



# Nonlocal problems with Hardy–Littlewood–Sobolev critical exponent and Hardy potential

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**Abstract.** We are concerned with a Brezis–Nirenberg-type problem for a critical Choquard equation, in the sense of Hardy–Littlewood–Sobolev inequality, and with the Hardy potential in a smooth bounded domain. By exploiting variational methods, we obtain existence results, which extend to different perturbation terms. Some estimates of independent interest about a nonlocal minimization problem are also derived.

**Mathematics Subject Classification.** 35J20, 35J25, 35J60.

**Keywords.** Choquard equation, Hardy potential, Critical exponent, Variational method.

## 1. Introduction

In this paper, we consider the following nonlocal problem:

$$\begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = \left( \int_{\Omega} \frac{|u(y)|^{2^*_{\alpha}}}{|x-y|^{\alpha}} dy \right) |u|^{2^*_{\alpha}-1} + \lambda f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (1.1)$$

where  $\Omega \subset \mathbb{R}^N (N \geq 3)$  is a smooth bounded domain with smooth boundary  $\partial\Omega$ ,  $0 \in \Omega$ ,  $0 < \alpha < N - (N - 4)_+$ ,  $0 < \mu < \bar{\mu} = \frac{(N-2)^2}{4}$ ,  $2^*_{\alpha} := \frac{2N-\alpha}{N-2}$  is the upper critical exponent in the sense of the Hardy–Littlewood–Sobolev inequality. Moreover, the problem features the so-called Hardy potential. This potential arises in many physical contexts such as nonrelativistic quantum mechanics, molecular physics and quantum cosmology (see [17, 19, 31]). Since the Hardy potential does not belong to Kato class ([38]), equation (1.1) can be seen as doubly critical problem, which brings some new difficulties and analytical challenges, especially with respect to the lack of compactness.

When  $f(u) = \lambda u$  and  $\alpha \rightarrow N$ , equation (1.1) reduces to the following elliptic problem:

$$\begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = |u|^{2^*-1} + \lambda u, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \quad (1.2)$$

where  $2^* := \frac{2N}{N-2}$  is the critical Sobolev exponent. The existence of solution of equation (1.2) for  $\mu < \bar{\mu}$  depends on the spectrum  $\sigma_{\mu}$  of the operator  $-\Delta u - \frac{\mu}{|x|^2}$  with Dirichlet boundary conditions. Jannelli [30] proved that equation (1.2) possesses at least one positive solution, provided that  $\mu \in (0, \bar{\mu} - 1]$  and  $\lambda \in (0, \lambda_1)$  or  $\Omega = B(0, 1)$  is a ball,  $\mu \in (\bar{\mu} - 1, \bar{\mu})$  and  $\lambda \in (\lambda^*, \lambda_1)$ , where  $\lambda^* \in (0, \lambda_1)$  is a suitable constant depending on  $\mu$  and  $\lambda_1$  is the first eigenvalue of the operator  $-\Delta u - \frac{\mu}{|x|^2}$  with zero Dirichlet boundary condition. Ferrero and Gazzola [18] showed that equation (1.2) has at least one positive solution for

$N \geq 4$ ,  $\lambda > 0$  and  $\lambda \notin \sigma_\mu$ ; if  $\mu \geq 0$ ,  $\mu \in (\bar{\mu} - 1, \bar{\mu})$  there exists  $\lambda_k \in \sigma_\mu$  such that  $\lambda \in (\lambda_k - S_\mu |\Omega|^{-2/N}, \lambda_k)$  then equation (1.2) has  $\nu_k$  pairs of nontrivial solutions, where  $\nu_k$  denotes the multiplicity of  $\lambda_k$  and

$$S_\mu := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx - \mu \int_{\mathbb{R}^N} \frac{u^2}{x^2} dx}{\left( \int_{\mathbb{R}^N} |u|^{2^*} dx \right)^{2/2^*}}.$$

Cao and Han [3] proved that for  $N \geq 5$ ,  $\mu \in [0, \bar{\mu} - (\frac{N+2}{N})^2]$  and  $\lambda \in (0, \lambda_1)$ , equation (1.2) has a nontrivial solution. Cao and Yan [6] showed the existence of infinitely many solutions for  $N \geq 5$ ,  $\mu \in [0, \bar{\mu} - 4]$  and  $\lambda \in (0, \lambda_1)$ . If  $N \geq 7$  and  $\mu \in [0, \bar{\mu} - 4]$ , Cao and Peng [5] proved that equation (1.2) admits at least a pair of sign-changing solutions for  $\lambda \in (0, \lambda_1)$ . Chen and Zou [11] also considered equation (1.2) and showed that for  $N \geq 7$  and  $\mu \in [0, \bar{\mu} - 4]$ , equation (1.2) has infinitely many sign-changing solutions for  $\lambda > 0$ . Furthermore, if  $\lambda \in [\lambda_n, \lambda_{n+1})$  for some  $n \geq 1$ , equation (1.2) has a ground state solution. Other results can be found in [4, 7, 10, 13, 15, 16, 18, 23, 39] and the references quoted therein. We also refer the interested reader to [2, 8, 9, 26, 29] and the discussion in the sequel concerning the results of (1.2) in the case of  $\mu = 0$  where equation (1.1) reduces to the well-known Brézis–Nirenberg problem

$$\begin{cases} -\Delta u = |u|^{2^*-1} + \lambda u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

which has been extensively studied by many researchers after, particularly, the celebrated papers [2] carried out by Brézis and Nirenberg.

The nonlocal elliptic equation (1.1) is related to the following nonlinear Choquard equation:

$$-\Delta u + V(x)u = \left( \int_{\mathbb{R}^N} \frac{F(u(y))}{|x-y|^\alpha} dy \right) f(u(x)), \tag{1.4}$$

where  $0 < \alpha < N$  and  $F$  is the primitive of  $f$ . Equation (1.4) arises from the study of Bose–Einstein condensation and can be used to depict the finite range many-body interactions between particles. When  $N = 3$ ,  $\alpha = 1$  and  $f(u) = u$ , equation (1.4) was proposed by Choquard [33] as an approximation to Hartree–Fock theory for a one component plasma and was also introduced by Pekar [37] to describe the quantum theory of a polaron at rest. See also [1, 12, 14, 20, 24, 25, 32, 35, 40], and the references therein for more mathematical and physical background as well as the existence of solutions of equation (1.4).

The appearance of the nonlocal term  $(|x|^{-\alpha} * F(u))f(u)$  gives rise to some mathematical difficulties for the study of Choquard equation, which has received increasing attention from many authors. Recently, Gao and Yang in [22] considered equation (1.1) with  $\mu = 0$  and  $f(u) = \lambda u$ , that is, the following Brézis–Nirenberg-type problem of the nonlinear Choquard equation

$$\begin{cases} -\Delta u = \left( \int_{\Omega} \frac{|u(y)|^{2_\alpha^*}}{|x-y|^\alpha} dy \right) |u|^{2_\alpha^*-1} + \lambda u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.5}$$

and obtained some existence results corresponding to the well-known results in [2] by exploiting the concentration properties of the Talenti–Aubin functions and the variational methods. In [22], the compactness of the  $(PS)_c$  sequence associated with (1.5) holds true, provided that  $c$  is strictly less than a threshold value related to the best constant of the Hardy–Littlewood–Sobolev inequality. Moreover, the authors showed the non-existence of solution by using the Pohožaev identity. Later, they [21] also established some existence and multiplicity results for (1.5) with  $\mu = 0$ . In [28], the author proved that (1.5) has infinitely many solutions by applying the truncation method. Yang, Ye and Zhao [42] obtain the existence and asymptotic behavior of the solutions of (1.5) by using the Lyapunov–Schmidt reduction procedure. Pan, Wen and Yang [36] consider the qualitative analysis the blow-up solutions of (1.5).

