Journal of Dynamics and Differential Equations
https://doi.org/10.1007/s10884-022-10200-y

Unbounded Solutions to a System of Coupled Asymmetric Oscillators at Resonance

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Received: 28 April 2022 / Revised: 14 July 2022 / Accepted: 28 July 2022

Abstract

We deal with the following system of coupled asymmetric oscillators

$$\begin{cases} \ddot{x}_1 + a_1 x_1^+ - b_1 x_1^- + \phi_1(x_2) = p_1(t) \\ \ddot{x}_2 + a_2 x_2^+ - b_2 x_2^- + \phi_2(x_1) = p_2(t), \end{cases}$$

where $\phi_i : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous and bounded, $p_i : \mathbb{R} \to \mathbb{R}$ is continuous and 2π-periodic and the positive real numbers a_i, b_i satisfy

$$\frac{1}{\sqrt{a_i}} + \frac{1}{\sqrt{b_i}} = \frac{2}{n}, \quad \text{for some } n \in \mathbb{N}.$$

We define a suitable function $L : \mathbb{T}^2 \to \mathbb{R}^2$, appearing as the higher-dimensional generalization of the well known resonance function used in the scalar setting, and we show how unbounded solutions to the system can be constructed whenever L has zeros with a special structure. The proof relies on a careful investigation of the dynamics of the associated (four-dimensional) Poincaré map, in action-angle coordinates.

Keywords Systems of ODEs · Asymmetric oscillators · Unbounded solutions · Resonance

Mathematics Subject Classification 34C11 · 34C15

1 Introduction

In this paper, we investigate the existence of unbounded solutions for a system of coupled asymmetric oscillators of the type

Under the auspices of INdAM-GNAMPA, Italy. In particular, the first author acknowledges the support of the GNAMPA Project 2020 "Problemi ai limiti per l'equazione della curvatura media prescritta".

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Published online: 16 August 2022

$$\begin{cases} \ddot{x}_1 + a_1 x_1^+ - b_1 x_1^- + \phi_1(x_2) = p_1(t) \\ \ddot{x}_2 + a_2 x_2^+ - b_2 x_2^- + \phi_2(x_1) = p_2(t), \end{cases} \tag{1.1}$$

where, as usual, $x^\pm = \max\{\pm x, 0\}$ and, for $i = 1, 2$, $\phi_i : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous and bounded, $p_i : \mathbb{R} \to \mathbb{R}$ is continuous and 2π-periodic. As for the positive real numbers a_i, b_i, we assume that

$$\frac{1}{\sqrt{a_i}} + \frac{1}{\sqrt{b_i}} = \frac{2}{n}, \qquad \text{for some } n \in \mathbb{N}, \tag{1.2}$$

thus implying that each oscillator is at resonance with respect to the same curve of the Fucik spectrum [11].

The study of unbounded solutions for oscillators at resonance is a classical topic in the qualitative theory of ordinary differential equations and we refer to [15] for an excellent survey on this subject. In order to motivate our contribution, the crucial reference to be recalled here is the seminal paper [2] by Alonso and Ortega. It is proved therein (cf. [2, Theorem 4.1]) that, for the scalar asymmetric oscillator

$$\ddot{x} + ax^+ - bx^- = p(t), \qquad x \in \mathbb{R}, \tag{1.3}$$

with $1/\sqrt{a} + 1/\sqrt{b} = 2/n$, all large solutions are unbounded (either in the past or in the future) whenever the 2π-periodic function

$$\Phi(\theta) = \int_0^{2\pi} C\left(\frac{\theta}{n} + t\right) p(t)\, dt, \qquad \theta \in \mathbb{R} \tag{1.4}$$

has zeros, all simple (in the above formula, C stands for the asymmetric cosine function, cf. Sect. 2.1). The function Φ, sometimes referred to as resonance function, was previously introduced by Dancer [7] to investigate the 2π-periodic solvability of equation (1.3). In the linear case ($a = b = n^2$), the function Φ has (simple) zeros if and only if $\int_0^{2\pi} p(t)e^{-\mathrm{i}nt}\, dt \neq 0$: in this case, as well known, all the solutions of $\ddot{x} + n^2 x = p(t)$ are unbounded; instead, 2π-periodic and unbounded solutions to (1.3) can coexist in the genuinely asymmetric case $a \neq b$. The proof of this result was obtained by a careful investigation of the dynamics of the associated Poincaré map: more precisely, the zeros of the function Φ were shown to give rise to invariant sets for the discrete dynamical system associated with (1.3) and eventually to the existence of unbounded orbits. Generalization of this approach, requiring the introduction of suitable resonance functions, were later provided for forced asymmetric oscillators

$$\ddot{x} + ax^+ - bx^- + \phi(x) = p(t), \qquad x \in \mathbb{R},$$

with $\phi : \mathbb{R} \to \mathbb{R}$ a bounded function (see [6, 9]) and, more in general, for planar system of the type

$$Jz' = \nabla H(z) + R(z) + e(t), \qquad z \in \mathbb{R}^2,$$

where J is the standard symplectic matrix, $H : \mathbb{R}^2 \to \mathbb{R}$ is positive and positively homogeneous of degree 2 and $R : \mathbb{R}^2 \to \mathbb{R}^2$ is bounded (see [8, 10]). We also refer to [1, 4, 5, 12–14, 16, 17] for related results.

In spite of this extensive bibliography, the existence of unbounded solutions for systems of coupled oscillators seems to be an essentially unexplored topic. To the best of our knowledge, the only available results are the ones contained in the recent paper [3], dealing however with systems of equations looking like weakly coupled perturbations of linear oscillators (i.e. $a_i = b_i = n_i^2$ for $i = 1, 2$) and not being applicable to the more general setting of (1.1).

The aim of the present paper is to extend the approach of [2] in this higher-dimensional framework. As expected, this is a quite delicate task, since it leads to the study of the dynamics of a four-dimensional map; nonetheless, we will succeed in providing some partial generalizations of the results in [2]. In more details, our strategy and results can be described as follows.

In Sect. 2 we pass to an appropriate set of action-angle coordinates and we perform an asymptotic expansion, at infinity, of the Poincaré map associated with (1.1), cf. (2.26). In doing this, we are led to define a resonance function defined on the two-dimensional torus,

$$L : \mathbb{T}^2 \to \mathbb{R}^2, \qquad (\theta_1, \theta_2) \mapsto (L_1(\theta_1, \theta_2), L_2(\theta_1, \theta_2))$$

which can be thought as the higher-dimensional generalization of the resonance function Φ defined in (1.4), see (2.25)-(2.28). We notice that when system (1.1) is uncoupled (that is, $\phi_1 = \phi_2 = 0$), then $L(\theta_1, \theta_2) = (L_1(\theta_1), L_2(\theta_2))$ and, up to a constant, $L_i = \Phi$ with $p = p_i$.

In Sect. 3 we investigate the dynamics of this four-dimensional Poincaré map and we construct invariant sets, giving rise to unbounded orbits. As in the two-dimensional setting, the zeros of the function L are shown to play a role; however, due to the coupling terms in system (1.1), we need here to assume that the Jacobian matrix JL has a special structure at the zeros. More precisely, we introduce the notion of $\mathcal{D}^{\pm}$-matrix, cf. Definition 3.1: again, we observe that such a condition is satisfied by diagonal matrices with concordant sign diagonal entries and, hence, by the matrix JL when system (1.1) is uncoupled and the functions L_i have simple zeros, as in the main result of [2]. This is a quite technical part of the proof, involving, among other things, a delicate estimate for the 2-norm of a two-parameter family of suitable matrices, which are perturbations of the identity by $\mathcal{D}^{\pm}$-matrices, cf. Lemma 3.2.

In Sect. 4 we finally give our main result for the existence of unbounded solutions to system (1.1), Theorem 4.1. It provides a positive measure set of initial conditions giving rise to unbounded orbits to (1.1), whenever the function L has a zero $\omega \in \mathbb{T}^2$ such that the Jacobian matrix $JL(\omega)$ is a $\mathcal{D}^{\pm}$-matrix. Notice that this can be interpreted as a kind of local version of the main result in [2]. Indeed, we do not claim that every large solution of (1.1) is unbounded: due to the higher-dimensional setting, obtaining this global information seems to be a very hard task, even in the case when all the zeros of L are such that the Jacobian at each zero is a $\mathcal{D}^{\pm}$-matrix. We mention that the condition for JL to be a $\mathcal{D}^{\pm}$-matrix can be, in general, not easy to verify. To this end, we discuss some situations in which this can be done and Theorem 4.1 can thus be applied. The first, quite natural, possibility that we present is a semi-perturbative result (cf. Corollary 4.3), dealing with the case in which the L^{∞}-norms of the coupling terms ϕ_1, ϕ_2 are not too big: it is worth noticing that this provides a genuinely asymmetric (non-quantitative) generalization of a result obtained in [3] for coupled linear oscillators. Other results, more global in nature but focusing on specific choices for the parameters a_i, b_i or the forcing terms p_i, are given by Corollary 4.6 and Corollary 4.7. It seems that various other situations could be treated at the expenses of longer computations.

We finally mention that it should be possible, with the same approach, to consider also the more general case of resonance with respect to different curves of the Fucik spectrum, that is, $1/\sqrt{a_i} + 1/\sqrt{b_i} = 2/n_i$ with $n_i \in \mathbb{N}$. Also, the possibility of coupling more oscillators in a cyclic way $\phi_{i+1} = \phi_i$ could be considered. All these generalizations, however, seem to require substantial technical modifications of the proofs and they are thus postponed to future investigations.

Notation. Throughout the paper, the symbol $\| \cdot \|$ will be used for the Euclidean norm of a vector in the plane. Also, for the index $i = 1, 2$, we will adopt the cyclic agreement $i + 1 = 1$ for $i = 2$.

2 Coupled Asymmetric Oscillators: Some Preliminary Estimates

In this section, we perform some preliminary estimates for the solutions of system (1.1), with the final goal of obtaining an asymptotic expansion for its Poincaré-map in action-angle coordinates (see Sect. 2.2).

From now on, as in the Introduction we will always assume that, for $i = 1, 2$, the positive real numbers a_i, b_i satisfy (1.2), the function $p_i : \mathbb{R} \to \mathbb{R}$ is continuous and 2π-periodic and the function $\phi_i : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous and bounded. Furthermore, we also suppose that there exist

$$\lim_{x \to \pm\infty} \phi_i(x) := \phi_i(\pm\infty); \tag{2.1}$$

moreover, without loss of generality,

$$\phi_i(-\infty) = -\phi_i(+\infty). \tag{2.2}$$

2.1 Remarks on the Asymmetric Cosine and Related Functions

We collect here some results on various functions related to the asymmetric cosine function C_i, $i = 1, 2$, which is defined as the solution of

$$\begin{cases} \ddot{x}_i + a_i x_i^+ - b_i x_i^- = 0 \\ x_i(0) = 1, \ \dot{x}_i(0) = 0, \end{cases} \tag{2.3}$$

with a_i and b_i as in (1.2). We recall that, for every $i = 1, 2$, the function C_i is even, $2\pi/n$-periodic and its explicit expression in $[-\pi/n, \pi/n]$ is

$$C_i(t) = \begin{cases} \cos\sqrt{a_i}t & \text{if } |t| \le \dfrac{\pi}{2\sqrt{a_i}} \\ -\sqrt{\dfrac{a_i}{b_i}} \sin\sqrt{b_i}\left(|t| - \dfrac{\pi}{2\sqrt{a_i}}\right) & \text{if } \dfrac{\pi}{2\sqrt{a_i}} \le |t| \le \dfrac{\pi}{n}. \end{cases} \tag{2.4}$$

For future reference, let us observe that C_i, $i = 1, 2$, when $a_i \neq b_i$ admits the Fourier series expansion

$$C_i(t) = \sum_{h=0}^{+\infty} c_{h,i} \cos hnt, \quad t \in \mathbb{R}, \tag{2.5}$$

where

$$c_{0,i} = \frac{n}{\pi} \frac{b_i - a_i}{b_i\sqrt{a_i}}$$

and, for $h \ge 1$,

$$c_{h,i} = \begin{cases} \dfrac{2n}{\pi} \dfrac{b_i - a_i}{b_i - h^2n^2} \dfrac{\sqrt{a_i}}{a_i - h^2n^2} \cos\dfrac{hn\pi}{2\sqrt{a_i}} & \text{if } a_i \neq h^2n^2, \ b_i \neq h^2n^2, \\ \dfrac{1}{2h} & \text{otherwise} \end{cases} \tag{2.6}$$

(see [2, Lemma 4.2]).

