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Exponential time integration for delay differential equations via pseudospectral discretization

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Abstract: We propose the use of exponential Runge-Kutta methods for the time integration of delay differential equations. The approach is based on their reformulation as abstract differential equations and the reduction of the latter to finite-dimensional systems of ordinary differential equations via pseudospectral discretization. We substantiate our results by means of some illustrative numerical simulations with EXPINT, a MATLAB package for exponential integrators.

Keywords: time delay systems, model reduction, exponential time-integration, numerical simulations, semigroup and operator theory.

1. INTRODUCTION

The time integration of Cauchy problems for delay differential equations (DDEs) has been deeply studied in the past few decades, leading to the development of continuous and functional continuous Runge-Kutta (RK) methods (Bellen and Zennaro (2003); Bellen et al. (2009)). The reformulation of DDEs as abstract differential equations (Diekmann et al. (1995)) constitutes the basis of a class of numerical schemes that have been proposed (see also (Maset (2003))).

Exponential time integrators, which constitute the topic of the present paper, were originally developed for semilinear ordinary differential equations (ODEs) but then extended to abstract differential equations defined in Banach spaces of functions (Hochbruck and Ostermann (2010)). Thus, they could in principle be applied to the abstract reformulation of DDEs directly. Indeed, as observed in Andò and Vermiglio (2024), exponential RK methods for DDEs reduce to functional continuous RK schemes.

In this paper, we focus on a different approach, namely on applying the methods to the finite-dimensional ODEs obtained via pseudospectral discretization (Breda et al. (2016)) of the abstract reformulation of DDEs. Indeed, the resulting ODEs are defined by a semilinear right-hand side with stiff linear part, which makes the choice of exponential methods particularly well-suited. Here, we focus in particular on those of RK type.

The paper is organized as follows. In section 2 we provide an overview of exponential RK methods, as well as the relevant theoretical convergence results. Section 3 is devoted to the reformulation of DDEs as abstract differential equations, and the pseudospectral discretization of the latter. Section 4 aims at exploring several aspects of the application of exponential RK methods for DDEs, by means of a range of numerical simulations on various prototype models. Finally, we conclude by suggesting possible extensions of the work in section 5.

2. EXPONENTIAL RUNGE-KUTTA METHODS

An initial value problem for a semilinear ODE has the form

$$\begin{cases} U'(t) = AU(t) + F(t, U(t)), \ t \in [0, t_f] \\ U(0) = U_0, \end{cases} \tag{1}$$

where $U_0 \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}$ is a typically matrix, while $F : [0, t_f] \times \mathbb{R}^d \to \mathbb{R}^d$ is generally non-linear and locally Lipschitz continuous with respect to its second argument in a strip along the exact solution U. In other words, for all $R \in [0, \overline{R}]$ there exists $L > 0$ such that

$$|F(t, W) - F(t, V)| \le L|W - V|, \quad t \in [0, t_f]$$

for any $W, V \in \mathbb{R}^d$ such that $|W - U|, |V - U| \le R$. The idea behind exponential methods is to integrate (1) via its reformulation as an equivalent integral equation (IE) obtained by the variation of constants formula, which reads

$$U(t) = T(t)U_0 + \int_0^t T(t-s)F(s, U(s))ds, \ t \in [0, t_f], \tag{2}$$

where $T(t) := e^{At}, \ t \in [0, t_f]$.

An exponential time differencing method is defined by a time mesh

$$t_{n+1} = t_n + h_n, \ n = 0, 1, \dots, N-1, \ t_0 = 0, \ t_N = t_f$$

indicating the points where the approximation $U_n \approx U(t_n)$ of the exact solution U of (1) is computed. In each

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subinterval $[t_n, t_{n+1}]$, the approximation is constructed starting from

$$U(t_{n+1}) = T(h_n)U(t_n) + \int_0^{h_n} T(h_n - s)F(t_n + s, U(t_n + s))ds \tag{3}$$

by means of efficient algorithms for the product of the exponential matrix with a vector (Moler and Van Loan (2003)) and exponential quadrature rules (Hochbruck and Ostermann (2010)).

