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Dimensional analysis: reasoning with units of measurement as a tool for improved problem-solving competency

Jöran Petersson [a], Maria Antonietta Lepellere [b] and Rosie Lopez-Conde [c]

[a]Faculty of Education and Society, Malmö University, Malmö, Sweden; [b]Department of Agricultural, Food, Environmental and Animal Sciences, University of Udine, Udine, Italy; [c]College of Graduate Studies and Teacher Education Research, Philippine Normal University, Manila, Philippines

ABSTRACT

The key idea of dimensional analysis is to see units of measurement as a crucial source of information for solving mathematical problems. Yet, this powerful tool is hardly at all mentioned in mathematics education research within secondary schools. Hence, this study explores dimensional analysis in secondary school with respect to its occurrence in high-stakes tests and whether teaching it could make students use it as a tool in mathematical problem solving. A first result from inspecting released national tests is that knowledge of the idea behind dimensional analysis seems useful on a substantial proportion of the test items. A second result from inspecting students' archived responses to test items is that a large proportion of the students do not make use of the information contained in units of measurement. A third result is that an intervention on teaching the elements of dimensional analysis made students significantly improved their results from pre-test to post-test. Accordingly, we recommend teaching the key idea in dimensional analysis from lower secondary school as a tool for mathematical problem solving.

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KEYWORDS
Dimensional analysis; intervention; mathematics; secondary school; test items; test results

1. Introduction

1.1. A brief history of dimensional analysis

Dimensional analysis emphasises that a (physical) quantity is a composition of both a value and a unit of measurement. One of MacHale's (2022, p. 134) collected jokes about mathematics is on what could happen if not doing so, namely $1€ = 100 \text{ cent} \rightarrow \sqrt{1€} = \sqrt{100} \text{ cent} \rightarrow 1€ = 10 \text{ cent}$. This apparent contradiction resolves when involving the units as follows $1 \text{ m}^2 = 100 \text{ dm}^2 \rightarrow \sqrt{1 \text{ m}^2} = \sqrt{100} \cdot \sqrt{\text{dm}^2} = 10 \text{ dm}$. This example illustrates that the purpose of dimensional analysis is to make use of the information conveyed by units of measurement when determining how to calculate a quantity. Another example adapted to school mathematics is to calculate velocity in units of m/s. From the resulting unit being metres per second, simply conclude that the calculation is metres divided by seconds. Specifically, the Latin word 'per' means 'through', which, in some languages, is used as a mathematical synonym for 'division'. As a contrast, a formula triangle,

CONTACT Jöran Petersson joran.petersson@mau.se

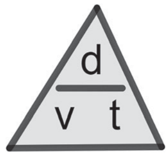

Figure 1. A formula triangle.

as shown in Figure 1, is sometimes taught or learnt without connection to the information carried by the units of measurement. If so, it could become a guesswork for the student in where to put the letters t for time, v for speed and d for distance in the formula triangle, with errors as a likely consequence, despite that a formula triangle might be a way to at least escape errors due to algebraic manipulations (Foster, 2021).

The concept of dimensional analysis traces back to 1883 with the dimensionless Reynolds's number used in fluid mechanics (Rivadulla, 2017). A formalisation of the idea is known as the Buckingham Pi-theorem (Buckingham, 1914), though it was formulated even earlier by Bertrand (1878) and Vaschy (1892). For this reason, it sometimes gets the longer name Bertrand-Vaschy-Buckingham-Pi theorem. The suffix 'Pi' stands for the product (of dimensions), which the following example illustrates. Look for a relation between velocity in meters per second, distance (m), time (s) and mass (kg). The dimensional product equation becomes Equation (1) with an obvious solution $\{1 = a; -1 = b; 0 = c\}$.

$$[\mathrm{m/s}]^1 = \mathrm{m}^a \cdot \mathrm{s}^b \cdot \mathrm{kg}^c \tag{1}$$

Here, the powers of each physical unit state that the mass is not involved since power $c = 0$, length in metres, is in the numerator since $a = 1$ and time in seconds is in the denominator since $b = -1$. Hence, the formula becomes metres divided by seconds. This example illustrates that proportionality is a special case of dimensional analysis in which the variables get integer values ± 1 or 0. In other words, dimensional analysis is a generalisation of proportionality.

1.2. Dimensional analysis in tertiary education

In tertiary education, dimensional analysis is a standard tool in physics and chemistry for deriving formulas (Brown, 2009; Deakin, 2013; Krantz, 2000; Misic et al., 2010; Sznitman et al., 2017), in chemistry for conversion between units of measurements (Saitta et al., 2011) and in mathematics for modelling (Fay & Joubert, 2002). Though these subjects typically are mathematically intensive, Rice and Bell (2005) have also tried dimensional analysis in tertiary nursing education, which is mathematically much less intensive. They reported improved success in the students' achievement in medical dosage calculation. In addition, they reported increased self-perceived confidence in dosage calculations. For example, the students gave comments such as '[...] makes sense; not just plain memorization' and '[...] don't have to remember multiple formulas' (Rice & Bell, 2005, p. 317). Greenfield et al. (2006), Koohestani and Baghcheghi (2010) and Montazer et al. (2022) made similar studies in which the experiment group was taught dimensional analysis and concluded with a recommendation to teach dimensional analysis in dosage calculation for nursing education. In each of these three studies, the reference group and the experiment group had similar results on the pre-test, but during the post-test, the experimental groups performed better, and with statistical significance, than the reference groups. So, the conclusion by Fay and Joubert (2002, p. 280) that dimensional analysis 'promotes better understanding of the role of units and dimensions in mathematical modelling problems' seems to hold within tertiary education, irrespective of the level of the mathematics content in the education.

1.3. Dimensional analysis in secondary education

From a mathematics education perspective, a frequently cited book on problem solving is Pólya (1945). However, what seems to be less known is that Pólya also mentioned dimensional analysis as a 'test by dimensions' (Pólya, 1945, p. 202f) which he, for the case of geometry, described as checking that the dimensions are the same for the right-hand and left-hand sides of an equation, as shown in Equation (2) and that all terms in a sum have the same dimension, e.g. $\mathrm{kg} + \mathrm{kg} = \mathrm{kg}$. Another example of dimensional analysis is Vergnaud (1979, p. 265), who stated that quantities 'are not pure numbers but magnitudes of various kinds' and continued: 'It is necessary to take these dimensions into account not only to analyse tasks but also to lead pupils to some important theories such as dimensional analysis'. Despite these two examples by frequently read and cited sources, dimensional analysis seems rare in research in secondary education and – surprisingly – in particular, in research on secondary school mathematics teaching.

One possible explanation for this is that the original version of dimensional analysis through the dimensional product equation in the Bertrand-Vaschy-Buckingham Pi-theorem is mathematically demanding for secondary school students since it involves powers, and it results in a system of linear equations, though often with obvious solutions for the case of direct proportionalities, as illustrated earlier with the relation between speed, time and distance. The 'factor-label method' is a mathematically simpler version of dimensional analysis, adapted for secondary school mathematics. It does not express the Bertrand-Vaschy-Buckingham Pi-theorem in a product of dimensions with powers, but instead as a chain product of units starting with a given initial unit and incrementally multiplied with conversion factors that cancel unwanted units until only the wanted unit remains, as exemplified in Equation (2), illustrating the factor-label method for mole calculations of making water. Hence, the idea of the factor-label method is the same as Pólya's (1945) test by dimensions since the unit cancelling factors together ensure the unit symmetry on the left- and right-hand sides, as illustrated in Equation (2) and Figure 2. Equation (2) illustrates the key idea of dimensional analysis is unit symmetry in the sense that both sides of an equation must have the same unit of measurement.

