to \mathrm{Kn}$ defined in the proof of Theorem 4.3.2 and used also in the proof of Theorem 4.3.11. By (the proof of) Proposition 2.4.11 there exists a strictly increasing sequence $\{C_\alpha \mid \alpha < \omega_1\} \subseteq \mathsf{CO}$ without upper bound with respect to $\trianglelefteq^{<\omega}_{\mathsf{CO}}$. Notice moreover that each C_α has the property that $C_\alpha \cong_{\mathsf{CO}} C_\alpha \setminus A$ for any finite $A \subseteq C_\alpha$. The sequence $\{F_{\mathrm{Kn}}(C_\alpha) \mid \alpha < \omega_1\} \subseteq \mathrm{CKn}$ is then strictly $\precsim^{<\omega}_{\mathrm{CKn}}$-increasing, and we claim that it is also unbounded in CKn. Suppose towards a contradiction that there is $K \in \mathrm{Kn}$ such that $F_{\mathrm{Kn}}(C_\alpha) \precsim^{<\omega}_{\mathrm{CKn}} K$ for all $\alpha < \omega_1$. Then $I\Sigma_K$ is infinite and thus the circular order C^K_f is, up to isomorphism, an element of CO. Pick now $\ell \in C^K_f$ and define $L \in \mathsf{LO}$ by setting $x \leq_L y$ if and only if $C^K_f(\ell, x, y)$ and $x = \ell$ when $y = \ell$. Notice that $C^K_f = C[L]$. Then the circular order $C = C[L + L^*]$ is such that $C^K_f \trianglelefteq_{\mathrm{c}} C$ and $(C^K_f)^* \trianglelefteq_{\mathrm{c}} C$. By

[4] If not specified otherwise, we always choose the natural orientation of $F_{\mathrm{Kn}}(C)$, witnessed by f.

Lemma 4.3.14 and the fact that each $C_f^{F_{\mathrm{Kn}}(C_\alpha)} \cong_{\mathsf{CO}} C_\alpha$, this would imply that for every $\alpha < \omega_1$ there exists a finite $A_\alpha \subseteq C_\alpha$ such that $C_\alpha \setminus A_\alpha \trianglelefteq^{<\omega}_{\mathsf{CO}} C$. As $C_\alpha \cong_{\mathsf{CO}} C_\alpha \setminus A$, the circular order C would be a $\trianglelefteq^{<\omega}_{\mathsf{CO}}$-upper bound for $\{C_\alpha \mid \alpha < \omega_1\}$, yielding the desired contradiction.

We now prove that $\mathfrak{b}(\precsim^{<\omega}_{\mathrm{CKn}}) > \aleph_0$. Let $\{K_i \mid i \in \mathbb{N}\} \subseteq \mathrm{CKn}$ be a countable family of knots. By definition of CKn, each K_i can be written as $C[(\bar{B}_i, f_i)]$ for some proper arc $(\bar{B}_i, f_i)$ (we are using a canonical embedding of the solid torus in S^3). Then the knot $C[\bigoplus_{\mathbb{N}}(\bar{B}_i, f_i)]$ is a $\precsim^{<\omega}_{\mathrm{CKn}}$-upper bound for $\{K_i \mid i \in \mathbb{N}\}$.

To prove that $\mathfrak{d}(\precsim^{<\omega}_{\mathrm{CKn}}) \geq 2^{\aleph_0}$ we follow the same strategy of the proof of Theorem 4.2.10. Consider the $\trianglelefteq^{<\omega}_{\mathsf{CO}}$-antichain $\{C_S \mid S \subseteq \mathbb{N}\}$ defined in the proof of Proposition 2.4.12(a) and, using Lemma 4.3.14 (removing finitely many elements from C_S does not affect the argument) and the proof of Proposition 2.4.13, prove that for every knot $K \in \mathrm{CKn}$, $\{F_{\mathrm{Kn}}(C_S) \mid F_{\mathrm{Kn}}(C_S) \precsim^{<\omega}_{\mathrm{CKn}} K\}$ is countable. The proof is then completed using Theorem 4.3.2. □

Corollary 4.3.16. *Every knot $K \in \mathrm{CKn}$ is the bottom of an $\precsim^{<\omega}_{\mathrm{CKn}}$-unbounded chain of length ω_1.*

Proof. Given $K \in \mathrm{CKn}$, let $(\bar{B}, f)$ be a proper arc such that $K = C[(\bar{B}, f)]$. As in the proof of Theorem 4.3.15 let $\{C_\alpha \mid \alpha < \omega_1\} \subseteq \mathsf{CO}$ be an unbounded strictly $\trianglelefteq^{<\omega}_{\mathsf{CO}}$-increasing sequence in CO, so that $\{F_{\mathrm{Kn}}(C_\alpha) \mid \alpha < \omega_1\} \subseteq \mathrm{CKn}$ is unbounded and strictly $\precsim^{<\omega}_{\mathrm{CKn}}$-increasing in CKn. For every $\alpha < \omega_1$, let $(\bar{B}_\alpha, f_\alpha)$ be a proper arc obtained by cutting $F_{\mathrm{Kn}}(C_\alpha)$ in a point which is not an isolated singularity, so that in particular $F_{\mathrm{Kn}}(C_\alpha) = C[(\bar{B}_\alpha, f_\alpha)]$. Let $K_0 = K$ and, for $0 < \alpha < \omega_1$, $K_\alpha = C[(\bar{B}, f) \oplus (\bar{B}_\alpha, f_\alpha)]$. For every $\alpha < \beta < \omega_1$ we have $K_\alpha \precsim^{<\omega}_{\mathrm{CKn}} K_\beta$ (even though it might happen that $(\bar{B}_\alpha, f_\alpha) \not\precsim_{\mathrm{Ar}} (\bar{B}_\beta, f_\beta)$, in which case we need a circular sum of proper arcs with more than one element to witness $K_\alpha \precsim^{<\omega}_{\mathrm{CKn}} K_\beta$) and $F_{\mathrm{Kn}}(C_\alpha) \precsim^{<\omega}_{\mathrm{CKn}} K_\alpha$. Hence the sequence $(K_\alpha)_{\alpha<\omega_1}$ is $\precsim^{<\omega}_{\mathrm{CKn}}$-unbounded. By the same argument used in the proof of Corollary 4.2.11 we can extract from $(K_\alpha)_{\alpha<\omega_1}$ a strictly $\precsim^{<\omega}_{\mathrm{CKn}}$-increasing subsequence of length ω_1 starting with K. □

We finally deal with minimal elements and basis w.r.t. $\precsim^{<\omega}_{\mathrm{CKn}}$. In contrast with the case of proper arcs, it is not interesting to consider the restriction of $\precsim^{<\omega}_{\mathrm{CKn}}$ to the collection of tame knots because by Proposition 4.3.8 tame knots are all $\approx^{<\omega}_{\mathrm{CKn}}$-equivalent. Let thus consider $\precsim^{<\omega}_{\mathrm{WCKn}}$.

Lemma 4.3.17. *Let $\{(\bar{B}_i^p, f_i^p) \mid i \in \mathbb{N}\}$ be a family of (oriented) prime arcs, and let $K^*_S = C[\bigoplus_{i\in S}(\bar{B}_i^p, f_i^p)]$ for some infinite $S \subseteq \mathbb{N}$.*

(a) *If $K^*_S = C^h[\bigoplus_{i\leq k}(\bar{B}_i, f_i)]$ for some $k \in \mathbb{N}$ and $h \colon T \to S^3$, then there is a unique $j \leq k$ such that $(\bar{B}_j, f_j)$ is wild; moreover, either $(\bar{B}_j, f_j) \equiv_{\mathrm{Ar}} \bigoplus_{i\in S'}(\bar{B}_i^p, f_i^p)$ or $(\bar{B}_j, f_j) \equiv_{\mathrm{Ar}} \bigoplus^{\partial}_{i\in S'}(\bar{B}_i^p, f_i^p)$ for some $S' \subseteq S$ with $S \setminus S'$ finite.*

(b) *The knot K^*_S is $\precsim^{<\omega}_{\mathrm{WCKn}}$-minimal in WCKn.*

(c) *If $K^*_{S_0} \precsim^{<\omega}_{\mathrm{WCKn}} K^*_{S_1}$ then $S_0 =^* S_1$, where $=^*$ is the identity modulo a finite set.*

Proof. (a) Let $j \leq k$ be such that $(h(\bar{B}_j), h \circ f_j)$ is wild and contains the unique singularity x of K^*_S. There is at least one such j because otherwise K^*_S would be tame, and it is unique because the singularity x is "one-sided", i.e. it is witnessed only on one side while the other side is tame. Thus $(\bar{B}_j, f_j)$, being equivalent to $(h(\bar{B}_j), h \circ f_j)$ via $h \restriction \bar{B}_j$, is wild, while all other $(\bar{B}_i, f_i)$ with $i \neq j$ are tame because so are the proper arcs $(h(\bar{B}_i), h \circ f_i)$. Moreover, by construction $(h(\bar{B}_j), h \circ f_j)$ is either of the form $\bigoplus_{i\in S'}(\bar{B}_i^p, f_i^p)$ (if $x \in \operatorname{Int} h(\bar{B}_j)$) or $\bigoplus^{\partial}_{i\in S'}(\bar{B}_i^p, f_i^p)$ (if $x \in \partial\, h(\bar{B}_j)$), for some $S' \subseteq S$ omitting finitely many elements of S: since $(\bar{B}_j, f_j) \equiv_{\mathrm{Ar}} (h(\bar{B}_j), h \circ f_j)$ we are done.

(b) Suppose that $K \in \mathrm{WCKn}$ is such that $K \precsim^{<\omega}_{\mathrm{WCKn}} K^*_S$ but $K \not\equiv_{\mathrm{Kn}} K^*_S$ (otherwise we are done), and let $\{(\bar{B}_j, f_j) \mid j \leq k\}$, $\{(\bar{B}'_\ell, f'_\ell) \mid \ell \leq k'\}$ and $c \colon C[\mathbf{k+1}] \to C[\mathbf{k'+1}]$ witness this. By part (a) there is a unique ℓ such that $(\bar{B}'_\ell, f'_\ell)$ is wild, and necessarily ℓ is in the range of c because otherwise K would be tame. Let $j = c^{-1}(\ell)$. Since $(\bar{B}_j, f_j) \equiv_{\mathrm{Ar}} (\bar{B}'_\ell, f'_\ell)$, using Remark 4.3.9 one easily gets

$$K \approx^{<\omega}_{\mathrm{WCKn}} C[(\bar{B}_j, f_j)] \approx^{<\omega}_{\mathrm{WCKn}} C[(\bar{B}'_\ell, f'_\ell)] \approx^{<\omega}_{\mathrm{WCKn}} K^*_S.$$

(c) Let $\{(\bar{B}_j, f_j) \mid j \leq k\}$, $\{(\bar{B}'_\ell, f'_\ell) \mid \ell \leq k'\}$ and $c \colon C[\mathbf{k+1}] \to C[\mathbf{k'+1}]$ witness $K^*_{S_0} \precsim^{<\omega}_{\mathrm{WCKn}} K^*_{S_1}$. Apply part (a) to both $K^*_{S_0}$ and $K^*_{S_1}$ to isolate the unique $j \leq k$ and $\ell \leq k'$ such that the proper arcs $(\bar{B}_j, f_j)$ and $(\bar{B}'_\ell, f'_\ell)$ are wild, so that necessarily $c(j) = \ell$ and $(\bar{B}_j, f_j) \equiv_{\mathrm{Ar}} (\bar{B}'_\ell, f'_\ell)$. Let also $S'_0 \subseteq S_0$ and $S'_1 \subseteq S_1$ be such that

$$(\bar{B}_j, f_j) \equiv_{\mathrm{Ar}} \bigoplus\nolimits^{(\partial)}_{i\in S'_0} (\bar{B}^p_i, f^p_i), \qquad (\bar{B}_\ell, f_\ell) \equiv_{\mathrm{Ar}} \bigoplus\nolimits^{(\partial)}_{i\in S'_1} (\bar{B}^p_i, f^p_i),$$

and both $S_0 \setminus S'_0$ and $S_1 \setminus S'_1$ are finite. Then $S'_0 = S'_1$ because $\bigoplus^{(\partial)}_{i\in S'_0} (\bar{B}^p_i, f^p_i) \equiv_{\mathrm{Ar}} \bigoplus^{(\partial)}_{i\in S'_1} (\bar{B}^p_i, f^p_i)$, hence $S_0 =^* S_1$. □

Theorem 4.3.18. (a) *There are $2^{\aleph_0}$-many $\precsim^{<\omega}_{\mathrm{WCKn}}$-incomparable $\precsim^{<\omega}_{\mathrm{WCKn}}$-minimal elements in* WCKn. *In particular, all bases for $\precsim^{<\omega}_{\mathrm{WCKn}}$ are of maximal size.*

(b) *There is a strictly $\precsim^{<\omega}_{\mathrm{WCKn}}$-decreasing ω-sequence in* WCKn *which is not $\precsim^{<\omega}_{\mathrm{WCKn}}$-bounded from below.*

Proof. Fix an enumeration without repetitions $\{(\bar{B}^p_i, f^p_i) \mid i \in \mathbb{N}\}$ of all prime arcs.

(a) As in the proof of Theorem 4.2.13(c), let $\mathcal{P}$ be a family of size $2^{\aleph_0}$ consisting of infinite subsets of $\mathbb{N}$ with pairwise finite intersections. For every $S \in \mathcal{P}$ consider the knot K^*_S defined in Lemma 4.3.17. By Lemma 4.3.17(b) each K^*_S is $\precsim^{<\omega}_{\mathrm{WCKn}}$-minimal in WCKn, and if $S, S' \in \mathcal{P}$ are distinct then K^*_S and $K^*_{S'}$ are $\precsim^{<\omega}_{\mathrm{WCKn}}$-incomparable by Lemma 4.3.17(c).

(b) Let $K_n = C[\bigoplus_{i\geq n}(\bigoplus^\partial_{\mathbb{N}} (\bar{B}^p_i, f^p_i))]$ (notice that each $i \geq n$ is associated to an element of $I\Sigma_{K_n}$). We prove that $\{K_n \mid n \in \mathbb{N}\}$ is the desired ω-chain.

Let $n < n'$. Clearly, $K_{n'} \precsim^{<\omega}_{\mathrm{WCKn}} K_n$. Suppose now, towards a contradiction, that $K_n \precsim^{<\omega}_{\mathrm{WCKn}} K_{n'}$, as witnessed by $\{(\bar{B}_j, f_j) \mid j \leq k\}$, $\{(\bar{B}'_\ell, f'_\ell) \mid \ell \leq k'\}$ and the embedding $c\colon C[\mathbf{k}+\mathbf{1}] \to C[\mathbf{k}'+\mathbf{1}]$. Then there exist $j \leq k$ and $m \in \mathbb{N}$ such that $(\bar{B}_j, f_j)$ contains the tail $\bigoplus^\partial_{t\geq m} (\bar{B}^p_n, f^p_n)$ of $\bigoplus^\partial_{\mathbb{N}} (\bar{B}^p_n, f^p_n)$. But $(\bar{B}_j, f_j) \equiv_{\mathrm{Ar}} (\bar{B}'_{c(j)}, f'_{c(j)})$, and hence $(\bar{B}'_{c(j)}, f'_{c(j)})$ should contain (a proper arc equivalent to) $\bigoplus^\partial_{t\geq m} (\bar{B}^p_n, f^p_n)$, a contradiction. Hence $K_{n'} \prec^{<\omega}_{\mathrm{WCKn}} K_n$.

Suppose now that $K \in \mathrm{WCKn}$ bounds from below $\{K_n \mid n \in \mathbb{N}\}$. Notice that $K \precsim^{<\omega}_{\mathrm{WCKn}} K_0$ implies $I\Sigma_K \neq \emptyset$, so that we can fix $x \in I\Sigma_K$. Let $\{(\bar{B}_j, f_j) \mid j \leq k\}$, $\{(\bar{B}'_\ell, f'_\ell) \mid \ell \leq k'\}$ and the embedding $c\colon C[\mathbf{k}+\mathbf{1}] \to C[\mathbf{k}'+\mathbf{1}]$ witness $K \precsim^{<\omega}_{\mathrm{WCKn}} K_0$. Then there exists $j \leq k$ such that $x \in \bar{B}_j$ and $(\bar{B}_j, f_j)$ is wild. Since $(\bar{B}_j, f_j) \equiv_{\mathrm{Ar}} (\bar{B}'_{c(j)}, f'_{c(j)})$, the proper arc $(\bar{B}'_{c(j)}, f'_{c(j)})$ is also wild and contains an element of $I\Sigma_{K_0}$, which belongs to $\bigoplus^\partial_{\mathbb{N}} (\bar{B}^p_n, f^p_n)$ for some $n \geq 0$. This implies that there exists $m \in \mathbb{N}$ such that $C[\bigoplus^\partial_{t\geq m} (\bar{B}^p_n, f^p_n)] \precsim^{<\omega}_{\mathrm{WCKn}} K$. But by the argument of the previous paragraph $C[\bigoplus^\partial_{t\geq m} (\bar{B}^p_n, f^p_n)] \not\precsim^{<\omega}_{\mathrm{WCKn}} K_{n+1}$, hence $K \not\precsim^{<\omega}_{\mathrm{WCKn}} K_{n+1}$, which is a contradiction. □

When $K \in \mathrm{Kn}$, we say that $x \in K$ is **isolable in** K if there exists a subarc $(\bar{B}, f)$ of K such that $x \in I\Sigma_{(\bar{B},f)}$. Notice that every $x \in I\Sigma_K$ is isolable in K, but some point which is isolable in K can fail to belong to $I\Sigma_K$ because it is an accumulation point of other singular points only from one side. It is immediate that the set of points isolable in K is countable.

Theorem 4.3.19. *Every $\precsim^{<\omega}_{\mathrm{WCKn}}$-antichain is contained in a $\precsim^{<\omega}_{\mathrm{WCKn}}$-antichain of size $2^{\aleph_0}$. In particular, there are no maximal $\precsim^{<\omega}_{\mathrm{WCKn}}$-antichains of size smaller than $2^{\aleph_0}$, and every $K \in \mathrm{WCKn}$ belongs to a $\precsim^{<\omega}_{\mathrm{WCKn}}$-antichain of size $2^{\aleph_0}$.*

Proof. Let $\{(\bar{B}^p_i, f^p_i) \mid i \in \mathbb{N}\}$, $\mathcal{P}$ and $K_S \in \mathrm{WCKn}$, with $S \in \mathcal{P}$, be as in the proof of Theorem 4.3.18(a). Following the proof of Proposition 2.2.9, it is enough to prove that the set $\{S \in \mathcal{P} \mid K_S \precsim^{<\omega}_{\mathrm{WCKn}} K\}$ is countable for each $K \in \mathrm{WCKn}$. Suppose that $S \subseteq \mathbb{N}$ is such that $K_S \precsim^{<\omega}_{\mathrm{WCKn}} K$, as witnessed by $\{(\bar{B}_j, f_j) \mid j \leq k\}$, $\{(\bar{B}'_\ell, f'_\ell) \mid \ell \leq k'\}$ and the embedding $c\colon C[\mathbf{k}+\mathbf{1}] \to C[\mathbf{k}'+\mathbf{1}]$. There exist $j \leq k$ and m such that $\bigoplus^\partial_{i\in S, i\geq m} (\bar{B}^p_i, f^p_i) \precsim_{\mathrm{Ar}} (\bar{B}_j, f_j)$, and hence $\bigoplus^\partial_{i\in S, i\geq m} (\bar{B}^p_i, f^p_i) \precsim_{\mathrm{Ar}} (\bar{B}'_{c(j)}, f'_{c(j)})$; thus $\bigoplus^\partial_{i\in S, i\geq m} (\bar{B}^p_i, f^p_i)$ is a subarc of K. Therefore there exists $x_S \in \Sigma_K$ which is determined by a tail of $\bigoplus^\partial_{i\in S} (\bar{B}^p_i, f^p_i)$. Notice that x_S is isolable in K. If $S, S' \in \mathcal{P}$ are distinct and $x_S = x_{S'}$ then by Lemma 4.2.12(b) (and recalling that S and S' have finite intersection) the images of $\bigoplus_{i\in S, i\geq m} (\bar{B}^p_i, f^p_i)$ and $\bigoplus_{i\in S', i\geq m'} (\bar{B}^p_i, f^p_i)$ approach x_S from opposite sides. Hence, $|\{S \in \mathcal{P} \mid x_S = x\}| \leq 2$ for every x isolable in K. Since the set of isolable points in K is countable, $\{S \in \mathcal{P} \mid K_S \precsim^{<\omega}_{\mathrm{WCKn}} K\}$ is countable as well. □

5

Classification of 3-manifolds and Cantor sets of $\mathbb{R}^3$

5.1 Preliminaries

In this section we introduce the basic notions and theorems that we need in Chapter 5. Regarding notions about Polish spaces and Borel reducibility, refer to Section 1.1.

Given a metric space (X, d), by

$$B_X(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}, \text{ and}$$
$$\bar{B}_X(x, \varepsilon) = \{y \in X \mid d(x, y) \leqslant \varepsilon\}$$

we always denote, respectively, the open and closed ball in X with center $x \in X$ and radius $\varepsilon \in \mathbb{R}_+$, where $\mathbb{R}_+$ is the space of positive real numbers. We drop X from the lower case when it is clear from the context, and briefly write B^n and $\bar{B}^n$ to denote respectively the open and closed unit ball of the Euclidean n-space $\mathbb{R}^n$.

If X is a topological space and $Y \subseteq X$, then by $\text{int}(Y)$ and ∂Y we denote, respectively, the interior and boundary of Y. Moreover, if $Z \subseteq Y$, we respectively denote by $\text{int}_Y(Z)$ and $\partial_Y(Z)$ the relative interior and boundary of Z w.r.t. the topology induced by X on Y.

When X and Y are sets, $f\colon X \to Y$ is a function and $X_0 \subseteq X$, we write $f[X_0]$ instead of $\{f(x) \mid x \in X_0\}$.

If X is a topological space and β is a basis for a topology $\mathcal{T}$ on X, we often use the notation $\langle \beta \rangle$ instead of $\mathcal{T}$.

Given the product $\prod_{i \in I} X_i$ of a family of topological spaces $(X_i)_{i \in I}$, for each $j \in I$ we denote by $\text{pr}_{X_j}\colon \prod_{i \in I} X_i \to X_j$ the projection function on the j-th coordinate.

An important Polish space that we often use is given by the following.

Example 5.1.1. Let X be a topological space, and consider the space $K(X)$ of all compact subsets of X equipped with the **Vietoris topology**, i.e., the one generated by the sets of the form

$$\{K \in K(X) \mid K \subseteq U\}, \text{ and}$$
$$\{K \in K(X) \mid K \cap U \neq \varnothing\},$$

for U open in X.

Let now (X, d) be a metric space with $d \leqslant 1$, i.e. $d(x, y) \leqslant 1$ for every $x, y \in X$. We define the **Hausdorff metric** on $K(X)$, d_H , as follows:

$$d_H(K, L) = \begin{cases} 0 & \text{if } K = L = \varnothing \\ 1 & \text{if exactly one of } K, L \text{ is } \varnothing \\ \max\{\delta(K, L), \delta(L, K)\} & \text{if } K, L \neq \varnothing \end{cases},$$

where $\delta(K, L) = \max_{x \in K} d(x, L)$ and $d(x, L) = \inf\{d(x, y) \mid y \in L\}$. It is easy to see that the Hausdorff metric is compatible with the Vietoris topology.

If X is Polish, so is $K(X)$, and if in addition X is compact, so is $K(X)$ (see [Kec95, Theorems 4.25 and 4.26]).

The next proposition recalls some properties of basic relations defined on $K(X)$.

Proposition 5.1.2. *[Kec95, Exercise 4.29] Let X be a metric space.*

(i) *The relation "$x \in K$" is closed, i.e., $\{(x, K) \mid x \in K\}$ is closed in $X \times K(X)$.*

(ii) *The relation "$K \subseteq L$" is closed, i.e., $\{(K, L) \mid K \subseteq L\}$ is closed in $K(X)^2$.*

(iii) *The relation "$K \cap L \neq \varnothing$" is closed in $K(X)^2$.*

(iv) *If Y is metrizable, then the map $(K, L) \mapsto K \times L$ from $K(X) \times K(Y)$ into $K(X \times Y)$ is continuous.*

In addition, we will use the following:

Proposition 5.1.3. *The map $K(X) \times K(X) \to K(X)$ defined by $(K_0, K_1) \mapsto K_0 \cap K_1$ is continuous.*

Proof. Let $(K_0, K_1) \in K(X)^2$. Let $\varepsilon \in \mathbb{R}_+$. We will find $\delta \in \mathbb{R}_+$ such that for all $(K_0', K_1') \in K(X)^2$, if $d(K_i, K_i') < \delta$ for $i \in \{0, 1\}$, then $d(K_0 \cap K_1, K_0' \cap K_1') < \varepsilon$. For all $t \in \mathbb{R}_+$ define $U_t = \{x \in X \mid d(x, K_0 \cap K_1) < t\}$. Define the following sets:

$$K_0^{\geqslant \varepsilon/2} = K_0 \setminus U_{\varepsilon/2} \qquad K_1^{\geqslant \varepsilon/2} = K_1 \setminus U_{\varepsilon/2}$$
$$K_0^{\leqslant \varepsilon/2} = K_0 \cap \bar{U}_{\varepsilon/2} \qquad K_1^{\leqslant \varepsilon/2} = K_1 \cap \bar{U}_{\varepsilon/2}.$$

All of these sets are compact, and

$$K_0^{\geqslant \varepsilon/2} \cap K_1^{\geqslant \varepsilon/2} = \varnothing, \text{ so } d(K_0^{\geqslant \varepsilon/2}, K_1^{\geqslant \varepsilon/2}) > 0$$

Let

$$\delta = \min \left\{ d(K_0^{\geqslant \varepsilon/2}, K_1^{\geqslant \varepsilon/2})/2, \varepsilon/3 \right\}.$$

Suppose now $(K_0', K_1') \in K(X)^2$ is such that $d(K_i, K_i') < \delta$ for $i \in \{0, 1\}$, and let $x \in K_0' \cap K_1'$. Let $y_i \in K_i$ for $i \in \{0, 1\}$ be such that $d(x, y_i) < \delta$. So we must have $d(y_0, y_1) < 2\delta \leqslant d(K_0^{\geqslant \varepsilon/2}, K_1^{\geqslant \varepsilon/2})$. This means that for at least one i, y_i belongs to $K_i^{\leqslant \varepsilon/2}$. W.l.o.g. assume $i = 0$ and $y_0 \in K_0^{\leqslant \varepsilon/2}$ which by definition means that $d(y_0, K_0 \cap K_1) \leqslant \varepsilon/2$. Now

$$d(x, K_0 \cap K_1) \leqslant d(x, y_0) + d(y_0, K_0 \cap K_1) < \delta + \varepsilon/2 \leqslant \varepsilon/3 + \varepsilon/2 = 5\varepsilon/6.$$

Since x was arbitrary, this implies that $d(K_0 \cap K_1, K_0' \cap K_0') \leqslant 5\varepsilon/6 < \varepsilon$. □

Recall Definition 1.1.1 of a standard Borel space. A particularly important construction of a standard Borel space that we use in this chapter is given in the following example.

Example 5.1.4. Given a topological space X, the collection $F(X)$ of all its closed subsets can be equipped with the σ-algebra $\mathcal{B}_{F(X)}$ generated by the sets of the form

$$\{F \in F(X) \mid F \cap U \neq \varnothing\},$$

for $U \subseteq X$ nonempty open. It turns out that if X is Polish, then $(F(X), \mathcal{B}_{F(X)})$ is a standard Borel space, called **Effros Borel space** (see [Kec95, Theorem 12.6]).

The following results are basic facts about the Effros Borel space.

Proposition 5.1.5. *[Kec95, Exercise 12.11] Let X be Polish.*

(i) *$K(X)$ is a Borel set in $F(X)$.*

(ii) *The relation "$F_0 \subseteq F_1$" in $F(X)^2$ is Borel.*

(iii) *The class of regular closed sets in X is Borel in $F(X)$.*

(iv) *For each $F \in F(X)$ and $K \in K(X)$, the relation "$F \cap K = \varnothing$" is Borel in $F(X) \times K(X)$.*

Theorem 5.1.6. *[Kec95, Theorem 12.13] Let X be Polish. There is a sequence of Borel functions $d_n : F(X) \to X$, such that for nonempty $F \in F(X)$, $\{d_n(F)\}$ is dense in F.*

Definition 5.1.7. When X is Polish, we denote by $D(X)$ the countable dense set of X obtained in a Borel way by applying Theorem 5.1.6.

We often make use of a **universal Urysohn space** $\mathbb{U}$, referring the reader to [Gao09, Section 1.2] for the relevant definitions and proofs. Given any Polish metric space X, using the Katětov construction one can canonically construct a Polish metric space $\mathbb{U}_X$ such that for all Polish metric spaces X and Y

- $\mathbb{U}_X$ contains (a canonical isometric copy of) X, and every isometry $\iota : X \to Y$ can be extended to an isometry $\iota^* : \mathbb{U}_X \to \mathbb{U}_Y$;
- $\mathbb{U}_X$ has the so-called Urysohn property, whence $\mathbb{U}_X$ is isometric to $\mathbb{U}_Y$ for all Polish metric spaces X, Y.

Let now $\mathbb{U}$ be the space $\mathbb{U}_{\mathbb{R}}$: by the Urysohn property, a metric space is Polish if and only if it is isometric to a closed subspace of $\mathbb{U}$. It is thus natural to regard the Effros Borel space $F(\mathbb{U})$ of closed subspaces of $\mathbb{U}$ as the standard Borel space of all Polish metric spaces.

Definition 5.1.8. Let $\mathcal{U}$ be the set of all open subsets of the Urysohn space $\mathbb{U}$. The topology on $\mathcal{U}$ is induced by the bijective map $F(\mathbb{U}) \to \mathcal{U}$ given by $F \mapsto \mathbb{U} \setminus F$.

We say that a Polish metric space X is **Heine-Borel** if any closed bounded subset of X is compact. One can equivalently express the property of being Heine-Borel in $\mathbb{U}$ in the following form:

Proposition 5.1.9. *A metric space $X \subseteq \mathbb{U}$ is Heine-Borel if and only if $\bar{B}(x, n) \cap X$ is compact for all $x \in \mathbb{U}$, $n \in \mathbb{N}$.*

We also need the following result.

Proposition 5.1.10. *[MR17, Proposition 2.3] The class of Heine-Borel Polish metric space is a standard Borel space.*

Recall that a subset A of a topological space X is K_σ if $A = \bigcup_n K_n$, where $K_n \in K(X)$.

Theorem 5.1.11. *[Kec95, Theorem 5.3] Let X be Hausdorff and locally compact. Then the following statements are equivalent:*

- *X is metrizable and K_σ;*
- *X is Polish.*

A relevant result involging Heine-Borel metric spaces is stated in the next theorem.

Theorem 5.1.12. *[WJ87, Vau37] If X is a K_σ, locally compact, metrizable space, then there is a compatible metric on X which is Heine-Borel.*

5.1.1 Spaces of Embeddings

In this section we define some standard Borel spaces of functions which are useful in the sequel. Recall that an embedding is a homeomorphism onto its image.

Definition 5.1.13. Let $X = (X, d)$ and $Y = (Y, d')$ be metric spaces. Let $\operatorname{PartEmb}(X, Y)$ be the set of all $F \in K(X \times Y)$ such that

$$\forall \varepsilon \in \mathbb{R}_+ \exists \delta \in \mathbb{R}_+ \forall ((x, y), (x', y')) \in (F \times F)\big(d(x, x') < \delta \to d'(y, y') \leqslant \varepsilon\big)$$
$$\text{and } \forall \varepsilon \in \mathbb{R}_+ \exists \delta \in \mathbb{R}_+ \forall ((x, y), (x', y')) \in (F \times F)\big(d'(y, y') < \delta \to d(x, x') \leqslant \varepsilon\big).$$

Lemma 5.1.14. *Suppose $(X, d), (Y, d')$ are metric spaces. Then $F \in \operatorname{PartEmb}(X, Y)$ if and only if there is a compact $C \subseteq X$ and an embedding $f : C \to Y$ such that F is the graph $\operatorname{graph}(f)$ of f. Thus, the set $\operatorname{PartEmb}(X, Y)$ parametrizes in a natural way all partial embeddings from X to Y with compact domain.*

Proof. Suppose $f : C \to Y$ is such an embedding. Then the conditions of Definition 5.1.13 are simply saying that both f and its inverse are uniformly continuous, so they are satisfied by the compactness of C (from which in turn the compactness of $f[C]$ also follows).

Suppose $F \in \operatorname{PartEmb}(X, Y)$. Let $C = \operatorname{pr}_X(F)$, $D = \operatorname{pr}_Y(F)$. Then C and D are compact by the compactness of F. The conditions imply that for all $x \in C$ there is unique $y \in D$ with $(x, y) \in F$ and vice versa, so the map $f : X \to Y$ such that $F = \operatorname{graph}(f)$ is indeed a bijection. But then the conditions imply that both f and its inverse are continuous, so we are done. □

Lemma 5.1.15. *If X and Y are Polish, then $\operatorname{PartEmb}(X, Y)$ is a Borel subset of $K(X \times Y)$.*

Proof. For any fixed $\varepsilon, \delta \in \mathbb{Q}_+$ the sets

$$Z^0_{\varepsilon\delta} = \{((x, y), (x', y')) \in (X \times Y)^2 \mid d(x, x') < \delta \to d(y, y') \leqslant \varepsilon\}$$

$$Z^1_{\varepsilon\delta} = \{((x, y), (x', y')) \in (X \times Y)^2 \mid d(y, y') < \delta \to d(x, x') \leqslant \varepsilon\}$$

are closed in $(X \times Y)^2$, so the sets $K(Z^k_{\varepsilon\delta})$ are Borel for $k \in \{0, 1\}$ in $F((X \times Y)^2)$ by Proposition 5.1.5(i). Let $\xi : K(X \times Y) \to K((X \times Y)^2)$ be the map $F \mapsto F \times F$, which is Borel by Proposition 5.1.2(iv). Now it is easy to check that $\operatorname{PartEmb}(X, Y) = \bigcap_{k \in \{0,1\}} \bigcap_\varepsilon \bigcup_\delta \xi^{-1}[K(Z^k_{\varepsilon\delta})]$, which is a Borel set. □

Definition 5.1.16. For compact metric space X and a metric space Y, let $\operatorname{Emb}(X, Y) = \{F = \operatorname{graph}(f) \in \operatorname{PartEmb}(X, Y) \mid \operatorname{dom}(f) = X\}$ be the set of embeddings of X into Y.

Lemma 5.1.17. *If X is compact, then $\operatorname{Emb}(X, Y)$ is a closed subset of $\operatorname{PartEmb}(X, Y)$. Thus, it is a standard Borel space.*

Proof. Suppose $F = \operatorname{graph}(f) \in \operatorname{PartEmb}(X, Y) \setminus \operatorname{Emb}(X, Y)$ and let $C = \operatorname{dom} f$. Let $x \in X \setminus C$ and let $\varepsilon = \frac{1}{2} d(x, C)$. Let U_ε be the ε-neighbourhood of F in $K(X \times Y)$ (w.r.t. the Hausdorff metric). Then $U_\varepsilon \cap \operatorname{PartEmb}(X, Y)$ is contained in $\operatorname{PartEmb}(X, Y) \setminus \operatorname{Emb}(X, Y)$. Hence, $\operatorname{PartEmb}(X, Y) \setminus \operatorname{Emb}(X, Y)$ is an open set of $\operatorname{PartEmb}(X, Y)$, and thus $\operatorname{Emb}(X, Y)$ is closed.

The second assertion follows by applying Lemma 5.1.15. □

Lemma 5.1.18. *Suppose $(X, d), (Y, d')$ are metric spaces, and $L \in [1, \infty)$. Let*

$$\operatorname{PartEmb}_L(X, Y) \subseteq \operatorname{PartEmb}(X, Y)$$

be the set of partial embeddings which are L-bilipschitz.