Guo and Tang [27] consider the following nonlocal problem with Hardy potential

$$-\Delta u - \mu \frac{u}{|x|^2} = \left( \int_{\mathbb{R}^N} \frac{|u(y)|^{2_\alpha^*}}{|x-y|^\alpha} dy \right) |u|^{2_\alpha^*-1}, \quad x \in \mathbb{R}^N. \tag{1.6}$$

They proved the existence and symmetry of solutions by developing a suitable version of concentration-compactness lemma and the moving plane method. Up to a constant, the solutions of equation (1.6) are the minimizers of the problem:

$$S_{H,\alpha} := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx - \mu \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx}{\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u|^{2_\alpha^*}(x)|u(y)|^{2_\alpha^*}}{|x-y|^\alpha} dx dy \right)^{1/2_\alpha^*}},$$

where  $D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$ . Assume that  $u_\mu$  is a positive solutions of equation (1.6), define

$$u_{\mu,\varepsilon} = \varepsilon^{\frac{2-N}{2}} u_\mu\left(\frac{x}{\varepsilon}\right),$$

then  $u_{\mu,\varepsilon}$  is also the positive solution of equation (1.6). In particular, due to the homogeneity of equation (1.6),  $u_{\mu,\varepsilon}$  are still the minimizers for  $S_{H,\alpha}$  with

$$\int_{\mathbb{R}^N} |\nabla u_{\mu,\varepsilon}|^2 dx - \mu \int_{\mathbb{R}^N} \frac{|u_{\mu,\varepsilon}|^2}{|x|^2} dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_{\mu,\varepsilon}(x)|^{2_\mu^*} |u_{\mu,\varepsilon}(y)|^{2_\mu^*}}{|x-y|^\alpha} dx dy = S_{H,\alpha}^{\frac{2N-\alpha}{N-\alpha+2}}. \tag{1.7}$$

Furthermore, Guo and Tang [27] obtained the asymptotic behavior of the extremal function  $u_\mu$ . More precisely, there exist positive constants  $C_1$  and  $C_2$ , such that

$$\frac{C_1}{(|x|^{\gamma'/\sqrt{\mu}} + |x|^{\gamma/\sqrt{\mu}})\sqrt{\mu}} \leq u_\mu(x) \leq \frac{C_2}{(|x|^{\gamma'/\sqrt{\mu}} + |x|^{\gamma/\sqrt{\mu}})\sqrt{\mu}}, \tag{1.8}$$

where  $N \geq 3$ ,  $0 < \alpha < N - (N - 4)_+$ ,  $\mu < \bar{\mu} = \frac{(N-2)^2}{4}$ ,  $\gamma = \sqrt{\mu} + \sqrt{\mu - \mu}$  and  $\gamma' = \sqrt{\mu} - \sqrt{\mu - \mu}$ .

Our first result is about  $S_{H,\alpha}$  and its counterpart on bounded domains, for which we get the following estimates.

**Theorem 1.1.** (1) *Let  $N \geq 3$ ,  $0 < \alpha < N - (N - 4)_+$ ,  $0 < \mu < \bar{\mu} = \frac{(N-2)^2}{4}$ , then  $S_{H,\alpha}$  satisfies*

$$\frac{1}{C(N, \alpha)^{1/2_\alpha^*}} S_\mu < S_{H,\alpha} < \frac{1}{C(N, \alpha)^{1/2_\alpha^*}} S,$$

where

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left( \int_{\mathbb{R}^N} |u|^{2^*} dx \right)^{2/2^*}} \text{ and } S_\mu := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx - \mu \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx}{\left( \int_{\mathbb{R}^N} |u|^{2^*} dx \right)^{2/2^*}}.$$

(2) *Let  $N \geq 4$ ,  $0 < \alpha < N - (N - 4)_+$ . If  $\mu < 0$ , then  $S_{H,\alpha}(\Omega) = \frac{1}{C(N,\alpha)^{1/2_\alpha^*}} S$  and is not attained for any  $\Omega$ , where*

$$S_{H,\alpha}(\Omega) := \inf_{u \in D_0^{1,2}(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx - \mu \int_\Omega \frac{u^2}{|x|^2} dx}{\left( \int_\Omega \int_\Omega \frac{|u|^{2_\alpha^*}(x)|u(y)|^{2_\alpha^*}}{|x-y|^\alpha} dx dy \right)^{1/2_\alpha^*}}.$$

We next turn to the existence of nontrivial solution to the following equation,

$$\begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = \left( \int_\Omega \frac{|u(y)|^{2_\alpha^*}}{|x-y|^\alpha} dy \right) |u(x)|^{2_\alpha^*-1} + \lambda u, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.9}$$

and we obtain the following conclusions.

- Theorem 1.2.** (1) *Let  $N \geq 3$ ,  $0 < \alpha < N$  and  $0 < \mu \leq \bar{\mu}$ , Then, equation (1.9) has a solution for all  $\lambda \in (0, \lambda_1)$ .*  
 (2) *Let  $N \geq 3$ ,  $0 < \alpha < N - (N - 4)_+$  and  $0 < \mu \leq \bar{\mu} - 1$ , Then, equation (1.9) has at least one nontrivial solution for all  $\lambda \in (0, \lambda_1)$ .*

For the following critical Choquard equation with superlinear perturbation,

$$\begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = \left( \int_{\Omega} \frac{|u(y)|^{2^*_{\alpha}}}{|x-y|^{\alpha}} dy \right) |u|^{2^*_{\alpha}-1} + \lambda u^q, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.10}$$

our third main result is the following:

**Theorem 1.3.** *Let  $N \geq 3$ ,  $0 < \alpha < N - (N - 4)_+$ ,  $0 < \mu < \bar{\mu}$  and  $1 < q < 2^* - 1$ , Then, equation (1.10) has at least one nontrivial solution, provided that*

- $N > \frac{2(2\alpha+q+1)}{2\alpha+q-1}$  and  $\lambda > 0$ , or
- $N < \frac{2(2\alpha+q+1)}{2\alpha+q-1}$  and  $\lambda > \lambda_0$ , for  $\lambda_0$  large enough.

Analogously, we have an existence result for critical Choquard equation with superlinear nonlocal perturbation.

**Theorem 1.4.** *Let  $N \geq 3$ ,  $0 < \alpha < N - (N - 4)_+$ ,  $0 < \mu < \bar{\mu}$  and  $1 < p < 2^*_{\alpha} - 1$ , Then, equation*

$$\begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = \left( \int_{\Omega} \frac{|u(y)|^{2^*_{\alpha}}}{|x-y|^{\alpha}} dy \right) |u|^{2^*_{\alpha}-1} + \lambda \left( \int_{\Omega} \frac{|u(y)|^p}{|x-y|^{\alpha}} dy \right) |u|^{p-1}, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.11}$$

has at least one nontrivial solution, provided that

- $N > \frac{2\alpha-\alpha+2p}{\alpha+p-2}$  and  $\lambda > 0$ , or
- $N \leq \frac{2\alpha-\alpha+2p}{\alpha+p-2}$  and  $\lambda$  is sufficiently large.

Regarding the aforementioned three results, we primarily employ variational methods. Associated with equation (1.1), we consider the following energy functional

$$I := \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{\mu}{2} \int_{\Omega} \frac{|u|^2}{|x|^2} dx - \frac{1}{22^*_{\alpha}} \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2^*_{\alpha}} |u(y)|^{2^*_{\alpha}}}{|x-y|^{\alpha}} dx dy - \lambda \int_{\Omega} F(u) dx,$$

where  $F(u) = \int_0^u f(t) dt$ . It follows that the critical points of  $I$  are solutions of equation (1.1). In comparison with the previous works, the investigation of (1.1) presents significantly enhanced complexity owing to the synergistic interaction between the critical Hardy term and the upper critical growth of Choquard-type nonlinear term. It is commonly acknowledged that the key step in addressing critical problems lies in obtaining a precise critical level associated with the sharp constant  $S_{H,\alpha}$  within which one can prove a local Palais–Smale ((PS) for short) condition. Due to the absence of a specific expression for the extremal function  $u_{\mu}(x)$ , more intricate strategies are demanded to reconcile the competing effects between the Hardy potential and the nonlocal convolution term. It is worth emphasizing that when  $\mu = 0$ , Gao and Yang [22] have derived the specific expression for the extremal function  $u_{\mu}(x)$  of  $S_{H,\alpha}$ . Their seminal work provides a foundational reference point for subsequent investigations, yet the generalization to  $\mu \neq 0$  remains an active area of research requiring innovative analytical frameworks. The present study contributes to this endeavor by the variational approach that systematically accounts for the interplay between local singular effects and non-local convolution dynamics.

The paper is organized as follows: In sect. 2, we collect some basic results needed in the sequel, in Sect. 3 we derive the estimates about the minimization problem and in the last three sections we obtain the existence results for the different perturbation terms.

