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# Succinctness issues for $\mathsf{LTL_f}$ and safety and cosafety fragments of LTL

Alessandro Artale [a], Luca Geatti [b,*], Nicola Gigante [a], Andrea Mazzullo [c], Angelo Montanari [b]

[a] Free University of Bozen-Bolzano, Piazza Università, 1, Bolzano, 39100, Italy
[b] University of Udine, Via delle Scienze, 206, Udine, 33100, Italy
[c] University of Trento, Via Sommarive 9, Povo (Trento), 38123, Italy

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A B S T R A C T

Linear Temporal Logic over finite traces ($\mathsf{LTL_f}$) has proved itself to be an important and effective formalism in formal verification as well as in artificial intelligence. Pure past $\mathsf{LTL_f}$ (pLTL) is the variant of $\mathsf{LTL_f}$ featuring only past temporal modalities, and is naturally interpreted at the end of a finite trace. It is known that each property definable in $\mathsf{LTL_f}$ is also definable in pLTL, and *vice versa* (they are expressively equivalent). The same goes for the *safety* and *cosafety* fragments of Linear Temporal Logic over infinite traces (LTL), when compared to G(pLTL) and F(pLTL) formulas, respectively, that is, pLTL formulas prefixed by a *globally* and an *eventually* modality. However, despite being extensively used in practice, to the best of our knowledge, there is no systematic study of their succinctness. Moreover, when considering (co)safety fragments of LTL devoid of binary temporal modalities, there are no known characterizations based on pLTL.

In this paper, we investigate succinctness issues for $\mathsf{LTL_f}$ and (co)safety fragments of LTL when compared with their pure past counterparts. First, we provide a pure past characterization of the (co)safety fragments of LTL devoid of binary temporal modalities. Then, we prove that the (co)safety fragments of LTL have pure past counterparts that can be exponentially more succinct. Finally, we show that the same holds for $\mathsf{LTL_f}$ with respect to pLTL, and viceversa: $\mathsf{LTL_f}$ and pLTL are incomparable when succinctness is concerned.



## 1. Introduction

In this paper, we investigate succinctness issues for Linear Temporal Logic over finite traces ($\mathsf{LTL_f}$) and the safety and cosafety fragments of Linear Temporal Logic over infinite traces (LTL).

$\mathsf{LTL_f}$ is a modal logic that extends Propositional Logic with *temporal modalities* for reasoning about time and it is interpreted over *finite* sequences of states, called *traces* or *words*. It semantically differs from standard LTL, which is interpreted over infinite traces, and it is extensively used in many areas of Artificial Intelligence (AI), including automated synthesis [1–3], planning [4–6], and business process management [7,8], where a finite temporal dimension is often involved. In the last years, the *pure past* version of $\mathsf{LTL_f}$ (pLTL for short), whose formulas involve only past modalities and are evaluated at the last state of a finite trace, gained momentum in AI as well. It is well known that pLTL and $\mathsf{LTL_f}$ are *expressively*

* Corresponding author.
*E-mail address:* luca.geatti@uniud.it (L. Geatti).

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*equivalent* [9–11], i.e., all properties expressible in $\mathsf{LTL_f}$ are also expressible in pLTL and *vice versa*. Since the 80s, pLTL has been advocated as a suitable declarative, logic programming language [12,13]. What is hard to deny is that certain properties, like, for instance, those characterizing planning problems ("*to reach a goal while always obeying to a safety rule*") are more naturally expressed using past modalities [9] ($\texttt{goal} \wedge \mathsf{H}\ \texttt{safety\_rule}$). In any case, the most important feature of pLTL is arguably to enjoy a compilation into deterministic finite automata of singly exponential size [11,14], a result that cannot be achieved for $\mathsf{LTL_f}$ [15].

As for LTL, an important class of properties that it allows one to formulate is the set of *safety properties*, *i.e.*, properties expressing the fact that "something bad never happens", like, for instance, a deadlock or a simultaneous access to a critical section. This class of properties is captured by the Safety-LTL logic, defined as LTL in negation normal form (NNF) devoid of the *until* (U) operator. There is a tight connection between such a safety fragment of LTL and the pure past logic pLTL: Safety-LTL is expressively equivalent to G(pLTL), that is the set of formulas of type $\mathsf{G}(\alpha)$ where $\alpha$ is a pLTL formula and G is the *globally* modality. Formulas of G(pLTL) constrain a formula $\alpha$ of pLTL to be true *at each time point* of an infinite trace.

Dual to the safety fragment is the *cosafety fragment* of LTL, which expresses properties of the form: "something good will eventually happen", and it is captured by coSafety-LTL, defined as LTL in NNF devoid of the *release* (R) modality. In analogy to the case of safety, coSafety-LTL is expressively equivalent to F(pLTL), that is, the set of formulas of type $\mathsf{F}(\alpha)$, where $\alpha$ is a pLTL formula and F is the *eventually* modality, forcing $\alpha$ to be true in at least a prefix of an infinite trace.

While Safety-LTL and coSafety-LTL admit a pLTL-based characterization in terms of G(pLTL) and F(pLTL), respectively, to the best of our knowledge, there is no similar investigation in the literature for smaller safety and cosafety fragments. Among them, we would like to mention the logics $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$ and $\mathsf{LTL}[\mathsf{X},\mathsf{G}]$ [16]. $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$ is the fragment of coSafety-LTL devoid of *binary temporal* modalities. There is no study that systematically addresses the problem of finding which pLTL-based fragment of F(pLTL) captures $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$. Similarly, there is no pLTL-based characterization of $\mathsf{LTL}[\mathsf{X},\mathsf{G}]$, the fragment of Safety-LTL devoid of binary temporal modalities.

In spite of the relevance of $\mathsf{LTL_f}$ and of all the above-mentioned safety and cosafety fragments of LTL, to the best of our knowledge, there is no systematic study of their *succinctness* with respect to their pLTL-based counterparts, that is, the study of those properties (if any) which are definable in one logic with formulas of small, polynomial size, but such that all formulas in the other logic would require exponential size or more. The importance of addressing succinctness issues is twofold. On the one hand, the study of succinctness is an important theoretical tool, that joins the investigation of computational complexity and expressive power (see, for instance, the work by Hella and Vilander [17] comparing first-order logic with basic modal logic and $\mu$-calculus in terms of succinctness by means of formula size games). On the other hand, it may help in choosing the right formalism when dealing with problems like model checking and reactive synthesis.

The main contributions of the paper can be summarized as follows. First, we look for a pLTL-based characterization of $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$ and $\mathsf{LTL}[\mathsf{X},\mathsf{G}]$. As for $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$, one might be tempted to consider the logic $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y},\mathsf{O}])$, that is, the fragment of F(pLTL) featuring as past modalities the past counterparts of modalities X and F, namely, modalities Y (*yesterday*) for X and O (*once*) for F, as a good candidate. We prove that it is not, as there are formulas in $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$ that cannot be expressed in it. The main contribution of this part is to show that the addition of modality $\widetilde{\mathsf{Y}}$ (*weak tomorrow*) to $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y},\mathsf{O}])$ suffices to obtain a fragment, that we denote by $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y},\widetilde{\mathsf{Y}},\mathsf{O}])$, which is expressively equivalent to $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$. Interestingly, the proof of this result is based on an original notion of *closure under pseudo-stuttering*, which can be viewed as a generalization of the stuttering condition given by Sistla in [18]. An analogous result holds for $\mathsf{LTL}[\mathsf{X},\mathsf{G}]$: we prove that $\mathsf{G}(\mathsf{pLTL}[\mathsf{Y},\widetilde{\mathsf{Y}},\mathsf{H}])$, where modality H (*historically*) is the past counterpart of G, is expressively equivalent to $\mathsf{LTL}[\mathsf{X},\mathsf{G}]$.

Then, we focus on succinctness issues for the considered safety and cosafety fragments of LTL with respect to their pLTL-based counterparts. We first show that G(pLTL) *can be exponentially more succinct than* Safety-LTL, that is, there exists a family of properties which can be expressed in G(pLTL) by means of formulas of size $n$ such that the size of all Safety-LTL formulas defining them is at least exponential in $n$. By a duality argument, we derive a similar result for the cosafety case: F(pLTL) can be exponentially more succinct than coSafety-LTL. Whether, conversely, Safety-LTL (resp., coSafety-LTL) can be exponentially more succinct than G(pLTL) (resp., F(pLTL)) remain – to the best of our knowledge – an open question. We address the same problem for safety and cosafety fragments devoid of binary temporal modalities proving that $\mathsf{G}(\mathsf{pLTL}[\mathsf{Y},\widetilde{\mathsf{Y}},\mathsf{H}])$ *and* $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y},\widetilde{\mathsf{Y}},\mathsf{O}])$ *can be exponentially more succinct than* $\mathsf{LTL}[\mathsf{X},\mathsf{G}]$ *and* $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$, respectively. In a recent work [19], Geatti et al. show that, for these fragments, the *vice versa* holds as well.

Finally, we prove that pLTL *can be exponentially more succinct than* $\mathsf{LTL_f}$ and *vice versa*. This result has three important consequences: 1. it shows that, despite being expressively equivalent, $\mathsf{LTL_f}$ and pLTL are *incomparable* when succinctness is concerned; 2. it confirms the conjecture formulated in [20], derived from the complexity gap between the realizability problem of $\mathsf{LTL_f}$, which is 2EXPTIME-complete, and pLTL, which is EXPTIME-complete; 3. it proves that *any* translation from $\mathsf{LTL_f}$ to pLTL (and *vice versa*), for which we only know a triply exponential upper bound [11], has at least an exponential complexity in the size of the initial formula. Table 1 summarises the main results of the paper.

The paper is organized as follows. In Section 2, we provide the necessary background. Section 3 studies the pLTL-based counterparts of $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$ and $\mathsf{LTL}[\mathsf{X},\mathsf{G}]$. In Section 4, we investigate the succinctness of Safety-LTL and coSafety-LTL with respect to their pure past counterparts G(pLTL) and F(pLTL), respectively, and we do the same for the fragments devoid of binary temporal modalities. Section 5 proves that pLTL can be exponentially more succinct than $\mathsf{LTL_f}$, and *vice versa*. Finally, Section 6 summarizes the achieved results and points out some future research directions.

This paper extends a previous conference paper [21]. In particular, the completely new parts are those on the expressive equivalence of $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$ and $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y},\widetilde{\mathsf{Y}},\mathsf{O}])$, and its safety counterpart (Section 3), and on their succinctness (in Section 4).

**Table 1**
Overview of the results of the paper.

| Ref. | Past | | Future |
|---|---|---|---|
| Theorem 1 | $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y},\mathsf{O}])$ | is strictly less expressive than | $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$ |
| Theorem 2 | $\mathsf{G}(\mathsf{pLTL}[\widetilde{\mathsf{Y}},\mathsf{H}])$ | is strictly less expressive than | $\mathsf{LTL}[\widetilde{\mathsf{X}},\mathsf{G}]$ |
| Theorem 3 | $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y},\widetilde{\mathsf{Y}},\mathsf{O}])$ | is expressively equivalent to | $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$ |
| Theorem 4 | $\mathsf{G}(\mathsf{pLTL}[\mathsf{Y},\widetilde{\mathsf{Y}},\mathsf{H}])$ | is expressively equivalent to | $\mathsf{LTL}[\mathsf{X},\mathsf{G}]$ |
| Theorem 5 | $\mathsf{G}(\mathsf{pLTL}[\mathsf{Y},\widetilde{\mathsf{Y}},\mathsf{H}])$ | can be exp. more succinct than | $\mathsf{LTL}[\mathsf{X},\mathsf{G}]$ |
| Theorem 5 | $\mathsf{G}(\mathsf{pLTL})$ | can be exp. more succinct than | Safety-LTL |
| Theorem 6 | $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y},\widetilde{\mathsf{Y}},\mathsf{O}])$ | can be exp. more succinct than | $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$ |
| Theorem 6 | $\mathsf{F}(\mathsf{pLTL})$ | can be exp. more succinct than | coSafety-LTL |
| Theorem 7 | pLTL | can be exp. more succinct than | $\mathsf{LTL}_\mathsf{f}$ |
| Theorem 8 | | and *vice versa* | |

## 2. Background

In this section, we give the necessary background on the temporal logic of linear time and finite state automata.

### 2.1. Linear temporal logic with past

Let $\mathcal{AP}$ be a set of *atomic propositions*. A formula $\phi$ of Linear Temporal Logic with Past (LTL+P , for short) is built as follows:

$$\begin{aligned} \phi &:= p \mid \neg p \mid \phi \vee \phi \mid \phi \wedge \phi && \text{Boolean connectives} \\ &\mid \mathsf{X}\phi \mid \widetilde{\mathsf{X}}\phi \mid \phi \mathsf{U} \phi \mid \phi \mathsf{R} \phi && \text{future modalities} \\ &\mid \mathsf{Y}\phi \mid \widetilde{\mathsf{Y}}\phi \mid \phi \mathsf{S} \phi \mid \phi \mathsf{T} \phi && \text{past modalities} \end{aligned}$$

where $p \in \mathcal{AP}$, $\mathsf{X}$ stands for *next*, $\widetilde{\mathsf{X}}$ for *weak next*, $\mathsf{U}$ for *until*, $\mathsf{R}$ for *releases*, $\mathsf{Y}$ for *yesterday*, $\widetilde{\mathsf{Y}}$ for *weak yesterday*, $\mathsf{S}$ for *since*, and $\mathsf{T}$ for *triggers*. Note that, *w.l.o.g.*, the proposed definition of LTL+P assumes formulas to be already in Negation Normal Form (NNF), that is, negations are applied only to proposition letters. For any formula $\phi$, the *size* of $\phi$, denoted by $|\phi|$, is the size of the (smallest) syntax tree of $\phi$.

Let $\sigma \in (2^{\mathcal{AP}})^+ \cup (2^{\mathcal{AP}})^\omega$ be a *word* (or *trace*) *over the alphabet* $2^{\mathcal{AP}}$. We define the *length* of $\sigma$ as $|\sigma| = n$, if $\sigma = \langle \sigma_0, \ldots, \sigma_{n-1} \rangle \in (2^{\mathcal{AP}})^+$ (in this case we say that $\sigma$ is a *finite* trace), and as $|\sigma| = \omega$, if $\sigma \in (2^{\mathcal{AP}})^\omega$ (in this case we say that $\sigma$ is an *infinite* trace). For any $1 \le i < j < |\sigma|$, we define $\sigma_{[i,j]}$ as the interval of $\sigma$ from position $i$ to position $j$. We call any subset of $(2^{\mathcal{AP}})^*$ a *language of finite words over* $2^{\mathcal{AP}}$. Similarly, a *language of infinite words over* $2^{\mathcal{AP}}$ is any subset of $(2^{\mathcal{AP}})^\omega$.

