Available online at www.sciencedirect.com

ScienceDirect

Procedia Engineering 101 (2015) 34 – 42

Procedia Engineering

www.elsevier.com/locate/procedia

3rd International Conference on Material and Component Performance under Variable Amplitude Loading, VAL2015

Basic principles of spectral multi-axial fatigue analysis

Denis Benasciutti[a,*], Frank Sherratt[b], Alessandro Cristofori[c]

[a]*University of Udine, Dept. Electrical Managment Mechanical Engineering (DIEGM), via delle Scienze 208, 33100 Udine, Italy*
[b]*Engineering consultant, Rugby, CV23 0LG, United Kingdom*
[c]*University of Ferrara, Dept. of Engineering, via Saragat 1, 44122 Ferrara, Italy*

Abstract

This paper presents a discussion on multi-axial fatigue criteria for random stresses which are formulated as spectral methods in the frequency-domain. Spectral solutions to the multi-axial problem can be viewed as a natural step forward from uni-axial spectral methods. The paper attempts to identify and critically analyze some underlying principles that join together different multi-axial spectral methods. General remarks on advantages and drawbacks of multi-axial spectral methods from the literature are pointed out and commented on.

Peer-review under responsibility of the Czech Society for Mechanics

Keywords: multi-axial fatigue, random stress, Power Spectral Density, spectral methods

1. Introduction

Life prediction under multi-axial loading has been an active research topic for more than fifty years and activity has increased in the last decade. One reason may be the increased use of Finite Element (FE) analysis to design loaded components. A normal FE suite gives the full stress tensor at many points in a loaded body. FE-based fatigue calculations may become highly time-consuming in mechanical components under multi-axial loadings, due to the processing time needed to analyze random stress time histories simulated in finely discretized 3D FE models having

* Corresponding author. Tel.: +39-0432-558048; fax: +39-0432-558251.
E-mail address: denis.benasciutti@uniud.it

Peer-review under responsibility of the Czech Society for Mechanics
doi:10.1016/j.proeng.2015.02.006

hundreds of thousands of nodes. Allowing for the effect of all the components in the stress tensor must be accomplished with as little increase as possible in computation time.

Frequency-domain spectral methods seem to offer this, and have been suggested as a valid alternative to time-domain ones. Over slightly more than a decade the role of spectral analysis in the fatigue field has also changed. Random loading, traditionally called "vibration", can now be measured on loaded components and quantified in various ways, particularly its Power Spectral Density (PSD). Ways of estimating life from PSD have been developed, confirmed in the laboratory and incorporated in legal standards, mainly for uni-axial random loading. Deriving spectral solutions to the multi-axial problem is an obvious (although not often simple) step forward and this is a major activity for several research groups. Underlying principles are beginning to emerge, which the paper attempts to codify and to critically analyze. Experience gained in the development of uni-axial random fatigue is considered and then extended to multi-axial case.

2. Motivation for spectral multi-axial fatigue analysis

When estimating fatigue damage and service life under random loadings, especially at early design stage, designers and engineers have to choose the approach that mixes high accuracy and short analysis times. While the use of algorithms in time-domain has been gradually consolidated in design, the spectral methods in frequency-based fatigue analysis have often been viewed with mistrust. When approaching spectral fatigue analysis for the first time, designers are often tempted to question whether frequency-domain methods do really provide some advantages to classical time-domain analysis. An answer could be given by looking at the following issues: i) type of loading (deterministic or random), ii) analysis environment (FE), iii) number of stress components (multi-axial fatigue).

Regarding the loads, a distinction has to be made between deterministic and random loadings. While deterministic loads are exactly known from present or past values, random loads are inherently uncertain (this is "aleatory" variability) and future values can only be estimated in terms of probability. For example, typical questions in structural durability could be: which is the probability that a load will exceed a given threshold "x" in future time T? How many cycles have amplitudes higher than a given value?

From the designer's perspective, the matter is to understand which approach (time- or frequency-domain) is best suited for fatigue analysis of multi-axial random loadings, especially at the design phase within a FE environment.