Similarly to the construction of classical RK methods, the quadrature formulae to approximate the integral on the right-hand side of (3) have, for a timestep h, the form

$$\int_0^h T(h-s)f(s)ds \approx h\sum_{i=1}^{\nu} b_i(hA)f(c_ih),$$

$$\int_0^{c_ih} T_0(c_ih-s)f(s)ds \approx h\sum_{j=1}^{\nu} a_{ij}(hA)f(c_jh), \quad i = 1\ldots,\nu,$$

for $f(s) := F(s, U(s))$ and nodes $c_i := \sum_{j=1}^{\nu} a_{ij}(0)$. The coefficients $b_i(z)$, $a_{ij}(z)$ are linear combinations of the functions

$$\varphi_k(z) = \int_0^1 e^{(1-c)z}\frac{c^{k-1}}{(k-1)!}dc, \; k \geq 0,$$

for which we get

$$\begin{cases} \varphi_0(chA) = T(ch), \\ (ch)^k\varphi_k(chA) = \displaystyle\int_0^{ch} T(ch-s)\frac{s^{k-1}}{(k-1)!}ds, \; k \geq 1, \end{cases}$$

for $c \in [0, 1]$. A ν-stage exponential RK method is defined by the formulae

$$\begin{cases} U_{n+1} = T(h_n)U_n + h_n\displaystyle\sum_{i=1}^{\nu} b_i(h_nA)F(t_{n,i}, U_{n,i}), \\ U_{n,i} = T(c_ih_n)U_n + h_n\displaystyle\sum_{j=1}^{\nu} a_{ij}(h_nA)F(t_{n,j}, U_{n,j}), \\ \quad i = 1, \ldots, \nu, \end{cases} \tag{4}$$

where $t_{n,i} = t_n + c_ih_n$ and $U_{n,i}$ is meant to approximate the value of $U(t_{n,i})$.

Just like classical RK methods, exponential RK methods can be represented by Butcher tableaux (Butcher (1987)). This work focuses on *explicit* methods, whose Butcher tableaux have the form

$$\begin{array}{c|ccccc|c} 0 & 0 & 0 & \ldots & 0 & 0 & I_d \\ c_2 & a_{21}(hA) & 0 & \ldots & 0 & 0 & T(c_2h) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_\nu & a_{\nu 1}(hA) & a_{\nu 2}(hA) & \ldots & a_{\nu,\nu-1}(hA) & 0 & T(c_\nu h) \\ \hline & b_1(hA) & \ldots & \ldots & b_{\nu-1}(hA) & b_\nu(hA) & T(h) \end{array}$$

An example is given by the exponential Euler method defined by

$$\begin{array}{c|c|c} 0 & 0 & I_d \\ \hline & \varphi_1(hA) & \varphi_1(hA) \end{array} \tag{5}$$

for which we have the following convergence result.

Theorem 1. (Hochbruck and Ostermann (2005)). Assume that $f(t) := F(t, U(t))$, $t \in [0, t_f]$, belongs to $C^1([0, t_f]; \mathbb{R}^d)$. If $\{U_n\}_{n=1,\ldots,N}$ is the solution obtained by the exponential Euler method, then

$$|U(t_n) - U_n| \leq C(t_f) \max_{0\leq t\leq t_f} |f'(t)| \cdot h_{\max} \tag{6}$$

for $n = 1, \ldots, N$, $h_{\max} := \max_{n=0,\ldots,N-1} h_n$, where $C(t_f)$ depends on t_f but is independent of n and h_n.

As is the case for any time-integration method, we are interested in the order conditions guaranteeing the convergence of higher-order exponential RK methods. In (Luan and Alhsmy (2024)), sufficient conditions on the coefficients to reach stiff order of convergence up to 6 are provided. We report in Table 1 those up to order 4, which can be also found in (Hochbruck and Ostermann (2005)). The conditions are expressed in terms of the operators

$$\begin{aligned} \psi_k(hA) &:= \varphi_k(hA) - \sum_{j=1}^{\nu} b_j(hA)\frac{c_j^{k-1}}{(k-1)!} \\ \psi_{ki}(hA) &:= \varphi_{i,k}(hA) - \sum_{j=1}^{i-1} a_{ij}(hA)\frac{c_j^{k-1}}{(k-1)!} \end{aligned} \tag{7}$$

where $\varphi_{i,k}(hA) := \varphi_k(c_ihA)$, $i = 1, \ldots, \nu$, $k \geq 1$.