(a) *If X is compact, then $\operatorname{PartEmb}_L(X, Y)$ and $\operatorname{Emb}_L(X, Y) = \operatorname{Emb}(X, Y) \cap \operatorname{PartEmb}_L(X, Y)$ are compact.*

(b) *If X is Polish and locally compact, then $\operatorname{PartEmb}_L(X, Y)$ is K_σ.*

Proof. (a) Let X be compact. By Lemma 5.1.17 it is enough to prove that $\mathrm{PartEmb}_L(X,Y)$ is compact. For this it is enough to show that $\mathrm{PartEmb}_L(X,Y)$ is closed in $K(X\times Y)$. Suppose $(H_i)_i \subseteq \mathrm{PartEmb}_L(X,Y)$ is a Cauchy sequence, and let $H \in K(X\times Y)$ be the limit of $(H_i)_i$ in $K(X\times Y)$. We want to show that H is in fact L-bilipschitz and belongs to $\mathrm{PartEmb}_L(X,Y)$. Suppose $(x,y),(x',y') \in H\times H$. It is enough to show that $L^{-1}d(y,y') \leqslant d(x,x') \leqslant Ld'(y,y')$, because this implies simultaneously both that H satisfies the definition of being in $\mathrm{PartEmb}(X,Y)$ and that it is L-bilipschitz. From the fact that $H_i \stackrel{i\to\infty}{\longrightarrow} H$ it is easy to obtain sequences $(x_i,y_i)_i$ and $(x_i',y_i')_i$ converging to (x,y) and (x',y') respectively such that $(x_i,y_i),(x_i',y_i') \in H_i$. Since each H_i is L-bilipschitz, we have $L^{-1}d'(y_i,y_i') \leqslant d(x_i,x_i') \leqslant Ld'(y_i,y_i')$, a property which is preserved in the limit.

(b) Suppose that X is Polish and locally compact. Then by Theorem 5.1.11 X is K_σ and by applying Theorem 5.1.12 we obtain a compatible metric on X with respect to which X is Heine-Borel. It is now easy to see that we can write X in the form $X=\bigcup_{i\in\mathbb{N}} C_i$, where C_i is the closure of an open set U_i such that

$$U_0 \subseteq C_0 \subseteq U_1 \subseteq C_1 \subseteq \dots$$

Then C_i is compact for every i, and

$$\mathrm{PartEmb}_L(X,Y) = \bigcup_{i\in\mathbb{N}} \mathrm{PartEmb}_L(C_i,Y) \text{ and}$$
$$\mathrm{Emb}_L(X,Y) = \bigcup_{i\in\mathbb{N}} \mathrm{Emb}_L(C_i,Y).$$

Thus, $\mathrm{PartEmb}_L(X,Y)$ and $\mathrm{Emb}_L(X,Y)$ are K_σ by (a). ☐

Lemma 5.1.19. *The set* $B=\{(C,\mathrm{graph}(f)) \in K(X)\times \mathrm{PartEmb}(X,Y) \mid C\subseteq \mathrm{dom}(f)\}$ *is closed.*

Proof. If $(C,\mathrm{graph}(f)) \notin B$, pick $x \in C\setminus \mathrm{dom}(f)$ and $\varepsilon>0$ such that $d(x,\mathrm{dom}(f))>2\varepsilon$, which exists by the compactness of $\mathrm{dom}(f)$. Then the ε-neighbourhood of $(C,\mathrm{graph}(f))$ is an open neighbourhood of $(C,\mathrm{graph}(f))$ outside B. Thus, the complement of B is open, equivalently B is closed. ☐

5.1.2 Stabilizing function sequences

Definition 5.1.20. Given a sequence of functions $f_k\colon X\times Y\to Y$, denote by $\hat f_k\colon X\to Y$ the function obtained by iterating as:

$$\hat f_0(x)=f_0(x) \quad \text{ and } \quad \hat f_k(x)=f_k(x,\hat f_{k-1}(x)),\ \forall k\geq 1.$$

Notice that if X,Y are standard Borel spaces, and each f_k is Borel then each $\hat f_k$ is Borel as well.

Let now X be a nonempty set and $n\in\mathbb{N}$. We denote by X^n the set of finite sequences $(x(0),\dots,x(n-1))=(x_0,\dots,x_{n-1})$ of length n from X. We allow the case $n=0$, in which case $A^0=\{\varnothing\}$, where $\varnothing$ denotes here the empty sequence. Finally, let $X^{<\mathbb{N}}=\bigcup_{n\in\mathbb{N}} X^n$ (resp. $X^{\mathbb{N}}$) be the set of all finite (resp. infinite) sequences from X.

Definition 5.1.21. Let X,Y be standard Borel spaces and $B\subset X\times Y^{<\mathbb{N}}$ a Borel subset such that for all $x\in X$, $(x,\varnothing)\in B$. For all $k\in\mathbb{N}$ denote by $Y^{<\mathbb{N}}_{\geqslant k}$ the set of all finite sequences of elements of Y which have length at least k. Suppose $f_k\colon B\to Y^{<\mathbb{N}}_{\geqslant k}$ is a Borel map for each k. We say that $(f_k)_k$ is **stabilizing for** B, if

(a) for all $(x,y)\in B$ and all k, $(x,f_k(x,y))\in B$,

(b) for all $x\in X$ and for every i the sequence $(\hat f_k(x)_i)_{k>i}$ is eventually constant, where $\hat f_k(x)_i$ is defined for $i<k$ to be the i-th element of $\hat f_k(x)\in Y^{<\mathbb{N}}_{\geqslant k}$.

Given a stabilizing sequence $(f_k)_k$ as above, let $\lim_{k\to\infty} f_k$ be the function $F\colon X \to Y^{\mathbb{N}}$ defined by

$$F(x) = (\hat{f}(x)_i)_{i\in\mathbb{N}}$$

where $\hat{f}(x)_i = \lim_{k\to\infty} f_k(x)_i$ is the unique y_i such that $y_i = \hat{f}_k(x)_i$ for co-finitely many k.

Lemma 5.1.22 (Stabilization)**.** *Let X, Y be standard Borel spaces, $B \in X \times Y^{<\mathbb{N}}$ Borel such that*

$$\forall x((x, \varnothing) \in B), \tag{5.1.1}$$

and for all k let $f_k\colon B \to Y^{<\mathbb{N}}_{\geqslant k}$ be a Borel map. Assume that the sequence $(f_k)_k$ is stabilizing for B. Then the following hold for $F = \lim_{k\to\infty} f_k$:

(a) *F is Borel,*

(b) *for all $x \in X$ and all $i \in \mathbb{N}$ we have $(x, F(x)\restriction i) \in B$*

(c) *for all i there are k and $n \geq k$ such that $F(x) \restriction i \in \{f_k(b, y_0, \ldots, y_{k-1}, \ldots, y_{n-1}) \restriction i \mid (b, y_0, \ldots, y_{k-1}, \ldots, y_{n-1}) \in B\}$.*

Proof. For all $x \in X$, $F(x)\restriction i = \hat{f}(x)_i = \hat{f}_k(x)_i$ for all large enough k. Using (5.1.1), the definition of $\hat{f}_k$, and Definition 5.1.21(a), one can prove (b) and (c) by induction on i.

Let us prove (a). Let $O \subseteq Y^{\mathbb{N}}$ be open of the form $O = \prod_{i=0}^{n} O_i \times \prod_{i=n+1}^{\infty} Y$ (where $O_i \subseteq Y$ is open in some admissible topology on Y). It is enough to show that $F^{-1}[O]$ is Borel. Now

$$F^{-1}[O] = \bigcap_{i\leqslant n} \{x \in X \mid \hat{f}(x)_i \in O_i\},$$

so it is enough to show that if $U \subseteq Y$ is open, then $\hat{f}(x)_i^{-1}[U]$ is Borel for every $i \leq n$. Fix i. Then

$$\begin{aligned}
\hat{f}(x)_i^{-1}[U] &= \{x \in X \mid \hat{f}(x)_i \in U\} \\
&= \{x \in X \mid \forall k \in \mathbb{N}(k > i \to (\exists j \in \mathbb{N}\ \forall m > j\ \hat{f}_m(x)_i \in U))\} \\
&= \bigcap_{k>i} \bigcup_{j\in\mathbb{N}} \bigcap_{m>j} \{x \in X \mid \hat{f}_m(x)_i \in U)\} \\
&= \bigcap_{k>i} \bigcup_{j\in\mathbb{N}} \bigcap_{m>j} (\mathrm{pr}_i \circ \hat{f}_k)^{-1}[U]
\end{aligned}$$

where pr_i is the projection to the i-th coordinate which is Borel, so the set $(\mathrm{pr}_i \circ \hat{f}_k)^{-1}[U]$ is also Borel. □

Lemma 5.1.23. *Let X, Y be standard Borel spaces and suppose $f_k\colon X \times Y^k \to Y$ are Borel maps for all $k \in \mathbb{N}$. Then the map $f\colon X \to Y^{\mathbb{N}}$ defined by $f(x) = \bar{y}$, where*

$$y_0 = f_0(x) \quad \text{and} \quad y_{n+1} = f_{n+1}(x, y_0, \ldots, y_n),$$

is Borel.

Proof. Given $(f_k)_k$ as in the assumption, let $f'_k\colon X \times Y^{<\mathbb{N}} \to Y^{<\mathbb{N}}$ be defined by

$$f'_k(x, y_0, \ldots, y_{n-1}) = \begin{cases} (y_0, \ldots, y_{n-1}, f_k(x, y_0, \ldots, y_{n-1})) & \text{if } n = k, \\ \varnothing & \text{otherwise.} \end{cases}$$

Then $(f'_k)_k$ is stabilizing for $B = X \times Y^{<\mathbb{N}}$, and $\lim_{k\to\infty} f'_k = f$, so we can apply Lemma 5.1.22. □

5.2 Manifolds

An n-**manifold** is a separable metric space each point of which has a neighbourhood homeomorphic to the Euclidean n-space $\mathbb{R}^n$. It is standard to check that a separable metric space M is an n-manifold if and only if there exists a locally finite countable **atlas** on M. An atlas is an indexed family of **charts** $\bar\varphi = (\varphi_i)_{i\in I}$ such that each φ_i is an embedding from the closed ball $\bar B^n$ into M, the set $\{\varphi_i[B^n] \mid i \in I\}$ is an open cover of M, and the collection $\{\varphi_i[\bar B^n] \mid i \in I\}$ is locally finite, meaning that for all i the set

$$\{j \in \mathbb{N} \mid \varphi_i[\bar B^n] \cap \varphi_j[\bar B^n] \neq \varnothing\}$$

is finite. Since M is separable, local finiteness implies that the atlas is countable and without loss of generality we can always assume that $I = \mathbb{N}$, so call such atlas as defined above a **locally finite atlas**.

Our goal in this section is to code the collection of n-manifolds as a standard Borel space. Recall that if X is a Polish space, we can consider a countable dense set $D(X)$ of X defined as in Definition 5.1.7.

Definition 5.2.1. Define the set of locally finite collections of charts into $\mathbb{U}$:

$$\mathfrak{L}_0 = \{\bar\varphi \in \mathrm{Emb}(\bar B^n, \mathbb{U})^{\mathbb{N}} \mid \forall i \in \mathbb{N} \exists j \in \mathbb{N} \forall k \in \mathbb{N}(k > j \to \varphi_k[\bar B^n] \cap \varphi_i[\bar B^n] = \varnothing)\},$$

the space of collections of charts into $\mathbb{U}$ which cover each other's boundaries:

$$\mathfrak{L}_1 = \{\bar\varphi \in \mathrm{Emb}(\bar B^n, \mathbb{U})^{\mathbb{N}} \mid \forall k \in \mathbb{N} \exists \delta \in \mathbb{Q}_+ \forall x \in D(\varphi_k[\partial \bar B^n]) \exists i \in \mathbb{N}(B(x,\delta) \subset \varphi_i[B^n])\}$$

and the space of n-manifold atlases as:

$$\mathfrak{M}_n = \{\bar\varphi \in \mathfrak{L}_0 \cap \mathfrak{L}_1 \mid \varphi_i[B^n] \text{ is open in } \bigcup_i \varphi_i[B^n]\}.$$

Let $\approx_{\mathfrak{M}_n}$ be the relation on $\mathfrak{M}_n$ where $\bar\varphi \approx_{\mathfrak{M}_n} \bar\varphi'$ if and only if $\bigcup_{i\in\mathbb{N}} \varphi_i[B^n]$ is homeomorphic to $\bigcup_{i\in\mathbb{N}} \varphi'_i[B^n]$. For convenience, given $\bar\varphi \in \mathfrak{M}_n$, we will denote

$$M(\bar\varphi) = \bigcup_{i\in\mathbb{N}} \varphi_i[B^n],$$

the manifold associated with φ (see the following Lemma). Note that by the property of being in $\mathfrak{L}_1$, we have

$$\bigcup_{i\in\mathbb{N}} \varphi_i[B^n] = \bigcup_{i\in\mathbb{N}} \varphi_i[\bar B^n] \tag{5.2.1}$$

Lemma 5.2.2. *Let M be an n-manifold and $(\psi_i)_{i\in\mathbb{N}}$ a locally finite atlas on M. Then there is $\bar\varphi \in \mathfrak{M}_n$ such that M is isometric to $M(\bar\varphi)$ via an isometry $\iota\colon M \to \mathbb{U}$ such that for all $i \in \mathbb{N}$ we have $\varphi_i = \iota \circ \psi_i$. Conversely, for every $\bar\varphi \in \mathfrak{M}_n$, the space $M(\bar\varphi)$ is an n-manifold with $\bar\varphi$ constituting a locally finite atlas for it.*

Proof. By the universality of $\mathbb{U}$, M can be isometrically embedded into $\mathbb{U}$ by an isometry ι. Define $\varphi_i = \iota \circ \psi_i$, which proves the first part. For the converse, suppose that $\bar\varphi \in \mathfrak{M}_n$. As a subset of $\mathbb{U}$, $M(\bar\varphi)$ is separable in the induced metric. Let $x \in M(\bar\varphi)$. By (5.2.1) there is $i \in \mathbb{N}$ such that $x \in \varphi_i[B^n]$. Now, by definition of $\mathfrak{M}_n$, $\varphi_i[B^n]$ is an open neighbourhood of x in $M(\bar\varphi)$, and it is homeomorphic to B^n, so $M(\bar\varphi)$ is indeed an n-manifold, and $\bar\varphi$ is a locally finite atlas by the property of being in $\mathfrak{L}_0$. □

Lemma 5.2.3. *$\mathfrak{M}_n$ is a Borel subset of $\mathrm{Emb}(\bar B^n, \mathbb{U})^{\mathbb{N}}$, and so it is a standard Borel space.*

Proof. First note that $\mathfrak{L}_0$ and $\mathfrak{L}_1$ are Borel sets because they are defined using countable quantifiers, and by Proposition 5.1.5 all the conditions are Borel. Now it is enough to show that the set of those $\bar{\varphi} \in \mathrm{Emb}(B^n, \mathbb{U})^{\mathbb{N}}$ which satisfy the condition

$$\varphi_i[B^n] \text{ is open in } M(\bar{\varphi}) \tag{5.2.2}$$

is Borel. We will show that (5.2.2) is equivalent to the following condition $(*)$:

$(*)$ for all $\varepsilon \in \mathbb{Q}_+$ there is $\delta \in \mathbb{Q}_+$ such that for all $x \in \varphi_i[B^n(0, 1-\varepsilon) \cap \mathbb{Q}^n]$, all $j \in \mathbb{N}$, and all $y \in \varphi_j[B^n \cap \mathbb{Q}^n] \setminus \varphi_i[\bar{B}^n]$ we have $d(x, y) > \delta$.

This condition is clearly Borel, because all quantifiers range over countable sets. Let us prove that (5.2.2) $\Rightarrow$ $(*)$. Suppose (5.2.2) holds. Let $\varepsilon \in \mathbb{Q}_+$. Since $\varphi_i[\bar{B}^n(0, 1-\varepsilon)]$ is compact in $\varphi_i[B^n]$ which in turn is open in $M(\bar{\varphi})$,

$$\delta = d\Big(M(\bar{\varphi}) \setminus \varphi_i[B^n], \varphi_i[\bar{B}^n(0, 1-\varepsilon)]\Big) > 0. \tag{5.2.3}$$

Let $x \in \varphi_i[B^n(0, 1-\varepsilon) \cap \mathbb{Q}^n]$ and $y \in \varphi_j[B^n \cap \mathbb{Q}^n] \setminus \varphi_i[\bar{B}^n]$ for some $j \in \mathbb{N}$. Then by (5.2.3) clearly $d(x, y) > \delta$.

Now let us prove $(*)$ $\Rightarrow$(5.2.2). Assume $(*)$ and suppose $x' \in \varphi_i[B^n]$. Let $\varepsilon \in \mathbb{Q}_+$ be such that

$$\varepsilon < d(\varphi_i^{-1}(x'), \partial B^n).$$

Let $\delta \in \mathbb{Q}_+$ be as given by $(*)$. Let $j \in \mathbb{N}$ be arbitrary. We claim that

$$B(x', \delta/3) \cap \varphi_j[B^n] \subseteq \varphi_i[\bar{B}^n] \tag{5.2.4}$$

for all $j \in \mathbb{N}$. This is sufficient, because by the arbitrariness of j, it implies that $B(x', \delta/3) \cap \varphi_i[B^n] = B(x', \delta/3) \cap M(\bar{\varphi})$ is an open neighbourhood of x' in $\varphi_i[B^n]$. We will prove the following statement, which is equivalent to (5.2.4):

$$(\varphi_j[B^n] \setminus \varphi_i[\bar{B}^n]) \cap B(x', \delta/3) = \varnothing. \tag{5.2.5}$$

If $\varphi_j[B^n] \setminus \varphi_i[\bar{B}^n]$ is empty, we are done. Otherwise pick an arbitrary $y' \in \varphi_j[B^n] \setminus \varphi_i[\bar{B}^n]$. We will show that $d(x', y') > \delta/3$. Since $\varphi_i[\bar{B}^n]$ is compact, the set

$$\varphi_j^{-1}[\varphi_j[B^n] \setminus \varphi_i[\bar{B}^n]] = B^n \setminus \varphi_j^{-1}[\varphi_i[\bar{B}^n]]$$

is open, so there is $q \in (B^n \cap \mathbb{Q}^n) \setminus \varphi_j^{-1}[\varphi_i[\bar{B}^n]]$ such that $d(\varphi_j(q), y') < \delta/3$. Let $y = \varphi_j(q)$, so we have $y \in \varphi_j[B^n \cap \mathbb{Q}^n] \setminus \varphi_i[\bar{B}^n]$. By the choice of ε, we have $x' \in \varphi_i[B^n(0, 1-\varepsilon)]$, and so there is $x \in \varphi_i[B^n(0, 1-\varepsilon) \cap \mathbb{Q}^n]$ so that $d(x, x') < \delta/3$. Now the conditions of $(*)$ are satisfied for $\varepsilon, \delta, x, j$ and y, so we have $d(x, y) > \delta$. Thus, $d(x', y') \geqslant d(x, y) - d(x, x') - d(y, y') > \delta - \delta/3 - \delta/3 = \delta/3$. $\square$

5.3 PL-geometry

5.3.1 Heine-Borel Simplicial Complexes

Definition 5.3.1. A k**-simplex** $\Delta^k = [v_0, \dots, v_k]$ in the Euclidean n-space $\mathbb{R}^n$ for $k \leqslant n$ is the convex hull of $\bigcup_{i \leq k} \{v_i\}$, where $v_0, \dots, v_k$ are $k+1$ points in $\mathbb{R}^n$ such that no $(k-1)$-hyperplane in $\mathbb{R}^n$ contains all of them (by convention, a 0-hyperplane is a singleton and (-1)-hyperplane is empty). We say that k is the **dimension** of Δ^k and denote it by $\dim(\Delta^k)$. The convex hull of a subcollection of $\bigcup_{i \leq k} \{v_i\}$ of size $m+1$ is an m**-face** of Δ^k. It is also an m-simplex. A 0-simplex and a 1-simplex are also called a **vertex** and an **edge** respectively.

The **standard** n**-simplex**, denoted by Δ^n, is the convex hull of the set of all the $n+1$ unit vectors of $\mathbb{R}^{n+1}$.

Definition 5.3.2. Let X be a metric space. An **n-simplex** in X is the image of an isometry $\iota\colon \Delta \to X$ for some n-simplex $\Delta \subseteq \mathbb{R}^n$. The isometry determines its **faces**. If κ denotes an n-simplex, we use $V(\kappa)$ to denote the set of vertices of κ.

A map f from an n-simplex κ to an n'-simplex κ' in X is **linear** if there are simplexes $\Delta \subseteq \mathbb{R}^n$, $\Delta' \subseteq \mathbb{R}^{n'}$, surjective isometries $\iota\colon \Delta \to \kappa$ and $\iota'\colon \Delta' \to \kappa'$, and an affine map $h\colon \mathbb{R}^n \to \mathbb{R}^{n'}$ such that $f = \iota' \circ h \circ \iota^{-1}$. The **rank** of such f is the rank of h, which is independent from the choice of ι, ι'. It is **full rank** if and only if it is a bijection.

Definition 5.3.3. Given a metric space X, let $\mathfrak{S}_n(X) \subseteq K(X)$ be the set of n-simplexes in X, and let $\mathfrak{S}_{<n}(X) = \bigcup_{k<n} \mathfrak{S}_k(X)$, and $\mathfrak{S}(X) = \bigcup_{k\in\mathbb{N}} \mathfrak{S}_k(X)$.

Lemma 5.3.4. *For a Polish metric space X, the set of those $(\kappa, \kappa') \in \mathfrak{S}(X)^2$ such that κ is a face of κ' is closed.*

Proof. If κ is not a face of κ', then it has a positive distance (in $K(X)$) to all the faces of κ, so it has an open neighbourhood each element of which has the same property. Hence, the set of pairs $(\kappa, \kappa') \in \mathfrak{S}(X)^2$ such that κ is not a face of κ' is open. □

Definition 5.3.5. Let X be a Polish metric space and let $D(X)$ be defined as in Definition 5.1.7. A **Heine-Borel simplicial complex** in X is a sequence of simplexes $T = (\kappa_i)_{i\in I} \in \mathfrak{S}(X)^I$, for $I \subseteq \mathbb{N}$, such that the following hold:

(1) each face of an element of T is an element of T,

(2) if two elements of T intersect, the intersection is a face of each,

(3) for all $x \in D(X)$ and all $\varepsilon \in \mathbb{Q}_+$, the set

$$\{i \in I \mid B(x, \varepsilon) \cap \kappa_i \neq \varnothing\}$$

is finite.

(4) no two simplexes are the same.

We denote by $V(T) = \bigcup_{\kappa\in T} V(\kappa)$ the set of vertices of T and by $R(T) = \bigcup T$ the topological realization associated with T. If κ is a simplex of T, then the **star** of κ, denoted by $\mathrm{Star}(\kappa)$, is the subsequence of T of those simplexes that intersect κ. The **closed star** of κ is the subsequence of all simplexes κ' such that κ' is a face of some simplex in $\mathrm{Star}(\kappa)$. We say that T is a **triangulation** of X if $R(T) = X$. Note that the third condition is equivalent to saying that any bounded set intersects only finitely many simplexes which is a strong version of local finiteness and which implies that $R(T)$ is Heine-Borel. Denote the set of Heine-Borel finite simplicial complexes in X by $\mathrm{SC}^{fin}(X)$ and the set of infinite ones by $\mathrm{SC}^{\infty}(X)$. If $X = \mathbb{U}$, denote them by just $\mathrm{SC}^{fin} = \mathrm{SC}^{fin}(\mathbb{U})$ and $\mathrm{SC}^{\infty} = \mathrm{SC}^{\infty}(\mathbb{U})$.

Lemma 5.3.6. (1) *The sets $\mathfrak{S}_n(\mathbb{U})$ and $\mathfrak{S}(\mathbb{U})$ are Borel.*

(2) *There is a Borel function mapping a simplex $\kappa \in \mathfrak{S}_n(\mathbb{U})$ to a pair $(\mathbf{v}, \iota)$ such that $\mathbf{v} \in \mathbb{R}^{n+1}$ is such that its convex hull is an n-simplex and $\iota\colon \Delta \to \kappa$ is an isometry.*

(3) *There is a Borel function mapping a simplex $\kappa \in \mathfrak{S}_n(\mathbb{U})$ to a finite sequence $(\kappa_i)_{i\leqslant m} \subseteq \mathfrak{S}_{\leqslant n}(\mathbb{U})$ such that $\{\kappa_i \mid i \leqslant m\}$ is the set of all faces of κ.*

Proof. We start proving (1). Recall that $\mathrm{PartEmb}_1(X, Y)$ is the set of partial isometries from X to Y (Lemma 5.1.18). Let $V \subseteq \mathbb{R}^{n\times(n+1)}$ be the set of those $\mathbf{v} = (\bar{v}_0, \ldots, \bar{v}_n)$ whose convex hull is an n-simplex. Then

$$V = \{\mathbf{v} \in \mathbb{R}^{n\times(n+1)} \mid \forall i \leqslant n\ d\big(v_i, P((v_j)_{j\neq i})\big) > 0\}$$

where $P((v_j)_{j\neq i})$ is the hyperplane passing through all the vertices other than v_i. It is easy to see from the above expression that V is open. Denote by $\Delta(\mathbf{v})$ the convex hull of $\{\bar{v}_0, \ldots, \bar{v}_n\}$. Let

B be the set of those triples $(\mathbf{v}, \iota, \kappa) \in V \times \mathrm{PartEmb}_1(\mathbb{R}^n, \mathbb{U}) \times K(\mathbb{U})$ such that $\mathrm{dom}\,\iota = \Delta(\mathbf{v})$ and $\kappa = \iota[\Delta(\mathbf{v})]$. It is easy to check that conditions "$\mathrm{dom}(\iota) \neq \Delta(\mathbf{v})$" and "$\kappa \neq \iota[\Delta(\mathbf{v})]$" are open, and hence B is a closed subset of $V \times \mathrm{PartEmb}_1(\mathbb{R}^n, \mathbb{U}) \times K(\mathbb{U})$. This implies that for any $\kappa \in K(\mathbb{U})$, the section $B_\kappa = \{(\mathbf{v}, \iota) \in V \times \mathrm{PartEmb}_1(\mathbb{R}^n, \mathbb{U}) \mid (\mathbf{v}, \iota, \kappa) \in B\}$, given by $B \cap (V \times \mathrm{PartEmb}_1(\mathbb{R}^n, \mathbb{U}) \times \{\kappa\})$, is a closed subset of $V \times \mathrm{PartEmb}_1(\mathbb{R}^n, \mathbb{U})$. Since V is an open set of a K_σ metric space, and hence V is K_σ as well, and by Lemma 5.1.18 $\mathrm{PartEmb}_1(\mathbb{R}^n, \mathbb{U})$ is K_σ, we obtain that each section B_κ is also K_σ. Note that $\mathrm{pr}_3(B)$, the projection of B to the third coordinate, equals $\mathfrak{S}_n(\mathbb{U})$. Now by the Arsenin-Kunugui Theorem [Kec95, Thm 18.18] $\mathrm{pr}_3(B) = \mathfrak{S}_n(\mathbb{U})$ is Borel which, together with the fact that $\mathfrak{S}(\mathbb{U}) = \bigcup_{k \in \mathbb{N}} \mathfrak{S}_k(\mathbb{U})$, proves (1). Further, by the same theorem, there is a Borel function

$$f \colon \mathfrak{S}_n(\mathbb{U}) \to V \times \mathrm{PartEmb}(\mathbb{R}^n, \mathbb{U})$$

such that if $(\mathbf{v}, \iota) = f(\kappa)$, then $(\mathbf{v}, \iota, \kappa) \in B$ which proves (2). Finally for any fixed $I \subseteq \{0, \ldots, n\}$, the map $((v_i)_{i \leqslant n}, \iota) \mapsto ((v_i)_{i \in I}, \iota \restriction \Delta((v_i)_{i \in I}))$ is also Borel which proves (3). □

Lemma 5.3.7. *Let X be a Polish metric space. Then $\mathfrak{S}(X)^n \cap \mathrm{SC}^{fin}$ is an intersection of a closed and an open set in $\mathfrak{S}(X)^n$. The set SC^∞ is Borel in $\mathfrak{S}(X)^{\mathbb{N}}$.*

Proof. A an element $(\kappa_i)_{i<n} \in \mathfrak{S}(X)^n$ belongs to SC^{fin} if and only if conditions (1), (2) and (4) of Definition 5.3.5 are satisfied, because condition (3) is only relevant for infinite complexes. We will show that conditions (1) and (2) are closed and condition (4) is open.

(1) Let $C_{ij} = \{(\kappa_i)_{i<n} \in \mathfrak{S}(X)^n \mid \kappa_i \text{ is a face of } \kappa_j\}$. This set is closed by Lemma 5.3.4. But the set of those $(\kappa_i)_{i<n}$ satisfying 5.3.5(1) is a finite boolean combination of those sets, so it is also closed.

(2) Let $C'_{ijk} = \{(\kappa_i)_{i<n} \in \mathfrak{S}(X)^n \mid \kappa_i \cap \kappa_j = \kappa_k\}$. It follows from Proposition 5.1.3 that set C'_{ijk} is closed. Again, the set of those $(\kappa_i)_{i<n}$ satisfying 5.3.5(2) is a finite boolean combination of those sets, so is also closed.

(4) The set $O_{ij} = \{(\kappa_i)_{i<n} \in \mathfrak{S}(X)^n \mid \kappa_i \neq \kappa_j\}$ is easily seen to be open, and the required set is a finite intersection of those.

Concerning SC^∞, the same arguments for conditions (1), (2) and (4) show that they are Borel in $\mathfrak{S}(X)^{\mathbb{N}}$. Condition (3) is Borel because it boils down to countable quantification and Propositions 5.1.2 and 5.1.3. □

Definition 5.3.8. Let $T' = (\kappa'_i)_{i<N'}$ and $T = (\kappa_i)_{i<N}$ be either finite ($N, N' \in \mathbb{N}$) or infinite ($N, N' = \mathbb{N}$) Heine-Borel simplicial complexes in the Urysohn space $\mathbb{U}$. Then we define the following terminology:

1. T' is a **subset-complex** of T, if for all $\kappa' \in T'$ there is $\kappa \in T$ such that $\kappa' = \kappa$.

2. T' is a **subdivision** of T, if $R(T') = R(T)$ and for all $\kappa' \in T'$ there is $\kappa \in T$ such that $\kappa' \subseteq \kappa$.

3. T' is a **subcomplex** of T, if there is a subdivision T'' of T such that T' is a subset-complex of T''. Following [Moi52, Moi77] we call subcomplex also a **polyhedron** or a **polyhedral set**.

4. T' is a **finitary subdivision** of T, if it is a subdivision of T and either they are both finite or there are $n, n' \in \mathbb{N}$ such that for all $i \in \mathbb{N}$ we have $\kappa_{n+i} = \kappa'_{n'+i}$.

5. Given $n \in \mathbb{N}$, we say that T is an n-**extension** of T', if T' is a subset-complex of T and for all $v \in V(T) \setminus V(T')$ and $v' \in V(T')$ we have $d(v, v') > n$. Note that T is a 0-extension of T' if and only if T' is a subset-complex of T.

6. An **isomorphism** from T to T' is a homeomorphism h from $R(T)$ to $R(T')$ such that for all $\kappa \in T$ there is $\kappa' \in T'$ such that $h \restriction \kappa$ is linear (Definition 5.3.2) and $h[\kappa] = \kappa'$. If such isomorphism exists, then we say that T and T' are **isomorphic**.

7. A map $h\colon R(T) \to R(T')$ is a **PL-homeomorphism** if it is an isomorphism between some subdivisions T_0 and T_0' of T and T' respectively. If such map exists, we say that T and T' are **PL-homeomorphic**, and we write $T \approx_{\mathrm{PL}} T'$.

8. A **PL-embedding** from T to T' (or from $R(T)$ to $R(T')$) is a PL-homeomorphism from T to a subcomplex of T'.

9. If L is a subcomplex in K and A_0, A_1 are subsets of K, we say that L **separates** A_0 from A_1 in K, if there is no subcomplex C in K such that C is connected, and $C \cap A_0 \neq \varnothing \neq C \cap A_1$, but $C \cap L = \varnothing$.

5.3.2 Algebraic complexes

Let $\mathbb{A} \subseteq \mathbb{R}$ be the countable set of all algebraic numbers, namely numbers that are roots of non-zero polynomials in one variable with integer coefficients.

Definition 5.3.9 (Algebraic complex)**.** For a metric space (X, d), we say that a simplex κ in X is **algebraic** if there is a simplex $\Delta \subseteq \mathbb{R}^n$ with $V(\Delta) \subseteq \mathbb{A}^n$ such that κ is isometric to Δ. If κ is an algebraic simplex, Δ is as above, and $g\colon \Delta \to \kappa$ is an isometry, we say that (Δ, g) is a **witness** that κ is algebraic. We say that a Heine-Borel simplicial complex T is **algebraic** if every simplex of T is algebraic. We call algebraic Heine-Borel simplicial complexes just **algebraic complexes** for short.

Definition 5.3.10 (Algebraic dense set)**.** Given an algebraic complex $T = (\kappa_i)_{i\in I}$, define $\mathbb{A}(T) = \bigcup_{i\in I} g_i[\mathbb{A}^{n_i} \cap \Delta_i]$ where for each $i \in I$, (Δ_i, g_i) is a witness that κ_i is algebraic and n_i is such that $\Delta_i \subset \mathbb{R}^{n_i}$. We prove below (Lemma 5.3.12) that $\mathbb{A}(T)$ is well-defined.

Lemma 5.3.11. *Suppose that κ and κ' are two algebraic n-simplexes in $\mathbb{U}$ witnessed by (Δ, g) and (Δ', g') and $h\colon \kappa \to \kappa'$ is a linear bijection. Let m, m' be such that $\Delta \subseteq \mathbb{R}^m$ and $\Delta' \subseteq \mathbb{R}^{m'}$. Then $h[g[\mathbb{A}^m \cap \Delta]] = g'[\mathbb{A}^{m'} \cap \Delta']$.*

Proof. By definition of an n-simplex we must have $m, m' \geqslant n$. Enumerate $V(\Delta)$ and $V(\Delta')$ as $\{v_0, \dots, v_n\}$ and $\{v_0', \dots, v_n'\}$ respectively. Since there is a linear bijection between any two n-simplexes, and such a linear bijection preserves the vertices, by Definition 5.3.2 we have that $((g')^{-1} \circ h \circ g)$ is linear and w.l.o.g. we can assume that $((g')^{-1} \circ h \circ g)(v_i) = v_i'$ for all $i \leqslant n$. Let A and A' be the $(m \times n)$ and $(m' \times n)$ matrices whose columns are given by the vectors $(v_i - v_0)$ and $(v_i' - v_0')$ respectively. These matrices have coefficients in $\mathbb{A}$ which is a subfield of $\mathbb{R}$, so there is $A^+ \in \mathbb{A}^{n\times m'}$ which is a left inverse of A. Then the map $x \mapsto A'A^+(x - v_0) + v_0'$ equals $(g')^{-1} \circ h \circ g$ on $x \in \{v_0, \dots, v_n\}$ and by linear extension on the entire set Δ. Since all the coefficients in A', A^+, v_0, and v_0' are algebraic, if x is algebraic, so is $((g')^{-1} \circ h \circ g)(x) = A'A^+(x - v_0) + v_0'$. Thus, $h[g[\mathbb{A}^m \cap \Delta]] \subseteq g'[\mathbb{A}^{m'} \cap \Delta']$. By a symmetric argument (take a left inverse of A'), we have $h[g[\mathbb{A}^m \cap \Delta]] \supseteq g'[\mathbb{A}^m \cap \Delta]$. ∎

Lemma 5.3.12. *The set $\mathbb{A}(T)$ is well-defined. Namely, it is independent of the choices of Δ_i and g_i of Definition 5.3.10.*

Proof. Suppose that $(\Delta_i, g_i)_{i\in I}$ and $(\Delta_i', g_i')_{i\in I}$ are two sequences both witnessing that the simplexes in $(\kappa_i)_{i\in I}$ are algebraic. For each $i \in I$, let $h_i\colon \kappa_i \to \kappa_i$ be the identity map. Then, by Lemma 5.3.11 we obtain that for every i,

$$g_i[\mathbb{A}^{n_i} \cap \Delta_i] = h_i[g_i[\mathbb{A}^{n_i} \cap \Delta_i]] = g_i'[\mathbb{A}^{n_i} \cap \Delta_i'],$$

which implies

$$\bigcup_{i\in I} g_i[\mathbb{A}^{n_i} \cap \Delta_i] = \bigcup_{i\in I} g_i'[\mathbb{A}^{n_i} \cap \Delta_i'].$$

∎

Definition 5.3.13. T' is an **algebraic subdivision** of T, if T' is a subdivision of T and $V(T') \subseteq \mathbb{A}(T)$.

