A Road Map of Interval Temporal Logics and Duration Calculi

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ABSTRACT. We survey main developments, results, and open problems on interval temporal logics and duration calculi. We present various formal systems studied in the literature and discuss their distinctive features, emphasizing on expressiveness, axiomatic systems, and (un)decidability results.

KEYWORDS: interval temporal logic, duration calculus, expressiveness, axiomatic system, decidability.

1. Introduction

Interval-based temporal logics stem from four major scientific areas:

Philosophy. The philosophical roots of interval temporal logics can be traced back to Zeno and Aristotle. The nature of Time has always been a core subject in philosophy, and in particular, the discussion whether instants or time periods should be regarded as the primary objects of temporal logic has a distinct philosophical flavour. Some of the modern formal treatments of interval-based structures of time include: [HAM 72] providing a philosophical analysis of interval ontology and interval-based tense logic; [HUM 79] which elaborates on Hamblin's work, introducing a sequent calculus for an interval tense logic over precedence and sub-interval relations; [SE 80], a follow-up on Humberstone's work, discussing and analyzing persistence (preservation of truth in sub-intervals) and homogeneity; [BUR 82] proposing axiomatic systems for interval-based tense logics of the rational reals, studied earlier in [ROE 80]. A comprehensive study and logical analysis of point-based

and interval-based ontologies, languages, and logics can be found in [BEN 91].

Linguistics. Interval-based logical formalisms have featured in the study of natural languages since the seminal work of Reichenbach [REI 47]. These are suitable frameworks for modeling progressive tenses and express various language constructions involving time periods and event duration which cannot be adequately grasped by point-based temporal languages. Recent temporal languages and logics have been proposed and studied in [DOW 79, RIC 88], to mention a few. The linguistic aspects of interval logics will not be treated here, apart from some discussion of the expressive power about various interval-based temporal languages.

Artificial intelligence. Interval temporal languages and logics have sprung up from *expert systems, planning systems, theories of actions and change, natural language analysis and processing*, etc. as formal tools for temporal representation and reasoning in artificial intelligence. Some of the notable contributions in that area include: [ALL 83] proposing the thirteen Allen relations between intervals in a linear ordering and a temporal logic for reasoning about them; [ALL 85] providing an axiomatization and a representation result for interval structures based on the *meets* relation between intervals, further studied and developed in [LAD 87], which also provides a completeness theorem and algorithms for satisfiability checking for Allen's calculus represented as a first-order theory; [GAL 90] critically analyzing Allen's framework and arguing the necessity of considering points and intervals on a par, and [P4] developing interval-based theory of actions and events. A comprehensive survey on temporal representation and reasoning in artificial intelligence can be found in [CHI 00].

Computer science. One of the first applications of interval temporal logics to computer science, viz. for specification and design of hardware components, was proposed in [HAL 83, MOS 83] and further developed in [MOS 84, MOS 94, MOS 98, MOS 00a]. Later, other systems and applications of interval logics were proposed in [BOW 00, CHA 98, DIL 92a, DIL 92b, DIL 96a, DIL 96b, RAS 99]. Model checking tools and techniques for interval logics were developed and applied in [CAM 96, PEN 98]. Particularly suitable interval logics for specification and verification of real-time processes in computer science are the *duration calculi* (see [CHA 91, CHA 94, CHA 99, HAN 92, HAN 97, SØR 90]) introduced as extensions of interval logics, allowing representation and reasoning about time durations for which a system is in a given state. For an up-to-date survey on duration calculi see [CHA 04].

Intervals can be regarded as primitive entities or as defined in terms of their endpoints. Accordingly, interval-based temporal logics can be divided into two main classes: 'pure' interval logics, where the semantics is entirely interval-based, that is, formulas are directly evaluated with respect to intervals, and 'non-pure' interval

logics, where the semantics is essentially point-based and intervals are only auxiliary entities. An important family of 'non-pure' interval logics is that of the logics in which the *locality* principle is imposed. Such a principle states that an atomic proposition is true at an interval if and only if it is true at the beginning point of that interval.

In this survey we outline (without claiming completeness) some developments, results, and open problems on interval temporal logics and duration calculi, focusing on 'pure' interval logics and on those non-pure ones which drop locality. We present various formal systems studied in the literature and discuss their distinctive features, emphasizing on expressiveness, axiomatic systems, and decidability results. Since duration calculi are discussed in more details in [CHA 04], we will present this topic in a rather succinct way, while going in more detail on interval logics, mainly on propositional level.

The paper is organized as follows. In Section 2 we introduce the basic syntactic and semantic ingredients of interval temporal logics and duration calculi, including interval temporal structures, operators, and languages, as well as their syntax and semantics. In Section 3 we discuss propositional interval logics, in Section 4 we present a general tableau method for them, while in Section 5 we briefly survey first-order interval logics and duration calculi. Section 6 contains some concluding remarks and directions for future research.

2. Preliminaries

2.1. Temporal ontologies, interval structures and relations between intervals

Interval temporal logics are subject to the same ontological dilemmas as the instant-based temporal logic, viz.: should the time structure be considered *linear* or *branching*? *Discrete* or *dense*? *With* or *without beginning*? etc. In addition, however, new dilemmas arise regarding the nature of the intervals:

- Should intervals include their end-points or not?
- Can they be unbounded?
- Are point-intervals (i.e. with coinciding endpoints) admissible or not?
- How are points and intervals related? Which is the primary concept? Should an interval be identified with the set of points in it, or there is more into it?

The last question is of particular importance for the semantics of interval logics.

Given a strict partial ordering $\mathcal{D} = \langle D, < \rangle$, an interval in D is a pair $[d_0, d_1]$ such that $d_0, d_1 \in D$ and $d_0 < d_1$. $[d_0, d_1]$ is a strict interval if $d_0 < d_1$. Often we will refer to all intervals on D as *non-strict intervals*, to distinguish from the latter. In particular, intervals $[d, d]$ will be called *point-intervals*. A point d belongs to an interval $[d_0, d_1]$ if $d_0 \leq d \leq d_1$ (i.e. the endpoints of an interval are included in it). The set of all non-strict intervals on D will be denoted by $I(D)^+$, while the set of all strict intervals will be denoted by $I(D)^-$. By $I(D)$ we will denote either of these. For the purpose of this survey, we will call a pair $\langle D, I(D) \rangle$ an *interval structure*.

In all systems considered here the intervals will be assumed *linear*, although this restriction can often be relaxed without essential complications. Thus, we will concentrate on partial orderings with the *linear interval property*:

$$x < y \quad (x < z_1 < z_2 < y \quad x < z_2 < y \quad z_1 < z_2 \quad z_1 = z_2 \quad z_2 < z_1) ,$$

that is, orderings in which every interval is linear. Clearly every linear ordering falls here. An example of a non-linear ordering with this property

Figure 1. Interval structure with the linear interval property

while a non-example is:

Figure 2. Interval structure violating the linear interval property

An interval structure is:

- **linear**, if every two points are comparable;
- **discrete**, if every point with a successor/predecessor has an immediate successor/predecessor along every path starting from/ending in it; that is,

$$x < y \quad (x < z < z < y \quad w(x < w < w < y < z < w) ,$$

and

$$x < y \quad (x < z < z < y \quad w(x < w < w < y < w < z)$$

– **dense**, if for every pair of different comparable points there exists another point in between:

$$x < y \quad \text{and} \quad z(x < z < y)$$

– **unbounded above** (resp. **below**), if every point has a successor (resp. predecessor);

– **Dedekind complete**, if every non-empty and bounded above set of points has a least upper bound.

Besides interval logics over the classes of linear, (un)bounded, discrete, dense, and Dedekind complete interval structures, we will be discussing those interpreted on the single structures $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ with their usual orderings.

It is well known that there are 13 different binary relations between intervals on a linear ordering (and quite a few more on a partial ordering [83]: *equals*, *ends*, *during*, *begins*, *overlaps*, *meets*, *before*, together with their inverses.

Figure 3. Allen's relations

These relations lead to a rich interval algebra, the so-called Allen's Interval Algebra, which will not be discussed in detail here. A survey of Interval Algebra and of a number of its tractable fragments, including Vilain and Kautz's Point Algebra [VIL 86], van Beek's Continuous Endpoint Algebra [BEE 90], and Nebel and Bürckert's ORD-Horn Algebra [NEB 95], can be found in [CHI 00].

Another natural binary relation between intervals, defined in terms of Allen's relations, is the one of *sub-interval* which comes in three versions. Given a partial ordering D and intervals $[s_0, s_1]$ and $[d_0, d_1]$ in it:

– $[s_0, s_1]$ is a *sub-interval* of $[d_0, d_1]$ if $d_0 \leq s_0$ and $s_1 \leq d_1$. The relation of sub-interval will be denoted by ;

- $[s_0, s_1]$ is a *proper sub-interval* of $[d_0, d_1]$, denoted $[s_0, s_1] \text{ @ } [d_0, d_1]$, if $[s_0, s_1] \subset [d_0, d_1]$ and $[s_0, s_1] \neq [d_0, d_1]$;
- $[s_0, s_1]$ is a *strict sub-interval* of $[d_0, d_1]$, denoted $[s_0, s_1] \text{ @ } [d_0, d_1]$, if $d_0 < s_0$ and $s_1 < d_1$.

Amongst the multitude of ternary relations between intervals there is one of particular importance for us, which corresponds to the binary operation of concatenation of meeting intervals. Such a ternary interval relation, which has been introduced by Venema in [VEN 91], can be graphically depicted as follows:

Figure 4. The ternary relation A

It is denoted by A and it is defined as follows:

- A_{ijk} if i meets j , i begins k , and j ends k , that is, k is the concatenation of i and j .

2.2. Propositional interval temporal languages and models

The generic language of propositional interval logics includes the set of propositional letters AP , the classical propositional connectives $\neg$ (all others, including the propositional constants $\top$ and $\perp$, are definable as usual), and a set of *interval temporal operators (modalities)* specific for each logical system.

There are two different natural semantics for interval logics, namely, a *strict* one, which excludes point-intervals, and a *non-strict* one, which includes them. A *non-strict interval model* is a pair $M^+ = \langle D, V \rangle$, where D is a partial ordering and $V : I(D)^+ \rightarrow P(AP)$ is a valuation assigning to each interval a set of atomic propositions considered true at it. Respectively, a *strict interval model* is a structure $M^- = \langle D, V \rangle$ defined likewise, where $V : I(D)^- \rightarrow P(AP)$. When we do not wish to specify the strictness, we will write simply M , assuming either version.

Allen's relations give rise to respective unary modal operators, thus defining the modal logic of time intervals HS introduced by Halpern and Shoham in [HAL 91]. Some of these modal operators are definable in terms of others; it suffices to choose as basic the modalities corresponding to the relations *begins*, *ends*, and their inverses. Thus, the formulas of HS are generated by the following abstract syntax:

$$\models p \mid \neg \mid \mid \mid B \mid E \mid \bar{B} \mid \bar{E}.$$

The formal semantics of these modal operators (given in [HAL 91] in terms of non-strict models) is defined as follows:

- $(B) M^+, [d_0, d_1] \models B \phi$ if $M^+, [d_0, d_2] \models \phi$ for some d_2 such that $d_0 \leq d_2 < d_1$;
 $(E) M^+, [d_0, d_1] \models E \phi$ if $M^+, [d_2, d_1] \models \phi$ for some d_2 such that $d_0 < d_2 \leq d_1$;
 $(\bar{B}) M^+, [d_0, d_1] \models \bar{B} \phi$ if $M^+, [d_0, d_2] \models \phi$ for some d_2 such that $d_1 < d_2$;
 $(\bar{E}) M^+, [d_0, d_1] \models \bar{E} \phi$ if $M^+, [d_2, d_1] \models \phi$ for some d_2 such that $d_2 < d_0$.

A useful new symbol is the *nodal constant* for point-intervals interpreted in non-strict models as follows:

- $(\cdot) M^+, [d_0, d_1] \models \cdot$ if $d_0 = d_1$.

Note that the constant is definable as either $[B]$ or $[E]$, so it is only needed in weaker languages. The presence of the language allows one to interpret the strict semantics into the non-strict one by means of the translation

- $(p) = p$ for $p \in AP$;
- $(\neg \phi) = \neg (\phi)$
- $(\phi \wedge \psi) = (\phi) \wedge (\psi)$
- $(\phi \rightarrow \psi) = (\neg (\phi) \vee (\psi))$ for any (unary) interval diamond-modality.

The interpretation is effected by the following claim, proved by a straightforward induction on ϕ :

PROPOSITION 1. — *For every interval model M , proper interval $[d_0, d_1]$ in M , and formula ϕ :*

$$M^-, [d_0, d_1] \models \phi \text{ iff } M^+, [d_0, d_1] \models (\phi).$$

Usually, but not always, the non-strict semantics is taken as default.

Venema introduced in [VEN 91] three binary modalities D , A , and T , associated with the ternary relation A , with the following non-strict semantics:

- $(C) M^+, k \models C \phi$ if there exist two intervals s, j such that $Aij k$ and $M^+, i \models \phi$, and $M^+, j \models \phi$, that is,
 $M^+, [d_0, d_1] \models C \phi$ if $M^+, [d_0, d_2] \models \phi$, and $M^+, [d_2, d_1] \models \phi$ for some $d_2 \in D$ such that $d_0 \leq d_2 \leq d_1$.
 $(D) M^+, j \models D \phi$ if there exist two intervals s, k such that $Aij k$ and $M^+, i \models \phi$, and $M^+, k \models \phi$, that is,
 $M^+, [d_0, d_1] \models D \phi$ if $M^+, [d_2, d_0] \models \phi$, and $M^+, [d_2, d_1] \models \phi$ for some $d_2 \in D$ such that $d_2 \leq d_0$.
 $(T) M^+, i \models T \phi$ if there exist two intervals s, k such that $Aij k$ and $M^+, j \models \phi$, and $M^+, k \models \phi$, that is,
 $M^+, [d_0, d_1] \models T \phi$ if $M^+, [d_1, d_2] \models \phi$, and $M^+, [d_0, d_2] \models \phi$ for some $d_2 \in D$ such that $d_1 \leq d_2$.

The interval logic CDT based on these modal operators will be discussed in the next section.

Similar modal operators for a relation of any arity were introduced in [NIS 80], and a logic with the 'Chop' operator was studied in [ROS 86].

The semantics of interval temporal logics is sometimes subject to restrictions justified by specific applications for which a logical system is designed, such as:

- *locality*, meaning that all atomic propositions are point-wise and that an interval is defined as truth at its initial point;
- *homogeneity*, requiring that truth of a formula at an interval implies that formula at every sub-interval of it.

A different kind of restriction is imposed by the so-called *split-structures* (see Section 3.3). In the split-structures not all sub-intervals of an interval are 'available' but only those two which are determined by the 'split-point' in that interval.

We will not assume any semantic restrictions, unless otherwise specified.

2.3. First-order languages and models for interval logics and duration calculi.

The first-order languages for interval logics extend the propositional ones essentially the same way as in classical logic, but accounting for the fact that the first-order domain may change over time. Formally, these languages are built as usual from variables, constants, and functional symbols. Constants and functional symbols are classified as *global* (or *rigid*), whose interpretation does not depend on the time, and *temporal* (or *flexible*), whose interpretation can vary over time. Predicate symbols are also classified as global or temporal, are denoted by p^i , ..., where $i, j, \dots$ represent the arities. The abstract syntax of formulas of a generic first-order interval language includes the clauses

$$\models p^n(t_1, \dots, t_n) \mid x \mid \dots,$$

where $t_1, \dots, t_n$ are terms, plus the clauses for the specific interval modal operators.

A formula is *global* (rigid) if it only contains global constant, functional, and predicate symbols.

Among the constants, there is a specific and important one present in most of the first-order languages for interval logics and duration calculi, viz. the flexible constant I denoting the *length of the current interval*. Often it is combined with a structure of an additive group (typically, the additive group of reals) as part of the temporal domain, which allows for computing lengths of concatenated intervals, etc.