In the next sections we will use the integrals of C_i over the sets $J^{\pm}_{i+1}$ defined by

$$J^{+}_{i+1} = \left\{ t \in [0, 2\pi] : \; C_{i+1}\left(\frac{\theta_{i+1}}{n} + t\right) > 0 \right\}, \quad J^{-}_{i+1} = \left\{ t \in [0, 2\pi] : \; C_{i+1}\left(\frac{\theta_{i+1}}{n} + t\right) < 0 \right\} \tag{2.7}$$

where $\theta_{i+1} \in \mathbb{R}$. It is immediate to observe that the fact that C_i and C_{i+1} are both $2\pi/n$-periodic implies that

$$\int_{J^{\pm}_{i+1}} C_i\left(\frac{\theta_i}{n} + t\right) dt, \quad \theta_i \in \mathbb{R},$$

does not change if we replace $[0, 2\pi]$ in the definition of $J^{\pm}_{i+1}$ by any interval of lenght 2π. In particular, in the computation of the integral of C_i on J^{+}_{i+1}, we can replace J^{+}_{i+1} by the set

$$\bigcup_{k=0}^{n-1} \left(-\frac{\pi}{2\sqrt{a_i}} - \frac{\theta_{i+1}}{n} + 2k\frac{\pi}{n}, \frac{\pi}{2\sqrt{a_i}} - \frac{\theta_{i+1}}{n} + 2k\frac{\pi}{n} \right),$$

thus obtaining that

$$\int_{J^{+}_{i+1}} C_i\left(\frac{\theta_i}{n} + t\right) dt = n\Lambda_i(\theta_i - \theta_{i+1}), \quad \forall\, \theta_i, \theta_{i+1} \in \mathbb{R}, \quad i = 1, 2, \tag{2.8}$$

where $\Lambda_i : \mathbb{R} \to \mathbb{R}$ is defined by

$$\Lambda_i(t) = K_i\left(\frac{t}{n} + \frac{\pi}{2\sqrt{a_{i+1}}}\right) - K_i\left(\frac{t}{n} - \frac{\pi}{2\sqrt{a_{i+1}}}\right), \quad \forall\, t \in \mathbb{R}, \tag{2.9}$$

being K_i the primitive of C_i such that $K_i(0) = 0$.
A crucial point in our analysis will be the study of the resolubility of the equation

$$\Lambda_i(t) = \alpha_i, \tag{2.10}$$

where α_i is given by

$$\alpha_i = \frac{1}{\sqrt{a_i}} - \frac{\sqrt{a_i}}{b_i}, \quad i = 1, 2, \tag{2.11}$$

which is related to C_i by

$$\int_0^{2\pi} C_i\,(t)\; dt = 2n\alpha_i, \quad i = 1, 2. \tag{2.12}$$

In particular, we will be interested in the situation where (2.10) has simple solutions; in order to face this problem, let us first concentrate on the range of the function Λ_i. Introducing the function $\Sigma_i : \mathbb{R} \to \mathbb{R}$ defined by

$$\Sigma_i(t) = n\Lambda_i'(t), \quad \forall\, t \in \mathbb{R}, \tag{2.13}$$

is it possible to prove the following result.

Lemma 2.1 *The function Λ_i given in (2.9) is even, 2π-periodic, decreasing in $(0, \pi)$ and increasing in $(-\pi, 0)$.*

Proof Let us first observe that we have

$$\Sigma_i(t) = C_i\left(\frac{t}{n} + \frac{\pi}{2\sqrt{a_{i+1}}}\right) - C_i\left(\frac{t}{n} - \frac{\pi}{2\sqrt{a_{i+1}}}\right) = C_i^n\left(t + \frac{n\pi}{2\sqrt{a_{i+1}}}\right) - C_i^n\left(t - \frac{n\pi}{2\sqrt{a_{i+1}}}\right)$$

for all $t \in \mathbb{R}$, where $C_i^n : \mathbb{R} \to \mathbb{R}$ is defined by

$$C_i^n(t) = C_i\left(\frac{t}{n}\right), \quad \forall\, t \in \mathbb{R}.$$

The function C_i^n is continuous, 2π-periodic, even and strictly decreasing in $[0, \pi)$; as a consequence, Σ_i is continuous, 2π-periodic and odd. As far as the sign of Σ_i is concerned, let us observe that $\Sigma_i(t) = 0$ if and only if $t = k\pi$ for some $k \in \mathbb{Z}$. Indeed, $\Sigma_i(t) = 0$ if and only if:

$$\text{either } t + \frac{n\pi}{2\sqrt{a_{i+1}}} = t - \frac{n\pi}{2\sqrt{a_{i+1}}} + 2k\pi \quad \text{or}$$
$$t + \frac{n\pi}{2\sqrt{a_{i+1}}} = -t + \frac{n\pi}{2\sqrt{a_{i+1}}} + 2k\pi \quad \text{for some } k \in \mathbb{Z}.$$

Now, since $a_{i+1} > n^2/4$, the first alternative cannot hold, and the second one implies that $t = k\pi$. Therefore Σ_i has constant sign in $(0, \pi)$ and a straightforward argument shows that

$$\Sigma_i\left(\pi - \frac{n\pi}{2\sqrt{a_{i+1}}}\right) < 0.$$

From the above described properties of Σ_i we immediately deduce the thesis. □

From now on, in order to simplify the notation, let us continue with the case $i = 1$; the case $i = 2$ is completely analogous.
From Lemma 2.1 we deduce that equation (2.10), with $i = 1$, admits simple solutions if and only if

$$\Lambda_1(\pi) < \alpha_1 < \Lambda_1(0);$$

in general, the validity of this condition depends on the original pairs (a_1, b_1) and (a_2, b_2). Hence, let us define the resolubility set

$$\mathcal{R} = \left\{(a_1, a_2) \in \left(\frac{n^2}{4}, +\infty\right) \times \left(\frac{n^2}{4}, +\infty\right) \subset \mathbb{R}^2 :\ \Lambda_1(\pi) < \alpha_1 < \Lambda_1(0)\right\}. \tag{2.14}$$

The complete description of the open set $\mathcal{R}$ is quite difficult; by means of long computations it is possible to show that the vertical sections $\mathcal{R} \cap \{(a_1^*, a_2) : a_2 > n^2/4\}$, with $a_1^* > n^2/4$, $a_1^* \neq n^2$, are bounded. On the other hand, the study of the horizontal sections $\mathcal{R} \cap \{(a_1, a_2^*) : a_1 > n^2/4\}$, with $a_2^* > n^2/4$, $a_2^* \neq n^2$, is much more complicated. However, the following simple result holds true.

Lemma 2.2 *The set $\mathcal{R}$ contains the half-lines $\{(n^2, a_2) : a_2 > n^2/4\}$ and $\{(a_1, n^2) : a_1 > n^2/4\}$.*

Proof Let us first assume that $a_1 = n^2$ (and, thus, $b_1 = n^2$) and fix $\sqrt{a_2} > n/2$; we then have

$$C_1(t) = \cos(nt), \quad K_1(t) = \frac{1}{n}\sin(nt), \quad \Lambda_1(t) = \frac{2}{n}\sin\frac{\pi n}{2\sqrt{a_2}}\cos t, \quad \forall t \in \mathbb{R}.$$

A simple computation proves that $\Lambda_1(\pi) < 0 < \Lambda_1(0)$; noticing that $\alpha_1 = 0$, by (2.11), this shows that $(n^2, a_2) \in \mathcal{R}$.
On the other hand, if $a_2 = n^2$, we have

$$\Lambda_1(0) = 2\int_0^{\pi/2n} C_1(t)dt, \quad \Lambda_1(\pi) = 2\int_{\pi/2n}^{\pi/n} C_1(t)dt.$$

Recalling (2.12), we deduce that $\Lambda_1(\pi) = 2\alpha_1 - \Lambda_1(0) < \Lambda_1(0)$ and, thus, $\alpha_1 < \Lambda_1(0)$. From these relations we also obtain $\Lambda_1(\pi) = 2\alpha_1 - \Lambda_1(0) < \alpha_1$, proving that $(a_1, n^2) \in \mathcal{R}$. □

2.2 Asymptotic Analysis

We now perform an asymptotic expansion of the Poincaré map associated to (1.1). We adapt the argument of the proof of [2, Theorem 4.1] to our case: we write (1.1) as a first order system in $(x_1, x_2, y_1, y_2) = (x_1, x_2, \dot{x}_1, \dot{x}_2)$ and use the change of variables

$$\begin{cases} x_i = \gamma_i r_i C_i\left(\dfrac{\theta_i}{n}\right) \\ y_i = \gamma_i r_i S_i\left(\dfrac{\theta_i}{n}\right) \end{cases} \quad \text{with } \gamma_i = \sqrt{2n/a_i}, \tag{2.15}$$

where $S_i(t) = C_i'(t)$ and C_i is defined in Subsection 2.1.

It is straightforward to see that (1.1) is formally equivalent to

$$\begin{cases} \dot{\theta}_1 = n - \dfrac{\gamma_1}{2r_1} C_1\left(\dfrac{\theta_1}{n}\right)\left[p_1(t) - \phi_1\left(\gamma_2 r_2 C_2\left(\dfrac{\theta_2}{n}\right)\right)\right] \\ \dot{\theta}_2 = n - \dfrac{\gamma_2}{2r_2} C_2\left(\dfrac{\theta_2}{n}\right)\left[p_2(t) - \phi_2\left(\gamma_1 r_1 C_1\left(\dfrac{\theta_1}{n}\right)\right)\right] \\ \dot{r}_1 = \dfrac{\gamma_1}{2n} S_1\left(\dfrac{\theta_1}{n}\right)\left[p_1(t) - \phi_1\left(\gamma_2 r_2 C_2\left(\dfrac{\theta_2}{n}\right)\right)\right] \\ \dot{r}_2 = \dfrac{\gamma_2}{2n} S_2\left(\dfrac{\theta_2}{n}\right)\left[p_2(t) - \phi_2\left(\gamma_1 r_1 C_1\left(\dfrac{\theta_1}{n}\right)\right)\right] \end{cases} \tag{2.16}$$

We denote by $(\theta_1, \theta_2, r_1, r_2)(t)$ the solution of (2.16) satisfying $(\theta_1, \theta_2, r_1, r_2)(0) = (\theta_{1,0}, \theta_{2,0}, r_{1,0}, r_{2,0})$ and study the behavior of $(\theta_1, \theta_2, r_1, r_2)(2\pi)$ as $\min\{r_{1,0}, r_{2,0}\} \to +\infty$. We also set $\theta_0 = (\theta_{1,0}, \theta_{2,0})$, $r_0 = (r_{1,0}, r_{2,0})$ and remark that $\theta_0 \in \mathbb{R}^2$ and $r_{1,0}, r_{2,0} > 0$.