## 2. Notations and preliminary results

Before proving our results, we consider the work space  $H_0^1(\Omega)$  which is the completion of  $C_0^\infty(\Omega)$  with the norm

$$\|u\| := \left( \int_{\Omega} |\nabla u|^2 dx \right)^{\frac{1}{2}}.$$

For all  $\mu \in [0, \bar{\mu})$ , we endow the scalar product of the Hilbert space  $H = H_0^1(\Omega)$ ,

$$\langle u, v \rangle_{\mu} = \int_{\Omega} \nabla u \nabla v dx - \mu \int_{\Omega} \frac{uv}{|x|^2} dx$$

and define the norm

$$\|u\|_{\mu}^2 = \langle u, u \rangle_{\mu} = \int_{\Omega} |\nabla u|^2 dx - \mu \int_{\Omega} \frac{|u|^2}{|x|^2} dx.$$

By the Hardy inequality [23]

$$\int_{\Omega} \frac{|u|^2}{|x|^2} dx \leq \frac{1}{\bar{\mu}} \int_{\Omega} |\nabla u|^2 dx, \quad \forall u \in H,$$

it is not difficult to derive that the norm  $\|\cdot\|_{\mu}$  is equivalent to the usual normal  $\|u\| = \left( \int_{\Omega} |\nabla u|^2 dx \right)^{1/2}$ . It is well known that  $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$  continuously for  $p \in [1, 2^*]$ , compactly for  $p \in [1, 2^*)$ . Moreover, write  $\mathcal{L}_{\mu} := -\Delta - \frac{\mu}{|x|^2}$  with  $0 < \mu < \bar{\mu} := \frac{(N-2)^2}{4}$ . Then, according to [15, 16], the spectrum  $\sigma_{\mu} \subset (0, +\infty)$  of  $\mathcal{L}_{\mu}$  is discrete and each eigenvalue  $\lambda_i$  ( $i = 1, 2, \dots$ ) is isolated and has finite multiplicity. Moreover, the smallest eigenvalue  $\lambda_1$  is simple and  $\lambda_i \rightarrow \infty$  as  $i \rightarrow \infty$ .

We will need to make use of the well-known Hardy–Littlewood–Sobolev inequality [34]:

**Proposition 2.1.** (Hardy–Littlewood–Sobolev inequality) *Let  $t, r > 1$ , and  $0 < \alpha < N$  with  $1/t + 1/r + \alpha/N = 2$ ,  $f \in L^t(\mathbb{R}^N)$  and  $g \in L^r(\mathbb{R}^N)$ . There exists a sharp constant  $C(t, r, N, \alpha) > 0$ , independent of  $f$  and  $g$ , such that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^{\alpha}} dx dy \leq C(t, r, N, \alpha) \|f\|_t \|g\|_r. \tag{2.1}$$

Here,  $\|f\|_t = \left( \int_{\mathbb{R}^N} |f|^t \right)^{1/t}$ . If  $t = r = \frac{2N}{2N-\alpha}$ , then

$$C(t, r, N, \alpha) = C(N, \alpha) = \pi^{\frac{\alpha}{2}} \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{2N-\alpha}{2})} \left( \frac{\Gamma(\frac{N}{2})}{\Gamma(N)} \right)^{\frac{\alpha}{N}-1}.$$

In this case, the equality in (2.1) holds if and only if  $f \equiv Cg$  and

$$g(x) = A \left( a + |x - x_0|^2 \right)^{\frac{\alpha-2N}{2}}$$

for some  $A \in \mathbb{C}, x_0 \in \mathbb{R}^N$  and  $0 \neq a \in \mathbb{R}$ .

If  $f(x) = h(x) = u^p(x)$  in (2.1), the following integral

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{\alpha}} dx dy$$

is well-defined provided

$$\frac{2N - \alpha}{N} \leq p \leq \frac{2N - \alpha}{N - 2}.$$

$2_{*\alpha} := \frac{2N - \alpha}{N}$  is called the lower critical exponent and  $2^*_\alpha := \frac{2N - \alpha}{N - 2}$  the upper critical exponent in the sense of the Hardy–Littlewood–Sobolev inequality.

By the Hardy–Littlewood–Sobolev inequality, for all  $u \in D^{1,2}(\mathbb{R}^N)$ , we get

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\alpha} |u(y)|^{2^*_\alpha}}{|x - y|^\alpha} dx dy \leq C(N, \alpha) \|u\|_{2^*_\alpha}^{2 \cdot 2^*_\alpha} < +\infty, \tag{2.2}$$

where  $C(N, \alpha)$  is defined as in Proposition 2.1.

### 3. Estimates

Regarding  $S_{H,\alpha}$ , we get the following estimate.

**Lemma 3.1.** *Let  $N \geq 3$ ,  $0 < \alpha < N - (N - 4)_+$ ,  $0 < \mu < \bar{\mu} = \frac{(N-2)^2}{4}$ , then  $S_{H,\alpha}^*$  defined in (1) satisfies*

$$\frac{1}{C(N, \alpha)^{1/2^*_\alpha}} S_\mu \leq S_{H,\alpha} < \frac{1}{C(N, \alpha)^{1/2^*_\alpha}} S,$$

where

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*_\alpha} dx\right)^{2/2^*_\alpha}} \text{ and } S_\mu := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx - \mu \int_{\mathbb{R}^N} \frac{u^2}{x^2} dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*_\alpha} dx\right)^{2/2^*_\alpha}}.$$

*Proof.* On the one side, by the Hardy–Littlewood–Sobolev inequality, we have

$$\begin{aligned} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u|^{2^*_\alpha}(x) |u|^{2^*_\alpha}(y)}{|x - y|^\alpha} dx dy\right)^{1/2^*_\alpha} &\leq C(N, \alpha)^{1/2^*_\alpha} \|u\|_{2^*_\alpha}^2 \\ &\leq C(N, \alpha)^{1/2^*_\alpha} S_\mu^{-1} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx - \mu \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx\right), \end{aligned}$$

which implies that

$$\frac{S_\mu}{C(N, \alpha)^{1/2^*_\alpha}} \leq S_{H,\alpha}.$$

On the other side, by (2.2), it is easy to see that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\bar{U}(x)|^{2^*_\alpha} |\bar{U}(y)|^{2^*_\alpha}}{|x - y|^\alpha} dx dy = C(N, \alpha) \|\bar{U}\|_{2^*_\alpha}^{2 \cdot 2^*_\alpha}$$

if and only if

$$\bar{U} = A \left(a + |x - x_0|^2\right)^{\frac{2-N}{2}}$$

for some  $A \in \mathbb{R}, x_0 \in \mathbb{R}^N$  and  $0 \neq a \in \mathbb{R}$ , where  $\bar{U}$  is a minimizer for  $S$ . It follows from  $0 < \mu < \bar{\mu}$  that

$$\begin{aligned} S_{H,\alpha} &= \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx - \mu \int_{\mathbb{R}^N} \frac{u^2}{x^2} dx}{\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u|^{2^*_\alpha}(x)|u(y)|^{2^*_\alpha}}{|x-y|^\alpha} dx dy \right)^{1/2^*_\alpha}} \\ &\leq \frac{\int_{\mathbb{R}^N} |\nabla \bar{U}|^2 dx - \mu \int_{\mathbb{R}^N} \frac{\bar{U}^2}{x^2} dx}{\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\bar{U}|^{2^*_\alpha}(x)|\bar{U}(y)|^{2^*_\alpha}}{|x-y|^\alpha} dx dy \right)^{1/2^*_\alpha}} \\ &< \frac{1}{C(N, \alpha)^{1/2^*_\alpha}} \frac{\int_{\mathbb{R}^N} |\nabla \bar{U}|^2 dx}{\left( \int_{\mathbb{R}^N} |\bar{U}|^{2^*} dx \right)^{2/2^*}}, \end{aligned}$$

which means  $S_{H,\alpha} < \frac{1}{C(N,\alpha)^{1/2^*_\alpha}} S$ , where

$$S = \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left( \int_{\mathbb{R}^N} |u|^{2^*} dx \right)^{2/2^*}}.$$

Therefore,  $\frac{1}{C(N,\alpha)^{1/2^*_\alpha}} S_\mu \leq S_{H,\alpha} < \frac{1}{C(N,\alpha)^{1/2^*_\alpha}} S$ . □

Denote  $S_{H,\alpha}(\Omega)$  as follows:

$$S_{H,\alpha}(\Omega) := \inf_{u \in D_0^{1,2}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 dx - \mu \int_{\Omega} \frac{u^2}{x^2} dx}{\left( \int_{\Omega} \int_{\Omega} \frac{|u|^{2^*_\alpha}(x)|u(y)|^{2^*_\alpha}}{|x-y|^\alpha} dx dy \right)^{1/2^*_\alpha}}.$$

Regarding  $S_{H,\alpha}(\Omega)$  we get the following estimate.