The *satisfaction* of an LTL+P formula $\phi$ by $\sigma$ at time $0 \le i < |\sigma|$, denoted by $\sigma, i \models \phi$, is defined as follows:

- $\sigma, i \models p$ iff $p \in \sigma_i$;
- $\sigma, i \models \neg p$ iff $p \notin \sigma_i$;
- $\sigma, i \models \phi_1 \vee \phi_2$ iff $\sigma, i \models \phi_1$ or $\sigma, i \models \phi_2$;
- $\sigma, i \models \phi_1 \wedge \phi_2$ iff $\sigma, i \models \phi_1$ and $\sigma, i \models \phi_2$;
- $\sigma, i \models \mathsf{X}\phi$ iff $i + 1 < |\sigma|$ and $\sigma, i + 1 \models \phi$;
- $\sigma, i \models \widetilde{\mathsf{X}}\phi$ iff either $i + 1 = |\sigma|$ or $\sigma, i + 1 \models \phi$;
- $\sigma, i \models \mathsf{Y}\phi$ iff $i > 0$ and $\sigma, i - 1 \models \phi$;
- $\sigma, i \models \widetilde{\mathsf{Y}}\phi$ iff either $i = 0$ or $\sigma, i - 1 \models \phi$;
- $\sigma, i \models \phi_1 \mathsf{U} \phi_2$ iff there exists $i \le j < |\sigma|$ such that $\sigma, j \models \phi_2$, and $\sigma, k \models \phi_1$ for all $k$, with $i \le k < j$;
- $\sigma, i \models \phi_1 \mathsf{S} \phi_2$ iff there exists $j \le i$ such that $\sigma, j \models \phi_2$, and $\sigma, k \models \phi_1$ for all $k$, with $j < k \le i$;
- $\sigma, i \models \phi_1 \mathsf{R} \phi_2$ iff either $\sigma, j \models \phi_2$ for all $i \le j < |\sigma|$, or there exists $i \le k < |\sigma|$ such that $\sigma, k \models \phi_1$ and $\sigma, j \models \phi_2$ for all $i \le j \le k$;
- $\sigma, i \models \phi_1 \mathsf{T} \phi_2$ iff either $\sigma, j \models \phi_2$ for all $0 \le j \le i$, or there exists $k \le i$ such that $\sigma, k \models \phi_1$ and $\sigma, j \models \phi_2$ for all $i \ge j \ge k$.

We say that $\sigma$ is a *model* of $\phi$ (written as $\sigma \models \phi$) iff $\sigma, 0 \models \phi$. The *language of infinite* (resp., *finite*) *traces* of $\phi$, denoted by $\mathcal{L}(\phi)$, is the set of traces $\sigma \in (2^{\mathcal{AP}})^\omega$ (resp., $\sigma \in (2^{\mathcal{AP}})^+$) such that $\sigma \models \phi$. Notice that, given $\sigma \in (2^{\mathcal{AP}})^\omega$, it holds that $\sigma, i \models \mathsf{X}\phi$ iff $\sigma, i \models \widetilde{\mathsf{X}}\phi$, for every $i < \omega$.

We use the standard shortcuts $\top := p \vee \neg p$ and $\bot := p \wedge \neg p$, for some $p \in \mathcal{AP}$, the future modalities $\mathsf{F}\phi := \top \mathsf{U} \phi$ (*eventually*) and $\mathsf{G}\phi := \bot \mathsf{R} \phi$ (*globally*), and the past modalities $\mathsf{O}\phi := \top \mathsf{S} \phi$ (*once*) and $\mathsf{H}\phi := \bot \mathsf{T} \phi$ (*historically*).

From now on, given a linear temporal logic $\mathbb{L}$, with some abuse of notation, we denote by $\mathbb{L}$ also the set of formulas of $\mathbb{L}$. A *pure future* (resp., *past*) *formula* is an LTL+P formula devoid of occurrences of past (resp., future) modalities. We denote by LTL (resp., pLTL) the set of pure future (resp., pure past) formulas, and we denote by $\mathsf{LTL}_\mathsf{f}$ the logic LTL interpreted on finite traces. Similarly, we denote by $\mathsf{LTL_f{+}P}$ the logic $\mathsf{LTL{+}P}$ interpreted on finite traces. Notice that if $\phi$ belongs to $\mathsf{pLTL}$, then we interpret $\phi$ on *finite words* only, and we say that $\sigma \in (2^{\mathcal{AP}})^+$ is a model of $\phi$ if and only if $\sigma, |\sigma|-1 \models \phi$, that is, each $\phi$ in $\mathsf{pLTL}$ is interpreted in the *last* state of a finite word (*i.e.*, $\mathsf{pLTL}$ is the pure past fragment of $\mathsf{LTL_f{+}P}$ ).

Finally, given a set $S \subseteq \{\mathsf{X}, \widetilde{\mathsf{X}}, \mathsf{F}, \mathsf{U}, \mathsf{G}, \mathsf{R}\}$ (resp., $S \subseteq \{\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{O}, \mathsf{S}, \mathsf{H}, \mathsf{T}\}$) of future (resp., past) modalities, we denote by $\mathsf{LTL}[S]$ (resp., $\mathsf{pLTL}[S]$) the set of $\mathsf{LTL}$ (resp., $\mathsf{pLTL}$) formulas whose temporal modalities belong to $S$ (if $S$ is the set of all future or past modalities, respectively, we omit it). In the following, we define the *syntactic safety fragment of* $\mathsf{LTL}$, denoted by $\mathsf{Safety\text{-}LTL}$, as the fragment $\mathsf{LTL}[\mathsf{X}, \mathsf{R}]$ [22–24] (observe that, given the equivalence of $\widetilde{\mathsf{X}}$ and $\mathsf{X}$ over infinite traces, this fragment can be considered as a notational variant of $\mathsf{LTL}[\widetilde{\mathsf{X}}, \mathsf{R}]$). Similarly, we define the *syntactic cosafety fragment of* $\mathsf{LTL}$, denoted by $\mathsf{coSafety\text{-}LTL}$, as the set of $\mathsf{LTL}[\mathsf{X}, \mathsf{U}]$ formulas.

Finally, we denote by $\mathsf{G}(\mathsf{pLTL}[S])$ (resp., $\mathsf{F}(\mathsf{pLTL}[S])$) the set of $\mathsf{LTL+P}$ formulas of the form $\mathsf{G}\alpha$ (resp., $\mathsf{F}\alpha$), with $\alpha \in \mathsf{pLTL}[S]$. Observe that the semantics of $\mathsf{G}(\mathsf{pLTL}[S])$ and $\mathsf{F}(\mathsf{pLTL}[S])$ formulas is given on infinite words, but with the subformulas $\alpha \in \mathsf{pLTL}[S]$ always evaluated at the last time point of finite prefixes of infinite traces.

Let us first introduce the notion of *expressive equivalence* of two logics.

**Definition 1.** Let $\Sigma := 2^{\mathcal{AP}}$. Two linear temporal logics $\mathbb{L}$ and $\mathbb{L}'$ interpreted on finite (resp., infinite) traces are *expressively equivalent* iff, for any language $\mathcal{L} \subseteq \Sigma^+$ (resp., $\mathcal{L} \subseteq \Sigma^\omega$), there exists a formula $\phi \in \mathbb{L}$ such that $\mathcal{L}(\phi) = \mathcal{L}$ iff there exists a formula $\phi' \in \mathbb{L}'$ such that $\mathcal{L}(\phi') = \mathcal{L}$.

It holds that $\mathsf{LTL_f}$ and $\mathsf{pLTL}$ are expressively equivalent, as stated by the following proposition.

**Proposition 1** *([9–11]).* $\mathsf{LTL_f}$ *and* $\mathsf{pLTL}$ *are expressively equivalent.*

A fundamental result by Chang et al. [23], based on the work done by Zuck [10], establishes the expressive equivalence of $\mathsf{Safety\text{-}LTL}$ and $\mathsf{G}(\mathsf{pLTL})$, and of $\mathsf{coSafety\text{-}LTL}$ and $\mathsf{F}(\mathsf{pLTL})$, when interpreted over infinite traces.

We now define what it means that a linear temporal logic $\mathbb{L}$ *can be exponentially more succinct than* a linear temporal logic $\mathbb{L}'$. We use the $\Omega$-notation $f(n) \in \Omega(g(n))$ to denote that the function $f$ is asymptotically bounded from below by $g$. Similarly, we use the $\mathcal{O}$-notation $f(n) \in \mathcal{O}(g(n))$ to denote that $f$ is asymptotically bounded from above by $g$.

**Definition 2** *([25]).* Let $\mathbb{L}$ and $\mathbb{L}'$ be two linear temporal logics. We say that $\mathbb{L}$ *can be exponentially more succinct than* $\mathbb{L}'$ *over infinite* (resp., *finite*) *traces* iff there exist a set of atomic propositions $\mathcal{AP}$ and a family of languages $\{\mathcal{L}_n\}_{n>0} \subseteq (2^{\mathcal{AP}})^\omega$ (resp., $\{\mathcal{L}_n\}_{n>0} \subseteq (2^{\mathcal{AP}})^*$) such that, for any $n > 0$,

- there exists a formula $\phi \in \mathbb{L}$ over $\mathcal{AP}$ such that its language over infinite (resp., finite) traces is $\mathcal{L}_n$ and $|\phi| \in \mathcal{O}(n)$, and
- for all formulas $\phi' \in \mathbb{L}'$ over $\mathcal{AP}$, if the language of $\phi'$ over infinite (resp., finite) traces is $\mathcal{L}_n$, then $|\phi'| \in 2^{\Omega(n)}$.

### 2.2. Finite-state automata

A *Nondeterministic Finite-state Automaton* ($\mathsf{NFA}$, for short) is a tuple $A = (2^{\mathcal{AP}}, Q, I, \Delta, F)$, where $2^{\mathcal{AP}}$ is a finite (nonempty) *alphabet*, $Q$ is a finite set of *states*, $I \subseteq Q$ is the set of *initial states*, $\Delta \subseteq Q \times 2^{\mathcal{AP}} \times Q$ is the *transition relation*, and $F \subseteq Q$ is the set of *final states*. We define the *size of* $\mathcal{A}$, denoted by $|\mathcal{A}|$, as the number of its states $|Q|$.

A *run* $\pi$ of $\mathcal{A}$ over the word $\sigma = \langle \sigma_0, \sigma_1, \ldots, \sigma_{n-1}\rangle \in (2^{\mathcal{AP}})^*$ is a finite sequence of states $\pi = \langle q_0, q_1, \ldots, q_n\rangle$ such that $(q_i, \sigma_i, q_{i+1}) \in \Delta$, for all $0 \le i < n-1$. A run $\pi = \langle q_0, q_1, \ldots, q_n\rangle$ is *accepting* if $q_n$ is a final state of $\mathcal{A}$, that is, $q_n \in F$.

Given an $\mathsf{NFA}$ $\mathcal{A} = (2^{\mathcal{AP}}, Q, I, \Delta, F)$, a word $\sigma \in (2^{\mathcal{AP}})^*$ is *accepted by* $\mathcal{A}$ iff there exists an accepting run of $\mathcal{A}$ over $\sigma$. The *language of* $\mathcal{A}$, denoted by $\mathcal{L}(\mathcal{A})$, is the set of (finite) words accepted by $\mathcal{A}$.

For every $\mathsf{LTL_f{+}P}$ formula $\phi$, with $|\phi| = n$, over a set of proposition letters $\mathcal{AP}$, we can effectively build an $\mathsf{NFA}$ whose language is exactly $\mathcal{L}(\phi)$ and whose size is at most exponential in $n$.

**Proposition 2** *([15,11,26,27]). For any formula $\phi$ of* $\mathsf{LTL_f{+}P}$ *, with $|\phi| = n$, there exists an* $\mathsf{NFA}$ *$\mathcal{A}$ such that $\mathcal{L}(\phi) = \mathcal{L}(\mathcal{A})$ and $|\mathcal{A}| \in 2^{\mathcal{O}(n)}$.*

This construction for $\mathsf{LTL_f{+}P}$ does not have a direct reference in the literature, but it follows from what showed in [15,11] for $\mathsf{LTL_f}$ and $\mathsf{pLTL}$, respectively, as well as from the results obtained in [26] and [27].

## 3. Expressiveness of cosafety and safety fragments of LTL devoid of binary temporal modalities

In this section, we investigate the expressiveness of cosafety and safety fragments devoid of binary temporal modalities. First, we show that over infinite words $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y}, \mathsf{O}])$ is not expressively equivalent to $\mathsf{LTL}[\mathsf{X}, \mathsf{F}]$, proving that there are $\mathsf{LTL}[\mathsf{X}, \mathsf{F}]$ formulas that are not equivalent to any $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y}, \mathsf{O}])$ formula. An analogous negative result holds, by duality, for $\mathsf{G}(\mathsf{pLTL}[\widetilde{\mathsf{Y}}, \mathsf{H}])$ and $\mathsf{LTL}[\mathsf{X}, \mathsf{G}]$. We then prove that $\mathsf{LTL}[\mathsf{X}, \mathsf{F}]$ and $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{O}])$ are expressively equivalent over infinite words, that is, any language $\mathcal{L}$ is definable in $\mathsf{LTL}[\mathsf{X}, \mathsf{F}]$ if and only if it is definable in $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{O}])$. The proof is based on the novel notion of *closure under pseudo-stuttering*. We then dualize the result obtaining that $\mathsf{LTL}[\mathsf{X},\mathsf{G}]$ is expressively equivalent to $\mathsf{G}(\mathsf{pLTL}[\mathsf{Y},\widetilde{\mathsf{Y}},\mathsf{H}])$.

### 3.1. Expressive non-equivalence of $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y},\mathsf{O}])$ (resp., $\mathsf{G}(\mathsf{pLTL}[\widetilde{\mathsf{Y}},\mathsf{H}])$) and $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$ (resp., $\mathsf{LTL}[\mathsf{X},\mathsf{G}]$)

We first prove that $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y},\mathsf{O}])$ is *not* expressively equivalent to $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$, showing that there exists a formula of $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$whose language is not definable in $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y},\mathsf{O}])$. The proof is based on the following characterization of languages recognized by $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y},\mathsf{O}])$ formulas.

**Proposition 3.** *Let $\phi := \mathsf{F}\psi$ be a formula of $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y},\mathsf{O}])$over a set of atomic propositions $\mathcal{AP}$ and let $\Sigma := 2^{\mathcal{AP}}$. It holds that $\mathcal{L}(\phi) = \Sigma^* \cdot \mathcal{L}(\psi) \cdot \Sigma^\omega$.*

**Proof.** By the semantics of modality $\mathsf{F}$, it holds that $\mathcal{L}(\phi) = \mathcal{L}(\psi) \cdot \Sigma^\omega$ (recall that, since $\psi \in \mathsf{pLTL}[\mathsf{Y},\mathsf{O}]$, the models of $\psi$ are, by definition, finite traces $\sigma$ such that $\sigma, |\sigma| - 1 \models \psi$). It suffices to show that $\mathcal{L}(\psi) = \Sigma^* \cdot \mathcal{L}(\psi)$ for all formulas $\psi$ of $\mathsf{pLTL}[\mathsf{Y},\mathsf{O}]$. Proving that $\mathcal{L}(\psi) \subseteq \Sigma^* \cdot \mathcal{L}(\psi)$ is trivial: for all $\sigma \in \mathcal{L}(\psi)$, we choose $\varepsilon \in \Sigma^*$, having that $\varepsilon \cdot \sigma$, that is, $\sigma$, belongs to $\Sigma^* \cdot \mathcal{L}(\psi)$.