The boundedness of p_i and ϕ_i implies that $\dot{r}_i$ is uniformly bounded and, hence, we have

$$r_i = r_{i,0} + O(1) \quad \text{and} \quad r_i^{-1} = r_{i,0}^{-1} + O(r_{i,0}^{-2}) \quad \text{as } r_{i,0} \to +\infty, \tag{2.17}$$

where these and all the following estimates hold uniformly w.r.t. $t \in [0, 2\pi]$, $\theta_{1,0}$, $\theta_{2,0}$ and $r_{i+1,0}$. We deduce that

$$\begin{aligned} \dot{\theta}_i &= n - \frac{\gamma_i}{2} C_i\left(\frac{\theta_i}{n}\right)\left(\frac{1}{r_{i,0}} + O(r_{i,0}^{-2})\right)\left[p_i(t) - \phi_i\left(\gamma_{i+1} r_{i+1} C_{i+1}\left(\frac{\theta_{i+1}}{n}\right)\right)\right] \\ &= n - \frac{\gamma_i}{2r_{i,0}} C_i\left(\frac{\theta_i}{n}\right)\left[p_i(t) - \phi_i\left(\gamma_{i+1} r_{i+1} C_{i+1}\left(\frac{\theta_{i+1}}{n}\right)\right)\right] + O(r_{i,0}^{-2}), \end{aligned} \quad \text{as } r_{i,0} \to +\infty. \tag{2.18}$$

This relation implies that

$$\frac{\theta_i}{n} = \frac{\theta_{i,0}}{n} + t + O(r_{i,0}^{-1}), \qquad \text{as } r_{i,0} \to +\infty;$$

and, thus:

$$\begin{aligned} C_i\left(\frac{\theta_i}{n}\right) &= C_i\left(\frac{\theta_{i,0}}{n} + t\right) + O(r_{i,0}^{-1}) \\ S_i\left(\frac{\theta_i}{n}\right) &= S_i\left(\frac{\theta_{i,0}}{n} + t\right) + O(r_{i,0}^{-1}) \end{aligned} \qquad \text{as } r_{i,0} \to +\infty, \tag{2.19}$$

since C_i and S_i are smooth enough. By replacing (2.19) in the last two equations of (2.16) we get

$$\dot{r}_i = \frac{\gamma_i}{2n} S_i\left(\frac{\theta_{i,0}}{n} + t\right)\left[p_i(t) - \phi_i\left(\gamma_{i+1} r_{i+1} C_{i+1}\left(\frac{\theta_{i+1}}{n}\right)\right)\right] + O(r_{i,0}^{-1}) \qquad \text{as } r_{i,0} \to +\infty.$$

As a consequence, we infer that

$$\begin{aligned} r_i(2\pi) = r_{i,0} &+ \frac{\gamma_i}{2n}\int_0^{2\pi} S_i\left(\frac{\theta_{i,0}}{n} + t\right) p_i(t)\, dt \\ &- \frac{\gamma_i}{2n}\int_0^{2\pi} S_i\left(\frac{\theta_{i,0}}{n} + t\right)\phi_i\left(\gamma_{i+1} r_{i+1} C_{i+1}\left(\frac{\theta_{i+1}}{n}\right)\right) dt + F_{i,i}(\theta_0, r_0) \end{aligned} \tag{2.20}$$

where

$$\lim_{r_{i,0} \to +\infty} F_{i,i}(\theta_0, r_0) = 0 \quad \text{uniformly w.r.t. } \theta_1, \theta_2, \text{ and } r_{i+1,0}.$$

Now, we deduce from (2.19) that $C_{i+1}(\theta_{i+1}(t)/n) \to C_{i+1}(\theta_{i+1,0}/n + t)$ uniformly w.r.t. $t \in [0, 2\pi]$, $\theta_{1,0}$, $\theta_{2,0}$ and $r_{i,0}$, as $r_{i+1,0} \to +\infty$ and, setting

$$\begin{aligned} J_{i+1,0}^{+} &= \left\{t \in [0, 2\pi] : \; C_{i+1}\left(\frac{\theta_{i+1,0}}{n} + t\right) > 0\right\} \\ J_{i+1,0}^{-} &= \left\{t \in [0, 2\pi] : \; C_{i+1}\left(\frac{\theta_{i+1,0}}{n} + t\right) < 0\right\}, \end{aligned}$$

we have that

$$t \in J_{i+1,0}^{\pm} \implies \lim_{r_{i+1,0} \to +\infty} \phi_i\left(\gamma_{i+1} r_{i+1} C_{i+1}\left(\frac{\theta_{i+1}}{n}\right)\right) = \phi_i(\pm\infty),$$

where these two limits are not uniform w.r.t. $t \in [0, 2\pi]$. However, using that ϕ_i is bounded and $C_{i+1}(\theta_{i+1}(t)/n)$ converges uniformly, it is possible to show that:

$$\lim_{r_{i+1,0} \to +\infty} \int_{J_{i+1,0}^{\pm}} S_i\left(\frac{\theta_{i,0}}{n} + t\right)\left[\phi_i(\pm\infty) - \phi_i\left(\gamma_{i+1} r_{i+1} C_{i+1}\left(\frac{\theta_{i+1}}{n}\right)\right)\right] dt = 0$$

uniformly w.r.t. $\theta_{1,0}$, $\theta_{2,0}$ and $r_{i,0}$. Therefore, we can write equation (2.20) in the following way:

$$\begin{aligned} r_i(2\pi) = r_{i,0} + \frac{\gamma_i}{2n}\int_0^{2\pi} S_i\left(\frac{\theta_{i,0}}{n}+t\right)p_i(t)\,dt \\ - \frac{\gamma_i}{2n}\left(\phi_i(+\infty)\int_{J^+_{i+1,0}} S_i\left(\frac{\theta_{i,0}}{n}+t\right)dt + \phi_i(-\infty)\int_{J^-_{i+1,0}} S_i\left(\frac{\theta_{i,0}}{n}+t\right)dt\right) \\ + F_{i,i}(\theta_0, r_0) + F_{i,i+1}(\theta_0, r_0), \end{aligned} \tag{2.21}$$

where also

$$\lim_{r_{i+1,0}\to+\infty} F_{i,i+1}(\theta_0, r_0) = 0 \qquad \text{uniformly w.r.t. } \theta_{1,0}, \theta_{2,0} \text{ and } r_{i,0}. \tag{2.22}$$

We now substitute (2.17) and (2.19) in (2.18), obtaining

$$\dot\theta_i = n - \frac{\gamma_i}{2r_{i,0}}C_i\left(\frac{\theta_{i,0}}{n}+t\right)\left[p_i(t) - \phi_i\left(\gamma_{i+1}r_{i+1}C_{i+1}\left(\frac{\theta_{i+1}}{n}\right)\right)\right] + O(r_{i,0}^{-2}), \qquad \text{as } r_{i,0}\to+\infty.$$

Integrating on $[0, 2\pi]$ and making similar considerations as done for $r_i(2\pi)$, we deduce that

$$\begin{aligned} \theta_i(2\pi) = \theta_{i,0} + 2n\pi - \frac{\gamma_i}{2r_{i,0}}\int_0^{2\pi} C_i\left(\frac{\theta_{i,0}}{n}+t\right)p_i(t)\,dt \\ + \frac{\gamma_i}{2r_{i,0}}\left(\phi_i(+\infty)\int_{J^+_{i+1,0}} C_i\left(\frac{\theta_{i,0}}{n}+t\right)dt + \phi_i(-\infty)\int_{J^-_{i+1,0}} C_i\left(\frac{\theta_{i,0}}{n}+t\right)dt\right) \\ + \frac{1}{r_{i,0}}\left(G_{i,i}(\theta_0, r_0) + G_{i,i+1}(\theta_0, r_0)\right), \end{aligned} \tag{2.23}$$

where

$$\lim_{r_{i,0}\to+\infty} G_{i,i}(\theta_0, r_0) = \lim_{r_{i+1,0}\to+\infty} G_{i,i+1}(\theta_0, r_0) = 0 \tag{2.24}$$

uniformly w.r.t. the other variables.
Recalling (2.2), we observe that (2.8) and (2.12) imply that $\theta_i(2\pi)$, $i = 1, 2$, can be written as

$$\begin{aligned} \theta_i(2\pi) = \theta_{i,0} + 2n\pi - \frac{\gamma_i}{2r_{i,0}}\int_0^{2\pi} C_i\left(\frac{\theta_{i,0}}{n}+t\right)p_i(t)\,dt \\ + \frac{1}{r_{i,0}}\gamma_i n\phi_i(+\infty)(\Lambda_i(\theta_{i,0}-\theta_{i+1,0}) - \alpha_i) + \frac{1}{r_{i,0}}\left(G_{i,i}(\theta_0, r_0) + G_{i,i+1}(\theta_0, r_0)\right), \end{aligned}$$

where α_i, Λ_i are defined in (2.11), (2.9).
For $i = 1, 2$, let us now denote

$$\begin{aligned} \Phi_i(\theta_{i,0}) = -\frac{\gamma_i}{2}\int_0^{2\pi} C_i\left(\frac{\theta_{i,0}}{n}+t\right)p_i(t)\,dt, \\ L_i(\theta_0) = \Phi_i(\theta_{i,0}) + \gamma_i n\phi_i(+\infty)(\Lambda_i(\theta_{i,0}-\theta_{i+1,0}) - \alpha_i), \end{aligned} \tag{2.25}$$

for every $\theta_0 = (\theta_{1,0}, \theta_{2,0}) \in \mathbb{R}^2$. Then, we can summarize (2.21), (2.22), (2.23) and (2.24) as follows:

$$\begin{cases} \theta_i(2\pi) = \theta_{i,0} + 2\pi n + \dfrac{1}{r_{i,0}} \left[L_i(\theta_0) + G_{i,i}(\theta_0, r_0) + G_{i,i+1}(\theta_0, r_0) \right] \\ r_i(2\pi) = r_{i,0} - \dfrac{\partial L_i}{\partial \theta_{i,0}}(\theta_0) + F_{i,i}(\theta_0, r_0) + F_{i,i+1}(\theta_0, r_0) \end{cases} \quad \text{for } i = 1, 2, \tag{2.26}$$

where

$$\begin{aligned} &\lim_{r_{i,0}\to+\infty} F_{i,i}(\theta_0, r_0) = \lim_{r_{i,0}\to+\infty} G_{i,i}(\theta_0, r_0) = 0 && \text{uniformly w.r.t. } r_{i+1,0} \text{ and } \theta_0, \\ &\lim_{r_{i+1,0}\to+\infty} F_{i,i+1}(\theta_0, r_0) = \lim_{r_{i+1,0}\to+\infty} G_{i,i+1}(\theta_0, r_0) = 0 && \text{uniformly w.r.t. } r_{i,0} \text{ and } \theta_0. \end{aligned} \tag{2.27}$$

The functions L_1, L_2 will be meant as the components of the vector valued function

$$L(\theta_0) = (L_1(\theta_0), L_2(\theta_0)), \qquad \theta_0 = (\theta_{1,0}, \theta_{2,0}) \in \mathbb{R}^2. \tag{2.28}$$

which we will call *resonance function* for system (1.1). Notice that, due to the 2π-periodicity in both the variables, we can interpret L as a function defined on the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$. This function will play a crucial role in the statement of our main result (see Sect. 4).