Table 1. Stiff order conditions for explicit exponential RK methods, where J and K denote arbitrary bounded operators on $\mathbb{R}^d$.

No.	Order	Order condition
1	1	$\psi_1(hA) = 0$
2	2	$\psi_2(hA) = 0$
3	2	$\psi_{1i}(hA) = 0, \quad i = 1, \ldots, \nu$
4	3	$\psi_3(hA) = 0$
5	3	$\sum_{i=1}^{\nu} b_i(hA)J\psi_{2i}(hA) = 0, \quad i = 1, \ldots, \nu$
6	4	$\psi_4(hA) = 0$
7	4	$\sum_{i=1}^{\nu} b_i(hA)J\psi_{3i}(hA) = 0, \quad i = 1, \ldots, \nu$
7	4	$\sum_{i=1}^{\nu} b_i(hA)J\sum_{j=2}^{i-1} a_{ij}(hA)J\psi_{2j}(hA) = 0, \quad i = 1, \ldots, \nu$
9	4	$\sum_{i=1}^{\nu} b_i(hA)c_iK\psi_{2i}(hA) = 0, \quad i = 1, \ldots, \nu$

The following theorem formalizes the relationship between stiff order conditions and global order of convergence.

Theorem 2. (Hochbruck and Ostermann (2005)). Assume that $f(t) := F(t, U(t))$, $t \in [0, t_f]$, is p-times differentiable with bounded derivatives. If an exponential RK method satisfies the stiff order conditions up to order $p - 1$, $\sum_{i=1}^{\nu} b_ic_i^{p-1} = \frac{1}{p}$, and the remaining stiff order conditions of order p hold with $b_i(0)$ (instead of $b_i(hA)$), $i = 1, ..., \nu$, then the error bound

$$|U(t_n) - U_n| \leq C(t_f) \cdot h^p$$

holds uniformly in $nh \in [0, t_f]$ where $C(t_f)$ depends on t_f but is independent of n and h.

3. PSEUDOSPECTRAL DISCRETIZATION

In this section, we show a way to exploit the results above on exponential methods in order to integrate Cauchy problems for DDEs, which have the form

$$\begin{cases} x'(t) = F(t, x_t), & t \in [0, t_f], \\ x_0(\theta) = \phi(\theta), & \theta \in [-\tau, 0], \end{cases} \tag{8}$$

where x_t defined as

$$x_t(\theta) := x(t + \theta), \quad \theta \in [-\tau, 0],$$

is the *history function* at time t and belongs (together with ϕ) to the Banach space $C := C([-\tau, 0], \mathbb{R}^d)$, equipped with the uniform norm $\|\phi\|_C := \max_{\theta \in [-\tau, 0]} |\phi(\theta)|$. $F : [0, +\infty) \times C \to \mathbb{R}^d$ is generally a nonlinear map (see, e.g., (Diekmann et al. (1995))). In the following, we also assume that it satisfies the same Lipschitz property as F in (1).

The first step consists in reformulating (8) as an *abstract* Cauchy problem, as follows. The well-posedeness of (8) allows us to define the shift semigroup $\{\mathcal{T}(t)\}_{t \geq 0}$ as

$$[\mathcal{T}(t)\varphi](\theta) = \begin{cases} \varphi(t + \theta), \ t + \theta \in [-\tau, 0] \\ \varphi(0), \ t + \theta \geq 0 \end{cases}$$

for all $t \geq 0$. Consider now the map $j : C \to X := \mathbb{R}^d \times L^\infty([-\tau, 0], \mathbb{R}^d)$ defined by

$$u(t) := jx_t = (x(t); x_t), \quad t \geq 0. \tag{9}$$

and the shift operator $\mathcal{T}_0(t) : X \to X$ whose restriction to jC is $\mathcal{T}_0(t) := j\mathcal{T}(t)j^{-1}$. The infinitesimal generator $\mathcal{A}_0 : \mathrm{dom}(\mathcal{A}_0) \subset X \to X$ of the semigroup $\{\mathcal{T}_0(t)\}_{t \geq 0}$ is differentiation, i.e., it is defined as

$$\mathcal{A}_0(\alpha; \psi) := (0; \psi')$$

with

$D(\mathcal{A}_0) = \{(\alpha; \psi) \in X | \psi$ is Lipschitz-continuous, $\psi(0) = \alpha\}$.