**Lemma 3.2.** *Let  $N \geq 4, 0 < \alpha < N - (N - 4)_+$ . If  $\mu < 0$ , then  $S_{H,\alpha}(\Omega) = \frac{1}{C(N,\alpha)^{1/2^*_\alpha}} S$  and is not attained for any  $\Omega$ , where*

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left( \int_{\mathbb{R}^N} |u|^{2^*} dx \right)^{2/2^*}}.$$

*Proof.* Assume that  $U(x) = \frac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}$ . Let  $\phi \in C_0^\infty(\Omega)$  be a negative function such that  $\phi(x) = 1, \forall x \in B(0, \delta), \phi(x) = 0, \forall x \in \Omega \setminus B(0, 2\delta)$  and  $0 \leq \phi(x) \leq 1, \forall x \in \Omega$ , where  $2\delta < |\xi|$  and  $B(0, 3|\xi|) \subset \Omega$ . For  $\varepsilon > 0$ , we define,

$$u_\varepsilon(x) := \psi(x)U_\varepsilon(x) \text{ where } U_\varepsilon(x) := \varepsilon^{\frac{2-N}{2}}U\left(\frac{x}{\varepsilon}\right).$$

It follows from [22] that, as  $\varepsilon \rightarrow 0^+$ ,

$$\int_{\Omega} |\nabla u_\varepsilon|^2 dx = S^{\frac{N}{2}} + O(\varepsilon^{N-2}) = C(N, \alpha)^{\frac{N(N-2)}{2(2N-\alpha)}} S_{H,L}^{\frac{N}{2}} + O(\varepsilon^{N-2}),$$

$$\int_{\Omega} |u_\varepsilon|^2 dx = \begin{cases} O(\varepsilon^2 |\ln \varepsilon|), & \text{if } N = 4, \\ O(\varepsilon^2), & \text{if } N \geq 5, \end{cases}$$

and

$$\int_{\Omega} \int_{\Omega} \frac{|u_\varepsilon(x)|^{2^*_\alpha} |u_\varepsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha} dx dy \geq C(N, \alpha)^{\frac{N}{2}} S_{H,L}^{\frac{2N-\alpha}{2}} - O(\varepsilon^{\frac{2N-\alpha}{2}}) = C(N, \alpha) S^{\frac{2N-\alpha}{2}} - O(\varepsilon^{\frac{2N-\alpha}{2}}).$$

Denote  $v_\varepsilon(x) = u_\varepsilon(x + 2\xi)$ , by the invariance by translation of the integral,

$$\int_{\Omega} \frac{|v_\varepsilon(x)|^2}{|x|^2} = \int_{\Omega} \frac{|u_\varepsilon(x + 2\xi)|^2}{|x|^2} \leq \frac{1}{|\xi|^2} \int_{\Omega} |u_\varepsilon(x)|^2.$$

Combining this and  $\mu < 0$ ,

$$\begin{aligned} S_{H,\alpha}(\Omega) &\leq \frac{\int_{\Omega} |\nabla v_\varepsilon|^2 dx - \mu \int_{\Omega} \frac{v_\varepsilon^2}{x^2} dx}{\left( \int_{\Omega} \int_{\Omega} \frac{|v_\varepsilon(x)|^{2_\alpha^*} |v_\varepsilon(y)|^{2_\alpha^*}}{|x-y|^\alpha} dx dy \right)^{1/2_\alpha^*}} \\ &\leq \frac{\int_{\Omega} |\nabla u_\varepsilon(x)|^2 dx - \frac{\mu}{|\xi|^2} \int_{\Omega} |u_\varepsilon(x)|^2 dx}{\left( \int_{\Omega} \int_{\Omega} \frac{|u_\varepsilon(x)|^{2_\alpha^*} |u_\varepsilon(y)|^{2_\alpha^*}}{|x-y|^\alpha} dx dy \right)^{1/2_\alpha^*}}. \end{aligned}$$

Now we distinguish the following cases:

(i) In the case  $N = 4$ ,

$$\begin{aligned} S_{H,\alpha}(\Omega) &\leq \frac{\int_{\Omega} |\nabla u_\varepsilon(x)|^2 dx - \frac{\mu}{|\xi|^2} \int_{\Omega} |u_\varepsilon(x)|^2 dx}{\left( \int_{\Omega} \int_{\Omega} \frac{|u_\varepsilon(x)|^{2_\alpha^*} |u_\varepsilon(y)|^{2_\alpha^*}}{|x-y|^\alpha} dx dy \right)^{1/2_\alpha^*}} \\ &\leq \frac{S^2 + O(\varepsilon^2) + O(\varepsilon^2 |\ln \varepsilon|)}{\left( C(N, \alpha) S^{\frac{8-\alpha}{2}} - O(\varepsilon^{\frac{8-\alpha}{2}}) \right)^{\frac{2}{8-\alpha}}} \\ &\leq \frac{1}{C(N, \alpha)^{\frac{2}{8-\alpha}}} S. \end{aligned}$$

(ii) In the case  $N \geq 5$ ,

$$\begin{aligned} S_{H,\alpha}(\Omega) &\leq \frac{\int_{\Omega} |\nabla u_\varepsilon(x)|^2 dx - \frac{\mu}{|\xi|^2} \int_{\Omega} |u_\varepsilon(x)|^2 dx}{\left( \int_{\Omega} \int_{\Omega} \frac{|u_\varepsilon(x)|^{2_\alpha^*} |u_\varepsilon(y)|^{2_\alpha^*}}{|x-y|^\alpha} dx dy \right)^{1/2_\alpha^*}} \\ &\leq \frac{S^{\frac{N}{2}} + O(\varepsilon^{N-2}) + O(\varepsilon^2)}{\left( C(N, \alpha) S^{\frac{2N-\alpha}{2}} - O(\varepsilon^{\frac{2N-\alpha}{2}}) \right)^{1/2_\alpha^*}} \\ &\leq \frac{1}{C(N, \alpha)^{1/2_\alpha^*}} S. \end{aligned}$$

Thus,  $S_{H,\alpha}(\Omega) \leq \frac{1}{C(N,\alpha)^{1/2_\alpha^*}} S$ . We deduce from [22] and  $\mu < 0$  that  $\frac{1}{C(N,\alpha)^{1/2_\alpha^*}} S \leq S_{H,\alpha}(\Omega)$ . Then  $S_{H,\alpha}(\Omega) = \frac{1}{C(N,\alpha)^{1/2_\alpha^*}} S$ . Moreover, the infimum  $S_{H,\alpha}(\Omega)$  cannot be attained because such a minimizer would contradict the fact that  $S$  is never achieved except when  $\Omega = \mathbb{R}^N$ .  $\square$

#### 4. Linear perturbation

In this section, we consider the existence of the solutions to the following equation

$$\begin{cases} -\Delta u(x) - \mu \frac{u}{|x|^2} = \left( \int_{\Omega} \frac{|u(y)|^{2_\alpha^*}}{|x-y|^\alpha} dy \right) |u|^{2_\alpha^*-1} + \lambda u, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \quad (4.1)$$

The energy functional corresponding to (4.1) is

$$\mathcal{I}(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{\mu}{2} \int_{\Omega} \frac{|u|^2}{|x|^2} dx - \frac{\lambda}{2} \int_{\Omega} |u|^2 dx - \frac{1}{2 \cdot 2_{\alpha}^*} \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2_{\alpha}^*} |u(y)|^{2_{\alpha}^*}}{|x-y|^{\alpha}} dx dy.$$

It is not difficult to check that the functional  $\mathcal{I}$  satisfies the Mountain–Pass Geometry, that is

**Lemma 4.1.** *The functional  $\mathcal{I}$  has the mountain pass geometry:*

- (1) *There exist  $\beta, \rho > 0$  such that  $\mathcal{I}(u) \geq \beta$  whenever  $\|u\| = \rho$ .*
- (2) *There is an  $e \in H_0^1(\Omega)$  with  $\|e\| \geq \rho$  such that  $\mathcal{I}(e) < 0$ .*

*Proof.* (1) By  $0 < \mu < \bar{\mu}$ , the Sobolev embedding and the Hardy–Littlewood–Sobolev inequality, for all  $u \in H_0^1(\Omega) \setminus \{0\}$  we have

$$\begin{aligned} \mathcal{I}(u) &= \frac{1}{2} \|u\|_{\mu}^2 - \frac{\lambda}{2} \int_{\Omega} |u|^2 dx - \frac{1}{2 \cdot 2_{\alpha}^*} \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2_{\alpha}^*} |u(y)|^{2_{\alpha}^*}}{|x-y|^{\alpha}} dx dy \\ &\geq \left( \frac{1}{2} - \frac{\lambda}{2\lambda_1} \right) \|u\|_{\mu}^2 - \frac{1}{2 \cdot 2_{\alpha}^*} \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2_{\alpha}^*} |u(y)|^{2_{\alpha}^*}}{|x-y|^{\alpha}} dx dy \\ &\geq \frac{1}{2} \left( 1 - \frac{\lambda}{\lambda_1} \right) \|u\|_{\mu}^2 - \frac{1}{2 \cdot 2_{\alpha}^*} C_2 \|u\|_{\mu}^{2(\frac{2N-\alpha}{N-2})}. \end{aligned}$$

It follows from  $2 < 2(\frac{2N-\alpha}{N-2})$  that we can choose some  $\alpha, \rho > 0$  such that  $\mathcal{I}(u) \geq \alpha$  for  $\|u\| = \rho$ .