The time-domain approach looks directly at the random stress time histories – measured (strain gages) or simulated (FE) – and it computes the quantities relevant to fatigue by using step-by-step algorithms, e.g. rainflow counting of uni-axial stress time histories, critical plane search with multi-axial stresses. Statistical inference on observed sample data is often used to estimate, for example, the probability distribution of counted cycles or extrapolate the loading spectra to component service life.

The length of time histories and the computational analysis time are the main drawbacks of time-domain approach applied to random loadings. For instance, very long stress time-histories are needed to gather sufficiently large samples that can assure statistical convergence and small scatter of estimators. The simulation of large stress time histories can increase dramatically the simulation time in large or even medium FE models, especially if material non-linearity or contact elements are included. In such FE models, also the algorithms implementing critical plane multi-axial fatigue criteria may require processing times that become really prohibitive (e.g. thousands of planes have to be searched at every node in the FE model).

The finite length of observed time histories gives raise to another problem: no information is actually gathered outside the observed sample and the observation time interval. This issue is particularly relevant for service life assessment, which may be controlled by the occurrence probability of large amplitude cycles or overloads. Obviously, the longer the observed stress time-history, the higher the information gathered. Unfortunately, the computational problems previously pointed out make the choice of time-history length a trade-off between accuracy and processing time. Extrapolation strategies could be used to infer about rare events.

The previous issues could become even more critical when dealing with multi-axial stresses, since almost all multi-axial criteria have been developed as time-domain algorithms. On the other hand, the reformulation from time- to frequency-domain is not straightforward as a statistical redefinition of multi-axial criteria is often needed.

The limitations of probabilistic analysis in time-domain can be overcome by moving to frequency-domain, where the multi-axial random stress is characterized by a Power Spectral Density (PSD) functions matrix $\mathbf{S}(\omega)$, which is formally the Fourier transform of the autocorrelation matrix $\mathbf{R}(\delta)$ (Wiener-Khintchine theorem):

$$\mathbf{S}(\omega) = \Im\left\{ \mathbf{R}(\delta) \right\} \tag{1}$$

Expression (1) allows the time-domain properties of multi-axial random stress to be transferred from time-domain into frequency-domain. Broadly speaking, it can be said that time-domain and frequency-domain represents the two opposite sides of the same coin (which is the random signal).

Compared to time-domain analysis, frequency-based spectral analysis has several advantages. Shorter time-histories are sufficient to get an estimate the spectrum, since suitable averaging techniques (e.g. Welch's method) can be used to reduce the scatter in estimated PSD. Secondly, the scatter in the estimated PSD is less relevant, since spectral methods refer to moments λ_n and bandwidth parameters α_m, which are integral averages of PSD. For stationary Gaussian processes, some quantities relevant to fatigue (e.g. peaks/valleys distribution, frequency of peaks and up-crossings, irregularity factor) can be estimated directly from the PSD by simple analytical expressions [1].

Finally, the main advantage of spectral analysis is the possibility to get analytical expressions to estimate the probability distribution of rainflow cycles and the fatigue damage under Palmgren-Miner hypothesis, directly from PSD data. For multi-axial random stresses, spectral analysis also allows time-domain criteria to be reformulated in frequency-domain. This makes the spectral approach very powerful with FE analysis, especially when combined with frequency-domain analysis at the early design stage (quick estimation of the overall structural response).

3. Classification of multi-axial fatigue criteria

Multi-axial spectral methods can be classified similarly to time-domain multi-axial criteria: equivalent stresses, critical plane criteria, invariants-based criteria, energy-based criteria. This Section provides a brief overview on some selected multi-axial spectral methods.

3.1. Criteria based on equivalent stress

The idea of using equivalent stresses in multi-axial fatigue is rather old and it is suggested by the theories for static loads (e.g. von Mises criterion). The quadratic nature of von Mises stress, however, brings about several problems with random stresses: its PSD is not consistent with that of the stress components (frequency doubling and shifting), von Mises stress is always non-Gaussian and with positive mean, even with zero-mean and Gaussian stress components [2]. These limitations have been overcome by Preumont et al. with the definition of an "equivalent von Mises stress" (EVMS), in which the PSD is computed by applying the von Mises coefficients to the stress tensor PSDs [3]. An example for a biaxial stress is ($\mathbf{Q}$ is a matrix of numbers) [3]:

$$S_{\mathrm{vm}}(\omega) = \mathrm{trace}\left\{\mathbf{Q}\mathbf{S}(\omega)\right\} = S_{\mathrm{xx}}(\omega) + S_{\mathrm{xx}}(\omega) - \mathrm{Re}\left\{S_{\mathrm{xx,yy}}(\omega)\right\} + 3S_{\mathrm{xy}}(\omega) \tag{2}$$