We have to prove that $\Sigma^* \cdot \mathcal{L}(\psi) \subseteq \mathcal{L}(\psi)$. We first prove, by induction on the structure of $\psi$, that for all $\sigma \in \mathcal{L}(\psi)$ and for all $\sigma' \in \Sigma^*$, it holds that $\sigma' \cdot \sigma \in \mathcal{L}(\psi)$.

*Base Case.* Let $\psi = p$, for some $p \in \mathcal{AP}$, and let $\sigma \in \mathcal{L}(\psi)$. Since $p \in \mathcal{AP}$, and, by hypothesis, $\sigma \in \mathcal{L}(p)$, we have that $\sigma' \cdot \sigma \in \mathcal{L}(p)$ for all $\sigma' \in \Sigma^*$.

*Inductive step.* We distinguish a few cases.

**Case:** $\psi = \psi_1 \wedge \psi_2$. By hypothesis, $\sigma \in \mathcal{L}(\psi_1 \wedge \psi_2)$, and thus it holds that $\sigma \in \mathcal{L}(\psi_1)$ and $\sigma \in \mathcal{L}(\psi_2)$. By inductive hypothesis, $\sigma' \cdot \sigma \in \mathcal{L}(\psi_1)$, for all $\sigma' \in \Sigma^*$, and $\sigma' \cdot \sigma \in \mathcal{L}(\psi_2)$, for all $\sigma' \in \Sigma^*$. This is equivalent to $\sigma' \cdot \sigma \in \mathcal{L}(\psi_1)$ and $\sigma' \cdot \sigma \in \mathcal{L}(\psi_2)$, for all $\sigma' \in \Sigma^*$, that is $\sigma' \cdot \sigma \in \mathcal{L}(\psi_1) \cap \mathcal{L}(\psi_2)$. Therefore, $\sigma' \cdot \sigma \in \mathcal{L}(\psi_1 \wedge \psi_2)$ for all $\sigma' \in \Sigma^*$.

**Case:** $\psi = \psi_1 \vee \psi_2$. By hypothesis, $\sigma \in \mathcal{L}(\psi_1 \vee \psi_2)$, and thus it holds that either $\sigma \in \mathcal{L}(\psi_1)$ or $\sigma \in \mathcal{L}(\psi_2)$. By inductive hypothesis, either $\sigma' \cdot \sigma \in \mathcal{L}(\psi_1)$, for all $\sigma' \in \Sigma^*$, or $\sigma' \cdot \sigma \in \mathcal{L}(\psi_2)$, for all $\sigma' \in \Sigma^*$. Since $\mathcal{L}(\psi_1) \subseteq \mathcal{L}(\psi_1) \cup \mathcal{L}(\psi_2)$ and $\mathcal{L}(\psi_2) \subseteq \mathcal{L}(\psi_1) \cup \mathcal{L}(\psi_2)$, this is equivalent to $\sigma' \cdot \sigma \in \mathcal{L}(\psi_1) \cup \mathcal{L}(\psi_2)$, for all $\sigma' \in \Sigma^*$. Therefore, $\sigma' \cdot \sigma \in \mathcal{L}(\psi_1 \vee \psi_2)$, for all $\sigma' \in \Sigma^*$.

**Case:** $\psi = \mathsf{Y}\psi_1$. By hypothesis $\sigma \in \mathcal{L}(\mathsf{Y}\psi_1)$, and thus it holds that $|\sigma| > 1$ and $\sigma'' \in \mathcal{L}(\psi_1)$, where $\sigma''$ is the word obtained from $\sigma$ by removing the last state. By inductive hypothesis, $\sigma' \cdot \sigma'' \in \mathcal{L}(\psi_1)$, for all $\sigma' \in \Sigma^*$. By the semantics of modality $\mathsf{Y}$, for all $a \in \Sigma$, it holds that $\sigma' \cdot \sigma'' \cdot a \models \mathsf{Y}\psi_1$. In particular, if we choose $a$ as the last state of $\sigma$, we have that $\sigma' \cdot \sigma \models \mathsf{Y}\psi_1$, for all $\sigma' \in \Sigma^*$.

**Case:** $\psi = \mathsf{O}\psi_1$. By hypothesis $\sigma \in \mathcal{L}(\mathsf{O}\psi_1)$, and thus it holds that $\sigma'' \in \mathcal{L}(\psi_1)$, where $\sigma''$ is some (not necessarily proper) *prefix* of $\sigma$. By inductive hypothesis, $\sigma' \cdot \sigma'' \in \mathcal{L}(\psi_1)$, for all $\sigma' \in \Sigma^*$. By the semantics of modality $\mathsf{O}$, for all $\tau \in \Sigma^*$, it holds that $\sigma' \cdot \sigma'' \cdot \tau \models \mathsf{O}\psi_1$. In particular, if we choose $\tau$ as the suffix of $\sigma$ obtained by removing $\sigma''$ from $\sigma$, we have that $\sigma' \cdot \sigma \models \mathsf{O}\psi_1$, for all $\sigma' \in \Sigma^*$.

We now prove that $\Sigma^* \cdot \mathcal{L}(\psi) \subseteq \mathcal{L}(\psi)$. Let $\sigma$ be a word in $\Sigma^* \cdot \mathcal{L}(\psi)$. It holds that $\sigma = \sigma' \cdot \sigma''$ for some $\sigma' \in \Sigma^*$ and $\sigma'' \in \mathcal{L}(\psi)$. Since we proved that, for all $\tau \in \mathcal{L}(\psi)$ and $\tau' \in \Sigma^*$, it holds that $\tau' \cdot \tau \in \mathcal{L}(\psi)$, in the particular case in which $\tau' = \sigma'$ and $\tau = \sigma''$, we have that $\sigma' \cdot \sigma'' \in \mathcal{L}(\psi)$, *i.e.*, $\sigma \in \mathcal{L}(\psi)$. $\square$

Following Proposition 3, to prove that $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y},\mathsf{O}])$is not expressively equivalent to $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$ it suffices to construct an $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$ formula whose language is *not* of the form $\Sigma^* \cdot \mathcal{L} \cdot \Sigma^\omega$. This is done by the next proposition.

**Proposition 4.** *There exists a formula $\phi$ of $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$ such that $\mathcal{L}(\phi)$ is not definable in $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y},\mathsf{O}])$.*

**Proof.** Consider the $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$ formula $\phi := \mathsf{p}$ over the set of atomic propositions $\mathcal{AP} := \{p\}$. Let $\Sigma := 2^{\mathcal{AP}}$. We have that $\mathcal{L}(\phi) = \{p\} \cdot \Sigma^\omega$. Clearly, $\mathcal{L}(\phi) \neq \Sigma^* \cdot \mathcal{L} \cdot \Sigma^\omega$ for every $\mathcal{L} \subseteq \Sigma^*$. Therefore, by Proposition 3, $\mathcal{L}(\phi)$ is *not* expressible in $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y},\mathsf{O}])$. $\square$

The following theorem directly follows.

**Theorem 1.** $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y},\mathsf{O}])$*is not expressively equivalent to* $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$.

By duality, we get the following result.

**Theorem 2.** $\mathsf{G}(\mathsf{pLTL}[\widetilde{\mathsf{Y}},\mathsf{H}])$*is not expressively equivalent to* $\mathsf{LTL}[\mathsf{X},\mathsf{G}]$.

**Proof.** Towards a contradiction, suppose that the statement does not hold, that is, suppose that $\mathsf{G}(\mathsf{pLTL}[\widetilde{\mathsf{Y}},\mathsf{H}])$and $\mathsf{LTL}[\mathsf{X},\mathsf{G}]$ are expressively equivalent. In particular, we have that, for any alphabet $\Sigma := 2^{\mathcal{AP}}$ and for any language $\mathcal{L} \subseteq \Sigma^\omega$, given a

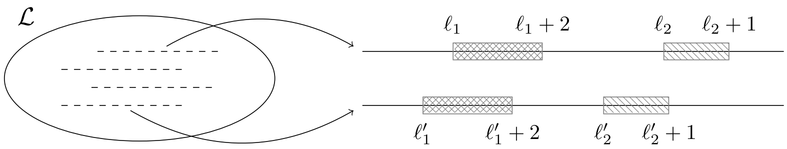


**Fig. 1.** Visualization of pseudo-stuttering. On the left, the $\mathcal{L}(\phi)$ language. On the right, two words, each with its own interval of positions where wstutt$(\cdot,\cdot)$ cannot be applied. Formulas $p \wedge \mathsf{Y}q \wedge \mathsf{YY}p$ and $q \vee (p \wedge \mathsf{Y}q)$ hold in each word at the end of the intervals of length $c_1 = 2$ and $c_2 = 1$, respectively.

formula $\phi \in \mathsf{LTL}[\mathsf{X},\mathsf{G}]$ such that $\mathcal{L}(\phi) = \mathcal{L}$, there exists a formula $\psi \in \mathsf{G}(\mathsf{pLTL}[\widetilde{\mathsf{Y}},\mathsf{H}])$ such that $\mathcal{L}(\psi) = \mathcal{L}$. Let $\phi$ be a formula of $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$, as in Proposition 4, and let $\overline{\phi}$ be the NNF (Negation Normal Form) of $\neg\phi$. Given the equivalence of $\mathsf{X}$ and $\widetilde{\mathsf{X}}$ over infinite traces, it holds that $\overline{\phi} \in \mathsf{LTL}[\mathsf{X},\mathsf{G}]$, and $\mathcal{L}(\overline{\phi}) = \overline{\mathcal{L}(\phi)}$. By our assumption, we have that there exists a formula $\psi$ of $\mathsf{G}(\mathsf{pLTL}[\widetilde{\mathsf{Y}},\mathsf{H}])$such that $\mathcal{L}(\psi) = \overline{\mathcal{L}(\phi)}$. By taking the NNF of $\neg\psi$, we obtain a formula $\overline{\psi}$ of $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y},\mathsf{O}])$ such that $\mathcal{L}(\overline{\psi}) = \overline{\mathcal{L}(\psi)} = \mathcal{L}(\phi)$, contradicting Proposition 4. $\square$

As we will see below, it follows from Theorem 3 and Theorem 4 that $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$ and $\mathsf{LTL}[\mathsf{X},\mathsf{G}]$ are indeed *strictly more* expressive than $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y},\mathsf{O}])$ and $\mathsf{G}(\mathsf{pLTL}[\widetilde{\mathsf{Y}},\mathsf{H}])$, respectively.

### 3.2. *Closure under pseudo-stuttering*

We now prove that $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$ is expressively equivalent to $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y},\widetilde{\mathsf{Y}},\mathsf{O}])$. In [16], a transformation of $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$ formulas into equivalent $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y},\widetilde{\mathsf{Y}},\mathsf{O}])$ ones is given, and thus it suffices to show the opposite direction, that is, for every $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y},\widetilde{\mathsf{Y}},\mathsf{O}])$ formula there exists an equivalent $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$ one. We prove this result in two steps: (i) we show that all languages definable in $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y},\widetilde{\mathsf{Y}},\mathsf{O}])$ enjoy a suitable model-theoretic property $\mathbb{P}$; (ii) we prove that all languages for which the property $\mathbb{P}$ holds are definable in $\mathsf{LTL}[\mathsf{X},\mathsf{F}]$. We define the property $\mathbb{P}$ as a generalization of the stuttering property introduced by Sistla in [18], and we call it *closure under pseudo-stuttering*. Intuitively, the pseudo-stuttering condition imposes that a finite number of finite intervals of a trace remain untouched, while all the other positions can be arbitrarily modified, without altering the membership to the language.

Let $\Sigma := 2^{\mathcal{AP}}$, where $\mathcal{AP}$ is a finite set of atomic propositions. The definition of the property of closure under pseudo-stuttering is based on the following operator wstutt$(\cdot,\cdot)$.

**Definition 3** *(The* wstutt$(\cdot,\cdot)$ *operator).* For any $\sigma := \langle \sigma_0, \ldots, \sigma_i, \ldots \rangle \in \Sigma^+ \cup \Sigma^\omega$ and any $0 \le i < |\sigma|$, we define the set of words wstutt$(\sigma, i) \subseteq \Sigma^+ \cup \Sigma^\omega$ as follows:

$$\mathrm{wstutt}(\sigma, i) := \bigcup_{a \in \Sigma} \langle \sigma_0, \ldots, a, \sigma_i, \ldots \rangle$$

**Definition 4** *(Pseudo-stuttering).* Let $\mathcal{L} \subseteq \Sigma^+ \cup \Sigma^\omega$. We say that $\mathcal{L}$ is *closed under pseudo-stuttering* iff there exist $n \in \mathbb{N}$ and $c_1, \ldots, c_n \in \mathbb{N}$ such that, for all $\sigma \in \mathcal{L}$, there exist $l_1, \ldots, l_n \in \mathbb{N}$ such that, for all $k \notin \bigcup_{1 \le i \le n} [l_i, l_i + c_i]$ and $0 \le k < |\sigma|$, it holds that wstutt$(\sigma, k) \subseteq \mathcal{L}$.

We point out that Definition 3 applies to languages of both finite and infinite words. Moreover, we notice that, in Definition 4, $n$ may be equal to 0 and positions $l_1, \ldots, l_n$, as well as the sizes $c_1, \ldots, c_n$ of the intervals, need not to be different from each other.

*Example.* Consider the set of atomic propositions $\mathcal{AP} := \{p, q\}$ and the following formula $\phi$:

$$\mathsf{F}((q \vee (p \wedge \mathsf{Y}q)) \wedge \mathsf{O}(p \wedge \mathsf{Y}q \wedge \mathsf{YY}p))$$

The language of $\phi$ is $\mathcal{L}(\phi) := \Sigma^* \cdot I \cdot \Sigma^* \cdot K \cdot \Sigma^\omega$, where $K := \Sigma_q \cup (\Sigma_q \cdot \Sigma_p)$ and $I := \Sigma_p \cdot \Sigma_q \cdot \Sigma_p$, with $\Sigma_q := \{\{q\}, \{p, q\}\}$ and $\Sigma_p := \{\{p\}, \{p, q\}\}$. The language $\mathcal{L}(\phi)$ is closed under pseudo-stuttering. In fact, if we choose $n = 2$, $c_1 = 2$ and $c_2 = 1$, then Definition 4 is fulfilled. Fig. 1 visualizes how Definition 4 applies to $\mathcal{L}(\phi)$.