A specific additional feature of the syntax of duration calculi is the special category of terms called *state expressions* which are used to represent the duration for which a system stays in a particular state.

The semantics of first-order interval formulas is a combination of the standard semantics of a first-order (temporal) logic with the semantics of the specific underlying propositional interval logic.

3. Propositional Interval Logics

As already noted, every interval logic L has two versions, namely, the *strict* version L^+ and the *non-strict* one L^- , and when writing just L we will mean either one, as specified in the text.

3.1. Monadic interval logics

In this section we introduce and analyze the most well-known and/or interesting interval logics involving only unary modal operators, starting from the weakest. We will assume that the semantic structures are of the most general type we consider, viz. interval structures over partial orderings with the linear interval property, unless otherwise specified.

3.1.1. The sub-interval logic D

The logic D is the logic of the sub-interval relation. Since it allows one to look inside the current interval only, from the linear interval hypothesis it follows that we can restrict ourselves to the class of linear structures.

The abstract syntax of the simplest version of D is:

$$\models p \mid \mid \mid D,$$

but one could also include in the language the modal constant

The sub-interval relation and the temporal logics associated with it were studied, from the perspective of philosophical temporal logics [HAM 72, ROE 80], [HUM 79] (together with precedence), and [BEN 91]. In the computer science literature, it was apparently first mentioned in [HAL 91] and its expressiveness (interpreted over linear non-strict models) discussed in [LOD 00].

Besides the strict and non-strict versions, the logic D has essential semantic variations, depending on which sub-interval relation ($\subseteq$, or $\subset$) is assumed. Accordingly, the truth definition for D is based on the clause:

$$(\models D) \quad M, [d_0, d_1] \models D \quad \text{if there exists a sub-interval } [d_2, d_3] \text{ of } [d_0, d_1] \text{ such that } M, [d_2, d_3] \models .$$

At present, we are not aware of any specific published results about expressiveness, axiomatic systems, and decidability for any variants of logic D , but we note that they all involve non-trivial valid formulas expressible in D , associated with 'length vs depth'. To give some idea, here is an infinite scheme of valid formulas of the logic D , with a strict sub-interval relation, which says that if an interval contains sufficiently many distinct sub-intervals (and hence, sufficiently many distinct points), then it contains a chain of nested sub-intervals of pre-defined length:

$$\bigwedge_{i=1}^{d(n)} D @_{p_i} \bigwedge_{j=i}^{2^n-1} p_j \rightarrow D^n,$$

$$\text{for } d(n) = \frac{2^n-1}{2} + 1$$

3.1.2. The logics $\overline{B}\overline{B}$ and $\overline{E}\overline{E}$

Interval logics make it possible to express properties of intervals of time points, rather than single time points. In most cases, this feature prevents one from the possibility of reducing interval-based temporal logics to point-based ones without resorting to any kind of projection principle. However, there are a few exceptions where such a reduction can be defined thanks to a suitable choice of the interval modalities, thus allowing one to benefit from the good computational properties of point-based logics. This is the case of the logics $\overline{B}\overline{B}$ and $\overline{E}\overline{E}$ (and of their fragments).

The logic $\overline{B}\overline{B}$ is generated by the following abstract syntax:

$$\models p \mid \neg \mid \mid \mid B \mid \overline{B} ,$$

while $\overline{E}\overline{E}$ is obtained from $\overline{B}\overline{B}$ by substituting E for B and $\overline{E}$ for $\overline{B}$. In the following, we restrict our attention to $\overline{B}\overline{B}$. However, all definitions and results can be easily adapted to $\overline{E}\overline{E}$.

The decidability, as well as other logical properties, of $\overline{B}\overline{B}$ can be obtained by translating it into the propositional temporal logic of linear time Lin-PTL with temporal modalities F (sometime in the future) and P (sometime in the past), which has the finite model property and is decidable (see e.g. [GAB 94]). The formulas of Lin-PTL are defined by

$$f \models p \mid f \mid f \mid g \mid Pf \mid Ff ,$$

and a model for Lin-PTL is a pair $\langle D, V \rangle$, where $D = \langle D, < \rangle$ is a linearly ordered set and $V : D \rightarrow \mathcal{P}(AP)$ is a valuation function. The semantics is standard:

- $M, d \models p$ if $p \in V(d)$;
- $M, d \models \neg f$ if it is not the case that $M, d \models f$;
- $M, d \models f \mid g$ if $M, d \models f$ and $M, d \models g$;
- $M, d \models Pf$ if there exists d' such that $d' < d$ and $M, d' \models f$;
- $M, d \models Ff$ if there exists d' such that $d < d'$ and $M, d' \models f$.

The formulas of $\overline{B}\overline{B}$ are simply translated into formulas of Lin-PTL by a mapping which replaces B by P and $\overline{B}$ by F .

Now, for every model $M = \langle D, V \rangle$ of $\overline{B}\overline{B}$, where $D = \langle D, < \rangle$, and point $d \in D$, we construct a model for Lin-PTL $M[d] = \langle [d], V \rangle$, where $[d] = \{d' \in D \mid d \leq d'\}$ and the valuation V is defined as follows: for all $d' \in [d]$ and $p \in AP$: $p \in V(d')$ iff $p \in V(d')$. Conversely, every model $M = \langle D, V \rangle$ for Lin-PTL based on a linear ordering with a least element can be obtained in such a way from some model of $\overline{B}\overline{B}$.

LEMMA 2. — For every model $M = \langle D, V \rangle$ of $\overline{B}\overline{B}$, with $D = \langle D, < \rangle$, point $d \in D$, and formula φ of $\overline{B}\overline{B}$:

$$M, [d, d] \models \varphi \text{ iff } M[d], d \models \varphi$$

for any $d \in D$.

PROOF. — Structural induction on φ . For propositional variables the claim holds by definition. The cases of the propositional connectives are straightforward.

Let $\varphi = B\psi$. By definition, $\varphi = P(\psi)$, and, by hypothesis, $M[d, d] \models B\psi$, that is, there exists d' such that $d' < d$ and $M[d, d'] \models \psi$. By the inductive hypothesis, $M[d], d' \models \psi$, and thus $M[d], d \models P(\psi)$.

The case $\varphi = \overline{B}\psi$ is similar.

The claim of the lemma now follows immediately. $\square$

COROLLARY 3. — *A formula $B\overline{B}\psi$ is satisfiable in a model M of $B\overline{B}$ iff ψ is satisfiable in some model $M[d]$.*

Given a linear ordering L , we denote by ${}^+L$ the ordering obtained from L by adding a new least element. Accordingly, if C is a class of linear orderings, we define ${}^+C = \{{}^+L \mid L \in C\}$.

Consequently, we obtain the following theorem.

THEOREM 4. — *The satisfiability problem for the logic $B\overline{B}$, interpreted in a given class of interval structures over a class of linear orderings C , is reducible to the satisfiability problem for the logic Lin-PTL interpreted over the class ${}^+C$.*

Thus, for instance, the decidability of $B\overline{B}$ over the class of all linear orderings follows.

3.1.3. The logic B

The logic BE features the two modalities B and E , and its formulas are generated by the following abstract syntax:

$$\varphi ::= p \mid \neg \varphi \mid \varphi_1 \mid \varphi_2 \mid B\varphi \mid E\varphi.$$

As we have already shown, the modal constants definable as $[B]$. Accordingly, the point-intervals that respectively begin and end the current interval can be captured as follows:

- $[BP]$, $(\varphi) \rightarrow B(\varphi)$, and
- $[EP]$, $(\varphi) \rightarrow E(\varphi)$.

BE is strictly more expressive than (the non-strict version) D . On the one hand, if we assume the sub-interval relation to be the strict one (the other two cases can be dealt with in a similar way), the modality D can be defined as follows:

- $D\varphi$, $B\varphi \vee E\varphi$.

On the other hand, the undecidability of B and E in D can be easily proved as follows. Let $I(D)^+, @, V$ be a D -model, where $I(D)^+$ is the set of all non-strict intervals over D , $@$ is the strict sub-interval relation over $I(D)^+$, and V is the valuation function. The notions of p -morphism and bisimulation between D -models are defined in the usual way for modal logic (see e.g. [BLA 01]), and they satisfy the standard truth-preservation properties. Given two linearly ordered sets $D = \{d_0, d_1\}$, with $d_0 < d_1$, and $D = \{d_0\}$, we take into consideration two D -models $M^+ = I(D)^+, @, V$ and $M^+ = I(D)^+, @, V$ such that:

- 1) $I(D)^+ = \{[d_0, d_0], [d_1, d_1], [d_0, d_1]\}$ and $I(D)^+ = \{[d_0, d_0]\}$;
- 2) the valuations of all intervals in both models are equal to $\{p\}$.

Let $R \subseteq D \times D$ be the relation $\{(d_0, d_0), (d_1, d_0)\}$. It is immediate to show that such a relation induces a bisimulation $\tilde{R} \subseteq I(D)^+ \times I(D)^+$ between M^+ and M^+ . First, all

intervals of both models are evaluated (to), and thus any pair of $\mathbb{Z}$ -related intervals satisfies the same atomic propositions. Second, the strict interval relation is empty in both models, and thus the back and the forth conditions are trivially satisfied. Since $M^+, [d_0, d_1]$ satisfies $B \ p$ (resp., $E \ p$), while $M^-, [d_0, d_1]$ does not, it immediately follows that $B \ p$ (resp., $E \ p$) cannot be defined in D .

BE is expressive enough to capture some relevant conditions on the underlying interval structure, as originally pointed out by Halpern and Shoham in the context of the logic HS [HAL 91]), from where the examples below are adapted. First, one can constrain an interval structure to be discrete by means of the formula:

$$-d \quad , \quad \parallel \quad (B \parallel \quad E \parallel \quad) ,$$

where $\parallel$ is true over an interval $[d_0, d_1]$ if and only if $d_0 < d_1$ and there are no points between d_0 and d_1 . Such a condition can be expressed in BE as follows:

$$\parallel \quad , \quad B \quad [B \parallel B] \quad .$$

It is not difficult to show that an interval structure is discrete if and only if the formula *discrete* is valid in it. Furthermore, one can easily force an interval structure to be dense by constraining the formula

$$-d \quad , \quad \parallel \quad .$$

to be valid. Finally, one can constrain an interval structure to be Dedekind complete by means of the formula

$$-d \quad , \quad (B \parallel \quad [P] \ q \ [E \ B] \ q \ B \parallel \quad) \\ B \ [E \ D \parallel \quad)$$

where $\parallel$ is true over an interval $[d_0, d_1]$ if and only if its endpoints satisfy a given proposition letter q (the cell delimiters), all sub-intervals satisfy a proposition letter p (the cell content), and there exists at least one sub-interval satisfying p , that is,

$$\parallel \quad , \quad [B] \ q \ [P] \ q \ [D] \ p \ D \ p .$$

BE also allows one to define a modality $[All]$, referring to all sub-intervals of the given interval, which in that logic is essentially equivalent to the *universal modality* over the submodel generated by the current interval:

$$-[All] \quad , \quad [B] \ [E] \ [B \parallel E] \quad .$$

As for (un)decidability results, Lodaya [LOD 00] proves the following theorem, which tailors the undecidability proof for HS provided by Halpern and Shoham (cf. Theorem 12) to BE.

THEOREM 5. — *The satisfiability problem for BE-formulas interpreted over non-strict dense linear structures is not decidable.*

Undecidability is proved by reducing the non-halting problem of a Turing Machine (TM) on a blank tape to the satisfiability problem for BE. According to Halpern and Shoham's approach, any computation of a TM is modeled as an infinite sequence of configurations of the machine, called instantaneous descriptions (IDs for short). Each ID is a finite sequence of tape cells that contain a unique symbol, and one of the cells has additional information representing the position and the state of the machine. A suitable proposition is used to talk about consecutive IDs, e.g., to relate

the n -th cell of a given ID to the same cell of the successive ID. By exploiting such a proposition, the transition function of a TM can be expressed by examining a group of three cells belonging to a given ID and determining the cell of the same three cells in the successive ID. A suitable interval formula, parameterized by a TM, can then be built in such a way that such a formula is satisfiable if and only if the TM does not halt on a blank tape. As a matter of fact, most of Halpern and Shoham's proof is carried out in the BE fragment. The other modalities are only used to specify the sequence of IDs and to express the relationships between consecutive IDs. Dayala shows how to treat the entire infinite computation as being inside a dense interval, which makes it possible to use the D modality to express the relationships between consecutive IDs as well as to talk about sequences of IDs.

Since density is expressible in BE by a constant formula, we have the following corollary of Theorem 5.

COROLLARY 6. — *The satisfiability problem for BE over the class of all non-strict linear structures is not decidable.*

The satisfiability of a formula in a dense model is indeed equivalent to the satisfiability of $[All] \perp$ in any non-strict model.

We conclude our description of BE by remarking that a number of meaningful problems, such as the decidability of the satisfiability problem for BE-formulas interpreted over special classes of linear orderings, or over models, and the definition of sound and complete axiomatic systems for BE, are, at the best of our knowledge, still open.

3.1.4. Propositional neighbourhood logics

The interval logics based on the *meets* relation and its inverse *meets-by* are called *neighbourhood logics*. Notably, first-order neighbourhood logics were introduced and studied by Zhou and Hansen in [CHA 98], while their propositional variants, interpreted over linear structures (both strict and non-strict), were studied only quite recently by Goranko, Montanari, and Sciavicco [GOR 03b].

The language of propositional neighbourhood logics includes the modal operators $\Box$ and $\Diamond$ borrowed from [CHA 98]. Its formulas are generated by the following abstract syntax:

$$\varphi ::= p \mid \neg \varphi \mid \Box \varphi \mid \Diamond \varphi \mid \varphi \Box \varphi \mid \varphi \Diamond \varphi.$$

The dual operators $\Box$ and $\Diamond$ are defined in the usual way. To make it easier to distinguish between the two semantics from the syntax, we reserve this notation for the case of non-strict propositional neighbourhood logics, generically denoted by PNL^+ , while for the strict ones, denoted by PNL . $\Box$ and $\bar{\Box}$ are used instead of $\Box$ and $\Diamond$, respectively. The class of non-strict propositional neighbourhood logics extended with the modal constant $\top$ will be denoted by PNL^+ .

The modalities $\Box$ and $\bar{\Box}$ were originally introduced in the logic HS [HAL 91] as derived operators. The semantics of HS admits point intervals and hence, according to our classification, it is non-strict. However, the modalities $\Box$ and $\bar{\Box}$ only refer to strict intervals, and thus the semantics of the fragment $\bar{\Box}\bar{\Box}$ can be considered essentially strict.

The formal semantics of the modal operators $\Box$ and $\Diamond$ is defined as follows:

$(\rightarrow) M^+, [d_0, d_1] \rightarrow r$ if there exists d_2 such that $d_1 < d_2$ and $M^+, [d_1, d_2] \rightarrow r$;
 $(\leftarrow) M^+, [d_0, d_1] \leftarrow l$ if there exists d_2 such that $d_2 < d_0$ and $M^+, [d_2, d_0] \leftarrow l$,

while the semantic clauses for the operators Δ and $\bar{\Delta}$ are:

$(\Delta) M^-, [d_0, d_1] \Delta A$ if there exists d_2 such that $d_1 < d_2$ and $M^-, [d_1, d_2] \Delta A$;
 $(\bar{\Delta}) M^-, [d_0, d_1] \bar{\Delta} A$ if there exists d_2 such that $d_2 < d_0$ and $M^-, [d_2, d_0] \bar{\Delta} A$.