The EVMS by Preumont is not exactly the von Mises stress at all; instead, it is a stress that has the qualities they were looking for: zero-mean Gaussian process, with variance equal to the same variance of stress components. Despite its simple definition, which gave wide use in academia and industry, the EVMS by Preumont is based on some inherent assumptions that can lead to large errors for some types of materials [2]. A modified version of Premont's definition, given by Niesłony (see [4]), has partly solved this drawback and it has been shown to provide good estimates compared to experimental results for bending-torsion random loading, see Fig. 1.

Another example of equivalent stress that is worth mentioning is the normal or resolved shear stress on the critical plane, which is used in the critical plane approach by Macha et al., which is discussed in Section 3.2.

The use of uni-axial equivalent stress opens the possibility to apply uni-axial spectral methods (e.g. Rayleigh formula, Dirlik expression, Tovo-Benasciutti (TB) method, or others) for estimating the fatigue damage and life of multi-axial stress. The accuracy of the multi-axial spectral method obviously depends on the particular definition of the equivalent stress that is adopted, as well as on the accuracy of the uni-axial spectral method that provides the estimated damage. Although different spectral methods are available in the literature, simulation studies by various independent researchers [1,5,6] have confirmed that some uni-axial spectral methods (Dirlik [7], TB method [1]) have an accuracy far superior to others, especially for wide-band frequency spectra. The significant point here, though, is not which uni-axial formula gives good life predictions but whether or not a particular proposal for 'equivalence' allows those expressions to make good estimates. Niesłony et al. [4] report tests showing successful use of Dirlik and TB formulae following application of one particular equivalence criterion. Fig. 1 shows the results.

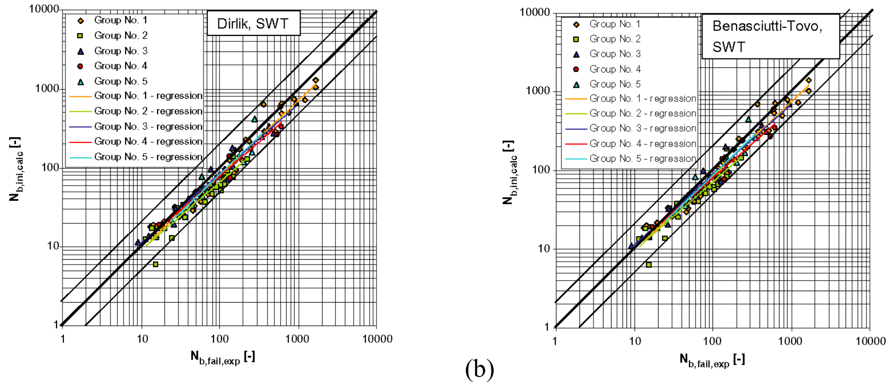

Fig. 1. Comparison of experimental results with theoretical estimations obtained by the equivalent strain related to the SWT method, applied with two different spectral methods: (a) Dirlik; (b) Tovo-Benasciutti (TB). (Reprinted from Niesłony et al., 2012 [4])

3.2. Critical plane criteria

These criteria look at the plane where crack nucleation and propagation is likely to occur, which is the plane of highest shearing stress amplitude during crack initiation and highest normal stress amplitude during propagation. Therefore, these criteria are interested in the stresses acting on the critical plane: normal stress σ, shear stress τ, or their combinations. Most multi-axial damage expressions in critical plane criteria then contain a normal stress, σ, and a shearing stress, τ. The ratio σ/τ will then depend on the instantaneous values of the loads on the two (or three) axes, in addition to the variation due to planar orientation.