$$20\,\mathrm{g\,O_2} \cdot \frac{1\,\mathrm{mole\,O_2}}{32\,\mathrm{g\,O_2}} \cdot \frac{2\,\mathrm{mole\,H_2O}}{1\,\mathrm{mole\,O_2}} \cdot \frac{18\,\mathrm{g\,H_2O}}{1\,\mathrm{mole\,H_2O}} = 22.5\,\mathrm{g\,H_2O} \tag{2}$$

The search phrases 'factor-label method' and 'dimensional analysis' combined with 'secondary school' were applied to the databases Google Scholar and ERIC (Education Resources Information Center). In ERIC, the only search hit of relevance for secondary school was Lassri and Lassri (2022), who investigated second-year baccalaureate physics students' familiarity with dimensional analysis presented in a style like Equation (1) in which the exponents of each letter in a physics formula is checked and not like the factor-label style in Equation (2), in which the unit of measurement is balanced. Google Scholar

$$\boxed{\frac{1\,\cancel{\text{km}}}{1\,\cancel{\text{h}}}} \cdot \boxed{\frac{1000\text{ m}}{1\,\cancel{\text{km}}}} \cdot \boxed{\frac{1\,\cancel{\text{h}}}{60\,\cancel{\text{min}}}} \cdot \boxed{\frac{1\,\cancel{\text{min}}}{60\text{ s}}} = \boxed{\frac{1000\text{ m}}{3\,600\text{ s}}}$$

Figure 2. A card game for unit conversion.

gave 142 search hits for 'dimensional analysis' of which none were on the mathematical concept of dimensional analysis. The phase 'factor-label method' in Google Scholar gave 89 hits, of which one was of relevance for secondary school mathematics, while the rest were on secondary school chemistry. The single reference to secondary education in mathematics is Hoile (1974), who designed a package for students in year 10, of teaching conversion between metric and non-metric units. Surprisingly, it seems that if students encounter dimensional analysis in secondary school, they do so by their chemistry teacher and presented as the factor-label method, mainly for mole calculations (Gabel & Sherwood, 1983) and unit conversions (Garrett, 1983), exemplified in Figure 2. Summers (1960) wrote a brief wish-list of core concepts in which he compared what is and what should be in secondary school textbooks in chemistry. One core concept on his list is the factor-label method, which is stressed in tertiary chemistry education, while secondary school chemistry textbooks typically use proportionality (Summers, 1960). The factor-label method scored high also in Deters (2003) forty-year-younger wish-list based on a survey of what knowledge college professors think high school students need to bring into college chemistry. However, the use of dimensional analysis in chemistry education might depend on local curricular traditions since pre-service teachers in chemistry used other methods than dimensional analysis when solving chemistry problems (Danjuma, 2011).

Besides the many non-empirical studies on using dimensional analysis in secondary school chemistry, there are a scattered handful of empirical studies on using the factor-label method. Garrett (1983) designed a card game study similar to that in Figure 2 to help students learning the factor-label method as a procedure. On the other hand, the procedure might also prohibit them from reasoning, since the card game might convert the original problem into a symbol manipulation task. Such worries concern Navidi and Baker (1984) and Cardulla (1987), and both these papers suggested not introducing the factor label method until after the students have learnt to reason about stoichiometry and unit analysis. Alternatives to having each factor in Figure 2 on cards as in Garrett (1983) are to have them on the toy LEGO® (Molnár & Molnár-Hamvas, 2011) or on dominoes visualised in a computer game (Ellis, 2013) for teaching dimensional analysis in secondary school. Specifically, Ellis (2013) reported that, during an intervention study, her experimental group developed significantly better than her reference group.

Despite the generality of dimensional analysis in also handling proportionality situations, the research found on dimensional analysis in secondary school was strictly confined to stoichiometry calculations in chemistry and unit conversions. Further, the research on teaching dimensional analysis in secondary school mathematics seems severely neglected, and Hoile's fifty-year-old quotation still holds: 'there appears to be no materials or research available on the topic of factor label method in the secondary mathematics curriculum' (1974, p. 9). This is in stark contrast to the abundant research on teaching proportionality

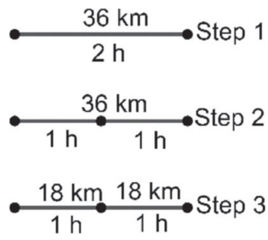

Figure 3. Solving steps when using a unitisation diagram.

despite the close relation, portrayed earlier, between dimensional analysis and proportionality. A speculation why dimensional analysis seems common in educational research on secondary school chemistry but is effectively absent in mathematics and physics is the following. The complexity of secondary school chemistry calculations, as illustrated in Equation (2) and the referred research with chains of proportional relations, seems to require some kind of tool for designing the calculations. This is not the case in mathematics and physics, in which one-step proportionality often is enough.

1.4. Proportionality

Knowledge of proportionality is useful for mathematically describing several phenomena in different fields of human activity (Modestou & Gagatsis, 2010, p. 36). For this reason, proportionality occurs in mathematics teaching from early primary as doubles and halves of integers to late secondary school as continuous quantities of units of measurement (Lamon, 2007). Consequently, a solid knowledge of proportionality will lead students to success in many topics in mathematics (Lobato & Ellis, 2010; Miyakawa & Winsløw, 2009).

Among students, there are several ways to handle proportionality problems, and Vergnaud (1983) is an influential chapter on characterising students' responses to rational relations of the kind $a/b = x/c$ where a and x have the same dimension, while b and c typically have one dimension different from that of a and x. One response category is the calculation $(c/b) \cdot a = x$, in which the bracket denotes the calculation made first. Vergnaud (1983, p. 145ff) denoted this calculation a scalar operator since c/b is dimensionless. The scalar operator complies with the definition of rational relations in Euclid's Elements Book Five, definitions 5 and 6 (Euclid & Heath, 1956, p. 114; Fine, 1917, p. 73). A second response category is the calculation $(a/b) \cdot c = x$, and Vergnaud denoted this calculation as a function operator since a/b it has a composite dimension. An intermediate between the scalar and function operations is the unit operator, where the student transforms the original left-hand side into $a/b = d/1$ where $d = a/b$ and then treats it as a scalar operator, as illustrated in Figure 3 for solving the problem: What distance x km will you have cycled at $c = 3$ h if you have cycled so far $a = 36$ km in $b = 2$ h?

Figure 3, Beckmann and Izsák (2015; see also Izsák & Beckmann, 2019), denotes a tape diagram and strip diagram, and when they use the term unit, they follow Vergnaud's terminology for scalar unit operation. Though the numerical calculations in the unit operation illustrated in Figure 3 are the same as in the function operation, the difference is in the treatment of the units, where $d = 18$ km is a distance, while $a/b = 18$ km/h is a speed, which

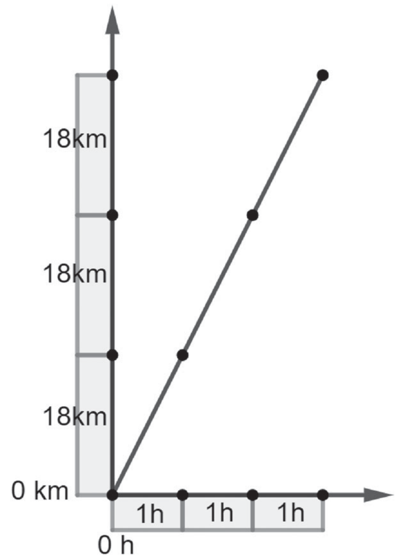

Figure 4. Unitisations diagram in a coordinate system.

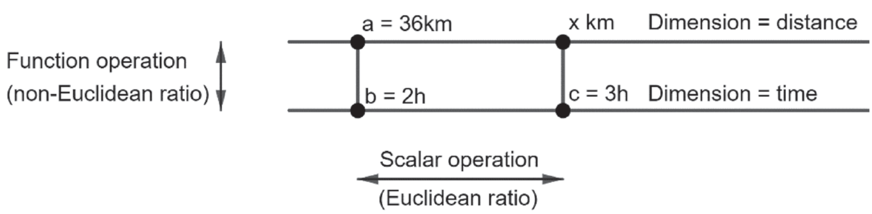

Figure 5. Diagram for scalar and function operations.

is a composite dimension. Kaput and West (1994) found that students in year 6 spontaneously use the unit operation. Bentley and Yates (2017) actively taught Vergnaud's unit operation through the steps in Figure 3 and found it promising for year 7. Beckmann and Izsák (2015) developed further the strip diagram in Figure 3 into a coordinate diagram, as shown in Figure 4.