Conversely, the language of the $\mathsf{F}(\mathsf{pLTL})$ formula $\mathsf{F}(p\mathsf{S}q \wedge \mathsf{O}r)$ is *not* closed under pseudo-stuttering. To see it, consider any word $\sigma$ in the language in which $q$ is fulfilled before $r$. Let $i$ and $j$ be the positions of $\sigma$ in which $q$ and $r$ are fulfilled, respectively. Since in the interval between $i$ (excluded) and $j$ (included) proposition $p$ must hold, the application of wstutt$(\cdot,\cdot)$ inside this interval is forbidden. But since there are infinitely many words in the language of $\mathsf{F}(p\mathsf{S}q \wedge \mathsf{O}r)$ for which such an interval is of increasing length, there is no way to choose an appropriate length for the interval as requested by Definition 4.

*Differences with standard stuttering.* The definition of *pseudo-stuttering* differs from that of *stuttering* [18] in two respects:

- pseudo-stuttering can both repeat a letter at a certain position, like standard stuttering, and change the letter at that position;

- there is a finite number of intervals (whose number and whose length is given *a priori* to the language) where pseudo-stuttering is forbidden; in contrast, in standard stuttering, all positions can in principle be stuttered.

### 3.3. Closure under pseudo-stuttering of $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{O}])$-definable languages

We now prove that all languages definable in $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{O}])$ are closed under pseudo-stuttering. We first prove that all languages (of finite words) definable in $\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{O}]$are closed under pseudo-stuttering. Recall that formulas of $\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{O}]$are interpreted at the last position of a finite word.

**Lemma 1.** *For every formula $\phi \in \mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{O}]$, it holds that $\mathcal{L}(\phi)$ is closed under pseudo-stuttering.*

**Proof.** Let $\phi$ be a formula of $\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{O}]$over a set of atomic propositions $\mathcal{AP}$, and let $\Sigma \coloneqq 2^{\mathcal{AP}}$. We proceed by induction on the structure of $\phi$.

*Base case.* Let $\phi = p$, for some $p \in \mathcal{AP}$. We prove that $\mathcal{L}(\phi)$ is closed under pseudo-stuttering. Following Definition 4, we let $n = 1$ and $c_1 = 0$. Let $\sigma \in \mathcal{L}(\phi)$. Since $\phi = p$, it holds that for every $k < |\sigma| - 1$, that is, for all positions but the last one, if any, and for all $\sigma' \in \mathsf{wstutt}(\sigma, k)$, we have that $\sigma' \models \phi$. Therefore, with $l_1 = |\sigma| - 1$, we have that for each $k \notin [l_1, l_1 + c_1]$, *i.e.*, for each $k < |\sigma| - 1$, it holds that $\mathsf{wstutt}(\sigma, k) \models \phi$. The case for $\phi = \neg p$ is analogous.

*Inductive step.* We proceed by structural induction.

**Case:** $\phi = \phi_1 \wedge \phi_2$. By inductive hypothesis, we have that both $\mathcal{L}(\phi_1)$ and $\mathcal{L}(\phi_2)$ are closed under pseudo-stuttering, that is (in the case of $\phi_1$),

$$\begin{aligned}&\exists n_1 \exists c^1_1 \ldots c^1_{n_1} \,.\, \forall \sigma \in \mathcal{L}(\phi_1) \,.\, \exists l^1_1 \ldots l^1_{n_1} \,.\\ &\forall k \notin \bigcup_{1 \le i \le n_1} [l^1_i, l^1_i + c^1_i] \,.\, (k < |\sigma| \to \mathsf{wstutt}(\sigma, k) \subseteq \mathcal{L}(\phi_1))\end{aligned} \tag{1}$$

and (in the case of $\phi_2$)

$$\begin{aligned}&\exists n_2 \exists c^2_1 \ldots c^2_{n_2} \,.\, \forall \sigma \in \mathcal{L}(\phi_2) \,.\, \exists l^2_1 \ldots l^2_{n_2} \,.\\ &\forall k \notin \bigcup_{1 \le i \le n_2} [l^2_i, l^2_i + c^2_i] \,.\, (k < |\sigma| \to \mathsf{wstutt}(\sigma, k) \subseteq \mathcal{L}(\phi_2))\end{aligned} \tag{2}$$

We take $n \coloneqq n_1 + n_2$ and, for all $1 \le i \le n$, we take $c_i \coloneqq c^1_i$ if $i \le n_1$, and $c_i \coloneqq c^2_{i-n_1}$ otherwise. We prove that, for all $\sigma \in \mathcal{L}(\phi_1 \wedge \phi_2)$, the trace $\sigma$ satisfies the conditions in Definition 4. Let $\sigma$ be any trace in $\mathcal{L}(\phi_1 \wedge \phi_2)$. By the semantics of conjunction, we have that $\sigma \in \mathcal{L}(\phi_1) \cap \mathcal{L}(\phi_2)$. We know that there exist positions $l^1_1 \ldots l^1_{n_1}$ and $l^2_1 \ldots l^2_{n_1}$ with the properties of Eqs. (1) and (2), respectively. For all $1 \le i \le n$, we define $l_i \coloneqq l^1_i$ if $i \le n_1$, or $l_i \coloneqq l^2_{i-n_1}$ otherwise. Since the intervals in $\bigcup_{1 \le i \le n_1} [l^1_i, l^1_i + c^1_i]$ are also in $\bigcup_{1 \le i \le n} [l_i, l_i + c_i]$, by Eq. (1), for all $k \notin \bigcup_{1 \le i \le n} [l_i, l_i + c_i]$, with $k < |\sigma|$, we have $\mathsf{wstutt}(\sigma, k) \subseteq \mathcal{L}(\phi_1)$. Similarly, since the intervals in $\bigcup_{1 \le i \le n_2} [l^2_i, l^2_i + c^2_i]$ are also in $\bigcup_{1 \le i \le n} [l_i, l_i + c_i]$, by Eq. (2), for all $k \notin \bigcup_{1 \le i \le n} [l_i, l_i + c_i]$, with $k < |\sigma|$, we have $\mathsf{wstutt}(\sigma, k) \subseteq \mathcal{L}(\phi_2)$.

It follows that, for all $k \notin \bigcup_{1 \le i \le n} [l_i, l_i + c_i]$, with $k < |\sigma|$, we have $\mathsf{wstutt}(\sigma, k) \subseteq \mathcal{L}(\phi_1) \cap \mathcal{L}(\phi_2)$, that is, $\mathsf{wstutt}(\sigma, k) \subseteq \mathcal{L}(\phi_1 \wedge \phi_2)$. Therefore, $\mathcal{L}(\phi_1 \wedge \phi_2)$ is closed under pseudo-stuttering.

**Case:** $\phi = \phi_1 \vee \phi_2$. By inductive hypothesis, we have that both $\mathcal{L}(\phi_1)$ and $\mathcal{L}(\phi_2)$ are closed under pseudo-stuttering, that is, Eqs. (1) and (2) hold. As for the case of conjunctions, we take $n \coloneqq n_1 + n_2$ and, for all $1 \le i \le n$, we take $c_i \coloneqq c^1_i$ if $i \le n_1$, and $c_i \coloneqq c^2_{i-n_1}$ otherwise. We prove that, for all $\sigma \in \mathcal{L}(\phi)$, the trace $\sigma$ satisfies the conditions in Definition 4. Let $\sigma$ be any trace in $\mathcal{L}(\phi)$. Since $\phi = \phi_1 \vee \phi_2$, we have that either $\sigma \models \phi_1$ or $\sigma \models \phi_2$. Without loss of generality, suppose that $\sigma \models \phi_1$. We know that there exist positions $l^1_1 \ldots l^1_{n_1}$ with the properties of Eq. (1). For each $1 \le i \le n$, we define $l_i \coloneqq l^1_i$ if $i \le n_1$, and $l_i \coloneqq 0$ otherwise. Notice that setting $l_i$ to 0 for $i > n_1$ is an arbitrary choice: any other value would still be correct. Since the intervals in $\bigcup_{1 \le i \le n_1} [l^1_i, l^1_i + c^1_i]$ are also in $\bigcup_{1 \le i \le n} [l_i, l_i + c_i]$, by Eq. (1), for all $k \notin \bigcup_{1 \le i \le n} [l_i, l_i + c_i]$, with $k < |\sigma|$, we have $\mathsf{wstutt}(\sigma, k) \subseteq \mathcal{L}(\phi_1)$. Since $\mathcal{L}(\phi_1) \subseteq \mathcal{L}(\phi_1) \cup \mathcal{L}(\phi_2) = \mathcal{L}(\phi_1 \vee \phi_2)$, we have that $\mathsf{wstutt}(\sigma, k) \subseteq \mathcal{L}(\phi_1 \vee \phi_2)$, for all $k \notin \bigcup_{1 \le i \le n} [l_i, l_i + c_i]$ and $k < |\sigma|$. The case $\sigma \models \phi_2$ is completely symmetric, and thus we can conclude that $\mathcal{L}(\phi_1 \vee \phi_2)$ is closed under stuttering.

**Case:** $\phi = \mathsf{Y}\phi_1$. By inductive hypothesis, we have that $\mathcal{L}(\phi_1)$ is closed under pseudo-stuttering, that is, Eq. (1) holds. We define $n = n_1 + 1$ and, for every $1 \le i \le n$, we set $c_i = c^1_i$ if $i \le n_1$, and $c_i = 0$ otherwise. In other words, we concatenated a point-interval to the right. We prove that, for all $\sigma \in \mathcal{L}(\mathsf{Y}\phi_1)$, the trace $\sigma$ satisfies the conditions in Definition 4. Let $\sigma \in \mathcal{L}(\mathsf{Y}\phi_1)$. By the semantics of modality $\mathsf{Y}$, it holds that $|\sigma| > 1$ and $\sigma' \in \mathcal{L}(\phi_1)$, where $\sigma'$ is the trace obtained from $\sigma$ by removing the last state.

We know that there exist positions $l^1_1 \ldots l^1_{n_1}$ such that for every $k \notin \bigcup_{1 \le i \le n_1} [l^1_i, l^1_i + c^1_i]$, with $k < |\sigma'|$, it holds $\mathsf{wstutt}(\sigma', k) \subseteq \mathcal{L}(\phi_1)$. By the semantics of modality $\mathsf{Y}$ and by definition of $\mathsf{wstutt}(\cdot, \cdot)$, we have that $\mathsf{wstutt}(\sigma' \cdot a, k) \subseteq \mathcal{L}(\mathsf{Y}\phi_1)$, for all $a \in \Sigma$ and all $k \notin \bigcup_{1 \le i \le n_1} [l^1_i, l^1_i + c^1_i] \cup \{|\sigma'|\}$, with $k < |\sigma'| + 1$. For each $1 \le i \le n$, we define $l_i \coloneqq l^1_i$ if $i \le n_1$, and $l_i \coloneqq |\sigma| - 1$ otherwise. In such a way, we let the new interval to start at the very last position of $\sigma$. Since

$\sigma = \sigma' \cdot a$, for some $a \in \Sigma$, and since $\bigcup_{1\le i\le n}[l_i, l_i + c_i] = \bigcup_{1\le i\le n_1}[l_i^1, l_i^1 + c_i^1] \cup \{|\sigma'|\}$, we have that $\mathsf{wstutt}(\sigma, k) \subseteq \mathcal{L}(\mathsf{Y}\phi_1)$, for all $k \notin \bigcup_{1\le i\le n}[l_i, l_i + c_i]$ such that $k < |\sigma|$. Therefore, $\mathcal{L}(\mathsf{Y}\phi_1)$ is closed under pseudo-stuttering.

**Case:** $\phi = \widetilde{\mathsf{Y}}\phi_1$. By inductive hypothesis, we have that $\mathcal{L}(\phi_1)$ is closed under pseudo-stuttering, that is, Eq. (1) holds. We define $n = n_1 + 1$ and, for every $1 \le i \le n$, we set $c_i := c_i^1$ if $i \le n_1$, and $c_i := 0$ otherwise, that is, we concatenated a point-interval to the right. We prove that, for all $\sigma \in \mathcal{L}(\widetilde{\mathsf{Y}}\phi_1)$, the trace $\sigma$ satisfies the conditions of Definition 4. Let $\sigma \in \mathcal{L}(\widetilde{\mathsf{Y}}\phi_1)$. By the semantics of modality $\widetilde{\mathsf{Y}}$, it holds that either $|\sigma| = 1$ or $\sigma' \in \mathcal{L}(\phi_1)$, where $\sigma'$ is the trace obtained from $\sigma$ by removing the last state. The proof for the latter case is identical to the proof for $\mathsf{Y}\phi_1$, so we focus on the former case. For each $1 \le i \le n$, we let $l_i = 0$. Since $|\sigma| = 1$, we have that there is no $k < |\sigma|$ such $k \notin \bigcup_{1\le i\le n}[l_i, l_i + c_i]$, and thus the implication is trivially true. Therefore, $\mathcal{L}(\widetilde{\mathsf{Y}}\phi_1)$ is closed under pseudo-stuttering.

**Case:** $\phi = \mathsf{O}\phi_1$. By inductive hypothesis, we have that $\mathcal{L}(\phi_1)$ is closed under pseudo-stuttering, that is, Eq. (1) holds. We put $n = n_1$ and, for each $1 \le i \le n$, we set $c_i = c_i^1$. We prove that, for all $\sigma \in \mathcal{L}(\mathsf{O}\phi_1)$, the trace $\sigma$ satisfies the conditions of Definition 4. Let $\sigma \in \mathcal{L}(\mathsf{O}\phi_1)$. By the semantics of modality $\mathsf{O}$, it holds that $\sigma' \in \mathcal{L}(\phi_1)$, where $\sigma'$ is some prefix of $\sigma$. We know that there exist positions $l_1^1 \dots l_{n_1}^1$ such that for all $k \notin \bigcup_{1\le i\le n_1}[l_i^1, l_i^1 + c_i^1]$, with $k < |\sigma'|$, it holds that $\mathsf{wstutt}(\sigma', k) \subseteq \mathcal{L}(\phi_1)$. For each $1 \le i \le n$, we let $l_i = l_i^1$. By the semantics of modality $\mathsf{O}$, for all $\sigma'' \in \Sigma^*$ and all $k \notin \bigcup_{1\le i\le n}[l_i, l_i + c_i]$, with $k < |\sigma'| + |\sigma''|$, it holds that $\mathsf{wstutt}(\sigma' \cdot \sigma'', k) \subseteq \mathcal{L}(\mathsf{O}\phi_1)$. Since $\sigma = \sigma' \cdot \sigma''$ for some $\sigma'' \in \Sigma^*$, it holds that $\mathsf{wstutt}(\sigma, k) \subseteq \mathcal{L}(\mathsf{O}\phi_1)$ for every $k \notin \bigcup_{1\le i\le n}[l_i, l_i + c_i]$, with $k < |\sigma|$. Therefore, $\mathcal{L}(\mathsf{O}\phi_1)$ is closed under pseudo-stuttering. □

**Lemma 2.** *For every formula $\phi \in \mathsf{F}(\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{O}])$, it holds that $\mathcal{L}(\phi)$ is closed under pseudo-stuttering.*