Thus, (8) can be reformulated as the semilinear abstract Cauchy problem

$$\begin{cases} u'(t) = \mathcal{A}_0 u(t) + \mathcal{F}(t, u(t)), & t \in [0, t_f], \\ u(0) = \phi, \end{cases} \tag{10}$$

and, as proved in (Diekmann et al. (1995)), it is equivalent to

$$u(t) = \mathcal{T}_0(t)\phi + \int_0^t \mathcal{T}_0(t - s)\mathcal{F}(s, u(s))\, \mathrm{d}s, \tag{11}$$

for $t \in [0, t_f]$ and $u(t) = jx_t$, where the nonlinear $\mathcal{F}(t, (\psi(0); \psi)) = (F(t, \psi); 0)$ inherits the regularity of F.

We now show how to formulate a discretized version of (10) correspondingly, assuming that $d = 1$ for the sake of simplicity. The idea is to reduce (10) to a system of ODEs via pseudospectral discretization (Breda et al. (2016)).

Given a discretization index M, representing the degree of the interpolating polynomials, consider the $M + 1$ Chebyshev extreme points, defined as

$$\theta_j := \frac{\tau}{2}\left(\cos\left(\frac{j\pi}{M}\right) - 1\right), \quad j = 0, \ldots, M. \tag{12}$$

Let us introduce the restriction operator $R_M : X \to \mathbb{R}^{M+1}$ mapping a function to the vector of its values at the nodes, i.e.,

$$R_M(\alpha; \varphi) = (\alpha; \varphi(\theta_1); \ldots; \varphi(\theta_M)),$$

where we use the MATLAB notation with semicolons to indicate column vectors. We also introduce the prolongation operator $P_{M,0} : \mathbb{R}^{M+1} \to X_{M,0} \subset D(\mathcal{A}_0)$ mapping a vector to the polynomial interpolating the nodes at its values, i.e.,

$$P_{M,0}(\alpha; \Phi_M) = \left(\alpha; \theta \mapsto \sum_{i=1}^{M} \ell_{M,i}(\theta)\Phi_{M,i} + \ell_{M,0}(\theta)\alpha\right),$$

where $\{\ell_{M,i}\}_{i=0,\ldots,M}$ is the Lagrange basis relevant to the nodes (12), defined as

$$\ell_{M,i}(\theta) := \prod_{\substack{k=0 \\ k \neq i}}^{M} \frac{\theta - \theta_{M,k}}{\theta_{M,i} - \theta_{M,k}}, \qquad i = 0, \ldots, M.$$

By defining the finite-dimensional counterparts of $\mathcal{A}_0$ and $\mathcal{F}$ as

$$\begin{aligned} A_{0,M} &:= R_M \mathcal{A}_0 P_{M,0}, \\ F_M(t, \cdot) &:= R_M \mathcal{F}(t, P_{M,0}(\cdot)), \ t \in [0, t_f], \end{aligned} \tag{13}$$

the discretized version of (10) is defined by

$$\begin{cases} U'(t) = A_{0,M}U(t) + F_M(t, U(t)), & t \in [0, t_f], \\ U(0) = U_0, \end{cases} \tag{14}$$

where $U_0 := R_M\phi$, and the corresponding integral equation reads

$$U(t) = T_{0,M}(t)U_0 + \int_0^t T_{0,M}(t - s)F_M(s, U(s))\, \mathrm{d}s, \tag{15}$$

where $T_{0,M}(t) = \mathrm{e}^{A_{0,M}t}$, $t \in [0, T]$. Note that, given

$$d_M := \begin{pmatrix} \ell'_{M,0}(\theta_{M,1}) \\ \vdots \\ \ell'_{M,0}(\theta_{M,M}) \end{pmatrix} \in \mathbb{R}^M,$$

and

$$D_M := \begin{pmatrix} \ell'_{M,1}(\theta_{M,1}) & \ldots & \ell'_{M,M}(\theta_{M,1}) \\ \vdots & \ddots & \vdots \\ \ell'_{M,1}(\theta_{M,M}) & \ldots & \ell'_{M,M}(\theta_{M,M}) \end{pmatrix} \in \mathbb{R}^{M \times M}$$

we can write

$$A_{0,M} = \left(\begin{array}{c|c} 0 & 0_M \\ \hline d_M & D_M \end{array}\right) \tag{16}$$