The unitisation diagrams in Figures 3–4 are limited to the value *b* being suitably small integers. For the case of *b* being a decimal number, Okazaki and Koyama (2005) overcame this limitation by emphasising the algebraic relation rather than the numerical calculation for the quotient. Their original schema is essentially the same double number line, as shown in Treffers (1993) and builds directly on Vergnaud's (1983) scalar case and hence corresponds to the horizontal lines in Figure 5. Further, it is easy to extend Figure 5 to Vergnaud's function operation by adding the vertical lines. Hence, the horizontal lines in Figure 5 illustrate that Vergnaud's scalar proportion is a quotient within the same dimension, while the vertical lines illustrate that Vergnaud's function proportion is a quotient between different dimensions.

2. Research questions

Teaching and learning Vergnaud's scalar operation through unitisation has been well conceptualised in Figures 3–4 by, for example, Beckmann and Izsák (2015), Bentley and Yates (2017) and Izsák and Beckmann (2019) and for the continuous case in Figure 5 by Okazaki and Koyama (2005). Yet, a question is to what extent these scalar diagrams in Figures 3–5 support students in how to use units of measurement as a means to justify their numerical calculations in general proportion situations. In other words, how can the teaching support students in developing knowledge about the elements of dimensional analysis suggested in Pólya (1945) and Vergnaud (1979, 1983)? So, the following three research questions constitute a sequence of: First exploring what place dimensional analysis takes in high-stakes exams, then probing if its idea seems used among students and finally, if students are receptive to its teaching. These three research questions were answered in three separate studies.

- For Study 1: How useful is elementary knowledge in dimensional analysis in national tests?
- For Study 2: What knowledge do students display in large-scale tests with respect to informing their calculations from the units stated in a problem?
- For Study 3: How do secondary school students respond to an intervention focusing on elementary dimensional analysis for proportionality problems in mathematics?

3. Study 1 – The usefulness of dimensional analysis in national tests

Study 1 explored the proportion of tasks in the national test for which knowledge in elementary dimensional analysis could be useful for correctly responding to tasks in national tests. Data sources were released from national tests in Italy, Sweden and the USA.

3.1. Methods

3.1.1. Choosing data for Study 1

There are several reasons for choosing national tests as a data source for research question 1. First, their purpose is to measure how well students attain the intended curriculum and to shape the implemented teaching along with the intended curriculum (Ferretti et al., 2018; Madaus & Russell, 2010; Palm et al., 2011). Second, national tests reach a large proportion of a cohort since they are mandatory in several countries, for example, Sweden and Italy. Moreover, year 8 in Italy and year 9 in Sweden are the last years in compulsory school, and in the USA, year 8 typically completes junior high school. So, the national tests in these school years should cover most of the curriculum content. These two arguments lead to a third argument. Since released national tests constitute a condensed expression of both intended and implemented curriculum, inspecting their test items forms a compact data source that requires a modest work effort. Alternatives, such as inspecting selected chapters in textbooks as in Charalambous et al. (2010) or even whole textbooks as in Petersson et al. (2022), require much more work.

The tests analysed were from 2013 and 2017 (NAEP, 2024; PRIM, 2024; Prove Invalsi, 2024). During data collection, 2017 was the latest year with released tests. The year 2013

Table 1. A framework for examining whether dimensional analysis is useful in a task.

Expressions with values v_j and units u_k	Equation: One unknown value to determine	Comparison: (When) does it hold?
Direct proportionality	$\frac{\{v_1\vert u_1\}}{\{v_2\vert u_2\}} = \frac{\{v_3\vert u_1\}}{\{v_4\vert u_2\}}$ or $\frac{\{v_1\vert u_1\}}{\{v_2\vert u_2\}} = \left\{v_3\vert\frac{u_1}{u_2}\right\}$	$\frac{\{v_1\vert u_1\}}{\{v_2\vert u_2\}} < \frac{\{v_3\vert u_1\}}{\{v_4\vert u_2\}}$ or $\frac{\{v_1\vert u_1\}}{\{v_2\vert u_2\}} < \left\{v_3\vert\frac{u_1}{u_2}\right\}$
Linear relation	$\frac{\{v_1 - v_3\vert u_1\}}{\{v_2 - v_4\vert u_2\}} = \left\{v_5\vert\frac{u_1}{u_2}\right\}$	$\frac{\{v_1 - v_3\vert u_1\}}{\{v_2 - v_4\vert u_2\}} < \left\{v_5\vert\frac{u_1}{u_2}\right\}$

was chosen since it is between 2017 and the previous curricular reform. Specifically, Italy had a curricular reform in 2010 (Menghini, 2016) and Sweden had one in 2011 (Skolverket, 2018). Sub-items in a test were counted as separate test items, for example, 1a and 1b as counted as two items.

3.1.2. Analysing data for Study 1

To answer research question 1, the authors determined the proportion of tasks in national tests for which knowledge in dimensional analysis could help the student in giving a correct response. For each task, the usefulness of dimensional analysis was operationalised through the framework in Table 1, which builds directly on Vergnaud (1979) and Pólya (1945). First, and along with Vergnaud (1979, p. 265) quoted earlier in this paper, the framework in Table 1 recognises magnitudes as pairs $\{v|u\}$ of a value v and a unit u. For example, the magnitude 18 km/h has a value $v = 18$ and a unit $u = \text{km/h}$. Second, and along with Pólya's (1945, p. 202f) 'test by dimensions', the framework in Table 1 uses unit symmetry defined through the requirement that the left-hand and right-hand sides of an expression have the same dimensions for the units. Third, the relation between the right-hand and left-hand sides could be either an equation $\{=\}$, with one unknown magnitude to determine, or a comparison $\{<\}$, in which the task is to investigate if this relation holds or does not hold. Fourth, Study 1 confines to linear models and direct proportionalities, though it is possible to extend the framework in Table 1 to, for example, quadratic and exponential models. Fifth, the case of equal units $u_1 = u_2$, resulting in a dimensionless quotient u_1/u_2 such as a percentage, is allowed. Sixth, in Study 1, the dimensional analysis is considered useful in a task if it solves at least a part of the task for the cases displayed in Table 1.

For example, test item 79 in the Delaware state NAEP 2013 (National Assessment of Educational Progress) asked, Sally can buy 20 pencils for \$0.99. What is the greatest number of pencils Sally can buy for \$3.00?' This task could be solved using a proportional comparison $\frac{\{x|\text{pencils}\}}{\{3|\$\}} < \frac{\{20|\text{pencils}\}}{\{0.99|\$\}}$. Further, the models in Table 1 have in common that they can be illustrated graphically as coordinates $(\{v_1|u_1\}; \{v_2|u_2\})$ in a coordinate system corresponding to Figure 4. A test item that does not meet the requirement in Table 1 with respect to explicit units is test item 55 in the same NAEP 2013. It asked: 'In which of the following equations does the value of y increase by 6 units when x increases by 2 units?' It could be formulated as a difference quotient $\frac{v_1 - v_3}{v_2 - v_4} = v_5$, but there are no explicitly stated units in this task. Some students might treat this task as if it had an imagined y-unit and a separate imagined x-unit. However, since there are no explicit units available as support in the problem description, other students might not impose imagined units on the test item. This latter example illustrates that some tasks must be excluded since they are purely algebraic or numerical.

Table 2. Frequency of test items in which knowledge of dimensional analysis is useful.