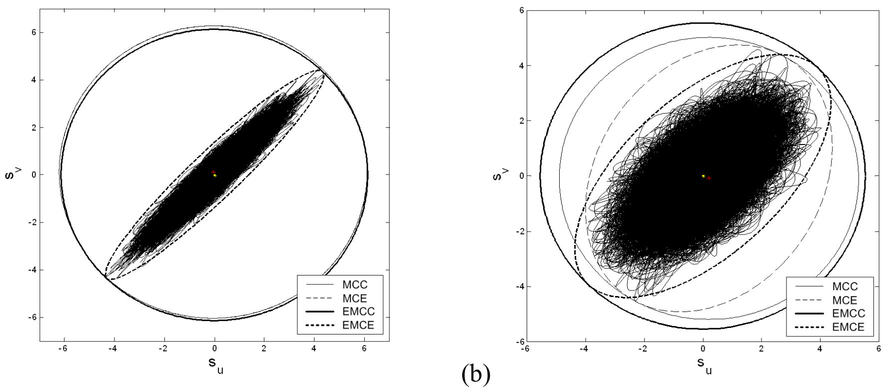

Fig. 2. Comparison of MCC/MCE with their spectral definitions for random stress paths with different correlation degree: (a) r=0.95; (b) r=0.6. (Reprinted from Benasciutti and Cristofori, 2008 [9])

The shear stress on the critical plane is a vector process $\boldsymbol{\tau}(t)$ that describes a curve Ψ called stress path, which is closed (for periodic stresses) or entangled (for random stresses). The definition of the amplitude τ_a of the curve Ψ is non trivial and it is generally solved by conventional rules, as the Minimum Circumscribed Circle (MCC) or Ellipse (MCE), where τ_a is the radius or semi-axis of circumscribed circle/ellipse. A statistical definition of MCC/MCE methods, however, is necessary when analyzing random stresses. For example, an expected radius $E[R]$ based on the "maximum variance" concept is an intuitive measure of the expected amplitude $E[\tau_a]=E[R]$ on the critical plane. The critical plane is then defined as the plane with the maximum expected radius $E[C_a]=E[R]$.

Spectral analysis provides a simple theoretical framework to compute $E[\tau_a]$ from the stress PSD matrix $\mathbf{S}(\omega)$ [8]. This approach has been extended in [9] also to the MCE concept, to account for non-proportional stress paths. The estimations by the spectral definitions of MCC/MCE only depend on the statistical properties of the multi-axial random stress and they can mitigate the high statistical scatter of time-domain algorithms, as confirmed by numerical simulations (Fig. 2) [9]. For example, the time-domain MCE algorithm can provide a wrong measure of the degree of non-proportionality (for example, in Fig. 2(a) the ellipse is close to the circle), which is not observed in the spectral definition.

The spectral definition of MCC has been used by Preumont et al. to reformulate in frequency-domain the critical plane criterion of Matake, which provides computational advantages compared to time-domain algorithm [8].

An easier approach, instead, is to look at a resolved stress on the critical plane, which is an equivalent uni-axial stress that is computed by a linear combination of stress tensor components s_i (normal and shear stress) [10]:

$$s_{eq}(t)=\sum_{i=1}^{6} a_i\, s_i(t) \qquad\qquad S_{eq}(\omega)=\sum_{h=1}^{6}\sum_{k=1}^{6} a_h a_k S_{hk}(\omega) \tag{3}$$

as a function of direction cosines a_i. Linearity assures that the PSD of the equivalent stress is a linear combination of frequency spectra $S_{hk}(\omega)$ of stress tensor components s_i. Compared to Matake's spectral method by Preumont, this approach seems conceptually easier, as it is more similar to the multi-axial spectral methods based on "equivalent stress" discussed in Section 3.1.

3.3. Invariants-based criteria

This class of multi-axial criteria refers to the second invariant of the deviator stress tensor. Crossland's criterion is probably the most known and it has been first reformulated in frequency-domain by Preumont et al. [8], by adopting the same statistical definition of MCC method used for Matake's criterion, see Section 3.2.