Propositional neighbourhood logics are quite expressive. For example, PNL allows one to characterize various classes of linear structures:

(A-SPNL^u) $[A]p \rightarrow A p$, in conjunction with its mirror image, defines the class of *unbounded* structures;

(A-SPNL^{de}) $(\Delta A \rightarrow A p \rightarrow A \Delta A p) \wedge (\bar{\Delta} A \rightarrow A p \rightarrow A \bar{\Delta} A p)$, in conjunction with its mirror image, defines the class of *dense* structures, extended with the 2-element linear ordering

(A-SPNL^{di}) $[A] \rightarrow [\bar{A}] A$, $A \rightarrow (\Delta A \rightarrow [\bar{A}] A) \wedge (\bar{\Delta} A \rightarrow [A] p \rightarrow [\bar{A}] p \rightarrow [A] p) \wedge [\bar{A}] A \rightarrow (A \rightarrow p \rightarrow [A] A p)$, in conjunction with its mirror image, defines the class of *discrete* structures;

(A-SPNL^c) $A \rightarrow A [\bar{A}] p \rightarrow A [A] [\bar{A}] p \rightarrow A (\Delta [\bar{A}] p \rightarrow [A] A \rightarrow [\bar{A}] p)$ defines the class of *Dedekind complete* structures.

Moreover, the language of PNL over unbounded structures is powerful enough to express the *difference* $\boxminus$ operator:

$$\boxminus q, [\bar{A}] \bar{A} [A] q \rightarrow [\bar{A}] A [A] q \rightarrow [A] A [\bar{A}] q \rightarrow [A] \bar{A} [\bar{A}] q,$$

saying that q is true at every interval different from the current one, and consequently to simulate *nominals* (the application of the operator $\boxminus$ to q constrains q to hold over the current interval and nowhere else):

$$n(q), q \boxminus q.$$

It follows (see, e.g., [GAR 93]) that every universal property of strict unbounded linear structures can be expressed in PNL.

Sound and complete axiomatic systems for propositional neighbourhood logics have been obtained in [GOR 03b].

THEOREM 7. — *The following axiomatic system is sound and complete for the logic PNL⁺ of non-strict linear structures:*

(A-NT) enough propositional tautologies;

1. The 2-element linear ordering cannot be separated in the language of PNL.

(A-NK) the K axioms for r and l ;

(A-NNF0) $r p \rightarrow r p$, and its inverse;

(A-NNF1) $p \rightarrow r l p$, and its inverse;

(A-NNF2) $r l p \rightarrow r l p$, and its inverse;

(A-NNF3) $r l p \rightarrow l r r p \rightarrow l l r p$, and its inverse;

(A-NNF4) $r r r p \rightarrow r r p$, and its inverse;

(A-NNF $^+$) $r q \rightarrow r p_1 \rightarrow \dots \rightarrow r p_n \rightarrow r (r q \rightarrow r p_1 \rightarrow \dots \rightarrow r p_n)$, and its inverse, for each $n \geq 1$.

The rules of inference are Modus Ponens, Uniform Substitution and r and l Generalization. Interestingly, some of these axioms, including in nite scheme (A-NNF $^+$), were not included in the axiomatization of the first-order neighbourhood logic given in [BAR 00] as they could be derived using the n -order axioms.

THEOREM 8. — [GOR 03b] A sound and complete axiomatic system for the logic PNL^+ can be obtained from that for PNL^+ by adding the following axioms:

(A-1) $l \rightarrow r$;

(A-2) $r (r p) \rightarrow r (p)$, and its inverse;

(A-3) $r p \rightarrow r q \rightarrow r (r p \rightarrow r q)$, and its inverse.

Once r, l are substituted by $A, \bar{A}$, and r, l accordingly by $[A], [\bar{A}]$, the axioms for PNL^- are very similar to those for PNL (accordingly modified to reflect the fact that point-intervals are now excluded), except for the scheme (A-NNF $^+$) which is no longer valid.

THEOREM 9. — [GOR 03b] The following axiomatic system is sound and complete for the logic PNL^- of strict linear models:

(A-ST) enough propositional tautologies;

(A-SK) the K axioms for $[A]$ and $[\bar{A}]$;

(A-SNF1) $p \rightarrow [A] \bar{A} p$, and its inverse;

(A-SNF2) $A \bar{A} p \rightarrow [A] \bar{A} p$, and its inverse;

(A-SNF3) $(\bar{A} \bar{A} \rightarrow A \bar{A} p) \rightarrow p \bar{A} A A p \bar{A} \bar{A} A p$, and its inverse;

(A-SNF4) $A A A p \rightarrow A A p$, and its inverse.

Let us denote by PNL^- , with $\{u, de, di, c, ude, udi, uc\}$, PNL^- interpreted respectively over unbounded, dense, discrete, Dedekind complete, dense and unbounded, discrete and unbounded, and Dedekind complete and unbounded linear structures, respectively. Likewise, PNL^+ denotes the respective class of non-strict models.

THEOREM 10. — [GOR 03b] The following hold:

1) For every $\gamma_1, \gamma_2 \in \{u, de, di, c, ude, udi, uc\}$, $PNL^{\gamma_1-} \subseteq PNL^{\gamma_2-}$ iff the class of linear orders characterized by the condition γ_2 is strictly contained in the class of linear orders characterized by the condition γ_1 .

2) $PNL^{ude-} \subseteq PNL^+$, where the inclusion is in terms of the obvious translation between the two languages (which replaces the strict modalities with the non-strict ones, and vice versa).

3) $PNL^+ = PNL^{u+} = PNL^{de+} = PNL^{ude+} = PNL^{di+} = PNL^{udi+}$.

Note that the logic PNL^{di-} does not yet characterize the interval structure Z of because the formula

$$A p \wedge [A] p \wedge A p \wedge [A] [A] p \wedge A p \wedge [A] A \wedge A p$$

is valid in Z , but not in PNL^{di-} since it fails in a PNL^{di-} -model based on $\omega + Z$.

THEOREM 11. — [GOR 03b] The axiomatic system for PNL^- extended with (A- $SPNL^u$) (resp. (A- $SPNL^{de}$), (A- $SPNL^{di}$), (A- $SPNL^{ude}$), and (A- $SPNL^{udi}$)) is sound and complete for the class of unbounded (resp. dense, discrete, dense unbounded, and discrete unbounded) structures.

Finally, we point out that most of the decidability problems related to propositional neighbourhood logics and their fragments are still open.

3.1.5. The logic $\mathcal{H}$

The most expressive propositional interval logic with unary modal operators studied in the literature is Halpern and Shoham's logic HS introduced in [HAL 91]. HS contains (as primitive or definable) all unary modalities introduced earlier. As mentioned in Section 2, HS features the modalities E , E and their inverses $\bar{B}$, $\bar{E}$, which suffice to define all other modal operators, so that it can be regarded as the temporal logic of Allen's relations. Unlike most other previously studied interval logics, HS was originally interpreted in non-strict models not over linear orderings, but over all partial orderings with the linear interval property, and all results about HS stated below apply to that class of models, unless otherwise specified.

Formally, HS-formulas are generated by the following grammar:

$$\varphi ::= p \mid \neg \varphi \mid \varphi \wedge \psi \mid B \varphi \mid E \varphi \mid \bar{B} \varphi \mid \bar{E} \varphi.$$

Furthermore, as pointed out by Venema in [VEN 90], the neighbourhood modalities A and $\bar{A}$ are definable in the non-strict semantics as follows:

- $A \varphi$, $[EP] \bar{B} \varphi$, and
- $\bar{A} \varphi$, $[BP] \bar{E} \varphi$.

HS can express linearity of the interval structure by means of the following formula:

$$\neg \text{linear}, \\ (A p \wedge [A] p \wedge B p \wedge \bar{B} p) \wedge (\bar{A} p \wedge [\bar{A}] p \wedge E p \wedge \bar{E} p),$$

as well as all conditions that can be expressed in its fragment

As expected, HS is a highly undecidable logic. In [HAL 91] the authors have obtained important results about non-axiomatizability, undecidability and complexity of

the satisfiability in HS for many natural classes of models. Their idea for proving undecidability is based on using an infinitely ascending sequence in the model to simulate the halting problem for Turing Machines. An *infinitely ascending sequence* is an infinite sequence of points $d_0, d_1, d_2, \dots$ such that $d_i < d_{i+1}$ for all i . Any unbounded above ordering contains an infinite ascending sequence. A class of ordered structures contains an infinite ascending sequence if at least one of the structures in the class does.

THEOREM 12. — *The validity problem in HS interpreted over any class of ordered structures with an infinitely ascending sequence is r.e.-hard.*

From Theorem 12, it immediately follows that HS is undecidable for the class of all (non-strict) models, the class of all linear models, the class of all discrete linear models, the class of all dense linear models, the class of finite and unbounded linear models, etc.

THEOREM 13. — *The validity problem in HS interpreted over any class of Dedekind complete ordered structures having an infinitely ascending sequence is Σ_1^1 -hard.*

For instance, the validity in HS in any of the orderings of the natural numbers, integers, or reals is not recursively axiomatizable.

Undecidability occurs even without existence of infinite ascending sequences. We say that a class of ordered structures has *unboundedly ascending sequences* if for every n there is a structure in the class with an ascending sequence of length at least n .

THEOREM 14. — *The validity problem in HS interpreted over any class of Dedekind complete ordered structures having unboundedly ascending sequences is co-r.e. hard.*

Another proof of undecidability of HS, using a tiling problem, can be found in [MAR 99], see also [GAB 00].

In [VEN 90] (see also [MAR 97]) Venema has shown that HS interpreted over a linear ordering is at least as expressive as the universal second-order logic (where second-order quantification is only allowed over monadic predicates) and there are cases where it is strictly more expressive. As a corollary, it can be proved that HS is strictly more expressive than every point-based temporal logic on linear orderings.

In the same paper Venema provided an interesting geometrical interpretation of HS, using which he obtained sound and complete axiomatizations for HS with respect to relevant classes of structures. Here is the idea. An interval can be viewed as an ordered pair of coordinates over a $D_1 < \times D_2 <$ plane, where $D_1 <$ is supposed to be linear. Since the ending point of an interval must be greater than or equal to the starting point, only the north-west half-plane is considered. Clearly, this geometrical interpretation has a good meaning only when formulas are interpreted over linear frames. The geometrical operators are defined as follows:

- $\downarrow$, B (holds at a point right below the current one);
- $\uparrow$, $\overline{B}$ (holds at a point right above the current one);
- $\rightarrow$, E (holds somewhere to the right of the current point);
- $\leftarrow$, $\overline{E}$ (holds somewhere to the left of the current point);

same vertical line); λ holds at a point with the same longitude, i.e. on the

horizontal line).

Notice that, in order to obtain the mirror image (inverse) of a formula written in the geometrical notation, one should simultaneously replace $\perp$ by $-$ and all $\vdash$ by $-$, and vice versa. Using this geometrical interpretation, we have provided sound and complete axiomatic systems for HS over the class of all structures, the class of all linear structures, the class of all discrete structures and $\mathbf{Q}$. The basic axiomatic system **(A-HS)** for HS includes the following axioms and their mirror-image

(A-HS1) enough propositional tautologies;

(A-HS2a) $\vdash (p \rightarrow q) \rightarrow (\neg p \rightarrow \neg q)$;

(A-HS2b) $\vdash (p \rightarrow q) \rightarrow (\neg p \rightarrow \neg q)$;

(A-HS3a) $\vdash p \rightarrow p$

(A-HS3b) $\frac{1}{2} \frac{1}{p} \frac{1}{p}$ $\frac{1}{2} \frac{1}{p}$;

(A-HS4a) $\vdash p \rightarrow p$;

(A-HS4b) $\vdash p \rightarrow p$;

(A-HS5) $\vdash \neg \exists x (x \neq x) \rightarrow \neg \exists x (x \neq x)$;

$$(A-HS6) \quad \begin{cases} \text{if } \alpha \in \mathcal{A} \text{ then } \alpha \in \mathcal{A} \end{cases} ;$$

(A-HS7a) $\vdash -p \rightarrow -\vdash p$;

(A-HS7b) $\vdash -p \rightarrow -p$

(A-HS7c) $-|p \quad | -p;$

(A-HS8) $(\vdash p \rightarrow q) \rightarrow [(\vdash (p \rightarrow q) \rightarrow (p \rightarrow q)) \rightarrow (\vdash p \rightarrow q)]$,

and the following inference rules: Modus Ponens, Generalization for $\forall$, $\exists$, $\neg$, and $\rightarrow$, and a pair of additional, un-orthodox rules which guarantee that all vertical and horizontal lines in the model are syntactically represented

$$\frac{\text{hor}(p)}{\quad} \quad \frac{\text{ver}(q)}{\quad},$$

where p, q do not occur in φ , respectively, and

$$-\text{hor}(\cdot), \quad - \quad - \quad - \quad (\quad - \quad - \quad) \quad (\quad - \quad - \quad);$$
$$-\text{ver}(\quad), \quad \quad \quad -(\quad) -(\quad).$$

The formula $\text{hor}(\varphi)$ holds at an interval $[d_0, d_1]$ if and only if φ holds at any $[d_2, d_1]$ where $d_2 \leq d_0$ and nowhere else. Geometrically, it represents a horizontal line on which φ is true, and only there. Likewise, $\text{ver}(\varphi)$ says that φ is true exactly at the points of some vertical line.

THEOREM 15. — *The axiomatic system (A-HS) is sound and complete for the class of all non-strict interval structures.*

THEOREM 16. — *A sound and complete axiomatic system for the class of discrete structures can be obtained from (A-HS) by adding the following axiom:*

(A-HS²) discrete.

A sound and complete axiomatic system for the class of linear structures can be obtained from (A-HS) by replacing axiom (A-HS8) by the following axiom:

(A-HS^{lin}) $(\varphi \rightarrow \text{hor}(\varphi)) \rightarrow (\text{hor}(\varphi) \rightarrow \varphi), (\varphi \rightarrow \text{ver}(\varphi)) \rightarrow (\text{ver}(\varphi) \rightarrow \varphi).$

A sound and complete axiomatic system for $\mathbb{Q}$ can be obtained from the system for the class of linear structures by adding the following axiom:

(A-HS^Q) dense.

In conclusion, we note that, besides $\overline{DB}, \overline{BE}, \overline{BE}, \overline{BE}$, and $\overline{AA}$, there exist other interesting fragments of HS, such as, for instance, $\overline{DB}$, where $\overline{D}$ is the transpose of D ($\overline{D}$ was already mentioned in [HAL 91]), and $\overline{AD}$, which have not been investigated so far. Moreover, to the best of our knowledge, the strict logic HS^- has not been studied yet either, and thus no complete axiomatic systems and decidability/undecidability results have been explicitly established for it.

3.2. Interval logics with binary operators

3.2.1. The chop operator and (Local) Propositional Interval Logics.

Arguably, the most natural binary interval modality is the chop operator C . As proved in [MAR 97], such an operator is not definable in HS. The logic that features the operator C and the modal constant $\perp$, interpreted according to the non-strict semantics, is the propositional fragment of the first-order Interval Temporal Logic (ITL) introduced by Moszkowski in [MOS 83] (cf. Section 5.1), usually denoted by PITL. PITL-formulas are defined as follows:

$$\varphi ::= p \mid \neg \varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid C \varphi \psi.$$

The modalities B and E are definable in PITL as follows:

- $B\varphi ::= \varphi \wedge C\varphi$, and
- $E\varphi ::= \varphi \vee C\varphi$.

As a matter of fact, the study of PITL was originally confined to the class of discrete linear orderings with finite time, with the chop operator paired with a next operator, denoted by N , instead of C . Intervals in such structures will be identified with the (finite) sequences of points occurring in them. For a, φ holds at a given (discrete) interval $\alpha = s_1 s_2 \dots s_n$, with $n \geq 1$, if φ holds at the interval $\beta = s_2 \dots s_n$

(if any). It is immediate to see that, over discrete linear structures, the modal constant $\perp$ and the next operator are inter-changeable. On the one hand, $\perp \equiv \text{next } \perp$; on the other hand, for any ϕ , $\text{next } \phi \equiv \phi$, I1C.

The logic PITL is quite expressive, as the following result [MOS 83] testifies. THEOREM 17. — *The satisfiability problem for PITL interpreted over the class of non-strict discrete structures is undecidable.*

The proof is actually an adaptation of a theorem by Chandra & CHA 85] showing the undecidability of the satisfiability problem for propositional process logic. Given two context-free grammars G_1 and G_2 , one can build up a PITL-formula which is satisfiable if and only if the intersection of the languages generated by the two grammars is not empty. Since the latter problem is not decidable [HOP 79]), the claim immediately follows.