Year	Italy	Sweden	USA
2013	14 of 45 (31%)	18 of 39 (46%)	13 of 53 (25%)
2017	8 of 46 (17%)	11 of 46 (24%)	11 of 37 (30%)

3.2. Results

The result in Study 1 is that knowledge in dimensional analysis, for at least the proportional and linear cases as defined in Table 1, is considerably useful in national tests. Specifically, the national tests in each of the three countries had a substantial proportion of such tasks, as shown in Table 2. As a comparison, the Swedish national test required at least 25% correct for passing the test, while the highest grade required 85% correct (Prim, 2024). For the USA, the level 'NAEP basic' corresponds to about 50% correct and 'NAEP advanced' about 67% correct (NAEP, 2025). For Italy, passing corresponds to 55% correct, and the highest grade is 92% of the test scores.

4. Study 2 – students' use of dimensional analysis in a large-scale test

Study 2 explored students' responses to a test with respect to whether their calculations matched their use of units. In 2022, the Swedish compulsory school curriculum was revised. One change in the mathematics curriculum (Lgr22, 2024) was to introduce the concept of composite units, which refers to a quotient of two different dimensions, for example, speed. Therefore, Study 2 used data with the aim of capturing a situation from well before this change.

4.1. Methods

Along with research question 2, Study 2 analysed students' responses to two national test items for which knowledge in dimensional analysis would be helpful. The analysis was to categorise how well the unit they stated in their responses matched their calculations. So, along with Vergnaud's (1979) suggestion, dimensional analysis informed the analysis.

4.1.1. The sample of archived responses

The data in Study 2 are 186 students' written responses to tasks in a national test. One reason for choosing these data is the properties of the sample. In the ninth and last year of Swedish compulsory school, students take a mandatory national test in mathematics. From each national test, the Swedish national test group in mathematics collects and archives a random sample based on birth dates. Hence, the data in Study 2 constitute a random sample from a nationwide student population.

4.1.2. The test tasks

The context for the test tasks was time for lining up dominoes and knocking them over. These tasks occurred in the national test year 2009, tasks 9a and 9b in part C being a released test (PRIM, 2024). One task is a rate problem whose answer has the dimension [dominoes per time], and the other task is an inverse rate problem whose answer has the

Table 3. Assessment categories for calculations in the lining up dominoes task.

Category	Calculations in response correspond to:
Correct	A correct solution to the task.
Incomplete	For the lining up task, a quotient (only dominoes)/(only time factors) with missing time factor(s).
Inverse dimension	A consequently inverse quotient; for the lining up task (only time factors)/(only dominoes).
Erroneous dimension	A quotient of time factors or a product of time factors and dominos, or a subtraction of different dimensions.
Other	Responses not fitting other categories.
No response	No response given.

dimension [time per domino]. Along with the purpose of Study 2, they provided an opportunity to investigate the students' ability to consistently match the calculated dimension with the unit stated in the problem, see solutions in Equations (3) and (4).

Lining up dominos task: Ma Lihua lined up 303 628 dominoes. It took her 6 weeks and she worked 12 h every day. On average, how many dominoes did Ma Lihua line up in one minute?

Solution: Since Wanted unit $= \frac{\text{dominos}}{\text{minute}}$, calculations that solve this task are as in Equation (3):

$$\frac{303\ 628 \text{ dominos}}{(6\text{weeks}) \cdot \left(7\frac{\text{days}}{\text{week}}\right) \cdot \left(12\frac{\text{h}}{\text{day}}\right) \cdot \left(60\frac{\text{minutes}}{\text{h}}\right)} \approx 10 \text{ dominos/minute.} \tag{3}$$

Knocking over dominos task: It took four minutes for the dominoes to fall and only six dominoes remained standing after the attempt. How many milliseconds did it take on average for a domino to fall?

Solution: Since Wanted unit = milliseconds/domino, calculations that solve this task are as in Equation (4):

$$\frac{(4 \text{ minutes}) \cdot \left(60\frac{\text{s}}{\text{minute}}\right) \cdot \left(1\ 000\frac{\text{ms}}{\text{s}}\right)}{(\text{all } 303\ 628 - 6 \text{ remaining standing}) \text{ dominos}} \approx 0.8 \text{ ms/domino} \tag{4}$$

During this part of the national test, the students had access to calculators. Furthermore, both tasks are what Tourniaire and Pulos (1985) denoted 'rates', since the quotients in both tasks are ratios of dissimilar objects and hence with different dimensions (Karplus et al., 1983).

4.1.3. Method for analysing the archived responses

Equation (3) gives the solution for the lining up dominoes task as a quotient $\frac{\text{dominos}}{\text{a product of time factors}}$. Each student's response was assessed into the categories of how well they handled their calculation and how well the stated unit matched their calculations in line with Table 1 in Study 1. Tables 3 and 4 show categories for the lining up dominoes task. The same categorisation works for the knocking over dominoes task, but with the numerator and denominator switched, as shown in Equation (4) and hence inverted units. Besides categorising correct responses, Vergnaud (1983) also categorised incorrect calculations, such as confusing a quotient $\frac{c}{b}$ with its inverse $\frac{b}{c}$ or its product $b \cdot c$ and subtracting different dimensions, such as length minus time. In line with this, Tables 3 and 4 have categories also for incorrect responses. These errors correspond to what Pólya's (1945) test by dimension would prevent.

Table 4. Assessment categories for the relation between units and calculations.

Category	Unit stated in response:
Complete matching	The unit fully matches the calculations.
Incomplete matching	The unit partly matches the dimension of the calculations. For example, only 'dominos' with omitted or erroneous 'per time'.
Inverse matching	The unit corresponds to inverted calculations. This category includes incomplete inverse matching, such as 'time' instead of 'time per domino'.
Not matching	Other non-matching units.
No unit	Unit omitted – only numerical response.
No response	No response given.

The hierarchical categorisations in Tables 3 and 4, from fully correct to no response, allowed us to present the results in a cross-table of calculation and units according to the categories in Tables 3 and 4. This allowed exploring the relation between students' ability in seeing a relation between calculations made and units used on a more fine-grained level than only correct and incorrect answers. In other words, to measure what Brown (2009) expressed as letting the units tell what calculations to do. Consequently, Spearman's rho for these ordinal data (see e.g. Cohen et al., 2018, table 39.2) measured the strength of the correlation between, on the one hand, correctness in calculations and, on the other hand, giving their answer a unit matching their actual calculations. Since the calculation of Spearman's rho needs numerical rank values, the rank values for Table 3 were Correct $= 4$ (highest rank), Incomplete but otherwise correct $= 3$, Inverse dimension $= 2$, while Erroneous dimension and Other were aggregated into rank $= 1$. No response got the rank $= 0$ (lowest rank). Similarly, the rows in Table 4 got the rank values from 5 (complete matching) down to 0 (no response). The results section gives frequency and examples of students' responses in these categories.

4.2. Results

4.2.1. Students' responses to the lining up dominoes task

Table 5 shows that 85 (46%) students gave a response with both correct calculations and a completely matching unit, while 32 (17%) did not respond at all. When it comes to the relation between calculations and units, there were a total of 94 (51%) in whose responses the stated unit matched the calculation. For the other students, the stated unit did not match the calculation. Spearman's rho ≈ 0.89 for Table 5 corresponds to a strong correlation in how well the given units matched the calculations made in the responses. In other words, for this case, the Spearman correlation tells that if a student did well on the calculation, they typically chose a unit that describes their calculation, and also, if the student did not do well on the calculations, they typically stated a unit that does not match their calculation. A note is that if the 'no response' were excluded from the correlation calculation, Spearman's rho ≈ 0.68 would correspond to a medium strength in the correlation.