More recently, a new stress invariants multi-axial criterion has been proposed first in time-domain [11] and then in frequency-domain [12,13]. The approach is called "Projection-by-Projection", since the damage of the stress path $d(\Psi)$ is computed by summing-up the damage values of stress projections Ψ_i in a "principal reference frame" of maximum variance. The method provides a closed form analytical expression to compute the fatigue damage of multi-axial stress, which depends on the uni-axial spectral method used to estimate damage $d(\Psi_i)$ for each projection. For example, by using the TB method, the damage is:

$$d_{TB}^{PbP}(\Psi)=\left[\sum_{i=1}^{5}\left(d(\Psi_i)\right)^{2/k_{ref}}\right]^{\frac{k_{ref}}{2}}=c_{ref}^{-1}\Gamma\left(1+\frac{k_{ref}}{2}\right)\left[\sum_{i=1}^{5}\left(2\lambda_{0,i}\right)\sqrt[k_{ref}]{\eta_{TB,i}\nu_{0,i}}\right]^{\frac{k_{ref}}{2}} \tag{4}$$

Nevertheless, the theoretical framework of PbP method can easily be extended to other uni-axial spectral methods (e.g. Dirlik expression). The PbP criterion has been applied to study an L-shaped steel beam under band-limited acceleration [13]. In some way, also the PbP method can be interpreted as an "equivalence criterion" that transforms a multi-axial stress into uni-axial random stresses in the principal reference frame.

4. Special features of multi-axial spectral methods

4.1. Importance of material fatigue properties

A multi-axial state of stress becomes uni-axial when only one stress component is present. The estimation by any multi-axial criterion must always be consistent with that obtained by the SN line for simple uni-axial stress (e.g. pure torsion, pure axial/bending loading). Incorrect estimations are expected whenever this condition is not met.

An interesting and quite emblematic example is the "equivalent von Mises stress" criterion of Preumont et al. [3]. It has recently been shown [2] that, for a pure random torsion loading, Preumont's criterion gives the following ratio:

$$r_{\mathrm{d}} = \frac{d_{\mathrm{eq}}}{d_{\tau}} = \frac{\tau_{\mathrm{A}}^{k_{\tau}}}{\left(\sigma_{\mathrm{A}}/\sqrt{3}\right)^{k_{\sigma}}} \frac{\Gamma\left(1+k_{\sigma}/2\right)}{\Gamma\left(1+k_{\tau}/2\right)} \left(\sqrt{2V_{\tau}}\right)^{k_{\sigma}-k_{\tau}} \tag{5}$$

to the damage d_τ calculated on the SN line for torsion loading (which is the "true" estimation). Symbols σ_A, τ_A are the reference fatigue strengths at $N_A=2\times10^6$ cycles, k_σ, k_τ are the inverse slopes of SN lines (Fig. 3(a)), while V_τ is the variance of the applied shear stress by random torsion. A damage ratio r_d of unity means a correct estimation, regardless of the parameter values in Eq. (5). However, Benasciutti [2] showed that a good accuracy is obtained only when $k_\sigma=k_\tau$ and $\sigma_A=\sqrt{3}\tau_A$, which however is an exception rather than the rule in typical material (see [2,14]). Very large errors are obtained for typical combinations of material fatigue parameters, see Fig. 3(b). On the other hand, the accuracy of Preumont's approach for more complex multi-axial stresses could not be higher than the accuracy that characterizes a simple random torsion loading.