Since PITL is strictly more expressive than BE over the class of discrete linear structures, the above result does not transfer to the latter. On the contrary, the undecidability of the satisfiability problem for PITL over dense structures as well as over all linear structures immediately follows from the undecidability of BE over such structures.

COROLLARY 18. — *The satisfiability problem for PITL-formulas interpreted over the class of (non-strict) dense linear structures is undecidable.*

COROLLARY 19. — *The satisfiability problem for PITL interpreted over the class of (non-strict) linear structures is undecidable.*

The propositional counterpart of the fragment of ITL that includes the chop operator, has not been investigated yet, as far as we know.

Decidable variants of PITL, interpreted over finite or infinite discrete structures, have been obtained by imposing the so-called *locality projection principle* [MOS 83]. Such a locality constraint states that each propositional variable is true over an interval if and only if it is true at its first state. This allows one to collapse all the intervals starting at the same state into the single interval consisting of the first state only.

Let Local PITL (LPITL for short) be the logic obtained by imposing the locality projection principle to PITL. The syntax of LPITL coincides with that of PITL, while its semantic clauses are obtained from PITL ones by modifying the truth definition of propositional variables as follows:

(loc-PS1) $M^+, [d_0, d_1] \models p$ iff $p \in V(d_0)$.

where the valuation function V has been adapted to evaluate propositional variables over points instead of intervals.

Various extensions of LPITL have been proposed in the literature. In [MOS 83], Moszkowski focused his attention on the extension of LPITL (over finite time) with quantification over propositional variables, and he proved decidability of the resulting logic, denoted by QLPITL, by reducing its satisfiability problem to that of the point-based Quantified Propositional Temporal Logic QPTL. Interpreted over discrete linear structures with an initial point, QLPITL is translated into QPTL

over finite time, the decidability of which can be proved by a simple adaptation of the standard proof for QPTL over infinite time.

THEOREM 20. — QPTL is at least as expressive as QLPITL interpreted over the class of (non-strict) discrete linear structures.

Since the translation of QLPITL into QPTL is effective and QPTL is (non-elementarily) decidable, we have the following result.

COROLLARY 21. — The satisfiability problem for the logic QLPITL, interpreted over the class of (non-strict) discrete linear structures is (non-elementarily) decidable.

The (non-elementary) decidability of LPITL immediately follows from Corollary 21. A lower bound for the satisfiability problem for LPITL, and thus for any extension of it, has been given by Kozen (see [MOS 83]).

THEOREM 22. — Satisfiability for LPITL is non-elementary.

In several papers [MOS 83, MOS 94, MOS 98, MOS 00a, MOS 03], Moskowski explored the extension of LPITL with the so-called *up-star* modality, denoted by $\uparrow$. For any ϕ , $\uparrow\phi$ holds over a given (discrete) interval if and only if the interval can be chopped into zero or more parts such that ϕ holds over each of them. The resulting logic, which we denote by LPITL $\uparrow$ is interpreted over either finite or infinite discrete linear structures. A sound and complete axiomatic system for LPITL $\uparrow$ with finite time is given in [MOS 03].

THEOREM 23. — The following axiomatic system is sound and complete for the class of (non-strict) discrete linear structures:

(A-CLPITL1) enough propositional tautologies;

(A-CLPITL2) $(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \psi)$;

(A-CLPITL3) $(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \psi)$;

(A-CLPITL4) $\phi \rightarrow (\phi \rightarrow \psi) \rightarrow \psi$;

(A-CLPITL5) $\phi \rightarrow \psi$;

(A-CLPITL6) $\phi \rightarrow \psi$;

(A-CLPITL7) $\phi \rightarrow (\phi \rightarrow \psi)$, with $\phi \in AP$;

(A-CLPITL8) $(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \psi)$;

(A-CLPITL9) $\phi \rightarrow \psi$;

(A-CLPITL10) $(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \psi)$;

(A-CLPITL11) $(\phi \rightarrow \psi) \rightarrow \psi$,

together with Modus Ponens and the following inference rules:

$$\frac{}{(\phi \rightarrow \psi)}, \quad \frac{}{(\phi \rightarrow \psi)}.$$

All axioms have a fairly natural interpretation. In particular, locality is basically dealt with by Axiom (A-CLPITL7).

The chop-star operator is a special case of a more general one called the *projection* operator. Such a binary operator, denoted proj , yields general repetitive behaviour: for any given pair of formulas ϕ, ψ , $\phi \text{ proj } \psi$ holds over an interval if such an interval can be partitioned into a series of sub-intervals each of which satisfies ϕ , while ψ (called the *projected formula*) holds over the new interval formed from the end points of these sub-intervals. Let us denote by $\text{LPITL}_{\text{proj}}$ the extension of LPITL with the projection operator proj . By taking advantage from such an operator, $\text{LPITL}_{\text{proj}}$ can express meaningful iteration constructs, such as *do-while* loops:

- δ n times $\delta \phi, \phi \text{ proj } \psi$ (n);
- $\delta \phi \text{ proj } \psi, (\phi \text{ proj } \psi) \text{ proj } \psi$ (C($\delta \phi \text{ proj } \psi$), ψ),

where the formula ϕ occurring in the $\delta \phi$ loop typically is a point formula, that is, a formula whose satisfaction is totally determined from the state of the satisfying interval, and, for all $Q, \delta \psi(n)$ constrains the length of the current interval to be exactly n . $\delta \psi(n)$ is defined as follows:

- $\delta \psi(n), \delta \psi(n+1)$.

Furthermore, the chop-star operator can be easily defined in terms of projection operator as follows:

- $\phi \text{ chop-star } \psi, \phi \text{ proj } \psi$.

$\text{LPITL}_{\text{proj}}$ was originally proposed by Moszkowski in [MOS 83] and later systematically investigated by Bowman and Thompson [BOW 98, BOW 06]. In particular, a tableau-based decision procedure and a sound and complete systematic system for $\text{LPITL}_{\text{proj}}$, interpreted over finite discrete structures, is given in [BW 03].

The core of the tableau method is the definition of suitable normal forms for all operators of the logic. These normal forms provide inductive conditions of the operators. Then, in the style of [WOL 85], a tableau decision procedure to check satisfiability of $\text{LPITL}_{\text{proj}}$ formulas is established. Although the method has been developed at the propositional level, the authors advocate its validity for first-order $\text{LPITL}_{\text{proj}}$.

The normal form for $\text{LPITL}_{\text{proj}}$ formulas has the following general format:

$$(\phi_e) \vee \bigwedge_i (\phi_i \text{ proj } \psi_i)$$

where ϕ_e and ϕ_i are point formulas and ψ_i is an arbitrary $\text{LPITL}_{\text{proj}}$ formula. The first disjunct states when a formula is satisfied over a point interval, while the second one states the possible ways in which a formula can be satisfied over a strict interval, namely, a point formula must hold at the initial point and then an arbitrary formula must hold over the remainder of the interval. This normal form embodies a recipe for evaluating $\text{LPITL}_{\text{proj}}$ formulas: the first disjunct is the base case, while the second disjunct is the inductive step. Bowman and Thompson showed that any $\text{LPITL}_{\text{proj}}$ formula can be equivalently transformed into this normal form.

In [BOW 03], Bowman and Thomson also provided a sound and complete axiomatic system for $\text{LPITL}_{\text{proj}}$, interpreted over discrete linear structures. Let ϕ, ψ be arbitrary formulas and AP . The proposed system includes the following axioms:

(A-LPITL1) enough propositional tautologies;

(A-LPITL2) $\phi \rightarrow \phi$;

(A-LPITL3) $\phi \rightarrow \phi$;

(A-LPITL4) $(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \psi)$;

(A-LPITL5) $(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \psi)$;

(A-LPITL6) $(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \psi)$;

(A-LPITL7) $\phi \rightarrow (\psi \rightarrow \phi)$;

(A-LPITL8) $\phi \rightarrow (\psi \rightarrow \phi)$;

(A-LPITL9) $(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \psi)$, with $\phi \in \text{AP}$;

(A-LPITL10) $\phi \rightarrow \phi$;

(A-LPITL11) $\phi \rightarrow \phi$;

(A-LPITL12) $\phi \rightarrow (\psi \rightarrow \phi)$;

(A-LPITL13) $\phi \rightarrow (\psi \rightarrow \phi)$;

(A-LPITL14) $\phi \rightarrow (\psi \rightarrow \phi)$.

The inference rules, besides Modus Ponens and generalization, include the following rule:

$$\frac{}{\phi \rightarrow \phi}.$$

THEOREM 24. — *The above axiomatic system is sound and complete for the class of (non-strict) discrete structures.*

Finally, Kono [KON 95] presents a tableau-based decision procedure for QLPITL with *projection*, which has been successfully implemented. The method generates a deterministic state diagram as a verification result. Although it has been argued that the associated axiomatic system is unsound (see [MOS 03]), it work actually inspired Bowman and Thompson's one.

3.2.2. The logics $\mathbb{D}$ and $\mathbb{D}^+$

The most expressive propositional interval logic over (strict) linear orderings proposed in the literature is Venema's CDT [VEN 91]. A generalization of CDT to (non-strict) partial orderings with the linear interval property, called BCDT has been recently investigated by Goranko, Montanari, and Sciacova [GOR 03a]. The language of CDT and BCDT contains the three binary operators $\mathbb{D}$, and $\mathbb{T}$, together with the modal constant. Formulas of CDT are generated by the following abstract grammar:

$$\models \quad | p | \quad | \quad | \quad | C | D | T .$$

The semantics of both CDT and BCDT is non-strict.

The following result links the expressiveness of CDT in terms of definable binary operators to that of the fragment $FO_3[x_i, x_j]$ of first-order logic over linear orderings with at most three variables, at most two of which, x_i and x_j are free [VEN 91].

THEOREM 25. — *Every binary modal operator definable in $FO_3[<[x_i, x_j]]$ has an equivalent in CDT, and vice versa.*

As for the relationships with the other propositional interval logics, interpreted over linear orderings, CDT is strictly more expressive than HTL, since the latter is not able to access any interval which is not a sub-interval of the current interval. Moreover, it is immediate to show that CDT subsumes HS:

- $\neg \neg \in (\quad)C$;
- $\neg \neg \in (\quad)D$;
- $\neg \neg \in (\quad)T$;
- $\neg \neg = C(\quad)$.

A sound and complete axiomatic system for CDT over (non-strict) linear structures has been defined by Venema in [VEN 91]. Let us denote $\mathbb{C}(\quad)$ as in the case of HS. The axiomatic system for CDT includes the following axioms, and their inverses (obtained by exchanging the arguments of all occurrences, and replacing each occurrence of $\mathbb{D}$ by $\mathbb{T}$ and vice versa):

(A-CDT1) enough propositional tautologies;

(A-CDT2a) $(\quad)C \quad C \quad C$;

(A-CDT2b) $(\quad)T \quad T \quad T$;

(A-CDT2c) $T(\quad) \quad T \quad T$;

(A-CDT3a) $(\quad T)C$;

(A-CDT3b) $(\quad T)D$;

(A-CDT3c) $T(\quad C)$;

(A-CDT4) C ;

- (A-CDT5a) C ;
 (A-CDT5b) T ;
 (A-CDT5c) $\neg T$;
 (A-CDT6) $[()C ()C] ()C$;
 (A-CDT6a) $(C)C C(C)$;
 (A-CDT6b) $T(C) (C(T) (T)T)$;
 (A-CDT6c) $C(T) T(C)$;
 (A-CDT7d) $(T)C (D)T C(D)$;

and the following derivation rules: Modus Ponens, Generalization:

$$\frac{}{(C)'} \quad \frac{}{(T)'} \quad \frac{}{(T)'}, \text{ and their inverses,}$$

and the Consistency rule: if AP and p does not occur in , then

$$\frac{}{p}.$$

THEOREM 26. — *The above axiomatic system is sound and complete for the class of (non-strict) linear orderings.*

THEOREM 27. — *A sound and complete axiomatic system for the class of (non-strict) dense linear orderings can be obtained from the system for the class of (non-strict) linear orderings by adding the following axiom:*

$$(A-CDT^d) \quad (C).$$

A sound and complete axiomatic system for the class of (non-strict) discrete linear orderings can be obtained from the system for the class of (non-strict) linear orderings by adding the following axiom:

$$(A-CDT^z) \quad (I1C) (CI) ;$$

A sound and complete axiomatic system for $\mathbb{Q}$ can be obtained from the system for the class of (non-strict) linear orderings by adding the following axiom:

$$(A-CDT^Q) \quad ((C) (T) (D)).$$

In [VEN 91], Venema has also developed a sound and complete deduction system for CDT, similar to the natural deduction system for relation algebras earlier developed by Maddux [MAD 92].

Finally, as a consequence from previous results for HS and TL, the satisfiability (resp. validity) for CDT is not decidable over almost all interesting classes of linear orderings, including all, dense, discrete, etc. Again, strict versions of CDT and BCDT⁺ have not been explicitly studied yet, but it is natural to expect that similar results apply there, too.

3.3. Restricted interval logics: split logics

Split Logics (SLs for short) can be viewed as an attempt of finding expressive, yet decidable, propositional interval logics without resorting to any locality principle. We have already seen that, in the interval logic setting, decidability can be gained by reducing the set of modal operators (this is the case of BdE) or by imposing locality conditions (this is the case of LPITL). In the case of SLs, decidability is achieved by imposing suitable constraints on the interval structures over which formulas are interpreted. In the following, we briefly describe the basic features of SLs, and we provide a short summary of the relevant results about them.

SLs have been proposed by Montanari, Sciavicco, and Vianca in [MON 02] as the interval logic counterparts of the monadic first-order (MFO) theories of time granularity studied in [MON 96, FRA 02a] (as a matter of fact, there exist also interesting connections between SLs and the propositional tense logic proposed by Ahmed and Venkatesh in [AHM 93]). SLs are propositional interval logics equipped with operators borrowed from HS and CDT, but interpreted over specific structures, called *split structures*. Models based on split structures are called *split models*. The distinctive feature of split structures is that every interval can be 'chopped' in at most one way (obviously, there is no way to constrain the length of the two resulting sub-intervals). In [MON 02], the authors show that such a restriction does not prevent SLs from the possibility of expressing a number of meaningful temporal properties. Furthermore, they prove the decidability of various SLs by embedding them into decidable MFO theories of time granularity as well as their completeness with respect to the guarded fragment of these theories.

Formulas of SLs are generated by the following abstract syntax:

$$\models p \mid \mid \mid D \mid \bar{D} \mid F \mid \bar{F} \mid C \mid D \mid T .$$

A split structure is a pair $\langle D, H(D) \rangle$, where $H(D)$ is proper subset of D (a precise characterization of $H(D)$ can be found [MON 02]). A *split model* is a pair $M = \langle D, V \rangle$, where $V : H(D) \rightarrow \mathcal{P}(AP)$. The semantic clauses for the modalities D , $\bar{D}$, F , and $\bar{F}$ are the following ones (the semantic clauses for C , D , and T have already been given):

- (D) $M, [d_0, d_1] \models D$ if there exist d_2, d_3 such that $[d_2, d_3] \subseteq [d_0, d_1]$, and $M, [d_2, d_3] \models \perp$;
- ($\bar{D}$) $M, [d_0, d_1] \models \bar{D}$ if there exist d_2, d_3 such that $[d_0, d_1] \supseteq [d_2, d_3]$, and $M, [d_2, d_3] \models \perp$;
- (F) $M, [d_0, d_1] \models F$ if there exist d_2, d_3 such that $d_1 < d_2$, $d_2 < d_3$, and $M, [d_2, d_3] \models \perp$;
- ($\bar{F}$) $M, [d_0, d_1] \models \bar{F}$ if there exist d_2, d_3 such that $d_3 < d_2$, $d_2 < d_0$, and $M, [d_3, d_2] \models \perp$.