Table 5 shows that a substantial proportion of the students did not connect their calculation with a matching unit. Figure 6 gives an example of the latter. It shows a response corresponding to $\frac{303\,628\text{ dominoes}}{6\cdot 7\cdot 12\cdot\text{hours}}$. The missing factor 60 min in the denominator makes the calculations, in Figure 6, incomplete. Also, the response unit is incomplete since the calculated unit is *dominoes per hour,* while the response unit is '602 om dagen', which translates

Table 5. Dimension in calculation versus unit in responses to the lining up task (dominoes/time).

	Unit matched calculations						
Calculations	Complete	Incomplete	Inverse	No match	No unit	Total	
Correct	85	9	0	0	2	96	(52%)
Incomplete	7	25	0	0	6	38	(20%)
Inverse dimension	0	0	0	0	1	1	(1%)
Erroneous dimension	0	0	0	6	2	8	(4%)
Other	2	2	0	7	0	11	(6%)
No response	0	0	0	0	32	32	(17%)
Total	94 (51%)	36 (19%)	0 (0%)	13 (7%)	43 (23%)	186	

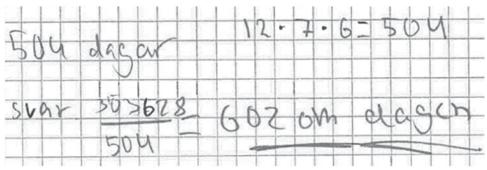

Figure 6. Incomplete matching between the calculated and stated unit for the lining up dominoes task.

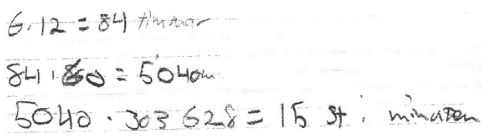

Figure 7. Irrelevant calculations with a non-matching unit for the lining up dominoes task.

into '602 a-day'. Hence, the response in Figure 6 was categorised as incomplete calculations and incomplete matching between units and calculations.

For the lining up dominoes task, only one response corresponded to inverse calculations through the calculations $(6 \cdot 12 \cdot 60\,\text{minutes})/(303\,628\,\text{dominoes})$, which omitted the factor 7 days and gave no unit in the response. One example of erroneous calculation dimension with a non-matching unit is Figure 7 in which the student multiplied dominoes and time instead of dividing the former by the latter. Moreover, the response '15 pieces per minute' in Figure 7 has only the first two digits in common with the stated product 5040minutes · dominoes, which evaluates to $15 \cdot 10^8$. A plausible interpretation is that the student disregarded the exponent. Further, the response in Figure 7 shows no connection between calculations and the stated unit.

Other examples of calculations with erroneous dimensions are to divide time factors by each other, which six (3%) students did. One such example is the nonsense calculation $(6 \cdot 7 \cdot 12)/60$, hence dividing 504 h with 60 min. In addition, this student stated the unit 'pieces per minute' for this quotient and hence shows no matching between calculations and stated unit. However, in two cases, even students with calculations categorised as other showed the relation between calculation and unit. One example of this is the response '$7 \cdot 6 \cdot 12 = 504$ hours in 6 weeks'. Here, the stated unit matches the calculation, though it includes only time factors, but not the dominoes. The latter made the calculation fit the response category other.

Table 6. Dimension in calculation versus unit in responses to the knocking over task (time/dominoes).

	Unit matched calculations						
Calculations	Complete	Incomplete	Inverse	No match	No unit	Total	
Correct	15	5	0	0	0	20	(11%)
Incomplete	6	12	1	7	2	28	(15%)
Inverse dimension	9	1	26	15	6	57	(31%)
Erroneous dimension	0	0	0	7	0	7	(4%)
Other	0	0	0	6	1	7	(4%)
No response	0	0	0	0	67	67	(36%)
Total	30 (16%)	18 (10%)	27 (15%)	35 (18%)	76 (41%)	186	

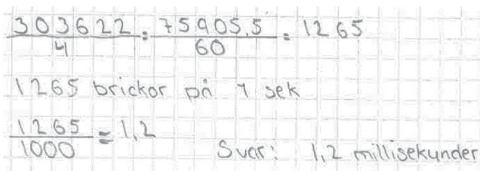

Figure 8. Inverse calculation and inversely matching unit for the knocking over dominoes task (time/domino).

4.2.2. Students' responses to the knocking over dominoes task

Table 6 shows that only 15 (8%) students gave a response with both correct calculations and a completely matching unit. Furthermore, of the 119 who responded with a calculation for the task, only thirty students gave a completely matching unit to their calculations. Spearman's rho ≈ 0.89 for Table 6 corresponds to a strong correlation in how well the given units match the calculations. A note is that if the 'no response' were excluded from the correlation calculation, Spearman's rho ≈ 0.60 would correspond to a medium strength in the correlation.

Interestingly, Table 6 shows that about one-third of the students calculated dominos/time, which is the inverse rate for the knocking over dominoes task. Nevertheless, 26 of these 57 gave their calculations the unit time/domino instead of dominoes/time. Only 9 of these students gave a unit that completely matched their calculations.

Figure 8 exemplifies the case of inverse calculations and a unit whose inverse matches the calculation. This means that the calculated dimension in Figure 8 is ≈ 1.2 dominoes per millisecond, while the stated unit instead is 1.2 milliseconds. Particularly, the student in Figure 8 showed awareness of the relation between calculation and unit in the second row through stating '1265 dominos in 1 s', and yet stated milliseconds as a time unit in the third row. Apparently, this and similar responses showed ignorance about the relation between performed calculations and stated units.

Figure 9 is an example of a response categorised as incomplete in both calculation and unit. The calculation $\frac{4\text{minutes}}{303622\text{dominoes}}$ in Figure 9 omitted both the factor 60 s per minute and the conversion to milliseconds. The translated response in Figure 9 states that 'Answer: It took 0.000 013 s for one domino to fall'. The unit given in the answer is seconds per domino, while the calculated unit is minutes per domino. Since both calculation and unit have the dimension time per domino but with omitted time factors, the unit given in the response was categorised as incomplete matching.

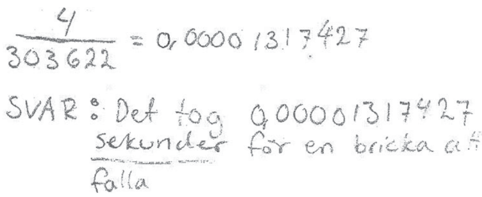

Figure 9. Incomplete calculations and incomplete unit matching for the knocking over dominoes task.

5. Study 3 – secondary school students' responses to an intervention

5.1. Methods

To answer research question 3 on the effects of teaching elementary dimensional analysis, an intervention study was conducted in Italy and Sweden in secondary schools. Its effects were evaluated through a statistical comparison of the students' results on a pre-test and a post-test.

5.1.1. The sample of students

The 36 participants in Study 3 were from classrooms in Italy and Sweden. The youngest group consisted of 12 students. We name this group compulsory school, since they were studying in the last year (year 8) of Italian compulsory school. In Italy, proportions are taught in year 7. No review was done by the teacher before the pre-test, but their preparation from the previous year was quite good (see Lepellere & Gasparo, 2024).

The second group consisted of seven students. We name this group remedial, since they all had received the grade 'failed' in mathematics when leaving Swedish compulsory school (year 9). During the intervention, they were studying a separate remedial course in mathematics in a vocational programme (year 10) in the upper secondary school, and all seven students gave their consent to participate in this study.

The third group consisted of 17 students. We name this group nursing, since they were studying a nursing vocational programme in a Swedish upper secondary school during the intervention. Of these students, four were remedial students since they had received a grade 'failed' in mathematics when leaving compulsory school. All nursing students were studying in the same class and the intervention lasted for two weeks with 6 lessons of 60 min each, including pre-test and post-test. The class had 19 students, but two were absent when the consents were signed and collected. Hence, these two students were excluded from the results in this study.