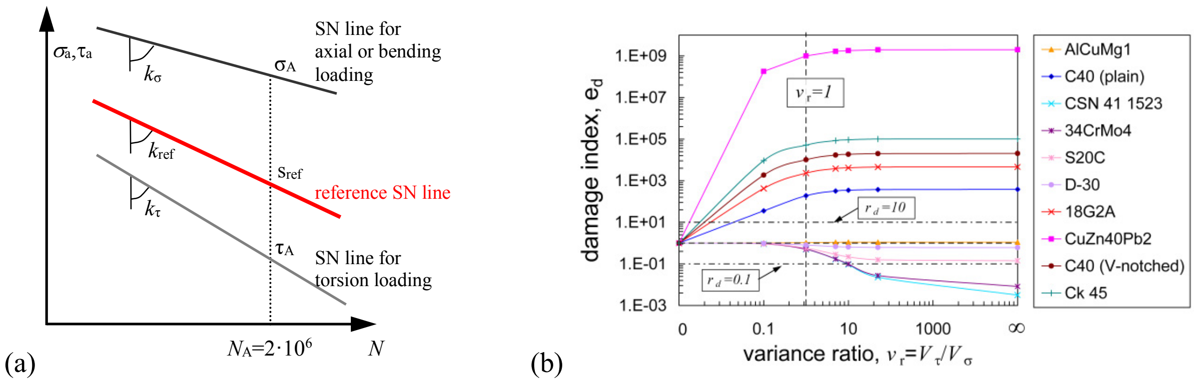

Fig. 3. (a) the Modified Wöhler Diagram; (b) damage ratio r_d in Eq. (5) for different types of materials. (Reprinted from Benasciutti, 2014 [2])

The errors of Preumont's criterion pointed out in [2] can be explained by considering that only the SN line for normal stress is used in fatigue damage computation for the equivalent stress, while the torsion fatigue properties are completely neglected. A simple way to overcome this limitation and to include any combination of material fatigue properties could be the use of a "reference SN line" that is calibrated on the SN lines for both normal and shear stress in a so-called Modified Wöhler Diagram (MWD), see Fig. 3(a), invented by Susmel and Lazzarin [15]. As an example, the PbP criterion in Eq. (4) adopts this formalism and it is shown to provide more consistent results [13].

4.2. Rotation of principal stress directions: non-proportional loadings

The multi-axial case is further complicated when principal axes of stress at a point rotate. A very simple case of this occurs in a cylindrical component carrying torsion, M_T, about the longitudinal x axis and bending, M_B, about the y one. Conventional calculations then give the normal and shearing stresses at any point on the face with normal x. At any instant the orientation of the principal stresses will depend on the ratio of normal stress to shearing stress on that face. This in turn will depend on the ratio of the applied loads, M_T/ M_B at that instant. If in the time domain both torsion and bending follow constant amplitude sinusoidal cycles of the same frequency, and both are zero at time zero, this ratio will not vary in the time domain. Introducing a phase shift φ by, for instance, starting with the torsion input at maximum value when bending is zero, causes the ratio to vary with time, thus giving rotating principal axes.

Load regimes which do not cause rotation of principal axes are termed proportional and ones which do are non-proportional. In constant amplitude loading, the phase shift φ between two harmonic (sinusoidal) loadings with the same frequency gives a simple measure of the degree of non-proportionality. A phase shift φ=0 is for in-phase (proportional) loading, while $\varphi\neq 0$ (typically φ=90°) is for out-of-phase (non-proportional) loading. A random loading, though, is formed by superposition of several harmonics with different phase shift and frequencies, usually spread over a wide range. By using a single phase angle φ, the random loading would be inherently non-proportional. However, in random loading a single phase shift φ is unsuitable to correctly quantify the degree of proportionality and it has to be replaced by its statistical counterpart, i.e. the correlation coefficient. Given two random loadings (x_h, x_k), the correlation coefficient is $r_{hk}=C_{hk}/\sqrt{(V_h V_k)}$, where C_{hk} is the covariance and V_h, V_k the variances. Proportional loading have r_{hk}=+1 (principal axes of stress are fixed in "average"), while non-proportional ones have $r_{hk}\neq 1$ (principal axes rotate) – typical values are r_{hk}=0 and r_{hk}=-1.

As for constant amplitude loading, the rotation of principal axes is strictly related to the degree of proportionality. In random loading, though, the angle θ of principal axes is randomly distributed over time. This means that, even in a proportional loading (r_{hk}=+1), the principal axes are never exactly fixed (the angle θ is not constant) and they are fixed only in "average" (with only small changes, yet), while in non-proportional loadings (r_{hk}=0) the values of angle θ are spread over a wider interval, compared the two distributions in Fig. 4(b) and (c).