The modal constant can also be introduced as a useful shorthand.

In the following we sketch the correspondence between split and MFO theories of time granularity. In particular, we enlighten the relationship that exists between split structures and the temporal structures for granularity, called layered (or granular) structures [MON 96]. Layered structures are the single 'at' temporal domain of linear, point-based temporal logics by a (possibly infinite) set of temporal layers. Each layer is a discrete, linear, point-based domain bounded in the past and in finite in the future. The relationships between time points belonging to the same layer are governed by the usual order relation, while those between points belonging to different layers are expressed by means of suitable projection relations. A formal definition of layered structures can be found in [MON 96, FRA 02a]. Here we give an intuitive account of them. The domain of layered structures is a set $\bigcup_{i \in I} T^i$, where $I \subseteq \mathbb{Z}$, which consists of many copies of (possibly infinitely many), denoted T^i , each one being a layer of the structure. If there is a finite number of layers, the structure is called *n-layered* (n-LS), otherwise, the structure is called *layered*. Among *layered* structures, we consider the *upward unbounded* layered structure (UULS), which consists of a finest layer and an infinite sequence of finer and coarser layers, and the *downward unbounded* one (DULS), which consists of a coarsest layer and an infinite sequence of finer and finer ones. In all cases, layers are totally ordered according to their degree of 'coarseness/ fineness', and a point of a given layer is associated with points of the immediately finer layer, if any (refinability). This accounts for a view of layered structures as (possibly infinite) sequences of (possibly infinite) complete k-ary trees. In the case of the UULS, there is only one infinite tree built up from leaves, which form the finest layer of the structure. In the case of the DULS (resp. n-LS), the infinite sequence of infinite trees (resp. finite) is ordered according to the ordering of the roots, which form the coarsest layer of the structure. In [MON 96, FRA 02a], monadic second-order (MSO) theories of layered structures have been systematically studied and the decidability of a number of them has been proved.

SLs can be viewed as the interval logic counterparts of the order fragments of the MSO theories of 2-reducible layered structures. More precisely, we focus our attention on the theories $\text{MFO}[\bigcup_{i \in I} T^i, <_1, <_2, 0, 1]$, interpreted over the 2-reducible n-LS, $\text{MFO}[\bigcup_{i \in I} T^i, <_1, <_2, 0, 1]$, interpreted over the 2-reducible DULS, and $\text{MFO}[\bigcup_{i \in I} T^i, <_1, <_2, 0, 1]$ interpreted over the 2-reducible UULS. The symbols in the square brackets are (pre)interpreted as follows: $0(x, y)$ (resp. $1(x, y)$) is a binary projection relation such that x is the first (resp. second) point in the refinement of x ; $<_1$ is a strict partial order such that $x <_1 y$ if x belongs to a tree that precedes the tree to which y belongs to; $x <_2 y$ holds if y is a descendant of x . As for split structures, we consider (i) the class of bounded below, unbounded above, dense, and with maximal intervals split structures, (ii) the class of bounded below, unbounded above, discrete, and with maximal intervals split structures, and (iii) the class of bounded below, unbounded above, discrete split structures. A split structure with maximal intervals is a split structure $(D, H(D))$, such that, for every $[d_0, d_1] \in H(D)$ there exists $[d_2, d_3] \in H(D)$ such that $[d_0, d_1] \subseteq [d_2, d_3]$ and there is no $[d_4, d_5] \in H(D)$ such that $[d_2, d_3] \subset [d_4, d_5]$ (the interval $[d_2, d_3]$ is called a *maximal interval*).

THEOREM 28. — *The following results hold:*

1) SL interpreted over the class of bounded below, unbounded above, dense, and with maximal intervals split structures can be embedded into $\text{MFO}[\tau_i, \tau_1, \tau_2, 0, 1]$ interpreted over the 2-refinable DULS;

2) SL interpreted over the class of bounded below, unbounded above, discrete, and with maximal intervals split structures can be embedded into $\text{MFO}[\tau_i, \tau_1, \tau_2, 0, 1]$ interpreted over the 2-refinable n-LS;

3) SL interpreted over the class of bounded below, unbounded above, discrete split structures can be embedded into $\text{MFO}[\tau_i, \tau_1, \tau_2, 0, 1]$ interpreted over the 2-refinable UULS.

Since such MFO theories of time granularity are decidable, have the following corollary.

COROLLARY 29. — *The satisfiability problem for SL formulas, interpreted over the above classes of split structures, is decidable.*

4. A general tableau method for propositional interval logics

In this section we describe a sound and complete tableau method for BCDT^+ , developed by Goranko, Montanari and Sciavicco in [GOR03], which combines features of tableau methods for modal logics with constraint management and the classical tableau method for first-order logic. The proposed method can be adapted to variations and subsystems of BCDT , thus providing a general tableau method for propositional interval logics.

First, some basic terminology. A *finite tree* is a finite directed connected graph in which every node, apart from one (the *root*), has exactly one incoming arc. A *successor* of a node n is a node n' such that there is an edge from n to n' . A *leaf* is a node with no successors. A *path* is a sequence of nodes $n_0, \dots, n_k$ such that, for all $i = 0, \dots, k-1$, n_{i+1} is a successor of n_i ; a *branch* is a path from the root to a leaf. The *height* of a node n is the maximum length (number of edges) of a path from n to a leaf. If n, n' belong to the same branch and the height of n is less than or equal to the height of n' , we write $n \leq n'$.

Let $C = (C, <)$ be a finite partial order. A *labelled formula*, with label in C , is a pair $(\phi, [c_i, c_j])$, where $\phi \in \text{BCDT}^+$ and $[c_i, c_j] \in I(C)^+$.

For a node n in a tree, the *decoration* $\text{dec}(n)$ is a triple $(\phi, [c_i, c_j], C, u_n)$, where C is a finite partial order, $(\phi, [c_i, c_j])$ is a labelled formula, with label in C , and u_n is a *local flag function* which associates the value 0 or 1 with every branch B containing n . Intuitively, the value $u_n(B)$ for a node n with respect to a branch B means that n can be expanded on B (in fact, n must be expanded on B , sooner or later, in order to saturate the current decorated tree). For the sake of simplicity we will often assume the interval $[c_i, c_j]$ to consist of the elements $c_i < c_{i+1} < \dots < c_j$, and sometimes, with a little abuse of notation, we will write $C = \{c_i < c_k, c_m < c_j, \dots\}$. A *decorated tree* is a tree in which every node has a decoration. For every decorated tree, we define a *global flag function* u acting on pairs (node, branch through that node) as $u(n, B) = u_n(B)$. Sometimes, for convenience, we will include in the decoration of the nodes the global flag function instead of the local ones. Any branch B in a

decorated tree, we denote $\mathbf{C}_B$ the ordered set in the decoration of the leaf B , and for any node n in a decorated tree, we denote $\mathbf{b}_T(n)$ the formula in its decoration. If B is a branch, then $B \cdot n$ denotes the result of the expansion of B with the node n (addition of an edge connecting the leaf B to n). Similarly, $B \cdot n_1 | \dots | n_k$ denotes the result of the expansion of B with k immediate successor nodes $n_1, \dots, n_k$ (which produces k branches extending from B). A tableau for BCDT^* will be defined as a special decorated tree. We note again that it remains finite throughout the construction of the tableau.

DEFINITION 30. — *Given a decorated tree T , a branch B in T , and a node $n \in B$ such that $\mathbf{b}_T(n) = (\mathbf{C}_i, [\mathbf{c}_i, \mathbf{c}_j], \mathbf{C}_B, u)$, with $u(n, B) = \emptyset$, the branch-expansion rule for B and n is defined as follows (in all the considered cases, $u(n', B) = \emptyset$ for all new pairs (n', B') of nodes and branches).*

– If $\mathbf{C}_B = \emptyset$, then expand the branch to $B \cdot n_0$, with $\mathbf{b}_T(n_0) = (\mathbf{C}_i, [\mathbf{c}_i, \mathbf{c}_j], \mathbf{C}_B, u)$.

– If $\mathbf{C}_B = \langle \mathbf{c}_0, \mathbf{c}_1 \rangle$, then expand the branch to $B \cdot n_0 \cdot n_1$, with $\mathbf{b}_T(n_0) = (\mathbf{C}_i, [\mathbf{c}_i, \mathbf{c}_j], \mathbf{C}_B, u)$ and $\mathbf{b}_T(n_1) = (\mathbf{C}_i, [\mathbf{c}_i, \mathbf{c}_j], \mathbf{C}_B, u)$.

– If $\mathbf{C}_B = \langle \mathbf{c}_0, \mathbf{c}_1 \rangle$, then expand the branch to $B \cdot n_0 | n_1$, with $\mathbf{b}_T(n_0) = (\mathbf{C}_i, [\mathbf{c}_i, \mathbf{c}_j], \mathbf{C}_B, u)$ and $\mathbf{b}_T(n_1) = (\mathbf{C}_i, [\mathbf{c}_i, \mathbf{c}_j], \mathbf{C}_B, u)$.

– If $\mathbf{C}_B = \langle \mathbf{c}_0, \mathbf{c}_1 \rangle$ and $\mathbf{c}$ is the least element of $\mathbf{C}_B$, with $\mathbf{c}_i \leq \mathbf{c} \leq \mathbf{c}_j$, which has not been used yet to expand the node n on B , then expand the branch to $B \cdot n_0 | n_1$, with $\mathbf{b}_T(n_0) = (\mathbf{C}_i, [\mathbf{c}_i, \mathbf{c}], \mathbf{C}_B, u)$ and $\mathbf{b}_T(n_1) = (\mathbf{C}_i, [\mathbf{c}, \mathbf{c}_j], \mathbf{C}_B, u)$.

– If $\mathbf{C}_B = \langle \mathbf{c}_0, \mathbf{c}_1 \rangle$, $\mathbf{c}$ is a minimal element of $\mathbf{C}_B$ such that $\mathbf{c} \leq \mathbf{c}_i$, and there exists $\mathbf{c}' \in [\mathbf{c}, \mathbf{c}_i]$ which has not been used yet to expand the node n on B , then take the least such $\mathbf{c}' \in [\mathbf{c}, \mathbf{c}_i]$ and expand the branch to $B \cdot n_0 | n_1$, with $\mathbf{b}_T(n_0) = (\mathbf{C}_i, [\mathbf{c}, \mathbf{c}'], \mathbf{C}_B, u)$ and $\mathbf{b}_T(n_1) = (\mathbf{C}_i, [\mathbf{c}', \mathbf{c}_j], \mathbf{C}_B, u)$.

– If $\mathbf{C}_B = \langle \mathbf{c}_0, \mathbf{c}_1 \rangle$, $\mathbf{c}$ is a maximal element of $\mathbf{C}_B$ such that $\mathbf{c}_j \leq \mathbf{c}$, and there exists $\mathbf{c}' \in [\mathbf{c}_j, \mathbf{c}]$ which has not been used yet to expand the node n on B , then take the greatest such $\mathbf{c}' \in [\mathbf{c}_j, \mathbf{c}]$ and expand the branch to $B \cdot n_0 | n_1$, so that $\mathbf{b}_T(n_0) = (\mathbf{C}_i, [\mathbf{c}_j, \mathbf{c}'], \mathbf{C}_B, u)$ and $\mathbf{b}_T(n_1) = (\mathbf{C}_i, [\mathbf{c}_i, \mathbf{c}'], \mathbf{C}_B, u)$.

– If $\mathbf{C}_B = \langle \mathbf{c}_0, \mathbf{c}_1 \rangle$, then expand the branch to $B \cdot (n_i \cdot m_i) | \dots | (n_j \cdot m_j) | (n_i \cdot m_i) | \dots | (n_{j-1} \cdot m_{j-1})$, where:

1) for all $\mathbf{c}_k \in [\mathbf{c}_i, \mathbf{c}_j]$, $\mathbf{b}_T(n_k) = (\mathbf{C}_i, [\mathbf{c}_i, \mathbf{c}_k], \mathbf{C}_B, u)$ and $\mathbf{b}_T(m_k) = (\mathbf{C}_i, [\mathbf{c}_k, \mathbf{c}_j], \mathbf{C}_B, u)$;

2) for all $i \leq k \leq j-1$, let $\mathbf{C}_k$ be the interval structure obtained by inserting a new element $\mathbf{c}$ between $\mathbf{c}_k$ and $\mathbf{c}_{k+1}$ in $[\mathbf{c}_i, \mathbf{c}_j]$, $\mathbf{b}_T(n_k) = (\mathbf{C}_i, [\mathbf{c}_i, \mathbf{c}], \mathbf{C}_k, u)$, and $\mathbf{b}_T(m_k) = (\mathbf{C}_i, [\mathbf{c}, \mathbf{c}_j], \mathbf{C}_k, u)$.

– If $\mathbf{C}_B = \langle \mathbf{c}_0, \mathbf{c}_1 \rangle$, then repeatedly expand the current branch, once for each minimal element $\mathbf{c}$ (where $[\mathbf{c}, \mathbf{c}_i] = \{\mathbf{c} = \mathbf{c}_0 < \mathbf{c}_1 < \dots < \mathbf{c}_i\}$), by adding the decorated sub-tree $(n_0 \cdot m_0) | \dots | (n_i \cdot m_i) | (n_1 \cdot m_1) | \dots | (n_i \cdot m_i) | (n_0 \cdot m_0) | \dots | (n_i \cdot m_i)$ to its leaf, where:

1) for all c_k such that $c_k \in [c, c_i]$, $(n_k) \in (\phi_0, [c_k, c_i], C_B, u)$ and $(m_k) \in (\phi_1, [c_k, c_j], C_B, u)$;

2) for all $0 < k < i$, let C_k be the interval structure obtained by inserting a new element c immediately before c_k in $[c, c_i]$, and $(n_k) \in (\phi_0, [c, c_i], C_k, u)$ and $(m_k) \in (\phi_1, [c, c_j], C_k, u)$;

3) for all $0 < k < i$, let C_k be the interval structure obtained by inserting a new element c in C_B , with $c < c_k$, which is incomparable with all existing predecessors of c_k , $(n_k) \in (\phi_0, [c, c_i], C_k, u)$, and $(m_k) \in (\phi_1, [c, c_j], C_k, u)$.

– If $\phi \in (\phi_0 T \phi_1)$, then repeatedly expand the current branch, once for each maximal element c (where $[c_j, c] = \{c_j < c_{j+1} < \dots < c_n = c\}$), by adding the decorated sub-tree $(n_j \cdot m_j) | \dots | (n_n \cdot m_n) | (n_j \cdot m_j) | \dots | (n_{n-1} \cdot m_{n-1}) | (n_j \cdot m_j) | \dots | (n_n \cdot m_n)$ to its leaf, where:

1) for all c_k such that $c_k \in [c_j, c]$, $(n_k) \in (\phi_0, [c_j, c_k], C_B, u)$ and $(m_k) \in (\phi_1, [c_i, c_k], C_B, u)$;

2) for all $j < k < n$, let C_k be the interval structure obtained by inserting a new element c immediately after c_k in $[c_j, c]$, and $(n_k) \in (\phi_0, [c_j, c], C_k, u)$ and $(m_k) \in (\phi_1, [c_i, c], C_k, u)$;

3) for all $j < k < n$, let C_k be the interval structure obtained by inserting a new element c in C_B , with $c_k < c$, which is incomparable with all existing successors of c_k , $(n_k) \in (\phi_0, [c_j, c], C_k, u)$, and $(m_k) \in (\phi_1, [c_i, c], C_k, u)$.

Finally, for any node $m (= n)$ in B and any branch B' extending B , let $u(m, B')$ be equal to $u(m, B)$, and for any branch B' extending B , $u(n, B') = 1$, unless $\phi = (\phi_0 C \phi_1)$, $\phi = (\phi_0 D \phi_1)$, or $\phi = (\phi_0 T \phi_1)$ (in such cases $u(n, B') = 0$).