5.1.2. The intervention

The intervention in the participating classes consisted of their teachers' demonstrations of dimensional analysis applied to proportionality problems and of giving exercises to the students that provided further learning opportunities for it. As shown by Lassri and Lassri (2022), the interventions did not teach dimensional analysis as balancing exponential powers in formulas. Instead, the demonstrations emphasised how to match the units of measurement when setting up a dimension quotient or dimension product. Equations (5) and (6) exemplify this through having the same unit in the numerators of both left-hand and right-hand sides for each equation, and the same holds for the denominators.

Further, Equations (5) and (6) illustrate that there typically are multiple ways of doing this; for the tube weight task, Equation (5) has the unknown in the numerator, while for the book weight task, Equation (6) has the unknown in the denominator.

> *Tube weight task*: The plastic tubes in the picture are of a similar kind and are 48 and 57 cm long. The longer weighs 177 grams. What is the weight of the shorter one? [The task included a picture of two plastic tubes].
>
> Solution: Dimension equation for solving the tube weight task is as shown in Equation (5):

$$\frac{177\,\text{gram}}{57\,\text{cm}} = \frac{X\,\text{gram}}{48\,\text{cm}} \tag{5}$$

> *Book weight task*: 18 books of those in the picture weigh 11,23 kg. What is the weight of 26 of those books? [The task included a picture of a pile of books].
>
> Solution: Dimension equation for solving the book weight task is as shown in Equation (6):

$$\frac{18\,\text{books}}{11.23\,\text{kg}} = \frac{26\,\text{books}}{X\,\text{kg}} \tag{6}$$

5.1.3. The pre-test and post-test

Both the pre-test and post-test consisted of one task for each student. To prevent features in the test tasks from affecting the change in performance from pre-test to post-test, the test situations were arranged as follows. In the compulsory and remedial groups, about half of the students got the task *tube weight task* in the pre-test and the *book weight task* in the post-test. The other half of the students instead got the *book weight task* in the pre-test and the *tube weight task* in the post-test. These two halves were done random. The routine was the same for the nursing group, but with the two tasks 'heat stroke' and 'surgery' on medical dose-rate calculation, whose solutions are shown in Equations (7) and (8). To illustrate that the dimension equations for the dose-rate problems can be formulated in multiple ways, Equation (7) shows an equation which balances the number of drops on each side of the equals sign, while Equation (8) uses a factor-label approach.

> *Heat stroke task*: A patient has heat stroke and is in urgent need of fluids. He is prescribed 1500 ml of glucose infusion. You put an infusion for 5 h. The drop device gives 20 drops/ml. What drop rate is needed for the infusion? (drops/minute).
>
> Solution: Dimension equation for solving the heat stroke task:

$$1500\,\text{ml} \cdot 20\frac{\text{drops}}{\text{ml}} = 5 \cdot 60\,\text{minutes} \cdot x\frac{\text{drops}}{\text{minute}} \tag{7}$$

> *Surgery task*: A patient will undergo surgery for a tumour in the small intestine. He is prescribed a 330 ml infusion of Flagyl. The infusion takes 1 h, and the drop-device gives 20 drops/ml. Determine the drop-rate! (drops/minute).
>
> Solution: Dimension equation for solving the surgery task:

$$x\,\frac{\text{drops}}{\text{minute}} = 20\,\frac{\text{drops}}{\text{ml}} \cdot \frac{330\,\text{ml}}{1\,\text{h}} \cdot \frac{1\,\text{h}}{60\,\text{minutes}} \tag{8}$$

5.1.4. Evaluating the test results

The students' responses to the test tasks were assessed in the same way as Study 2. That is, the responses were classified according to Table 3 for calculations and Table 4 for units, and these were combined into a cross-table similar to Tables 5 and 6 with Spearman's rho for measuring their correlation. In addition to this, the sum of the rank-code for calculation and unit from the pre-test and post-test was compared statistically using a Mann–Whitney *U* test (synonym: Wilcoxon rank sum test) for testing the hypothesis of whether there was a change in achievement and hence in knowledge in dimensional analysis during the interventions. This statistical test also works when students are absent in either the pre-test or post-test. Jamieson (2004) recommends ordinal statistical tests for ordinal data, though Norman (2010) argues that averages, including the proportion of correct responses, are continuous data for which tests like a t-test apply. Hence, an unpaired two-sided t-test would also work as an alternative statistical test. The effect size of the outcome was calculated with the Wendt formula, which is suitable for the Mann–Whitney *U* test (Kerby, 2014).

5.2. Results

Before presenting extracts from the students' responses, we give the statistical results. From pre-test to post-test, all three groups demonstrated an increased proportion of correct results. In the compulsory school group, there were twelve students, who all participated in the pre-test, but one was absent during the post-test. There were 7 responses with correct calculations in the pre-test, and all 11 responses had both calculation and unit correct in the post-test. For this increase between the pre-test and post-test, a Mann–Whitney *U* test gave a significant difference ($p = 0.019$) with a strong effect size ($r_{Wendt} \approx 0.97$). In the remedial group, there were seven students, who all participated in both pre-test and post-test. During the pre-test, only one single response was correct, but with an erroneous unit while the post-test gave four responses with correct calculation, of which one also had a correct unit. For this increase in achievement, a Mann–Whitney *U* test gave a significant difference ($p = 0.037$) with a strong effect size ($r_{Wendt} \approx 0.93$). In the nursing group, the number of correct responses increased from 1 to 11, which is statistically significant ($p = 0.0007$) with a strong effect size ($r_{Wendt} \approx 0.98$). Further, in this group, there were two students who gave their consent to participate in this study and took both tests but were absent during the intervention. These two students were counted as taking the pre-test twice, and they both failed on both tests. However, excluding these two students gave statistically significant results still with $p = 0.0017$.

For all three groups taken together, the correlation between the students' calculations and units was Spearman's rho ≈ 0.63 in the pre-test and Spearman's rho ≈ 0.71 in the post-test, both indicating a medium strength correlation. Though the correlations were similar in the pre-test and post-test, Figure 10 illustrates the students' progress in the following way: During the pre-test, in the left diagram in Figure 10, their achievements displayed a large achievement spread with respect to both calculations and units, where the ball in the lower left corner corresponds to 'no response'. In contrast, the right diagram in Figure 10 displays a large ball in the upper right corner, which corresponds to a large proportion of responses being correct in both calculations and units.

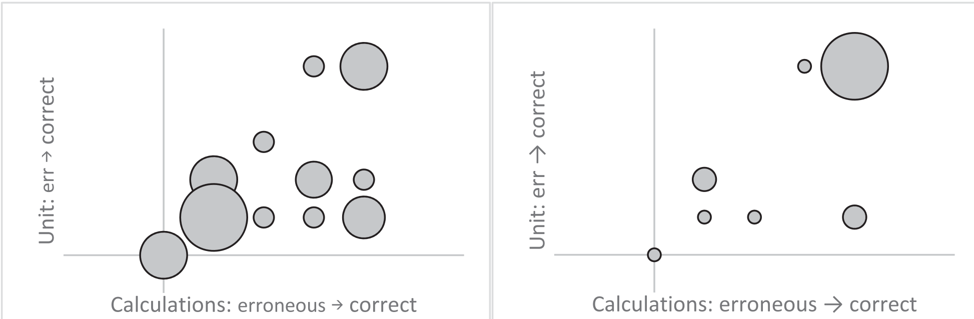

Figure 10. Bubble diagram displaying the correlation between calculation and unit in Pre-test (left) and Post-test (right).

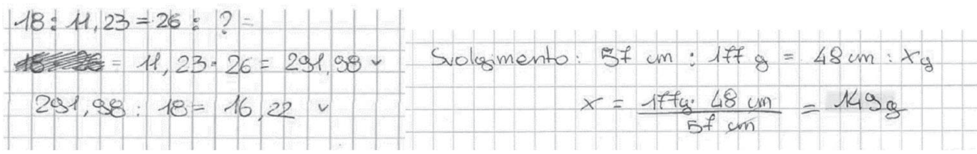

Figure 11. Pre-test response to the book weight task (left figure) and post-test to the tube weight task (right figure) by student 9 in the compulsory school group.