Observe that, by discretizing the problem as (14), one is not able to exploit any linearity in $\mathcal{F}$. Alternatively, for DDEs of the form

$$\begin{cases} x'(t) = Lx(t) + G(t, x_t), t \in [0, t_f], \\ x_0 = \phi, \end{cases} \tag{17}$$

where L is a stiff matrix, one can define

$$A_{L,M} = \left(\begin{array}{c|c} L & 0_M \\ \hline d_M & D_M \end{array}\right) \tag{18}$$

and

$$G_M(\cdot, U) = (G(\cdot, P_{M,0}U); 0_M), \tag{19}$$

so that the abstract formulation of (17) can be discretized as

$$\begin{cases} U'(t) = A_{L,M}U(t) + G_M(t, U(t)), t \in [0, t_f], \\ U(0) = U_0. \end{cases} \tag{20}$$

Similarly, the corresponding integral equation reads

$$U(t) = T_{L,M}(t)U_0 + \int_0^t T_{L,M}(t-s)G_M(s,U(s))\,\mathrm{d}s \quad (21)$$

where $T_{L,M}(t) = \mathrm{e}^{A_{L,M}t}$, $t \in [0, t_f]$.

As we will show in section 4, it is advisable to resort to the decomposition in (20) when the linear part of F in (8) is stiff.

4. NUMERICAL RESULTS

In this section, we test the performance of exponential RK methods on some prototype DDE models. More specifically, we use EXPINT, a MATLAB package for exponential integrators (Berland et al. (2007)), to analyze the role of the discretization index M, to underline the importance of choosing the discretization of (21) when dealing with stiff DDEs (17) and to provide experimental validation of the theoretical convergence order for some selected exponential RK methods. In particular, we consider the Euler method (5) of (nonstiff) order 1, the Heun method of order 2, the method ((Hochbruck and Ostermann, 2005, (5.8))) of order 3 and the method ((Hochbruck and Ostermann, 2005, (5.19))) of order 4, having Butcher tableaux respectively

$$\begin{array}{c|ccc|c} 0 & 0 & 0 & & I \\ 1 & \varphi_1 & 0 & & \varphi_1 \\ \hline & \varphi_1-\varphi_2 & \varphi_2 & & \varphi_1 \end{array}, \quad (22)$$

$$\begin{array}{c|cccc|c} 0 & 0 & 0 & 0 & & I \\ \frac{1}{2} & \frac{1}{2}\varphi_{1,2} & 0 & 0 & & \varphi_{1,2} \\ \frac{2}{3} & \frac{2}{3}\varphi_{1,3}-\frac{8}{9}\varphi_{2,3} & \frac{8}{9}\varphi_{2,3} & 0 & & \varphi_{1,3} \\ \hline & \varphi_1-\frac{3}{2}\varphi_2 & 0 & \frac{3}{2}\varphi_2 & & \varphi_1 \end{array} \quad (23)$$

and

$$\begin{array}{c|cccccc|c} 0 & 0 & 0 & 0 & 0 & 0 & & I \\ \frac{1}{2} & \frac{1}{2}\varphi_{1,2} & 0 & 0 & 0 & 0 & & \varphi_{1,2} \\ \frac{1}{2} & \frac{1}{2}\varphi_{1,2}-\varphi_{2,2} & \varphi_{2,2} & 0 & 0 & 0 & & \varphi_{1,2} \\ 1 & \varphi_1-2\varphi_2 & \varphi_2 & \varphi_2 & 0 & 0 & & \varphi_1 \\ \frac{1}{2} & \frac{1}{2}\varphi_{1,2}-2a-b & a & a & b & 0 & & \varphi_{1,2} \\ \hline & \varphi_1-3\varphi_2+4\varphi_3 & 0 & 0 & -\varphi_2+4\varphi_3 & 4\varphi_2-8\varphi_3 & & \varphi_1 \end{array}, \quad (24)$$

for $a := \frac{1}{2}\varphi_{2,2} - \varphi_3 + \frac{1}{4}\varphi_2 - \frac{1}{2}\varphi_{3,2}$ and $b := \frac{1}{4}\varphi_{2,2} - a$. Finally, we include the results of some numerical simulations where we vary one parameter of the model, in order to highlight the role of numerical simulations in the analysis of the dynamics of DDEs.