Let us briefly explain the expansion rules for $\phi_0 C \phi_1$ and $(\phi_0 C \phi_1)$ (similar considerations hold for the other temporal operators). The rule for the (existential) formula $\phi_0 C \phi_1$ deals with the two possible cases: either there exists c_B such that $c_i < c_k < c_j$ and ϕ_0 holds over $[c_i, c_k]$ and ϕ_1 holds over $[c_k, c_j]$ or such an element c_k must be added. The (universal) formula $\phi_0 C \phi_1$ states that, for all $c_i < c_j$, ϕ_0 does not hold over $[c_i, c_j]$ or ϕ_1 does not hold over $[c_i, c_j]$. As a matter of fact, the expansion rule imposes such a condition for a single element in C_B (the least element which has not been used yet), and it does not change the *rank* (it remains equal to 0). In this way, all elements will be eventually taken into consideration, including those elements in between c_i and c_j that will be added to C_B in some subsequent steps of the tableau construction.

Let us define now the notions of open and closed branch. We say a node n in a decorated tree $\mathcal{T}$ is *available on a branch* B to which it belongs if and only if $u(n, B) = 0$. The branch-expansion rule is *applicable* to a node n on a branch B if the node is available on B and the application of the rule generates at least one *source* node with a new labelled formula. This second condition is *added* to avoid looping of the application of the rule on formulas $(\phi_0 C \phi_1)$, $(\phi_0 D \phi_1)$, and $(\phi_0 T \phi_1)$.

DEFINITION 31. — A branch B is closed if some of the following conditions holds:

(i) there are two nodes $n, m \in B$ such that $(n) \in (\phi, [c_i, c_j], C, u)$ and $(m) \in (\psi, [c_i, c_j], C, u)$ for some formula ϕ and $c_i, c_j \in C \setminus C'$;

(ii) there is a node n such that $(n) = (\langle \langle \dots, [c_i, c_j] \rangle, C, u) \text{ and } c_i = c_j$; or

(iii) there is a node n such that $(n) = (\langle \langle \dots, [c_i, c_j] \rangle, C, u) \text{ and } c_i = c_j$.

If none of the above conditions hold, the branch is open.

DEFINITION 32. — *The branch-expansion strategy for a branch B in a decorated tree T is defined as follows:*

1) Apply the branch-expansion rule to a branch B only if it is open;

2) If B is open, apply the branch-expansion rule to the closest to the root available node in B for which the branch-expansion rule is applicable.

DEFINITION 33. — *A tableau for a given formula φ in BCDT^+ is any finite decorated tree T obtained by expanding the three-node decorated tree built up from an empty-decoration root and two leaves with decorations $(\langle \langle \dots, [c_b, c_e] \rangle, \{c_b < c_e\}, u)$ and $(\langle \langle \dots, [c_b, c_b] \rangle, \{c_b\}, u)$, where the value of u is $\mathbb{Q}$, through successive applications of the branch-expansion strategy to the existing branches.*

It is easy to show that if φ is in BCDT^+ , T is a tableau for φ , and C is the ordered set in the decoration of T , then $\langle C, < \rangle$ is an interval structure.

THEOREM 34 (SOUNDNESS AND COMPLETENESS). — *If φ is in BCDT^+ and a tableau T for φ is closed, then φ is not satisfiable. Moreover, if φ is in BCDT^+ is a valid formula, then there is a closed tableau for φ .*

5. First-Order Interval Logics and Duration Calculi

Research on interval temporal logics in computer science was originally motivated by problems in the field of specification and verification of hardware protocols, rather than by abstract philosophical or logical issues. Not simply, it focused on first-order, rather than propositional, interval logics. In this section, we summarize some of the most-important developments in first-order interval logics and duration calculi, referring the interested reader to respectively [MOS 03] for CHA 04] for more details.

5.1. The logic ITL

First-order ITL, interpreted over discrete linear orders with finite time intervals, was originally developed by Halpern, Manna, and Moszkowski [MOS 83, HAL 83]. The language of ITL includes terms, predicates, Boolean connectives, first-order quantifiers, and the temporal modalities $\Box$ and $\Diamond$. Terms are built on variables, constants, and function symbols in the usual way. Constants and function symbols are classified as *global/rigid* and *temporal/flexible*. Terms are usually denoted by $t_1, \dots, t_n$. Predicate symbols are also partitioned into global and temporal ones. They are denoted by $p^i, q^j, \dots$, where p^i is a predicate of arity i , q^j is a predicate of arity j , and so on. The abstract syntax of ITL formulas is:

$$\varphi ::= \langle p^n(t_1, \dots, t_n) \mid x \mid \langle \varphi_1 \mid \varphi_2 \mid \varphi_3 \mid C \rangle.$$

The semantics of ITL-formulas is a combination of the standard semantics of a first-order temporal logic with the semantics of PITL. An account of possible uses and applications is e.g. [MOS 86].

In [DUT 95a] Dutertre studies the fragment of ITL which we will denote here by ITL_D , involving only the *chop* operator. First, ITL_D is considered over abstract, Kripke-style models $M^+ = \langle W, R, I \rangle$, where W is a set of worlds (abstract intervals), R is a ternary relation corresponding to Venema's ternary relation A (cf. Section 2.1, and I is a first-order interpretation. Further, Dutertre considers a more concrete semantics, over interval structures with associated time measure represented by a special temporal variable which takes values in a commutative group $D, +, -, 0$. The language is assumed to have the flexible constant and the rigid symbols 0 and $+$, respectively interpreted as the neutral element and addition in $D, +, 0$. The semantics of ITL_D -formulas is a combination of the semantics of ITL (without out), and the interpretation d in a model M^+ for an interval $[d_0, d_1]$ is $d_1 - d_0$.

As for the expressive power of ITL_D , note that one can easily define the modal constant true (cf. Section 2.2) by means of

$$\text{true} = \neg (I \neq 0).$$

Hence, the HS modalities corresponding to *begins* and *ends* are also definable in the language, and thus, from the results of Section 3.1.3, we can conclude that ITL_D is at least as expressive as PITL. The undecidability of ITL_D clearly follows.

Dutertre developed a sound and complete axiomatic system ITL_D (the details of the soundness and completeness proof can be found in [DUT]). In addition to the standard axioms of first-order classical logic, including the axioms of identity and the axioms describing the properties for the temporal domain D , Dutertre's system involves the following specific axioms for ITL_D :

$$(A\text{-ITL1}) \quad (C \rightarrow \neg C) \rightarrow C;$$

$$(A\text{-ITL2}) \quad (C \rightarrow \neg C) \rightarrow (C \rightarrow C);$$

$$(A\text{-ITL3}) \quad (C \rightarrow C) \rightarrow (C \rightarrow C);$$

$$(A\text{-ITL4}) \quad (C \rightarrow \neg C) \text{ if } C \text{ is a rigid formula};$$

$$(A\text{-ITL5}) \quad (C \rightarrow \neg C) \text{ if } C \text{ is a rigid formula};$$

$$(A\text{-ITL6}) \quad (C(x) \rightarrow \neg C(x)) \rightarrow (C(x) \rightarrow \neg C(x)) \text{ if } x \text{ is not free in } C;$$

$$(A\text{-ITL7}) \quad (C(x) \rightarrow \neg C(x)) \rightarrow (C(x) \rightarrow \neg C(x)) \text{ if } x \text{ is not free in } C;$$

$$(A\text{-ITL8}) \quad (I = x) \rightarrow (I = x);$$

$$(A\text{-ITL9}) \quad (C(I = x) \rightarrow \neg C(I = x)) \rightarrow (C(I = x) \rightarrow \neg C(I = x));$$

$$(A\text{-ITL10}) \quad (I = x + y) \rightarrow (I = x) \rightarrow (I = y);$$

$$(A\text{-ITL11}) \quad (C(I \neq 0) \rightarrow \neg C(I \neq 0));$$

$$(A\text{-ITL12}) \quad (I \neq 0 \rightarrow C(I \neq 0)).$$

The inference rules are Modus Ponens, Generalization, Substitution, and the following Monotonicity rule:

$$\frac{}{C \quad C'}$$

together with the symmetric one. It should be noted that no restrictions apply to the instantiation with $\exists$ -exible terms in quantified formulas.

As in the propositional case, variants of ITL obtained by dropping the locality constraint have been explored in the literature. Sound and complete axiomatic systems for local variants of ITL for finite and infinite time have been established in [DUT 95a, DUT 95b, MOS 00b], while automata-theoretic techniques for proving completeness of ITL have been applied in [MOS 00a, MOS 03].

For more details about completeness and decidability results for ITL see [MOS 03]. See also [MOS 86] and [DUA 96], for applications of ITL to temporal logic programming, and [MOS 96b, MOS 98], where the ITL-based programming language Tempura is described in detail.

5.1.1. Some extensions and variations of ITL

An extension of ITL with projection has been studied in [GUE 95], where a complete axiomatic system for it has been established. A projection extension of ITL has been studied in [GUE 00d].

An interesting variation of ITL is the Signed Interval Logic (SIL) introduced by Rasmussen [RAS 99, RAS 02]. The semantics of SIL is based on *signed intervals*, i.e., intervals provided with a *direction* (forward or backward). A sound and complete axiomatic system for SIL was established in [RAS 99], a natural deduction system in [RAS 01b], and a sequent calculus in [RAS 01a].

Dillon, Kutty, Moser, Melliar-Smith, and Ramakrishna introduced and study in a series of publications [RAM 92, DIL 92a, DIL 92b, DIL 93, DIL 94a, DIL 94b, DIL 95, MOS 96a, DIL 96a, DIL 96b, DIL 94a] the so-called Future Interval Logics. These employ the locality principle and feature 'interval models' encoded by pairs of formulas and refer to intervals whose endpoints satisfy these formulas. Notably, these logics are more tractable and have lower complexity than ITL. Complexity results for Future Interval Logic have been obtained by Aaby Narayana [AAB 85], while applications of these logics have been explored in Aaby Narayana's PhD thesis [RAM 93].

5.2. The logic NL

The logic ITL has an intrinsic limitation: its modalities do not allow one to 'look' outside the current interval (modalities with this characteristic are called *contracting* modalities). To overcome such a limitation, Zhou and Harizanov [ZHA 91] proposed the first-order logic of *left* and *right* neighbourhood modalities, called *neighbourhood logic* (NL for short), whose propositional fragment has been analyzed in Section 3.1.4.

Right-left neighbourhood
tract syntax of NL formulas

$$r \mid x,$$

and r are defined as $r = \frac{1}{N} \sum_{i=1}^N r_i$ and r_i is defined exactly as in [1]. The group of the real numbers $\mathbb{R}$ is a set of first-order

ssive particular, it allows

$$= y) \quad) ,$$

tions. Consequently, NL
ing linear ordering, such

what follows, the
(resp., μ) when stands
following axioms:

$$y) \quad) \quad (1 =$$

The inference rules are, in the following rule for

In [BAR 97], NL has been extended to a 'two-dimensional' version called NL², where two modalities $\mathbf{u}$ and $\mathbf{d}$ have been added and interpreted as 'up' and 'down' neighbourhoods. NL can be used to specify super-dense computations, taking vertical time as virtual time, and horizontal time as real time.

The relationship between the Neighbourhood Logic and the fragments of Allen's Interval Algebra has been studied in [PUJ 97].

5.3. Duration calculi

Duration Calculus (DC for short) is an interval temporal logic endowed with the additional notion of *state*. Each state is denoted by means of a state expression, and it is characterized by a *duration*. The duration of a state is (the length of) the time period during which the system remains in the state. DC has been successfully applied to the specification and verification of real-time systems. Furthermore, it has been used to express the behaviour of communicating processes sharing a processor and to specify their scheduler, as well as to specify the requirements of a debugger [SØR 90].

DC has originally been developed as an extension of Moszkowski's ITL, and thus denoted by DC/ITL. Since the seminal work by Zhou, Hoare, and Ravn [CHA 91], various meaningful fragments of DC/ITL have been isolated and analyzed. Recently, an alternative Duration Calculus, based on the logic NL, and denoted by DC/NL, has been proposed by Roy in [ROY 97]. As a matter of fact, results for DC/ITL and its fragments transfer to DC/NL and its fragments. In the following we introduce the basic notions and we summarize the main results about DC. Further details can be found in [CHA 04].

5.3.1. The calculus DC/ITL

Zhou, Hoare, and Ravn's DC/ITL is based on Moszkowski's ITL interpreted over the class of non-strict interval structures based on its only interval modality *chop*. Its distinctive feature is the notion of state. States are represented by means of a new syntactic category, called *state expression*, which is defined as follows: the constants 0 and 1 are state expressions, a state variable is a state expression, and, for any pair of state expressions S and T , $S \wedge T$ and $S \vee T$ are state expressions (the other Boolean connectives are defined in the usual way). Furthermore, given a state expression S , the duration of S is denoted by $\mathbf{R}S$. DC/ITL terms are defined as in ITL, provided that temporal variables are replaced by state expressions. DC/ITL formulas are generated by the following abstract syntax:

$$\models p^n(r_1, \dots, r_n) \mid \mid \mid \mid \mid \mid \mid C \mid x$$

where $r_1, \dots, r_n$ are terms, p^n is an n -ary (global) predicate, C is the *chop* modality, and x is a (global) variable.

Any state (expression) S is associated with a total function $\phi_S : \mathbb{R} \rightarrow \{0, 1\}$, which has a finite number of discontinuity points only. For a time point t , the state expression interpretation is defined as follows:

$$- \mathbf{I}(\mathbf{D} \ t) = 0 \ ;$$

- $I \models t \neq$;
- $I \models S(t) = S(t)$;
- $I \models S(t) \neq \neg I \models S(t)$;
- $I \models S \vee T(t) \neq$ if $I \models S(t) \neq$ or $I \models T(t) \neq$, otherwise.

The semantics of a duration S in a given (non-strict) model, with respect to an interval $[d_0, d_1]$, can be defined using the Riemann definite integral $\int_{d_0}^{d_1} I \models S(t) dt$. The semantics of the other syntactic constructs is given in the case of ITL.

A number of useful abbreviations can be defined in DC/ITL. In particular, S stands for: "S holds almost everywhere over a strict interval", and it is defined as follows:

$$R \models S, (R \models S = R) \quad (R \models 1 = R).$$

$R \models 1$ is usually abbreviated by true , and it can be viewed as the length of the current interval; namely, true , which holds over point-intervals, can be defined as $\text{true} = S$.

The satisfiability problem for both first-order DC/ITL (fulDC/ITL) and its fragment devoid of first-order quantification (Propositional DC/ITL) has been shown to be undecidable. First-order DC/ITL, provided with, at least, the functional symbol and the predicate symbol, with the usual interpretation, has been completely axiomatized in [HAN 92]. The axiomatic system includes the following specific axioms:

$$(A\text{-DC1}) \quad R \models \infty;$$

$$(A\text{-DC2}) \quad R \models S \rightarrow Q;$$

$$(A\text{-DC3}) \quad R \models S + T = (R \models S) + (R \models T);$$

$$(A\text{-DC4}) \quad (R \models S = x) \wedge (R \models S = y) \rightarrow (R \models S = x + y);$$

$$(A\text{-DC5}) \quad R \models S = R \models T \text{ provided that } S \rightarrow T \text{ holds in propositional logic}$$

and the following inference rule (where $S_1, \dots, S_n$ are state expressions and $W_{i=1}^n S_i$ is the disjunction of $S_1, \dots, S_n$):

$$\frac{H(\neg), H(\neg) \quad H(\bigvee_{i=1}^n (C \wedge S_i))}{H(\neg)},$$

in conjunction with its inverse (obtained by exchanging the ordering of the formulas in every chop), where $H(\neg)$ represents the formula obtained from $H(X)$ by replacing every occurrence of X in H by $\neg$.

Duration calculus on abstract domains has been studied and axiomatized in [GUE 98].