5.2.1. Students who succeeded on both tests

Seven students in the compulsory school group succeeded on both the pre-test and post-test. For six students, there was a qualitative development through not showing units in the pre-test, as illustrated in Figure 11 (left figure), while in the post-test (right figure), all six included units in both formulating the equation and making calculations for all values. Hence, during the intervention, these students seem to have consolidated an already good ability in solving proportionality problems by also making explicit use of the given units. The same holds for the single student in the nursing group, who succeeded on both tests. That student correctly wrote units to all values in all calculations in the post-test, while in the pre-test, only wrote units in the answer and in the intermediate result '300 ml in 1 h'.

Also in the remedial group, there was one single student who succeeded on both tests. In the pre-test, this student solved the tube weight task in the following way. First, this student set up the proportional relation '$\frac{177}{57} \approx 3.1$ cm' though the unit of this proportionality constant should be g/cm instead of the stated 'cm'. Next, this student multiplied '$48 \cdot 3.1 = 145.7$ cm' and here the unit should be grams instead of the stated 'cm'. In the post-test, this student solved the book weight task through setting up the dimensional equation '$\frac{18}{11{,}23} = \frac{26}{x}$' though without units, and correctly solved this equation and gave the correct unit in the answer. These examples show progression during the intervention in handling the units among students who showed satisfactory skills already in the pre-test.

5.2.2. Viewing proportionality as Euclidean or as dimensional

Since the *tube weight task* (and the *book weight task*) compare two situations with a proportional relation, it is possible to consider them as two data points (57 cm; 177 grams) and (48 cm; x grams) in a graphic or algebraic proportionality relation grams = constant · cm

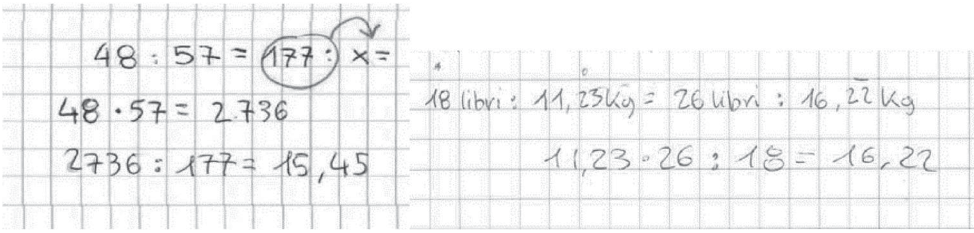

Figure 12. Pre-test response (left figure) to the tube weight task and post-test (right figure) to the book weight task by student 2 in the compulsory school group.

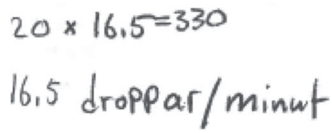

Figure 13. Dimension error in the post-test response to surgery task by nursing Student 9.

where the constant has the dimension gram per cm. If instead regarded as an Euclidean and hence dimensionless ratio relation where x and 177 grams relates to each other as do 48 cm and 57 cm, there seems to be a risk of confusing the order of the data with each other, as illustrated in the pre-test response in Figure 12, where the arrow and circle the value 177 is the teacher's note on that the quotient instead should be $x/177$. This confusion led the student to instead solve the relation (48 cm; 177 grams) and (57 cm; x grams). This confusion no longer occurred in the compulsory school group after the intervention on dimensionality. This error occurred once in the remedial group in the post-test, where the student formulated a dimensionless ratio equation without units, but erroneously as '$\frac{48}{57} = \frac{177}{x}$' instead of $\frac{48}{57} = \frac{x}{177}$. Despite the erroneous response on the post-test, this student made some progress during the intervention from a nonsense calculation in the pre-test to at least formulating an equation with dimensionless quotients, though with an inverted ratio relation.

5.2.3. Dimension errors

In Figure 12, another error in the pre-test is that this student did not divide, but instead multiplied $48 \cdot 57$, which, with units, corresponds to an area in cm^2. Student 8 in the compulsory school group made the same kind of error in the pre-test; specifically, he wrote '$177 : 57 = 48 : x$' where the colon symbol is a division symbol. Figure 13 shows a similar error made by nursing Student 9 in the post-test. With units, this calculation corresponds to $20\frac{\text{drops}}{\text{ml}} \cdot 16.5\frac{\text{ml}^2}{\text{drops}} = 330\,\text{ml}$ where the unit ml^2/drops is obviously invalid.

While Figures 12 (left) and 13 show dimension errors through multiplication, Figures 14 and 15 show dimension errors through subtraction. Though these calculations were given without units or other explanations, the only reasonable interpretation of Figure 14 is that this calculation corresponds to $1500\,\text{ml} - 20\ \text{drops}/\text{ml}$. Figure 15 should correspond to the not written calculation $57\,\text{cm} - 48\,\text{cm} = 9\,\text{cm}$ followed by a written calculation corresponding to $177\,\text{grams} - 9\,\text{cm} = 168$ and the numerically correct answer 'kortaste

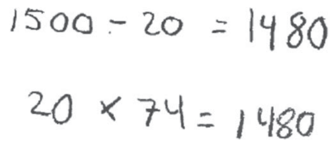

Figure 14. Subtracting different units in the pre-test response to the heat stroke task by nursing Student 9.

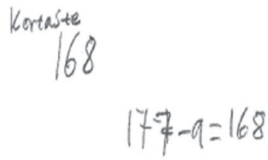

Figure 15. Subtracting different units in the pre-test response to the tube weight task by the remedial student 2.

[shortest] 168' though with nonsense units. In the post-test, this student in Figure 15 formulated a correct dimension equation and gave a correct response.

The frequency of dimension errors decreased from pre-test to post-test. In the pre-test, there were eleven such errors (4 when adding and 7 when multiplying), while in the post-test, there were three cases (1 when adding and 2 when multiplying).

6. Discussion

6.1. Discussion of Study 1

Table 2 in Study 1 showed that knowledge in dimensional analysis would be beneficial in roughly a quarter of the national test items in each test and country. This means that to pass the Swedish national test, it would be enough to have solid knowledge about dimensional analysis, and it would help considerably in the NAEP in the USA and 'prove invalsi' in Italy as well. This result makes it surprising that dimensional analysis, hitherto in practice, is absent in mathematics education research for the secondary school, despite it being advocated several decades ago by well-recognised and frequently cited scholars in this area, such as Pólya (1945) for problem solving and Vergnaud (1979, 1983) for proportionality. On the other hand, this is in line with the research review, which gave the impression that Vergnaud's (1983) scalar procedure using dimensionless ratios is well conceptualised in mathematics education on proportionality (Beckmann & Izsák, 2015; Bentley & Yates, 2017; Izsák & Beckmann, 2019; Kaput & West, 1994; Okazaki & Koyama, 2005) in contrast to Vergnaud's function procedure being quotients of different dimensions. Further, the literature review gave the impression that dimensional analysis in the secondary school is strictly confined to certain areas in chemistry education through the factor-label method (Ellis, 2013; Garrett, 1983; Molnár & Molnár-Hamvas, 2011).

6.2. Discussion of Study 2

The lining up task (dominoes per time) and the knocking over task (time per dominoes) asked for a rate and an inverse rate, and for both these tasks, with data in Tables 5 and 6, Spearman's rho coefficients were similar and of medium strength. This means that the students' knowledge in letting their calculations follow the task instructions correlates with their knowledge in giving their answer a unit that matches their actual calculations.

For the lining up dominoes task, Table 5 gives the impression that most students know how to make use of the information conveyed by the given task, since 72% responded with calculations that at least to some extent matched the unit asked for in the given task and 70% responded with a unit that at least to some extent matched their calculations. This is in line with the idea of dimensional analysis (Brown, 2009; Fay & Joubert, 2002; Pólya, 1945). However, for the knocking over domino task, Table 6 shows that this knowledge seems superficial since only 26% responded in the categories correct or incomplete calculations. Further, only 26% gave a unit to their calculations in the knocking over domino task which at least to some extent matched their calculations. Hence, they didn't show the solid knowledge in proportionality that could help them to succeed in mathematics, as suggested by Lobato and Ellis (2010) and Miyakawa and Winsløw (2009).