Lemma 5.3.14. *If T is algebraic and T' is an algebraic subdivision of T, then T' is algebraic and $\mathbb{A}(T') = \mathbb{A}(T)$.*

Proof. Let us first prove the following:

Claim 5.3.14.1. Let κ and κ' be algebraic simplexes of dimension n and n' respectively witnessed by (Δ, g) and (Δ', g') where $\Delta \subseteq \mathbb{R}^m$ and $\Delta' \subseteq \mathbb{R}^{m'}$ for some $m, m' \in \mathbb{N}$. Assume that κ' is a face of κ. Then $g'[\mathbb{A}^{m'} \cap \Delta'] = g[\mathbb{A}^m \cap \Delta] \cap \kappa'$.

Proof of the Claim. Let $\Delta^* = g^{-1}[\kappa']$ and define $g^* = g \restriction \Delta^*$. Then g^* is an isometry, and Δ^* is algebraic, so (Δ^*, g^*) is also a witness that κ' is algebraic. By Lemma 5.3.11 we have

$$g'[\mathbb{A}^{m'} \cap \Delta'] = g^*[\mathbb{A}^m \cap \Delta^*] = g[\mathbb{A}^m \cap \Delta \cap \Delta^*] = g[\mathbb{A}^m \cap \Delta] \cap \kappa'.$$

The second-to-last equality follows from the definition of g^* and the last from the fact that g is one-to-one being an isometry and that $g[\Delta^*] = \kappa'$. □

Assume w.l.o.g. that $T = (\kappa_i)_{i\in\mathbb{N}}$ and $T' = (\kappa'_i)_{i\in\mathbb{N}}$, and fix corresponding witnesses $(\Delta_i, g_i)_{i\in\mathbb{N}}$, $(\Delta'_i, g'_i)_{i\in\mathbb{N}}$ respectively with $\Delta_i \subseteq \mathbb{R}^{n_i}$ and $\Delta' \subseteq \mathbb{R}^{n'_i}$. From Claim 5.3.14.1 and condition (2) of Definition 5.3.5 it follows that for all $i, j \in \mathbb{N}$ we have

$$g_i[\mathbb{A}^{n_i} \cap \Delta_i] \cap \kappa_j \subseteq g_j[\mathbb{A}^{n_j} \cap \Delta_j]. \tag{5.3.1}$$

Fix $i \in \mathbb{N}$. We will show that κ'_i is algebraic. Since T' is a subdivision of T there exist j such that $\kappa'_i \subseteq \kappa_j$. Since $V(\kappa'_i) \subseteq \mathbb{A}(T)$, from (5.3.1) it follows that $V(\kappa'_i) \subseteq \mathbb{A}(\kappa_j) = g_j[\mathbb{A}^{n_j} \cap \Delta_j]$, so

$$(g_j^{-1}[\kappa'_i], g_j \restriction (g_j^{-1}[\kappa'_i])) \tag{5.3.2}$$

is a witness that κ'_i is algebraic. By the arbitrariness of κ'_i, T' is algebraic. It immediately also follows that $\mathbb{A}(\kappa'_i) \subseteq \mathbb{A}(\kappa_j)$. On the other hand since $R(T') = R(T)$, for each $j \in \mathbb{N}$ and each $x \in \mathbb{A}(\kappa_j)$ there is $\kappa'_i \subseteq \kappa_j$ with $x \in \mathbb{A}(\kappa'_i)$, because it is witnessed as shown in (5.3.2). □

Lemma 5.3.15. *If T, T' are algebraic and $h\colon T \to T'$ is an isomorphism, then $h[\mathbb{A}(T)] = \mathbb{A}(T')$.*

Proof. Let $\kappa \in T$. Then there is $\kappa' \in T'$ such that $h \restriction \kappa$ is a linear bijection onto κ'. Let (Δ, g) and (Δ', g') witness that κ and κ' are algebraic. By Lemma 5.3.11 we have $h[\mathbb{A}(\kappa)] = \mathbb{A}(\kappa')$. Thus, $\mathbb{A}(T) \subseteq \mathbb{A}(T')$. By symmetry, $\mathbb{A}(T) = \mathbb{A}(T')$. □

Definition 5.3.16 ($\mathbb{A}$PL-homeomorphism)**.** If T and T' are algebraic, we say that they are **$\mathbb{A}$PL-homeomorphic**, if there are algebraic subdivisions T_0 and T'_0 of T and T' respectively which are isomorphic. This is denoted $T \approx_{\mathbb{A}\mathrm{PL}} T'$.

Lemma 5.3.17. *If T and T' are algebraic and h witnesses that they are $\mathbb{A}$PL-homeomorphic, then $h[\mathbb{A}(T)] = \mathbb{A}(T')$.*

Proof. Let T_0 and T'_0 be the algebraic subdivisions of T and T' respectively which are isomorphic via h. Then by Lemma 5.3.14 $\mathbb{A}(T) = \mathbb{A}(T_0)$ and $\mathbb{A}(T') = \mathbb{A}(T'_0)$ and by Lemma 5.3.15 it follows that $h[\mathbb{A}(T_0)] = \mathbb{A}(T'_0)$. Thus, $h[\mathbb{A}(T)] = \mathbb{A}(T')$. □

Lemma 5.3.18. *A simplicial complex T is algebraic if and only if the length of every 1-simplex in T is an algebraic number.*

Proof. Suppose T is algebraic and let $\kappa \in T$ be a 1-simplex. By definition of being algebraic, there are $a, b \in \mathbb{A}^n$ such that the length of κ equals $|b - a|$ which is algebraic since both a and b are.

Suppose every 1-simplex is algebraic. Let us show by induction on n that all n-simplexes in T are algebraic. The basic case $n = 1$ is already assumed, so suppose that the claim holds for $n = k$ and prove it for $n = k + 1$. Let κ be an n-simplex and κ' its face of dimension k. There is now a witness (Δ', g') that κ' is algebraic. W.l.o.g. assume that $\Delta' \subseteq \mathbb{R}^k$, and denote the vertices of Δ' by $\{v_0, \ldots, v_k\} \subseteq \mathbb{A}^k$ assuming w.l.o.g. that $v_0 = \bar{0}$. Let $\xi_0, \ldots, \xi_{k+1}$ be the vertices of κ such that $g'(v_i) = \xi_i$ for all $i \leqslant k$. We will show that there is $x \in \mathbb{A}^k \times \mathbb{A}$ such that

$$g' \cup \{(x, \xi_{k+1})\} \tag{5.3.3}$$

is an isometry from $\{v_0, \ldots, v_k, x\}$ to $\{\xi_0, \ldots, \xi_{k+1}\}$. Then (Δ, g) will be a witness that κ is algebraic, where Δ is the convex hull of $\{v_0, \ldots, v_k, x\}$ and g is a linear interpolation of the function in (5.3.3).

Let κ'_i be the face opposite ξ_i, thus in particular $\kappa' = \kappa'_{k+1}$. Let the i-th *height* of κ, denoted h_i, be the distance $d(\xi_i, \kappa'_i)$. This distance is a solution to a polynomial in $d(\xi_i, \xi_j)$, $i, j \in \{0, \ldots, k+1\}$, because h_i equals nC_{k+1}/C^i_k where C_{k+1} is the $(k+1)$-dimensional measure of κ, and C^i_k is the k-dimensional measure of κ'. The squares of C_{k+1} and C^i_k can be obtained using the Cayley-Menger determinant which is a polynomial in the edge lengths of the simplex.

Fix $x \in \mathbb{R}^n \setminus \mathbb{R}^k$. For $i \leqslant k$, let $P_i(x)$ be the hyperplane passing through $\{v_j \mid j \neq i\} \cup x$, and let $P_{k+1}(x)$ be the hyperplane parallel to $\mathbb{R}^k$ which passes through x. Thus, x is in the intersection of all these hyperplanes whose expressions are algebraic in x. This intersection equals $\{x\}$ by the assumption that the vectors $v_i - v_0$, $i \leqslant k$, are independent. Then (5.3.3) is the required isometry if and only if $d(v_i, P_i) = h_i$ for all $i \leqslant k$ and $d(x, \mathbb{R}^k) = h_{k+1}$. This effectively expresses x as a solution to a number of polynomial equations with algebraic coefficients, so we are done. □

Lemma 5.3.19. *Suppose T, T' are algebraic complexes. Then $T \approx_{\mathrm{PL}} T'$ if and only if $T \approx_{\mathbb{A}\mathrm{PL}} T'$.*

Proof. It is enough to show that for any algebraic complex T and every subdivision T_0 of T there is an algebraic subdivision T_1 of T which is isomorphic to T_0. By [Lic99, Thm 4.5] any subdivision of T can be obtained by a sequence of *stellar moves*, so it is enough to show this by induction on stellar move sequences. We will use terminology of [Lic99] in this proof in cursive font. For details the reader is referred to [Lic99]. Suppose T is algebraic and T_0 is obtained from T by one stellar move. If the stellar move is a *weld*, then no new edges are introduced, so by Lemma 5.3.18 T_0 remains algebraic. If it is a *subdivision*, then the choice of the vertex a at which it is *starred* is arbitrary as long as it is within the same simplex. So one may choose it so that its distance to all vertices of that simplex are algebraic. This is possible because it is enough to consider only the vertices of the closed star of a (Definition 5.3.5), and we can assume w.l.o.g. that this star is a subset of some $\mathbb{R}^n$ the vertices of that star are in $\mathbb{A}^n$. □

Lemma 5.3.20. *There is a countable set $\mathbb{A}\mathrm{SC}^{fin} \subseteq \mathrm{SC}^{fin}$ such that:*

1. *Every simplicial complex in $\mathbb{A}\mathrm{SC}^{fin}$ is algebraic.*
2. *(Completeness) If T is any finite simplicial complex, then there is $T' \in \mathbb{A}\mathrm{SC}^{fin}$ which is isomorphic to T.*
3. *(Upward closure) If $n \in \mathbb{N}$, $T \in \mathbb{A}\mathrm{SC}^{fin}$, and T' is a 0-extension of T, then there is $T'' \in \mathbb{A}\mathrm{SC}^{fin}$ which is an algebraic n-extension of T and there is $h\colon R(T'') \to R(T')$ which is an isomorphism from T'' to T' such that $h \restriction R(T)$ is the identity.*
4. *(Downward closure) If $T \in \mathbb{A}\mathrm{SC}^{fin}$ and $T' \subseteq T$, then $T' \in \mathbb{A}\mathrm{SC}^{fin}$.*
5. *(Closure under subdivisions) If $T \in \mathbb{A}\mathrm{SC}^{fin}$ and T' is a subdivision of T, then there is $T'' \in \mathbb{A}\mathrm{SC}^{fin}$ which is also a subdivision of T and the identity $R(T') \to R(T'')$ constitutes an isomorphism between these two.*

6. *(Closure under barycentric subdivisions) If $T \in \mathbb{A}\mathrm{SC}^{fin}$ and T' is a barycentric subdivision of T, then $T' \in \mathbb{A}\mathrm{SC}^{fin}$. Note that if T is algebraic, then any of its barycentric subdivisions is also algebraic.*

7. *(Permutation) If $(\kappa_i)_{i \in I} \in \mathbb{A}\mathrm{SC}^{fin}$, and $p\colon I \to I$ is a bijection, then we have $(\kappa_{p(i)})_{i \in I} \in \mathbb{A}\mathrm{SC}^{fin}$.*

Proof. Close under the listed properties. Note that 2 follows from 3 and 4, because by 4 the empty set belongs to $\mathbb{A}\mathrm{SC}^{fin}$ and by 3 one can now extend the empty set to an isomorphic copy of any complex. □

Remark 5.3.21. Closeness under barycentric subdivision ensures arbitrarily fine subdivisions, while the Upward closure ensures that the resulting simplicial complexes we construct in Section 5.5 are Heine-Borel.

Lemma 5.3.22. *The following sets are Borel:*

$$\begin{aligned}
&A_0 = \{(K', K) \in (\mathrm{SC}^{fin})^2 \mid K' \text{ is a subset-complex of } K\},\\
&A_1 = \{(K', K) \in (\mathrm{SC}^{fin})^2 \mid K' \text{ is a subdivision of } K\},\\
&A_2 = \{(K', K) \in (\mathrm{SC}^{fin})^2 \mid K' \text{ is a subcomplex of } K\},\\
&A_3 = \{(K', K, U_0, U_1) \in (\mathrm{SC}^{fin})^2 \times \mathcal{U}^2 \mid K' \text{ is a subcomplex of } K \text{ and } K' \text{ separates } U_0 \text{ and } U_1\},
\end{aligned}$$

where $\mathcal{U}$ is the space of the open subsets of $\mathbb{U}$ defined in Definition 5.1.8.

Proof. All the sets A_0, A_1, A_2, A_3 can be expressed using quantification over $\mathbb{A}\mathrm{SC}^{fin}$, its finite subsets, and relations that were shown to be Borel in Sections 5.1. □

Definition 5.3.23. Let

$$\mathbb{A}\mathrm{SC}^{\infty}_{*} = \{\bar{\kappa} \in \mathrm{SC}^{\infty} \mid \forall k \exists k' > k(\bar{\kappa}\restriction k' \in \mathbb{A}\mathrm{SC}^{fin})\}$$

where all quantifiers range over $\mathbb{N}$. Let

$$\mathbb{A}\mathrm{SC}^{\infty} = \{\bar{\kappa} \in \mathbb{A}\mathrm{SC}^{\infty}_{*} \mid \forall i \forall m \exists j \forall k > j(d(\kappa_i, \kappa_k) > m)\} \tag{5.3.4}$$

where all the quantifiers also range over $\mathbb{N}$.

The space $\mathbb{A}\mathrm{SC}^{\infty}$ is a Borel subset of SC^{∞}, so it is a standard Borel space.

Lemma 5.3.24. *If $T \in \mathbb{A}\mathrm{SC}^{\infty}$, then $R(T)$ is Heine-Borel.*

Proof. Here we use the characterization of being Heine-Borel stated in Proposition 5.1.9. Fix $x_0 \in \mathbb{U}$ and let $n \in \mathbb{N}$. If $\bar{B}(x_0, n) \cap R(T) = \varnothing$, then there is nothing to prove, so assume that the intersection is non-empty, and let i be such that $\kappa_i \cap \bar{B}(x_0, n) \neq \varnothing$. Applying Definition 5.3.23(5.3.4) we obtain j such that for all $k > j$ we have $d(\kappa_i, \kappa_k) > n$, and hence $d(x_0, \kappa_k) > n$. Now $\bar{B}(x_0, n) \cap R(T) = \bar{B}(x_0, n) \cap \bigcup_{i \leqslant j} \kappa_i$ which is compact. □

Corollary 5.3.25. *If $T \in \mathbb{A}\mathrm{SC}^{\infty}$, then $R(T) \in F(\mathbb{U})$.*

Proof. We show that $\mathbb{U} \setminus R(T)$ is open. Let $x \in \mathbb{U} \setminus R(T)$. For some $n \in \mathbb{N}$, $B(x, n) \cap R(T) \neq \varnothing$, and by Lemma 5.3.24 it is compact, so $B(x, \delta)$ is an open neighbourhood of x outside of $R(T)$, where $\delta = d(x, B(x, n) \cap R(T))$. □

5.3.3 Spaces of PL-embeddings

Recall that for compact X, $\mathrm{Emb}(X,Y)$ is the space of embeddings from X to Y. If $X = R(T)$ and $Y = R(T')$ for some finite simplicial complexes T, T', let $\mathrm{Emb}^{PL}(X,Y) \subseteq \mathrm{Emb}(X,Y)$ be the set of all PL-embeddings. In this section we show that $\mathrm{Emb}^{PL}(X,Y)$ is K_σ (Lemma 5.3.32).

Definition 5.3.26. Given a map $f\colon X \to Y$ between metric spaces (X,d) and (Y,d'), denote by $BL(f)$ the **bilipschitz constant of** f,

$$BL(f) = \inf\{c \in \mathbb{R}_{\geqslant 1} \mid \forall x_0, x_1 \in X(c^{-1}d(x_0,x_1) \leqslant d'(f(x_0),f(x_1)) \leqslant cd(x_0,x_1))\}.$$

With the convention that the infimum of the empty set is ∞, we have that $BL(f)$ is finite iff f is bilipschitz, and $BL(f) = 1$ iff f is an isometry. Also, given any functions $f\colon X \to Y$ and $g\colon Y \to Z$ such that f is a bijective isometry, we have $BL(g \circ f) = BL(g)$.

Given an n-simplex $\kappa \subseteq X$, there is always a linear bijection $\Delta^n \to \kappa$. Additionally, every linear bijection from Δ^n to itself is an isometry.

Let κ be an n-simplex, and suppose $f, g\colon \Delta^n \to \kappa$ are two linear bijections onto κ. Then $f = g \circ (g^{-1} \circ f)$ and by the above obsevations, $(g^{-1} \circ f)$ is an isometry and $BL(f) = BL(g)$. Note also that all linear bijections $\Delta^n \to \kappa$ are bilipschitz, so $BL(f) < \infty$. We can thus give the following definition.

Definition 5.3.27. Let κ be an n-simplex. We set

$$BL(\kappa) = BL(f) \in \mathbb{R}_{\geqslant 1},$$

for some (all) linear bijection $f\colon \Delta^n \to \kappa$.

Definition 5.3.28. Suppose X is Polish and d is a Polish metric on X. Let d_H be the Hausdorff metric on $K(X)$ induced by d. Let d_S be the metric on $\mathfrak{S}_n(X)$ defined by

$$d_S(\kappa,\kappa') = d_H(\kappa,\kappa') + |BL(\kappa) - BL(\kappa')|.$$

The following is easy to verify:

Lemma 5.3.29. *The metrics d_H and d_S generate the same topology on $\mathfrak{S}_n(X)$, and if $(x_i)_i$ is a Cauchy sequence with respect to d_S, then it is a Cauchy sequence with respect to d_H.* □

Lemma 5.3.30. *If X is a compact metric space, then $\mathfrak{S}_n(X)$ is Heine-Borel in the metric d_S. In particular $\mathfrak{S}_n(X)$ is locally compact and K_σ. Hence also $\mathfrak{S}(X) = \bigcup_{n\in\mathbb{N}} \mathfrak{S}_n(X)$ is K_σ.*

Proof. Let d be a compatible metric on X. It is enough to show that every d_S-bounded sequence in $\mathfrak{S}_n(X)$ has a d_S-convergent subsequence. Suppose $(\kappa_i)_{i\in\mathbb{N}}$ is bounded in d_S. For each κ_i fix a linear map $f_i\colon \Delta^n \to \kappa_i$. Since $(\kappa_i)_{i\in\mathbb{N}}$ is bounded in d_S, there is L such that f_i is L-bilipschitz for all i, so we have $(f_i)_{i\in\mathbb{N}} \subseteq \mathrm{Emb}_L(\Delta^n, X)$. By Lemma 5.1.18 the space $\mathrm{Emb}_L(\Delta^n, X)$ is compact, and hence there is a subsequence of $(f_i)_{i\in\mathbb{N}}$ which converges to some L-bilipschitz $f\colon \Delta^n \to X$, which is linear as well. By moving further to a subsequence using the compactness of the interval $[1,L]$, we can make sure that $(BL(f_i))_{i\in\mathbb{N}}$ is also a converging sequence, as well as the sequence $(BL(\kappa_i))_{i\in\mathbb{N}}$. But then it is easy to see that $f[\Delta^n]$ is in fact an n-simplex which is the limit of $(\kappa_i)_{i\in\mathbb{N}}$ in d_S. □

Lemma 5.3.31. *If X is compact, then the set $\mathfrak{T} = \mathfrak{T}(X)$ of all triangulations of X is K_σ (recall Definition 5.3.5).*

Proof. Let $\mathfrak{T}_n$ be the set of triangulations of X which consists of exactly n simplexes. Since $\mathfrak{T} = \bigcup_{n\in\mathbb{N}} \mathfrak{T}_n$, it is enough to show that $\mathfrak{T}_n$ is K_σ.

By Lemma 5.3.30 $\mathfrak{S}(X)^n$ is K_σ. By Lemma 5.3.7 the set of simplicial complexes $\mathrm{SC}^{fin} \cap \mathfrak{S}(X)^n$ is then also K_σ. It is now enough to show that the set of those $(\kappa_i)_{i<n} \in \mathfrak{S}(X)^n$ such that $\bigcup_{i<n} \kappa_i = X$ is closed. But this is an easy consequence of compactness of each κ_i and finiteness of the union. □

Lemma 5.3.32. *Let $X = R(T)$ and $Y = R(T')$ for some finite simplicial complexes T, T'. Then $\mathrm{Emb}^{PL}(X,Y)$ is K_σ.*

Proof. Given $L \in [1,\infty)$, let $\mathrm{Emb}^{PL}_L(X,Y)$ be the set of all L-bilipschitz piecewise linear $g\colon X \to Y$. Clearly

$$\mathrm{Emb}^{PL}(X,Y) = \bigcup_{L\in[1,\infty)} \mathrm{Emb}^{PL}_L(X,Y) = \bigcup_{L\in\mathbb{N}} \mathrm{Emb}^{PL}_L(X,Y),$$

so it is enough to show that each $\mathrm{Emb}^{PL}_L(X,Y)$ is K_σ. Let $\mathfrak{T}(X)$ be the set of all triangulations of X. Let

$$Z = \{(g,T) \in \mathrm{Emb}_L(X,Y) \times \mathfrak{T}(X) \mid \forall \kappa \in \mathfrak{T}(X)(g \restriction \kappa \text{ is linear})\}.$$

It is easy to see that the complement of Z, given by those $(g,T) \in \mathrm{Emb}_L(X,Y) \times \mathfrak{T}(X)$ for which there exists $\kappa \in \mathfrak{T}(X)$ such that $g \restriction \kappa$ is not linear, is open. Hence Z is a closed subset of the set $\mathrm{Emb}_L(X,Y) \times \mathfrak{T}(X)$ which in turn is K_σ by Lemmas 5.3.31 and 5.1.18. Thus, Z is also K_σ and hence also the projection of Z to the first coordinate is K_σ. But this projection is exactly equal to $\mathrm{Emb}^{PL}_L(X,Y)$. □

5.3.4 Continuous complexes

Definition 5.3.33. A **finite continuous complex in** $\mathbb{U}$ is a sequence $(\kappa_j,h_j)_{j<i}$ such that $(\kappa_j)_{j<i} \in \mathbb{A}\mathrm{SC}^{fin}$, and $h = \bigcup_{j<i} h_j$ is a homeomorphism throwing $R((\kappa_j)_{j<i})$ into $\mathbb{U}$. If $K = (\kappa_j,h_j)_{j<i}$, we abuse notation by denoting $K = h[R((\kappa_j)_{j<i})]$. An **infinite continuous complex** is defined in the same way except that $\mathbb{A}\mathrm{SC}^{fin}$ is replaced by $\mathbb{A}\mathrm{SC}^{\infty}$. Let $\mathfrak{C}^{fin}$ and $\mathfrak{C}^{\infty}$ be the sets of finite and infinite continuous complexes, respectively.

Note that a simplicial complex $(\kappa_i)_{i<n} \in \mathbb{A}\mathrm{SC}^{fin} \cup \mathbb{A}\mathrm{SC}^{\infty}$ can always be canonically identified with the continuous complex in $\mathbb{U}$ given by $(\kappa_i,h_i)_{i<n}$ where each h_i is the identity map $\kappa_i \to \kappa_i$.

Definition 5.3.34. The following terms from Definition 5.3.8 are also applicable to continuous complexes in a natural way: *subset-complex*, *subdivision*, and *subcomplex*. A **subset-complex** of a continuous complex $K = (\kappa_j,h_j)_{j<i}$ is any continuous complex $K = (\kappa'_j,h'_j)_{j\in i'}$ such that $(\kappa'_j)_{j<i'}$ is a subset-complex of $(\kappa_j)_{j<i}$ and $h' = h \restriction R((\kappa'_j)_{j<i'})$, where $h' = \bigcup_{j<i'} h'_j$ and $h = \bigcup_{j<i} h_j$. A continuous complex $K = (\kappa'_j,h'_j)_{j\in i'}$ is a **subdivision** of a continuous complex $K = (\kappa_j,h_j)_{j<i}$, if $(\kappa'_j)_{j<i'}$ is a subdivision of $(\kappa_j)_{j<i}$, and $h = h'$, where $h' = \bigcup_{j<i'} h'_j$ and $h = \bigcup_{j<i} h_j$. A continuous complex K' is a **subcomplex** of a continuous complex K, if K' is a subcomplex of a subdivision of K.

Lemma 5.3.35. *The sets $\mathfrak{C}^{fin}, \mathfrak{C}^{\infty}$ are a standard Borel spaces.*

Proof. The space $\mathfrak{C}^{\infty}$ is the subset of

$$\mathbb{A}\mathrm{SC}^{\infty} \times \mathrm{PartEmb}(\mathbb{U},\mathbb{U})^{\mathbb{N}}$$

satisfying a number of conditions which are all easily seen to be Borel. Similarly one sees that $\mathfrak{C}^{fin}$ is Borel. □

5.3.5 Combinatorial and continuous combinatorial manifolds

Definition 5.3.36. A complex $(\kappa'_j)_{j<i} \in \mathrm{SC}^{fin}$ is a **combinatorial n-manifold with boundary**, if every closed star of every vertex (which is a 0-simplex, recall Definition 5.3.5) is PL-homeomorphic to the standard n-simplex Δ^n. By [Moi52] for 3-manifolds, being a triangulated manifold is the same as being a combinatorial manifold. Let $\mathfrak{M}^{\mathrm{PL}}_3 \subset \mathrm{SC}^{fin} \cup \mathrm{SC}^{\infty}$ be the set of finite and infinite combinatorial 3-manifolds with boundary. Let $\mathbb{A}\mathfrak{M}^{fin}_3 = \mathbb{A}\mathrm{SC}^{fin} \cap \mathfrak{M}^{\mathrm{PL}}_3$ and $\mathbb{A}\mathfrak{M}^{\infty}_3 = \mathbb{A}\mathrm{SC}^{\infty} \cap \mathfrak{M}^{\mathrm{PL}}_3$.

Lemma 5.3.37. *The set $\mathfrak{M}^{\mathrm{PL}}_3$ is a standard Borel space.*

Proof. Let Z be the set of those $(K, S, f) \in \mathrm{SC}^{fin} \times \mathrm{SC}^{fin} \times \mathrm{Emb}(\Delta^n, \mathbb{U})$ for which S is a closed star of some vertex in K and $\mathrm{ran}(f) = S$. To say that S is a closed star of a vertex of $K = (\kappa_i)_{i \in I}$ is equivalent to:

- there is $i \in I$ such that κ_i is a singleton, and S is the union of all κ_j such that κ_j is a face of some κ_k whose intersection with κ_i is non-empty.

This boils down to countable quantification and Borel expressions of Propositions 5.1.2 and 5.1.3, thus the condition on S is Borel. The condition on f is closed, so Z is a Borel set.

Given fixed $(K, S) \in \mathrm{SC}^{fin} \times \mathrm{SC}^{fin}$, consider the section

$$Z_{K,S} = \{f \in \mathrm{Emb}(\Delta^n, \mathbb{U}) \mid (K, S, f) \in Z\}.$$

Let $\mathrm{Emb}^{\mathrm{PL}}_*(\Delta^n, S) \subseteq \mathrm{Emb}^{\mathrm{PL}}(\Delta^n, S)$ be the subset consisting of bijections. This is a closed subset (use compactness of the range of the embeddings). So it is K_σ by Lemma 5.3.32. But we have $Z_{K,S} = \mathrm{Emb}^{\mathrm{PL}}_*(\Delta^n, S)$, so by the Arsenin-Kunugui Theorem [Kec95, Theorem 18.18] the projection of Z to the first two coordinates is Borel. The set $\mathfrak{M}_3^{PL}$ is obtained now by a universal countable quantification over the vertices of K. □

For a simplicial complex to be a 3-manifold is equivalent to being a combinatorial 3-manifold [Moi52, Theorem 1].

Lemma 5.3.38. *$\mathbb{A}\mathfrak{M}_3^\infty$ is a Borel set.*

Proof. By definition $\mathbb{A}\mathfrak{M}_3^\infty = \mathbb{A}\mathrm{SC}^\infty \cap \mathfrak{M}_3^{\mathrm{PL}}$, so it is an intersection of two Borel sets (see Definition 5.3.23). □

5.4 A Borel version of a theorem of E. Moise

In this section we prove a Borel version of [Moi52, Theorem 2] (Lemma 5.4.7). Recall that $\mathcal{U}$ denotes the space of open subsets of $\mathbb{U}$ (Definition 5.1.8).

Lemma 5.4.1. *There is a Borel map $\mathrm{Exh}_1 \colon \mathbb{A}\mathrm{SC}^{fin} \times \mathcal{U} \to (\mathbb{A}\mathrm{SC}^{fin})^{\mathbb{N}}$ such that if $(C_i)_{i \in \mathbb{N}} = \mathrm{Exh}_1(K, U)$, then*

- *for all i we have $C_{i+1} \subseteq \mathrm{int}(C_i) \subseteq K \cap U$,*
- *$K \cap U = \bigcup_{i \in \mathbb{N}} C_i$,*

where the interior is taken in K, i.e. $\mathrm{int}\, C = \{x \in C \mid \exists \varepsilon \in \mathbb{Q}_+ (B(x, \varepsilon) \cap K \subset C)\}$. We say that (C_i) is an exhausting sequence *for K, U.*

Proof. Take repeated barycentric subdivisions of K until there is at least one simplex entirely contained in $U \cap K$. Let C_0 be that simplex. Obtain C_{i+1} from C_i by taking further repeated barycentric subdivisions of K until the maximum diameter of a simplex is less than $d(C_i, \partial_K U)$. Here the boundary is computed in K, $\partial_K U = \{x \in K \mid \forall \varepsilon \in \mathbb{Q}_+ (B(x, \varepsilon) \cap K \cap U \neq \varnothing \neq B(x, \varepsilon) \cap K \setminus U)\}$. Then let C_{i+1} be a complex consisting of simplexes in this subdivision and which is maximal with respect to the condition $C_{i+1} \subset U \cap K$. □

Lemma 5.4.2. *There is a Borel map $\mathrm{Exh}_2 \colon \mathbb{A}\mathrm{SC}^{fin} \times \mathcal{U} \to (\mathbb{A}\mathrm{SC}^{fin})^{\mathbb{N}} \times (\mathbb{A}\mathrm{SC}^{fin})^{\mathbb{N}}$ such that if $((C_i)_{i<\mathbb{N}}, (U_i)_{i<\mathbb{N}}) = \mathrm{Exh}_2(K, U)$, then*

- *C_i is a 3-manifold with boundary for all i.*
- *$C_i \cap C_j = \varnothing$ for $i \neq j$, unless $|i - j| = 1$ in which case $C_i \cap C_j$ is a 2-manifold.*
- *U_i is a closed polyhedral neighbourhood of $C_i \cap C_{i+1}$,*
- *$U_i \cap U_j = \varnothing$ for all $i \neq j$,*

- $U_i \cap (C_{i-1} \cup C_{i+2}) = \varnothing$.

Proof. Use similar technique as in the proof of Lemma 5.4.1. □

Definition 5.4.3. In this section we will use the concept of an *approximation* to mean that embeddings are approximated by embeddings. Thus, given an embedding $f\colon X \to Y$ and $\varepsilon \in \mathbb{R}_+$, an **$\varepsilon$-approximation** of f is a function f' such that $f'\colon X \to Y$ is also an embedding and for all $x \in X$ we have $d(f(x), f'(x)) < \varepsilon$. If f' is piecewise linear (the PL-structures on X and Y should be clear from the context), then we say that f' is a **PL-ε-approximation**.

As in the case of $\mathbb{A}\mathfrak{M}_3^{fin}$, one can define $\mathbb{A}\mathfrak{M}_2^{fin} \subset \mathbb{A}\mathrm{SC}^{fin}$ and show that it is a standard Borel space.

Lemma 5.4.4. *Let Z be the subspace of*

$$(\mathbb{A}\mathfrak{M}_3^{fin})^5 \times \mathbb{A}\mathfrak{M}_2^{fin} \times \mathrm{PartEmb}(\mathbb{U}, \bar{B}^3) \times \mathbb{R}_+$$

consisting of those $(K, K_1, K_2, \mathfrak{R}, U, L, f, \varepsilon)$ such that $K, K_1, K_2, \mathfrak{R}$, and U are combinatorial 3-manifolds with boundary, and L is a 2-manifold with boundary such that

- *K_1, K_2, U and L are subcomplexes of K,*
- *$K_1 \cup K_2 = K$,*
- *$K_1 \cap K_2 = L$,*
- *$L \subset \mathrm{int}(U)$,*
- *$\mathrm{dom}(f) = K$, $\mathrm{Im}(f) \subset \mathfrak{R}$.*

Then Z is a Borel set and there exists a Borel function $\mathrm{Delta}\colon Z \to \mathbb{R}_+$ such that if $\delta = \mathrm{Delta}(K, K_1, K_2, \mathfrak{R}, U, L, f, \varepsilon)$, then the following holds for all $f_1', f_2' \in \mathrm{PartEmb}(\mathbb{U}, \bar{B}^3)$:

(∗) *if f_1' and f_2' are PL-δ-approximations of $f \restriction K_1$ and $f \restriction K_2 \cup U$ respectively, then there is a PL-ε-approximation f' of f such that $f' \restriction K_1 = f_1'$ and $f' \restriction (K_2 \setminus U) = f_2' \restriction K_2 \setminus U$.*

Proof. To be a subcomplex of a complex is expressed by a countable quantification and the subset relation within $\mathbb{A}\mathfrak{M}_3$. The other conditions for Z are Borel Proposition 5.1.2, so Z is Borel. Let $A \subseteq Z \times \mathbb{R}_+$ be the set of those $(K, K_1, K_2, L, \mathfrak{R}, U, f, \varepsilon, \delta)$ such that (∗) is satisfied. We claim that A is Borel, that the sections $A_z = \{\delta \mid (z, \delta) \in A\}$ are non-empty and K_σ. The conclusion of the Lemma will then follow by an application of the Arsenin-Kunugui Theorem [Kec95, Theorem 18.18].

The condition (∗) is closed downward, so the sections are intervals in $\mathbb{R}_+$, and therefore K_σ. They are non-empty by [Moi52, Lemma 4].

Let us show that A is Borel. Let C_0 be the set of all $(z, f_1', f_2', f', \delta) \in F$, where $F = Z \times \mathrm{PartEmb}(\mathbb{U}, \bar{B}^3)^3 \times \mathbb{R}_+$, such that f_1' and f_2' are PL-δ-approximations of $f \restriction K_1$ and $f \restriction K_2 \cup U$ respectively. Let C_1 be the set of all $(z, f_1', f_2', f', \delta) \in F$ such that f' is a PL-ε-approximation f' of f, and C_2 is the set of those where $f' \restriction K_1 = f_1'$ and $f' \restriction (K_2 \setminus U) = f_2' \restriction K_2 \setminus U$.