Various interesting fragments of DC have been investigated by Zhou, Hansen, and Sestoft in [CHA 93a]. First, they consider the possibility of interpreting DC formulas over different classes of structures. In particular, a fragment of DC interpreted over N is the set of DC formulas interpreted over N and reevaluated with respect to intervals, that is, intervals whose endpoints are in N . The fragment of DC interpreted over Q is similarly defined. Then, the authors take into consideration some syntactic sub-fragments of the above calculi and they study the decidability of

their satisfiability problem. It turns out that the fragment of propositional DC whose formulas are built up from primitive formulas of the type S only have a decidable satisfiability problem when interpreted over N , Q , and R . A validity checking procedure for some of these fragments was developed in [SKA 94]. By adding to the set of primitive formulas those of the form $t = k$, the problem remains decidable over N , but it becomes undecidable over the other classes of structures. The same fragment at the first-order level is undecidable in all the considered cases. Finally, the fragment of propositional DC whose formulas are built up from primitive formulas of the type $S = T$ only is also undecidable.

As for the complexity of the satisfiability problem, in [RAB 89] Rabinovich reports a result by Sestoft (personal communication) stating that the satisfiability problem for the fragment of DC whose formulas are built up from primitive formulas of the type S only, interpreted over N , has a non-elementary complexity. Rabinovich shows that the satisfiability problem for the same fragment interpreted over R , also is non-elementarily decidable, by providing a linear time reduction from the equivalence problem for star-free expressions to the validity problem in the considered fragment of DC.

In [CHE 00], Chetcuti-Sperandio and Fariñas del Cerro isolate another fragment of propositional DC by imposing suitable syntactic restrictions. Formulas of such a fragment are generated by the following abstract syntax:

$$\varphi ::= \text{true} \mid \neg \varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \rightarrow \varphi \mid \varphi \text{ until } \varphi \mid \varphi \text{ release } \varphi \mid \varphi \text{ until }^R \varphi \mid \varphi \text{ release }^R \varphi \mid C, \varphi$$

where k is a constant $P \in \{<, =, >\}$, and I is R or S , for a given state S . The resulting logic is shown to be expressive enough to capture the Interval Algebra. The authors propose a sound, complete, and terminating tableau system for the logic, thus showing that its satisfiability problem is decidable. The tableau system is a mixed procedure, combining standard tableau techniques with each constraint network resolution algorithms.

5.3.2. Some extensions and variations of Duration Calculus

In [CHA 98] (see also [ROY 97]) Duration Calculus and the first-order neighbourhood logic (NL) have been combined into the (clearly, undecidable) DC/NL which has been completely axiomatized by merging the axiomatizations for DC and NL. The fragment of DC/NL obtained by restricting the formulas to be built up only from primitive formulas of the type S has been proved to be decidable, while the extension of the latter with primitive formulas of the type $t = k$ is undecidable, as already mentioned.

Duration Calculus with infinite intervals has been studied in [CHA 95]. Other extensions of Duration Calculus include: Extended Duration Calculus for real-time systems [CHA 93b], Mean Value Calculus of Durations [CHA, 94], Duration Calculus with Iteration [HUN 99c, GUE 00c], Duration Calculus with Projection [GUE 02, GUE 03], higher-order Duration Calculus [GUE 00a, NAI 00], probabilistic Duration Calculus for continuous time [HUN 99b].

Another variation of DC is Pandya's Interval Duration Logic [PAN 96] the models of which are timed state sequences in dense time structures.

Applications of Duration Calculus to real-time and hybrid systems have been developed in [HUN 99a, HUN 02, HUO 02, SIE 01, THA 01].

Automatic verification and model-checking tools for interval logics and duration calculi have been developed and analyzed in [KON 92, SKA 94, NI 94, CAM 96, YON 02] and program synthesis from DC specifications has been studied in [SIE 01].

Finally, in [FRÄ 96, FRÄ 02b, FRÄ 98] Fränzle describes model-checking methods for DC and he argues that, despite its undecidability, the class of models is restricted to the possible behaviours of embedded real systems, model-checking procedures are feasible for rich subsets of Duration Calculus and related logics.

For further details, recent results, and applications of DC [CHA 04].

6. Summary and concluding remarks

In this survey paper, we have attempted to give a general picture of the extensive and rather diverse research done in the areas of interval logics and duration calculi. Among all important issues in the field, we have mainly focused on expressiveness, proof systems, and decidability/undecidability results.

To summarize, sound and complete axiomatic systems on *point-based* level are known for CDT, with respect to certain classes of linear orders, for HS, with respect to the class of partial orderings with the linear interval property, for the family of logics in PNL, with respect to various classes of linear orderings, both the strict and non-strict semantics, and for ITL and NL with respect to *interval* semantics, while the problem of finding an axiomatic system for specific linear orderings is still largely unexplored.

Furthermore, sound and complete tableau systems have been developed for BCDT and for some local variants of ITL. Given the generality of BCDT, the tableau method for such a logic is in fact a tableau method for a large variety of propositional interval logics.

The satisfiability/validity problem has been shown to be *undecidable* for HS, CDT, ITL, and NL, with respect to most classes of structures. As a matter of fact, rather weak subsystems of HS turn out to be (highly) undecidable for some classes of structures. Decidable fragments have been obtained by imposing *syntactic* restrictions on the expressive power or the semantics of the logics (as *model* by imposing the locality projection principle).

Finally, we point out once more that, to the best of our knowledge, the problems of constructing axiomatic systems, tableau systems, and decidability proofs have not been explicitly addressed yet for the strict semantics *variants* of most of the existing interval logics (with the exceptions of PNL and its subsystems).

In conclusion, the single major challenge in the area of *interval* temporal logics is to identify expressive enough, yet decidable, fragments of logics which are genuinely interval-based, that is, not explicitly translated to point-based logics and not invoking locality or other semantic restrictions reducing the interval-based semantics to the point-based one.

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7. References

- [AAB 85] AABY A., NARAYANA K., "Propositional Temporal Interval Logic is PSPACE Complete", *Proc. of 9th International Conference on Automated Deduction*, vol. 193 of LNCS, Springer, 1985, p. 218–237.
- [AHM 93] AHMED M., VENKATESH G., "A Propositional Dense Time Logic (Based on Nested Sequences)", *Proc. of TAPSOFT*, 1993, p. 584–598.
- [ALL 83] ALLEN J., "Maintaining Knowledge about Temporal Intervals", *Communications of the ACM*, vol. 26, num. 11, 1983, p. 832–843.
- [ALL 85] ALLEN J., HAYES P., "A Common-sense Theory of Time", *Proc. of the 9th International Joint Conference on Artificial Intelligence*, Morgan Kaufmann, 1985, p. 528–531.
- [ALL 94] ALLEN J., FERGUSON G., "Actions and Events in Interval Temporal Logic", *Journal of Logic and Computation*, vol. 4, num. 5, 1994, p. 531–579.
- [BAR 97] BARUA R., CHAOCHEN Z., "Neighbourhood Logics: NL and NL 2", report num. 120, 1997, UNU/IIST, Macau.
- [BAR 00] BARUA R., ROY S., CHAOCHEN Z., "Completeness of Neighbourhood Logic", *Journal of Logic and Computation*, vol. 10, num. 2, 2000, p. 271–295.
- [BEE 89] VAN BEEK P., "Approximation Algorithms for Temporal Reasoning", *Proc. of the 11th International Joint Conference on Artificial Intelligence*, Morgan Kaufmann, 1989, p. 1291–1296.
- [BEN 91] BENTHEM J. V., *The Logic of Time (2nd Edition)*, Kluwer Academic Press, 1991.
- [BLA 01] BLACKBURN P., DE RIJKE M., VENEMA V., *Modal Logic*, vol. 53 of *Cambridge Tracts in Theoretical Computer Science*, Cambridge Univ. Press, 2001.
- [BOW 98] BOWMAN H., THOMPSON S., "A Tableau Method for Interval Temporal Logic with Projection", *Proc. of the International Conference Tableaux 1998*, num. 1397 LNCS, Springer, 1998, p. 108–134.
- [BOW 00] BOWMAN H., CAMERON H., KING P., THOMPSON S., "Specification and Prototyping of Structured Multimedia Documents using Interval Temporal Logic", BARRINGER H., FISHER M., GABBAY D., G. GOUGH, Eds., *Advances in Temporal Logic*, vol. 16 of *Applied Logic*, Kluwer Academic, 2000, p. 435–453.
- [BOW 03] BOWMAN H., THOMPSON S., "A Decision Procedure and Complete Axiomatization of Finite Interval Temporal Logic with Projection", *Journal of Logic and Computation*, vol. 13, num. 2, 2003, p. 195–239.

- [BUR 82] BURGESS J., "Axioms for Tense Logic II. Time Periods", *Notre Dame Journal of Formal Logic*, vol. 23, num. 4, 1982, p. 375–383.
- [CAM 96] CAMPOS S., GRUMBERG O., "Selective Quantitative Analysis and Interval Model Checking: Verifying Different Facets of a System", LAR R., HENZINGER T., Eds., *Proc. of the 8th International Conference on Computer Aided Verification*, num. 1102 LNCS, Springer, 1996, p. 257–268.
- [CHA 85] CHANDRA A., HALPERN J., MEYER A., PARIKH R., "Equations Between Regular Terms and an Application to Process Logic", *SIAM Journal on Computing*, vol. 14, num. 4, 1985, p. 935–942.
- [CHA 91] CHAOCHEN Z., HOARE C., RAVN A. P., "A Calculus of Durations", *Information Processing Letters*, vol. 40, num. 5, 1991, p. 269–276.
- [CHA 93a] CHAOCHEN Z., HANSEN M., SESTOFT P., "Decidability and Undecidability Results for Duration Calculus", *Proc. of the 10th Symposium on Theoretical Aspects of Computer Science*, num. 665 LNCS, Springer, 1993, p. 58–68.
- [CHA 93b] CHAOCHEN Z., RAVN A. P., HANSEN M. R., "An Extended Duration Calculus for Real-time Systems", *Hybrid Systems*, num. 736 LNCS, p. 36–59, Springer, 1993.
- [CHA 94] CHAOCHEN Z., XIAOSHAN L., "A Mean Value Calculus of Durations", ESCOE A. W., Ed., *A Classical Mind: Essays in Honour of C.A.R. Hoare*, p. 431–451, Prentice Hall International Series in Computer Science, 1994.
- [CHA 95] CHAOCHEN Z., HUNG D. V., XIAOSHAN L., "A Duration Calculus with Infinite Intervals", REICHEL H., Ed., *Fundamentals of Computation Theory*, num. 965 LNCS, Springer-Verlag, 1995, p. 16–41.
- [CHA 98] CHAOCHEN Z., HANSEN M. R., "An Adequate First Order Interval Logic", DE ROEVER W., LANGMAAK H., PNUELI A., Eds., *Compositionality: the Significant Difference*, num. 1536 LNCS, Springer, 1998, p. 584–608.
- [CHA 99] CHAOCHEN Z., "Duration Calculus, a Logical Approach to Real-Time Systems", *Proc. of the Annual Conference of the European Association for Computer Science Logic*, num. 1548 LNCS, Springer, 1999, p. 1–7.
- [CHA 04] CHAOCHEN Z., HANSEN M. R., *Duration Calculus. A Formal Approach to Real-Time Systems*, EATCS Series of Monographs in Theoretical Computer Science, Springer, 2004.
- [CHE 00] CHETCUTI-SERANDIO N., DEL CERRO L. F., "A Mixed Decision Method for Duration Calculus", *Journal of Logic and Computation*, vol. 10, num. 6, 2000, p. 877–895.
- [CHI 00] CHITTARO L., MONTANARI A., "Temporal Representation and Reasoning in Artificial Intelligence: Issues and Approaches", *Annals of Mathematics and Artificial Intelligence*, vol. 28, num. 1–4, 2000, p. 47–106.
- [DIL 92a] DILLON L., KUTTY G., MOSER L. E., MELLIAR-SMITH P., RAMAKRISHNA Y. S., "Graphical Specifications for Concurrent Software Systems", *Proc. of the 14th International Conference on Software Engineering*, ACM Press, 1992, p. 214–224.
- [DIL 92b] DILLON L., KUTTY G., MOSER L., MELLIAR-SMITH P. M., RAMAKRISHNA Y., "An Automata-Theoretic Decision Procedure for Future Interval Logic", SHYAMA-SUNDAR R., Ed., *Proc. of the 12th Foundations of Software Technology and Theoretical Computer Science*, vol. 652 of LNCS, Springer, 1992, p. 51–67.
- [DIL 93] DILLON L., KUTTY G., MOSER L., MELLIAR-SMITH P. M., RAMAKRISHNA Y., "A Real-Time Interval Logic and Its Decision Procedure", *Proc. of the 13th Conference on*

Foundations of Software Technology and Theoretical Computer Science, vol. 761 of LNCS, Springer, 1993, p. 173–192.

- [DIL 94a] DILLON L., KUTTY G., MOSER L., MELLIAR-SMITH P. M., RAMAKRISHNA Y., “A Graphical Interval Logic for Specifying Concurrent Systems”, *ACM Transactions on Software Engineering and Methodology*, vol. 3, num. 2, 1994, p. 131–165.
- [DIL 94b] DILLON L., MELLIAR-SMITH P., MOSER L., KUTTY G., RAMAKRISHNA Y., “Completeness and Soundness of Axiomatizations for Temporal Logics Without Next”, *Fundamenta Informaticae*, vol. 21, 1994, p. 257–305.
- [DIL 94c] DILLON L., MELLIAR-SMITH P., MOSER L., KUTTY G., RAMAKRISHNA Y., “First-Order Future Interval Logic”, *Proc. of the First International Conference on Temporal Logic, Bonn, Germany*, vol. 827 of LNCS, Springer, 1994, p. 195–209.
- [DIL 95] DILLON L., MELLIAR-SMITH P., MOSER L., KUTTY G., RAMAKRISHNA Y., “Axiomatizations of Interval Logics”, *Fundamenta Informaticae*, vol. 24, num. 4, 1995, p. 313–331.
- [DIL 96a] DILLON L., MELLIAR-SMITH P., MOSER L., KUTTY G., RAMAKRISHNA Y., “Interval Logics and their Decision Procedures. Part I: Interval Logic”, *Theoretical Computer Science*, vol. 166, num. 1–2, 1996, p. 1–47.
- [DIL 96b] DILLON L., MELLIAR-SMITH P., MOSER L., KUTTY G., RAMAKRISHNA Y., “Interval Logics and their Decision Procedures. Part II: Real Time Interval Logic”, *Theoretical Computer Science*, vol. 170, num. 1–2, 1996, p. 1–46.
- [DOW 79] DOWTY D., *Word Meaning and Montague Grammar*, Reidel, Dordrecht, 1979.
- [DUA 96] DUAN Z., “An Extended Interval Temporal Logic and A Framing Technique for Temporal Logic Programming”, PhD thesis, University of Newcastle Upon Tyne, 1996.
- [DUT 95a] DUTERTRE B., “Complete Proof Systems for First Order Interval Temporal Logic”, *Proc. of the 10th International Symposium on Logic in Computer Science*, 1995, p. 36–43.
- [DUT 95b] DUTERTRE B., “On First Order Interval Temporal Logic”, report num. DSR-94-3, 1995, Royal Holloway, University of London.
- [FRÄ 96] FRÄNZLE M., “Controller Design from Temporal Logic: Undecidability need not matter”, Dissertation, Technische Fakultät der Christian-Albrechts-Universität Kiel, Germany, 1996, Available as Bericht Nr. 9710, Institut für Informatik und Praktische Mathematik, Christian-Albrechts-Universität Kiel, June 1997, and via WWW under URL <http://ca.informatik.uni-oldenburg.de/fraenzle/diss.ps.gz>.
- [FRÄ 98] FRÄNZLE M., “Model-Checking Dense-Time Duration Calculus”, ANSEN M., Ed., *Duration Calculus: A Logical Approach to Real-Time Systems. Workshop proceedings of the 10th European Summer School in Logic, Languages and Information (ESSLI X), Saarbrücken, Germany, August 1998. Final version to appear in a special issue of BCS FACS, 2004.*, 1998.
- [FRA 02a] FRANCESCHETTI M., “Dividing and Conquering the Layered Land”, PhD thesis, Department of Mathematics and Computer Science, University of Udine, Italy, 2002, PhD Thesis Series CS 2002/2.
- [FRÄ 02b] FRÄNZLE M., “Take it NP-easy: Bounded model construction for duration calculus”, OLDEROG E.-R., DAMM W., Eds., *Proc. of the International Symposium on Formal Techniques in Real-Time and Fault-Tolerant systems (FTRTFT 2002)*, vol. 2469 of LNCS, Springer, 2002, p. 245–264.