A side result is that Tables 5 and 6 showed that determining dominos per time, being a rate problem, seemed simpler than determining the inverse to a rate, namely time per domino. Though comparing the difficulty of a rate problem and an inverse rate problem is not the purpose of this paper, this side result calls for further research on this specific outcome. One plausible explanation is that exercises on rates with time in the denominator, such as speed, flow and change, are more frequent in the students' textbooks and workbooks than exercises with time in the numerator, such as lap time. So, the main result from Study 2 is that students had difficulties in matching units and calculations. To make students ready for handling a spectrum of proportionality situations, as described in Modestou and Gagatsis (2010), this result calls for future research on teaching the key idea of dimensional analysis, effected as teaching awareness of the role of units as guides of calculations.

6.3. Discussion of Study 3

Figure 10 illustrates two results. Among students who made correct calculations already in the pre-test, there was an increased quality with respect to handling units, as exemplified in Figure 11. During the pre-test, there were several dimension errors, as exemplified in Figures 14 and 15, though Figure 10 shows that the proportion of such errors decreased from pre-test to post-test. The authors' interpretation of the results is that knowing the key ideas in dimensional analysis proved to be a powerful tool for increasing students' achievement on proportionality tasks. This interpretation is in line with several other studies (Brown, 2009; Fay & Joubert, 2002; Greenfield et al., 2006; Koohestani & Baghcheghi, 2010; Montazer et al., 2022; Rice & Bell, 2005). To further highlight Brown's conclusion, a more meaningful and indicative terminology for school mathematics would be 'unit symmetry' while the academic terminology 'dimensional analysis' might seem odd for secondary school students since they might perceive it as only a geometric property.

One specific result was the one illustrated in Figure 12. Namely that formulating the proportional relation as a dimensionless quotient $\{v_1|u_1\}/\{v_3|u_1\} = \{v_2|u_2\}/\{v_4|u_2\}$ in which the units u_k are the same in the numerator and denominator, hence cancelling into a dimensionless quotient $v_1/v_3 = v_2/v_4$, might be connected to the mistake of confusing the relation with an erroneous quotient $v_3/v_1 = v_2/v_4$ in which two indices are swapped with each other. To instead formulate a non-Euclidean quotient of two different units, as in Table 1, should prevent such mistakes since it is easier to keep track of having values with the same units in the same position in the quotient. In other words, in a dimensionless quotient, the symmetry of the unit of measurement in the numerator and denominator is clear, but not which value belongs to which numerator and denominator. In a dimension quotient, both these aspects are visible. Finally, making statistical tests in 3 separate groups showed that teaching dimensional analysis gave significant improvement, even in small-scale interventions. Though Study 3 did not have a formal control group, Study 2, in some sense, plays this role since it was the final examination in compulsory school and showed less awareness of dimensional analysis than the post-test in Study 3, despite the low results in the pre-test of Study 3.

7. Conclusion

The focus for this article is to explore the terms for dimensional analysis in the secondary school for the intended curriculum through the content in national test items, for attained curriculum through students' test responses and for implemented curriculum through an intervention. In short, the findings in these three small studies on dimensional analysis in this article together showed the following. One deficiency in the attained curriculum of a random sample of year 9 students is that a large proportion of the students in Study 2 (Tables 5 and 6) seem unaware of using the information conveyed in units of measurement, though Study 3 showed that a short intervention, even among low-achieving students, can significantly address this shortage. The latter is good news since Study 1 (Table 2) showed that for a substantial proportion of national test items, knowledge in dimensional analysis would be beneficial when solving them. As shown in Table 2, this implies that a solid knowledge of only unit symmetry could itself be a considerable help towards a passing grade in high-stakes tests.

So, the authors, together with Hoile (1974), recommend dimensional analysis in the secondary school mathematics through Pólya's (1945, p. 202f) 'test by dimensions'. Namely, to let the units inform what calculations to do (Brown, 2009). Another way to state this is that reasoning with unit symmetry, as exemplified in Table 1 for figuring out what calculations to do, constitutes a praxeology (Chevallard, 2007) for solving a manifold of mathematical tasks, a conclusion confirmed in Rice and Bell (2005). One challenge is that using unit symmetry when formulating equations might require some algebraic manipulation for solving the equations. Such degeneration into symbol manipulation worried Navidi and Baker (1984) and Cardulla (1987). Yet, there is no point in putting symbol manipulation against reasoning, since the success story of mathematics largely builds on reasoning through symbol manipulation. Two examples of this are when François Viète replaced rhetorical and numerical algebra with symbolic algebra and when differential calculus replaced much of geometrical constructions with symbolic manipulation. Many students master problem solving precisely by reasoning with symbols, while some students do not because the symbols have lost meaning for those students. To help such students transfer the bare symbol manipulation into problem solving by encompassing reasoning, the teaching of unit symmetry must strive to let the units of measurement give sense to calculations. Suggestions for giving meaning to the symbols are to let students suggest more suitable units of measurement for snail speed, such as cm/minute and to compare a unit of measurement with its reciprocal, for example, jogging speed as the couple distance/time (e.g. km/h) and time/distance (e.g. minutes/km). This might help students to see that units of measurement belong to the calculations, and if the calculations result in a non-wanted unit, the calculations are incomplete or erroneous. That would make knowledge of unit symmetry a powerful tool for reasoning when solving problems. These latter aspects have not been investigated in the present study, but should be the topic for future research. Another limitation in the present study is that it focused only on students' test results. Hence, another topic for future research is to capture and explore classroom data on teachers' teaching and students' learning of unit symmetry. A third limitation is that unit symmetry applies only in problems involving actual units of measurement. For problems in pure calculus, this can be dealt with through introducing imagined x-units and y-units when describing, for example, derivatives and the equation of the straight line.

Besides the examples given in the results, other applications of unit symmetry are situations involving common magnitudes in physics and economics. Illustrative examples of this are derived units, such as energy $\mathrm{kWh} = \mathrm{kW} \cdot \mathrm{h}$, battery charge $\mathrm{mAh} = \mathrm{mA} \cdot \mathrm{h}$, volume $\mathrm{cm}^3 = \mathrm{cm}^2 \cdot \mathrm{cm}$, comparison price $\frac{€}{\mathrm{kg}}$ and $\frac{€}{\mathrm{litre}}$, density $= \frac{\mathrm{kg}}{\mathrm{m}^3}$ but also dimensionless (scalar) relations in percentage $\% = \frac{\text{part kg}}{\text{whole kg}}$ and in trigonometry $\tan(\text{angle}) = \frac{\text{vertical metres}}{\text{horizontal metres}}$. Further, unit symmetry holds also for problems involving the interpretation of derivatives and integrals, of which time-derivatives and time-integrals should be easy to grasp, for example, Equations (9) and (10).

$$\text{current} = \frac{d(\text{battery charge})}{d(\text{time})} \tag{9}$$

$$\text{battery charge} = \int \text{current} \cdot d(\text{time}) \tag{10}$$

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No potential conflict of interest was reported by the authors.

Data availability statement

For studies 1 and 2, the authors confirm that the data supporting the findings of this study are available within the article, though the raw data that support the findings of this study are available at

NAEP (2024), Prim (2024) and Prove Invalsi (2024). For study 3, the authors confirm that the data supporting the findings are available within the article and its supplementary materials.

ORCID

Jöran Petersson http://orcid.org/0000-0001-5609-0752
Maria Antonietta Lepellere http://orcid.org/0000-0002-0216-4135
Rosie Lopez-Conde http://orcid.org/0000-0002-7562-8076

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