**Proof.** Let $\phi := \mathsf{F}(\phi_1)$ be a formula of $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{O}])$ over a set of atomic propositions $\mathcal{AP}$ and let $\Sigma := 2^{\mathcal{AP}}$. Since, by Lemma 1, $\mathcal{L}(\phi_1)$ is closed under pseudo-stuttering, it holds that:

$$\begin{aligned}&\exists n_1 \exists c_1^1 \dots c_{n_1}^1 \,.\, \forall \sigma \in \mathcal{L}(\phi_1) \,.\, \exists l_1^1 \dots l_{n_1}^1 \,.\\ &\forall k \notin \bigcup_{1\le i\le n_1} [l_i^1, l_i^1 + c_i^1] \,.\, (k < |\sigma| \rightarrow \mathsf{wstutt}(\sigma, k) \subseteq \mathcal{L}(\phi_1))\end{aligned} \tag{3}$$

Let $n = n_1$ and, for all $1 \le i \le n$, let $c_i = c_i^1$. Consider a word $\sigma \in \mathcal{L}(\mathsf{F}(\phi_1))$. By the semantics of modality $\mathsf{F}$, it holds that there exists a position $l \in \mathbb{N}$ such that $\sigma_{[0,l]} \in \mathcal{L}(\phi_1)$, where $\sigma_{[0,l]}$ is the prefix of $\sigma$ up to position $l$. We know that there exist $l_1^1 \dots l_{n_1}^1$ such that, for all $k \notin \bigcup_{1\le i\le n_1}[l_i^1, l_i^1 + c_i^1]$, with $k < |\sigma|$, we have that $\mathsf{wstutt}(\sigma, k) \subseteq \mathcal{L}(\phi_1)$. Clearly, for all $\sigma' \in \Sigma^\omega$ and all $k \notin \bigcup_{1\le i\le n_1}[l_i^1, l_i^1 + c_i^1]$, it holds that $\mathsf{wstutt}(\sigma_{[0,l]} \cdot \sigma', k) \subseteq \mathcal{L}(\mathsf{F}(\phi_1))$. Since $\sigma = \sigma_{[0,l]} \cdot \sigma''$ for some $\sigma'' \in \Sigma^\omega$, we have that $\mathsf{wstutt}(\sigma, k) \subseteq \mathcal{L}(\mathsf{F}(\phi_1))$. Therefore, $\mathcal{L}(\phi)$ is closed under pseudo-stuttering. □

### 3.4. *Expressive equivalence of* $\mathsf{LTL}[\mathsf{X}, \mathsf{F}]$ *and* $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{O}])$

We now prove that every language closed under pseudo-stuttering is definable in $\mathsf{LTL}[\mathsf{X}, \mathsf{F}]$.

**Lemma 3.** *Let $\mathcal{AP}$ be a set of proposition letters and let $\Sigma := 2^{\mathcal{AP}}$. For every language $\mathcal{L} \subseteq \Sigma^\omega$ that is closed under pseudo-stuttering, there exists a formula $\phi$ of $\mathsf{LTL}[\mathsf{X}, \mathsf{F}]$ over $\mathcal{AP}$ such that $\mathcal{L}(\phi) = \mathcal{L}$.*

**Proof.** Let $\mathcal{L} \subseteq \Sigma^\omega$ be a language closed under pseudo-stuttering. By Definition 4, there exist $n \in \mathbb{N}$ and $c_1, \dots, c_n \in \mathbb{N}$ such that, for all $\sigma \in \mathcal{L}$, there exist $l_1^\sigma, \dots, l_n^\sigma$ such that, for all $k \notin \bigcup_{1\le i\le n}[l_i^\sigma, l_i^\sigma + c_i]$, it holds that $\mathsf{wstutt}(\sigma, k) \subseteq \mathcal{L}$.

To begin with, we make the following preprocessing step. For all $\sigma \in \mathcal{L}$ and all $1 \le i < j \le n$ such that $[l_i^\sigma, l_i^\sigma + c_i] \cap [l_j^\sigma, l_j^\sigma + c_j] \ne \emptyset$, we merge the two intervals into the bigger interval $[l_i^\sigma, l_i^\sigma + c_i] \cup [l_j^\sigma, l_j^\sigma + c_j]$ (notice that if some pair of indexes $i, j$ satisfies the condition, the condition is also satisfied by all pairs $i, k$, with $i < k < j$).

Once such a step has been completed, the resulting picture is as follows.

- All intervals are disjoint. In general, preprocessing may cause a decrease of both the value $n$ and the number of $c_i$, as well as of the number of $l_i^\sigma$, for a given $\sigma \in \mathcal{L}$.
- If, for some $\sigma$, we merged $[l_i^\sigma, l_i^\sigma + c_i]$ and $[l_j^\sigma, l_j^\sigma + c_j]$, then the resulting interval is of length at most $c_i + c_j$. This, in turn, may have caused other intervals (in a different word $\sigma'$) to overlap and thus it triggered the merge of two other intervals. In the worst case, we merged all $n$ intervals into one, big interval of size at most $c_1 + \dots + c_n$.
- In general, the merge of two intervals $[l_i^\sigma, l_i^\sigma + c_i]$ and $[l_j^\sigma, l_j^\sigma + c_j]$, with $i < j$, forces the replacement of $[l_i^\sigma, l_i^\sigma + c_i]$ by $[l_i^\sigma, l_i^\sigma + c_i']$, with $c_i' > c_i$. Now, by Definition 4, the sizes of the intervals must be the same for all $\overline{\sigma} \in \mathcal{L}$. Luckily, by Definition 3, if $\mathsf{wstutt}(\overline{\sigma}, k) \subseteq \mathcal{L}$ for all $k \notin [l_i, l_i + c_i]$, then $\mathsf{wstutt}(\overline{\sigma}, k) \subseteq \mathcal{L}$ for all $k \notin [l_i, l_i + \overline{c_i}]$, for any $\overline{c_i} > c_i$, and thus we can safely replace $c_i$ by $c_i'$.

We can thus conclude that, after the preprocessing, there exist $m \in \mathbb{N}$, with $m \le n$, and $d_1, \dots, d_m \in \mathbb{N}$ such that, for all $\sigma \in \mathcal{L}$, there exist *at most* $m$ positions $l_1^\sigma, \dots, l_m^\sigma$ such that all intervals $[l_i^\sigma, l_i^\sigma + d_i]$ are disjoint from each other, and for all $k \notin \bigcup_{1\le i\le m}[l_i^\sigma, l_i^\sigma + d_i]$, it holds that $\mathsf{wstutt}(\sigma, k) \subseteq \mathcal{L}$. We say "at most $m$ positions" because, for some $\sigma \in \mathcal{L}$, it may be the case that $[l_j^\sigma, l_j^\sigma + d_j] \subset [l_i^\sigma, l_i^\sigma + d_i]$, for some $i, j$ with $i < j$, and thus position $j$ must be ignored as $[l_j^\sigma, l_j^\sigma + d_j]$ has actually been incorporated into $[l_i^\sigma, l_i^\sigma + d_i]$ by the merging operation.

For each $1 \le i \le m$ (those words for which $i$ is missing are simply ignored), we define:

$$I_i \coloneqq \bigcup_{\sigma \in \mathcal{L}} \sigma_{[l_i^\sigma, l_i^\sigma + d_i]}$$

It is worth pointing out that the cardinality of each $I_i$ is *finite*. More precisely, $|I_i| \le |(2^{\mathcal{AP}})^{d_i}|$. Moreover, for each $1 \le i \le m$, we define $I_i^{adj}$ as the subset of $I_i$ consisting of those intervals which are adjacent to intervals in $I_{i-1}$ (the case of $I_1^{adj}$ needs a special treatment):

$$I_i^{adj} \coloneqq \begin{cases} \{\sigma' \in I_i \mid l_i^{\sigma'} = 0\} & \text{if } i = 1 \\ \{\sigma' \in I_i \mid l_i^{\sigma'} = l_{i-1}^{\sigma'} + d_{i-1} + 1\} & \text{otherwise} \end{cases}$$

The rationale for the construction of an LTL[X, F] formula is the following. For each $i$, we constrain each interval in $I_i$ with a sequence of nested occurrences of modality X. We start from the last $i$, that is, $i = m$, and move backwards. For each formula corresponding to an interval in $I_i \setminus I_i^{adj}$ that we created, we introduce an occurrence of modality F, because there is at least a position between $[l_{i-1}^\sigma, l_{i-1}^\sigma + d_{i-1}]$ and $[l_i^\sigma, l_i^\sigma + d_i]$ that is stuttered. However, not all intervals $\tau \in I_i$ can be combined in this way with all intervals in $\tau' \in I_{i-1}$, because $\tau$ and $\tau'$ might be extracted from different $\sigma \in \mathcal{L}$. To avoid this problem, we define $S \subseteq I_1 \times \cdots \times I_m$ as the set of "good" sequences of intervals. Formally, we say that a sequence $\langle \sigma^1, \ldots, \sigma^m \rangle$, where $\sigma^i \in I_i$, for each $1 \le i \le m$, is *good for* $\mathcal{L}$ if and only if there exists a word $\sigma \in \mathcal{L}$ such that all intervals in $\langle \sigma^1, \ldots, \sigma^m \rangle$ appear in $\sigma$ in the same order. We define $S \subseteq I_1 \times \cdots \times I_m$ as the set of good sequences for $\mathcal{L}$. $S$ is *finite* and its cardinality is at most $|((2^{\mathcal{AP}})^{d_{max}})^m|$, where $d_{max}$ is the greatest among $d_1, \ldots, d_m$.

As a preliminary step, given a state $s \in \Sigma$ $(= 2^{\mathcal{AP}})$, we define:

$$\texttt{state}(s) \coloneqq \bigwedge_{p \in s} p \wedge \bigwedge_{p \notin s} \neg p$$

Moreover, for all $\sigma \in \Sigma^+$ and all formulas $\phi$ in LTL[X, F], we define:

$$\texttt{interval}(\sigma, \phi) \coloneqq \bigwedge_{k=0}^{|\sigma|-1} \mathsf{X}^k(\texttt{state}(\sigma_k)) \wedge \mathsf{X}^{|\sigma|}\phi,$$

where $\mathsf{X}^i$ denotes the nesting of $i$ occurrences of modality X.

We are now ready to define the LTL[X, F] formula equivalent to $\mathcal{L}$. Let $S \subseteq I_1 \times \cdots \times I_m$ be the set of good sequences for $\mathcal{L}$. For each $\tau \coloneqq \langle \sigma^1, \ldots, \sigma^m \rangle \in S$ and each $1 \le i \le m+1$, we define:

$$\texttt{frm}(\tau, i) \coloneqq \begin{cases} \top & \text{if } i = m+1 \\ \texttt{interval}(\sigma^i, \texttt{frm}(\tau, i+1)) & \text{if } i \le m \text{ and } \sigma^i \in I_i^{adj} \\ \mathsf{F}(\texttt{interval}(\sigma^i, \texttt{frm}(\tau, i+1))) & \text{otherwise} \end{cases}$$

The formula $\phi$ is defined as follows:

$$\phi \coloneqq \bigvee_{\tau \coloneqq \langle \sigma^1, \ldots, \sigma^m \rangle \in S} \texttt{frm}(\tau, 1)$$

Formula $\phi$ clearly belongs to LTL[X, F]. We prove that $\mathcal{L}(\phi) = \mathcal{L}$. To this end, we preliminarily state the following property.

**Claim.** *Let $S$ be the set of good sequences for $\mathcal{L}$, for all $1 \le i \le m+1$, it holds that:*

$$\mathcal{L}(\bigvee_{\tau \in S} \texttt{frm}(\tau, i)) \coloneqq \{\sigma_{[h_{i-1}^\sigma, \infty]} \mid \sigma \in \mathcal{L}\}$$

*where $h_{i-1}^\sigma \coloneqq l_{i-1}^\sigma + d_{i-1} + 1$ if $i > 1$, and $h_{i-1}^\sigma \coloneqq 0$ otherwise.*

**Proof.** We proceed by induction on $i$.

*Base case.* For $i = m+1$, we have that $\texttt{frm}(\tau, m+1) = \top$, for all $\tau \in S$. Therefore, by definition of $\texttt{frm}(\cdot,\cdot)$, we have that $\mathcal{L}(\bigvee_{\tau \in S} \texttt{frm}(\tau, m+1)) = \Sigma^\omega$. Moreover, being $i = m+1$, $\{\sigma_{[h_{i-1}^\sigma, \infty]} \mid \sigma \in \mathcal{L}\}$ is equal to $\{\sigma_{[h_m^\sigma, \infty]} \mid \sigma \in \mathcal{L}\}$. Now, since, by definition, $h_m^\sigma$ is the position immediately after the end of the last interval of $\sigma$, and since, by hypothesis, $\mathcal{L}$ is closed under pseudo-stuttering, we have that $\text{wstutt}(\sigma, k) \subseteq \mathcal{L}$ for all $k \ge h_m^\sigma$. By definition of $\text{wstutt}(\cdot,\cdot)$, it follows that $\{\sigma_{[h_m^\sigma, \infty]} \mid \sigma \in \mathcal{L}\} = \Sigma^\omega$. Therefore $\mathcal{L}(\bigvee_{\tau \in S} \texttt{frm}(\tau, m+1)) \coloneqq \{\sigma_{[h_m^\sigma, \infty]} \mid \sigma \in \mathcal{L}\}$.

*Inductive step.* Let $i \leq m$. By definition of $\mathtt{frm}(\cdot,\cdot)$, we have that:

$$\bigvee_{\tau \in S} \mathtt{frm}(\tau, i) = \bigvee_{\sigma^i \in I_i^{adj}} \mathtt{interval}(\sigma^i, \mathtt{frm}(\tau, i+1)) \vee \bigvee_{\sigma^i \in I_i \setminus I_i^{adj}} \mathsf{F}(\mathtt{interval}(\sigma^i, \mathtt{frm}(\tau, i+1)))$$

By inductive hypothesis, $\mathcal{L}(\bigvee_{\tau \in S} \mathtt{frm}(\tau, i+1)) := \{\sigma_{[h_i^\sigma, \infty]} \mid \sigma \in \mathcal{L}\}$. By the semantics of the temporal modalities in $\mathtt{frm}(\tau, i)$, and by the inductive hypothesis, we have that:

$$\mathcal{L}(\bigvee_{\tau \in S} \mathtt{frm}(\tau, i)) = (\bigcup_{\sigma^i \in I_i^{adj}} \sigma^i) \cdot \{\sigma_{[h_i^\sigma, \infty]} \mid \sigma \in \mathcal{L}\} \cup \Sigma^* \cdot (\bigcup_{\sigma^i \in I_i \setminus I_i^{adj}} \sigma^i) \cdot \{\sigma_{[h_i^\sigma, \infty]} \mid \sigma \in \mathcal{L}\}$$

Moreover, since, by definition, $h_{i-1}^\sigma$ is the position immediately after the end of the $(i-1)^{th}$ interval of $\sigma$, and since, by hypothesis, $\mathcal{L}$ is closed under pseudo-stuttering, we have that:

$$\{\sigma_{[h_{i-1}^\sigma, \infty]} \mid \sigma \in \mathcal{L}\} = (\bigcup_{\sigma^i \in I_i^{adj}} \sigma^i) \cdot \{\sigma_{[h_i^\sigma, \infty]} \mid \sigma \in \mathcal{L}\} \cup \Sigma^* \cdot (\bigcup_{\sigma^i \in I_i \setminus I_i^{adj}} \sigma^i) \cdot \{\sigma_{[h_i^\sigma, \infty]} \mid \sigma \in \mathcal{L}\}$$

It follows that $\mathcal{L}(\bigvee_{\tau \in S} \mathtt{frm}(\tau, i)) = \{\sigma_{[h_{i-1}^\sigma, \infty]} \mid \sigma \in \mathcal{L}\}$. □

By definition, $h_{i-1}^\sigma = 0$, for $i = 1$, and thus $\mathcal{L}(\bigvee_{\tau \in S} \mathtt{frm}(\tau, 1)) = \{\sigma_{[0, \infty]} \mid \sigma \in \mathcal{L}\} = \mathcal{L}$. Since by definition $\phi = \bigvee_{\tau \in S} \mathtt{frm}(\tau, 1)$, we have that $\mathcal{L}(\phi) = \mathcal{L}$. □

From Lemmas 2 and 3, the next lemma directly follows.