(2) For some  $u_0 \in H_0^1(\Omega) \setminus \{0\}$ , we have

$$\mathcal{I}(tu_0) = \frac{t^2}{2} \|u_0\|_{\mu}^2 - \frac{\lambda t^2}{2} \int_{\Omega} |u_0|^2 dx - \frac{t^{2 \cdot 2_{\alpha}^*}}{2 \cdot 2_{\alpha}^*} \int_{\Omega} \int_{\Omega} \frac{|u_0(x)|^{2_{\alpha}^*} |u_0(y)|^{2_{\alpha}^*}}{|x-y|^{\alpha}} dx dy$$

for  $t >$  large enough. Then we may take an  $e := t_* u_0$  for some  $t_* > 0$  and (2) follows. □

**Lemma 4.2.** *Let  $0 < \mu < \bar{\mu}$ ,  $0 < \lambda < \lambda_1$  and  $\{u_n\}$  is a  $(PS)_c$  sequence of  $\mathcal{I}$ , then  $u_n$  is bounded. Moreover, assume that  $u_0 \in H_0^1(\Omega)$  is the weak limit of  $u_n$ , then  $u_0$  is a weak solution of equation (4.1).*

*Proof.* Suppose that  $\{u_n\}$  is a  $(PS)_c$  sequence of  $\mathcal{I}$ , that is,

$$\mathcal{I}(u_n) \rightarrow c \text{ and } \mathcal{I}'(u_n) \rightarrow 0 \text{ in } (H_0^1(\Omega))^{-1}.$$

By  $0 < \mu < \bar{\mu}$ ,  $0 < \lambda < \lambda_1$  and  $2 < 2(\frac{2N-\alpha}{N-2})$ , we have that, for some  $C_0 > 0$

$$\begin{aligned} c + o_n(1) &\geq \mathcal{I}(u_n) - \frac{1}{2 \cdot 2_{\alpha}^*} \langle \mathcal{I}'(u_n), u_n \rangle \\ &= \left( \frac{1}{2} - \frac{1}{2 \cdot 2_{\alpha}^*} \right) \left( \int_{\Omega} |u|^2 dx - \mu \int_{\Omega} \frac{|u|^2}{|x|^2} dx - \lambda \int_{\Omega} |u|^2 dx \right) \\ &\geq \left( \frac{1}{2} - \frac{1}{2 \cdot 2_{\alpha}^*} \right) C_0 \|u\|^2, \end{aligned}$$

which means that  $\{u_n\}$  is bounded in  $H_0^1(\Omega)$ . Up to a subsequence, there exists  $u_0 \in H_0^1(\Omega)$  such that  $u_n \rightharpoonup u_0$  in  $H_0^1(\Omega)$ ,  $u_n \rightharpoonup u_0$  in  $L^2(\Omega, |x|^{-2} dx)$ ,  $u_n \rightharpoonup u_0$  in  $L^{2^*}(\Omega)$  and  $u_n \rightarrow u_0$  a.e. in  $\Omega$  as  $n \rightarrow +\infty$ . Then,  $|u_n|^{2_{\alpha}^*} \rightharpoonup |u_0|^{2_{\alpha}^*}$  in  $L^{\frac{2N}{2N-\alpha}}(\Omega)$  as  $n \rightarrow +\infty$ . Since the Riesz potential of  $u$  defined by

$$I_{\alpha}(u)(x) = |x|^{-\alpha} * u = \int_{\Omega} \frac{u(y)}{|x-y|^{\alpha}} dy$$

is a linear continuous map from  $L^{\frac{2N}{2N-\alpha}}(\Omega)$  to  $L^{\frac{2N}{\alpha}}(\Omega)$ , by the Hardy–Littlewood–Sobolev inequality, we know that

$$\int_{\Omega} \frac{|u_n(y)|^{2^*_\alpha}}{|x-y|^\alpha} dy \rightharpoonup \int_{\Omega} \frac{|u_0(y)|^{2^*_\alpha}}{|x-y|^\alpha} dy \text{ in } L^{\frac{2N}{-\alpha}}(\Omega)$$

as  $n \rightarrow +\infty$ . Combining this with the following fact that,  $n \rightarrow +\infty$ ,

$$|u_n|^{2^*_\alpha-2}u_n \rightharpoonup |u_0|^{2^*_\alpha-2}u_0 \text{ in } L^{\frac{2N}{2N-\alpha}}(\Omega),$$

we have that

$$\int_{\Omega} \frac{|u_n(y)|^{2^*_\alpha}}{|x-y|^\alpha} dy |u_n|^{2^*_\alpha-2}u_n \rightharpoonup \int_{\Omega} \frac{|u_0(y)|^{2^*_\alpha}}{|x-y|^\alpha} dy |u_0|^{2^*_\alpha-2}u_0 \text{ in } L^{\frac{2N}{N+2}}(\Omega)$$

as  $n \rightarrow +\infty$ . Since  $\mathcal{I}'(u_n) \rightarrow 0$  in  $(H_0^1(\Omega))^{-1}$ , for any  $\phi \in H_0^1(\Omega)$  we have

$$o_n(1) = \int_{\Omega} \nabla u_n \nabla \phi dx - \int_{\Omega} \frac{u_n \phi}{|x|^2} dx - \lambda \int_{\Omega} u_n \phi dx - \int_{\Omega} \int_{\Omega} \frac{|u_n(y)|^{2^*_\alpha} |u_n(x)|^{2^*_\alpha-2} u_n(x) \phi(x)}{|x-y|^\alpha} dx dy.$$

Thus,

$$\int_{\Omega} \nabla u_0 \phi dx - \int_{\Omega} \frac{u_0 \phi}{|x|^2} dx - \lambda \int_{\Omega} u_0 \phi dx = \int_{\Omega} \int_{\Omega} \frac{|u_0(y)|^{2^*_\alpha} |u_0(x)|^{2^*_\alpha-2} u_0(x) \phi(x)}{|x-y|^\alpha} dx dy, \quad \forall \phi \in H_0^1(\Omega),$$

which yields that  $u_0$  is a weak solution of the equation (4.1). □

**Lemma 4.3.** *Let  $0 < \mu < \bar{\mu}$ ,  $0 < \lambda < \lambda_1$  and  $u_n$  is a  $(PS)_c$  sequence of  $\mathcal{I}$  with*

$$c < \frac{N+2-\alpha}{2(2N-\alpha)} S_{H,\mu}^{\frac{2N-\alpha}{N+2-\alpha}},$$

*then  $\{u_n\}$  has a convergent subsequence in  $H_0^1(\Omega)$ .*

*Proof.* Denote  $v_n := u_n - u_0$ , combining  $u_n \rightharpoonup u_0$  in  $H_0^1(\Omega)$  with the fact that the embedding  $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$  is continuous for  $q \in [1, 2^*]$  and compact for  $q \in [1, 2^*)$ ,  $v_n \rightharpoonup 0$  in  $L^{2^*}(\Omega)$ ,  $v_n \rightarrow 0$  in  $L^q(\Omega)$  for  $q \in [1, 2^*)$  and  $v_n \rightarrow 0$  a.e. in  $\Omega$  as  $n \rightarrow +\infty$ . By the Brezis–Lieb Lemma

$$\int_{\Omega} |\nabla u_n|^2 dx = \int_{\Omega} |\nabla v_n|^2 dx + \int_{\Omega} |\nabla u_0|^2 dx + o_n(1),$$

$$\int_{\Omega} \frac{|u_n|^2}{|x|^2} dx = \int_{\Omega} \frac{|v_n|^2}{|x|^2} dx + \int_{\Omega} \frac{|u_0|^2}{|x|^2} dx + o_n(1)$$

and

$$\int_{\Omega} \int_{\Omega} \frac{|u_n(y)|^{2^*_\alpha} |u_n(x)|^{2^*_\alpha}}{|x-y|^\alpha} dx dy = \int_{\Omega} \int_{\Omega} \frac{|v_n(y)|^{2^*_\alpha} |v_n(x)|^{2^*_\alpha}}{|x-y|^\alpha} dx dy + \int_{\Omega} \int_{\Omega} \frac{|u_0(y)|^{2^*_\alpha} |u_0(x)|^{2^*_\alpha}}{|x-y|^\alpha} dx dy + o_n(1).$$