Now for any fixed (z, δ), the sections $(C_0)_{(z,\delta)}$, $(C_1)_{(z,\delta)}$, and $(C_2)_{(z,\delta)}$ are respectively open, open, and closed in

$$F_z = F_z^1 \times F_z^2 \times F_z^3$$

where

$$F_z^1 = \mathrm{Emb}^{PL}(K_1, \mathfrak{R}), \quad F_z^2 = \mathrm{Emb}^{PL}(K_2 \cup \bar{U}, \mathfrak{R}), \text{ and } F_z^3 = \mathrm{Emb}^{PL}(K, \mathfrak{R}).$$

Let $C = (F \setminus C_0) \cup (C_1 \cap C_2)$. Note that

$$A = \{(z, \delta) \in Z \times \mathbb{R}_+ \mid \forall f_1' \in F_z^1 \forall f_2' \in F_z^2 \exists f' \in F_z^3 \big((z, \delta, f_1', f_2', f') \in C\big)\}$$

Let

$$C_\exists = \{(z, \delta, f_1', f_2') \in Z \times \mathbb{R}_+ \times F_z^1 \times F_z^2 \mid \exists f' \in F_z^3 \big((z, \delta, f_1', f_2', f') \in C\big)\}$$

Since the sections of C are K_σ, the set $C_\exists$ is Borel. Now for all (z,δ) the section

$$(C_\exists)_{(z,\delta)} = \{(f'_1, f'_2) \in F^1_z \times F^2_z \mid \exists f' \in F^3_z(f' \restriction K_1 \in F^1_z \wedge f' \restriction (K_2 \setminus U) \in F^2_z)\}$$

is an open set in $F^1_z \times F^2_z$. Then $A = (Z \times \mathbb{R}_+) \setminus \mathrm{pr}(F \setminus C)$, but $\mathrm{pr}(F \setminus C) = \bigcup_{(z,\delta)}(\mathrm{pr}(F \setminus C))_{(z,\delta)}$, where each section $(\mathrm{pr}(F \setminus C))_{(z,\delta)} = (C \setminus C_{(z,\delta)})$ is K_σ. Thus by the Arsenin-Kunugui Theorem [Kec95, Theorem 18.18] $\mathrm{pr}(F \setminus C)$ is Borel, and hence A is Borel as well. □

Lemma 5.4.5. *Let Z be the subspace of*

$$\mathbb{A}\mathrm{SC}^{fin} \times \mathrm{PartEmb}(\mathbb{U}, \bar{B}^3)$$

which consists of those (K, f) for which K is a 3-manifold with boundary, and $\mathrm{dom}\, f = K$. Then Z is Borel and there is a Borel map $\mathrm{Approx}\colon Z \times \mathbb{R}_+ \to \mathrm{PartEmb}(\mathbb{U}, \bar{B}^3)$ such that if $f' = \mathrm{Approx}(K, f, \varepsilon)$, then

(1) $\mathrm{dom}\, f' = \mathrm{dom}\, f = K$,

(2) $d(f'(x), f(x)) \leqslant \varepsilon$ *for all* $x \in \mathrm{dom}\, f$,

(3) $f' \in \mathrm{Emb}^{PL}(K, \bar{B}^3)$.

Proof. The set Z is easily seen to be Borel. The set A of those (K, f, ε, f') for which $(K, f, \varepsilon) \in Z \times \mathbb{R}_+$ and f' satisfies the conclusion is also easily seen to be Borel. Given fixed (K, f, ε), consider the section $\{f' \in \mathrm{PartEmb}(\mathbb{U}, \bar{B}^3) \mid (K, f, \varepsilon, f') \in A\}$. It is the intersection of three sets each corresponding to the conditions given by (1), (2), and (3). The first two conditions are closed and the third one is K_σ by Lemma 5.3.32. Applying the Arsenin-Kunugui Theorem [Kec95, Theorem 18.18] we have the intended result. □

Lemma 5.4.6. *Let Z_1 be the Z of Lemma 5.4.4. Let*

$$Z \subseteq Z_1 \times \mathbb{R}_+ \times \mathrm{PartEmb}(\mathbb{U}, \bar{B}^3)^2$$

consist of those $(K, K_1, K_2, \mathfrak{R}, U, L, f, \varepsilon, \delta, f'_1, f'_2)$ for which $\delta = \mathrm{Delta}(K, K_1, K_2, \mathfrak{R}, U, L, f, \varepsilon)$, and f'_1, f'_2 satisfy the assumption of $(*)$ of Lemma 5.4.4. Then Z is Borel and there is a Borel function

$$\mathrm{Fit}\colon Z \to \mathrm{PartEmb}(\mathbb{U}, \bar{B}^3)$$

such that if the map $f' = \mathrm{Fit}(K, K_1, K_2, \mathfrak{R}, U, L, f, \varepsilon, \delta, f'_1, f'_2)$, then f' satisfies the conclusion of $(*)$ of Lemma 5.4.4.

Proof. Again, it is easy to see that Z is Borel, because Delta is Borel being a δ-approximation is Borel and being a PL-map is Borel by Lemma 5.3.32. Let $A \subseteq Z \times \mathrm{PartEmb}(\mathbb{U}, \bar{B}^3)$ be the set of those sequences $(K, K_1, K_2, \mathfrak{R}, U, L, f, \varepsilon, \delta, f'_1, f'_2, f')$ where f' satisfies the conclusion. By similar arguments, it is also Borel. Given fixed $(K, K_1, K_2, \mathfrak{R}, U, L, f, \varepsilon, \delta, f'_1, f'_2) \in Z$, the section

$$\{f' \in \mathrm{PartEmb}(\mathbb{U}, \bar{B}^3) \mid (K, K_1, K_2, \mathfrak{R}, U, L, f, \varepsilon, \delta, f'_1, f'_2, f') \in A\}$$

is a closed subset of the K_σ set $\mathrm{Emb}^{PL}(K, \bar{B}^3)$ (by Lemma 5.3.32), so it is K_σ itself. The section is non-empty by the assumptions on Z and by Lemma 5.4.4. Thus, by Arsenin-Kunugui [Kec95, Theorem 18.18] we are done. □

The following is a Borel version of [Moi52, Theorem 2] where in place of Moise's K, U, and K' we have $K, K \cap U$, and $\bar{B}^3$, and instead of Moise's φ we have the function $\varphi(p) = \frac{1}{2} d(p, \partial_K U)$ (which also satisfies the condition $\varphi(p) > 0$ for $p \in U \cap K$).

Lemma 5.4.7. *Let Z be the subspace of*

$$\mathbb{A}\mathrm{SC}^{fin} \times \mathcal{U} \times \mathrm{PartEmb}(\mathbb{U}, \bar{B}^3)$$

which consists of those (K, U, f) for which K is a 3-manifold with boundary, and $\operatorname{dom} f \supseteq K \cap U$*. Then Z is Borel and there is a Borel map*

$$\eta\colon Z \to \mathrm{PartEmb}(\mathbb{U}, \bar{B}^3)$$

such that for all $(K, U, f) \in Z$, if $f' = \eta(K, U, f)$, then

- $\operatorname{dom} f' = \operatorname{dom} f$*,*
- *for all $x \in \operatorname{dom} f \setminus U$, $f'(x) = f(x)$,*
- *for all $x \in K \cap U$, $d(f'(x), f(x)) < \frac{1}{2}d(x, \partial_K U)$,*
- *for all $C \subset U \cap K$ which is a polyhedron in K, $f' \restriction C$ is PL.*

Proof. Let $(C_i, U_i)_{i\in\mathbb{N}} = \mathrm{Exh}_2(K, U)$ as given by Lemma 5.4.2. Denote $\varphi(x) = \frac{1}{2}d(x, \partial_K U)$. For each i, let ε_i be a positive number less than the greatest lower bound of $\varphi \restriction \bigcup_{j=1}^{i} C_i$. It is clear that such ε_i is obtained in a Borel way. For each i, let

$$\delta_i = \mathrm{Delta}(C, C_i, C_{i+1}, L, \operatorname{Im} f, U, f \restriction C, \varepsilon_i)$$

where $C = C_i \cup C_{i+1}$, $L = C_i \cap C_{i+1}$. Since Delta and Exh_2 are Borel, also the map $(K, U, f) \mapsto (C_i, U_i, \varepsilon_i, \delta_i)_{i<\mathbb{N}}$ is Borel. For each i, let $f_i' = \mathrm{Approx}(C_i, f \restriction C_i, \delta_i)$ as given by Lemma 5.4.5. By the Borelness of Approx, we have that $(K, U, f) \mapsto (C_i, U_i, \varepsilon_i, \delta_i, f_i'))_{i\in\mathbb{N}}$ is Borel. But then, also the map

$$(K, U, f) \mapsto (C, C_i, C_{i+1}, L, U_i, \varepsilon_i, \delta_i, f_i', f_{i+1}')_{i\in\mathbb{N}}$$

is Borel where C and L are as above. Applying Lemma 5.4.6 iteratively and using Lemma 5.1.23 in an appropriate way, we obtain a Borel map $(K, U, f) \mapsto (f_i)_{i\in\mathbb{N}}$ such that $f_i \subseteq f_{i+1}$ for all i, $\operatorname{dom} f_i = \bigcup_{j=1}^{i+1} C_i$, and $f_i \restriction C_i$ is a PL-ϵ_i-approximation of $f \restriction C_i$. Finally, let $f'' = \bigcup_i f_i$. By the second-to-last bullet point in the statement of the Lemma, f'' can be extended as identity to $\partial_K U \cap K$, and further to $\operatorname{dom} f$ as identity. This extension is the needed f'. It is not hard to see that the map $(f_i)_i \mapsto f'$ is continuous. □

5.5 From 3-manifolds to algebraic combinatorial 3-manifolds

The last theorem of this section (Theorem 5.5.12) says that 3-manifolds can be triangulated in a Borel way. This is a strengthening of the Moise-Bing theorem [Moi52, Bin83] which says that 3-manifolds can be triangulated, namely that for each 3-manifold one can assign a simplicial complex which is homeomorphic to that manifold. We will show in this section that this assignment can be a Borel function. We use as our basis the original proof of [Moi52, Theorem 3].

Here we denote $\mathfrak{M} = \mathfrak{M}_3$. For the entire section fix Δ' and Δ to be 3-simplexes in $\mathbb{R}^3$ such that $\Delta' \subset \operatorname{int}(\Delta)$ and $\Delta \subset B^3$.

Definition 5.5.1. Let $\mathfrak{N} \subset \mathfrak{M}$ be defined by

$$\mathfrak{N} = \{\bar{\varphi} \in \mathfrak{M} \mid \bigcup_{i\in\mathbb{N}} \varphi_i[\Delta'] = M(\bar{\varphi})\}.$$

Define the homeomorphism relation on $\mathfrak{N}$ to be induced by the one on $\mathfrak{M}$,

$$\approx_{\mathfrak{N}} \; = \; \approx_{\mathfrak{M}} \restriction \mathfrak{N}.$$

Proposition 5.5.2. $\mathfrak{N}$ *is a Borel subset of* $\mathrm{Emb}(\bar{B}^3, \mathbb{U})^{\mathbb{N}}$. □

Proof. Using the local finiteness of $\bar{\varphi}$ we have that $\bar{\varphi} \in \mathfrak{N}$ if and only if for all i there is $k \in \mathbb{N}$ such that $\varphi_i[\bar{B}^3] \subseteq \bigcup_{j<k} \varphi_j[\Delta']$. □

Theorem 5.5.3. *There is a Borel map $\xi_0 \colon \mathfrak{M} \to \mathfrak{N}$ which constitutes a Borel reduction $\approx_{\mathfrak{M}} \leqslant_B \approx_{\mathfrak{N}}$.*

Proof. Let $\bar{\varphi} \in \mathfrak{M}$. We will construct a sequence $\bar{\lambda} = (\lambda_i)_{i\in\mathbb{N}} \subset]0,1[$ such that the set

$$\{\varphi_i[B^3(0,\lambda_i)] \mid i \in \mathbb{N}\} \tag{5.5.1}$$

is a cover of $M(\bar{\varphi})$. Once we have that, we will construct a sequence $\bar{\psi}$ in which for each $i \in \mathbb{N}$, $\psi_i \colon \bar{B}^3 \to \bar{B}^3$ is a homeomorphism such that $\psi_i[\Delta'] = \bar{B}^3(0,\lambda_i)$. Then define, for each $i \in \mathbb{N}$, $\varphi_i' = \varphi_i \circ \psi_i$, whence $\bar{\varphi}' \in \mathfrak{N}$ and $M(\bar{\varphi}) = M(\bar{\varphi}')$. In particular the map $\bar{\varphi} \mapsto \bar{\varphi}'$ preserves the homeomorphism relation. This is the definition of ξ_0, i.e. we define $\xi_0(\bar{\varphi})$ to be $\bar{\varphi}'$.

Let us show that ξ_0 is a Borel function. The operation $(\varphi, \psi) \mapsto \varphi \circ \psi$ is Borel, so it remains to show that also the operations $\bar{\varphi} \mapsto \bar{\lambda}$ and $\bar{\lambda} \mapsto \bar{\psi}$ are Borel.

Define λ_i by induction. Suppose $(\lambda_0, \dots, \lambda_{k-1})$ have been defined such that

$$\{\varphi_i[B^3(0,\lambda_i)] \mid i < k\} \cup \{\varphi_i[B^3] \mid i \geqslant k\} \text{ is a cover of } M. \tag{5.5.2}$$

Let $\lambda_k' = \inf\{r \in \mathbb{R}_+ \mid \varphi_k^{-1}[Z_k] \subseteq B(0,r)\}$ where

$$Z_k = \varphi_k[\bar{B}^3] \setminus \left(\bigcup_{i<k} \varphi_i[B^3(0,\lambda_i)] \cup \bigcup_{i>k} \varphi_i[B^3] \right). \tag{5.5.3}$$

Note that $\varphi_k^{-1}[Z_k]$ is a compact subset of B^3 (by the property of $\bar{\varphi}$ being in $\mathfrak{L}_1$ of Definition 5.2.1), so $\lambda_k' < 1$. Let $\lambda_k = (1 + \lambda_k')/2$. We have

$$Z_k \subseteq \varphi_k[B^3(0,\lambda_k)]. \tag{5.5.4}$$

Now

$$\begin{aligned} M(\bar{\varphi}) &= (M(\bar{\varphi}) \setminus Z_k) \cup Z_k \\ &= \bigcup_{i<k} \varphi_i[B^3(0,\lambda_i)] \cup \bigcup_{i>k} \varphi_i[B^3] \cup Z_k && \text{by (5.5.3)} \\ &\subseteq \bigcup_{i<k} \varphi_i[B^3(0,\lambda_i)] \cup \bigcup_{i>k} \varphi_i[B^3] \cup \varphi_k[B^3(0,\lambda_k)] && \text{by (5.5.4)} \\ &= \bigcup_{i\leqslant k} \varphi_i[B^3(0,\lambda_i)] \cup \bigcup_{i>k} \varphi_i[B^3]. \end{aligned}$$

It is easy to verify that the maps

$$\begin{aligned} &(\bar{\varphi}, (\lambda_i)_{i<k}) \mapsto Z_k \\ &(\varphi_k, Z_k) \mapsto \lambda_k', && \text{and} \\ &\lambda_k' \mapsto \lambda_k \end{aligned}$$

are Borel, so the map $(\bar{\varphi}, (\lambda_i)_{i<k}) \mapsto \lambda_k$ is Borel. Now by Lemma 5.1.23, the map $\bar{\varphi} \mapsto \bar{\lambda}$ is Borel.

Let us show that if $\bar{\lambda}$ is defined in this way, then the set (5.5.1) is indeed a cover of $M(\bar{\varphi})$. This follows from local finiteness of $\bar{\varphi}$. Suppose that $x \in M(\bar{\varphi})$. Let n_x be such that $x \notin \varphi_m[B^3]$ for all $m > n_x$. Then by (5.5.2) we have

$$x \in \bigcup_{i=0}^{n_x} \varphi_i[B^3(0,\lambda_i)] \cup \bigcup_{i\geqslant n_x+1} \varphi_i[B^3],$$

which implies

$$x \in \bigcup_{i=0}^{n_x} \varphi_i[B^3(0,\lambda_i)] \subseteq \bigcup_{i\in\mathbb{N}} \varphi_i[B^3(0,\lambda_i)].$$

It is standard to construct a Borel function $\lambda \mapsto \psi_\lambda$, $\lambda \in]0,1[$, in which $\psi_\lambda \colon \bar{B}^3 \to \bar{B}^3$ is a homeomorphism such that $\psi[\Delta'] = \bar{B}(0,\lambda)$. Applying Lemma 5.1.23 again, one obtains the desired Borel map $\bar{\lambda} \mapsto \bar{\psi}$ which completes the proof. □

Definition 5.5.4. Let Y be the set of pairs (κ, h) where κ is an algebraic simplex such that the simplicial complex formed by the singleton (κ) is in $\mathbb{ASC}^{fin}$ and $h \in \mathrm{Emb}(\kappa, \mathbb{U})$.

Lemma 5.5.5. *Y is a standard Borel space.*

Proof. Since $\mathbb{ASC}^{fin}$ is countable, it is enough to see that $\mathrm{Emb}(\kappa,\mathbb{U})$ is a standard Borel space which it is by Lemma 5.1.17. □

Definition 5.5.6. Let $B \subset \mathfrak{N} \times Y^{<\mathbb{N}}$ consist of those

$$(\bar{\varphi}, (\kappa_i, h_i)_{i<n})$$

which satisfy:

1. $K = (\kappa_i, h_i)_{i<n} \in \mathfrak{C}^{fin}$ (Definition 5.3.33),
2. $K \subseteq M(\bar{\varphi})$.

Note that $(\bar{\varphi}, \varnothing) \in B$ for all $\bar{\varphi}$ setting $n = 0$.

Recall that an **irreducible n-manifold**, is one in which any embedded $(n-1)$-sphere bounds an embedded n-ball.

Definition 5.5.7 (Moise [Moi52])**.** Given an n-manifold with boundary X let $\partial^\mu(X)$ (Moise denotes this by $\beta'(X)$) be the set of points $x \in X$ which do not have a neighbourhood homeomorphic to $\mathbb{R}^n$.

Definition 5.5.8. Suppose $K = (\kappa_i, h_i)_{i<n}$ is a continuous complex and $L = (\lambda_j)_{j<n'} \in \mathbb{ASC}^{fin}$ a subcomplex of $(\kappa_i)_{i<n}$. Then let $L^K = (\lambda_j, h^K_j)_{j<n'}$ be defined by $h^K_j = (\cup_{i<n} h_i)\restriction\lambda_j$. Suppose now that $(\bar{\varphi}, K) \in B$ and $j \in \mathbb{N}$. We say that $L = (\lambda_i)_{i<n'} \in \mathbb{AM}^{fin}_2$ is a *j-separator for* (φ, K), if

(1) L^K is a 2-manifold with boundary,

(2) L^K separates $K \cap \partial^\mu(\varphi_{j+1}[\Delta])$ from $K \cap \partial^\mu(\varphi_{j+1}[\Delta'])$ (recall Definitions 5.3.8 and 5.3.34, and the definitions of Δ, Δ' from the beginning of this section),

(3) $\partial^\mu(L^K) \subset \partial^\mu(K)$,

(4) L^K is irreducible with respect to (1), (2) and (3).

Lemma 5.5.9. *There is a Borel map $L \colon B \times \mathbb{N} \to \mathbb{ASC}^{fin}$ such that for all $(\bar{\varphi}, K, j) \in \mathrm{dom}(L)$, $L(\bar{\varphi}, K, j)$ is a j-separator for (φ, K).*

Proof. Let A' be the set of tuples $(\bar{\varphi}, K, L) \in B \times \mathbb{N} \times \mathbb{ASC}^{fin}$ such that the conditions (1)–(3) of the definition of j-separator are satisfied for L. Then A' is a Borel by Lemma 5.3.22. Let $A \subseteq A'$ be the set of those which satisfy also condition (4). To say now that a tuple $(\bar{\varphi}, K, L)$ is in A, one has to say that "$(\bar{\varphi}, K, L)$ is in A' and for every closed polyhedral loop l in L^K, if l bounds a polyhderal disk D in K, and doesn't bound a disk in L^K, then replacing any component of $L \setminus l$ by D will yield an element not in A'." This only requires quantification over subcomplexes of K which is a countable set. To say that simplicial complex is a disk requires only countable quantification over the finite (algebraic) subdivisions of a 2-simplex.

The sections $\{L \in \mathbb{ASC}^{fin} \mid (\bar{\varphi}, K, L) \in B\}$ corresponding to fixed (φ, K, L) are countable. That they are non-empty is proved in the beginning of the proof of [Moi52, Theorem 3]. By the Lusin-Novikov Theorem [Kec95, Theorem 18.10] there is Borel uniformization $\eta \colon (\bar{\varphi}, K) \mapsto L$ as desired. □

Lemma 5.5.10. *Let B and Y be as in Definition 5.5.6. For each i there is a Borel function $f_i\colon B \to Y^{<\mathbb{N}}$ such that*

1. *$(f_i)_i$ is stabilizing for B (recall Definition 5.1.21),*
2. *If $(\bar\varphi, K) \in B$ where $K = (\kappa_k, h_k)_{k<n}$, and $K' = (\kappa'_k, h'_k)_{k<n'} = f_i(\bar\varphi, (\kappa_k, h_k)_{k<n})$, then*
   (a) *$K \cup \varphi_i[\Delta'] \subset K'$,*
   (b) *for all $v' \in V((\kappa'_k)_{k<n'}) \setminus V((\kappa_k)_{k<n})$ and all $v \in V((\kappa_k)_{k<n}) \cap V((\kappa'_k)_{k<n'})$ we have $d(v, v') \geqslant i$.*

Proof. Let $f_0\colon B \to Y$ be defined as follows. For any $(\bar\varphi, (\kappa_k, h_k)_{k<n})$, let

$$f_0((\bar\varphi, (\kappa_k, h_k)_{k<n})) = ((\varphi_0[\Delta']), \mathrm{id}_{\varphi_0[\Delta']}).$$

Thus, the value of f_0 is a simplicial complex with only one simplex which is identically mapped onto itself. Conditions (a) and (b) are clearly satisfied (note that $i = 0$).

Suppose f_i has been defined, and let us define f_{i+1}. Let $K_i = (\kappa_k, h_k)_{k<n} \in B$ be such that $(\bar\varphi, K_i) \in B$. Then let $L = L(\bar\varphi, (\kappa_k, h_k)_{k<n})$ be as given by Lemma 5.5.9. For $k \leqslant i+1$, let $\sigma_k = \varphi_k[\Delta]$, $\sigma'_k = \varphi_k[\Delta']$, $E_k = \varphi_k[B^3]$, and $E_{k+1} = \varphi_{k+1}[B^3]$. In the proof of [Moi52, Theorem 3], a continuous complex K_{i+1} with $K_i \cup \sigma'_{i+1} \subseteq K_{i+1}$ is defined using only K_i, L, σ_i, σ'_i, σ'_{i+1}, E_i, and E_{i+1}. Every step in that proof is readily seen to be constructive and hence Borel, except possibly for the step where [Moi52, Theorem 2] is used to obtain the function f'. But we have proved a Borel version of [Moi52, Theorem 2], namely our Lemma 5.4.7. Thus, the construction is, in fact, Borel. Denote the resulting complex by $K_{i+1} = (\kappa'_k, h'_k)_{k<n'}$. Take those indices $k < n'$ for which $\kappa_k \notin K_i$, take them out, and put them back one-by-one using clause (3) of Lemma 5.3.20 to satisfy condition 2(b). In the process redefine the corresponding $h_k \mapsto g \circ h_k$ where g is the appropriate linear bijection from the new to the old simplex. In this way we make sure that both 2(a) and 2(b) are satisfied.

By letting now $f_{i+1}(\bar\varphi, K_i) = K_{i+1}$, we obtain a sequence which satisfy condition 1 also by the property of Moise's construction, see the last paragraph of the proof of [Moi52, Theorem 3]. □

Theorem 5.5.11. *There is a Borel function $\xi_1\colon \mathfrak{N} \to \mathbb{A}\mathfrak{M}_3^\infty$ such that for all $\bar\varphi \in \mathfrak{N}$, $M(\bar\varphi) \approx R(\xi_1(\bar\varphi))$.*

Proof. Let $(f_i)_i$ be the sequence given by Lemma 5.5.10, and let $F\colon \mathfrak{N} \to Y^{\mathbb{N}}$ be the function $F = \lim_{i\to\infty} f_i$ given by Definition 5.1.21. Then F is Borel by Lemma 5.1.22(a), and $(x, F(x)\restriction i) \in B$ for all i by 5.1.22(b). In particular $F(\bar\varphi) = (\kappa_i, h_i)_{i\in\mathbb{N}}$, $(\kappa_i, h_i)_{i<j}$ is a continuous complex for all i, and $\cup_{i<j} h_i[\kappa_i] \subset M(\bar\varphi)$. By 5.1.22(c), for all i, $F(x)\restriction i$ satisfies the condition 2 of Lemma 5.5.10 Thus, by 2(b), $(\kappa_i)_{i\in\mathbb{N}}$ is a complex in $\mathbb{A}\mathrm{SC}^\infty$ and $h = \bigcup h_i$ is a homeomorphism throwing $R((\kappa_i)_{i\in\mathbb{N}})$ into $M(\bar\varphi)$. But by clause 2(a), and the definition of $\mathfrak{N}$, we have $M(\bar\varphi) \subseteq \mathrm{Im}(h)$, so the complex $R((\kappa_i)_{i\in\mathbb{N}})$ is homeomorphic to $M(\bar\varphi)$. So let $\xi_1(\bar\varphi) = (\kappa_i)_{i\in\mathbb{N}} = \mathrm{pr}_1(F(\bar\varphi))$ where $\mathrm{pr}_1((x_i, y_i)_{i\in\mathbb{N}}) = (x_i)_{i\in\mathbb{N}}$ is a projection operator. □

Theorem 5.5.12. *$\xi_1 \circ \xi_0$ is a reduction from homeomorphism on $\mathfrak{M}_3$ to $\mathbb{A}\mathrm{PL}$-homeomorphism on $\mathbb{A}\mathfrak{M}_3$, hence by Lemma 5.3.19 also to PL-homeomorphism on $\mathbb{A}\mathfrak{M}_3$.*

Proof. Let $\xi(\bar\varphi) = \xi_1(\xi_0(\bar\varphi))$ where ξ_1 is as given by Theorem 5.5.11 and ξ_0 is given by Theorem 5.5.3. By those theorems we have $R(\xi(\bar\varphi)) \approx M(\bar\varphi)$. Let us show that ξ also reduces the homeomorphism on $\mathfrak{M}_3$ to PL-homeomorphism on $\mathbb{A}\mathfrak{M}_3$. If $M(\bar\varphi) \approx M(\bar\varphi')$, then by the above clearly $R(\xi(\bar\varphi)) \approx R(\xi(\bar\varphi'))$. By [Moi52, Theorem 4] it follows that $R(\xi(\bar\varphi)) \approx_{\mathrm{PL}} R(\xi(\bar\varphi'))$. The other direction is trivial. □

5.6 Basis spaces and connection with algebraic combinatorial 3-manifolds

Definition 5.6.1. A **basis space** is a pair (X, β) such that X is a set and $\beta \subseteq \mathcal{P}(X)$ is a countable basis for a Polish topology on X.

A basis space (X, β) is **locally compact** if $(X, \langle\beta\rangle)$ is locally compact.

We say that two basis spaces (X, β) and (X', β') are **equivalent**, and write $(X, \beta) \equiv (X', \beta')$, if there is a bijection $h\colon X \to X'$ such that for all $b \in \beta$ we have $h[b] \in \beta'$ and for all $b' \in \beta'$ we have $h^{-1}[b'] \in \beta$.

Note that the map h of Definition 5.6.1 is always a homeomorphism from $(X, \langle\beta\rangle)$ to $(X, \langle\beta'\rangle)$.

We can parametrize all Heine-Borel metric basis spaces as follows. Consider the space $F(\mathbb{U}) \times F(\mathbb{U})^{\mathbb{N}}$, which is Polish in the product topology. Let $\mathfrak{B}$ be the subset of $F(\mathbb{U}) \times F(\mathbb{U})^{\mathbb{N}}$ consisting of all $(X, (X_i)_{i\in\mathbb{N}})$ such that $\{X \setminus X_i \mid i \in \mathbb{N}\}$ is a basis for *the* topology on X induced by $\mathbb{U}$ and X is Heine-Borel.

Proposition 5.6.2. *For each $(X, (X_i)_{i\in\mathbb{N}}) \in \mathfrak{B}$ the pair $(X, \{X \setminus X_i \mid i \in \mathbb{N}\})$ is a locally compact metric basis space. Conversely, if (X', β) is a locally compact metric basis space, then there is $(X, (X_i)_{i\in\mathbb{N}}) \in \mathfrak{B}$ such that $(X', \beta) \equiv (X, \{X \setminus X_i \mid i \in \mathbb{N}\})$.*

Proof. For the nontrivial direction, let (X', β) be a locally compact metric basis space. By Theorem 5.1.11 X' is K_σ, so we can apply Theorem 5.1.12 to obtain the existence of a metric d on X' which is compatible with $\langle\beta\rangle$ and Heine-Borel. Let $X = \iota(X', d)$, where ι is an isometric embedding ι of (X', d) in $\mathbb{U}$. Then X is Heine-Borel. For every $i \in \mathbb{N}$ we set $X_i = X \setminus \iota(b_i)$, with $b_i \in \beta$. Then the pair $(X, (X_i)_{i\in\mathbb{N}}) \in \mathfrak{B}$. □

Proposition 5.6.3. *$\mathfrak{B}$ is a Borel subset of $F(\mathbb{U}) \times F(\mathbb{U})^{\mathbb{N}}$. Thus, $\mathfrak{B}$ is a standard Borel space.*

Proof. Consider $(X, (X_i)_{i\in\mathbb{N}}) \in \mathfrak{B}$. By Proposition 5.1.10 the relation "X is Heine-Borel" is Borel. Let now $(d_i)_{i\in\mathbb{N}}$ be a dense sequence in $\mathbb{U}$ obtained applying Theorem 5.1.6, and fix the countable basis $\mathcal{B} = \{B(d_i, \varepsilon) \mid i \in \mathbb{N}, \varepsilon \in \mathbb{Q}_+\}$ of $\mathbb{U}$. Then $\{X \setminus X_i \mid i \in \mathbb{N}\}$ is a basis for the the topology on X induced by $\mathbb{U}$ if and only if

(a) $\forall i, j \in \mathbb{N}\ \exists k \in \mathbb{N}(X_k = X_i \cup X_j)$, and

(b) $\forall i \in \mathbb{N}, \forall \varepsilon \in \mathbb{Q}_+, \forall \varepsilon' \in \mathbb{Q}_+ \cap\ ]0, \varepsilon[\ \exists k \in \mathbb{N}\big((X \cap \bar{B}(d_i, \varepsilon')) \subseteq \bigcup_{j\leqslant k}(X \setminus X_j) \subseteq B(d_i, \varepsilon)) \cap X\big)$.

It is easy to see that (a) is a Borel condition. We now check that the relation (b) is Borel as well. Since all the quantifiers vary on countable sets, it is enough to show that the two set inclusions are Borel. Since X is Heine-Borel, $X \cap \bar{B}(d_i, \varepsilon')$ is compact and thus the condition $(X \cap \bar{B}(d_i, \varepsilon')) \subseteq \bigcup_{j\leqslant k}(X \setminus X_j)$, which is equivalent to

$$(X \cap \bar{B}(d_i, \varepsilon')) \cap \left(X \setminus \big(\bigcup_{j\leqslant k}(X \setminus X_j)\big)\right) = \varnothing,$$

is Borel. The second inclusion is Borel as a consequence of Proposition 5.1.5(ii). □

We write $(X, \beta) \in \mathfrak{B}$ to mean that the code of (X, β) is in $\mathfrak{B}$.

Definition 5.6.4. An element $(X, \beta) \in \mathfrak{B}$ is **complemented** if

(1) for all $b \in \beta$, either $\bar{b}$ or $X \setminus b$ is compact,

(2) for all $b \in \beta$, $X \setminus \bar{b} \in \beta$,

(3) for all $b \in \beta$, $\operatorname{int}(\bar{b}) = b$,

(4) $X \in \beta$,

(5) for all $b_0, b_1 \in \beta$ such that $\bar{b}_0 \subseteq b_1$ there is $b_2 \in \beta$ such that $\bar{b}_0 \subseteq b_2 \wedge \bar{b}_2 \subseteq b_1$.

We denote by $\mathfrak{B}^C \subseteq \mathfrak{B}$ the set of all complemented Heine-Borel metric basis spaces.

The next proposition shows that $\mathfrak{B}^C$ is a standard Borel space as well.

Proposition 5.6.5. *The set $\mathfrak{B}^C$ is a Borel subset of $\mathfrak{B}$, and hence it is a standard Borel space.*

Proof. Let $(X, \beta) \in \mathfrak{B}$. We need to check the conditions (1)-(5) of Definition 5.6.4 are Borel.

Since $K(\mathbb{U})$ is a Borel subset of $\mathbb{U}$, condition (1) is Borel. By Proposition 5.1.5(iii),(3) is a Borel relation as well. Consider now the relation (2). Then

$$X \setminus \bar{b} \in \beta \iff \exists b_0 \in \beta(X \setminus \bar{b} = b_0),$$

and by Proposition 5.1.5(ii) it follows that this condition is Borel. Similarly, one can show that (4) is Borel.

We now check that (5) is Borel as well: it is enough to prove that

$$\neg(\bar{b}_0 \subseteq b_1) \vee (\bar{b}_0 \subseteq b_2 \wedge \bar{b}_2 \subseteq b_1)$$

is Borel. To this aim, notice that for every b and b' in β the relation "$\bar{b} \subseteq b'$" is equivalent to "$\bar{b} \cap (X \setminus b') = \varnothing$", and by (1) either $\bar{b}$ or $X \setminus b' \subseteq X \setminus b$ is compact. Thus "$\bar{b} \subseteq b'$" is a Borel relation, and the same follows for (5). □

We denote by $\equiv_{\mathfrak{B}^C}$ the restriction of $\equiv$ to $\mathfrak{B}^C$.

Before stating the next result, it is useful to describe the topology which is defined on the spaces $\mathbb{ASC}^\infty$ and $\mathfrak{B}^C$. The basic open sets of the topology on $\mathbb{ASC}^\infty$ are of the form

$$\mathbf{N}_{(\kappa_i)_{i<j}} = \{T \in \mathbb{ASC}^\infty \mid T \restriction j = (\kappa_i)_{i<j}\},$$

where $(\kappa_i)_{i<j} \in \mathbb{ASC}^{fin}$.

Recall now that the topology on $\mathfrak{B}^C$ is that inherited from $F(\mathbb{U}) \times F(\mathbb{U})^{\mathbb{N}}$, i.e. the topology that has as basis the sets $\prod_{i \in \mathbb{N}} U_i$, where U_i is open in the i-th copy of $F(\mathbb{U})$ for all $i \in \mathbb{N}$, and $U_i = F(\mathbb{U})$ for all but finitely many $i \in \mathbb{N}$. Here we consider a finer topology on $\mathfrak{B}^C$ which is useful in the next theorem to show in an easier way that a function with range in $\mathfrak{B}^C$ is continuous, whence it follows that the function is continuous w.r.t. the topology inherited from $F(\mathbb{U}) \times F(\mathbb{U})^{\mathbb{N}}$ as well (which is coarser). The new topology on $\mathfrak{B}^C$ has as basic open sets those of the form $U \times \mathbf{N}_{(b_i)_{i<n}}$, where $(X, (b_i)_{i \in \mathbb{N}}) \in \mathfrak{B}^C$, U is an open neighboorhood of X in $F(\mathbb{U})$, $n \in \mathbb{N}$ and

$$\mathbf{N}_{(b_i)_{i<n}} = \{\beta \in F(\mathbb{U})^{\mathbb{N}} \mid (b_i)_{i<n} \sqsubseteq \beta\}.$$

Theorem 5.6.6. *There exists a Borel map $\xi_2 \colon \mathbb{ASC}^\infty \to \mathfrak{B}^C$ such that for all $T, T' \in \mathbb{ASC}^\infty$ we have that if $T \approx_{PL} T'$ then $\xi_2(T) \equiv_{\mathfrak{B}^C} \xi_2(T')$, and if $T \not\approx T'$ then $\xi_2(T) \not\equiv_{\mathfrak{B}^C} \xi_2(T')$.*

Proof. Fix an element $T \in \mathbb{ASC}^\infty$, and denote by $\mathcal{Q}$ the set of all the algebraic finitary subdivisions of T, which is countable. Define

$$\beta'_T = \bigcup_{T' \in \mathcal{Q}} \{\mathrm{int}_{R(T)}(\cup s) \mid s \subseteq T' \text{ finite}\},$$

where the interior is taken in $R(T)$. Let $\beta_T = \beta'_T \cup \{R(T) \setminus \bar{b} \mid b \in \beta'_T\}$.