3 Dynamics of Discrete Maps

In this section, we establish the abstract result that will be used to prove the existence of unbounded solutions to system (1.1).

3.1 $\mathcal{D}^{\pm}$-Matrices

We consider 2×2-matrices $A = (a_{ij})$, $i, j = 1, 2$.

Definition 3.1 A 2×2-matrix A is said to be a $\mathcal{D}^+$-matrix if

$$a_{11} < 0, \qquad a_{22} < 0, \qquad |a_{12}a_{22} + a_{11}a_{21}| < 2a_{11}a_{22}. \tag{3.1}$$

Analogously, a 2×2-matrix A is said to be a $\mathcal{D}^-$-matrix if

$$a_{11} > 0, \qquad a_{22} > 0, \qquad |a_{12}a_{22} + a_{11}a_{21}| < 2a_{11}a_{22}. \tag{3.2}$$

Notice that a diagonal matrix with negative entries (resp., positive entries) is a $\mathcal{D}^+$ matrix (resp., $\mathcal{D}^-$ matrix). Given a $\mathcal{D}^{\pm}$-matrix A and $\epsilon = (\epsilon_1, \epsilon_2) \in (0, +\infty)^2$, let us define

$$B_\epsilon = \begin{pmatrix} 1 + \epsilon_1 a_{11} & \epsilon_1 a_{12} \\ \epsilon_2 a_{21} & 1 + \epsilon_2 a_{22} \end{pmatrix}. \tag{3.3}$$

Moreover, for every $\epsilon_0 > 0$ and $\eta > 0$ let us define

$$C_{\epsilon_0,\eta} = \left\{ \epsilon = (\epsilon_1, \epsilon_2) \in (0, +\infty)^2 : \frac{a_{11}}{a_{22}} - \eta \le \frac{\epsilon_2}{\epsilon_1} \le \frac{a_{11}}{a_{22}} + \eta, \quad \|\epsilon\| \le \epsilon_0 \right\}. \tag{3.4}$$

We prove the following result.

Lemma 3.2 *Assume that A is a $\mathcal{D}^{\pm}$-matrix. Then, there exist $a_0 > 0$, $\epsilon_0 > 0$ and $\eta > 0$ such that*

$$\|B_\epsilon\|_2 \leq 1 - \frac{1}{2} a_0 \|\epsilon\|, \quad \forall\, \epsilon \in C_{\epsilon_0,\eta}. \tag{3.5}$$

Proof We give the proof in the case of $\mathcal{D}^{+}$-matrix; the other case is analogous. We recall that the matrix norm $\|B_\epsilon\|_2$ coincides with the square root of the maximum eigenvalue of the matrix $C_\epsilon = B_\epsilon^T B_\epsilon$.

Let us first observe that, for every ϵ, the elements on the diagonal of C_ϵ are given by

$$(1 + \epsilon_1 a_{11})^2 + \epsilon_2^2 a_{21}^2, \quad (1 + \epsilon_2 a_{22})^2 + \epsilon_1^2 a_{12}^2;$$

hence, we have

$$\mathrm{tr}(C_\epsilon) = 2 + 2(a_{11}\epsilon_1 + a_{22}\epsilon_2) + (a_{11}^2 + a_{12}^2)\epsilon_1^2 + (a_{22}^2 + a_{21}^2)\epsilon_2^2. \tag{3.6}$$

Hence, a simple computation shows that

$$\begin{aligned}(\mathrm{tr}(C_\epsilon))^2 &= 4 + 8(a_{11}\epsilon_1 + a_{22}\epsilon_2) + 4(a_{11}\epsilon_1 + a_{22}\epsilon_2)^2 + 4(a_{11}^2 + a_{12}^2)\epsilon_1^2 + 4(a_{22}^2 + a_{21}^2)\epsilon_2^2 + \\ &\quad + 4(a_{11}\epsilon_1 + a_{22}\epsilon_2)((a_{11}^2 + a_{12}^2)\epsilon_1^2 + (a_{22}^2 + a_{21}^2)\epsilon_2^2) + ((a_{11}^2 + a_{12}^2)\epsilon_1^2 + (a_{22}^2 + a_{21}^2)\epsilon_2^2)^2.\end{aligned} \tag{3.7}$$

On the other hand, we have

$$\begin{aligned}\det(C_\epsilon) = (\det B_\epsilon)^2 &= 1 + 2(a_{11}\epsilon_1 + a_{22}\epsilon_2) + (a_{11}\epsilon_1 + a_{22}\epsilon_2)^2 + 2\Delta\epsilon_1\epsilon_2 + \\ &\quad + 2(a_{11}\epsilon_1 + a_{22}\epsilon_2)\Delta\epsilon_1\epsilon_2 + \Delta^2\epsilon_1^2\epsilon_2^2,\end{aligned} \tag{3.8}$$

where $\Delta = a_{11}a_{22} - a_{12}a_{21}$.

Now, let us observe that the matrix C_ϵ is positive definite; as a consequence, the maximum eigenvalue of C_ϵ is given by

$$\lambda_+(\epsilon) = \frac{\mathrm{tr}(C_\epsilon) + \sqrt{(\mathrm{tr}(C_\epsilon))^2 - 4\det(C_\epsilon)}}{2}. \tag{3.9}$$

From (3.7) and (3.8), by means of simple computations we infer that

$$\sqrt{(\mathrm{tr}(C_\epsilon))^2 - 4\det(C_\epsilon)} = 2\sqrt{(1 + a_{11}\epsilon_1 + a_{22}\epsilon_2)d_2(\epsilon) + P_4(\epsilon)}, \tag{3.10}$$

where

$$d_2(\epsilon) = (a_{11}^2 + a_{12}^2)\epsilon_1^2 - 2\Delta\epsilon_1\epsilon_2 + (a_{21}^2 + a_{22}^2)\epsilon_2^2 = (a_{11}\epsilon_1 - a_{22}\epsilon_2)^2 + (a_{12}\epsilon_1 + a_{21}\epsilon_2)^2 \tag{3.11}$$

and

$$\begin{aligned}P_4(\epsilon) &= \frac{1}{4}\left[(a_{11}^2 + a_{12}^2)\epsilon_1^2 + (a_{22}^2 + a_{21}^2)\epsilon_2^2\right]^2 - \Delta^2\epsilon_1^2\epsilon_2^2 \\ &= \frac{1}{4}d_2(\epsilon)\left[(a_{11}^2 + a_{12}^2)\epsilon_1^2 + (a_{22}^2 + a_{21}^2)\epsilon_2^2 + 2\Delta\epsilon_1\epsilon_2\right] \\ &= d_2(\epsilon) \cdot O(\|\epsilon\|^2) \qquad \text{as } \epsilon \to 0.\end{aligned} \tag{3.12}$$

Using (3.6) and (3.9)–(3.12), we can estimate:

$$\begin{aligned}\lambda_+(\epsilon) &= 1 + a_{11}\epsilon_1 + a_{22}\epsilon_2 + O(\|\epsilon\|^2) + \sqrt{d_2(\epsilon)}\sqrt{1 + a_{11}\epsilon_1 + a_{22}\epsilon_2 + O(\|\epsilon\|^2)} \\ &= 1 + g(\epsilon) + O(\|\epsilon\|^2) \qquad \text{as } \epsilon \to 0,\end{aligned} \tag{3.13}$$

where:

$$g(\epsilon) = a_{11}\epsilon_1 + a_{22}\epsilon_2 + \sqrt{d_2(\epsilon)}.$$

Observe that g is a positively homogeneous function of degree 1. A simple computation shows that

$$g\left(-\frac{a_{22}}{\sqrt{a_{11}^2 + a_{22}^2}}, -\frac{a_{11}}{\sqrt{a_{11}^2 + a_{22}^2}}\right) = \frac{-2a_{11}a_{22} + |a_{12}a_{22} + a_{11}a_{21}|}{a_{11}^2 + a_{22}^2} := -4a_0 < 0, \tag{3.14}$$

since the matrix A is a $\mathcal{D}^+$-matrix. Using (3.14) we deduce that there exists $\eta > 0$ such that

$$g(\epsilon) < -2a_0, \tag{3.15}$$

for every ϵ such that $\|\epsilon\| = 1$ and

$$0 < \frac{a_{11}}{a_{22}} - \eta \leq \frac{\epsilon_2}{\epsilon_1} \leq \frac{a_{11}}{a_{22}} + \eta. \tag{3.16}$$

By homogeneity, we conclude that

$$g(\epsilon) < -2a_0\|\epsilon\|, \tag{3.17}$$

for every $\epsilon \in (0, +\infty)^2$ satisfying (3.16).

From (3.13) and (3.17) we deduce that there exists $\bar{\epsilon} > 0$ such that

$$\lambda_+(\epsilon) \leq 1 - a_0\|\epsilon\|, \tag{3.18}$$

for every $\epsilon \in C_{\bar{\epsilon},\eta}$. Let us now take $\epsilon_0 = \min\{\bar{\epsilon}, 1/a_0\}$; from (3.18) we immediately conclude that

$$\sqrt{\lambda_+(\epsilon)} \leq 1 - \frac{1}{2}a_0\|\epsilon\|,$$

for every $\epsilon \in C_{\epsilon_0,\eta}$. □

3.2 Invariant Sets and Unbounded Orbits of Discrete Maps

In (2.26) we have obtained an estimate for the Poincaré map $(\theta(0), r(0)) \mapsto (\theta(2\pi), r(2\pi))$ associated to the system (2.16) when both components $r_{1,0}$ and $r_{2,0}$ of $r(0)$ are large. Here we provide sufficient conditions under which the discrete dynamical systems generated by similar maps possess invariant sets that contain unbounded trajectories.

Few words are in order to clarify the setting in which the dynamical system is defined and represented. Equations (2.26) define a map $(\theta, r) \mapsto (u, \rho)$, with $\theta = (\theta_1, \theta_2)$, $r = (r_1, r_2)$, $u = (u_1, u_2)$ and $\rho = (\rho_1, \rho_2)$, such that:

$$\begin{cases} u = \theta + \begin{bmatrix} 2\pi n_1 \\ 2\pi n_2 \end{bmatrix} + \begin{bmatrix} L_1(\theta)/r_1 \\ L_2(\theta)/r_2 \end{bmatrix} + \begin{bmatrix} G_1(\theta, r)/r_1 \\ G_2(\theta, r)/r_2 \end{bmatrix} \\ \rho = r - \begin{bmatrix} \partial_1 L_1(\theta) \\ \partial_2 L_2(\theta) \end{bmatrix} + F(\theta, r), \end{cases} \tag{3.19}$$

where $n_1, n_2 \in \mathbb{N}$, $G(\theta, r) = (G_1(\theta, r), G_2(\theta, r))$ and $F(\theta, r) = (F_1(\theta, r), F_2(\theta, r))$ are continuous, $L(\theta) = (L_1(\theta), L_2(\theta))$ is a C^1-function with $\partial_j L_i = \partial L_i/\partial\theta_j$, and, moreover, L, G, F are all 2π-periodic w.r.t. θ_1 and θ_2. We recall that (θ_i, r_i) and (u_i, ρ_i) are modified polar coordinates in $\mathbb{R}^2$ according to (2.15) and, hence, there is a couple of well known issues to take into account.