Since  $\mathcal{I}(u_0) \geq 0$  and  $u_n \rightarrow u_0$  in  $L^q(\Omega)$  for  $q \in [1, 2^*)$ , we have

$$\begin{aligned}
 c \leftarrow \mathcal{I}(u_n) &= \frac{1}{2} \int_{\Omega} |\nabla v_n|^2 dx + \frac{1}{2} \int_{\Omega} |\nabla u_0|^2 dx - \frac{\mu}{2} \int_{\Omega} \frac{|v_n|^2}{|x|^2} dx - \frac{\mu}{2} \int_{\Omega} \frac{|u_0|^2}{|x|^2} dx - \frac{\lambda}{2} \int_{\Omega} |u_0|^2 dx \\
 &\quad - \frac{1}{2 \cdot 2_{\alpha}^*} \int_{\Omega} \int_{\Omega} \frac{|v_n(x)|^{2_{\alpha}^*} |v_n(y)|^{2_{\alpha}^*}}{|x-y|^{\alpha}} dx dy - \frac{1}{2 \cdot 2_{\alpha}^*} \int_{\Omega} \int_{\Omega} \frac{|u_0(x)|^{2_{\alpha}^*} |u_0(y)|^{2_{\alpha}^*}}{|x-y|^{\alpha}} dx dy \\
 &= \mathcal{I}(u_0) + \frac{1}{2} \int_{\Omega} |\nabla v_n|^2 dx - \frac{\mu}{2} \int_{\Omega} \frac{|v_n|^2}{|x|^2} dx - \frac{1}{2 \cdot 2_{\alpha}^*} \int_{\Omega} \int_{\Omega} \frac{|v_n(x)|^{2_{\alpha}^*} |v_n(y)|^{2_{\alpha}^*}}{|x-y|^{\alpha}} dx dy \\
 &\geq \frac{1}{2} \int_{\Omega} |\nabla v_n|^2 dx - \frac{\mu}{2} \int_{\Omega} \frac{|v_n|^2}{|x|^2} dx - \frac{1}{2 \cdot 2_{\alpha}^*} \int_{\Omega} \int_{\Omega} \frac{|v_n(x)|^{2_{\alpha}^*} |v_n(y)|^{2_{\alpha}^*}}{|x-y|^{\alpha}} dx dy.
 \end{aligned} \tag{4.2}$$

It follows from  $\langle \mathcal{I}'(u_0), u_0 \rangle = 0$  that

$$\begin{aligned}
 o_n(1) &= \langle \mathcal{I}'(u_n), u_n \rangle \\
 &= \int_{\Omega} |\nabla v_n|^2 dx + \int_{\Omega} |\nabla u_0|^2 dx - \int_{\Omega} \frac{|v_n|^2}{|x|^2} dx - \int_{\Omega} \frac{|u_0|^2}{|x|^2} dx + \lambda \int_{\Omega} |u_0|^2 dx \\
 &\quad - \int_{\Omega} \int_{\Omega} \frac{|v_n(x)|^{2_{\alpha}^*} |v_n(y)|^{2_{\alpha}^*}}{|x-y|^{\alpha}} dx dy - \int_{\Omega} \int_{\Omega} \frac{|u_0(x)|^{2_{\alpha}^*} |u_0(y)|^{2_{\alpha}^*}}{|x-y|^{\alpha}} dx dy \\
 &= \langle \mathcal{I}'(u_0), u_0 \rangle + \int_{\Omega} |\nabla v_n|^2 dx - \int_{\Omega} \frac{|v_n|^2}{|x|^2} dx - \int_{\Omega} \int_{\Omega} \frac{|v_n(x)|^{2_{\alpha}^*} |v_n(y)|^{2_{\alpha}^*}}{|x-y|^{\alpha}} dx dy.
 \end{aligned} \tag{4.3}$$

From (4.3), we know there exists a nonnegative constant  $m$  such that

$$\int_{\Omega} |\nabla v_n|^2 dx - \int_{\Omega} \frac{|v_n|^2}{|x|^2} dx \rightarrow m$$

and

$$\int_{\Omega} \int_{\Omega} \frac{|v_n(x)|^{2_{\alpha}^*} |v_n(y)|^{2_{\alpha}^*}}{|x-y|^{\alpha}} dx dy \rightarrow m$$

as  $n \rightarrow +\infty$ . By (4.2), we have

$$c \geq \frac{N - \alpha + 2}{2(2N - \alpha)} m. \tag{4.4}$$

We claim that  $m = 0$ . Otherwise, by the definition of the best constant  $S_{H,\mu}$ , we obtain

$$S_{H,\mu} \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n|^{2_{\alpha}^*}(x) |v_n|^{2_{\alpha}^*}(y)}{|x-y|^{\alpha}} dx dy \right)^{1/2_{\alpha}^*} \leq \int_{\mathbb{R}^N} |\nabla v_n|^2 dx - \mu \int_{\mathbb{R}^N} \frac{v_n^2}{|x|^2} dx,$$

which means  $m \geq S_{H,\mu}^{\frac{2N-\alpha}{N+2-\alpha}}$ . Then we deduce from (4.4) that

$$\frac{N - \alpha + 2}{2(2N - \alpha)} S_{H,\mu}^{\frac{2N-\alpha}{N+2-\alpha}} \leq \frac{N - \alpha + 2}{2(2N - \alpha)} m \leq c,$$

which contradicts with the fact that

$$c < \frac{N + 2 - \alpha}{2(2N - \alpha)} S_{H,\mu}^{\frac{2N-\alpha}{N+2-\alpha}}.$$

Thus  $m = 0$  and

$$\|v_n\|^2 = \int_{\Omega} |\nabla v_n|^2 dx - \int_{\Omega} \frac{|v_n|^2}{|x|^2} dx \rightarrow 0$$

as  $n \rightarrow +\infty$ . Therefore,  $\|u_n\| \rightarrow \|u_0\|$  as  $n \rightarrow +\infty$ . □

**Lemma 4.4.** *Let  $0 < \mu \leq \bar{\mu} - 1$ , then there exists  $\varepsilon > 0$  small enough such that*

$$\max_{t \geq 0} \mathcal{I}(t\bar{u}_\varepsilon) < \frac{N - \alpha + 2}{2(2N - \alpha)} S_{H,\mu}^{\frac{2N-\alpha}{N+2-\alpha}}.$$

*Proof.* Let us fix  $\delta > 0$  small enough such that  $B_{4\delta}(0) \subset \Omega$  and let  $\eta \in C_0^\infty(\mathbb{R}^N)$  be a cut-off function such that  $0 \leq \eta \leq 1$  in  $\mathbb{R}^N$ ,  $\eta = 1$  in  $B_\delta(0)$ ,  $\eta = 0$  in  $B_{2\delta}^c(0)$ .

Define

$$\bar{u}_\varepsilon(x) = \eta(x)u_{\mu,\varepsilon}(x), \text{ where } u_{\mu,\varepsilon} = \varepsilon^{\frac{2-N}{2}} u_\mu\left(\frac{x}{\varepsilon}\right).$$

Note that  $u_{\mu,\varepsilon}$  is also a positive solution of (4.1) and also a minimizer for  $S_{H,\alpha}$

$$\int_{\mathbb{R}^N} |\nabla u_{\mu,\varepsilon}|^2 dx - \mu \int_{\mathbb{R}^N} \frac{u_{\mu,\varepsilon}^2}{|x|^2} dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_{\mu,\varepsilon}(x)|^{2\alpha} |u_{\mu,\varepsilon}(y)|^{2\alpha}}{|x-y|^\alpha} dx dy = S_{H,\alpha}^{\frac{2N-\alpha}{N-\alpha+2}}. \tag{4.5}$$

Set  $a = \sqrt{1 - \frac{4\mu}{(N-2)^2}}$ , then  $\frac{\gamma'}{\sqrt{\bar{\mu}}} = 1 - a$  and  $\frac{\gamma}{\sqrt{\bar{\mu}}} = 1 + a$ . In the spirit of [2], we have

$$\int_{\Omega} \left( |\nabla \bar{u}_\varepsilon|^2 - \mu \frac{\bar{u}_\varepsilon^2}{|x|^2} \right) dx \leq S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} + O(\varepsilon^{(N-2)a}). \tag{4.6}$$