We define the map $\xi_2 \colon \mathbb{ASC}^\infty \to \mathfrak{B}^C$ by $\xi_2(T) = (R(T), \beta_T)$.

First let us show that for all $T \in \mathbb{ASC}^\infty$ we indeed have $(R(T), \beta_T) \in \mathfrak{B}^C$. By Lemma 5.3.24 and Corollary 5.3.25, $R(T) \in F(\mathbb{U})$ is Heine-Borel. Also it is standard to check that β_T is a basis for the topology on $R(T)$ induced by $\mathbb{U}$.

By definition of β_T, we can easily see that $(R(T), \beta_T)$ satisfies conditions (1)-(5) of Definition 5.6.4. Thus $(R(T), \beta_T) \in \mathfrak{B}^C$.

We now check that ξ_2 is continuous, and hence Borel. Let $T \in \mathbb{A}\mathrm{SC}^\infty$ and V be any neighbourhood of $\xi_2(T)$, i.e. V is of the form $\bigcup_{i\in I}(U \times N_{(b_i)_{i<n}})$, where $I \subseteq \mathbb{N}$ is finite, U is an open neighbourhood of $R(T)$ and $(b_i)_{i<n} \sqsubseteq \beta_T$. Then

$$O = \bigcup_{n\in I}\{T' \in \mathbb{A}\mathrm{SC}^\infty \mid R(T') = R(T) \ \wedge \ T'\restriction n = (b_i)_{i<n}\}$$

is an open neighbourhood of T such that $\xi_2[O] \subseteq V$. Thus ξ_2 is continuous.

Suppose now that $T_0, T_1 \in \mathbb{A}\mathrm{SC}^\infty$, and that $T_0 \approx_{PL} T_1$. Then by Lemma 5.3.19, we have $T_0 \approx_{\mathbb{A}\mathrm{PL}} T_1$ via some homeomorphism $h\colon R(T_0) \to R(T_1)$.

Let now $b \in \beta_{T_0}$. We want to show that $h[b] \in \beta_{T_1}$. By definition of $\mathbb{A}$PL-homeomorphism, there are algebraic subdivisions T_0' and T_1' of T_0 and T_1, respectively, such that T_0' and T_1' are isomorphic via h. By the closure under subdivisions we can assume without loss of generality, that $T_0', T_1' \in \mathbb{A}\mathrm{SC}^\infty$. Let $j \in \{0,1\}$, $(T'_{j,k})_{k\in\mathbb{N}}$ be a sequence of finitary algebraic subdivisions of T_j such that

(1) for all k there is n_k such that $T'_{j,k'}\restriction k = T'_j\restriction k$ for all $k' > n_k$.

We distinguish two cases.

- Suppose that b belongs to β'_{T_0}. Hence $b = \mathrm{int}(\cup s)$, with $s \subseteq T'$ for some algebraic finitary subdivision T' of T_0. Let $k \in \mathbb{N}$ be large enough such that $s \subseteq T'_{0,k}\restriction k$. By condition (1) there exists n_k such that $T'_{0,k'}\restriction k = T'_0\restriction k$ for all $k' > n_k$. Take one of these k'. Since both $T'_{0,k'}$ and T' are finitary algebraic subdivisions of T_0 we can consider a common finitary algebraic subdivision T''_0 of $T'_{0,k'}$ and T'. Then $\cup s = \cup s'$ for some finite $s' \subseteq T''_0$, so $b = \mathrm{int}(\cup s')$. Now $h[b] = h[\mathrm{int}(\cup s')] = \mathrm{int}[h(\cup s')] = \mathrm{int}(\cup h[s'])$, so it is enough to show that $h[s'_i]$ is in some algebraic finitary subdivision of T'_1 for all $s'_i \in s'$. Since the vertices of s'_i are in $\mathbb{A}(T''_0)$ and by Lemma 5.3.14 we have $\mathbb{A}(T''_0) = \mathbb{A}(T') = \mathbb{A}(T'_0)$, we can apply Lemma 5.3.15 to obtain that the vertices of $h[s'_i]$ are in $\mathbb{A}(T'_1)$. This means that $h[s'_i]$ is a simplex of some finitary algebraic subdivision T^i_1 of T_1, for every i. Let now $k \in \mathbb{N}$ be large enough such that $h[s'_i] \subseteq T'_{1,k}\restriction k$ for all i. By condition (1) there exists n_k such that $T'_{1,k'}\restriction k = T'_1\restriction k$ for all $k' > n_k$. Fix one of such k''s. Then $T'_{1,k'}$ is a finitary algebraic subdivision of T_1, and $h[\cup s'] = \cup_i h[s'_i] = \cup s''$ where $s'' \subseteq T'_{1,k'}$ is finite. Hence we have that $h[\cup s] = h[\cup s'] = \cup s''$ belongs to β'_{T_1}.
- If $b \in \{R(T_0) \setminus \bar{b} \mid b \in \beta'_{T_0}\}$, it is of the form $R(T_0) \setminus \cup s$, for some finite s contained in an algebraic finitary subdivision T' of T_0. Then by the previous argument we obtain that $h[R(T_0) \setminus \cup s] = R(T_1) \setminus h[\cup s]$, with $h[\cup s] \in \beta'_{T_1}$, and hence $h[R(T_0) \setminus \cup s] \in \beta_{T_1}$.

Simmetrically, one can show that $h^{-1}[b'] \in \beta_{T_0}$ for each $b' \in \beta_{T_1}$. Thus the map h witnesses $(R(T_0), \beta_{T_0}) \equiv_{\mathfrak{B}^C} (R(T_1), \beta_{T_1})$.

For the other direction, suppose that $(R(T_0), \beta_{T_0}) \equiv_{\mathfrak{B}^C} (R(T_1), \beta_{T_1})$ with witness $h\colon R(T_0) \to R(T_1)$. Then h is a homeomorphism with respect to the topologies generated by β_{T_0} and β_{T_1}. But these topologies coincide with their topologies inherited from $\mathbb{U}$, so we have $R(T_0) \approx R(T_1)$. ☐

Corollary 5.6.7. *$\xi_2\restriction\mathbb{A}\mathfrak{M}_3$ witnesses that $\approx_{\mathbb{A}\mathfrak{M}_3} \leqslant_B \equiv_{\mathfrak{B}^C}$.*

Proof. By [Moi52, Theorem 4] two triangulated manifolds are homeomorphic if and only if they are PL-homeomorphic, so the result follows from Theorem 5.6.6. ☐

5.7 Blurry Filters and Complemented Algebras

Definition 5.7.1. Let $L = \{\leqslant, K\}$ be a first-order vocabulary with one binary symbol $(\leqslant)$ and one unary symbol (K). A **sorted complemented algebra** is an L-model $\mathcal{A} = (A, \leqslant, K)$ such that

(a) $\mathcal{A}\restriction\{\leqslant\}$ is a partial order, i.e. reflexive, antisymmetric transitive relation,

(b) There are unique $\leqslant$-maximal and $\leqslant$-minimal elements denoted $\mathbf{1}$ and $\mathbf{0}$ respectively,

(c) For each $a \in A$ there is a unique $\neg a \in A$ such that the only element x satisfying $x \leqslant a$ and $x \leqslant \neg a$ is $x = \mathbf{0}$ and the only element y satisfying $y \geqslant a$ and $y \geqslant \neg a$ is $y = \mathbf{1}$,

(d) For all $a \in A$ we have $\neg\neg a = a$,

(e) There are no requirements on $K \subseteq A$.

Recall that by Mod_L we denote the set of all L-models. Let $\mathfrak{A} \subset \mathrm{Mod}_L$ be the subset of sorted complemented algebras. By [Kec95, Theorem 16.8] $\mathfrak{A}$ is a Borel set of $\mathrm{Mod}(L)$, so it is a standard Borel space. We write $(A, \leqslant, K) \in \mathfrak{A}$ to mean that the code of the sorted complemented algebra $(A, \leqslant, K)$ is in $\mathfrak{A}$.

Definition 5.7.2. Let $\mathcal{A} = (A, \leqslant, K)$ be a sorted complemented algebra. A set $F \subseteq A$ is a **filter** if

(i) $\mathbf{1} \in F$

(ii) for all a_0, a_1, if $a_0 \leqslant a_1$ and $a_0 \in F$, then $a_1 \in F$,

(iii) for all $a_0, a_1 \in F$ there is $a_2 \in F$ such that $a_2 \leqslant a_0$ and $a_2 \leqslant a_1$.

A filter is **proper** if additionally

(iv) $\mathbf{0} \notin F$.

A proper filter is **blurry** if

(v) for all $a \in A$, if $a \notin F$ and $\neg a \notin F$, then for all $a_0 \geqslant a$ with $a_0 \neq a$ we have $a_0 \in F$.

A filter is a K-**filter** if

(vi) $F \cap K \neq \varnothing$.

We denote the set of blurry K-filters on $\mathcal{A}$ by $\mathcal{F}(\mathcal{A})$.

Remark 5.7.3. Note that property (iii) holds for any finite collection, so for all $a_0, \dots, a_{n-1} \in F$ there is $a \leqslant \bigcup_{i=0}^{n-1} a_i$ with $a \in F$. In particular, if F is proper then $\bigcup_{i=0}^{n-1} a_i \neq \varnothing$.

We now define a map which connects complemented Heine-Borel metric basis spaces with sorted complemented algebras.

Definition 5.7.4. Let $\psi\colon \mathfrak{B}^C \to \mathrm{Mod}_L$ be the function which takes a Heine-Borel metric basis space (X, β) to the L-model $\psi(X, \beta) = (A, \leqslant, K)$ defined as follows:

- $A = \beta$,
- For all $b_0, b_1 \in \beta$ we have $b_0 \leqslant b_1 \iff (b_0 = b_1) \vee (\bar{b}_0 \subseteq b_1)$, and
- $K = \{b \in \beta \mid \bar{b} \text{ is compact}\}$.

Lemma 5.7.5. *For all $(X, \beta) \in \mathfrak{B}^C$ the model $\psi(X, \beta) = (\beta, \leqslant, K)$ is a sorted complemented algebra. Thus, the range of ψ is included in $\mathfrak{A}$.*

Proof. Reflexivity, antisymmetry, and transitivity of $\leqslant$ are easy to check, so this proves (a) of Definition 5.7.1. The $\leqslant$-minimal element is $\varnothing$ and the $\leqslant$-maximal is X (which are in β by the fact that $X \in \beta$ and (2) of Definition 5.6.4). For uniqueness, suppose $X' \subsetneq X$. Now pick $x \in X \setminus X'$ and an open basic neighbourhood $b \in \beta$ of x. Then $b \nleqslant X'$, and hence X' is not maximal. This proves (b) of Definition 5.7.1. So let us denote $\mathbf{1} = X$ and $\mathbf{0} = \varnothing$.

For $b \in \beta$, let $\neg b = X \setminus \bar{b}$. Then by (2) of Definition 5.6.4 $\neg b \in \beta$. Also $\bar{b} \cup \overline{\neg b} = X$, and $b \cap \neg b = \varnothing$. This means that $\mathbf{0} \leqslant b$, $\mathbf{0} \leqslant \neg b$, $b \leqslant \mathbf{1}$ and $\neg b \in \mathbf{1}$. It remains to show that $\mathbf{0}$ and $\mathbf{1}$ are unique with this property. Suppose $X' \subsetneq X$, $X' \in \beta$. By (3) of Definition 5.6.4 $\mathrm{int}(X \setminus X')$ is non-empty, and since $b \cup \neg b$ is dense, we have $(b \cup \neg b) \cap (X \setminus X') \neq \varnothing$. Therefore either $b \nleqslant X'$ or $\neg b \nleqslant X'$. On the other hand $b \cap \neg b = \varnothing$, so the only element b' with $b' \leqslant b$ and $b' \leqslant \neg b$ must be $b' = \mathbf{0}$. It remains to show that $\neg b$ is the unique element with these properties. Suppose $b' \in \beta$ is some other element satisfying condition (c) of Definition 5.7.1 (except for the uniqueness). Since $\varnothing$ is the unique open set whose closure is contained in both b and b', and they are both open, we have $b' \cap b = \varnothing$. So $b' \subseteq X \setminus b$, but since b' is open, we have in fact $b' \subseteq X \setminus \bar{b} = \neg b$. Suppose that $x \in \neg b = X \setminus \bar{b}$. Since $\bar{b}' \cup \bar{b} = X$, we have $x \in \bar{b}' \cup \bar{b}$, so $x \in \bar{b}'$ and so we have $\neg b \subseteq \bar{b}'$. By (3) of Definition 5.6.4 and the openess of $\neg b$, we have $\neg b \subseteq b'$.

Finally, by using (3) of Definition 5.6.4 one more time, we have for all $b \in \beta$ that $\neg\neg b = X \setminus \overline{(X \setminus \bar{b})} = b$. This proves (d) of Definition 5.7.1, and (e) of Definition 5.7.1 is trivial. □

Lemma 5.7.6. *Suppose that $(X, \beta) \in \mathfrak{B}^C$, $|X| \geqslant 3$, $\mathcal{A} = (\beta, \leqslant, K) = \psi(X, \beta)$, and $F \subseteq \beta$ is a blurry K-filter on $\mathcal{A}$. Then the following hold:*

(1) *If $x_0, x_1 \in \bigcap F$, then $x_0 = x_1$.*

(2) *$\bigcap F \neq \varnothing$.*

(3) *F is of the form $\{b \in \beta \mid x \in b\}$ for some $x \in X$.*

Proof. (1) Suppose $x_0 \neq x_1$. Since X has more than 2 elements, there is x_2 distinct from both x_0 and x_1. Since β generates a Polish topology, there are $b_k \in \beta$ with $x_k \in b_k$ for all $k \in \{0, 1, 2\}$ whose closures are mutually disjoint. Since x_0 and x_1 belong to all elements of F, we have $b_0 \notin F$ and $\neg b_0 \notin F$. Now consider $\neg b_1 = X \setminus \bar{b}_1$. We know that $\bar{b}_0 \cap \bar{b}_1 = \varnothing$, so $\bar{b}_0 \subseteq \neg b_1$. On the other hand $\neg b_1 \neq b_0$, because $b_2 \subseteq \neg b_1$. Since F is blurry and $b_0 \leqslant \neg b_1$, we have $\neg b_1 \in F$, but $x_1 \notin \neg b_1$, a contradiction.

(2) Let $c \in F \cap K$. Let $Z = \{\bar{b} \cap \bar{c} \mid b \in F\}$. Then $\bigcap Z$ is non-empty, because otherwise by compactness there would be $b_0, \dots, b_{n-1} \in F$ such that

$$b_0 \cap \dots \cap b_{n-1} \cap c = \varnothing,$$

which contradicts the properness of F (see Remark 5.7.3). So let $x \in \bigcap Z$. Let us show that $x \in \bigcap F$. Let $b \in F$. We want to show that $x \in b$. By the definition of Z we know that $x \in \bar{b} \cap \bar{c} \subseteq \bar{b}$. If $b = \bar{b}$, we are done. Otherwise b is open and not closed, so it must be infinite. By (1) of this Lemma, there must be $b' \leqslant b$ such that $b' \in F$, because otherwise $b \subseteq \bigcap F$ (use the definition of a filter). So since $x \in \bigcap Z$, we have

$$x \in \bar{b}' \cap \bar{c} \subseteq \bar{b}' \subseteq b,$$

so again $x \in b$.

(3) By (1) and (2) of this Lemma there is x such that $\bigcap F = \{x\}$. Thus $F \subseteq \{b \in \beta \mid x \in b\}$. Let us show the converse, namely that $\{b \in \beta \mid x \in b\} \subseteq F$. Suppose $b \in \beta$ is such that $x \in b$. We want to show that $b \in F$. Let $c \in F \cap K$ and let

$$Z = \{\bar{b}' \cap \bar{c} \cap \overline{\neg b} \mid b' \in F\}.$$

Clearly $\bigcap Z \subseteq \bigcap F$, because every element of Z is a subset of an element of F. On the other hand $x \notin \bigcap Z$, because $x \notin \overline{\neg b}$, so $\bigcap Z = \varnothing$. Since $\bar{c}$ is compact, there is a finite subset of Z whose intersection is empty, and let $b_0, \dots, b_{n-1} \in F$ witness that. So now we have

$$\bigcap_{i=0}^{n-1} \bar{b}_i \cap \bar{c} \cap \overline{\neg b} = \varnothing. \tag{5.7.1}$$

By Remark 5.7.3 there is $b_* \in F$ with $b_* \leqslant b_i$ for all $i \in \{0, \ldots, n-1\}$ and $b_* \leqslant c$, so

$$\begin{aligned}
\overline{b}_* &\subseteq \bigcap_{i=0}^{n-1} b_i \cap c \\
&= \bigcap_{i=0}^{n-1} b_i \cap c \cap (\overline{\neg b} \cup b) && \overline{\neg b} \cup b = X \\
&\subseteq \bigcap_{i=0}^{n-1} \overline{b}_i \cap \overline{c} \cap (\overline{\neg b} \cup b) && b_i \subseteq \overline{b}_i, b' \subseteq \overline{c} \\
&= \underbrace{\left(\bigcap_{i=0}^{n-1} \overline{b}_i \cap \overline{c} \cap \overline{\neg b}\right)}_{=\varnothing \text{ by (5.7.1)}} \cup \left(\bigcap_{i=0}^{n-1} \overline{b}_i \cap \overline{c} \cap b\right) \\
&= \bigcap_{i=0}^{n-1} \overline{b}_i \cap \overline{c} \cap b \\
&\subseteq b
\end{aligned}$$

This means that $b_* \leqslant b$, so by (ii) of Definition 5.7.2 we obtain $b \in F$.

□

Definition 5.7.7. Define the map φ which assigns to each sorted complemented algebra $\mathcal{A} = (A, \leqslant, K)$ the basis space $\varphi(\mathcal{A}) = (X_{\mathcal{A}}, \beta_{\mathcal{A}})$ such that $X_{\mathcal{A}} = \mathcal{F}(\mathcal{A})$ and $\beta_{\mathcal{A}} = \{U(b) \mid b \in A\}$ where $U(b) = \{F \in \mathcal{F}(\mathcal{A}) \mid b \in F\}$.

The following result is the analogue of the famous Stone's representation theorem. Here a sorted complemented algebra, the set of its blurry K-filters and its associated basis space play the role of a Boolean algebra, the set of its ultrafilters and its associated Stone space, respectively.

Theorem 5.7.8. *For all $(X, \beta) \in \mathfrak{B}^C$ we have that $(X, \beta) \equiv \varphi(\psi(X, \beta))$.*

Proof. Let $h \colon X \to X_{\psi(X,\beta)} = \mathcal{F}(\psi(X, \beta))$ be defined by

$$h(x) = \{b \in \beta \mid x \in b\}.$$

First we claim that $h(x) \in \mathcal{F}(\psi(X, \beta))$.

Claim 5.7.8.1. The set $h(x)$ is a blurry K-filter on $\psi(X, \beta)$ for every $x \in X$.

Proof. Fix $x \in X$. We show that $h(x)$ is a proper filter. First we can notice that $X \in h(x)$. Let now b_0 and b_1 be elements of β such that $b_0 \leq b_1$ and $b_0 \in h(x)$ and show that $b_1 \in h(x)$. By definition of $\leq$ we have that $b_0 = b_1$ or $\overline{b_0} \subseteq b_1$. If we are in the first case then we are done, otherwise it suffices to notice that $x \in b_0$ and so $x \in b_1$, whence it follows that $b_1 \in h(x)$. We now take $b_0, b_1 \in h(x)$ and we prove the existence of $b_2 \in h(x)$ such that $b_2 \leq b_0, b_1$. Consider the open neighbourhood $b_0 \cap b_1$ of x. Denote $b = b_0 \cap b_1$. Let us show that there exists $b_2 \subseteq b$ such that $x \in b_2$ and $\overline{b}_2 \subseteq b$. Fix a Polish metric d on X and assume towards a contradiction that there is no such b_2. Now pick $x_n \in \bar{B}(x, 1/n) \setminus b$. The sequence (x_n) witnesses that $x \in \overline{X \setminus b} = X \setminus b$, a contradiction. Thus $h(x)$ is a filter, and since $\varnothing \notin h(x)$ it is proper.

We now show that $h(x)$ is blurry. Let $b \in \beta$ and assume that $b \notin h(x)$ and $X \setminus \overline{b} \notin h(x)$. Let $b' \in \beta$ be such that $b' \geqslant b$ and $b' \neq b$, and hence $\overline{b} \subseteq b'$. We need to show that $b' \in h(x)$. Since $x \notin b$ and $x \notin X \setminus \overline{b}$, we have $x \in \partial b \subseteq \overline{b} \subseteq b'$. Thus $x \in b'$ and by definition of $h(x)$ we have $b' \in h(x)$.

We finally notice that by the local compactness of X we have that $h(x) \cap K \neq \varnothing$, and hence $h(x)$ is a K-filter on $\psi(X, \beta)$. □

We now show that h is a bijection such that $h[b] \in \beta_{\psi(X,\beta)}$ for every $b \in \beta$ and $h^{-1}[b] \in \beta$ for every $U(b) \in \beta_{\psi(X,\beta)}$. If $F \in \mathcal{F}(\psi(X,\beta))$, by Lemma 5.7.6.(3) it is of the form $F = \{b \in \beta \mid x_0 \in b\}$ for some $x_0 \in X$. By Lemma 5.7.6.(1)-(2) we have that $\bigcap F = \{x_0\}$, and so $h(x_0) = F$, which proves that h is onto. To see that h is one-to-one, let $x_0, x_1 \in X$ and assume that $h(x_0) = h(x_1) = F$. Then $x_i \in \bigcap F$ for both $i \in \{0,1\}$ and by 5.7.6.(1) it follows that $x_0 = x_1$.

Let now $b \in \beta$ and show that $h[b] \in \beta_{\psi(X,\beta)}$:

$$\begin{aligned} h[b] &= \{h(x) \mid x \in b\} \\ &= \{h(x) \mid b \in h(x)\} && \text{by definition of } h(x) \\ &= \{F \in \mathcal{F}(\psi(X,\beta)) \mid b \in F\} && \text{by surjectivity of } h(x) \\ &= U(b) && \text{by definition of } U(b). \end{aligned}$$

Finally we need to show that $h^{-1}[U(b)] \in \beta$ for every $b \in \beta$:

$$\begin{aligned} h^{-1}[U(b)] &= \{x \in X \mid h(x) \in U(b)\} \\ &= \{x \in X \mid b \in h(x)\} && \text{by definition of } U(b) \\ &= \{x \in X \mid x \in b\} && \text{by definition of } h(x) \\ &= b. \end{aligned}$$

□

Notice that by the previous result we obtain a homeomorphism between a Heine-Borel basis space (X,β) and $\varphi(\psi(X,\beta))$. Hence, in particular $\varphi(\psi(X,\beta)) \in \mathfrak{B}^C$.

Lemma 5.7.9. *Suppose (X,β) and (X',β') are equivalent locally compact basis spaces. Then $\psi(X,\beta) \cong \psi(X',\beta')$.*

Proof. If there is $h\colon X \to X'$ witnessing that (X,β) and (X',β') are equivalent, then define $\hat{h}\colon \beta \to \beta'$ by $\hat{h}(b) = h[b]$ which is easily seen to be an isomorphism from $\psi(X,\beta)$ to $\psi(X',\beta')$. □

Lemma 5.7.10. *Suppose $\mathcal{A} = (A, \leqslant, K)$ and $\mathcal{A}' = (A', \leqslant', K')$ are isomorphic sorted complemented algebras. Then $\varphi(\mathcal{A})$ is equivalent to $\varphi(\mathcal{A}')$, where φ is defined as in Definition 5.7.7.*

Proof. Suppose $f\colon A \to A'$ is an isomorphism from $\mathcal{A}$ to $\mathcal{A}'$. Define $g\colon \mathcal{F}(\mathcal{A}) \to \mathcal{F}(\mathcal{A}')$ by $g(F) = f[F]$. This is an equivalence between the basis spaces $\varphi(\mathcal{A})$ and $\varphi(\mathcal{A}')$. □

Corollary 5.7.11. *$\equiv_{\mathfrak{B}^C} \leqslant_B \cong_{\mathfrak{A}}$ witnessed by the map ψ of Definition 5.7.4.*

Proof. It is easy to check that ψ is Borel using Proposition 5.1.2. We now consider two elements (X,β) and (X',β') in $\mathfrak{B}^C$. If $\psi(X,\beta)$ and $\psi(X',\beta')$ are isomorpic complemented algebras, then by Lemma 5.7.10, $\varphi(\psi(X,\beta))$ and $\varphi(\psi(X',\beta'))$ are equivalent. Now by Theorem 5.7.8 also (X,β) and (X',β') must be equivalent. Suppose on the other hand that (X,β) and (X',β') are equivalent. Then by Lemma 5.7.9 the algebras $\psi(X,\beta)$ and $\psi(X',\beta')$ are isomorphic. □

5.8 Main results

5.8.1 Classification of 3-manifolds

It has been already shown that isomorphism on countable structures is a lower bound for the complexity of homeomorphism on n-manifolds for $n \geqslant 2$. Let us give a proof for the sake of completeness.

Theorem 5.8.1 (Folklore)**.** *The isomorphism on countable graphs is Borel reducible to homeomorphism relation on non-compact n-manifolds, for $n \geqslant 2$.*

Proof. By [CG01] the isomorphism on graphs is Borel reducible to the isomorphism on Boolean algebras which in turn is Borel reducible to the homeomorphism relation on compact subsets of the Cantor set $2^{\mathbb{N}}$. Let $f\colon 2^{\mathbb{N}} \to \mathbb{R}^n$ be the standard "1/3-Cantor set" embedding of the Cantor set into $\mathbb{R}^n$. It is now enough to show that for any two closed subsets $C, C' \subseteq 2^{\mathbb{N}}$, the complements $\mathbb{R}^n \setminus f[C]$ and $\mathbb{R}^n \setminus f[C']$ are homeomorphic iff C and C' are homeomorphic. If the complements are homeomorphic, then so are the Cantor sets by the remark after [CvM83, Theorem 4.1]. For the converse, note that since the embedding f is "standard", and $n \geqslant 2$, we have $f[C], f[C'] \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^n$. So, if $f[C]$ and $f[C']$ are homeomorphic, first apply [Moi77, §13/Theorem 7] to extend the homeomorphism to $\mathbb{R}^2$ and then further extend it canonically to $\mathbb{R}^n$. □

Using the results obtained in the previous sections, we are now able to show that isomorphism on countable graphs and homeomorphism on non-compact 3-manifolds have the same complexity.

Theorem 5.8.2. *3-manifolds are classifiable by countable structures in a Borel way, $\approx_{\mathfrak{M}_3} \leqslant_B \cong_{\mathfrak{A}}$.*

Proof. Let ψ be as given by Definition 5.7.4. By Corollary 5.7.11 it reduces $\equiv_{\mathfrak{B}^C}$ to $\cong_{\mathfrak{A}}$. Let ξ_2 be as given by Theorem 5.6.6. By Corollary 5.6.7 $\xi_2 \restriction \mathbb{A}\mathfrak{M}_3$ reduces $\approx_{PL}$ to $\equiv_{\mathfrak{B}^C}$. Let ξ_1 be given by 5.5.11. Then $\mathrm{Im}(\xi_1) \subseteq \mathrm{dom}(\mathbb{A}\mathfrak{M}_3)$ and by Theorem 5.5.12 $\xi_1 \circ \xi_0$ (where ξ_0 is from Proposition 5.5.3) reduces homeomorphism on $\mathfrak{M}_3$ to $\mathbb{A}\mathfrak{M}_3$. Now $\psi \circ \xi_2 \circ \xi_1 \circ \xi_0$ is the desired reduction. □

Corollary 5.8.3. *Homeomorphism on non-compact* 3*-manifolds is Borel bireducible with isomorphism on countable structures.*

5.8.2 Classification of wild Cantor sets in S^3

For convenience we consider the one-point compactification S^3 of $\mathbb{R}^3$ and we think of a Cantor set of $\mathbb{R}^3$ as a Cantor set of S^3.

Definition 5.8.4. A subset of S^3 is a **Cantor set** if and only if it is zero-dimensional, perfect and compact.

We denote the Polish space of Cantor sets in S^3 by $\mathfrak{C}(S^3)$.

Definition 5.8.5. We say that two Cantor sets C, $C' \subseteq S^3$ are **conjugate**, and write $C \approx_{\mathfrak{C}(S^3)} C'$, if there is a homeomorphism $h\colon S^3 \to S^3$ such that $h(C) = C'$.

We focus on the classification of Cantor sets in S^3 w.r.t. $\approx_{\mathfrak{C}(S^3)}$. First, recall the following result which states that $\approx_{\mathfrak{C}(S^3)}$ is at least as complicated as classification by countable structures.

Theorem 5.8.6. *[GKB13, Theorem 5.4] There exists a Borel reduction from the space of linear orders with the isomorphism relation to the space of Cantor sets with the conjugacy relation.*

As a consequence of the fact that, given $C, C' \in \mathfrak{C}(S^3)$ and a homeomorphism $h\colon S^3 \setminus C \to S^3 \setminus C'$, then h extends to a homeomorphism $h'\colon S^3 \to S^3$ (see the remark after the proof of [CvM83, Theorem 4.1]), one has the following proposition.

Proposition 5.8.7. *Two Cantor sets are conjugate if and only if their complements are homeomorphic.*

Using results of the previous sections, we determine the exact complexity of $\approx_{\mathfrak{C}(S^3)}$, answering Question 5.5 of [GKB13].

Theorem 5.8.8. *Conjugation between Cantor sets of S^3 is Borel bireducible with the isomorphism on countable structures.*

Proof. By Theorem 5.8.6 it sufficies to show that classifying Cantor sets in S^3 is at most as complex as classifying countable structures. We prove that $\approx_{\mathfrak{C}(S^3)} \leqslant_B \approx_{\mathfrak{M}_3}$. Then by Corollary 5.8.2 and the transitivity of $\leqslant_B$ we obtain the desired result. Let φ be the Borel map from $\mathfrak{C}(S^3)$ to $\mathfrak{M}_3$ defined by $\varphi(C) = S^3 \setminus C$. By Proposition 5.8.7 we immediately obtain that φ is a reduction. □

III

Generalized descriptive set theory and large cardinals

6

λ-Perfect Set Property and λ-Baire Property for λ-Σ^1_2 sets

6.1 Preliminaries

Let V denote the universe of all sets, as usual. If not specified, by M we denote an inner model, i.e., a transitive class that contains all ordinals and satisfies the axioms of ZFC.

We work in theories which are extensions of ZFC. Indeed, we add to axioms of ZFC some axiom which states the existence of a large cardinal.

If X and Y are topological spaces, we write $X \approx Y$ if they are homeomorphic.

Let now X be a nonempty set and $n \in \omega$. We denote by ${}^n X$ the set of finite sequences of length n from X. We indicate the length of a sequence s with $\mathrm{lh}(s)$. We allow the case $n = 0$, in which case ${}^0 X = \{\emptyset\}$, where $\emptyset$ denotes here the empty sequence. Finally, let ${}^{<\omega} X = \bigcup_{n\in\omega} {}^n X$ (resp. ${}^\omega X$) be the set of all finite sequences (resp. sequences of length ω) from X. When $s, t \in {}^{<\omega} X$, we write $s \sqsubseteq t$ if $s = t \restriction \mathrm{lh}(s)$.

6.1.1 Large cardinals

In this section we use notions and results from standard textbooks as [Jec03, Kan09] and [Dim18]. Let M and N be sets or classes. A function $j : M \to N$ is an **elementary embedding** if and only if for any formula φ and $a_1, \ldots, a_n \in M$, $M \models \varphi(a_1, \ldots, a_n)$ if and only if $N \models \varphi(j(a_1), \ldots, j(a_n))$. From now on, if we write $j : M \prec N$ we mean that j is an elementary embedding. If $j : M \prec N$ is an elementary embedding such that it is not the identity, and $M \models AC$ or $N \subseteq M$ then there is a least ordinal κ such that $j(\kappa) \neq \kappa$, called the **critical point** of j and denoted by $\mathrm{crt}(j)$.

From now on, we assume that every elementary embedding is not the identity.

Among large cardinals, a crucial role in the context of elementary embedding is played by measurable cardinals.

Definition 6.1.1 (Ulam, 1930 - Scott, 1961)**.** A cardinal κ is **measurable** if and only if there exists a κ-complete ultrafilter on κ.

Equivalently, κ is measurable if and only if there exist an inner model M and a $j : V \prec M$ such that $\mathrm{crt}(j) = \kappa$.

Definition 6.1.2. Let M, N be sets or classes such that $N \subseteq M$. We define the **critical sequence** $\langle \kappa_n \mid n \in \omega \rangle$ of $j : M \prec N$ by

- $\kappa_0 = \mathrm{crt}(j)$;
- $\kappa_{n+1} = j(\kappa_n)$.

We now deal with cardinals which are at the top of the hierarchy of large cardinals. By Kunen's Theorem ([Kun71]) it is known that there are no elementary embeddings from $V_{\lambda+2}$ to itself for every λ. The excluded cases give rise to large cardinals very close to inconsistency. In particular, we have:

- **I3**: There exists $j\colon V_\lambda \prec V_\lambda$, where λ is the supremum of its critical sequence.
- **I1**: There exists $j\colon V_{\lambda+1} \prec V_{\lambda+1}$.

Between these two axioms there is another large cardinal, which has found very few applications so far: I2.

- **I2**: There exists $j\colon V \prec M$, with $M \subseteq V$, such that $V_\lambda \subseteq M$ for some $\lambda = j(\lambda) > \mathrm{crt}(j)$.

Proposition 6.1.3. *[Kan09, Theorem 23.14(b)] Suppose that j witnesses I2. Then λ is the supremum of its critical sequence, and $j \restriction V_\lambda$ is an elementary embedding from V_λ to V_λ.*

One can also consider an axiom stronger than I1, which enlarges the domain of j.

- **I0**: There exists $j\colon L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with critical point less then λ.

We now describe the process of iterating ultrapowers. For any M-ultrafilter U over κ, we can recursively define structures $\langle M_\alpha, \in, U_\alpha\rangle$ for $\alpha < \tau$ where U_α is an M_α-ultrafilter over κ_α, and embeddings $i_{\alpha\beta}\colon \langle M_\alpha, \in, U_\alpha\rangle \prec \langle M_\beta, \in, U_\beta\rangle$ for $\alpha \leq \beta < \tau$ as follows.