The first one concerns the singularity of polar coordinates whenever the radius vanishes and will be easily dealt with since the invariant sets we are going to define will be contained in a region where $\min\{r_1, r_2\} \geq R > 0$.

The second issue is that (3.19) defines a lifting of the actual dynamical system that, indeed, acts on $\mathbb{T}^2 \times \mathbb{R}^2_+$, where, as usual, $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$ denotes the two-dimensional torus. More precisely, the coordinates (θ, r) and (u, ρ) should be projected to $\mathbb{T}^2 \times \mathbb{R}^2_+$ to determine the correct behavior of the dynamical system, but computations are more easily performed on the "flat" covering space $\mathbb{R}^2 \times \mathbb{R}^2_+$. To this aim, we denote by $\bar{\theta}_i$ the equivalence class of θ_i in $\mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}$ and, thus, we will have $\bar{\theta} = (\bar{\theta}_1, \bar{\theta}_2) \in \mathbb{T}^2$ for each $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$; the group metrics in $\mathbb{T}^1$ and $\mathbb{T}^2$ are respectively defined by

$$|\bar{\theta}_i - \bar{u}_i| = \min\{|\theta_i - u_i + 2n\pi| : n \in \mathbb{Z}\} \quad \text{and} \quad \|\bar{\theta} - \bar{u}\| = \sqrt{|\bar{\theta}_1 - \bar{u}_1|^2 + |\bar{\theta}_2 - \bar{u}_2|^2}. \tag{3.20}$$

It will be clear from the context when $|\cdot|$ and $\|\cdot\|$ are meant on either $\mathbb{R}$ and $\mathbb{R}^2$ or $\mathbb{T}^1$ and $\mathbb{T}^2$, respectively. In particular, we observe that $|\bar{\theta}_i - \bar{u}_i| = |\theta_i - u_i|$ if and only if $|\theta_i - u_i| \leq \pi$.

The invariant sets we obtain are built around a fixed $\bar{\omega} \in \mathbb{T}^2$ and depend of four other parameters as follows:

$$E_{R,\Theta,\lambda,\eta} = \left\{(\bar{\theta}, r) \in \mathbb{T}^2 \times \mathbb{R}^2_+ : r_1 \geq R,\ r_2 \geq R,\ \lambda - \eta \leq \frac{r_1}{r_2} \leq \lambda + \eta,\ \|\bar{\theta} - \bar{\omega}\| \leq \Theta\right\}, \tag{3.21}$$

where $R > 0, 0 < \Theta < \pi, \lambda > 0$ and $0 < \eta < \lambda$. We will denote by $f : E_{R,\Theta,\lambda,\eta} \to \mathbb{T}^2 \times \mathbb{R}^2_+$ the map which has (3.19) as a lifting. We remark that *all different choices of* $n_1, n_2 \in \mathbb{Z}$ *in* (3.19) *define good liftings of the map* f: we will use the choice $n_1 = n_2 = 0$ in the proof of the next result.

Theorem 3.3 *In the above setting, let us assume that there exists* $\omega \in \mathbb{R}^2$ *such that* $L(\omega) = 0$ *and suppose that the Jacobian* $JL(\omega)$ *is a* $\mathcal{D}^+$*-matrix. Moreover, assume that*

$$\lim_{\substack{r_i \to +\infty \\ i=1,2}} G(\theta, r) = 0 \quad \textit{and} \quad \lim_{\substack{r_i \to +\infty \\ i=1,2}} F(\theta, r) = 0 \qquad \textit{uniformly w.r.t. } \theta. \tag{3.22}$$

Then, there exist $R > 0$, $\Theta \in]0, \pi[$, $\lambda > 0$ *and* $\eta \in]0, \lambda[$ *such that:*

$$f(E_{R,\Theta,\lambda,\eta}) \subset E_{R,\Theta,\lambda,\eta}. \tag{3.23}$$

Proof We divide the proof into three parts.
Part 1. Choice of the constants R, Θ, λ and η. Let

$$\lambda = \frac{\partial_1 L_1(\omega)}{\partial_2 L_2(\omega)} > 0, \tag{3.24}$$

let $\eta, \epsilon_0 > 0$ be as in Lemma 3.2 and let $R_0 = 1/\epsilon_0$. Since $JL(\omega)$ is a $\mathcal{D}^+$-matrix we deduce that there exist $\Theta_0 \in]0, \pi[$ and $\gamma_i > 0$, $i = 1, 2$, such that

$$\partial_i L_i(\theta) \leq -\gamma_i < 0, \quad \text{for } i = 1, 2 \text{ and } \forall \theta \in \mathbb{R}^2 : \|\bar{\theta} - \bar{\omega}\| \leq \Theta_0. \tag{3.25}$$

Moreover, according to assumption (3.22), let $R_1 \geq R_0$ such that

$$F_i(\theta, r) \geq -\frac{\gamma_i}{2}, \quad \text{for } i = 1, 2 \text{ and } \forall\, \theta \in \mathbb{R}^2,\ r_1 \geq R_1,\ r_2 \geq R_1. \tag{3.26}$$

By the continuity of $JL(\theta)$ in $\theta = \omega$, a simple computation shows that there exists $\Theta_1 \in]0, \Theta_0]$ such that

$$\begin{aligned} &\frac{\partial_1 L_1(\theta)}{\lambda+\eta} - \partial_2 L_2(\theta) \geq \frac{-\partial_2 L_2(\omega)\eta}{2(\lambda+\eta)} \\ &\partial_2 L_2(\theta) - \frac{\partial_1 L_1(\theta)}{\lambda-\eta} \geq \frac{-\partial_2 L_2(\omega)\eta}{2(\lambda-\eta)} \end{aligned} \qquad \forall\theta : \|\bar{\theta} - \bar{\omega}\| \leq \Theta_1. \tag{3.27}$$

Moreover, using again assumption (3.22), we deduce that there exists $R_2 \geq R_1$ such that

$$\begin{aligned} &\left| F_2(\theta, r) - \frac{F_1(\theta, r)}{\lambda+\eta} \right| < \frac{-\partial_2 L_2(\omega)\eta}{2(\lambda+\eta)} \\ &\left| \frac{F_1(\theta, r)}{\lambda-\eta} - F_2(\theta, r) \right| < \frac{-\partial_2 L_2(\omega)\eta}{2(\lambda-\eta)} \end{aligned} \qquad \text{for each } r_1 \geq R_2,\ r_2 \geq R_2 \text{ and } \theta \in \mathbb{R}^2. \tag{3.28}$$

Now, let us write

$$L_i(\theta) = \langle \nabla L_i(\omega), \theta - \omega \rangle + \alpha_i(\theta)\|\theta - \omega\|, \tag{3.29}$$

for $i = 1, 2$ and $\theta \in \mathbb{R}^2$, with

$$\lim_{\theta \to \omega} \alpha(\theta) = 0, \quad i = 1, 2$$

with $\alpha(\theta) := (\alpha_1(\theta), \alpha_2(\theta))$. Then, we choose $\Theta \in]0, \Theta_1]$ such that

$$\|(\alpha_1(\theta), \alpha_2(\theta))\| \leq \frac{a_0}{4}, \qquad \text{if } \|\theta - \omega\| \leq \Theta, \tag{3.30}$$

where a_0 is given in Lemma 3.2.
Let us now define

$$L^* = \max\{\|L(\theta)\| : \|\theta - \omega\| \leq \Theta/2\}; \tag{3.31}$$

according to assumption (3.22), let $R_3 \geq R_2$ be such that

$$\|G(\theta, r)\| < \min\left\{L^*, \frac{a_0\,\Theta}{8}\right\} \quad \text{for every } \theta \in \mathbb{R}^2 \text{ and } r_1, r_2 \geq R_3. \tag{3.32}$$

Finally, let us fix

$$R \geq \max\left\{R_3, \frac{4L^*}{\Theta}\right\} \tag{3.33}$$

and consider the set $E_{R,\Theta,\lambda,\eta}$ corresponding to the chosen constants. From now on, we will simply denote this set by E.
Part 2. Invariance of E with respect to the radial components. Let us fix (θ, r) such that $(\bar{\theta}, r) \in E$ and consider $\rho = (\rho_1, \rho_2)$ given by (3.19). From conditions (3.25) and (3.26) we immediately deduce that

$$\rho_i \geq r_i + \frac{\gamma_i}{2} > r_i, \quad \text{for } i = 1, 2. \tag{3.34}$$

On the other hand, we have $r_1 \leq (\lambda+\eta)\, r_2$ and, then, we infer that

$$\begin{aligned}\frac{\rho_1}{\rho_2} &= \frac{r_1 - \partial_1 L_1(\theta) + F_1(\theta,r)}{r_2 - \partial_2 L_2(\theta) + F_2(\theta,r)}\\ &\leq (\lambda+\eta)\frac{r_2 - \dfrac{\partial_1 L_1(\theta)}{\lambda+\eta} + \dfrac{F_1(\theta,r)}{\lambda+\eta}}{r_2 - \partial_2 L_2(\theta) + F_2(\theta,r)}\\ &= (\lambda+\eta)\left(1 - \frac{\dfrac{\partial_1 L_1(\theta)}{\lambda+\eta} - \partial_2 L_2(\theta) + F_2(\theta,R) - \dfrac{F_1(\theta,r)}{\lambda+\eta}}{r_2 - \partial_2 L_2(\theta) + F_2(\theta,r)}\right).\end{aligned} \tag{3.35}$$

Let us now observe that (3.34) implies that $r_2 - \partial_2 L_2(\theta) + F_2(\theta,r) > 0$ in E; moreover, from the first relations in (3.27) and (3.28), we deduce that

$$\frac{\partial_1 L_1(\theta)}{\lambda+\eta} - \partial_2 L_2(\theta) + F_2(\theta,R) - \frac{F_1(\theta,r)}{\lambda+\eta} > 0. \tag{3.36}$$

From (3.35) we thus conclude that

$$\frac{\rho_1}{\rho_2} \leq \lambda+\eta. \tag{3.37}$$

In an analogous way, taking into account the second relations in (3.27) and (3.28), it is possible to prove that

$$\frac{\rho_1}{\rho_2} \geq \lambda-\eta. \tag{3.38}$$

From (3.34), (3.37) and (3.38) we deduce the invariance of the set E with respect to the radial components.

Part 3. Invariance with respect to the angular components. We have to show that, if $(\bar\theta, r) \in E$ then $\|\bar u - \bar\omega\| \leq \Theta$, where u is given in (3.19). By the definition of the metric on $\mathbb{T}^2$ in (3.20) and the choice $\Theta < \pi$, it is enough to work on the covering space and to prove that for a suitable lifting (3.19) we have $\|u-\omega\| \leq \Theta$, with $\theta \in \mathbb{R}^2$ such that $\|\theta-\omega\| \leq \Theta$, where these last two norms are Euclidean in the covering space $\mathbb{R}^2$ of $\mathbb{T}^2$. As already announced just before the statement of the theorem, the choice $n_1 = n_2 = 0$ in (3.19) will work here.