Indeed, by (4.5) we get that

$$\begin{aligned} & \int_{\Omega} \left( |\nabla \bar{u}_\varepsilon|^2 - \mu \frac{\bar{u}_\varepsilon^2}{|x|^2} \right) dx \\ &= \int_{\Omega} \eta^2 |\nabla u_{\mu,\varepsilon}|^2 + \int_{\Omega} |\nabla \eta|^2 u_{\mu,\varepsilon}^2 dx + 2 \int_{\Omega} \eta u_{\mu,\varepsilon} (\nabla \eta \cdot \nabla u_{\mu,\varepsilon}) dx - \mu \int_{\Omega} \frac{\eta^2 u_{\mu,\varepsilon}^2}{|x|^2} dx \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_{\mu,\varepsilon}(y)|^{2\alpha} |u_{\mu,\varepsilon}(x)|^{2\alpha} \eta^2(x)}{|x-y|^\alpha} dx dy + \mu \int_{\mathbb{R}^N} \frac{\eta^2 u_{\mu,\varepsilon}^2}{|x|^2} dx + \int_{\Omega} |\nabla \eta|^2 u_{\mu,\varepsilon}^2 dx - \mu \int_{\Omega} \frac{\eta^2 u_{\mu,\varepsilon}^2}{|x|^2} dx \\ &\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_{\mu,\varepsilon}(y)|^{2\alpha} |u_{\mu,\varepsilon}(x)|^{2\alpha}}{|x-y|^\alpha} dx dy + \int_{\Omega} |\nabla \eta|^2 u_{\mu,\varepsilon}^2 dx \\ &= S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} + \int_{\Omega} |\nabla \eta|^2 u_{\mu,\varepsilon}^2 dx \\ &:= S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} + \mathbb{A}, \end{aligned}$$

where

$$\begin{aligned}
 \mathbb{A} &:= \int_{\Omega} |\nabla \eta|^2 u_{\mu,\varepsilon}^2 dx \\
 &\leq C\varepsilon^{(2-N)} \int_{B_{2\delta} \setminus B_{\delta}} \frac{1}{\left(\left|\frac{x}{\varepsilon}\right|^{1-a} + \left|\frac{x}{\varepsilon}\right|^{1+a}\right)^{N-2}} dx \\
 &= C\varepsilon^{(N-2)a} \int_{B_{2\delta} \setminus B_{\delta}} \frac{1}{\left(|x|^{1-a}(\varepsilon^{2a} + |x|^{2a})\right)^{N-2}} dx \\
 &\leq C\varepsilon^{(N-2)a} \int_{B_{2\delta} \setminus B_{\delta}} \frac{1}{|x|^{(1+a)(N-2)}} dx \\
 &= O(\varepsilon^{(N-2)a}).
 \end{aligned}$$

On the one hand,

$$\begin{aligned}
 &\int_{\Omega} \int_{\Omega} \frac{|\bar{u}_{\varepsilon}(x)|^{2^*_{\alpha}} |\bar{u}_{\varepsilon}(y)|^{2^*_{\alpha}}}{|x-y|^{\alpha}} dx dy \\
 &\geq \int_{B_{\delta}} \int_{B_{\delta}} \frac{|\bar{u}_{\varepsilon}(x)|^{2^*_{\alpha}} |\bar{u}_{\varepsilon}(y)|^{2^*_{\alpha}}}{|x-y|^{\alpha}} dx dy \\
 &= \int_{B_{\delta}} \int_{B_{\delta}} \frac{|u_{\mu,\varepsilon}(x)|^{2^*_{\alpha}} |u_{\mu,\varepsilon}(y)|^{2^*_{\alpha}}}{|x-y|^{\alpha}} dx dy \\
 &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_{\mu,\varepsilon}(x)|^{2^*_{\alpha}} |u_{\mu,\varepsilon}(y)|^{2^*_{\alpha}}}{|x-y|^{\alpha}} dx dy - 2 \int_{\mathbb{R}^N \setminus B_{\delta}} \int_{B_{\delta}} \frac{|u_{\mu,\varepsilon}(x)|^{2^*_{\alpha}} |u_{\mu,\varepsilon}(y)|^{2^*_{\alpha}}}{|x-y|^{\alpha}} dx dy \\
 &\quad - \int_{\mathbb{R}^N \setminus B_{\delta}} \int_{\mathbb{R}^N \setminus B_{\delta}} \frac{|u_{\mu,\varepsilon}(x)|^{2^*_{\alpha}} |u_{\mu,\varepsilon}(y)|^{2^*_{\alpha}}}{|x-y|^{\alpha}} dx dy \\
 &:= S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} - 2\mathbb{B} - \mathbb{C},
 \end{aligned}$$

where

$$\mathbb{B} := \int_{\mathbb{R}^N \setminus B_{\delta}} \int_{B_{\delta}} \frac{|u_{\mu,\varepsilon}(x)|^{2^*_{\alpha}} |u_{\mu,\varepsilon}(y)|^{2^*_{\alpha}}}{|x-y|^{\alpha}} dx dy$$

and

$$\mathbb{C} := \int_{\mathbb{R}^N \setminus B_{\delta}} \int_{\mathbb{R}^N \setminus B_{\delta}} \frac{|u_{\mu,\varepsilon}(x)|^{2^*_{\alpha}} |u_{\mu,\varepsilon}(y)|^{2^*_{\alpha}}}{|x-y|^{\alpha}} dx dy.$$

After a simple calculation, we have

$$\begin{aligned}
 \mathbb{B} &\leq \int_{\mathbb{R}^N \setminus B_\delta} \int_{B_\delta} \frac{C\varepsilon^{(\alpha-2N)}}{\left(\left|\frac{x}{\varepsilon}\right|^{1-a} + \left|\frac{x}{\varepsilon}\right|^{1+a}\right)^{\frac{2N-\alpha}{2}} |x-y|^\alpha \left(\left|\frac{y}{\varepsilon}\right|^{1-a} + \left|\frac{y}{\varepsilon}\right|^{1+a}\right)^{\frac{2N-\alpha}{2}}} dx dy \\
 &\leq C\varepsilon^{(\alpha-2N)} \left( \int_{\mathbb{R}^N \setminus B_\delta} \frac{1}{\left(\left|\frac{x}{\varepsilon}\right|^{1-a} + \left|\frac{x}{\varepsilon}\right|^{1+a}\right)^N} dx \right)^{\frac{2N-\alpha}{2N}} \left( \int_{B_\delta} \frac{1}{\left(\left|\frac{y}{\varepsilon}\right|^{1-a} + \left|\frac{y}{\varepsilon}\right|^{1+a}\right)^N} dy \right)^{\frac{2N-\alpha}{2N}} \\
 &= C\varepsilon^{(\alpha-2N)} \left( \int_{\mathbb{R}^N \setminus B_\delta} \frac{\varepsilon^{(1+a)N}}{\left(|x|^{1-a}(\varepsilon^{2a} + |x|^{2a})\right)^N} dx \right)^{\frac{2N-\alpha}{2N}} \left( \int_{B_\delta} \frac{1}{\left(\left|\frac{y}{\varepsilon}\right|^{1-a} + \left|\frac{y}{\varepsilon}\right|^{1+a}\right)^N} dy \right)^{\frac{2N-\alpha}{2N}} \\
 &\leq C\varepsilon^{\frac{(2N-\alpha)a}{2}} \left( \int_{|x|>\delta} \frac{1}{|x|^{(1+a)N}} dx \right)^{\frac{2N-\alpha}{2N}} \left( \int_{|x|<\frac{\delta}{\varepsilon}} \frac{1}{\left(|x|^{1-a} + |x|^{1+a}\right)^N} dx \right)^{\frac{2N-\alpha}{2N}} \\
 &\leq C\varepsilon^{\frac{(2N-\alpha)a}{2}} \left( \int_0^{\frac{\delta}{\varepsilon}} \frac{r^{N-1}}{\left(|r|^{1-a} + |r|^{1+a}\right)^N} dr \right)^{\frac{2N-\alpha}{2N}} \\
 &= O\left(\varepsilon^{\frac{(2N-\alpha)a}{2}}\right)
 \end{aligned}$$

and

$$\begin{aligned}
 \mathbb{C} &\leq \int_{\mathbb{R}^N \setminus B_\delta} \int_{\mathbb{R}^N \setminus B_\delta} \frac{C\varepsilon^{(\alpha-2N)}}{\left(\left|\frac{x}{\varepsilon}\right|^{1-a} + \left|\frac{x}{\varepsilon}\right|^{1+a}\right)^{\frac{2N-\alpha}{2}} |x-y|^\alpha \left(\left|\frac{y}{\varepsilon}\right|^{1-a} + \left|\frac{y}{\varepsilon}\right|^{1+a}\right)^{\frac{2N-\alpha}{2}}} dx dy \\
 &\leq C\varepsilon^{(\alpha-2N)} \left( \int_{\mathbb{R}^N \setminus B_\delta} \frac{1}{\left(\left|\frac{x}{\varepsilon}\right|^{1-a} + \left|\frac{x}{\varepsilon}\right|^{1+a}\right)^N} dx \right)^{\frac{2N-\alpha}{N}} \\
 &= C\varepsilon^{(\alpha-2N)} \left( \int_{\mathbb{R}^N \setminus B_\delta} \frac{\varepsilon^{(1+a)N}}{\left(|x|^{1-a}(\varepsilon^{2a} + |x|^{2a})\right)^N} dx \right)^{\frac{2N-\alpha}{N}} \\
 &\leq C\varepsilon^{(2N-\alpha)a} \left( \int_{|x|>\delta} \frac{1}{|x|^{(1+a)N}} dx \right)^{\frac{2N-\alpha}{N}} \\
 &= O\left(\varepsilon^{(2N-\alpha)a}\right).
 \end{aligned}$$