As our first example, we consider the initial value problem ((Bellen and Zennaro, 2003, (1.2.3)))

$$\begin{cases} x'(t) = \lambda x(t) - \dfrac{\pi}{2}\mathrm{e}^{\lambda}x(t-1), \\ x(t) = \mathrm{e}^{\lambda t}\sin\left(\dfrac{\pi t}{2}\right), \quad t \in [-1, 0], \end{cases} \quad (25)$$

whose exact solution is $x(t) = \mathrm{e}^{\lambda t}\sin\left(\frac{\pi t}{2}\right), t \geq -1$. Our goal is to provide numerical evidence of the convergence

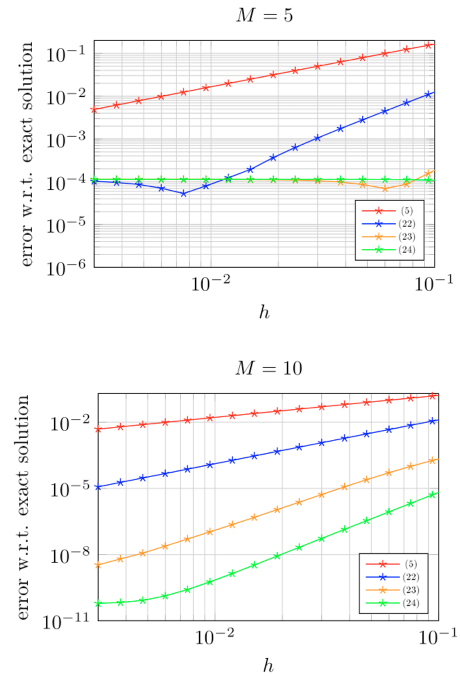

Fig. 1. Errors $|U_n - x(t_f)|$ at $nh = t_f = 4$ of the solution of the pseudospectral discretization of (15) for (25) with $\lambda = 1$, with $M = 5$ (top) and $M = 10$ (bottom).

rate for the pseudospectral discretization of (25) for different values of M, using the exponential methods (5), (22), (23) and (24). To integrate the pseudospectral discretization of (25) we performed tests using $M = 10$ and $M = 5$, $\lambda = 1$, $[0, 3]$ as timespan and timesteps $h = 3 \cdot 10^{-\alpha}$ with $\alpha = -3 + 0.1k$, $k = 0, \ldots, 20$ (thus, h does not have to be a submultiple of τ). Figure 1 confirms the $O(h^p)$ behavior until a certain treshold is reached (around 10^{-4} for $M = 5$ and 10^{-10} for $M = 10$), which can be interpreted as the error inherent to the discretization.

Next, consider the following class of Ikeda DDEs ((Erneux et al., 2004, (8)))

$$x'(t) = -\lambda x(t) + f(\mu, x(t-\tau)), \quad (26)$$

with λ and μ positive parameters, and f a function. We first analyze the Cauchy problem obtained by choosing

$$f(\mu, y) = \mu(1 - \sin(y)), \quad \tau = \frac{\pi}{2}, \quad (27)$$

together with the initial condition $x(t) = \cos(t)$, $t \in \left[-\frac{\pi}{2}, 0\right]$. Since the problem does not have an exact solution, we perform our simulations using $M = 50$, in order to minimize the impact on the discretization on the computed error. The reference solution is computed automatically by EXPINT, using the method (24) and a suitably small timestep h.