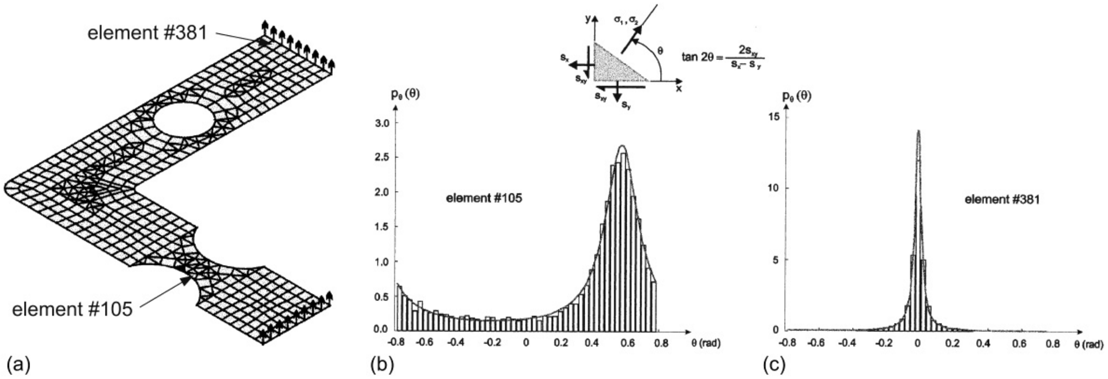

Fig. 4. Probability distribution of angle θ of principal directions at two different locations in a L-shaped beam under random excitation (see (a)); (b) non-proportional stress; (c) proportional stress. (Reprinted from Pitoiset et al., 2001 [8])

The degree of non-proportionality can vary from point to point in a mechanical component having a complex geometry or stress raisers. This creates a demand for methods for a rapid scan of FE models to measure the stress correlation and orientation of principal stress directions. Spectral methods seem to be very promising. For example, [8] developed a spectral approach to compute the probability distribution of the angle θ of principal stress directions, which has been applied to the L-shaped beam in Fig. 4. The comparison shows that the principal directions of stress in element no. 381 tend to be fixed (the distribution is peaked near θ=0), while in element no. 105 they rotate.

The information on the change of the principal directions of stress can be also useful to discriminate among multi-axial criteria. In fact, the literature seems to suggest that the validity of multi-axial criteria based on critical plane or stress invariants is generally related to evolution of principal axes. Critical plane criteria are more suitable for fixed principal directions, while invariants-based criteria are applicable to rotating principal stress directions.

On the other hand, the rotation of principal axes of stress has been found to cause changes in the rate of damage accumulation and to strongly affect the overall fatigue strength of engineering materials. The correlation also has a strong influence on the crack patterns. As documented in literature for constant amplitude loading, no general rule can be established, yet. For example, it has been observed that the presence of out-of-phase loading does not always result in a decrease of fatigue life, as commonly presumed (see [15]). This can be explained by the complex interaction involving the cyclic change of principal directions of stress, cyclic plasticity at microscopic level and intrinsic material ductility. Similar considerations seem to be true also for random loading. For example, some experimental data show either a decrease or even an increase in fatigue life for increasing stress correlation [10,16,17]. The above remarks make evident that "*only by running appropriate experiments can the actual material response to non-proportional loading be correctly evaluated*" [15].

5. Concluding remarks

Spectral multi-axial fatigue analysis has some advantages compared to time-domain approach: it gives a tool to scan critical plane orientation or to estimate fatigue damage from spectral properties of random stress, this proves to be particularly useful for a reduction of computational time in FE analysis. In addition, some analogies between uni-axial and multi-axial spectral methods, pointed out recently [18], open a new interesting perspective to improve uni-axial spectral methods. Some critical remarks, however, are worth mentioning. In spectral analysis, the stress is often modeled as a stationary Gaussian random process, as this greatly simplifies the theory. One may wonder whether this assumption is really satisfied by measured signals and to what extent the model fits reality. For examples, the stress can often deviate from Gaussianity (due to structural non-linearity), or from stationarity (e.g. overloads). Optimistically, the signal is almost stationary only over short-time intervals. Similarly, spectral methods are based on Palmgren-Miner rule, which many experimental data in literature have shown to be highly non-conservative (although this damage rule is currently included, for example, in Eurocode 3). However, extension to non-linear damage rules would greatly increase the complexity of spectral methods (for instance, how to account for sequence effects?). It has to be admitted very honestly that non-linear damage accumulation in spectral fatigue is a topic far from being solved. Practical reasons, however, often suggest some hypotheses as taken for granted, as no alternatives solutions are available. However, we recommend the designer to be always aware of the assumptions in the spectral method he/she is currently using; otherwise calculations would only be a fruitless exercise. Last, but not least, the importance of experimental testing with different types of multi-axial random loading, as a benchmark to validate the accuracy of spectral approaches, must always be emphasized.