**Lemma 4.** *Every language definable in* $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{O}])$ *is also definable in* $\mathsf{LTL}[\mathsf{X}, \mathsf{F}]$.

Pairing Lemma 4 with the transformation of $\mathsf{LTL}[\mathsf{X}, \mathsf{F}]$ formulas into equivalent $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{O}])$ ones given in [16], we have the next theorem.

**Theorem 3.** $\mathsf{LTL}[\mathsf{X}, \mathsf{F}]$ *and* $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{O}])$ *are expressively equivalent.*

From Theorem 3 and Lemma 3, it also follows that the set of languages definable in $\mathsf{LTL}[\mathsf{X}, \mathsf{F}]$ is the maximal set of languages closed under pseudo-stuttering.

Finally, by dualizing Theorem 3, we can prove the equivalence of $\mathsf{LTL}[\mathsf{X}, \mathsf{G}]$ and $\mathsf{G}(\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{H}])$.

**Theorem 4.** $\mathsf{LTL}[\mathsf{X}, \mathsf{G}]$ *and* $\mathsf{G}(\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{H}])$ *are expressively equivalent.*

**Proof.** Suppose by contradiction that this is not the case, that is, one of the following two alternatives hold:

(i) there exists a formula $\phi$ in $\mathsf{LTL}[\mathsf{X}, \mathsf{G}]$ such that all $\psi$ in $\mathsf{G}(\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{H}])$ are such that $\mathcal{L}(\phi) \neq \mathcal{L}(\psi)$;
(ii) there exists a formula $\phi$ in $\mathsf{G}(\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{H}])$ such that all $\psi$ in $\mathsf{LTL}[\mathsf{X}, \mathsf{G}]$ are such that $\mathcal{L}(\phi) \neq \mathcal{L}(\psi)$.

Without loss of generality, we focus on the former case; the latter one is analogous.

Let $\phi'$ be the formula obtained by transforming into NNF (Negation Normal Form) the formula $\neg\phi$. Clearly, $\phi'$ is a formula of $\mathsf{LTL}[\mathsf{X}, \mathsf{F}]$ and $\mathcal{L}(\phi') = \overline{\mathcal{L}(\phi)}$. By Theorem 3, there exists a formula $\psi'$ in $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{O}])$ such that $\mathcal{L}(\psi') = \mathcal{L}(\phi')$. Now, let $\psi$ be the formula obtained by turning into NNF the formula $\neg\psi'$. Clearly, $\psi$ is a formula of $\mathsf{G}(\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{H}])$ such that $\mathcal{L}(\psi) = \overline{\mathcal{L}(\psi')}$. Since $\mathcal{L}(\psi') = \mathcal{L}(\phi') = \overline{\mathcal{L}(\phi)}$, we have that $\mathcal{L}(\psi) = \mathcal{L}(\phi)$ (contradiction). Hence, alternative (i) must be false. Since an analogous proof can be given for alternative (ii), it follows that $\mathsf{LTL}[\mathsf{X}, \mathsf{G}]$ and $\mathsf{G}(\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{H}])$ are expressively equivalent. □

## 4. Succinctness of safety and cosafety fragments of LTL+P

In this section, we consider the succinctness of various safety and cosafety future fragments of LTL with respect to their past counterpart over infinite words. In particular, we prove that:

1. $\mathsf{G}(\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{H}])$ and $\mathsf{G}(\mathsf{pLTL})$ can be exponentially more succinct than $\mathsf{LTL}[\mathsf{X}, \mathsf{G}]$ and Safety-LTL, respectively (*the safety case*);
2. $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{O}])$ and $\mathsf{F}(\mathsf{pLTL})$ can be exponentially more succinct than $\mathsf{LTL}[\mathsf{X}, \mathsf{F}]$ and coSafety-LTL, respectively (*the cosafety case*).

### *4.1. The safety case*

The proof of the safety case follows from the result by Markey that LTL+P can be exponentially more succinct than LTL, when interpreted over infinite traces [25]. In the following, we show the details of the proof.

Let $\mathcal{AP} = \{p_0, \ldots, p_n\}$ be a finite set of proposition letters. Consider the family of languages $M_n$ over the alphabet $2^{\mathcal{AP}}$ proposed by Markey in [25]: for each $n > 0$, $M_n$ consists of all and only those infinite traces in which any position of the trace that agrees with the initial state on $p_1, \ldots, p_n$ also agrees on $p_0$, and *vice versa*. Formally, for each $n > 0$, we define:

$$M_n \coloneqq \{\sigma \in (2^{\mathcal{AP}})^\omega \mid \forall k > 0 \,.\, \forall i \in \{1, \ldots, n\} \,. \\ (p_i \in \sigma_k \leftrightarrow p_i \in \sigma_0) \leftrightarrow (p_0 \in \sigma_k \leftrightarrow p_0 \in \sigma_0)\}$$

In [25], Markey proves that, for any $n > 0$, any formula of LTL expressing $M_n$ is at least of size exponential in $n$. Since $\mathsf{LTL}[\mathsf{X}, \mathsf{G}]$ and Safety-LTL are proper subfragments of LTL, *i.e.*, each $\mathsf{LTL}[\mathsf{X}, \mathsf{G}]$ formula and each Safety-LTL formula are also LTL formulas, it follows that, for any $n > 0$, any formula of $\mathsf{LTL}[\mathsf{X}, \mathsf{G}]$ and any formula of Safety-LTL expressing $M_n$ is at least of size exponential in $n$.

**Lemma 5.** *For any $n > 0$ and any formula $\phi \in \mathsf{LTL}[\mathsf{X}, \mathsf{G}] \cup$ Safety-LTL, if $\mathcal{L}(\phi) = M_n$, then $|\phi| \in 2^{\Omega(n)}$.*

However, for each $n > 0$, there is a formula in $\mathsf{G}(\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{H}])$ (and thus a formula in $\mathsf{G}(\mathsf{pLTL})$) of size linear in $n$ that captures $M_n$:

$$\mathsf{G}((\bigwedge_{i=1}^{n} (p_i \leftrightarrow \mathsf{H}(\widetilde{\mathsf{Y}}\bot \to p_i))) \leftrightarrow (p_0 \leftrightarrow \mathsf{H}(\widetilde{\mathsf{Y}}\bot \to p_0)).)$$

Notice the crucial role of the subformula $\widetilde{\mathsf{Y}}\bot$ in hooking the initial state of the trace. The next theorem directly follows.

**Theorem 5.**

- $\mathsf{G}(\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{H}])$ *can be exponentially more succinct than* $\mathsf{LTL}[\mathsf{X}, \mathsf{G}]$*;*
- *G*(pLTL) *can be exponentially more succinct than Safety*-LTL*.*

### *4.2. The cosafety case*

We now dualize the previous result to the cosafety case, and obtain the succinctness lower bound of $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{O}])$ and $\mathsf{F}(\mathsf{pLTL})$ with respect to $\mathsf{LTL}[\mathsf{X}, \mathsf{F}]$ and coSafety-LTL, respectively.

We first prove a more general result on "dual" temporal logics (which are defined below), and then we instantiate the result to the specific case of $\mathsf{F}(\mathsf{pLTL})$ and $\mathsf{G}(\mathsf{pLTL})$ (and of $\mathsf{F}(\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{O}])$ and $\mathsf{G}(\mathsf{pLTL}[\mathsf{Y}, \widetilde{\mathsf{Y}}, \mathsf{H}])$). We define *dual logics* as follows.

**Definition 5** *(Dual logics).* Given two linear temporal logics $\mathbb{L}$ and $\overline{\mathbb{L}}$, we say that $\overline{\mathbb{L}}$ is a *dual logic* of $\mathbb{L}$ iff

1. for any formula $\phi \in \overline{\mathbb{L}}$, the transformation in *negation normal form* of $\neg\phi$ (denoted as $\mathrm{nnf}(\neg\phi)$) belongs to $\mathbb{L}$, and
2. for any formula $\phi \in \mathbb{L}$, the transformation in *negation normal form* of $\neg\phi$ (denoted as $\mathrm{nnf}(\neg\phi)$) belongs to $\overline{\mathbb{L}}$.

We would like to point out that being a dual logic is a symmetric property.

The following lemma proves that *duality* (as for Definition 5) preserves succinctness.

**Lemma 6** *(Duality Lemma). For any pair of linear temporal logics $\mathbb{L}$ and $\mathbb{L}'$, if $\mathbb{L}$ can be exponentially more succinct than $\mathbb{L}'$, then $\overline{\mathbb{L}}$ can be exponentially more succinct than $\overline{\mathbb{L}'}$, where $\overline{\mathbb{L}}$ (resp., $\overline{\mathbb{L}'}$) is a dual logic of $\mathbb{L}$ (resp., $\mathbb{L}'$).*

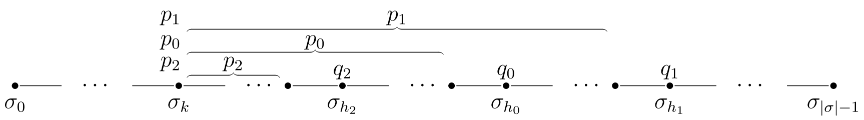


**Fig. 2.** Example of a word $\sigma$ in $C_2$.

**Proof.** Since by hypothesis $\mathbb{L}$ can be exponentially more succinct than $\mathbb{L}'$, there exists a formula $\phi \in \mathbb{L}$ of size $n$ such that, for all $\phi' \in \mathbb{L}'$, if $\mathcal{L}(\phi') = \mathcal{L}(\phi)$, then $|\phi'| \in 2^{\Omega(n)}$. Let $\overline{\phi}$ be the negation normal form of $\neg\phi$, *i.e.*, $\overline{\phi} = \text{nnf}(\neg\phi)$. By definition:

1. $\overline{\phi}$ belongs to $\overline{\mathbb{L}}$,
2. $\mathcal{L}(\overline{\phi}) = \mathcal{L}(\neg\phi)$, and
3. $|\overline{\phi}| \in \mathcal{O}(n)$.

Suppose, by contradiction, that the thesis does not hold, that is, for all formulas $\overline{\psi} \in \overline{\mathbb{L}}$ of size $s = |\overline{\psi}|$ there exists a formula $\overline{\psi'} \in \overline{\mathbb{L}'}$ such that $\mathcal{L}(\overline{\psi}) = \mathcal{L}(\overline{\psi'})$ and $|\overline{\psi'}|$ is less than exponential in $s$. In particular, for $\overline{\psi} := \overline{\phi}$, this means that there exists a formula $\overline{\psi'}$ in $\overline{\mathbb{L}'}$ such that $\mathcal{L}(\overline{\psi'}) = \mathcal{L}(\overline{\phi})$ and $|\overline{\psi'}|$ is less than exponential in $n$ (recall that $n$ is the size of $\overline{\phi}$).

Now, let $\chi'$ be the negation normal form of $\neg\overline{\psi'}$. It holds that:

1. $\chi'$ is a formula in $\mathbb{L}'$,
2. $\mathcal{L}(\chi') = \mathcal{L}(\phi)$, and
3. $|\chi'| \in \mathcal{O}(|\overline{\psi'}|)$.

Since the size of $\overline{\psi'}$ is less than exponential in $n$, the size of $|\chi'|$ is less than exponential in $n$ as well. Hence, $\mathcal{L}(\phi)$ can be defined in $\mathbb{L}'$ with a formula of size less than exponential in $n$, which contradicts the hypothesis. $\square$

Since, by definition, F(pLTL) and coSafety-LTL are dual logics of G(pLTL) and Safety-LTL, respectively, by Lemma 6 and Theorem 5, the next theorem follows.

**Theorem 6.**

- F(pLTL[Y, $\widetilde{\mathsf{Y}}$, O]) *can be exponentially more succinct than* LTL[X, F]*;*
- *F(*pLTL*) can be exponentially more succinct than coSafety-*LTL*.*

### *4.3. Open problems*

To complete the picture, we make a guess about the succinctness of the (co)safety fragments of LTL. To the best of our knowledge, it is still an open question whether coSafety-LTL (resp., Safety-LTL) can be exponentially more succinct than F(pLTL) (resp., G(pLTL)).

We conjecture that coSafety-LTL can be $n!$ ($n$ factorial) more succinct than F(pLTL). Let $\mathcal{AP} = \{p_i\}_{i=1}^n \cup \{q_i\}_{i=1}^n$ be a finite set of atomic propositions. Consider the following family of languages $C_n$ over the alphabet $\Sigma = 2^{\mathcal{AP}}$, where $n > 0$:

$$C_n := \{\sigma \in (2^{\mathcal{AP}})^\omega \mid \exists k \geq 0 \,.\, \bigwedge_{i=1}^{n} (\exists h \geq k \,.\, (q_i \in \sigma_h \wedge \forall l \,.\, k \leq l < h \,.\, p_i \in \sigma_l))\}.$$

For any $n > 0$, $C_n$ consists of the infinite traces for which there exists a time point $k$ such that, for each $i \in \{1, \dots, n\}$, $q_i$ will eventually be realized in the future of $k$ and $p_i$ holds until (and excluding) that point (cf. Fig. 2).

While $C_n$ is definable in coSafety-LTL with a formula of linear size in $n$ like $\mathsf{F}(\bigwedge_{\mathsf{i}=1}^{\mathsf{n}}(\mathsf{p_i}\mathsf{U}\mathsf{q_i}))$, using F(pLTL) one is forced to enumerate all possible orders among $q_1, \dots, q_n$ with a formula of the form:

$$\begin{aligned}\mathsf{F}\Big(\bigvee_{\pi\in\Pi} (q_{\pi(1)} \wedge & \mathsf{Y}(p_{\pi(1)})\mathsf{S}(p_{\pi(1)} \wedge q_{\pi(2)} \wedge \\ & \mathsf{Y}(p_{\pi(1)} \wedge p_{\pi(2)})\mathsf{S}(p_{\pi(1)} \wedge p_{\pi(2)} \wedge q_{\pi(3)} \wedge \dots \\ & \quad \mathsf{Y}(\bigwedge_{i=1}^{n-1} p_{\pi(i)})\mathsf{S}(q_{\pi(n)} \wedge \bigwedge_{i=1}^{n} p_{\pi(i)}))\dots)))\Big),\end{aligned}$$

where $\Pi$ is the set of permutations of $\{1, \dots, n\}$. This, in turn, forces the formula to be at least of size $n!$.