Set $M_0 = M$, $U_0 = U$, $\kappa_0 = \kappa$, and i_{00} the identity on M. Having defined M_α, U_α, κ_α, and $i_{\alpha\beta}$ for $\alpha < \beta < \delta$, we can have two cases:

1. δ is a successor ordinal, say $\delta = \gamma + 1$. If the ultrapower of M_γ by U_γ is well-founded, let M_δ be its transitive collapse and $U_\delta \subseteq M_\delta$ such that $j\colon \langle M_\gamma, \in, U_\gamma\rangle \prec \langle M_\delta, \in, U_\delta\rangle$ is the corresponding embedding. Set $\kappa_\delta = j(\kappa_\gamma)$, $i_{\gamma\delta} = j$, $i_{\alpha\delta} = j \circ i_{\alpha\gamma}$ for $\alpha < \gamma$, and $i_{\delta\delta}$ the identity on M_δ. If on the other hand the ultrapower is ill-founded, set $\delta = \tau$.
2. δ is a limit ordinal. If the direct limit of $\langle\langle\langle M_\gamma, \in, U_\gamma\rangle \mid \alpha < \delta\rangle, \langle i_{\alpha\beta} \mid \alpha \leq \beta\rangle\rangle$ is well-founded, let M_δ be its transitive collapse and $U_\delta \subseteq M_\delta$ such that for each $\alpha < \delta$ there is a direct limit embedding $i_{\alpha\delta}\colon \langle M_\alpha, \in, U_\alpha\rangle \prec \langle M_\delta, \in, U_\delta\rangle$ modulated by the transitive collapse. Set $\kappa_\delta = i_{\alpha\delta}(\kappa_\alpha)$ for some (and hence, any) $\alpha < \delta$ and $i_{\delta\delta}$ the identity on M_δ. If on the other hand the direct limit is ill-founded, set $\delta = \tau$.

If this definition proceeds through all the ordinals, set $\tau = \mathrm{Ord}$. We call $\langle M_\alpha, \in, U_\alpha, \kappa, i_{\alpha\beta}\rangle \mid \alpha \leq \beta < \tau\rangle$ the **iteration** of $\langle M, \in, U\rangle$; τ is the **length** of the iteration, and for $\alpha < \tau$, $\langle M_\alpha, \in, U_\alpha\rangle$ is the **α-th iterate** of $\langle M, \in, U\rangle$. Also, $\langle M, \in, U\rangle$ (and i_{01}) is **α-iterable** if there exists its α-th iterate.

In the following proposition we collect several basic properties of iterated ultrapowers which are useful later (see [Kan09, Lemmata 19.4-19.5, Corollary 19.7]).

Proposition 6.1.4. *Suppose that $\alpha < \beta < \tau$.*

1. $\mathrm{crt}(i_{\alpha\beta}) = \kappa_\alpha$ *and* $i_{\alpha\beta}(\kappa_\alpha) = \kappa_\beta$.
2. $i_{\alpha\beta}(x) = x$ *for every* $x \in V_{\kappa_\alpha} \cap M_\alpha$, $V_{\kappa_\alpha} \cap M_\alpha = V_{\kappa_\alpha} \cap M_\beta$, *and* $\mathcal{P}(\kappa_\alpha) \cap M_\alpha = \mathcal{P}(\kappa_\alpha) \cap M_\beta$.
3. *If β is a limit ordinal, then $\kappa_\beta = \sup\{\kappa_\gamma \mid \gamma < \beta\}$. Moreover, for any $X \in \mathcal{P}(\kappa_\beta) \cap M_\beta$,*
$$X \in U_\beta \iff \exists \alpha < \beta(\{\kappa_\gamma \mid \alpha \leq \gamma < \beta\} \subseteq X).$$
4. *If ν is a cardinal such that $|{}^\kappa\kappa \cap M| < \nu < \tau$, then $\kappa_\nu = i_{0\nu}(\kappa_0) = \nu$.*

In our setting, the Prikry forcing on a measurable cardinal κ turns out to be particularly useful. The reason is that κ becomes singular of cofinality ω in every generic extension, and hence we can develop a natural GDST on it. We recall here the definition and main properties of this forcing.

Definition 6.1.5. Let κ be a measurable cardinal, and U a normal measure on κ. Then $\mathbb{P}_U$, the **Prikry forcing** on κ via U, is the set of pairs (s, A) such that $s \in [\kappa]^{<\omega}$, $A \in U$ and $\min A > \max s$.

We say that $(s, A) < (t, B)$ if $s \sqsupseteq t$, $A \subseteq B$ and for any $n \in \mathrm{lh}(s) \setminus \mathrm{lh}(t)$, $s(n) \in B$.

If (s, A) and (t, B) are in the generic set G, then s and t must be compatible. Therefore by density $\bigcup\{s \mid \exists A(s, A) \in G\}$ is an ω-sequence cofinal in κ. So $(\mathrm{cof}(\kappa) = \omega)^{V[G]}$, but it is a very delicate forcing, as it does not add any bounded subset of κ ([Git10, Lemma 1.9]).

There is a convenient condition for an ω-sequence cofinal in κ to be generic.

Theorem 6.1.6. *(Mathias Condition, [Git10, Lemma 1.11, Theorem 1.12]) Let κ be a measurable cardinal, $\mathbb{P}_U$ the Prikry forcing on κ via the normal ultrafilter U and let $\langle \kappa_n \mid n \in \omega\rangle$ be a cofinal sequence in κ. Then $\langle \kappa_n \mid n \in \omega\rangle$ is generic for $\mathbb{P}_U$ if and only if for any $A \in U$ the set $\langle \kappa_n \mid n \in \omega\rangle \setminus A$ is finite.*

We often resort to the model M_ω, whose elements are now described more in detail. Suppose $j\colon \langle M_0, \in, U\rangle \prec \langle M_1, \in, U_1\rangle$ is ω-iterable and let $\langle \kappa_n \mid n \in \omega\rangle$ be its critical sequence. Then M_ω is the transitive collapse of the set of equivalence classes of (n, a) such that $a \in M_n$, where if $n < m$ we have that (n, a) is equivalent to (m, b) if and only if $j_{nm}(a) = b$. Thus, $j_{n\omega}(a)$ is the transitive collapse of the class of $[(n, a)]$. Finally, we have that $j_{n\omega}(\alpha) = \alpha$ for $\alpha \in \kappa_n$ and $j_{0\omega}(\kappa_0) = \kappa$, with $\kappa = \sup\{\kappa_n \mid n \in \omega\}$. Moreover, $j_{0\omega}(U_0)$ is a normal ultrafilter on κ, therefore κ is measurable in M_ω.

If we now consider the Prikry forcing $\mathbb{P}_U$ on κ_0 via U, then $j_{0\omega}(\mathbb{P}_U)$ is the Prikry forcing on κ via $j_{0\omega}(U)$, and $\langle \kappa_n \mid n \in \omega\rangle$ is $j_{0\omega}(\mathbb{P}_U)$-generic in M_ω. So it make sense to consider $M_\omega[\langle \kappa_n : n \in \omega\rangle]$.

We often make use of the canonical inner model $L[\mathcal{U}]$, due to Mitchell. We recall here its definition and main properties.

Definition 6.1.7. If $\langle A_\alpha \mid \alpha < \theta\rangle$ is a sequence of sets, we define the model $L[\langle A_\alpha \mid \alpha < \theta\rangle]$ as the model $L[A]$, where $A = \{(\alpha, X) \mid X \in A_\alpha\}$. Under this definition, $L[\langle A_\alpha \mid \alpha < \theta\rangle] = L[\langle B_\alpha \mid \alpha < \theta\rangle]$, where $B_\alpha = A_\alpha \cap L[\langle A_\alpha \mid \alpha < \theta\rangle]$ for all $\alpha < \theta$.

We consider the case of a strictly increasing ω-sequence κ_n of measurable cardinals with normal measure U_n. Then in $L[\langle U_n \mid n < \omega\rangle]$ each $U_n \cap L[\langle U_n \mid n < \omega\rangle]$ is again a normal measure on κ_n. We briefly denote by $L[\mathcal{U}]$ the model $L[\langle U_n \mid n < \omega\rangle]$, and collect in the next proposition the main property satisfied by $L[\mathcal{U}]$ that we use in the sequel.

Proposition 6.1.8. *[Jec03, Theorem 19.38] In $L[\mathcal{U}]$, the κ_n's are the only measurable cardinals, and the $U_n \cap L[\mathcal{U}]$'s are the only normal measures.*

We also recall the following results.

Theorem 6.1.9. *[Kan09, Theorem 3.3] There is a sentence σ_1 of $\mathcal{L}_\in(\dot{A})$ where $\dot{A}$ is unary such that for any set A and any transitive class N,*

$$\langle N, \in, A \cap N\rangle \models \sigma_1 \iff N = L[A] \vee N = L_\delta[A] \text{ for some limit } \delta > \lambda.$$

Also, there is a formula $\varphi_1(v_0, v_1)$ of $\mathcal{L}_\in(\dot{A})$ that in any $\langle L[A], \in, A \cap L[A]\rangle$ defines a well-ordering $<_{L[A]}$ such that for any limit $\delta > \lambda$, any $y \in L_\delta[A]$, and any x,

$$x <_{L[A]} y \iff x \in L_\delta[A] \wedge \langle L_\delta[A], \in, A \cap L_\delta[A]\rangle \models \varphi_1[x, y].$$

Fix now a homeomorphism f from $C(\vec{\lambda})$ to ${}^\lambda 2$ and let $\prec, \succ$ be the Gödel pairing function. Recall that for each $z \in C(\vec{\lambda})$, $f(z)$ can code a binary relation $E_z = \{(\alpha, \beta) \mid f(z)(\prec\alpha, \beta\succ) = 0\}$ defined on λ. We then consider the structure $M_z = \langle \lambda, E_z\rangle$. If M_z is well-founded and extensional, we can apply the Collapsing Lemma to obtain a unique transitive collapse $\mathrm{tr}(M_z)$ and a unique isomorphism $\pi_z\colon M_z \to \mathrm{tr}(M_z)$. Applying Theorem 6.1.9 to $\mathrm{tr}(M_z)$ and using the fact that π_z is a isomorphism, we have that for every $\alpha, \beta \in \lambda$,

$$\begin{aligned} \pi_z(\alpha) <_{\mathrm{tr}(M_z)} \pi_z(\beta) &\iff \langle \mathrm{tr}(M_z), \in, A \cap \mathrm{tr}(M_z)\rangle \models \sigma_1 \wedge \varphi_1[\pi_z(\alpha), \pi_z(\beta)] \\ &\iff M_z \models \sigma_1 \wedge \varphi_1[\alpha, \beta], \end{aligned}$$

where $\mathrm{tr}(M_z) = L[A]$ or $\mathrm{tr}(M_z) = L_\delta[A]$ for some limit $\delta > \lambda$.

We conclude this section recalling some notion and facts concerning $L(V_{\lambda+1})$ and I0.

Theorem 6.1.10. *[Kan09, Proof of Proposition 11.13] There exists a surjection $\Phi\colon \mathrm{Ord}\times V_{\lambda+1}\to L(V_{\lambda+1})$ which is definable in $L(V_{\lambda+1})$.*

Theorem 6.1.10 says that for every $A\in L(V_{\lambda+1})$ there exist $\alpha\in\mathrm{Ord}$ and $x\in V_{\lambda+1}$ such that $A=\Phi(\alpha,x)$. Thus, if we fix x we have that $\{A\in L(V_{\lambda+1})\mid \exists\alpha\in\mathrm{Ord}\ A=\Phi(\alpha,x)\}$ is a definable class in $L(V_{\lambda+1})$ and it is well-ordered: we set

$$A <_x B \tag{6.1.1}$$

if, when α is the least such that $A=\Phi(\alpha,x)$ and β is the least such that $B=\Phi(\beta,x)$, then $\alpha<\beta$. Notice that $<_x$ is definable in $L(V_{\lambda+1})$ as well.

We indicate with $\twoheadrightarrow$ the surjectivity of a function. Let

$$\Theta=\sup\{\gamma\mid \exists f\colon V_{\lambda+1}\twoheadrightarrow\gamma, f\in L(V_{\lambda+1})\}.$$

The role of Θ in $L(V_{\lambda+1})$ is exactly the same of its analogue in $L(\mathbb{R})$. It is used to quantify the "largeness" of a subset of $V_{\lambda+1}$. Indeed, while in the usual setting, under AC, to measure the largeness of a set one fix a bijection from this set to a cardinal or, equivalently, the order type of a well-ordering of the set, in the model $L(V_{\lambda+1})$ there is no Axiom of Choice, and hence one resorts to surjections instead of bijections, or, equivalently, to prewellorderings (briefly: pwo) instead of well-orders. Recall that a pwo is a binary relation which satisfies antireflexivity, transitivity, and such that every subset has a least element. It is easy to see that the preimage of a surjective function is a pwo. This creates a strong connection between subsets of $V_{\lambda+1}$ and ordinals in Θ, stated in the following proposition.

Proposition 6.1.11. *[Dim18, Lemma 5.6]*

1. *For every $\alpha<\Theta$, there exists in $L(V_{\lambda+1})$ a pwo with order type α, that is codeable as a subset of $V_{\lambda+1}$;*
2. *for every $Z\subseteq V_{\lambda+1}$, $Z\in L(V_{\lambda+1})$ there exists $\alpha<\Theta$ such that $Z\in L_\alpha(V_{\lambda+1})$.*

We now recall how to extend the notion of iterate for an embedding j witnessing that I0 holds at λ. Let

$$U=\{Z\in V_{\lambda+1}\mid j\restriction V_\lambda\in j(Z)\}$$

and let

$$j_U\colon L(V_{\lambda+1})\to \mathrm{Ult}(L(V_{\lambda+1}),U)$$

be the associated embedding, where $\mathrm{Ult}(L(V_{\lambda+1}),U)$ is the ultrapower of $L(V_{\lambda+1})$ by U. By [Woo11, Lemma 10] $\mathrm{Ult}(L(V_{\lambda+1}),U)$ is well-founded, j_U is an elementary embedding, and there is an elementary embedding $k_U\colon \mathrm{Ult}(L(V_{\lambda+1}),U)\to L(V_{\lambda+1})$ such that $j=k_U\circ j_U$. Thus, when $j=j_U$ we can use the notion of iteration previously defined and it is known that j is α-iterable for each α (apply [Woo11, Lemma 21] with $X=\emptyset$).

Definition 6.1.12 (Generic Absoluteness, [Cra17])**.** Suppose j witnesses that I0 holds at λ and j is iterable. Let $j_{0\omega}\colon L(V_{\lambda+1})\to M_\omega$ be the embedding into the ω-th iterate of $L(V_{\lambda+1})$ by j. We say that **generic absoluteness holds between** M_ω and $L_\alpha(V_{\lambda+1})$ if for some $\bar\alpha$ we have the following. Suppose $\mathbb{P}\in j_{0\omega}(V_\lambda)$, $x\in V$ is $\mathbb{P}$-generic over M_ω, and $(\mathrm{cof}(\lambda))^{M_\omega[x]}=\omega$. Then there is an elementary embedding $L_{\bar\alpha}(M_\omega[x]\cap V_{\lambda+1})\to L_\alpha(V_{\lambda+1})$ which is the identity below λ.

Recall that α is **good** if every element of $L_\alpha(V_{\lambda+1})$ is definable over $L_\alpha(V_{\lambda+1})$ from an element of $V_{\lambda+1}$. The good ordinals are cofinal in Θ (see [Lav01]).

Theorem 6.1.13. *[Cra17, Theorem 81] Suppose that I0 holds at λ as witnessed by j. Then for $\vec\lambda$ the critical sequence of j, if $\alpha<\Theta$ is good then for some $\bar\alpha<\lambda$ there is an elementary embedding $L_{\bar\alpha}(M_\omega[\vec\lambda]\cap V_{\lambda+1})\to L_\alpha(V_{\lambda+1})$ (which is the identity below λ).*

6.1.2 Generalized Descriptive Set Theory

In this section we introduce the basic definitions and results of generalized descriptive set theory that we will use in the sequel. The main references are [DMRon] and [AMR22]. We also highlight the connection between large cardinals and descriptive set theory.

We start from the generalization of the definition of a Polish space.

Definition 6.1.14. Let ν be an infinite cardinal. A topological space X is **ν-Polish** if it is completely metrizable and the least size of a well-ordered basis on X is $\leq \nu$.

When $\nu = \omega$ we obtain the classical notion of Polish space.

From now on, we denote by λ any singular cardinal of cofinality ω and by $\vec{\lambda} = (\lambda_i)_{i\in\omega}$ a strictly increasing cofinal sequence in λ.

In this work we focus on the following topological spaces:

1. the generalized Cantor space
$$^{\lambda}2$$
endowed with the bounded topology, i.e. the topology generated by the basic open sets
$$\boldsymbol{N}_s = \{x \in {}^{\lambda}2 \mid s \sqsubseteq x\},$$
for $s \in {}^{<\lambda}2$. It is homeomorphic to $X = \prod_{i\in\omega} {}^{\lambda_i}2$ equipped with the product of the discrete topologies on each ${}^{\lambda_i}2$. The metric d on X, defined by $d(x, y) = 0$ if $x = y$ and $d(x, y) = 2^{-n}$ with $n \in \omega$ the smallest such that $x \restriction \lambda_{n+1} \neq y \restriction \lambda_{n+1}$ if $x \neq y$, is complete.

2. The generalized Baire space
$$^{\omega}\lambda$$
endowed with the product of the discrete topologies on λ.

3. The space
$$C(\vec{\lambda}) = {}^{\omega}(\vec{\lambda}) = \prod_{i\in\omega} \lambda_i,$$
endowed with the product of the discrete topologies on λ_i.

4. The space
$$V_{\lambda+1}$$
endowed with the Woodin topology, i.e., the topology generated by the basic open sets
$$\boldsymbol{N}_{(\alpha,a)} = \{A \subseteq V_\lambda \mid A \cap V_\alpha = a\}$$
with $\alpha < \lambda$ and $a \subseteq V_\alpha$. We can endow $V_{\lambda+1}$ with the complete metric d defined by setting, for every $x, y \in V_{\lambda+1}$ distinct, $d(x, y) = 2^{-n}$ with $n \in \mathbb{N}$ the smallest such that $x \cap V_{\lambda_n} \neq y \cap V_{\lambda_n}$.

The space $C(\vec{\lambda})$ is λ-Polish. If in addition ${}^{<\lambda}2 = \lambda$, by Definition 6.1.14 it follows that ${}^{\lambda}2$ is λ-Polish, and if $\beth_\lambda = \lambda$ then $|V_\lambda| = \lambda$ and, again by Definition 6.1.14, $V_{\lambda+1}$ is λ-Polish as well. Moreover, under all these conditions on λ, one can show that all these spaces are homeomorphic.

Theorem 6.1.15. *[DMRon]*

(a) *For any strictly increasing sequence $(\lambda'_i)_{i\in\omega}$ of cardinals cofinal in λ,*
$$^{\omega}\lambda \approx \prod_{i\in\omega} \lambda'_i.$$
In particular ${}^{\omega}\lambda \approx C(\vec{\lambda})$.

(b) *If* ${}^{<\lambda}2 = \lambda$, *we further have*

$$^{\lambda}2 \approx {}^{\omega}\lambda \approx C(\vec{\lambda}).$$

(c) *If moreover* $|V_\lambda| = \lambda$, *then*

$$V_{\lambda+1} \approx {}^{\lambda}2 \approx {}^{\omega}\lambda \approx C(\vec{\lambda}).$$

One of the main goal of generalized descriptive set theory is the study of definable sets in λ-Polish spaces. Here, several hierarchies of formulæ figure in the analysis of definability: the descriptive set-theoretical, the effective one and the Lévy hierarchy.

Starting from the first, we recall the generalization of the usual notion of a Borel set, which corresponds to the case $\nu = \omega$.

Definition 6.1.16. Let X, Y be topological spaces and ν be an infinite cardinal. A set $B \subseteq X$ is ν^+**-Borel** if it belongs to the ν^+-algebra generated by the open sets of X. The collection of ν^+-Borel subsets of X is denoted by ν^+-$\mathbf{Bor}(X)$ or ν^+-$\boldsymbol{\Sigma}^1_0(X)$.

Thus ω_1-$\mathbf{Bor}(X)$ coincides with the collection of all classical Borel sets, i.e. it is the σ-algebra generated by the topology of X. We are interested in the case $\nu = \lambda$. Since λ is singular, the collection of λ^+-Borel subsets of X may equivalently be described as the smallest λ-algebra containing all open sets. For this reason, we drop the $+$ from the above terminology and notation and just speak e.g. of λ-Borel sets.

The λ^+-algebra of λ-$\mathbf{Bor}(X)$ can be naturally stratified as follows. Define by recursion on the ordinal $\alpha \geq 1$,

$$\begin{aligned}
&\lambda\text{-}\boldsymbol{\Sigma}^0_1(X) = \text{ open sets of } X,\\
&\lambda\text{-}\boldsymbol{\Sigma}^0_{\boldsymbol{\alpha}}(X) = \Big\{ \bigcup_{\alpha<\lambda} A_\alpha \mid X \setminus A_\alpha \in \lambda\text{-}\boldsymbol{\Sigma}^0_{\boldsymbol{\alpha'}}(X) \text{ for some } \alpha' < \alpha \Big\} \text{ for } \alpha > 1
\end{aligned}$$

and then set

$$\begin{aligned}
&\lambda\text{-}\boldsymbol{\Pi}^0_{\boldsymbol{\alpha}}(X) = \{A \subseteq X \mid X \setminus A \in \lambda\text{-}\boldsymbol{\Sigma}^0_{\boldsymbol{\alpha}}(X)\}\\
&\lambda\text{-}\boldsymbol{\Delta}^0_{\boldsymbol{\alpha}}(X) = \lambda\text{-}\boldsymbol{\Pi}^0_{\boldsymbol{\alpha}}(X) \cap \lambda\text{-}\boldsymbol{\Sigma}^0_{\boldsymbol{\alpha}}(X).
\end{aligned}$$

Arguing as in the classical case it is not hard to see that by AC the cardinal λ^+ is regular, and hence

$$\lambda\text{-}\mathbf{Bor}(X) = \bigcup_{1\leq\alpha<\lambda^+} \lambda\text{-}\boldsymbol{\Sigma}^0_{\boldsymbol{\alpha}}(X) = \bigcup_{1\leq\alpha<\lambda^+} \lambda\text{-}\boldsymbol{\Pi}^0_{\boldsymbol{\alpha}}(X) = \bigcup_{1\leq\alpha<\lambda^+} \lambda\text{-}\boldsymbol{\Delta}^0_{\boldsymbol{\alpha}}(X).$$

Definition 6.1.17. Let X be a λ-Polish space. A set $A \subseteq X$ is λ**-analytic** if it is a continuous image of some λ-Polish space Y .

The collection of all λ-analytic subsets of X is denoted by λ-$\boldsymbol{\Sigma}^1_1(X)$. As usual, when X is clear from the context we remove it from the notation above. We also set

$$\begin{aligned}
&\lambda\text{-}\boldsymbol{\Pi}^1_1(X) = \{A \subseteq X \mid X \setminus A \in \lambda\text{-}\boldsymbol{\Sigma}^1_1(X)\}\\
&\lambda\text{-}\boldsymbol{\Delta}^1_1(X) = \lambda\text{-}\boldsymbol{\Pi}^1_1(X) \cap \lambda\text{-}\boldsymbol{\Sigma}^1_1(X).
\end{aligned}$$

Sets in λ-$\boldsymbol{\Pi}^1_1(X)$ are called λ**-coanalytic**, while sets in λ-$\boldsymbol{\Delta}^1_1$ are called λ**-bianalytic**.

Definition 6.1.18. For $n \geq 1$, recursively define λ-$\boldsymbol{\Sigma}^1_{\boldsymbol{n+1}}(X)$ as follows. Let X be λ-Polish and $A \subseteq X$. Then $A \in \lambda$-$\boldsymbol{\Sigma}^1_{\boldsymbol{n+1}}(X)$ if and only if there is some λ-Polish space Y and a continuous function $f\colon Y \to X$ such that $A = f(B)$ for some $B \subseteq Y$ with $B \in \lambda$-$\boldsymbol{\Sigma}^1_{\boldsymbol{n}}(Y)$. We also let λ-$\boldsymbol{\Pi}^1_{\boldsymbol{n+1}}(X) = \{A \subseteq X \mid X \setminus A \in \lambda$-$\boldsymbol{\Sigma}^1_{\boldsymbol{n+1}}(X)\}$ and λ-$\boldsymbol{\Delta}^1_{\boldsymbol{n+1}}(X) = \lambda$-$\boldsymbol{\Pi}^1_{\boldsymbol{n+1}}(X) \cap \lambda$-$\boldsymbol{\Sigma}^1_{\boldsymbol{n+1}}(X)$.

One can show that $\bigcup_{n\geq 1} \lambda$-$\boldsymbol{\Sigma}^1_{\boldsymbol{n}}(X) = \bigcup_{n\geq 1} \lambda$-$\boldsymbol{\Pi}^1_{\boldsymbol{n}}(X) = \bigcup_{n\geq 1} \lambda$-$\boldsymbol{\Delta}^1_{\boldsymbol{n}}(X)$.

Definition 6.1.19. Let X be a λ-Polish space. A set $A \subseteq X$ is **λ-projective** if and only if $A \in \bigcup_{n \geq 1} \lambda\text{-}\boldsymbol{\Sigma}^1_n(X)$.

We now recall the definition of the Lévy hierarchy of formulæ.

Definition 6.1.20. For any first-order extension $\mathcal{L}_\in$ of the language of set theory, the **Lévy hierarchy** of formulæ of $\mathcal{L}_\in$ is formulated as follows. To simplify the notation we always avoid to write in any formula all the free variables.

A formula ϕ is Σ_0 (equivalently, Π_0 or Δ_0) if it belongs to the smallest collection of the atomic formulas of $\mathcal{L}_\in$ closed under negation, conjunction, disjunction and bounded quantification, i.e. quantifications of the form $\exists x \in y$ or $\forall x \in y$.

For $n \geq 0$, a formula ϕ is Σ_{n+1} if it is of the form $\exists y \psi$ where ψ is a Π_n-formula. A formula ϕ is Π_{n+1} if it is of the form $\forall y \psi$ where ψ is Σ_n-formula.

The following is a generalization of the **effective (lightface) hierarchy**.

Definition 6.1.21. Let $\mathcal{L}^2_\in$ be the language of set theory with first and second order variables. To simplify the notation we do not write also in this case the free variables of any formula.

A formula ϕ is Σ^1_0 if it does not contain second order quantifiers. For $n \geq 1$, ϕ is Σ^1_n (resp. Π^1_n) if it is of the form $\exists x_0 \forall x_1 \dots Q_{n-1} x_{n-1}\ \psi$ (resp. $\forall x_0 \exists x_1 \dots Q_{n-1} x_{n-1}\ \psi$), where ψ is Σ^1_0 and $x_0, \dots, x_{n-1}$ are second order variables.

The only model that we use to interpret such formulæ is $\langle V_\lambda, V_{\lambda+1} \rangle$. By an abuse of notation, we will therefore write $V_{\lambda+1} \models \phi$ instead of $\langle V_\lambda, V_{\lambda+1} \rangle \models \phi$.

Given $x \in V_{\lambda+1}$ and $n \geq 0$, we say that A is $\lambda\text{-}\Sigma^1_n(x)$ (resp. $\lambda\text{-}\Pi^1_n(x)$) if $A = \{y \in V_{\lambda+1} \mid V_{\lambda+1} \models \phi(x, y)\}$, where ϕ is a $\lambda\text{-}\Sigma^1_n$ (resp. $\lambda\text{-}\Pi^1_n$) formula.

In the case in which x is a finite set, say $x = \{x_0, \dots, x_n\}$, we write $\lambda\text{-}\Sigma^1_n(x_0, \dots, x_n)$ to mean $\lambda\text{-}\Sigma^1_n(\{x_0, \dots, x_n\})$.

Let X be a λ-Polish space definable with a parameter a in $V_{\lambda+1}$ and $x \in X$. For every $n \geq 0$, a set $A \subseteq X$ is $\lambda\text{-}\Sigma^1_n(a, x)$ (resp. $\lambda\text{-}\Pi^1_n(a, x)$) in X if $A = \{y \in V_{\lambda+1} \mid V_{\lambda+1} \models (y \in X \land \phi(x, y))\}$ is $\lambda\text{-}\Sigma^1_n(a, x)$ (resp. $\lambda\text{-}\Pi^1_n(a, x)$).

Notice that in the previous definition the complexity of A depends also on the complexity of X. Let A be the subset of X defined with the formula $\phi(a, x)$ as above. When $X \in \{V_{\lambda+1}, {}^\lambda 2, {}^\omega \lambda\}$ and $x = \emptyset$, then $a = \emptyset$ and we have that A is $\lambda\text{-}\Sigma^1_n$ (resp. $\lambda\text{-}\Pi^1_n$), while in the case $X = C(\vec{\lambda})$, in which one needs to fix a cofinal sequence $a = \vec{\lambda}$ in λ and use it as parameter, $A = \{y \in V_{\lambda+1} \mid V_{\lambda+1} \models (y \in C(\vec{\lambda}) \land \phi(y))\}$ and hence is $\lambda\text{-}\Sigma^1_n(\vec{\lambda})$ (resp. $\lambda\text{-}\Pi^1_n(\vec{\lambda})$). Therefore we do not have defined $\lambda\text{-}\Sigma^1_n$ (resp. $\lambda\text{-}\Pi^1_n$) subsets of $C(\vec{\lambda})$.

Definition 6.1.21 does not correspond to the exact generalization of the classical effective (lightface) hierarchy. Indeed, in the latter Σ^1_0 sets are exactly those which are arithmetical, while the concept of recursivity is not involved in the definition of $\lambda\text{-}\Sigma^1_0$ sets. However, many properties of the classical effective sets are preserved for sets of Definition 6.1.21, as the close connection between the effective hierarchy and the Lévy hierarchy. In the classical case it is shown that a set $A \subseteq V_{\omega+1}$ is Σ^1_2 if and only if there is it definable with a Σ_1-formula over the structure $(H_{\omega_1}, \in)$ of hereditarily countable sets (see [Jec03, Lemma 25.25]), and more in general Σ^1_{n+1} sets are exactly those that are Σ_n over $(H_{\omega_1}, \in)$. The generalization of this result for $\lambda\text{-}\Sigma^1_n$ sets is not trivial and strongly uses that λ is singular of cofinality ω and that $|V_\lambda| = \lambda$.

Under our assumptions, it is also possible to extend the tree representation of Σ^1_1 and Σ^1_2 sets of classical descriptive set theory to $\lambda\text{-}\Sigma^1_1$ and $\lambda\text{-}\Sigma^1_2$ sets of $V_{\lambda+1}$.

Definition 6.1.22. [Lav97, Section 1] Suppose that $\phi(x, y)$ is Σ^1_0, whose prenex form is

$$\forall a_0 \exists b_0 \forall a_1 \exists b_1 \dots \forall a_n \exists b_n \psi(x, y, a_0, b_0, \dots, a_n, b_n)$$

(so that ψ is quantifier-free), and fix a cofinal sequence $(\lambda_i)_{i \in \omega}$ in λ. We define the tree $T_{\phi(x,y)}$ with respect to $(\lambda_i)_{i \in \omega}$, that attemps to build $x^*, y^* \subseteq V_\lambda$ and Skolem functions $f_i(a_0, \dots, a_i)$ (for $0 \leq i \leq n$) witnessing that $\phi(x^*, y^*)$ holds. The m-th level of $T_{\phi(x,y)}$ is the set of (x_m, y_m, F, P) such that:

- $x_m, y_m \subseteq V_{\lambda_m}$;
- $F\colon (V_{\lambda_m})^{\leq n+1} \to V_{\lambda_m}$ is a partial function such that for all $d_0, \dots, d_n$ where F is defined then
$$\psi(x_m, y_m, d_0, F(d_0), d_1, F(d_0, d_1), \dots, d_n, F(d_0, \dots, d_n));$$
- $P\colon ((V_{\lambda_m})^{\leq n+1} \setminus \operatorname{dom} F) \to (\omega \setminus (m+1))$.

Intuitively, x_m and y_m approximate $x^* \cap V_{\lambda_m}$ and $y^* \cap V_{\lambda_m}$, F is an approximation of the Skolem function that would witness the first order part of ϕ, and P tells by which level of T the map F will be defined on the elements of V_{λ_m} that are not yet in $\operatorname{dom} F$.

We order $T_{\phi(x,y)}$ by setting $(x_m, y_m, F, P) < (x_{m'}, y_{m'}, F', P')$, where the first element in the m-th level and the second in the m'-th level, if $x_m \subseteq x_{m'}$, $y_m \subseteq y_{m'}$, $x_{m'} \cap V_{\lambda_m} = x_m$, $y_{m'} \cap V_{\lambda_m} = y_m$, $F \subseteq F'$, and if $P(\vec{d}) < m$ then $\vec{d} \in \operatorname{dom} F$, otherwise $P'(\vec{d}) = P(\vec{d})$.

Notice that if $T_{\phi(x,y)}$ has an infinite branch, the union of the x_m's and y_m's gives x and y, and the F's provide a Skolem function which is total because of the P's. One thus obtain the following result.

Proposition 6.1.23. *(Mostowski's tree representation, [Lav97, Theorem 1.1]) Let $\phi(x, y)$ be a Σ^1_0 formula and $y^* \subseteq V_\lambda$. Then $V_{\lambda+1} \models \exists x \phi(x, y^*)$ if and only if*
$$(T_{\phi(x,y)})_{y^*} = \{(x_m, y_m, F, P) \in T_{\phi(x,y)} \mid y_m \sqsubseteq y^*\}$$
has an infinite branch.

Both Definition 6.1.22 and Proposition 6.1.23 can be reformulated in the case of a Σ^1_0 formula $\phi(x_0, \dots, x_n, y_0, \dots, y_n)$ and parameters $y^*_0, \dots, y^*_n \subseteq V_\lambda$.

We now deal with λ-Σ^1_2 sets of $V_{\lambda+1}$.

Definition 6.1.24. [Lav97, Section 1] Let $\psi(z) \equiv \exists x \forall y \phi(x, y, z)$ be a Σ^1_2 formula. Fix a parameter $z^* \subseteq V_\lambda$. We define $(\mathcal{T}_{\phi(x,z)})_{z^*}$ as follows.

For each $m < \omega$ and $x_m \subseteq V_{\lambda_m}$, let
$$G_m(x_m) = \{(y_m, F, P) \mid (x_m, y_m, z \cap V_{\lambda_m}, F, P) \text{ is on the level } m \text{ of } (T_{\neg\phi(x,y,z)})_{z^*}\}$$
be the m-th level of $(T_{\neg\phi(x,y,z)})_{z^*}$. Then the m-th level of $(\mathcal{T}_{\phi(x,z)})_{z^*}$ is the set $\{(x_m, H) \mid x_m \subseteq V_{\lambda_m}, H\colon G_m(x_m) \to \lambda^+\}$. The order on $(\mathcal{T}_{\phi(x,z)})_{z^*}$ is given by setting $(x_m, H) < (x_{m'}, H')$, where the first element is in the m-th level and the second in the m'-th level, if $x_{m'} \cap V_{\lambda_m} = x_m$ and $H'(y_{m'}, F', P') < H(y_m, F, P)$ whenever $(x_m, y_m, z \cap V_{\lambda_m}, F, P) < (x_{m'}, y_{m'}, z \cap V_{\lambda_{m'}}, F', P')$ in $(T_{\neg\phi(x,y,z)})_{z^*}$.

Proposition 6.1.25. *(Shoenfield's tree representation, [Lav97, Theorem 1.1]) Let $\phi(x, y, z)$ be a Σ^1_0 formula and $z^* \subseteq V_\lambda$. Then $V_{\lambda+1} \models \exists x \forall y \phi(x, y, z^*)$ if and only if $(\mathcal{T}_{\phi(x,z)})_{z^*}$ has an infinite branch.*

If $(\mathcal{T}_{\phi(x,z)})_{z^*}$ has an infinite branch, and hence x^* is the union of the x_m's in the branch, then the H's assure that there are no possible infinite branches in $(T_{\neg\phi(x,y,z)})_{(x^*,z^*)}$ for every $y \subseteq V_\lambda$, because otherwise it would be possible to build a descending chain in λ^+.