Observe that (27) is stiff for large values of the parameter λ. We performed tests using the discretization of either (15) or (21), $\mu = 30$ and $[0, 4]$ as timespan. In the former case, we chose timesteps $h = 4 \cdot 10^{\alpha}$, with $\alpha = -3 + 0.05k$, $k = 0, \ldots, 25$, while in the latter we use $\alpha = -3 + 0.1k$, $k = 0, \ldots, 20$ (in particular, h is never a submultiple of τ). The reason for the discrepancy is clear from figures 2 and 3. Indeed, the former shows that for values of α greater

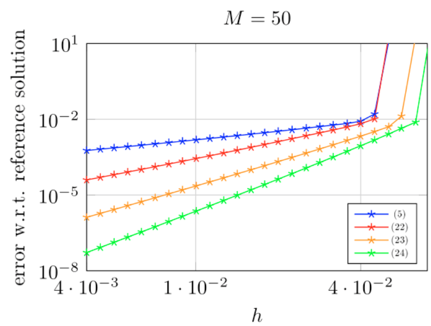

Fig. 2. Errors $|U_n - x(t_f)|$ at $nh = t_f = 4$ of the solution of the pseudospectral discretization of (15) for (27) with $\lambda = 40$.

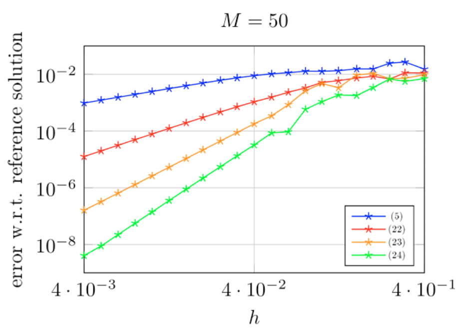

Fig. 3. Errors $|U_n - x(t_f)|$ at $nh = t_f = 4$ of the solution of the pseudospectral discretization of (21) for (27) with $\lambda = 40$.

or equal than -1.8 or -1.9 (according to the specific method considered) the error increases very quickly with h, meaning that the problem is too stiff to be integrated using $h > 4 \cdot 10^{-1.75}$ together with the discretization of (15). On the other hand, when considering the discretization of (21), one can afford using larger timesteps. In both cases, the $O(h^p)$ behavior of the error is confirmed. Figure 4, computed using the discretization of (21), the method (24) and $h = 10^{-2}$, shows the role of μ in the computation: the solution converges to an equilibrium using $\mu = 30$ and to a periodic solution using $\mu = 50$.

We obtain similar results by considering (26) with

$$f(\mu, y) = \mu \frac{1}{1 + y^n}, \quad \tau = 1, \tag{28}$$

where n is a positive parameter. We integrate in time (26) with (28) together with the initial condition $x(t) = \cos(t)$ using $M = 50$, since, again, we do not have an exact solution. Moreover, we fix $\lambda = 40$ and, due to the result obtained with (27), we use the discretization of (21). We choose $n = 3$ and either $\mu = 30$ or $\mu = 50$. Figure 5 shows the role of μ in the computation: again, we converge to an equilibrium using $\mu = 30$ and to a periodic solution using $\mu = 50$.

Lastly, we consider the equation ((Insperger and Stépán, 2000, (1)-(3)))

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = wh(t)(y(t - \tau) - y(t)), \tag{29}$$

where $h(t)$ is a periodic piecewise constant function defined as

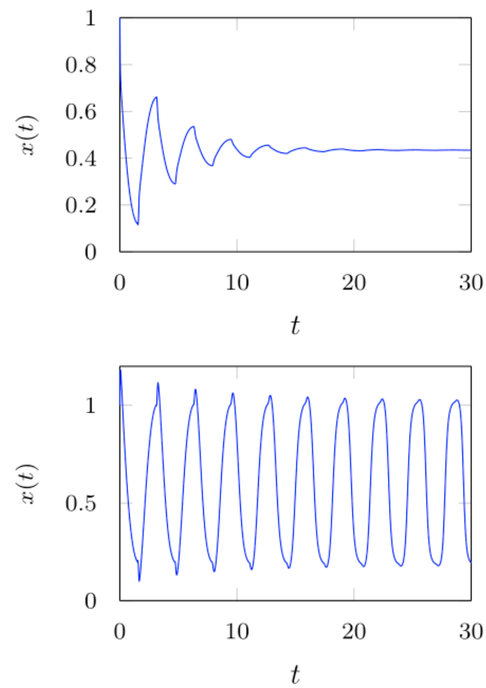

Fig. 4. Solutions of the pseudospectral discretization of (21) for (27) with $\lambda = 40$ and $\mu = 30$ (top), $\mu = 50$ (bottom) with $M = 50$, $t \in [0, 30]$, computed with method (24) and $h = 10^{-2}$.