**Conjecture 1.** *For any $n > 0$, the language $C_n$ is not expressible in F(*pLTL*) with a formula of size less than $n!$.*

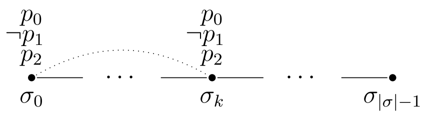


**Fig. 3.** Example of a word $\sigma$ in $A_2$.

We conjecture the same for the dual case of Safety-LTL and G(pLTL).

Finally, let us consider the succinctness of LTL[X, F] with respect to F(pLTL[Y, $\widetilde{\mathsf{Y}}$, O]). In [19], it has been recently shown that LTL[X, F] can be exponentially more succinct than F(pLTL[Y, $\widetilde{\mathsf{Y}}$, O]). Together with Theorem 6, this shows the incomparability of LTL[X, F] and F(pLTL[Y, $\widetilde{\mathsf{Y}}$, O]) when succinctness is concerned. Again, the same holds for the safety case of LTL[X, G] and G(pLTL[Y, $\widetilde{\mathsf{Y}}$, H]). Interestingly, [19] proves that an automata-theoretic approach cannot be used to show that LTL[X, F] can be exponentially more succinct than F(pLTL[Y, $\widetilde{\mathsf{Y}}$, O]).

## 5. LTL$_\mathsf{f}$ can be exponentially more succinct than pLTL, and *vice versa*

In this section, we prove that LTL$_\mathsf{f}$ can be exponentially more succinct than pLTL and *vice versa*. This shows that, despite being expressively equivalent, LTL$_\mathsf{f}$ and pLTL are *incomparable* when succinctness is concerned.

### *5.1.* pLTL *can be exponentially more succinct than* LTL$_\mathsf{f}$

We begin by proving that pLTL can be exponentially more succinct than LTL$_\mathsf{f}$. Let $\mathcal{AP} = \{p_0, p_1, \dots, p_n\}$ be a finite set of proposition letters. Consider the following family of languages over the alphabet $2^{\mathcal{AP}}$, where $n > 0$.

$$A_n := \{\sigma \in (2^{\mathcal{AP}})^+ \mid \exists k > 0 \,.\, (\bigwedge_{i=0}^{n} (p_i \in \sigma_k \leftrightarrow p_i \in \sigma_0))\} \tag{4}$$

For any $n > 0$, the language $A_n$ is the set of finite words over $2^{\mathcal{AP}}$ containing a position which agrees with the initial state on the evaluation of all proposition letters in $\mathcal{AP}$ (cf. Fig. 3).

We prove that all formulas of LTL$_\mathsf{f}$ defining $A_n$ are at least of size exponential in $n$. On the contrary, as shown by the following lemma, $A_n$ can be expressed in pLTL by formulas of linear size in $n$, for any $n > 0$.

**Lemma 7.** *For any $n > 0$, there exists a formula $\phi \in$ pLTL such that $\mathcal{L}(\phi) = A_n$ and $|\phi| \in \mathcal{O}(n)$.*

**Proof.** For any $n > 0$, let $\phi_{A_n}$ be the formula:

$$\mathsf{O}(\bigwedge_{i=0}^{n} (p_i \leftrightarrow \mathsf{YO}(\widetilde{\mathsf{Y}}\bot \wedge p_i)))$$

It is worth noticing the key role of the *weak yesterday* modality, and, in particular, of the subformula $\widetilde{\mathsf{Y}}\bot$, in hooking the initial state of a word. We prove that $\mathcal{L}(\phi_{A_n}) = A_n$. For any $\sigma \in (2^{\mathcal{AP}})^+$ and any $n > 0$, it holds that $\sigma \in A_n$ if and only if $\exists k > 0 \,.\, \bigwedge_{i=0}^{n} (\sigma_k \models p_i \leftrightarrow \sigma_0 \models p_i)$. This, in turn, is equivalent to $\exists k < |\sigma| \,.\, (k \neq 0 \wedge \bigwedge_{i=0}^{n} (\sigma_k \models p_i \leftrightarrow \sigma_0 \models p_i))$ and thus to $\exists k < |\sigma| \,.\, \bigwedge_{i=0}^{n} (\sigma_k \models p_i \leftrightarrow (\exists h \,.\, (h < k \wedge h = 0 \wedge \sigma_h \models p_i)))$. Therefore, $\sigma \models \mathsf{O}(\bigwedge_{i=0}^{n} (p_i \leftrightarrow \mathsf{YO}(\widetilde{\mathsf{Y}}\bot \wedge p_i)))$. Clearly, $|\phi_{A_n}| \in \mathcal{O}(n)$. $\square$

To prove that, for any $n > 0$, $A_n$ is not expressible in LTL$_\mathsf{f}$ with formulas of size less than $2^{\Omega(n)}$, we make use of an auxiliary family of languages. For each $n > 0$, we define the language $B_n$ over the alphabet $2^{\mathcal{AP}}$ with $\mathcal{AP} = \{p_0, p_1, \dots, p_n\}$ as follows:

$$B_n := \{\sigma \in (2^{\mathcal{AP}})^+ \mid \exists h \geq 0 \,.\, \exists k > h \,.\, (\bigwedge_{i=0}^{n} (p_i \in \sigma_k \leftrightarrow p_i \in \sigma_h))\}$$

For any $n > 0$, $B_n$ is the set of finite words over $2^{\mathcal{AP}}$ containing two (distinct) positions that agree on the interpretation of all the proposition letters in $\mathcal{AP}$ (cf. Fig. 4). Clearly, $A_n \subseteq B_n$, for any $n > 0$.

We now show that if $A_n$ was expressible in LTL$_\mathsf{f}$ in space less than exponential in $n$, then the property $B_n$ would be expressible in LTL$_\mathsf{f}$ in space less than exponential as well.

**Lemma 8.** *If there exists a formula of* LTL$_\mathsf{f}$ *for $A_n$ of size less than exponential in $n$, then there exists a formula of* LTL$_\mathsf{f}$ *for $B_n$ of size less than exponential in $n$.*

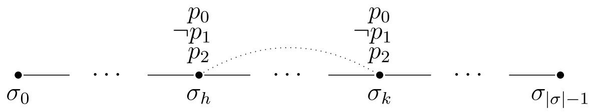


**Fig. 4.** Example of a word $\sigma$ in $B_2$.

**Proof.** Let $\psi_{A_n}$ be a formula of $\mathsf{LTL_f}$ for $A_n$ of size less than exponential in $n$. Consider the formula $\mathsf{F}(\psi_{\mathsf{A_n}})$: we prove that its language is exactly $B_n$. For any $\sigma \in (2^{\mathcal{AP}})^+$ and any $n > 0$, it holds that $\sigma \models \mathsf{F}(\psi_{\mathsf{A_n}})$ iff $\exists k \geq 0 \,.\, \sigma_{[k,-]} \models \psi_{A_n}$, where $\sigma_{[k,-]}$ is the suffix of $\sigma$ starting at $k$. This amounts to say that $\exists k \geq 0 \,.\, \exists h > k \,.\, (\bigwedge_{i=0}^{n}(\sigma_k \models p_i \leftrightarrow \sigma_h \models p_i))$. Equivalently, $\sigma \in B_n$. Moreover $\mathsf{F}(\psi_{\mathsf{A_n}})$ belongs to $\mathsf{LTL_f}$ and it is of size less than exponential in $n$. $\square$

We now show that there cannot exist formulas of $\mathsf{LTL_f}$, and, in general, of $\mathsf{LTL_f}$+P , defining $B_n$ whose size is less than exponential in $n$. In order to prove it, we first show that any NFA accepting $B_n$ is of size at least *doubly exponential* in $n$.

**Lemma 9.** *For any $n > 0$ and any* NFA *$\mathcal{A}$ over the alphabet $2^{\mathcal{AP}}$, if $\mathcal{L}(\mathcal{A}) = B_n$, then $|\mathcal{A}| \in 2^{2^{\Omega(n)}}$.*

**Proof.** Let $n > 0$ and let $\langle a_0, \ldots, a_{2^n-1}\rangle$ be any permutation of the $2^n$ subsets of $\{p_1, \ldots, p_n\}$ (note that this set does not include the proposition letter $p_0 \in \mathcal{AP}$). Let $K$ be any subset of $\{0, \ldots, 2^n - 1\}$ and let $\overline{K}$ be the complement set of $K$. We define $b_i^K$ in this way: $b_i^K := a_i$, if $i \in \overline{K}$; and $b_i^K := a_i \cup \{p_0\}$, otherwise. We define $\sigma_K$ as the sequence $\langle b_0^K, b_1^K, \ldots, b_{2^n-1}^K\rangle$.

Suppose by contradiction that there is an NFA $\mathcal{A}$ for $B_n$ of size less than doubly exponential in $n$. Consider the words $\sigma_K \cdot \sigma_K$ (obtained by concatenating $\sigma_K$ with itself), $\sigma_{\overline{K}} \cdot \sigma_{\overline{K}}$ (the concatenation of $\sigma_{\overline{K}}$ with itself), and $\sigma_K \cdot \sigma_{\overline{K}}$ (the concatenation of $\sigma_K$ with $\sigma_{\overline{K}}$). By construction, $\sigma_K \cdot \sigma_K$ contains (at least) two positions that pairwise agree on the interpretation of all symbols in $\mathcal{AP}$ and thus it belongs to $B_n$. The same holds for $\sigma_{\overline{K}} \cdot \sigma_{\overline{K}}$ as well. Nevertheless, $\sigma_K \cdot \sigma_{\overline{K}}$ contains no such positions and so it does *not* belong to $B_n$. Hence, for any $K \subseteq \{0, \ldots, 2^n - 1\}$:

1. $\sigma_K \cdot \sigma_K$ is accepted by $\mathcal{A}$;
2. $\sigma_{\overline{K}} \cdot \sigma_{\overline{K}}$ is accepted by $\mathcal{A}$;
3. $\sigma_K \cdot \sigma_{\overline{K}}$ is *not* accepted by $\mathcal{A}$.

Now let $\pi$ (resp., $\pi'$) be any *accepting* run of $\mathcal{A}$ over the word $\sigma_K \cdot \sigma_K$ (resp., $\sigma_{\overline{K}} \cdot \sigma_{\overline{K}}$). Let $q$ (resp., $q'$) be the $2^n$-th state of $\pi$ (resp., $\pi'$). Suppose that $q = q'$ and let $\pi''$ be the sequence obtained by appending the suffix of $\pi'$ starting at its $2^n$-th state to the prefix of $\pi$ of length $2^n - 1$, *i.e.*, $\pi'' := \langle \pi_0, \ldots, \pi_{2^n-1}, \pi'_{2^n}, \pi'_{2^n+1}, \ldots\rangle$. By construction, $\pi''$ is an *accepting run* of $\mathcal{A}$ over the word $\sigma_K \cdot \sigma_{\overline{K}}$, which is a contradiction. Hence, the $2^n$-th states of $\pi$ and $\pi'$ must be distinct. This means that $\mathcal{A}$ has to contain at least a state for every choice of $K \subseteq \{0, \ldots, 2^n - 1\}$. Since there are $2^{2^n}$ of such possible choices, this means that $\mathcal{A}$ has to contain at least $2^{2^{\Omega(n)}}$ states. $\square$

By exploiting the singly exponential translation of $\mathsf{LTL_f}$+P formulas into equivalent NFAs(Proposition 2), we can prove that, for any $n > 0$, the language $B_n$ is not expressible in $\mathsf{LTL_f}$+P, and, in particular, in $\mathsf{LTL_f}$, in space less than exponential.

**Lemma 10.** *For any formula $\phi \in \mathsf{LTL}_f + P$, if $\mathcal{L}(\phi) = B_n$ then $|\phi| \in 2^{\Omega(n)}$.*

**Proof.** Suppose by contradiction that this does not hold, *i.e.*, there exists a formula $\phi \in \mathsf{LTL_f} + P$ such that $\mathcal{L}(\phi) = B_n$ and $|\phi|$ is less than exponential in $n$. Then, by Proposition 2, it holds that there exists an NFA $\mathcal{A}$ such that $\mathcal{L}(\mathcal{A}) = B_n$ and $|\mathcal{A}|$ is *less than doubly exponential* in $n$, which is a contradiction with Lemma 9. $\square$

From Lemmas 8 and 10, it follows that the family of languages $A_n$ cannot be expressed in $\mathsf{LTL_f}$ with formulas of size less than exponential in $n$.

**Lemma 11.** *For any $n > 0$ and any formula $\phi \in \mathsf{LTL}_f$, if $\mathcal{L}(\phi) = A_n$, then $|\phi| \in 2^{\Omega(n)}$.*

The following theorem is a direct consequence of Lemmas 7 and 11.

**Theorem 7.** pLTL *can be exponentially more succinct than* $\mathsf{LTL}_f$.

*Comparison with Markey's proof about LTL.+P and* LTL In [25], Markey proves that LTL+P can be exponentially more succinct than LTL. In particular, he exploits the result by Etessami et al. [28] that there are no Büchi automata of size less than doubly exponential for the family of languages $I_n$ (for all $n > 0$), defined as the language of *infinite* traces in which any two positions that agree on $p_1, \ldots, p_n$, agree also on $p_0$.

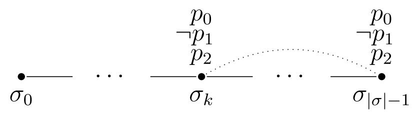


**Fig. 5.** Example of a word $\sigma$ in $A_2^-$.

In principle, one may think of exploiting $I_n$ interpreted over finite trace (let us denote it by $I_n^{<\omega}$) to prove that any NFA recognizing $I_n^{<\omega}$ is at least of doubly exponential size in $n$, and use it as a base to prove that pLTL can be exponentially more succinct than LTL$_f$. This would require to restate and reprove the theorem by Etessami et al. [28] over finite traces. While we believe it possible, we followed a simpler (and more useful) path by showing that there is another family of properties, in our case $B_n$ (which is arguably simpler than $I_n$ and $I_n^{<\omega}$), for which each NFA explodes double-exponentially.

### 5.2. LTL$_f$ *can be exponentially more succinct than* pLTL

We now show that LTL$_f$ can be exponentially more succinct than pLTL. Together with Theorem 7, this shows that LTL$_f$ and pLTL are *incomparable* with respect to succinctness.