Definition 6.1.26. [Lav97, Section 1] Let M be an inner model, with $V_\lambda \subseteq M$. For every $n \geq 1$, we say that M is Σ^1_n **correct** at λ if for any Σ^1_n formula φ and $x^* \subseteq V_\lambda$ with $x^* \in M$, $M \models \varphi(x^*)$ iff $V \models \varphi(x^*)$.

The following theorem gives us absoluteness for Σ^1_2 formulæ between V and any superstructure of the ω-th iterate of V by some elementary embedding j.

Theorem 6.1.27. *[Lav97, Theorem 1.4] If M_ω is the ω-th iterate of V by j, then $M_\omega[\langle \kappa_n \mid n < \omega \rangle]$ is Σ^1_2 correct at λ.*

Moreover, for every inner model N such that $M_\omega[\langle \kappa_n \mid n < \omega \rangle] \subseteq N \subseteq V$ we have that is Σ^1_2 correct at λ.

Let now $\prec,\succ$ be the Gödel pairing function. Recall that each $x \in {}^{\lambda}2$ can code a binary relation $E_x = \{(\alpha,\beta) \mid x(\prec\alpha,\beta\succ) = 0\}$ defined on λ. We then consider the structure $M_x = \langle \lambda, E_x\rangle$. If M_x is well-founded and extensional, we can apply the Collapsing Lemma to obtain a unique transitive collapse $\mathrm{tr}(M_x)$ and a unique isomorphism $\pi_x \colon M_x \to \mathrm{tr}(M_x)$.

Proposition 6.1.28. *Let λ be such that ${}^{<\lambda}2 = \lambda$. Then the set*

$$\mathrm{WF}_\lambda = \{x \in {}^{\lambda}2 \mid x \textit{ codes a well-founded relation on } \lambda\}$$

is λ-Π^1_1.

Proof. We follow the proof of [Jec03, Lemma 25.9]. Let $E_x = \{(\alpha,\beta) \mid x(\prec\alpha,\beta\succ) = 0\}$ be the binary relation on λ coded by $x \in {}^{\lambda}2$. Then E_x is well-founded if and only if there is no $z \colon \omega \to \lambda$ such that $z(\alpha+1) E_x z(\alpha)$ for all $\alpha < \omega$. Thus, since each map z is a subset of V_λ, we have

$$\begin{aligned} x \in \mathrm{WF} &\iff \forall z \in {}^{\omega}\lambda\ \exists \alpha < \lambda\ \neg\big(z(\alpha+1) E_x z(\alpha)\big) \\ &\iff \forall z \in {}^{\omega}\lambda\ \exists \alpha < \lambda\ \big(x(\prec\alpha+1,\alpha\succ)) \neq 0\big), \end{aligned}$$

which is expressed by a Π^1_1-formula[1]. □

We can now prove the following result.

Theorem 6.1.29. *A set $A \subseteq V_{\lambda+1}$ is λ-Σ^1_2 if and only if there is a Σ_1-formula ϕ such that $A = \{x \in V_{\lambda+1} \mid \langle H_{\lambda^+}, \in\rangle \models \phi(x)\}$.*

Proof. We mimic the proof of [Jec03, Lemma 25.25] replacing ω with λ.

First, suppose that $A = \{x \in V_{\lambda+1} \mid \langle H_{\lambda^+}, \in\rangle \models \phi(x)\}$, where ϕ is a Σ_1-formula, i.e. $\phi(x) \equiv \exists y \psi(x,y)$. Since ψ is Σ_0, it is absolute for transitive models, and hence in particular it is absolute for $\mathrm{trcl}(\{x,y\})$ which, by AC, has size λ. Then

$$\begin{aligned} x \in A &\iff (\exists \text{ a transitive set } M \text{ of size } \lambda)(\exists y \in M)(M \models \psi(x,y)) \\ &\iff (\exists \text{ well-founded extensional relation } E \text{ on } \lambda) \\ &\qquad \exists\alpha\exists\beta(\pi_E(\beta) = x \wedge \langle\lambda, E\rangle \models \psi(\alpha,\beta)), \end{aligned}$$

where π_E is the transitive collapse of $\langle \lambda, E\rangle$ to $\langle M, \in\rangle$. Now, recalling that each binary relation E on λ can be coded by an element $z \in {}^{\lambda}2 \subseteq V_{\lambda+1}$, obtaining the relation E_z, we have

$$\begin{aligned} x \in A \iff (\exists z \in {}^{\lambda}2)\big(&z \in \mathrm{WF}_\lambda \wedge M_z \models \text{extensionality} \\ &\wedge\ \exists\alpha\exists\beta(\pi_{E_z}(\beta) = x \wedge M_z \models \psi(\alpha,\beta))\big). \end{aligned} \tag{$\star$}$$

Using Proposition 6.1.28 it is now easy to see that $(\star)$ is Σ^1_2, and hence A is λ-Σ^1_2.

For the converse, suppose that A is λ-Σ^1_2, i.e. $A = \{x \in V_{\lambda+1} \mid V_{\lambda+1} \models \exists y \phi(x,y)\}$, where $\phi(x,y)$ is Π^1_1. Let $\Phi(x,M)$ be the formula expressing that M is a transitive model of size λ, $x, \lambda \in M$, $M \models \mathrm{cof}(\lambda) = \omega$, and M satisfies enough axioms to know that well-founded trees have a rank function, and $M \models \exists y \phi(x,y)$. In particular, this suffices to show that Proposition 6.1.23 holds in M.

If $x \in A$, using the Reflection Theorem, we obtain the existence of a transitive model M of size λ such that $\Phi(x,M)$ holds.

Vice versa, suppose there exists a transitive model M such that $\Phi(x,M)$ holds. Let $x^* \in M$ be such that $M \models \exists y \phi(x^*,y)$. Then by Proposition 6.1.23 there is a tree $T^M_{\psi(x^*)}$ in M, where $\psi(x^*) = \exists y \phi(x^*,y)$, such that $T^M_{\psi(x^*)}$ has an infinite branch in M if and only if $M \models \exists y \phi(x^*,y)$. But then $T^M_{\psi(x^*)}$ has an infinite branch in M, and since $T^M_{\psi(x^*)} \subseteq T^V_{\psi(x^*)}$, the same tree defined in V, $T^V_{\psi(x^*)}$ has an infinite branch and therefore $V_{\lambda+1} \models \exists y \phi(x^*,y)$, so $x \in A$.

We therefore proved that $x \in A$ if and only if exists M such that $\Phi(x,M)$ holds, where $\Phi(x,M)$ is a Σ_1-formula over $\langle H_{\lambda^+}, \in\rangle$, as desired. □

[1] Notice that z and α are respectively a second and first order variable.

Corollary 6.1.30. *A set A is λ-Σ^1_2 in V if and only if A is Σ_1-definable in V, i.e. there is a Σ_1-formula ϕ such that $A = \{x \in V \mid \phi(x)\}$.*

The correlation of the effective hierarchy with the Borel and projective hierarchies can instead be made through the expedient of relativization to parameters in $V_{\lambda+1}$.

Proposition 6.1.31. *Let $n \geq 0$. A set $A \subseteq V_{\lambda+1}$ is λ-$\boldsymbol{\Sigma^1_n}$ if and only if there exists $x \in V_{\lambda+1}$ such that A is λ-$\Sigma^1_n(x)$. Hence, λ-$\boldsymbol{\Sigma^1_n} = \bigcup_{x \in V_{\lambda+1}} \lambda$-$\Sigma^1_n(x)$.*

Proof. We argue by induction on n. Let $n = 0$ and $A \subseteq V_{\lambda+1}$ be λ-$\Sigma^1_0(x)$ for some $x \in V_{\lambda+1}$, i.e. $A = \{y \in V_{\lambda+1} \mid V_{\lambda+1} \models \phi(x, y)\}$, where $\phi(x, y)$ is Σ^1_0. Hence, in its prenex form $\phi(x, y)$ is

$$\forall x_0 \exists y_0 \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \psi(x, y, x_0, y_0, \ldots, x_n, y_n),$$

where $x_0, \ldots, x_n, y_0, \ldots, y_n$ are first order variables and ψ is a formula without any quantifiers, i.e. a combination of formulæ of the form "$x_i \in x_j$", "$x_i \in z$","$z \in x_i$", "$z \in y$" or "$y \in z$", where x_i, x_j are first order variables and z a second order variable. Let us consider

$$A_\psi = \{y \in V_{\lambda+1} \mid V_{\lambda+1} \models \psi(x, y, a_0, b_0, \ldots, a_n, b_n)\},$$

Then all the sets of the form $\{y \in V_{\lambda+1} \mid a_i \in a_j\}$, $\{y \in V_{\lambda+1} \mid a_i \in x\}$, $\{y \in V_{\lambda+1} \mid a_i \in y\}$ and $\{y \in V_{\lambda+1} \mid x \in a_i\}$ are either the whole space $V_{\lambda+1}$ or empty, while $\{y \in V_{\lambda+1} \mid y \in a_i\} = a_i = \bigcup_{a \in a_i}\{a\}$ and $\{y \in V_{\lambda+1} \mid y \in x\} = x = \bigcup_{a \in x}\{x\}$ are the union of $< \lambda$ closed sets, and $\{y \in V_{\lambda+1} \mid a_i \in y\} = \bigcup\{\mathbf{N}_{(\alpha,a)} \mid \alpha < \lambda, a_i \in a\}$, $\{y \in V_{\lambda+1} \mid x \in y\} = \bigcup\{\mathbf{N}_{(\alpha,a)} \mid \alpha < \lambda, x \in a\}$. Since A_ψ is the union/intersection of these sets, we have that A_ψ is a clopen set in $V_{\lambda+1}$ and hence $\phi(x)$ is a λ-$\boldsymbol{\Sigma^1_0}$, i.e. a Borel set.

For the converse, we first suppose that $A \subseteq V_{\lambda+1}$ is λ-$\boldsymbol{\Sigma^0_1}$. Then $A = \{y \in V_{\lambda+1} \mid V_{\lambda+1} \models \phi(x, y)\}$, with $\phi(x, y) \equiv \exists \alpha < \lambda (y \in \mathbf{N}_{(\alpha, x \cap V_\alpha)})$. Since ϕ is Σ^1_0 we have that A is λ-$\Sigma^1_0(x)$. The result for the whole class λ-$\boldsymbol{\Sigma^1_0}$ follows by a trivial induction.

Now suppose that the statement is true for $n > 0$. We show that it holds for $n + 1$ as well. Suppose that A is λ-$\Sigma^1_{n+1}(x)$ for some $x \in V_{\lambda+1}$, i.e. $A = \{y \in V_{\lambda+1} \mid V_{\lambda+1} \models \phi(x, y)\}$, where $\phi \equiv \exists x_0 \psi(x, y, x_0)$ with ψ a Π^1_n formula. Then the set $\{z_{(y,x_0)} \in V_{\lambda+1} \mid V_{\lambda+1} \models \psi(x, y, x_0)\}$ is λ-$\Pi^1_n(x)$, where each $z_{(y,x_0)}$ codes the pair (y, x_0), and hence by the inductive hypothesis it is λ-$\boldsymbol{\Pi^1_n}$. Being A the projection of this set, it follows that A is λ-$\boldsymbol{\Sigma^1_{n+1}}$. Symmetrically, one can show the other direction. □

The following definition generalizes the notions of isolated point and perfect space to arbitrary cardinals ν. As usual, setting $\nu = \omega$ one recovers (up to equivalence) the classical definitions.

Definition 6.1.32. [DMRon] Let X be a topological space and ν be an infinite cardinal. A point $x \in X$ is **ν-isolated** in X if there is an open neighborhood U of x with $|U| < \nu$. The space X is **ν-perfect** if it has no ν-isolated points. A **subspace** of X is **ν-perfect** (in X) if it is closed and ν-perfect as a subspace.

Recall that by topological embedding we mean a homeomorphism onto its image. The classical Perfect Set Property (briefly, PSP) for a set $A \subseteq X$ with X a Polish space states that either $|A| \leq \omega$ or there is a topological embedding of ${}^\omega 2$ into A. Since the Cantor space ${}^\omega 2$ is compact, it follows that the range of the topological embedding is necessarily closed in X. In the generalized setting, the latter condition does not hold and we need to require it.

Definition 6.1.33. [DMRon] Let λ be such that ${}^{<\lambda}2 = \lambda$ and X be λ-Polish. A set $A \subseteq X$ has the **λ-Perfect Set Property** (briefly, λ-PSP) if either $|A| \leq \lambda$, or ${}^\lambda 2$ topologically embeds into A as a closed-in-X set.

In the previous definition we can replace ${}^\lambda 2$ with one of ${}^\omega\lambda$ and $C(\vec{\lambda})$ since by Theorem 6.1.15(b) these spaces are all homeomorphic. Moreover, the second alternative of Definition 6.1.33 is equivalent to requiring that A contains a λ-perfect subspace of X by applying the following:

Theorem 6.1.34. *[DMRon] Let λ be such that ${}^{<\lambda}2 = \lambda$. Then ${}^{\lambda}2$ can be topologically embedded as a closed set into any nonempty λ-perfect λ-Polish space.*

We now generalize the basic notions of Baire category theory.

Definition 6.1.35. [DMRon] Let X be a topological space. We say that $A \subseteq X$ is **λ-meager** if it is a λ-union of nowhere dense sets. We say that A is **λ-comeager** if it is the complement of a λ-meager set, i.e., it contains a λ-intersection of open dense sets. We say that X is a **λ-Baire space** if every nonempty open set is not λ-meager. It is equivalent to say that the λ-intersection of open dense sets is dense.

In contrast with the classical case, the space $C(\vec{\lambda})$ is not λ-Baire with respect to the product topology.

Proposition 6.1.36. *[DMRSon] $C(\vec{\lambda})$ is the λ-union of nowhere dense sets.*

However one can consider another topology on $C(\vec{\lambda})$ which makes it λ-Baire. This topology is based on the diagonal Prikry forcing.

Definition 6.1.37. We call $\mathbb{P}_{\vec{\mathcal{U}}}$ the **diagonal Prikry forcing** on $\langle \lambda_n \mid n \in \omega \rangle$ with measures $\langle U_n \mid n \in \omega \rangle$, i.e. $p \in \mathbb{P}_{\vec{\mathcal{U}}}$ iff $p = (\alpha_0, \dots, \alpha_n, A_{n+1}, \dots)$ for some $n \in \omega$, with $\alpha_i \in \lambda_i$ for $0 \leq i \leq n$, and $A_j \in U_j$ for $j \geq n+1$. In this case, we call $s^p = \langle \alpha_0, \dots, \alpha_n \rangle$, $\mathrm{lh}(p) = \mathrm{lh}(s^p)$, and $A^p_j = A_j$. The sequence s^p is also called the **stem of** p. And we say that $p \leq q$ if:

- $\mathrm{lh}(p) \geq \mathrm{lh}(q)$;
- $s^q \sqsubseteq s^p$;
- for all i, $\mathrm{lh}(q) \leq i < \mathrm{lh}(p)$, $s^p(i) \in A^q_i$ and
- for all $j \geq \mathrm{lh}(p)$, $A^p_j \subseteq A^q_j$.

We say that $p \leq^* q$ if $p \leq q$ and $\mathrm{lh}(p) = \mathrm{lh}(q)$.

Definition 6.1.38. For any $p \in \mathbb{P}_{\vec{\mathcal{U}}}$, let $\mathbf{N}_p = \{x \in C(\vec{\lambda}) \mid \forall i < \mathrm{lh}(p)\ x(i) = s^p(i) \wedge \forall j \geq \mathrm{lh}(p)\ x(j) \in A^p_j\}$. The **Ellentuck-Prikry $\vec{\mathcal{U}}$-topology** on $C(\vec{\lambda})$ is the topology generated by $\{\mathbf{N}_p \mid p \in \mathbb{P}_{\vec{\mathcal{U}}}\} \cup \{\emptyset\}$. We can ignore the $\vec{\mathcal{U}}$ when it is clear from context, and we call the topology simply the EP topology.

An important combinatorial property of $\mathbb{P}_{\vec{\mathcal{U}}}$ is given by the following:

Theorem 6.1.39 (Strong Prikry condition)**.** *For any $D \subseteq \mathbb{P}_{\vec{\mathcal{U}}}$ open dense and for every $p \in \mathbb{P}_{\vec{\mathcal{U}}}$, there is a $q \leq^* p$ and an $n \in \omega$ such that for any $r \leq q$ of length at least n, $r \in D$.*

The proof is standard in the theory of Prikry-like forcings. For example, in this case it is an easy adaptation of the proof of Lemma 1.34 in [Git10].

Corollary 6.1.40. *For any $A \subseteq \mathbb{P}_{\vec{\mathcal{U}}}$ open and for every $p \in \mathbb{P}_{\vec{\mathcal{U}}}$, there is a $p^A \leq^* p$ such that if there is a $q \leq p^A$ with $q \in A$, then for any $r \leq p^A$ with stem as long as the stem of q, $r \in A$.*

Using this corollary and the strong Prikry condition, we can define a special element that we use later.

Definition 6.1.41. If $s \in \bigcup_{n \in \omega} \prod_{m \leq n} \lambda_m$, let $1_s = (s, \prod_{j \geq \mathrm{lh}(s)} \lambda_j) \in \mathbb{P}_{\vec{\mathcal{U}}}$. For any $A \subseteq \mathbb{P}_{\vec{\mathcal{U}}}$ open, let 1^A_s be as in the above corollary, i.e., 1^A_s has stem s and if there is a $q \leq 1^A_s$ with $q \in A$, then for any $r \leq 1^A_s$ with stem as long as the stem of q, $r \in A$.

Proposition 6.1.42 (λ-Baire Category)**.** *[DMRSon] The space $C(\vec{\lambda})$ endowed with the EP topology is a λ-Baire space. Moreover, every λ-comeager subset of $C(\vec{\lambda})$ contains a basic open set, and therefore a λ-perfect set.*

Definition 6.1.43. We say that $A \subseteq C(\vec{\lambda})$ has the **λ-Baire property** (briefly, λ-BP), if $A \,\Delta\, U$ is λ-meager for some open set $U \subseteq C(\vec{\lambda})$ (in the EP topology).

We now deal with Baire category theory in the product space $C(\vec{\lambda}) \times C(\vec{\lambda})$. In [DMRSon] it is shown that $C(\vec{\lambda}) \times C(\vec{\lambda})$, endowed with the product topology of the diagonal Prikry forcing by itself, is not $\mathfrak{c}$-Baire. One thus consider a slight different space: for any $x, y \in C(\vec{\lambda})$, set $x <^* y$ if and only if there exists $n < \omega$ such that $x(m) < y(m)$ for every $m > n$; we then define

$$C(\vec{\lambda}) \between C(\vec{\lambda}) = \{(x, y) \in C(\vec{\lambda}) \times C(\vec{\lambda}) \mid x <^* y \vee y <^* x\}.$$

Refer to [DMRSon] for the details.

Definition 6.1.44. Let U_n be a normal measure on λ_n for each $n \in \omega$, and consider on $C(\vec{\lambda}) \between C(\vec{\lambda})$ the **double diagonal Prikry forcing** $\mathbb{P}_{\vec{\mathcal{U}},2}$, which is defined as the set of sequences (k, p) of the form $(k, s^p, t^p, \langle A^p_i \mid i \in \omega, i \geq \mathrm{lh}(s^p)\rangle)$, where

- $k \in \{0, 1\}$;
- $s^p, t^p \in \bigcup_{n\in\omega} \prod_{m<n} \lambda_m$,
- $\mathrm{lh}(s^p) = \mathrm{lh}(t^p)$ and
- for every $i \in \omega$, $i \geq \mathrm{lh}(s^p)$, $A^p_i \in U_i$.

We call (s^p, t^p) the stem of p, and A^p_i is its i-th measure.

If $(k, p), (j, q) \in \mathbb{P}_{\vec{\mathcal{U}},2}$, we set $(k, p) \leq (j, q)$ iff

- $k = j$;
- $\mathrm{lh}(s^p) \geq \mathrm{lh}(s^q)$;
- $s^q \sqsubseteq s^p$ and $t^q \sqsubseteq t^p$;
- for all $i \geq \mathrm{lh}(p)$, $A^p_i \subseteq A^q_i$;
- for all i, $\mathrm{lh}(q) \leq i < \mathrm{lh}(p)$, $s^p(i), t^p(i) \in A^q_i$ and if $k = j = 0$ then $s^p(i) < t^p(i)$, otherwise $s^p(i) > t^p(i)$.

We say that $(k, p) \leq^* (j, q)$ if $(k, p) \leq (j, q)$ and $\mathrm{lh}(s^p) = \mathrm{lh}(s^q)$.

Definition 6.1.45. The **Ellentuck-Prikry product $\vec{\mathcal{U}}$-topology** on $C(\vec{\lambda}) \between C(\vec{\lambda})$, for short EP^2 topology, is the topology generated by the sets

$$\begin{aligned}\mathbf{N}_{(k,p)} =& \{(x, y) \in C(\vec{\lambda}) \between C(\vec{\lambda}) \mid s^p \sqsubseteq x, t^p \sqsubseteq y, \forall i \geq \mathrm{lh}(p)\big(x(i), y(i) \in A^p_i \\ & \wedge (k = 0 \to x(i) < y(i)) \wedge (k = 1 \to y(i) < x(i))\big)\}.\end{aligned}$$

One can prove that the strong Prikry property holds also for the double diagonal Prikry forcing.

Proposition 6.1.46. *(Strong Prikry property for $\mathbb{P}_{\vec{\mathcal{U}},2}$,[DMRSon]) For any D open dense set in $\mathbb{P}_{\vec{\mathcal{U}},2}$ and for any $(k, p) \in \mathbb{P}_{\vec{\mathcal{U}},2}$, there is a $(k, q) \leq^* (k, p)$ and there is a $n \in \omega$ such that for any $(k, r) \leq (k, q)$, $\mathrm{lh}(s^r) > n$, $r \in D$.*

By the Strong Prikry property, one can then prove the following.

Proposition 6.1.47. *[DMRSon] The space $C(\vec{\lambda}) \between C(\vec{\lambda})$ endowed with the EP^2 topology is a λ-Baire space. Moreover, every λ-comeager subset of $C(\vec{\lambda}) \between C(\vec{\lambda})$ contains a basic open set, and therefore a λ-perfect set.*

The next theorem is a generalization of the classical Kuratowski-Ulam theorem.

Theorem 6.1.48. *[DMRSon] For any $A \subseteq C(\vec{\lambda}) \between C(\vec{\lambda})$ with the λ-Baire property, A is λ-meager if and only if $\{x \in C(\vec{\lambda}) \mid A_{(0,x)}$ is λ-meager$\}$ is λ-comeager, if and only if $\{y \in C(\vec{\lambda}) \mid A_{(1,y)}$ is λ-meager$\}$ is λ-comeager.*

6.2 The λ-perfect set property

6.2.1 Limits of measurable cardinals

Our goal in this section is to prove that if we just assume the existence of an ω-strictly increasing sequence of measurable cardinals with limit λ then there exists an inner model with a strictly increasing ω-sequence of measurable cardinals and a Σ^1_2 set in it without the λ-PSP.

Theorem 6.2.1. *Let $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ be a strictly increasing sequence of measurable cardinals with limit λ, and let $\langle U_n \mid n < \omega \rangle$ be a sequence with the property that U_n is a normal ultrafilter on λ_n for all $n < \omega$. Assume that $V = L[\mathcal{U}]$, where*

$$\mathcal{U} = \{\langle n, A \rangle \mid n < \omega,\ A \in U_n\}.$$

If $\vec{\nu} = \langle \nu_n \mid n < \omega \rangle$ is a strictly increasing sequence of cardinals of uncountable cofinality with limit λ, then there exists $x \in H(\aleph_1)$ with the property that there is a $\Sigma^1_2(\vec{\nu}, x)$-subset of $C(\vec{\nu})$ of cardinality greater than λ that does not contain a λ-perfect subset.

Proof. Let $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ be a strictly increasing sequence of measurable cardinals with limit λ, let $\vec{\nu} = \langle \nu_n \mid n < \omega \rangle$ be a strictly increasing sequence of cardinals of uncountable cofinality with limit λ and let $\langle U_n \mid n < \omega \rangle$ be a sequence with the property that U_n is a normal ultrafilter on λ_n for all $n < \omega$ and $V = L[\mathcal{U}]$, where

$$\mathcal{U} = \{\langle n, A \rangle \mid n < \omega,\ A \in U_n\}.$$

We now build, using standard arguments about iterated measurable ultrapowers,

- a transitive class M,
- an elementary embedding $j : V \longrightarrow M$ with $j(\lambda) = \lambda$,
- a function $x : \omega \longrightarrow \omega$, and
- a sequence $\langle C_n \mid n < \omega \rangle$

such that the following statements hold for all $n < \omega$:

1. $j(\lambda_n) = \nu_{x(n)}$.
2. $\nu_{x(n+1)} > |\mathrm{H}(\nu_{x(n)})|$.
3. C_n is a closed unbounded subset of $\nu_{x(n)}$.
4. $j(U_n) = \{A \in M \cap \mathcal{P}(\nu_{x(n)}) \mid \exists \xi < \nu_{x(n)} \ \ C_n \setminus \xi \subseteq A\}$.

We start considering $\lambda_0^{\lambda_0}$. Since λ_1 is strong limit and $\vec{\lambda}$ has limit λ, we have that $\lambda_0^{\lambda_0} < \lambda_1 < \lambda$, and hence there exists $x(0) \in \omega$ such that $\lambda_0^{\lambda_0} < \nu_{x(0)}$. We now build the $\nu_{x(0)}$-th iterate $\langle M_{\nu_{x(0)}}, \in, U_{\nu_{x(0)}} \rangle$ of $\langle V, \in, U_0 \rangle$. By 4-3 of Proposition 6.1.4 it follows that $j^0_{0\nu_{x(0)}}(\lambda_0) = \nu_{x(0)}$, $C_0 = \{j^0_{0\alpha}(\lambda_0) \mid \alpha < \nu_{x(0)}\}$ is a closed unbounded subset of $\nu_{x(0)}$ and

$$U_{\nu_{x(0)}} = j^0_{0\nu_{x(0)}}(U_0) = \{A \in \mathcal{P}(\nu_{x(0)}) \mid \exists \xi < \nu_{x(0)} \ \ C_0 \setminus \xi \subseteq A\}.$$

At the $(n+1)$-th stage, we have already built the $\nu_{x(n)}$-th iterate $\langle M_{\nu_{x(n)}}, \in, U_{\nu_{x(n)}} \rangle$ and $i^n = j^n_{0\nu_{x(n)}} \circ \cdots \circ j^0_{0\nu_{x(0)}} : V \to M_{\nu_{x(n)}}$ such that

- $i^n(\lambda_n) = \nu_{x(n)}$, and
- $i^n(U_n) = U_{\nu_{x(n)}}$.

We now set $\mu_n = i^n(\lambda_{n+1})$, which is measurable in $M_{\nu_{x(n)}}$. Since $\lambda_n < \lambda_{n+1}$, by elementarity we have $\nu_{x(n)} < \mu_n$. Let's consider $\left(\mu_n^{\mu_n}\right)^{M_{\nu_{x(n)}}} < \lambda$. We then choose $x(n+1)$ as the smallest index such that $\nu_{x(n+1)} > \left(\mu_n^{\mu_n}\right)^{M_{\nu_{x(n)}}}$ and satisfies the condition that $\nu_{x(n+1)} > |\mathrm{H}(\nu_{x(n)})|$. Using the same argument as above, one then builds $j^{n+1}_{0\nu_{x(n+1)}}$, the $\nu_{x(n+1)}$-th iterate of $\langle M_{\nu_{x(n)}}, \in, U_{\nu_{x(n)}}\rangle$ via $i^n(U_{n+1})$ such that conditions 1-4 hold for $i^{n+1} = j^{n+1}_{0\nu_{x(n+1)}} \circ \cdots \circ j^0_{0\nu_{x(0)}}$ instead of j.

Then to obtain the desired objects it is enough to consider the direct limit j of the $j^n_{0\nu_{x(n)}}$'s. We have that $\nu_{x(n)} < \mu_n$ for every $n \in \omega$, and thus $\cdots \circ j^{n+2}_{0\nu_{x(n+2)}} \circ j^{n+1}_{0\nu_{x(n+1)}}(\nu_{x(n)}) = \nu_{x(n)}$, $j(\lambda_n) = \cdots \circ j^{n+2}_{0\nu_{x(n+2)}} \circ j^{n+1}_{0\nu_{x(n+1)}}(i^n(\lambda_n)) = \nu_{x(n)}$. The same holds for $j(U_n)$.

Now, set $\mathcal{V} = j(\mathcal{U})$ and define $\mathcal{N}$ to be the class of all pairs $\langle N, \vec{F}\rangle$ with the property that N is a transitive set of cardinality λ, $\vec{F} = \langle F_n \mid n < \omega\rangle$ is a sequence of length ω and there exists a sequence $\langle D_n \mid n < \omega\rangle$ such that the following statements hold:

(a) D_n is a closed unbounded subset of $\nu_{x(n)}$ for all $n < \omega$.

(b) If $n < \omega$, then F_n is an element of N, $\nu_{x(n)}$ is a regular cardinal in N and F_n is a normal ultrafilter in $\nu_{x(n)}$ in N.

(c) If $n < \omega$, then $F_n = \{A \in N \cap \mathcal{P}(\nu_{x(n)}) \mid \exists \xi < \nu_{x(n)}\ D_n \setminus \xi \subseteq A\}$.

(d) If $\mathcal{F} = \{\langle n, A\rangle \mid n < \omega,\ A \in F_n\}$, then $\mathcal{F} \in N$ and $N = L_{N\cap\mathrm{Ord}}[\mathcal{F}]$.

It is easy to see that the class $\mathcal{N}$ is definable by a Σ_1-formula with parameters $\vec{\nu}$ and x. Moreover, our assumptions ensures that for every $A \in M \cap \mathcal{P}(\lambda)$, there exists $\alpha < \lambda^+$ with $A \in L_\alpha[\mathcal{V}]$ and $\langle L_\alpha[\mathcal{V}], \langle j(U_n) \mid n < \omega\rangle\rangle \in \mathcal{N}$.

Claim 6.2.1.1. If $\langle N, \langle F_n \mid n < \omega\rangle\rangle \in \mathcal{N}$ and $\mathcal{F} = \{\langle n, A\rangle \mid n < \omega,\ A \in F_n\}$, then we have $\mathcal{F} \cap N = \mathcal{V} \cap L_{N\cap\mathrm{Ord}}[\mathcal{V}]$ and $N = L_{N\cap\mathrm{Ord}}[\mathcal{V}]$.

Proof of the Claim. Fix $\langle N, \langle F_n \mid n < \omega\rangle\rangle \in \mathcal{N}$, and let $\langle D_n \mid n < \omega\rangle$ be a sequence that witness that $\langle N, \langle F_n \mid n < \omega\rangle\rangle$ is contained in $\mathcal{N}$. Set $\gamma = N \cap \mathrm{Ord}$. By induction, we now show that $\mathcal{F} \cap L_\beta[\mathcal{F}] = \mathcal{V} \cap L_\beta[\mathcal{V}]$ holds for all $\beta \leq \gamma$. Hence, assume that $\beta \leq \gamma$ with $\mathcal{F} \cap L_\alpha[\mathcal{F}] = \mathcal{V} \cap L_\alpha[\mathcal{V}]$ for all $\alpha < \beta$. Then $L_\beta[\mathcal{F}] = L_\beta[\mathcal{V}]$. Pick $n < \omega$ and $A \in F_n$ with $\langle n, A\rangle \in L_\beta[\mathcal{F}]$. Then there exists $\xi < \nu_{x(n)}$ with $D_n \setminus \xi \subseteq A$. Since $C_n \cap D_n$ is unbounded in $\nu_{x(n)}$, we know that $A \cap C_n$ is unbounded in $\nu_{x(n)}$ and hence there is no $\zeta < \nu_{x(n)}$ with the property that $C_n \setminus \zeta \subseteq \lambda \setminus A$. In this situation, the fact that $j(U_n)$ is an ultrafilter on $\nu_{x(n)}$ in $L[\mathcal{V}]$ implies that $A \in j(U_n)$ and hence $\langle n, A\rangle \in j(\mathcal{U}) \cap L_\beta[\mathcal{V}]$. The dual argument then shows that we also have $\mathcal{V} \cap L_\beta[\mathcal{V}] \subseteq \mathcal{F} \cap L_\beta[\mathcal{F}]$. This completes the induction and we know that $\mathcal{F} \cap N = \mathcal{V} \cap L_\gamma[\mathcal{V}]$. This allows us to conclude that $N = L_\gamma[\mathcal{F}] = L_\gamma[\mathcal{F} \cap N] = L_\gamma[\mathcal{V} \cap L_\gamma[\mathcal{V}]] = L_\gamma[\mathcal{V}]$. □

Given a subset y of λ, let $\lhd_y$ denote the binary relation on λ defined by

$$\alpha \lhd_y \beta \iff \prec\alpha, \beta\succ \in y$$

for all $\alpha, \beta < \lambda$.[2] In addition, we define $\mathcal{WO}$ denote the collection of all subsets y of λ with the property that $\lhd_y$ is a well-ordering of λ. Note that, since $\lambda^+ = (\lambda^+)^M$, we know that for every $\lambda \leq \gamma < \lambda^+$, there exists $y \in M \cap \mathcal{WO}$ with the property that $\langle \lambda, \lhd_y\rangle$ has order-type γ and we let y_γ denote the $<_{L[\mathcal{V}]}$-least subset of λ with this property. The above claim then directly implies that the subset

$$Y \ = \ \{y_\gamma \mid \lambda \leq \gamma < \lambda^+\}$$

of $\mathcal{P}(\lambda)$ is definable by a Σ_1-formula with parameters $\vec{\nu}$ and x.

Next, we let $\vec{b}$ denote the $<_{L[\mathcal{V}]}$-least sequence $\langle b_\alpha \mid \alpha < \lambda\rangle$ in M with the property that

$$M \cap \mathcal{P}(\nu_{x(n)}) \ = \ \{b_\alpha \mid \alpha \in \mathrm{Lim} \cap \nu_{x(n+1)}\}$$

[2] Here, we let $\prec\cdot,\cdot\succ : \mathrm{Ord} \longrightarrow \mathrm{Ord}$ denote the Gödel pairing function.

holds all for $n < \omega$. The above claim allows us to define $\vec{b}$ as the unique sequence of length λ with the property that there exists $\langle N, \langle F_n \mid n < \omega \rangle\rangle$ in $\mathcal{N}$ and $F = \{\langle n, A\rangle \mid n < \omega, A \in F_n\}$ such that $\vec{b}$ is an element of $N = L_{N\cap\mathrm{Ord}}[\mathcal{V}]$ and, in N, this sequence is $<_{L_{N\cap\mathrm{Ord}}[\mathcal{V}]}$-least with the property stated in the above equation (this is possible because N contains all bounded subsets of λ that are contained in M). Since $\mathcal{N}$ is definable by a Σ_1-formula with parameters $\vec{\nu}$ and x, then $\{\vec{b}\}$ is definable by a Σ_1-formula with parameters $\vec{\nu}$ and x as well.

Given $\lambda \leq \gamma < \lambda^+$, we let z_γ denote the unique element of $C(\vec{\nu})$ with the property that the following statements hold:

- If $n < \omega$, then $z_\gamma(x(n+1))$ is the minimal limit ordinal below $\nu_{x(n+1)}$ with the property that $y_\gamma \cap \nu_{x(n)} = b_{z_\gamma(x(n+1))}$ holds.
- If k is a natural number that is not of the form $x(n+1)$ for some $n < \omega$, then $z_\gamma(k) = 0$.

Our earlier observations then show that the set

$$Z \;=\; \{z_\gamma \mid \lambda \leq \gamma < \lambda^+\}$$

is definable by a Σ_1-formula with parameters $\vec{\nu}$ and x. By Corollary 6.1.30, this shows that Z is a $\Sigma^1_2(\vec{\nu}, x)$-subset of $C(\vec{\nu})$.

Claim 6.2.1.2. The set Z does not contain a λ-perfect subset.