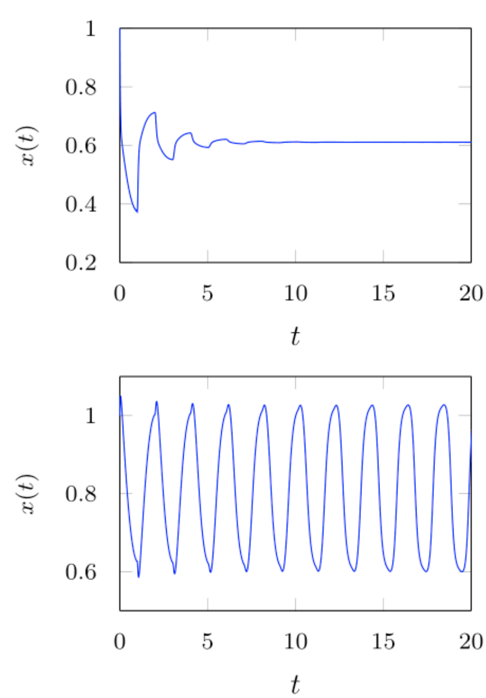

Fig. 5. Solutions of the pseudospectral discretization of (21) for (28) with $\lambda = 40$, $p = 3$, $\mu = 30$ (top) and $\mu = 50$ (bottom) with $M = 50$, $t \in [0, 20]$, computed with method (24) and $h = 10^{-2}$.

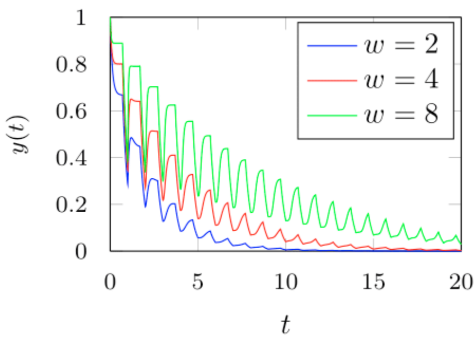

Fig. 6. Solutions of (31) via the pseudospectral discretization (20) of (31) for $m = 0.1$, $c = 140$, $k = H = 400$ and $\tau = 1$ with $M = 50$, $t \in [0, 20]$, computed with method (24) and $h = 10^{-2}$.

$$\begin{cases} h(t) = H & \text{if } \dfrac{t}{\tau} - \left\lfloor \dfrac{t}{\tau} \right\rfloor < 0.7 \\ h(t) = 0 & \text{otherwise.} \end{cases} \tag{30}$$

We rewrite (29) as the (non-autonomous) two-dimensional system

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -c/m & -k/m \end{pmatrix} x(t) + wh(t) \begin{pmatrix} 0 \\ x_1(t-\tau) - x_1(t) \end{pmatrix} \tag{31}$$

for $x(t) := (y(t); \dot{y}(t))$. We integrate the Cauchy problem obtained by discretizing (31) through (20) for

$$L = \begin{pmatrix} 0 & 1 \\ -c/m & -k/m \end{pmatrix}, \; G(t, x_t) = wh(t)(0; x_1(t-\tau) - x_1(t)),$$

coupled with the constant initial condition $x(0) = (1; 0)$. Figure 6 shows the results of three simulations obtained with the values $m = 0.1$, $c = 140$, $k = H = 400$ and $\tau = 1$ of the parameters, as well as $w = 2$, 4 or 8. The profile of the solution is characterized by damped oscillations, asymptotically approaching the zero equilibrium. The amplitude of such oscillations is determined by the value of w.

5. CONCLUSION

The present work of computational nature explores the use of exponential RK methods for the time integration of DDEs via the pseudospectral discretization of their abstract formulation. While it focuses on examples defined by discrete delays, it can be straightforwardly extended to distributed delays (which can, possibly, be state-dependent). Moreover, thanks to (Breda et al. (2016)), the same ideas can be applied to renewal equations (REs) and coupled REs/DDEs systems, often appearing in population dynamics models. Finally, it would be interesting to investigate the use of exponential methods other than those of RK type, such as exponential Rosenbrock methods, Magnus methods, linear multistep methods or general linear methods, some of which are included in the EXPINT package.

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