References

[1] D. Benasciutti, R. Tovo, Comparison of spectral methods for fatigue analysis of broad-band Gaussian random processes, Probab. Eng. Mech. 21(4) (2006) 287–299.
[2] D. Benasciutti, Some analytical expressions to measure the accuracy of the "equivalent von Mises stress" in vibration multiaxial fatigue, J. Sound Vib. 333(18) (2014) 4326–4340.
[3] X. Pitoiset, A. Preumont, Spectral methods for multiaxial random fatigue analysis of metallic structures, Int. J. Fatigue 22(7) (2000) 541–550.
[4] A. Niesłony, M. Růžička, J. Papuga, A. Hodr, M. Balda, J. Svoboda, Fatigue life prediction for broad-band multiaxial loading with various PSD curve shapes, Int. J. Fatigue 44 (2012) 74–88.
[5] Z. Gao, T. Moan, Frequency-domain fatigue analysis of wide-band stationary Gaussian processes using a trimodal spectral formulation, Int. J. Fatigue 30(10–11) (2008) 1944–1955.
[6] M. Mršnik, J. Slavič, M. Boltežar, Frequency-domain methods for a vibration-fatigue-life estimation - Application to real data, Int. J. Fatigue 47 (2013) 8–17.
[7] T. Dirlik, Application of computers in fatigue analysis, Ph.D. Thesis, Dept. of Engineering, University of Warwick, UK, 1985.

[8] X. Pitoiset, I. Rychlik, A. Preumont, Spectral methods to estimate local multiaxial fatigue failure for structures undergoing random vibrations, Fatigue Fract. Eng. Mater. Struct. 24 (2001) 715–727.
[9] D. Benasciutti, A. Cristofori, A frequency-domain formulation of MCE method for multi-axial random loadings, Fatigue Fract. Eng. Mater. Struct. 31(11) (2008) 937–948.
[10] T. Łagoda, E. Macha, A. Niesłony, Fatigue life calculation by means of the cycle counting and spectral methods multiaxial random loading, Fatigue Fract. Eng. Mater. Struct. 28 (2005) 409–420.
[11] A. Cristofori, L. Susmel, R. Tovo, A stress invariant based criterion to estimate fatigue damage under multiaxial loading, Int. J. Fatigue 30(9) (2008) 1646–1658.
[12] A. Cristofori, D. Benasciutti, R. Tovo, A stress invariant based spectral method to estimate fatigue life under multiaxial random loading, Int. J. Fatigue 33(7) (2011) 887–899.
[13] A. Cristofori, D. Benasciutti, "Projection-by-Projection" approach: a spectral method for multiaxial random fatigue. SAE Technical Paper 2014–01–0924.
[14] M. Kurek, T. Łagoda, Comparison of the fatigue characteristics for some selected structural materials under bending and torsion, Mater. Sci. 47(3) (2011) 334–344.
[15] L. Susmel, Multiaxial notch fatigue, Woodhead Publishing Limited, UK, 2009.
[16] Z. Marciniak, D. Rozumek, E. Macha, Fatigue lives of 18G2A and 10HNAP steels under variable amplitude and random non-proportional bending with torsion loading, Int. J. Fatigue 30(5) (2008) 800–813.
[17] T. Łagoda, E. Macha, Fatigue life under biaxial stress state with different cross-correlation coefficients of normal stress, Proc. 9th Int. Conf Fatigue of metallic and non-metallic materials and structures, vol. 3, 1371–1378.
[18] D. Benasciutti, A. Cristofori, R. Tovo, Analogies between spectral methods and multiaxial criteria in fatigue damage evaluation, Probab. Eng. Mech. 31 (2013) 39–45.