Let us split the set E into the following two subsets

$$E_1 = \left\{(\bar\theta, r) \in E : \|\bar\theta - \bar\omega\| \leq \frac{\Theta}{2}\right\}, \quad E_2 = \left\{(\bar\theta, r) \in E : \frac{\Theta}{2} \leq \|\bar\theta - \bar\omega\| \leq \Theta\right\}.$$

If $(\bar\theta, r) \in E_1$, then, using the first equation in (3.19), with $n_1 = n_2 = 0$, and also (3.31), (3.32) and (3.33), we deduce that

$$\|u-\omega\| \leq \|\theta-\omega\| + \frac{1}{R}\|L(\theta)\| + \frac{1}{R}\|G(\theta,r)\| \leq \|\theta-\omega\| + \frac{1}{R}L^* + \frac{1}{R}L^* \leq \frac{\Theta}{2} + \frac{2L^*}{R} \leq \Theta. \tag{3.39}$$

On the other hand, if $(\bar\theta, r) \in E_2$, we use (3.29) and write:

$$u-\omega = B(\theta-\omega) + \begin{bmatrix} \dfrac{\alpha_1(\theta)}{r_1} + \dfrac{G_1(\theta,r)}{r_1\|\theta-\omega\|} \\ \dfrac{\alpha_2(\theta)}{r_2} + \dfrac{G_2(\theta,r)}{r_2\|\theta-\omega\|} \end{bmatrix} \|\theta-\omega\|,$$

where the matrix B is given by

$$B = \begin{pmatrix} 1 + \partial_1 L_1(\omega)/r_1 & \partial_2 L_1(\omega)/r_1 \\ \partial_1 L_2(\omega)/r_2 & 1 + \partial_2 L_2(\omega)/r_2 \end{pmatrix}$$

and has the form (3.3) with $\epsilon = (1/r_1, 1/r_2)$. Using (3.30) and (3.32) we deduce that

$$\|u - \omega\| \leq \|B\|_2 \|\theta - \omega\| + \left(\|\alpha(\theta)\| \|\epsilon\| + \frac{2\|G(\theta, r)\|}{\Theta} \|\epsilon\| \right) \|\theta - \omega\| \leq \left(\|B\|_2 + \frac{a_0}{2} \|\epsilon\| \right) \Theta.$$

Now, $(\bar{\theta}, r) \in E$ implies that $\epsilon = (1/r_1, 1/r_2) \in C_{\epsilon_0, \eta}$, see (3.4), and we can use Lemma 3.2 to obtain that $\|B\|_2 \leq (1 - a_0\|\epsilon\|/2)$ and conclude that $\|u - \omega\| \leq \Theta$. □

Now, let $(\theta_0, r_0) \in E_{R,\Theta,\lambda,\eta}$, with $E_{R,\Theta,\lambda,\eta}$ given by Theorem 3.3; since $E_{R,\Theta,\lambda,\eta}$ is positively invariant, we can recursively define

$$(\theta_{n+1}, r_{n+1}) = f(\theta_n, r_n) \in E_{R,\Theta,\lambda,\eta}, \quad \forall\, n \geq 0.$$

From (3.34) we know that

$$(r_1)_i \geq (r_0)_i + \frac{\gamma_i}{2}, \quad i = 1, 2,$$

and iterating we infer that

$$(r_n)_i \geq (r_0)_i + n\frac{\gamma_i}{2}, \quad i = 1, 2, \quad n \geq 1.$$

This relation is sufficient to prove the final result of this section.

Theorem 3.4 *In the same setting of Theorem 3.3, for every $(\theta_0, r_0) \in E_{R,\Theta,\lambda,\eta}$ we have*

$$\lim_{n \to +\infty} (r_n)_i = +\infty, \quad i = 1, 2,$$

where $(\theta_{n+1}, r_{n+1}) = f(\theta_n, r_n)$, for every $n \geq 0$.

Remark 3.5 We observe that, in the case of a one-to-one map f as above, an analogous result can be proved when $JL(\omega)$ is a $\mathcal{D}^-$-matrix; indeed, in this situation there exist $R > 0$, $0 < \Theta < \pi$, $\lambda > 0$ and $0 < \eta < \lambda$ such that:

$$f^{-1}(E_{R,\Theta,\lambda,\eta}) \subset E_{R,\Theta,\lambda,\eta}.$$

Then, for every $(\theta_0, r_0) \in E_{R,\Theta,\lambda,\eta}$ it is possible to define

$$(\theta_{n-1}, r_{n-1}) = f(\theta_n, r_n),$$

for every $n \leq 0$, and we have

$$\lim_{n \to -\infty} (r_n)_i = +\infty, \quad i = 1, 2.$$

4 The Main Result and Some Corollaries

In this section we apply the theory developed in Sect. 3 in order to prove our main result, dealing with the existence of unbounded solutions to the system

$$\begin{cases} \ddot{x}_1 + a_1 x_1^+ - b_1 x_1^- + \phi_1(x_2) = p_1(t) \\ \ddot{x}_2 + a_2 x_2^+ - b_2 x_2^- + \phi_2(x_1) = p_2(t). \end{cases} \tag{4.1}$$

We recall that, for $i = 1, 2$, we are assuming the resonance condition

$$\frac{1}{\sqrt{a_i}} + \frac{1}{\sqrt{b_i}} = \frac{2}{n}, \quad \text{for some } n \in \mathbb{N}. \tag{4.2}$$

Moreover, the function $p_i : \mathbb{R} \to \mathbb{R}$ is continuous and 2π-periodic and the function $\phi_i : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous and bounded, with

$$\phi_i(-\infty) = -\phi_i(+\infty). \tag{4.3}$$

In this setting, and recalling the definition of the function L given in (2.25)-(2.28), the following result holds true.

Theorem 4.1 *Assume conditions (4.2) and (4.3); moreover, suppose that there exists $\omega \in \mathbb{R}^2$ such that $L(\omega) = 0$ and $JL(\omega)$ is a $\mathcal{D}^+$-matrix. Then, there exists an infinite measure set $E \subset \mathbb{R}^2 \times \mathbb{R}^2$ such that*

$$\lim_{t\to+\infty} (|x_i(t)|^2 + |x_i'(t)|^2) = +\infty, \qquad i = 1, 2, \tag{4.4}$$

for every solution x of (4.1) such that $(x(0), x'(0)) \in E$.

Proof The result follows from an application of Theorem 3.4, taking into account the fact that, from (2.26), the Poincaré map associated with (4.1) is of the form (3.19), with (2.27) implying (3.22).
More precisely, let $E \subset \mathbb{R}^2 \times \mathbb{R}^2$ be the set corresponding, via action-angle coordinates, to the set $E_{R,\Theta,\omega,\lambda,\eta}$ given in the statement of Theorem 3.4 and let x be a solution of (4.1) such that $(x(0), x'(0)) \in E$. Then, from Theorem 3.4 we infer that

$$\lim_{k\to+\infty} (|x_i(2k\pi)|^2 + |x_i'(2k\pi)|^2) = +\infty.$$

The thesis (4.4) follows from this relation and an application of Gronwall's lemma (see e.g. [2, Proof of Th. 41]), taking into account the boundedness of ϕ_i, for $i = 1, 2$. □

Remark 4.2 According to Remark 3.5, an analogous result for $t \to -\infty$ can be proved when $JL(\omega)$ is a $\mathcal{D}^-$-matrix.

In the rest of the section, we discuss some concrete situations in which the abstract condition on the zeros of the function L is verified, thus providing more explicit corollaries of Theorem 4.1, depending on the structure of the set of zeroes of the functions Φ_i, $i = 1, 2$, defined in (2.25).

The first situation we deal with is the one in which both Φ_1 and Φ_2 have a simple zero (in the scalar setting, this situation was the one treated by [2, Th. 4.1]). More precisely, we assume that there exists $\omega^* = (\omega_1^*, \omega_2^*) \in \mathbb{R}^2$ such that

$$\Phi_1(\omega_1^*) = \Phi_2(\omega_2^*) = 0, \qquad \Phi_i'(\omega_i^*) < 0, \quad i = 1, 2. \tag{4.5}$$

Under this assumption, the following result holds true.

Corollary 4.3 *Assume conditions (4.2), (4.3) and (4.5). Then, there exists $\phi^* = \phi^*(a_1, a_2, p_1, p_2) > 0$ such that, for every functions ϕ_i with $|\phi_i(+\infty)| < \phi^*$ ($i = 1, 2$), there exists an infinite measure set $E \subset \mathbb{R}^2 \times \mathbb{R}^2$ such that*

$$\lim_{t\to+\infty} (|x_i(t)|^2 + |x_i'(t)|^2) = +\infty, \qquad i = 1, 2,$$

for every solution x of (4.1) such that $(x(0), x'(0)) \in E$.

Proof Let us observe that, in view of Theorem 4.1 it is sufficient to prove that, under the given assumptions, there exist $\omega \in \mathbb{R}^2$ such that $L(\omega) = 0$ and $JL(\omega)$ is a $\mathcal{D}^+$-matrix.
Let us first recall, from (2.25), that we have

$$L_i(\theta) = \Phi_i(\theta_i) + \gamma_i n\phi_i(+\infty)(\Lambda_i(\theta_i - \theta_{i+1}) - \alpha_i), \quad \forall\, \theta \in \mathbb{R}^2, \tag{4.6}$$

where Λ_i is defined in (2.9). Let us define $H : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ by

$$H(\theta, v) = (\Phi_1(\theta_1) + \gamma_1 n v_1(\Lambda_1(\theta_1 - \theta_2) - \alpha_1), \Phi_2(\theta_2) + \gamma_2 n v_2(\Lambda_2(\theta_2 - \theta_1) - \alpha_2)), \quad \forall\, \theta \in \mathbb{R}^2,\ v \in \mathbb{R}^2. \tag{4.7}$$

From (4.5) we immediately infer that

$$H(\omega_1^*, \omega_2^*, 0, 0) = 0$$

and

$$J_\theta H(\omega_1^*, \omega_2^*, 0, 0) = \begin{pmatrix} \Phi_1'(\omega_1^*) & 0 \\ 0 & \Phi_2'(\omega_2^*) \end{pmatrix} \neq 0.$$

Hence, by the implicit function theorem, we deduce that there exists $\hat{\phi} > 0$ such that for every $(\phi_1(+\infty), \phi_2(+\infty)) \in \mathbb{R}^2$ with $|\phi_i(+\infty)| < \hat{\phi}$, $i = 1, 2$, there exists $\omega = \omega(\phi_1(+\infty), \phi_2(+\infty)) \in \mathbb{R}^2$ near ω^* such that

$$L(\omega) = 0.$$

Now, let us observe that

$$JL(\omega) = \begin{pmatrix} \Phi_1'(\omega_1) + \gamma_1\phi_1(+\infty)\Sigma_1(\omega_1 - \omega_2) & -\gamma_1\phi_1(+\infty)\Sigma_1(\omega_1 - \omega_2) \\ -\gamma_2\phi_2(+\infty)\Sigma_2(\omega_2 - \omega_1) & \Phi_2'(\omega_2) + \gamma_2\phi_2(+\infty)\Sigma_2(\omega_2 - \omega_1) \end{pmatrix},$$

where Σ_i is given in (2.13). The continuity of ω as function of $(\phi_1(+\infty), \phi_2(+\infty))$, ensured by the implicit function theorem, implies that

$$\lim_{|(\phi_1(+\infty), \phi_2(+\infty))| \to 0^+} JL(\omega) = \begin{pmatrix} \Phi_1'(\omega_1^*) & 0 \\ 0 & \Phi_2'(\omega_2^*) \end{pmatrix};$$

by (4.5) the limit matrix is a $\mathcal{D}^+$-matrix. As a consequence, there exists $\phi^* \in (0, \hat{\phi})$ such that for every $(\phi_1(+\infty), \phi_2(+\infty)) \in \mathbb{R}^2$ with $|\phi_i(+\infty)| < \phi^*$ the matrix $JL(\omega)$ is a $\mathcal{D}^+$-matrix, as well. The result is then proved. □

Remark 4.4 A dual result, ensuring the existence of solutions unbounded in the past, could be proved when (4.5) is replaced by

$$\Phi_1(\omega_1^*) = \Phi_2(\omega_2^*) = 0, \qquad \Phi_i'(\omega_i^*) > 0, \quad i = 1, 2.$$

We omit the details for briefness.