Proof of the Claim. Set $\mu_n = \nu_{x(n)}$ for all $n < \omega$. Moreover, let $f : {}^\omega\lambda \longrightarrow {}^\omega\lambda$ denote the unique function satisfying $f(y)(n) = y(x(n+1))$ for all $y \in {}^\omega\lambda$ and $n < \omega$. Then f is continuous and $f \restriction Z$ is an injection. Fix an enumeration $\vec{a} = \langle a_\alpha \mid \alpha < \lambda\rangle$ of $\mathrm{H}(\lambda)$ with the property that $a_\alpha = b_\alpha$ holds for all $\alpha \in \mathrm{Lim} \cap \lambda$, and define WO to consist of all $w \in {}^\omega\lambda$ with the property that there exists $y \in \mathcal{WO}$ such that $y \cap \mu_n = a_{w(n)}$ holds for all $n < \omega$. Given $\lambda \leq \gamma < \lambda^+$, we then have

$$a_{f(z_\gamma)(n)} \;=\; a_{z_\gamma(x(n+1))} \;=\; b_{z_\gamma(x(n+1))} \;=\; y_\gamma \cap \nu_{x(n)} \;=\; y_\gamma \cap \mu_n$$

and therefore y_γ witnesses that $f(z_\gamma)$ is an element of WO.

Now, assume, towards a contradiction, that Z contains a λ-perfect subset. Noticing that $C(\vec{\nu})$ is a λ-Polish space, by applying Theorem 6.1.34 we obtain the existence of a $\mathbf{\Sigma}^1_1$-subset P of $C(\vec{\nu})$ of cardinality 2^λ that is a subset of Z. The above computations then show that $f[P]$ is a $\mathbf{\Sigma}^1_1$-subset of ${}^\omega\lambda$ that is a subset of WO. Using [LM21, Lemma 4.5], we can now find an ordinal $\beta < \lambda^+$ with the property that that for every element w of $f[P]$ the corresponding well-ordering

$$\langle \lambda, \lhd_{\bigcup\{a_{w(n)} \mid n<\omega\}}\rangle$$

has order-type less than β. Since P has cardinality greater than λ, there exists $\beta \leq \gamma < \lambda^+$ with the property that $z_\gamma \in P$. But then $\bigcup\{a_{f(z_\gamma)(n)} \mid n < \omega\} = y_\gamma$ and the well-ordering $\langle \lambda, \lhd_{y_\gamma}\rangle$ has order-type greater than β, a contradiction. □

Since the set Z has cardinality λ^+, this completes the proof of the theorem. □

6.2.2 The λ-perfect set property for λ-Σ^1_2 sets

In this section we establish one of the main results of chapter 6, regarding the λ-PSP for λ-$\Sigma^1_2(\vec{\lambda})$ subsets of $C(\vec{\lambda})$. To this aim we first prove that if a tree belongs to an inner model M which is "large enough", then also the projection of its body is contained in M.

Lemma 6.2.2. *Let $\vec{\lambda} = \langle \lambda_n \mid n < \omega\rangle$ be a strictly increasing sequence of infinite cardinals with limit λ, let $\zeta > 0$ be an ordinal and let T be a subtree of ${}^{<\omega}\lambda \times {}^{<\omega}\zeta$ with the property that $p[T]$ does not contain a λ-perfect subset. If M is an inner model that contains $\vec{\lambda}$, T and V_λ, then $p[T] \subseteq M$.*

Proof. Given a subtree S of ${}^{<\omega}\lambda \times {}^{<\omega}\zeta$, we define S' to be the set of all $\langle t, u\rangle \in S$ with the property that for all $n < \omega$, there exists $\mathrm{dom}(t) < i < \omega$ such that the set

$$\{v \in {}^{i}\lambda \mid \exists w \in {}^{i}\zeta \ [t \subseteq v \wedge u \subseteq w \wedge \langle v, w\rangle \in S]\}$$

has cardinality at least λ_n. Then it is easy to see that for every such subtree S, the set S' is again a subtree of ${}^{<\omega}\lambda \times {}^{<\omega}\zeta$ with $S' \subseteq S$ and, if S is an element of M, then S' is also contained M. Now, let $\langle T_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ denote the unique sequence of subtrees of ${}^{<\omega}\lambda \times {}^{<\omega}\zeta$ with $T_0 = T$, $T_{\alpha+1} = T'_\alpha$ for all $\alpha \in \mathrm{Ord}$ and $T_\beta = \bigcap_{\alpha<\beta} T_\alpha$ for all $\beta \in \mathrm{Lim}$. Then it is easy to see that $T_\alpha \in M$ holds for all $\alpha \in \mathrm{Ord}$. Moreover, there exists $\alpha_* \in \mathrm{Ord}$ with $T_{\alpha_*} = T_\beta$ for all $\alpha_* \leq \beta \in \mathrm{Ord}$. Set $T_* = T_{\alpha_*}$.

Claim 6.2.2.1. $T_* = \emptyset$.

Proof of the Claim. Assume, towards a contradiction, that $T_* \neq \emptyset$. Let $S_{\vec{\lambda}}$ denote the subtree of ${}^{<\omega}\lambda$ consisting of all $s \in {}^{<\omega}\lambda$ with $s(i) < \lambda_i$ for all $i \in \mathrm{dom}(s)$. We inductively construct a system $\langle\langle s_u, t_u\rangle \in T_* \mid u \in S_{\vec{\lambda}}\rangle$ such that the following statements hold for all $u, v \in S_{\vec{\lambda}}$:

- If $u \subsetneq v$, then $s_u \subsetneq s_v$ and $t_u \subsetneq t_v$.
- If $\alpha < \beta < \lambda_{\mathrm{dom}(u)}$, then $\mathrm{dom}(s_{u^\frown\langle\alpha\rangle}) = \mathrm{dom}(s_{u^\frown\langle\beta\rangle})$ and $s_{u^\frown\langle\alpha\rangle} \neq s_{u^\frown\langle\beta\rangle}$.

First, define $s_\emptyset = t_\emptyset = \emptyset$. Now, assume that $u \in S_{\vec{\lambda}}$ and $\langle s_u, t_u\rangle \in T_*$ is already constructed. Since $\langle s_u, t_u\rangle \in T'_* = T_*$, we can find $\mathrm{dom}(s_u) < i < \omega$ and a sequence $\langle\langle s_\xi, t_\xi\rangle \in T_* \mid \xi < \lambda_{\mathrm{dom}(u)}\rangle$ with the property that for all $\xi < \rho < \lambda_{\mathrm{dom}(u)}$, we have $\mathrm{dom}(s_\xi) = \mathrm{dom}(s_\rho) = i$ and $s_\xi \neq s_\rho$. Given $\xi < \lambda_{\mathrm{dom}(u)}$, we then define $s_{u^\frown\langle\xi\rangle} = s_\xi$ and $t_{u^\frown\langle\xi\rangle} = t_\xi$. It then directly follows that the constructed sets satisfy all required properties. This completes the inductive construction of our system. If we now define

$$\pi : \prod_{i<\omega} \lambda_i \longrightarrow {}^{\omega}\lambda; \ x \longmapsto \bigcup\{s_{x\restriction i} \mid i < \omega\},$$

then our setup ensures that that π is a continuous injection. Moreover, we have that for all $x \in \prod_{i<\omega} \lambda_i$, $\langle \pi(x), \bigcup\{t_{x\restriction i} \mid i < \omega\}\rangle \in [T]$ and this shows that $\mathrm{ran}(\pi)$ is a subset of $p[T]$, contradicting our assumptions on T. □

Now, fix $\langle x, y\rangle \in [T]$. Then there is an $\alpha < \alpha_*$ with $\langle x, y\rangle \in [T_\alpha] \setminus [T_{\alpha+1}]$ and we can find $k < \omega$ with the property that $\langle x \restriction k, y \restriction k\rangle \notin T_{\alpha+1} = T'_\alpha$. Hence, there is $n < \omega$ with the property that for all $k < i < \omega$, the set

$$E_i \ = \ \{s \in {}^{i}\lambda \mid \exists t \in {}^{i}\zeta \ [x \restriction k \subseteq s \wedge y \restriction k \subseteq t \wedge \langle s, t\rangle \in T_\alpha]\}$$

has cardinality less than λ_n. Note that for all $k < i < \omega$, we have $x \restriction i \in E_i$. Moreover, since M contains the sequence $\langle E_i \mid k < i < \omega\rangle$ and each E_i has cardinality less than λ_n in M, we can find a sequence $\langle \tau : \lambda_n \longrightarrow E_i \mid k < i < \omega\rangle$ of surjections that is an element of M. If we pick $z \in {}^{\omega}\lambda_n$ with $\tau(z(i)) = x \restriction i$ for all $k < i < \omega$, then the fact that $V_\lambda \in M$ ensures that z is an element of M and hence we can conclude that x is also contained in M. □

The next theorem was shown by Laver and establishes that if M_ω is the ω-th iterate of V by an I2-elementary embedding j, then every transitive model N such that $M_\omega \subseteq N \subseteq V$ and $N \models \mathrm{cof}(\lambda) = \omega$ is Σ^1_2-correct at λ. We reformulate it to highlight the property that some λ-$\Sigma^1_2(\vec{\lambda})$ sets can be built as the projection in V of trees defined in N.

Theorem 6.2.3 ([Lav97, Theorem 1.4])**.** *Let $\vec{\lambda}$ be a strictly increasing sequence of infinite cardinals with limit λ, let j be an I2-elementary embedding with critical sequence $\vec{\lambda}$ and let x^* be an element of $V_{\lambda+1}$. If A is a $\Sigma^1_2(\vec{\lambda}, x^*)$-subset of $C(\vec{\lambda})$ and N is an inner model with $M_\omega \cup \{\lambda, x^*\} \subseteq N$, then there exist an ordinal ζ and a subtree T of ${}^{<\omega}\lambda \times {}^{<\omega}\zeta$ in N with the property that $p[T] = A$.*

Proof. Let $\psi(x^*) = \exists z \forall y \phi(z, y, x^*)$ be the $\Sigma^1_2(\vec{\lambda}, x^*)$ formula defining A. By Proposition 6.1.25 we have that

$$V \models \psi(x^*) \leftrightarrow \exists w \ (w, H) \in [(\mathcal{T}^V_{\phi(z,x)})_{x^*}]$$

and

$$N \models \psi(x^*) \leftrightarrow \exists w \ (w, H) \in [(\mathcal{T}^N_{\phi(z,x)})_{x^*}],$$

for some trees $\mathcal{T}^V_{\phi(z,x)} \subseteq V$ and $\mathcal{T}^N_{\phi(z,x)} \subseteq N$. We now show that $(p[(\mathcal{T}^V_{\phi(z,x)})_{x^*}])^V = (p[(\mathcal{T}^N_{\phi(z,x)})_{x^*}])^V$ for every $x^* \in N$, whence it follows that $A = (p[(\mathcal{T}^N_{\phi(z,x)})_{x^*}])^V$.

Notice that by Definition 6.1.22, for each $x^* \in V_{\lambda+1}$ such that $x^* \in N$ we have $(\mathcal{T}^N_{\phi(z,x)})_{x^*} \subseteq (\mathcal{T}^V_{\phi(z,x)})_{x^*}$, and thus $(p[(\mathcal{T}^N_{\phi(z,x)})_{x^*}])^V \subseteq (p[(\mathcal{T}^V_{\phi(z,x)})_{x^*}])^V$. For the converse, suppose that $N \models \neg\psi(x^*)$. Then by Proposition 6.1.25 it follows that $(\mathcal{T}^N_{\phi(z,x)})_{x^*}$ is well founded. We can thus consider the rank function $\rho\colon (\mathcal{T}^N_{\phi(z,x)})_{x^*} \to \mathrm{Ord}$ in N that witnesses it. Since $j \circ H \in N$ for each $H\colon c \to \mathrm{Ord}$ with $c \in V_\lambda$, and $j(\lambda^+) = \lambda^+$, we can define in V the map $G\colon (\mathcal{T}^V_{\phi(z,x)})_{x^*} \to (\mathcal{T}^N_{\phi(z,x)})_{x^*}$ by $G(a, H) = (a, j \circ H)$, and G is strict order-preserving. Hence $(\mathcal{T}^V_{\phi(z,x)})_{x^*}$ is well-founded as well and by Proposition 6.1.25 we get $V \models \neg\psi(x^*)$. We also obtain from the map G that $(p[(\mathcal{T}^V_{\phi(z,x)})_{x^*}])^V \subseteq (p[(\mathcal{T}^N_{\phi(z,x)})_{x^*}])^V$, as desired. □

We are now ready to prove the main result of this section.

Corollary 6.2.4. *Let $\vec{\lambda}$ be a strictly increasing sequence of infinite cardinals with limit λ, let j be an I2-elementary embedding with critical sequence $\vec{\lambda}$ and let x be an element of $V_{\lambda+1}$ with $(2^\lambda)^{M_\omega[\vec{\lambda},x]} < \lambda^+$. If A is a $\Sigma^1_2(\vec{\lambda}, x)$ subset of $C(\vec{\lambda})$ of cardinality greater than λ, then A contains a $\vec{\lambda}$-perfect subset.*

Proof. By Theorem 6.2.3 there exist an ordinal ζ and a tree $T \subseteq {}^{<\omega}\lambda \times {}^{<\omega}\zeta$ in $M_\omega[\vec{\lambda}, x]$ such that $A = p[T]$. Toward a contradiction, suppose that A does not contain any λ-perfect set. Then by Lemma 6.2.2 we have that $p[T] \subseteq M_\omega[\vec{\lambda}, x]$, and by our assumption that $(2^\lambda)^{M_\omega[\vec{\lambda},x]} < \lambda^+$ we obtain that $|A| = |p[T]| \leq \lambda$, a contradiction. □

Corollary 6.2.5. *Let $\vec{\lambda}$ be a strictly increasing sequence of infinite cardinals with limit λ, let j be an I2-elementary embedding with critical sequence $\vec{\lambda}$. If A is a $\Sigma^1_2(\vec{\lambda})$ subset of $C(\vec{\lambda})$ of cardinality greater than λ, then A contains a λ-perfect subset.*

Proof. Notice that $(2^\lambda)^{M_\omega[\vec{\lambda}]} < \lambda^+$. Then it is enough to apply Corollary 6.2.4 with $x = \emptyset$. □

We conclude this section showing that it is consistent that I2 does not suffice to guarantee that also the λ-$\mathbf{\Sigma^1_2}$ subsets of $C(\vec{\lambda})$ have the λ-PSP.

Corollary 6.2.6. *Let $\vec{\lambda}$ be a strictly increasing sequence of infinite cardinals with limit λ, let j be an I2-elementary embedding with critical sequence $\vec{\lambda}$ and let E be a subset of λ such that V_λ is a subset of $L[E]$ and $L[E]$ contains the sequence $\vec{\lambda}$ and the restriction of j to V_λ. Then the following statements hold true in $L[E]$:*

(1) *There is an I2-elementary embedding with critical sequence $\vec{\lambda}$.*

(2) *There is a subset A of $C(\vec{\lambda})$ which is λ-$\mathbf{\Sigma^1_2}$ and does not have the λ-PSP.*

Proof. (1) It easily follows from the fact that $V_\lambda \subseteq L[E]$.

(2) In $L[E]$, using the same argument of the proof of Theorem 6.2.1, one can build a set Z which is a λ-$\Sigma^1_2(E)$-subset of $C(\vec{\lambda})$, and hence a λ-$\mathbf{\Sigma^1_2}$-set, and does not have the λ-PSP. □

6.3 The λ-Baire property

6.3.1 The λ-Baire property for λ-Σ^1_2 sets

In this section we analyse the λ-Baire property of subsets for $C(\vec{\lambda})$ when λ is the limit of a strictly increasing sequence $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ of measurable cardinals.

We first adapt to our set-up a very standard result which is useful in the sequel.

Theorem 6.3.1 (Fuchs). *Let $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ be a strictly increasing sequence of measurable cardinals with limit λ, and let $\vec{\mathcal{U}} = \langle U_n \mid n < \omega \rangle$ be a sequence with the property that U_n is a normal ultrafilter on λ_n for all $n < \omega$. Let M be some transitive model of ZFC$^-$ and $\vec{\mathcal{U}} \subseteq M$. A sequence $x \in C(\vec{\lambda})$ is $\mathbb{P}^M_{\vec{\mathcal{U}}}$-generic if and only if for every sequence $< A_n \in U_n \mid n \in \omega >$ in M the set $\{n \in \omega \mid x(n) \notin A_n\}$ is finite.*

Proof. $\Rightarrow$) Let $x \in C(\vec{\lambda})$ be $\mathbb{P}^M_{\vec{\mathcal{U}}}$-generic and consider a sequence $< A_n \in U_n \mid n \in \omega >$ in M. Define

$$U = \{(s, (A'_n)_{n \geq N}) \mid N \in \omega, s \in \prod_{n<N} \lambda_n, A'_n \subseteq A_n, A'_n \in U_n\}.$$

We claim that U is dense in $\mathbb{P}^M_{\vec{\mathcal{U}}}$. Let $p = (\alpha_0, \dots, \alpha_{N-1}, A^p_N, \dots) \in \mathbb{P}^M_{\vec{\mathcal{U}}}$, with $N \geq 1$, and define $p' \in \mathbb{P}^M_{\vec{\mathcal{U}}}$ with $\mathrm{stem}(p') = \mathrm{stem}(p)$ and $A^{p'}_n = A^p_n \cap A_n \in U_n$ for every $n \geq N$. Then $p' \leq p$ and $p' \in U$, so U is dense.

Let now $G_x = \{p \in \mathbb{P}^M_{\vec{\mathcal{U}}} \mid x \in N_p \cap M\}$ be the filter induced by x. By genericity it follows that $G_x \cap U \neq \emptyset$. Take $p \in G_x \cap U$. Then $x \in N_p$, and so $x(n) \in A_n$ for every $n \geq |\mathrm{stem}(p)|$. Thus, the set $\{n \in \omega \mid x(n) \notin A_n\}$ is finite.

$\Leftarrow$) Let $x \in C(\vec{\lambda})$ be a sequence satisfying the property that for every sequence $\langle A_n \in U_n \mid n < \omega \rangle \subseteq M$, $x(n) \in A_n$ for all but finitely many n's. Let $U \in M \cap \mathbb{P}_{\vec{\mathcal{U}}}$ be a dense open set. Given $N < \omega$ and a sequence $s \in \prod_{n<N} \lambda_n$, let 1^U_s denote the condition below 1_s given by the strong Prikry property (recall Definition 6.1.41). Moreover, for each $N \leq n < \omega$, let A^s_n denote the ultrafilter set in the n-th coordinate of 1^U_s. Define

$$A^N_n = \bigcap_{s \in \prod_{n<N} \lambda_n} A^s_n,$$

for every $N \leq n < \omega$. Since U_n is λ_n-complete, A^N_n is an element of U_n. Finally, for each $n \in \omega$ define

$$A_n = \bigcap_{N \leq n} A^N_n,$$

which is an element of U_n. The resulting sequence $< A_n \mid n < \omega >$ is then contained in M and hence there exists $N < \omega$ with $x(n) \in A_n$ for all $n \geq N$. In particular, we have $x(n) \in A^{x \restriction N}_n$ for all $n \geq N$. But then the strong Prikry condition applied to 1_s with $s = x \restriction N$, yields $k < \omega$ such that the condition p with $\mathrm{stem}(p) = x \restriction (N+k)$ and $A^p_n = A_n$ for all $n \geq N+k$ is contained in $U \cap G_x$. Hence, G_x is $\mathbb{P}^M_{\vec{\mathcal{U}}}$-generic. □

In [DMRSon] it is shown that in ZFC every λ-analytic set has the λ-BP. In the next result we show that I2 is sufficient to get the λ-BP for $\Sigma^1_2(\vec{\lambda})$-sets of $C(\vec{\lambda})$ (therefore, for Σ^1_2-sets of $V_{\lambda+1}$).

Theorem 6.3.2. *Let j be an I2-elementary embedding with λ being the supremum of its critical sequence $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$. Then there exists a sequence $\mathcal{V} = \langle V_n \mid n < \omega \rangle$ such that each V_n is a normal ultrafilter on λ_n and every $\Sigma^1_2(\mathcal{V})$ subset of $C(\vec{\lambda})$ has the λ-BP w.r.t. the Ellentuck-Prikry topology induced by $\mathcal{V}$.*

Proof. Let U be a normal ultrafilter on λ_0, and let M_ω be the ω-th iterate of V by j. Then $\vec{\lambda}$ is $j_{0\omega}(\mathbb{P}_U)$-generic in M_ω, where $\mathbb{P}_U$ is the Prikry forcing on λ_0 via U. Therefore, we now consider the generic extension $M_\omega[\vec{\lambda}]$.

Since by elementarity of $j_{0\omega}$ each λ_n is measurable in $M_\omega[\vec{\lambda}]$, for every $n \in \omega$ we can pick a measure V_n on λ_n and take the sequence $\mathcal{V} = \langle V_n \mid n \in \omega \rangle$. Then we define $\mathrm{P}_\mathcal{V}$ as the diagonal Prikry forcing on λ via $\mathcal{V}$ in $M_\omega[\vec{\lambda}]$.

Let $A \in \Sigma^1_2(\vec{\lambda}, \mathcal{V})$. Define the open set

$$O = \bigcup_p \{N_p \mid p \Vdash^{M_\omega[\vec{\lambda}]}_{\mathrm{P}_\mathcal{V}} \text{“}\dot{x} \in \dot{A}\text{”}\}$$

and

$$C = \{x \in C(\lambda) \mid \text{“}x \text{ is } \mathrm{P}_\mathcal{V}\text{-generic over } M_\omega[\vec{\lambda}]\text{”}\}.$$

We claim that C is λ-comeager. Let $x \in C$. Then by Theorem 6.3.1 we have that for each sequence $\vec{A} = \langle A_n \in U_n \mid n < \omega \rangle$ in $M_\omega[\vec{\lambda}]$, x belongs to the open dense set $X_{\vec{A}} = \{x \in C(\vec{\lambda}) \mid \exists N < \omega \ \forall n > N \ (x(n) \in A_n)\}$. From the fact $(2^\lambda)^{M_\omega[\vec{\lambda}]} = \lambda$ it follows that there are only $\leq \lambda$-many of such $\vec{A}$ in $M_\omega[\vec{\lambda}]$, and hence C is λ-comeager.

Now, fix $x \in C$. We have that $x \in O$ if and only if there exists $p \in \mathrm{P}_\mathcal{V} \cap M_\omega[\vec{\kappa}]$ such that $p \Vdash^{M_\omega[\vec{\kappa}]}_{\mathrm{P}_\mathcal{V}}$ “$\dot{x} \in \dot{A}$” if and only if, using that x is a $\mathrm{P}_\mathcal{V}$-generic over $M_\omega[\vec{\kappa}]$ and $M_\omega[\vec{\kappa}][x]$ is Σ^1_2-correct at λ by Theorem 6.1.27, $x \in A$ (notice that $M_\omega[\vec{\kappa}][x] \subseteq V$). □

As in the case of the λ-PSP, we now prove that under the existence of only an ω-strictly increasing sequence of measurable cardinals there is a $\Sigma^1_2(\vec{\lambda})$-set of $C(\vec{\lambda})$ which does not have the λ-BP. The next result is a generalization of [Kan09, Corollary 13.10] in the classical case.

Proposition 6.3.3. *Let $\vec{\lambda} = \langle \lambda_n \mid n < \omega \rangle$ be a strictly increasing sequence of measurable cardinals with limit λ, and let $\langle U_n \mid n < \omega \rangle$ be a sequence with the property that U_n is a normal ultrafilter on λ_n for all $n < \omega$. Assume that $V = L[\mathcal{U}]$, where*

$$\mathcal{U} \ = \{\langle n, A \rangle \mid n < \omega, \ A \in U_n\}.$$

Then $(C(\vec{\lambda}) \text{¤} C(\vec{\lambda})) \cap <_{L[\mathcal{U}]}$ is a $\Sigma^1_2(\vec{\lambda}, \mathcal{U})$-subset of $C(\vec{\lambda}) \text{¤} C(\vec{\lambda})$ without the λ-BP.

Proof. First, notice that $(C(\vec{\lambda}) \text{¤} C(\vec{\lambda})) \cap <_{L[\mathcal{U}]}$ is λ-$\Sigma^1_2(\mathcal{U})$: indeed, by the observation after Theorem 6.1.9 we have

$$\begin{aligned} x <_{L[\mathcal{U}]} y \iff & \exists z \in C(\vec{\lambda}) \ \exists \alpha < \lambda \exists \beta < \lambda (M_z \text{ is well-founded and extensional} \\ & \wedge \pi_z(\alpha) = x \wedge \pi_z(\beta) = y \wedge M_z \models \sigma_1 \wedge \varphi_1[\alpha, \beta]), \end{aligned}$$

and by Proposition 6.1.28 this is a $\Sigma^1_2(\mathcal{U})$-formula.

Suppose now that $(C(\vec{\lambda}) \text{¤} C(\vec{\lambda})) \cap <_{L[\mathcal{U}]}$ has the λ-BP. For every $y \in C(\vec{\lambda})$, the set $\{x \in C(\vec{\lambda}) \mid x <_{L[\mathcal{U}]} y\}$ has size λ and hence it is λ-meager, and by Theorem 6.1.48 it follows that $(C(\vec{\lambda}) \text{¤} C(\vec{\lambda})) \cap <_{L[\mathcal{U}]}$ is λ-meager as well. However, the same argument could be applied to the complement of $(C(\vec{\lambda}) \text{¤} C(\vec{\lambda})) \cap <_{L[\mathcal{U}]}$, obtaining that $C(\vec{\lambda}) \text{¤} C(\vec{\lambda})$ is λ-meager, a contradiction. □

6.3.2 The λ-Baire property for λ-projective sets

Definition 6.3.4. Let κ be an infinite cardinal. We say a partially ordered set $\mathbb{P}$ is **κ-good** (in V) if it adds no bounded subsets of κ and for every generic filter G and for every $A \subseteq \mathrm{Ord}$ in $V[G]$ and of size $< \kappa$, there is a non-$\subseteq$-decreasing ω-sequence $\langle A_i \mid i < \omega \rangle$ such that $A = \bigcup_{i \in \omega} A_i$ and each A_i, $i < \omega$, is in V.

Recall that if P and Q are two λ-good forcings then the iteration forcing $P \star Q$ is λ-good as well. In [Shi15] Shi proved that λ-goodness guarantees the Generic Absoluteness (recall Definition 6.1.12), and that the standard Prikry forcing and the diagonal Prikry forcing are both λ-good. We use these facts to prove that under I0 every projective subset of $C(\vec{\lambda})$ in $L_1(V_{\lambda+1})$ has the λ-BP.

Theorem 6.3.5. *Let $\vec{\lambda}$ be a strictly increasing sequence of infinite cardinals with limit λ, let j be an I0-elementary embedding with critical sequence $\vec{\lambda}$. Then every subset of $C(\vec{\lambda})$ in $L_1(V_{\lambda+1})$ has the λ-BP.*

Proof. We work in $L(V_{\lambda+1})$. As in Theorem 6.3.2, consider the generic extension $M_\omega[\vec{\lambda}]$ obtained by using the Prikry forcing on $j_{0\omega}(\lambda_0)$ and define $\mathrm{P}_\mathcal{V}$ as the diagonal Prikry forcing on λ via $\mathcal{V} = \langle V_n \mid n \in \omega\rangle$ in $M_\omega[\vec{\lambda}]$, where V_n is a measure on λ_n for each n.

Let $A_{\phi,y} \subseteq C(\vec{\lambda})$ and $y \in M_\omega[\vec{\lambda}] \cap V_{\lambda+1}$ be such that $A_{\phi,y} = \{x \in C(\vec{\lambda}) \mid V_{\lambda+1} \models \phi(x,y)\}$, with $\phi(x,y) \in \bigcup_{n\in\mathbb{N}} \Sigma^1_n(y)$. As in Theorem 6.3.2, consider the open set

$$O = \bigcup_p \{N_p \mid p \Vdash^{M_\omega[\vec{\lambda}]}_{\mathrm{P}_\mathcal{V}} (V_{\check{\lambda}+1} \models \phi(\dot{x}_G, \check{y}))\},$$

where $\dot{x}_G$ is the $\mathrm{P}_\mathcal{V}$-name for the $\mathrm{P}_\mathcal{V}$-generic G over $M_\omega[\vec{\lambda}]$, and the λ-comeager set

$$C = \{x \in C(\vec{\lambda}) \mid \text{“}x \text{ is } \mathrm{P}_\mathcal{V}\text{-generic over } M_\omega[\vec{\lambda}]\text{”}\}.$$

Let $x \in C$. We have that $x \in O$ if and only if there exists $p \in \mathrm{P}_\mathcal{V} \cap M_\omega[\vec{\lambda}]$ such that $x \in N_p$ and $p \Vdash^{M_\omega[\vec{\lambda}]}_{\mathrm{P}_\mathcal{V}} (V_{\check{\lambda}+1} \models \phi(\dot{x}_G, \check{y}))$. Using that x is a $\mathrm{P}_\mathcal{V}$-generic over $M_\omega[\vec{\lambda}]$, we have that $M_\omega[\vec{\lambda}][x] \models (V_{\lambda+1} \models \phi(x,y))$ if and only if $M_\omega[\vec{\lambda}][x] \cap V_{\lambda+1} \models \phi(x,y)$, and by Generic Absoluteness on $M_\omega[\vec{\lambda}][x]$ (which is λ-good), $x \in A_{\phi,y}$. We thus obtain that $(A_{\phi,y} \cap C)\Delta O = \emptyset$ and $A_{\phi,y} \setminus C$ is λ-meager, and hence $A_{\phi,y} = (A_{\phi,y} \cap C) \cup (A_{\phi,y} \setminus C)$ has the λ-BP.

Therefore it follows that $A_{\phi,y} \in \bigcup_{n\in\mathbb{N}} \Sigma^1_n(\vec{\lambda}, y)$ has the λ-BP in V, for every $y \in M_\omega[\vec{\lambda}] \cap V_{\lambda+1}$. We hence have that for all formulæ ϕ and for all $y \in M_\omega[\vec{\lambda}] \cap V_{\lambda+1}$,

$$L(V_{\lambda+1}) \models A_{\phi,y} \text{ has the } \lambda\text{-BP}.$$

In order to apply Proposition 6.1.11.2 we now claim that the formula “$A_{\phi,y}$ has the λ-BP” can be expressed only using an existential quantifier on the set of subsets of $V_{\lambda+1}$. Notice that as a consequence of this it follows that the λ-BP is upward absolute. By definition of λ-BP, the set $A_{\phi,y}$ has the λ-BP if and only if there exist an open set $U \subseteq C(\vec{\lambda})$ and a sequence $\langle C_\alpha \mid \alpha < \lambda\rangle$ of nowhere dense sets in $C(\vec{\lambda})$ such that $A\Delta U = \bigcup_{\alpha<\lambda} C_\alpha$. Notice now that each open set V can be determined by the subset $\{p \mid N_p \subseteq V\}$ of $V_{\lambda+1}$. Moreover, since for every α, C_α is nowhere dense if and only if its closure $\bar{C}_\alpha$ is nowhere dense, we can assume that the sequence $\langle C_\alpha \mid \alpha < \lambda\rangle$ is such that C_α is closed for every $\alpha < \lambda$. Then $C(\vec{\lambda}) \setminus C_\alpha$ is open dense, and hence by the previous argument it is determined by a subset of $V_{\lambda+1}$. Then the same follows for each C_α. Hence, U and the sequence $\langle C_\alpha \mid \alpha < \lambda\rangle$ can be determined by a λ-sequence of subsets of $V_{\lambda+1}$, which in turn is a subset of $V_{\lambda+1}$. It is now easy to see that the claim holds.

Now we define the map $y \mapsto \alpha_y$, where if $y \in V_{\lambda+1}$ is such that $A_{\phi,y}$ has the λ-BP then, by applying Proposition 6.1.11.2, $\alpha_y < \Theta$ is the least such that $L_{\alpha_y}(V_{\lambda+1}) \models A_{\phi,y}$ has the λ-BP, otherwise $\alpha_y = 0$. This function is definable in $L(V_{\lambda+1})$. We want to prove that $\alpha = \sup\{\alpha_y \mid y \in V_{\lambda+1}\} < \Theta$. For any $x \in V_{\lambda+1}$, recalling the definition of $<_x$ in (6.1.1), we let $g_x(y)$ be the $<_x$-smallest surjection from $V_{\lambda+1}$ to α_y, if it exists, otherwise $g_x(y) = 0$. The map $x \mapsto g_x$ is also definable in $L(V_{\lambda+1})$. We now prove that the function f defined by

$$f(\langle x,y,z\rangle) = \begin{cases} g_x(y)(z) & \text{if } g_x(y) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

is a surjection from $V_{\lambda+1}$ to $\alpha = \sup\{\alpha_y \mid y \in V_{\lambda+1}\}$.

Let $\beta \in \alpha$. Then there exists $y \in V_{\lambda+1}$ such that $\beta < \alpha_y$. Since $\alpha_y < \Theta$, there is in $L(V_{\lambda+1})$ a surjection $g\colon V_{\lambda+1} \to \alpha_y$. Then by Theorem 6.1.10 there exist $x \in V_{\lambda+1}$ and $\gamma \in \mathrm{Ord}$ such that $\Phi(\gamma, x) = g$, where Φ is a surjection from $\mathrm{Ord} \times V_{\lambda+1}$ to $L(V_{\lambda+1})$, and we can thus consider the $<_x$-smallest surjection $g_x(y)$ from $V_{\lambda+1}$ to α_y. Therefore $g_x(y) \neq 0$, and it is a surjection from $V_{\lambda+1}$ to α_y. Hence there is $z \in V_{\lambda+1}$ such that $g_x(y)(z) = \beta$, and by construction $f(\langle x, y, z\rangle) = g_x(y)(z) = \beta$. Thus f is a surjection as well, and it is definable in $L(V_{\lambda+1})$. In particular, we have that $\alpha < \Theta$ and for all $y \in M_\omega[\vec{\lambda}] \cap V_{\lambda+1}$,

$$L_\alpha(V_{\lambda+1}) \models A_{\phi,y} \text{ has the } \lambda\text{-BP}.$$

By the fact that the sequence of good ordinals is cofinal in Θ (see Section 6.1.1 after Definition 6.1.12) and that the λ-BP is upward absolute, we can assume that α is good. Then by Theorem 6.1.13 there exist $\bar{\alpha} < \lambda$ and an elementary embedding $\pi\colon L_{\bar{\alpha}}(M_\omega[\vec{\lambda}] \cap V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ such that $\pi \restriction (M_\omega[\vec{\lambda}] \cap V_{\lambda+1}) = \mathrm{id}$. Thus,

$$\begin{aligned}
&\forall y \in M_\omega[\vec{\lambda}] \cap V_{\lambda+1},\ L_\alpha(V_{\lambda+1}) \models A_{\phi,\pi(y)} \text{ has the } \lambda\text{-BP} \\
&\iff \forall y \in M_\omega[\vec{\lambda}] \cap V_{\lambda+1},\ L_{\bar{\alpha}}(M_\omega[\vec{\lambda}] \cap V_{\lambda+1}) \models A_{\phi,y} \text{ has the } \lambda\text{-BP} \\
&\iff L_{\bar{\alpha}}(M_\omega[\vec{\lambda}] \cap V_{\lambda+1}) \models \forall y \in V_{\lambda+1}, A_{\phi,y} \text{ has the } \lambda\text{-BP} \\
&\iff L_\alpha(V_{\lambda+1}) \models \forall y \in V_{\lambda+1}, A_{\phi,y} \text{ has the } \lambda\text{-BP}.
\end{aligned}$$

We thus obtain that each $A \in \bigcup_{n\in\mathbb{N}} \bigcup_{y \in V_{\lambda+1}} \Sigma^1_n(y)$ has the λ-BP, i.e. each $A \in \bigcup_{n\in\mathbb{N}} \boldsymbol{\Sigma^1_n}$ has the λ-BP. $\square$

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