We first define the notions of *reverse language* and *reverse logic*. Given an alphabet $\Sigma = 2^{\mathcal{AP}}$ and a language $\mathcal{L} \subseteq \Sigma^+$ of finite words over $\Sigma$, we define the *reverse language* of $\mathcal{L}$ as the set:

$$\mathcal{L}^- = \{\sigma' \in \Sigma^+ \mid \sigma'_i = \sigma_{n-i}, \text{ for } \sigma = \sigma_0 \ldots \sigma_n \in \mathcal{L} \text{ and } 0 \le i \le n\}.$$

We then define *reverse logics* as follows.

**Definition 6** *(Reverse logics).* Given two linear temporal logics $\mathbb{L}$ and $\mathbb{L}^-$, we say that $\mathbb{L}^-$ is a *reverse logic* of $\mathbb{L}$ iff:

1. for any formula $\phi \in \mathbb{L}$, there is a formula $\phi' \in \mathbb{L}^-$ so that $\mathcal{L}(\phi) = \mathcal{L}(\phi')^-$ and $|\phi'| = |\phi|$;
2. for any formula $\phi' \in \mathbb{L}^-$, there is a formula $\phi \in \mathbb{L}$ so that $\mathcal{L}(\phi') = \mathcal{L}(\phi)^-$ and $|\phi| = |\phi'|$.

Clearly, being a reverse logic is a symmetric property: $\mathbb{L}$ is a reverse logic of $\mathbb{L}^-$ if and only if $\mathbb{L}^-$ is a reverse logic of $\mathbb{L}$.

As an example, consider the logic pLTL and any formula $\phi \in$ pLTL. By replacing any occurrence of temporal modalities Y, $\widetilde{\mathsf{Y}}$, S, and T in $\phi$ by X, $\widetilde{\mathsf{X}}$, U, and R, respectively, we obtain a formula $\phi'$ such that: (i) it belongs to LTL$_f$ ; (ii) its size is $|\phi|$; (iii) $\mathcal{L}(\phi) = \mathcal{L}(\phi')^-$. Therefore, LTL$_f$ is a reverse logic of pLTL, and *vice versa*.

The next lemma proves that, for any pair of linear temporal logics $\mathbb{L}$, $\mathbb{L}^-$ such that $\mathbb{L}$ is a reverse logic of $\mathbb{L}^-$, if a language $\mathcal{L}$ with a compact definition in $\mathbb{L}$ is not succinctly definable in $\mathbb{L}^-$, then $\mathcal{L}^-$ (the reverse language of $\mathcal{L}$) is compactly definable in $\mathbb{L}^-$, but its definitions exponentially blow-up in $\mathbb{L}$.

**Lemma 12** *(Reverse lemma). Let $\mathbb{L}$ and $\mathbb{L}^-$ be two linear temporal logics such that $\mathbb{L}^-$ is a reverse logic of $\mathbb{L}$. Moreover, let $\phi \in \mathbb{L}$ be such that, for every $\psi \in \mathbb{L}^-$, $\mathcal{L}(\psi) = \mathcal{L}(\phi)$ implies $|\psi| \in 2^{\Omega(|\phi|)}$. Then, there exists $\phi' \in \mathbb{L}^-$ such that: (i) $\mathcal{L}(\phi') = \mathcal{L}(\phi)^-$; (ii) $|\phi'| = |\phi|$; (iii) for every $\psi \in \mathbb{L}$, $\mathcal{L}(\psi) = \mathcal{L}(\phi')$ implies $|\psi| \in 2^{\Omega(|\phi'|)}$.*

**Proof.** Assume that $\phi \in \mathbb{L}$ is a formula such that, for any $\psi \in \mathbb{L}^-$, if $\mathcal{L}(\psi) = \mathcal{L}(\phi)$, then $|\psi| \in 2^{\Omega(|\phi|)}$, and, by contradiction, assume that for any formula $\phi' \in \mathbb{L}^-$ such that $\mathcal{L}(\phi') = \mathcal{L}(\phi)^-$ and $|\phi'| = |\phi|$, there exists a formula $\psi \in \mathbb{L}$ such that $\mathcal{L}(\psi) = \mathcal{L}(\phi')$ and $|\psi|$ is sub-exponential in $|\phi'|$. Since $\mathbb{L}^-$ is a reverse logic of $\mathbb{L}$, for any formula $\phi' \in \mathbb{L}^-$ such that $\mathcal{L}(\phi') = \mathcal{L}(\phi)^-$ and $|\phi'| = |\phi|$, by Definition 6, there exists a formula $\psi_R \in \mathbb{L}$ such that:

- $\mathcal{L}(\psi_R)^- = \mathcal{L}(\psi)$ and, since $\mathcal{L}(\psi) = \mathcal{L}(\phi')$ and $\mathcal{L}(\phi') = \mathcal{L}(\phi)^-$, then $\mathcal{L}(\psi_R) = \mathcal{L}(\phi)$;
- $|\psi_R| = |\psi|$ and thus $|\psi_R|$ is sub-exponential in $|\phi|$.

This contradicts the hypothesis. It follows that there must be a formula $\phi' \in \mathbb{L}^-$ such that $\mathcal{L}(\phi') = \mathcal{L}(\phi)^-$ and $|\phi'| = |\phi|$, and, for all $\psi \in \mathbb{L}$, if $\mathcal{L}(\psi) = \mathcal{L}(\phi')$, then $|\psi| \in 2^{\Omega(|\phi'|)}$. $\square$

From Lemma 12, we get a concrete family of languages that are definable by LTL$_f$ formulas of polynomial size, but such that any pLTL formula defining them requires at least an exponential amount of space.

For any $n > 0$, let $A_n$ be the language given in the previous section, and let $A_n^-$ be defined as follows (cf. Fig. 5):

$$A_n^- := \{\sigma \in (2^{\mathcal{AP}})^+ \mid \exists k < |\sigma| - 1 \,.\, (\bigwedge_{i=0}^{n} (p_i \in \sigma_k \leftrightarrow p_i \in \sigma_{|\sigma|-1}))\}$$

For every $n > 0$, $A_n^-$ can be expressed in $\mathsf{LTL_f}$ in space linear in $n$ by the following formula:

$$\mathsf{F}(\bigwedge_{i=0}^{n}(\mathsf{p_i} \leftrightarrow \mathsf{XF}(\widetilde{\mathsf{X}}\bot \wedge \mathsf{p_i}))).$$

Since $\mathsf{LTL_f}$ is a reverse logic of pLTL, by Lemmas 11 and 12, every formula of pLTL for $A_n^-$ requires an amount of space at least exponential in $n$. This leads immediately to the following lemma <nd theorem.

**Lemma 13.** *For any $n > 0$ and for any formula $\phi \in$ pLTL, if $\mathcal{L}(\phi) = A_n^-$ then $|\phi| \in 2^{\Omega(n)}$.*

**Theorem 8.** $\mathsf{LTL_f}$ *can be exponentially more succinct than* pLTL.

### 5.3. Some meaningful implications of the incomparability result

We have shown that the logics $\mathsf{LTL_f}$ and pLTL, despite being expressively equivalent (Proposition 1), are incomparable with respect to succinctness. We now point out some implications of such an incomparability result that are worth discussing.

*Succinctness and Realizability. Realizability* is the problem of establishing whether there is a strategy implementing a given formula. Formally, given a formula $\phi \in \mathsf{LTL_f}$ (resp., pLTL) over a set of variables $\mathcal{C} \cup \mathcal{U}$, where $\mathcal{C}$ and $\mathcal{U}$ are the sets of *controllable* and *uncontrollable* variables, respectively, the realizability problem for $\mathsf{LTL_f}$ (resp., pLTL) is the problem of establishing whether there exists a strategy $s : (2^{\mathcal{U}})^+ \to 2^{\mathcal{C}}$ such that, for all sequences $\langle \mathsf{U_0}, \mathsf{U_1}, \dots \rangle \in (2^{\mathcal{U}})^+$, it holds that there exists $k \in \mathbb{N}$ so that the prefix from 0 up to $k$ of $\langle \mathsf{U_0} \cup s(\langle \mathsf{U_0} \rangle), \mathsf{U_1} \cup s(\langle \mathsf{U_0}, \mathsf{U_1} \rangle), \dots \rangle$ is a model of $\phi$.

Even though $\mathsf{LTL_f}$ and pLTL have the same expressive power, the complexity of their realizability problem is different: while $\mathsf{LTL_f}$ realizability is 2EXPTIME-complete [1], pLTL realizability is EXPTIME-complete [20]. This is due to the fact that, starting from any $\mathsf{LTL_f}$ formula $\phi$ of size $n$, it is not possible to build a *Deterministic Finite Automaton* (DFA) recognizing $\mathcal{L}(\phi)$ of singly exponential size in $n$; on the contrary, this is possible for pLTL formulas, thanks to the fact that "since past already happened", there is no need to introduce nondeterminism [11,14,20].

In [20], the exponential gap between the two complexities, and the fact that $\mathsf{LTL_f}$ and pLTL are expressively equivalent, led to the conjecture that any translation from $\mathsf{LTL_f}$ formulas to equivalent pLTL ones requires at least an exponential blowup in the size of the resulting formulas. The results proved in this paper (in particular, Lemma 13) *confirm* this conjecture: any translation in pLTL of the $\mathsf{LTL_f}$ formula $\mathsf{F}(\bigwedge_{i=0}^{n}(\mathsf{p_i} \leftrightarrow \mathsf{XF}(\widetilde{\mathsf{X}}\bot \wedge \mathsf{p_i})))$, which defines the language $A_n^-$, requires at least an exponential blowup.

*Succinctness helps in choosing the most convenient formalism for realizability.* The succinctness results that we proved for $\mathsf{LTL_f}$ and pLTL may help in choosing the right formalism to express a given property when the time complexity of realizability is considered. As a matter of fact, consider the family of languages $A_n$ (Eq. (4)), and suppose one wants to solve the realizability problem for $A_n$, for a given partition of the variables $p_0, \dots, p_n$ into controllable and uncontrollable. There are two possibilities:

(i) to formalize $A_n$ in pLTL (in linear size) and to use pLTL realizability algorithms, which are singly exponential in the worst case, or
(ii) to formalize the language in $\mathsf{LTL_f}$, with at least an exponential blowup, by Lemma 11, and to use $\mathsf{LTL_f}$ realizability algorithms, which are doubly exponential in the worst case.

While the former option requires, in the worst case, only a *singly exponential* amount of time, the latter requires a *triply exponential* amount of time, in the worst case. This shows how the results on the succinctness of $\mathsf{LTL_f}$ and pLTL can tremendously help in choosing the best performing algorithm.

The fact that $\mathsf{LTL_f}$ can be exponentially more succinct than pLTL has an important implication as well. The realizability problem for the family of languages $A_n^-$ has the same worst-case time complexity (doubly exponential in $n$) irrespectively of whether we choose to formalize $A_n^-$ in $\mathsf{LTL_f}$ or in pLTL. In other words, the family of languages $A_n^-$ cancels out the advantages of the past in realizability.

*Translation of $\mathsf{LTL_f}$ into pLTL.* Recall that $\mathsf{LTL_f}$ and pLTL are expressively equivalent (Proposition 1). To the best of our knowledge, the most efficient translation of $\mathsf{LTL_f}$ into pLTL is the one reported in [11] that performs the following steps:

1. build an NFA for the original formula;
2. turn the NFA into a DFA via determinization;
3. build a pLTL formula from the DFA using the Krohn-Rhodes cascaded decomposition [29].

Since all three steps may introduce, in the worst case, an exponential blow up, the whole translation is *triply exponential* in the size of the initial formula. Maler and Pnueli proved that such a translation, in particular the third step, has an exponential lower bound [29]. In this respect, Theorem 8 proves that *any* translation from $\mathsf{LTL_f}$ to pLTL (not only the above one) has a lower bound at least exponential.

## 6. Conclusions

In this paper, we provided a pure past characterization of the (co)safety fragments of LTL devoid of binary temporal modalities, and we studied the succinctness of $\mathsf{LTL_f}$ and of (co)safety fragments of LTL with respect to their pure past counterparts.

We first investigated the expressiveness of cosafety and safety fragments of LTL devoid of binary temporal modalities. By using the novel notion of *pseudo-stuttering*, we proved that $\mathsf{LTL[X,F]}$ is expressively equivalent to $\mathsf{F(pLTL[Y,\widetilde{Y},O])}$. Interestingly, as a by product, we obtain that $\mathsf{LTL[X,F]}$ is the *maximal* fragment of LTL closed under pseudo-stuttering. By duality, we derived that $\mathsf{LTL[X,G]}$ is expressively equivalent to $\mathsf{G(pLTL[Y,\widetilde{Y},H])}$.

Then, we focused on succinctness. First, we dealt with safety and cosafety fragments, and compared them with their pLTL-based canonical forms. As for safety, we showed that $\mathsf{G(pLTL[Y,\widetilde{Y},H])}$ and $\mathsf{G(pLTL)}$ can be exponentially more succinct than $\mathsf{LTL[X,G]}$ and Safety-LTL, respectively. As for cosafety, we proved that $\mathsf{F(pLTL[Y,\widetilde{Y},O])}$ and $\mathsf{F(pLTL)}$ can be exponentially more succinct than, respectively, $\mathsf{LTL[X,F]}$ and coSafety-LTL. Together with the recent work in [19], this entails that $\mathsf{LTL[X,F]}$ and $\mathsf{F(pLTL[Y,\widetilde{Y},O])}$, as well as $\mathsf{LTL[X,G]}$ and $\mathsf{G(pLTL[Y,\widetilde{Y},H])}$, are incomparable with respect to succinctness.

Finally, we proved the incomparability of $\mathsf{LTL_f}$ and of pLTL with respect to succinctness. We first showed that a family of properties $A_n$ admits a formalization in pLTL with formulas of linear size, while all formulas in $\mathsf{LTL_f}$ for $A_n$ are at least of exponential size. By using the Reverse Lemma, we proved that the *vice versa* holds as well, *i.e.*, $\mathsf{LTL_f}$ can be exponentially more succinct than pLTL. This result confirmed the conjecture formulated in [20] about the lower bound to the complexity of translating $\mathsf{LTL_f}$ into pLTL.

The study of the maximal fragment of $\mathsf{LTL_f}$ that does not incur in the exponential blow-up in the translation into pLTL is surely a problem worth studying, for both its theoretical implications and its applications in reactive synthesis. Proving Conjecture 1 is an interesting future research direction as well, which may require more sophisticated techniques to prove the lower bound, like, e.g., Ehrenfeucht-Fraïssé games [30], Adler-Immermann games [31], or the proof system used in [19] to show the succinctness of $\mathsf{LTL[X,F]}$. Finally, while we know that the lower bound for the translation of $\mathsf{LTL_f}$ into pLTL is at least exponential, the only known upper bound is triply exponential. A tighter lower bound or more efficient algorithms for the problem are definitely worth investigating.

## CRediT authorship contribution statement

**Alessandro Artale:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **Luca Geatti:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **Nicola Gigante:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **Andrea Mazzullo:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **Angelo Montanari:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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