Remark 4.5 Let us analyze the result of Corollary 4.3 in the symmetric linear case $a_i = b_i = n^2$, $i = 1, 2$. In this situation, in the recent paper [3] the existence of unbounded solutions has been proved under the assumption

$$4|\phi_i(+\infty)| < |\widehat{p}_{i,n}|^2, \quad i = 1, 2, \tag{4.8}$$

where

$$\widehat{p}_{i,n} = \int_0^{2\pi} p_i(t)e^{int}\,dt \tag{4.9}$$

(see Theorem 3.1 in [3]). The assumption $|\phi_i(+\infty)| < \phi^*$ $(i = 1, 2)$, with $\phi^* = \phi^*(a_1, b_1, p_1, p_2)$, in Corollary 4.3 is then on the same spirit of (4.8).

Let us now focus on the situation where the function Φ_1 (or Φ_2) is identically zero, i.e.

$$\Phi_1(\theta_1) = 0, \quad \forall\, \theta_1 \in \mathbb{R}. \tag{4.10}$$

Incidentally, let us observe that in the linear symmetric case $a_1 = b_1 = n^2$ assumption (4.10) corresponds to the case when the number $\widehat{p}_{1,n}$ in (4.9) is zero. Instead, in the asymmetric case $a_1 \neq b_1$, condition (4.10) is more tricky to be checked. However, some examples in which it holds can be provided. For instance, if a_1 satisfies

$$\frac{\sqrt{a_1}}{n} = \frac{s}{1+2k} \qquad \text{for some } s, k \in \mathbb{N} \text{ and } s > k, \tag{4.11}$$

then the Fourier coefficient $c_{s,1}$ of C_1 vanishes (see (2.6)), and (4.10) holds when $p_1(t) = \cos snt$.

For the sake of brevity and clarity, we present here just a couple of corollaries in which (4.10) is assumed. In the first we suppose that a_2 is such that

$$c_{r,2} \neq 0 \quad \text{for some } r \in \mathbb{N}, \tag{4.12}$$

and that

$$p_2(t) = \mu \cos rnt, \quad \forall\, t \in \mathbb{R}, \tag{4.13}$$

with $\mu > 0$.

Corollary 4.6 *Let $a_i, b_i > 0$ satisfy, for $i = 1, 2$, assumption (4.2); moreover, suppose that*

$$(a_1, a_2) \in \mathcal{R}, \tag{4.14}$$

where $\mathcal{R}$ is defined in (2.14), and that (4.12) is fulfilled. Finally, assume that conditions (4.3), (4.10) and (4.13) are satisfied. Then, for every $\phi_1(+\infty) \neq 0$ and for every $\phi_2(+\infty) \in \mathbb{R}$ there exists $\mu^* > 0$ such that for every $\mu > \mu^*$ there exist two infinite measure sets $E^{\pm} \subset \mathbb{R}^2 \times \mathbb{R}^2$ such that:

- *for every solution x of (4.1) such that $(x(0), x'(0)) \in E^+$,*

$$\lim_{t\to+\infty} (|x_i(t)|^2 + |x_i'(t)|^2) = +\infty, \qquad i = 1, 2,$$

- *for every solution x of (4.1) such that $(x(0), x'(0)) \in E^-$*

$$\lim_{t\to-\infty} (|x_i(t)|^2 + |x_i'(t)|^2) = +\infty, \qquad i = 1, 2.$$

We observe that it is possible to find situations in which Corollary 4.6 applies. Indeed, let us first notice that Lemma 2.2 implies that (4.14) holds if (a_1, a_2) is close to (n^2, n^2). This happens, for instance if a_1 satisfies (4.11) with $s = 2k$ and k large enough, and if $\sqrt{a_2}$ is irrational and close to n. With these choices (4.10) holds with $p_1(t) = \cos(2knt)$, while (4.12) is trivially satisfied (see (2.6)).

Proof Let us first notice that, from (2.25) and (4.13), recalling the Fourier expansion of C_2 given in (2.5), we obtain

$$\Phi_2(\theta_2) = -\frac{\gamma_2}{2}\pi\mu c_{r,2}\cos r\theta_2, \quad \forall\, \theta_2 \in \mathbb{R}. \tag{4.15}$$

As a consequence, recalling (4.10), we obtain

$$\begin{aligned} L_1(\theta) &= \gamma_1 n\phi_1(+\infty)(\Lambda_1(\theta_1-\theta_2)-\alpha_1) \\ L_2(\theta) &= -\frac{\gamma_2}{2}\pi\mu c_{r,2}\cos r\theta_2 + \gamma_2 n\phi_2(+\infty)(\Lambda_2(\theta_2-\theta_1)-\alpha_2), \end{aligned} \tag{4.16}$$

for every $\theta \in \mathbb{R}^2$.
Now, let us look for solutions of $L(\theta) = 0$; from the relation $L_1(\theta) = 0$, recalling that $\phi_1(+\infty) \neq 0$, we deduce

$$\Lambda_1(\theta_1-\theta_2) = \alpha_1. \tag{4.17}$$

From Lemma 2.1, taking into account (4.14), we infer that there exists $\Lambda_1^* \in (0,\pi)$ such that

$$\begin{aligned} &\Lambda_1(t) = \alpha_1 \quad \Longleftrightarrow \quad t = \pm\Lambda_1^* + 2m\pi,\ m \in \mathbb{Z} \\ &\operatorname{sgn}(\Lambda_1'(\pm\Lambda_1^*)) = \mp 1. \end{aligned} \tag{4.18}$$

In particular, we choose $m = 0$; then, from (4.17) and (4.18) we obtain

$$\theta_1 - \theta_2 = \pm\Lambda_1^*. \tag{4.19}$$

Replacing the last equality in the expression of L_2 in (4.16) and recalling that Λ_2 is even and 2π-periodic, the equation $L_2(\theta) = 0$ reduces to

$$-\pi\mu c_{r,2}\cos r\theta_2 + 2n\phi_2(+\infty)(\Lambda_2(\Lambda_1^*)-\alpha_2) = 0, \tag{4.20}$$

i.e.

$$\cos r\theta_2 = \frac{2n\phi_2(+\infty)(\Lambda_2(\Lambda_1^*)-\alpha_2)}{\pi\mu c_{r,2}}. \tag{4.21}$$

Let now set

$$\hat{\mu} = \left|\frac{2n\phi_2(+\infty)(\Lambda_2(\Lambda_1^*)-\alpha_2)}{\pi c_{r,2}}\right|; \tag{4.22}$$

then, for every $\mu > \hat{\mu}$ the equation (4.21) can be solved and we obtain

$$\theta_2 = \frac{1}{r}\left(\pm\arccos\frac{2n\phi_2(+\infty)(\Lambda_2(\Lambda_1^*)-\alpha_2)}{\pi\mu c_{r,2}} + 2h\pi\right), \quad h \in \mathbb{Z}. \tag{4.23}$$

Choosing $h = 0$, we then conclude that, for every $\mu > \hat{\mu}$, the equation $L(\theta) = 0$ has the four solutions

$$\begin{aligned} \omega_\mu^{\pm,1} = \Bigg(&\Lambda_1^* + \frac{1}{r}\left(\pm\arccos\frac{2n\phi_2(+\infty)(\Lambda_2(\Lambda_1^*)-\alpha_2)}{\pi\mu c_{r,2}}\right), \\ &\frac{1}{r}\left(\pm\arccos\frac{2n\phi_2(+\infty)(\Lambda_2(\Lambda_1^*)-\alpha_2)}{\pi\mu c_{r,2}}\right)\Bigg) \end{aligned}$$

and

$$\omega_\mu^{\pm,2} = \left(-\Lambda_1^* + \frac{1}{r}\left(\pm\arccos\frac{2n\phi_2(+\infty)(\Lambda_2(\Lambda_1^*)-\alpha_2)}{\pi\mu c_{r,2}}\right),\right.$$

$$\frac{1}{r}\left(\pm \arccos \frac{2n\phi_2(+\infty)(\Lambda_2(\Lambda_1^*)-\alpha_2)}{\pi\mu c_{r,2}}\right)\bigg).$$

In order to apply Theorem 4.1, we claim that one of the above four solutions, to be named ω^+, is such that $JL(\omega^+)$ is a $\mathcal{D}^+$-matrix and another one, to be named ω^-, is such that $JL(\omega^-)$ is a $\mathcal{D}^-$-matrix. To do this, recalling (4.16) and the fact that $\Lambda_i' = \Sigma_i/n$ is 2π-periodic and odd, we observe that

$$\begin{aligned}
\partial_1 L_1(\omega_\mu^{\pm,i}) &= (-1)^{i+1}\gamma_1\phi_1(+\infty)\Sigma_1(\Lambda_1^*)\\
\partial_2 L_1(\omega_\mu^{\pm,i}) &= (-1)^{i}\gamma_1\phi_1(+\infty)\Sigma_1(\Lambda_1^*)\\
\partial_1 L_2(\omega_\mu^{\pm,i}) &= (-1)^{i}\gamma_2\phi_2(+\infty)\Sigma_2(\Lambda_1^*)\\
\partial_2 L_2(\omega_\mu^{\pm,i}) &= \pm\frac{\gamma_2}{2}\pi\mu c_{r,2} r \sin\arccos\frac{2n\phi_2(+\infty)(\Lambda_2(\Lambda_1^*)-\alpha_2)}{\pi\mu c_{r,2}} + (-1)^{i+1}\gamma_2\phi_2(+\infty)\Sigma_2(\Lambda_1^*),
\end{aligned} \tag{4.24}$$

for $i = 1, 2$. Now, since $\phi_1(+\infty) \neq 0$ and recalling (4.18), we have

$$\operatorname{sgn}(\partial_1 L_1(\omega_\mu^{\pm,i})) = (-1)^i \operatorname{sgn}(\phi_1(+\infty)); \tag{4.25}$$

moreover, there exists $\check{\mu} \geq \hat{\mu}$ such that for every $\mu > \check{\mu}$ we have

$$\operatorname{sgn}(\partial_2 L_2(\omega_\mu^{\pm,i})) = \pm \operatorname{sgn}(c_{r,2}). \tag{4.26}$$