- [GAB 94] GABBAY D., HODKINSON I., REYNOLDS M., *Temporal Logic, vol. 1: Mathematical Foundations and Computational Aspects*, Oxford Science Publications, 1994.
- [GAB 00] GABBAY D., REYNOLDS M., FINGER M., *Temporal Logic, vol. 2: Mathematical Foundations and Computational Aspects*, Oxford Science Publications, 2000.
- [GAL 90] GALTON A., "A Critical Examination of Allen's Theory of Action and Time", *Artificial Intelligence*, vol. 42, 1990, p. 159–198.
- [GAR 93] GARGOV G., GORANKO V., "Modal Logics with Names" *Journal of Philosophical Logic*, vol. 22, num. 6, 1993, p. 607–636.
- [GOR 03a] GORANKO V., MONTANARI A., SCIAVICCO G., "A General Tableau Method for Propositional Interval Temporal Logics" *Proc. of the International Conference Tableaux 2003*, LNAI, Springer, 2003, p. 102–116.
- [GOR 03b] GORANKO V., MONTANARI A., SCIAVICCO G., "Propositional Interval Neighborhood Temporal Logics" *Journal of Universal Computer Science*, vol. 9, num. 9, 2003, p. 1137–1167.
- [GUE 98] GUELEV D., "A Calculus of Durations on Abstract Domains: Completeness and Extensions", report num. 139, 1998, UNU/IIST, Macau.
- [GUE 00a] GUELEV D. P., "A Complete Fragment of Higher-Order Duration Calculus", *Proc. of the 20th Conference on Foundations of Software Technology and Theoretical Computer Science*, vol. 1974 of LNCS, 2000.
- [GUE 00b] GUELEV D., "A Complete Proof System for First Order Interval Temporal Logic with Projection", report num. 202, 2000, UNU/IIST, Macau.
- [GUE 00c] GUELEV D., "Interpolation and related results on the P-fragment of Duration Calculus with Iteration", report num. 203, 2000, UNU/IIST, Macau.
- [GUE 00d] GUELEV D., "Probabilistic Neighbourhood Logic" *Proc. of the 6th International Symposium on Formal Techniques in Real-Time and Fault-Tolerant Systems*, vol. 1926 of LNCS, Springer, 2000, p. 264–275.
- [GUE 02] GUELEV D. P., HUNG D. V., "Pre x and Projection onto State in Duration Calculus", *Proceedings of the 1st Workshop on the Theory and Practice of Timed Systems*, vol. 65(6) of *Electronic Notes in Theoretical Computer Science*, Elsevier Science, 2002.
- [GUE 03] GUELEV D., HUNG D. V., "Projection onto State in Duration Calculus: Relative Completeness", report num. 269, 2003, UNU/IIST, Macau.
- [HAL 83] HALPERN J., MANNA Z., MOSZKOWSKI B., "A Hardware Semantics Based on Temporal Intervals", *Proc. of the 10th International Colloquium on Automata, Languages and Programming*, num. 154 LNCS, Springer, 1983, p. 278–291.
- [HAL 91] HALPERN J., SHOHAM Y., "A Propositional Modal Logic of Time Intervals" *Journal of the ACM*, vol. 38, num. 4, 1991, p. 935–962.
- [HAM 72] HAMBLIN C., "Instants and Intervals", in: EASER J., HABER F., MUELLER G., Eds., *The Study of Time (Volume 1)*, Springer, 1972, p. 324–331.
- [HAN 92] HANSEN M. R., CHAOCHEN Z., "Semantics and Completeness of Duration Calculus", in: DE BAKKER J., HUIZING C., DE ROEVER W., ROZENBERG G., Eds., *Real-Time: Theory in Practice*, num. 600 LNCS, Springer, 1992, p. 209–225.
- [HAN 94] HANSEN M., "Model-Checking Discrete Duration Calculus" *Formal Aspects of Computing*, vol. 6, num. 6A, 1994, p. 826–845.

- [HAN 97] HANSEN M., CHAOCHEN Z., "Duration Calculus: Logical Foundations", *Formal Aspects of Computing*, vol. 9, 1997, p. 283–330.
- [HOP 79] HOPCROFT J., ULLMAN J., *Introduction to Automata Theory, Languages, and Computation*, Addison-Wesley, 1979.
- [HUM 79] HUMBERSTONE L., "Interval semantics for tense logics: some remarks", *Journal of Philosophical Logic*, vol. 8, 1979, p. 171–196.
- [HUN 99a] HUNG D. V., "Projections: A Technique for Verifying Real-Time Programs in Duration Calculus", report num. 178, 1999, UNU/IIST, Macau
- [HUN 99b] HUNG D. V., CHAOCHEN Z., "Probabilistic duration calculus for continuous time", *Formal Aspects of Computing*, vol. 11, num. 1, 1999, p. 21–44.
- [HUN 99c] HUNG D. V., GUELEV D., "Completeness and Decidability of a Fragment of Duration Calculus with Iteration", in: VIJAGARAJAN P., YAP R., Eds., *Advances in Computing Science*, vol. 1742 of LNCS, Springer-Verlag, 1999, p. 139–150.
- [HUN 02] HUNG D. V., "Real-time Systems Development with Duration Calculus: an Overview", report num. 255, 2002, UNU/IIST, Macau.
- [HUO 02] HUONG H. V., HUNG D. V., "Modelling Real-time Database Systems in Duration Calculus", report num. 260, 2002, UNU/IIST, Macau.
- [KAM 79] KAMP H., "Events, Instants and temporal reference", in: AUERLE R., EGLI U., VON STECHOV A., Eds., *Semantics from Different Points of View*, Springer, 1979, p. 376–417.
- [KON 92] KONO S., "Automatic Verification of Interval Temporal Logic", *Proceedings of the 8th British Colloquium for Theoretical Computer Science*, 1992.
- [KON 95] KONO S., "A Combination of Clausal and Non-Clausal Temporal Logic Programs", in: FISHER M., OWENS R., Eds., *Executable Modal and Temporal Logics*, num. 897 LNCS, Springer, 1995, p. 40–57.
- [LAD 87] LADKIN P., "The Logic of Time Representation", PhD thesis, University of California, Berkeley, 1987.
- [LOD 00] LODAYA K., "Sharpening the Undecidability of Interval Temporal Logic", *Proc. of 6th Asian Computing Science Conference*, num. 1961 LNCS, Springer, 2000, p. 290–298.
- [MAD 92] MADDUX R., "Relation Algebras of Every Dimension", *Journal of Symbolic Logic*, vol. 57, num. 4, 1992, p. 1213–1229.
- [MAR 97] MARX M., VENEMA Y., *Multi-Dimensional Modal Logics*, Kluwer Academic Press, 1997.
- [MAR 99] MARX M., REYNOLDS M., "Undecidability of Compass Logic", *Journal of Logic and Computation*, vol. 9, num. 6, 1999, p. 897–914.
- [MON 96] MONTANARI A., "Metric and Layered Temporal Logic for Time Granularity", PhD thesis, Institute for Logic, Language, and Computation, University of Amsterdam, 1996.
- [MON 02] MONTANARI A., SCIAVICCO G., VITACOLONNA N., "Decidability of Interval Temporal Logics over Split-Frames via Granularity", *Proc. the European Conference on Logic in Artificial Intelligence 2002*, num. 2424 LNAI, Springer, 2002, p. 259–270.
- [MOS 83] MOSZKOWSKI B., "Reasoning about Digital Circuits", PhD thesis, Department of Computer Science, Stanford University, Technical Report CS-83-970, Stanford, CA, 1983.

- [MOS 84] MOSZKOWSKI B., MANNA Z., "Reasoning in Interval Temporal Logic", LARKE E. M., KOZEN D., Eds., *Proc. of the 1983 Workshop on Logics of Programs*, num. 164 LNCS, Springer, 1984, p. 371–381.
- [MOS 86] MOSZKOWSKI B., *Executing Temporal Logic Programs*, Cambridge University Press, Cambridge, 1986.
- [MOS 94] MOSZKOWSKI B., "Some Very Compositional Temporal Properties", LARKE E., Ed., *Programming Concepts, Methods and Calculi*, IFIP Transaction, A-56, Elsevier Science B.V. (North-Holland), 1994, p. 238–245.
- [MOS 96a] MOSER L. E., MELLIAR-SMITH P. M., RAMAKRISHNA Y. S., KUTTY G., DILLON L. K., "Automated Deduction in a Graphical Temporal Logic", *Journal of Applied Non-Classical Logics*, vol. 6, num. 1, 1996, p. 29–48.
- [MOS 96b] MOSZKOWSKI B., "The Programming Language Tempura", *Journal of Symbolic Computation*, vol. 22, num. 5/6, 1996, p. 730–733.
- [MOS 98] MOSZKOWSKI B., "Compositional Reasoning using Interval Temporal Logic and Tempura", DE ROEVER W., LANGMAAK H., PNUELI A., Eds., *Compositionality: the Significant Difference*, num. 1536 LNCS, Springer, 1998, p. 439–464.
- [MOS 00a] MOSZKOWSKI B., "An Automata-Theoretic Completeness Proof for Interval Temporal Logic", MONTANARI U., ROLIM J., WELZL E., Eds., *Proc. of the 27th International Colloquium on Automata, Languages and Programming*, num. 1853 LNCS, Springer, 2000, p. 223–234.
- [MOS 00b] MOSZKOWSKI B., "A complete axiomatization of Interval Temporal Logic in finite time", *Proc. of the 15th Annual IEEE Symposium on Logic in Computer Science LICS 2000*, IEEE Computer Society Press, 2000, p. 242–251.
- [MOS 03] MOSZKOWSKI B., "A Hierarchical Completeness Proof for Interval Temporal Logic with Finite Time (Preliminary Version)", GRANKO V., MONTANARI A., Eds., *Proc. of the ESSLLI Workshop on Interval Temporal Logics and Duration Calculi*, 2003, p. 41–65.
- [NAI 00] NAIJUN Z., "Completeness of Higher-Order Duration Calculus", *Proc. of the European Conference on Computer Science Logics, Fischbachan, Munich, Germany*, 2000, p. 22–27.
- [NEB 95] NEBEL B., BÜRCKERT H., "Reasoning About Temporal Relations: a Maximal Tractable Subclass of Allen's Interval Algebra", *Journal of the ACM*, vol. 42, num. 1, 1995, p. 43–66.
- [NIS 80] NISHIMURA H., "Interval Logic with applications to Study of Tense and Aspect in English", *Publications of the Research Institute for Mathematical Sciences*, vol. 16, 1980, p. 417–459.
- [PAN 96] PANDYA P., "Weak Chop Inverses and Liveness in Mean-Value Calculus", *Proc. of the 4th International Conference on Formal Techniques in Real-Time and Fault-Tolerant Systems*, num. 1135 LNCS, Springer, 1996, p. 148–167.
- [PEN 98] PENIX J., PECHEUR C., HAVELUND K., "Using Model Checking to Validate AI Planner Domain Models", *23rd Annual Software Engineering Workshop, NASA Goddard*, 1998.
- [PUJ 97] PUJARI A. K., "Neighbourhood Logic & Interval Algebra", report num. 116, 1997, UNU/IIST.

- [RAB 98] RABINOVICH A., "Non-Elementary Lower Bound for Propositional Duration Calculus", *Information Processing Letters*, vol. 66, 1998, p. 7–11.
- [RAM 92] RAMAKRISHNA Y. S., MOSER L. E., DILLON L. K., MELLIAR-SMITH P. M., KUTTY G., "An Automata-Theoretic Decision Procedure for Propositional Temporal Logic with Since and Until", *Fundamenta Informaticae*, vol. 17, 1992, p. 271–282.
- [RAM 93] RAMAKRISHNA Y., "Interval Logics for Temporal Specification and Verification", PhD thesis, University of California, Santa Barbara, 1993.
- [RAS 99] RASMUSSEN T., "Signed Interval Logic", *Proc. of the Annual European Conference on Computer Science Logic*, num. 1683 LNCS, Springer, 1999, p. 157–171.
- [RAS 01a] RASMUSSEN T., "A Sequent Calculus for Signed Interval Logics", report num. IMM-TR-2001-06, 2001, Informatics and Mathematical Modelling, Technical Univ. of Denmark.
- [RAS 01b] RASMUSSEN T., "Labelled Natural Deduction for Interval Logics", RIBBOURG L., Ed., *Proc. of the Annual European Conference on Computer Science Logic CSL'2001*, num. 2142 LNCS, Springer, 2001, p. 308–323.
- [RAS 02] RASMUSSEN T., "Interval Logic. Proof Theory and Theorem Proving", PhD thesis, Technical Univ. of Denmark, Ph.D. Thesis, IMM-PHD-2002-2002.
- [REI 47] REICHENBACH H., *Elements of Symbolic Logic*, The Free Press, New York, 1947.
- [RIC 88] RICHARDS B., BETHKE I., "The Temporal Logic IQ", report num. EUCCS-RP-1988-1, 1988, Centre for Cognitive Science, University of Edinburgh.
- [ROE 80] ROEPER P., "Intervals and Tenses", *Journal of Philosophical Logic*, vol. 9, 1980, p. 451–469.
- [ROS 86] ROSNER R., RUEL A., "A choppy logic", *First Annual IEEE Symposium on Logic in Computer Science LICS'86*, IEEE Computer Society Press, 1986, p. 306–313.
- [ROY 97] ROY S., CHAOCHEN Z., "Notes on Neighbourhood Logic", report num. 97, 1997, UNU/IIST Technical Report.
- [SIE 01] SEWE F., HUNG D. V., "Deriving Real-Time Programs from Duration Calculus Specifications", *Proc. of the 11th Advanced Research Working Conference on Correct Hardware Design and Verification Methods CHARME 2001*, vol. 2144 of LNCS, Springer-Verlag, 2001, p. 92–97.
- [SKA 94] SKAKKEBÆK J., SESTOFT P., "Checking validity of duration calculus formulas", report , 1994, Technical report, ProCoS II, ESPRIT BRA 7074, report no. ID/DTH JUS 3/1, Department of Computer Science, Technical University of Denmark.
- [SØR 90] SØRENSEN E., RAVN A., RISCHER H., "Control Program for a Gas Burner. Part I: Informal Requirements, Process Case Study 1", report 90,19, ProCoS Report ID/DTH EVS2.
- [THA 01] THAE H. K., HUNG D. V., "Formal Design of Hybrid Control Systems: Duration Calculus Approach", *Proc. of the Twenty-Fifth Annual International Computer Software and Applications Conference COMPSAC 2001*, IEEE Computer Society Press, 2001, p. 423–428.
- [VEN 90] VENEMA Y., "Expressiveness and Completeness of an Interval Temporal Logic", *Notre Dame Journal of Formal Logic*, vol. 31, num. 4, 1990, p. 529–547.
- [VEN 91] VENEMA Y., "A Modal Logic for Chopping Intervals", *Journal of Logic and Computation*, vol. 1, num. 4, 1991, p. 453–476.

- [VIL 86] VILAIN M., KAUTZ H., "Constraint propagation algorithms for temporal reasoning", *Proc. of the 5th National Conference of the American Association for Artificial Intelligence (AAAI)*, AAAI Press, 1986, p. 377–382.
- [WOL 85] WOLPER P., "The Tableau Method for Temporal Logic: An Overview", *Logique et Analyse*, vol. 28, 1985, p. 119–136.
- [YON 02] YONG L., HUNG D. V., "Checking Temporal Duration Properties of Timed Automata", *Journal of Computer Science and Technology*, vol. 17, num. 6, 2002, p. 689–698.