Hence, for $\mu > \check{\mu}$, the choice of $\omega_\mu^{\pm,i}$ has to be made according to the signs of $\phi_1(+\infty)$ and $c_{r,2}$. For the sake of briefness, we discuss the case $\phi_1(+\infty) > 0$ and $c_{r,2} > 0$, the other ones being similar. We set $\omega^+ = \omega_\mu^{-,1}$ and $\omega^- = \omega_\mu^{+,2}$; hence, by construction, $JL(\omega^+)$ and $JL(\omega^-)$ satisfy the sign conditions on the diagonal coefficients in order to be a $\mathcal{D}^\pm$-matrix. As far as the third condition in Definition 3.1 is concerned, we have that

$$\begin{aligned}
\partial_1 L_1(\omega^\pm)\,\partial_1 L_2(\omega^\pm) + \partial_2 L_1(\omega^\pm)\,\partial_2 L_2(\omega^\pm) &= -2\gamma_1\gamma_2\phi_1(+\infty)\phi_2(+\infty)\Sigma_1(\Lambda_1^*)\Sigma_2(\Lambda_1^*)\\
&\quad + \gamma_1\frac{\gamma_2}{2}\pi\mu c_{r,2} rn \sin\arccos\frac{2n\phi_2(+\infty)(\Lambda_2(\Lambda_1^*)-\alpha_2)}{\pi\mu c_{r,2}}\phi_1(+\infty)\Sigma_1(\Lambda_1^*)
\end{aligned} \tag{4.27}$$

and

$$\begin{aligned}
2\partial_1 L_1(\omega^\pm)\,\partial_2 L_2(\omega^\pm) &= 2\gamma_1\gamma_2\phi_1(+\infty)\phi_2(+\infty)\Sigma_1(\Lambda_1^*)\Sigma_2(\Lambda_1^*)\\
&\quad - \gamma_1\gamma_2\pi\mu c_{r,2} rn \sin\arccos\frac{2n\phi_2(+\infty)(\Lambda_2(\Lambda_1^*)-\alpha_2)}{\pi\mu c_{r,2}}\phi_1(+\infty)\Sigma_1(\Lambda_1^*).
\end{aligned} \tag{4.28}$$

Hence, there exists $\mu^* \geq \check{\mu}$ such that for every $\mu > \mu^*$ the third condition in Definition 3.1 is satisfied; hence the values $\omega^\pm$ are such that $JL(\omega^\pm)$ is a $\mathcal{D}^\pm$ matrix. The thesis then follows from an application of Theorem 4.1. □

As a last application, we discuss the case when the oscillators are symmetric, i.e. $a_i = b_i = n^2$ for $i = 1, 2$, and (4.10) holds true; as already observed, this is equivalent to the assumption

$$\widehat{p}_{1,n} = 0, \tag{4.29}$$

where $\widehat{p}_{1,n}$ is as in (4.9). Let us observe that this situation is not covered by the results in [3].

Corollary 4.7 *Let $a_i = b_i = n^2$, for $i = 1, 2$, and suppose that conditions (4.3) and (4.29) are satisfied.*

Then, for every $\phi_1(+\infty) \neq 0$ *and for every* $\phi_2(+\infty) \in \mathbb{R}$ *such that*

$$|\phi_2(+\infty)| < \frac{3}{16}|\widehat{p}_{2,n}|, \tag{4.30}$$

with $\widehat{p}_{2,n}$ *as in* (4.9)*, there exist two infinite measure sets* $E^{\pm} \subset \mathbb{R}^2 \times \mathbb{R}^2$ *such that:*

- *for every solution* x *of* (4.1) *such that* $(x(0), x'(0)) \in E^+$*,*

$$\lim_{t\to+\infty}(|x_i(t)|^2 + |x_i'(t)|^2) = +\infty, \qquad i = 1, 2,$$

- *for every solution* x *of* (4.1) *such that* $(x(0), x'(0)) \in E^-$

$$\lim_{t\to-\infty}(|x_i(t)|^2 + |x_i'(t)|^2) = +\infty, \qquad i = 1, 2.$$

Proof First of all, let us observe that in this situation the functions C_i and Λ_i in (2.4) and (2.9) are given by

$$C_i(t) = \cos nt, \quad \Lambda_i(t) = \frac{2}{n}\cos t, \quad \forall\, t \in \mathbb{R}, \tag{4.31}$$

respectively, while the number α_i in (2.11) is zero. Moreover, the function Φ_2 in (2.25) becomes

$$\Phi_2(u) = -\frac{1}{\sqrt{2n}}\int_0^{2\pi} \cos(nt + u)p_2(t)dt, \quad \forall\, u \in \mathbb{R}; \tag{4.32}$$

this expression can be written as

$$\Phi_2(u) = -\frac{1}{\sqrt{2n}}|\widehat{p}_{2,n}|\cos(u + \psi_2), \quad \forall\, u \in \mathbb{R}, \tag{4.33}$$

for some $\psi_2 \in \mathbb{R}$.
From (4.31), (4.32), (4.33) and the assumption on $\widehat{p}_{1,n}$ we deduce that

$$\begin{aligned} L_1(\theta) &= 2\sqrt{\frac{2}{n}}\phi_1(+\infty)\cos(\theta_1 - \theta_2) \\ L_2(\theta) &= -\frac{1}{\sqrt{2n}}|\widehat{p}_{2,n}|\cos(\theta_2 + \psi_2) + 2\sqrt{\frac{2}{n}}\phi_2(+\infty)\cos(\theta_2 - \theta_1), \end{aligned} \tag{4.34}$$

for every $\theta \in \mathbb{R}^2$.
Recalling that $\phi_1(+\infty) \neq 0$, we can solve the equation $L_1(\theta) = 0$ to obtain

$$\theta_1 = \theta_2 \pm \frac{\pi}{2} + 2m\pi, \quad m \in \mathbb{Z}; \tag{4.35}$$

as a consequence the equation $L_2(\theta) = 0$ reduces to

$$|\widehat{p}_{2,n}|\cos(\theta_2 + \psi_2) = 0. \tag{4.36}$$

We now observe that assumption (4.30) implies that $\widehat{p}_{2,n} \neq 0$; hence, from (4.36) we infer that

$$\theta_2 = -\psi_2 \pm \frac{\pi}{2} + 2h\pi, \quad h \in \mathbb{Z}. \tag{4.37}$$

Choosing in particular $m = h = 0$, we conclude that the equation $L(\theta) = 0$ has the four solutions

$$\begin{aligned} \omega^{\pm,1} &= \left(-\psi_2 + \frac{\pi}{2} \pm \frac{\pi}{2}, -\psi_2 \pm \frac{\pi}{2}\right) \in \mathbb{R}^2, \\ \omega^{\pm,2} &= \left(-\psi_2 - \frac{\pi}{2} \pm \frac{\pi}{2}, -\psi_2 \pm \frac{\pi}{2}\right) \in \mathbb{R}^2. \end{aligned} \tag{4.38}$$

We now claim that one of the above four solutions, to be named ω^+, is such that $JL(\omega^+)$ is a $\mathcal{D}^+$-matrix and another one, to be named ω^-, is such that $JL(\omega^-)$ is a $\mathcal{D}^-$-matrix. To see this, let us observe that, from (4.38),

$$\begin{aligned}
\partial_1 L_1(\omega^{\pm,i}) &= \partial_1 L_1(\theta)_{|\theta=\omega^{\pm,i}} = -2\sqrt{\frac{2}{n}}\phi_1(+\infty)\sin(\theta_1-\theta_2)_{|\theta=\omega^{\pm,i}} = (-1)^i 2\sqrt{\frac{2}{n}}\phi_1(+\infty)\\
\partial_2 L_1(\omega^{\pm,i}) &= \partial_2 L_1(\theta)_{|\theta=\omega^{\pm,i}} = 2\sqrt{\frac{2}{n}}\phi_1(+\infty)\sin(\theta_1-\theta_2)_{|\theta=\omega^{\pm,i}} = (-1)^{i-1} 2\sqrt{\frac{2}{n}}\phi_1(+\infty)\\
\partial_1 L_2(\omega^{\pm,i}) &= \partial_1 L_2(\theta)_{|\theta=\omega^{\pm,i}} = 2\sqrt{\frac{2}{n}}\phi_2(+\infty)\sin(\theta_2-\theta_1)_{|\theta=\omega^{\pm,i}} = (-1)^i 2\sqrt{\frac{2}{n}}\phi_2(+\infty)\\
\partial_2 L_2(\omega^{\pm,i}) &= \partial_2 L_2(\theta)_{|\theta=\omega^{\pm,i}} = \frac{1}{\sqrt{2n}}|\widehat{p}_{2,n}|\sin(\theta_2+\psi_2) - 2\sqrt{\frac{2}{n}}\phi_2(+\infty)\sin(\theta_2-\theta_1)_{|\theta=\omega^{\pm,i}}\\
&= \pm\frac{1}{\sqrt{2n}}|\widehat{p}_{2,n}| + (-1)^{i-1} 2\sqrt{\frac{2}{n}}\phi_2(+\infty),
\end{aligned} \tag{4.39}$$

for $i = 1, 2$. Focusing for the sake of briefness on the case $\phi_1(+\infty) > 0$, we obtain from (4.30) that

$$\begin{aligned}
\operatorname{sgn}(\partial_1 L_1(\omega^{-,1}))\cdot\operatorname{sgn}(\partial_2 L_2(\omega^{-,1})) > 0\\
\operatorname{sgn}(\partial_1 L_1(\omega^{+,2}))\cdot\operatorname{sgn}(\partial_2 L_2(\omega^{+,2})) > 0.
\end{aligned} \tag{4.40}$$

Setting $\omega^+ = \omega^{+,2}$ and $\omega^- = \omega^{-,1}$, since

$$\partial_1 L_1(\omega^\pm)\,\partial_1 L_2(\omega^\pm) + \partial_2 L_1(\omega^\pm)\,\partial_2 L_2(\omega^\pm) = \frac{1}{n}\left(16\phi_1(+\infty)\phi_2(+\infty) - 2|\widehat{p}_{2,n}|\phi_1(+\infty)\right) \tag{4.41}$$

and

$$2\partial_1 L_1(\omega^\pm)\,\partial_2 L_2(\omega^\pm) = \frac{1}{n}\left(-16\phi_1(+\infty)\phi_2(+\infty) + 4|\widehat{p}_{2,n}|\phi_1(+\infty)\right), \tag{4.42}$$

from the same assumption (4.30) we deduce that

$$|\partial_1 L_1(\omega^\pm)\,\partial_1 L_2(\omega^\pm) + \partial_2 L_1(\omega^\pm)\,\partial_2 L_2(\omega^\pm)| < 2\partial_1 L_1(\omega^\pm)\,\partial_2 L_2(\omega^\pm) \tag{4.43}$$

as well. From (4.40) and (4.43) we conclude that $JL(\omega^\pm)$ is a $\mathcal{D}^\pm$-matrix. The thesis then follows from an application of Theorem 4.1. □

Acknowledgements The authors are grateful to Rafael Ortega for having proposed the subject of this investigation and for his enduring encouragement.

Funding Open access funding provided by Università degli Studi di Udine within the CRUI-CARE Agreement.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflicts of Interest The authors declare no conflicts of interest.

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