Thus,

$$\iint_{\Omega} \frac{|\bar{u}_\varepsilon(x)|^{2^*_\alpha} |\bar{u}_\varepsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha} dx dy \geq S_{H,\alpha}^{\frac{2N-\alpha}{N-\alpha+2}} - O\left(\varepsilon^{(2N-\alpha)a}\right). \tag{4.7}$$

On the other hand,

$$\begin{aligned}
 & \left| \int_{\Omega} \int_{\Omega} \frac{|\bar{u}_{\varepsilon}(x)|^{2^*} |\bar{u}_{\varepsilon}(y)|^{2^*}}{|x-y|^{\alpha}} dx dy - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_{\mu,\varepsilon}(x)|^{2^*} |u_{\mu,\varepsilon}(y)|^{2^*}}{|x-y|^{\alpha}} dx dy \right| \\
 & \leq \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(|\bar{u}_{\varepsilon}(x)|^{2^*} - |u_{\mu,\varepsilon}(x)|^{2^*}) |\bar{u}_{\varepsilon}(y)|^{2^*}}{|x-y|^{\alpha}} dx dy \right| \\
 & \quad + \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_{\mu,\varepsilon}(x)|^{2^*} (|\bar{u}_{\varepsilon}(y)|^{2^*} - |u_{\mu,\varepsilon}(y)|^{2^*})}{|x-y|^{\alpha}} dx dy \right| \\
 & \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{||\eta(x)|^{2^*} - 1| |u_{\mu,\varepsilon}(x)|^{2^*} |\eta(y)|^{2^*} |u_{\mu,\varepsilon}(y)|^{2^*}}{|x-y|^{\alpha}} dx dy \\
 & \quad + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_{\mu,\varepsilon}(x)|^{2^*} ||\eta(y)|^{2^*} - 1| |u_{\mu,\varepsilon}(y)|^{2^*}}{|x-y|^{\alpha}} dx dy \\
 & \leq \int_{\mathbb{R}^N \setminus B_{\delta}} \int_{\mathbb{R}^N} \frac{|u_{\mu,\varepsilon}(x)|^{2^*} |u_{\mu,\varepsilon}(y)|^{2^*}}{|x-y|^{\alpha}} dx dy + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N \setminus B_{\delta}} \frac{|u_{\mu,\varepsilon}(x)|^{2^*} |u_{\mu,\varepsilon}(y)|^{2^*}}{|x-y|^{\alpha}} dx dy \\
 & = 2 \int_{\mathbb{R}^N \setminus B_{\delta}} \int_{\mathbb{R}^N} \frac{|u_{\mu,\varepsilon}(x)|^{2^*} |u_{\mu,\varepsilon}(y)|^{2^*}}{|x-y|^{\alpha}} dx dy \\
 & \leq C \varepsilon^{(\alpha-2N)} \left( \int_{\mathbb{R}^N \setminus B_{\delta}} \frac{1}{(|\frac{x}{\varepsilon}|^{1-a} + |\frac{x}{\varepsilon}|^{1+a})^N} dx \right)^{\frac{2N-\alpha}{2N}} \left( \int_{\mathbb{R}^N} \frac{1}{(|\frac{y}{\varepsilon}|^{1-a} + |\frac{y}{\varepsilon}|^{1+a})^N} dy \right)^{\frac{2N-\alpha}{2N}} \\
 & \leq C \varepsilon^{\frac{(2N-\alpha)a}{2}} \left( \int_{\mathbb{R}^N \setminus B_{\delta}} \frac{1}{(|x|^{1-a}(\varepsilon^{2a} + |x|^{2a}))^N} dx \right)^{\frac{2N-\alpha}{2N}} \left( \int_{\mathbb{R}^N} \frac{1}{(|y|^{1-a} + |y|^{1+a})^N} dy \right)^{\frac{2N-\alpha}{2N}} \\
 & = O(\varepsilon^{\frac{(2N-\alpha)a}{2}}),
 \end{aligned}$$

which means that

$$\begin{aligned}
 \int_{\Omega} \int_{\Omega} \frac{|\bar{u}_{\varepsilon}(x)|^{2^*} |\bar{u}_{\varepsilon}(y)|^{2^*}}{|x-y|^{\alpha}} dx dy & \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_{\mu,\varepsilon}(x)|^{2^*} |u_{\mu,\varepsilon}(y)|^{2^*}}{|x-y|^{\alpha}} dx dy + O(\varepsilon^{\frac{(2N-\alpha)a}{2}}) \\
 & = S_{H,\alpha}^{\frac{2N-\alpha}{N-\alpha+2}} + O(\varepsilon^{\frac{(2N-\alpha)a}{2}}).
 \end{aligned} \tag{4.8}$$

By (4.7) and (4.8), we get

$$S_{H,\alpha}^{\frac{2N-\alpha}{N-\alpha+2}} - O(\varepsilon^{(2N-\alpha)a}) \leq \int_{\Omega} \int_{\Omega} \frac{|\bar{u}_{\varepsilon}(x)|^{2^*} |\bar{u}_{\varepsilon}(y)|^{2^*}}{|x-y|^{\alpha}} dx dy \leq S_{H,\alpha}^{\frac{2N-\alpha}{N-\alpha+2}} + O(\varepsilon^{\frac{(2N-\alpha)a}{2}}). \tag{4.9}$$

Moreover, by a straightforward computation it follows

$$\begin{aligned}
 \int_{\Omega} |\bar{u}_{\varepsilon}|^{q+1} dx &= \int_{|x| \leq 2\delta} |\eta(x)u_{\mu,\varepsilon}(x)|^{q+1} dx \\
 &\geq \int_{|x| \leq \delta} |u_{\mu,\varepsilon}(x)|^{q+1} dx \\
 &\geq C\varepsilon^{\frac{2-N}{2}(q+1)} \int_{|x| \leq \delta} \frac{1}{\left(\left|\frac{x}{\varepsilon}\right|^{1-a} + \left|\frac{x}{\varepsilon}\right|^{1+a}\right)^{\frac{N-2}{2}(q+1)}} dx \\
 &= C\varepsilon^{N-\frac{N-2}{2}(q+1)} \int_{|x| \leq \frac{\delta}{\varepsilon}} \frac{1}{\left(|x|^{1-a} + |x|^{1+a}\right)^{\frac{N-2}{2}(q+1)}} dx \\
 &\geq C\varepsilon^{N-\frac{N-2}{2}(q+1)} \int_{1 \leq |x| \leq \frac{\delta}{\varepsilon}} \frac{1}{\left(|x|^{1-a} + |x|^{1+a}\right)^{\frac{N-2}{2}(q+1)}} dx \\
 &\geq C\varepsilon^{N-\frac{N-2}{2}(q+1)} \int_{1 \leq |x| \leq \frac{\delta}{\varepsilon}} \frac{1}{|x|^{(1+a)\frac{N-2}{2}(q+1)}} dx \\
 &= C\varepsilon^{N-\frac{N-2}{2}(q+1)} \int_1^{\frac{\delta}{\varepsilon}} r^{N-(1+a)\frac{N-2}{2}(q+1)-1} dr \\
 &= \begin{cases} C\varepsilon^{N-\frac{N-2}{2}(q+1)}, & \text{if } N < (1+a)\frac{N-2}{2}(q+1), \\ C\varepsilon^{N-\frac{N-2}{2}(q+1)}|\ln \varepsilon|, & \text{if } N = (1+a)\frac{N-2}{2}(q+1), \\ C\varepsilon^{\frac{N-2}{2}(q+1)a}, & \text{if } N > (1+a)\frac{N-2}{2}(q+1). \end{cases} \tag{4.10}
 \end{aligned}$$

In particular, if  $q = 1$ , we have

$$\int_{\Omega} |\bar{u}_{\varepsilon}|^2 dx \geq \begin{cases} C\varepsilon^2, & \text{if } N < (N-2)(1+a), \\ C\varepsilon^2|\ln \varepsilon|, & \text{if } N = (N-2)(1+a), \\ C\varepsilon^{(N-2)a}, & \text{if } N > (N-2)(1+a). \end{cases} \tag{4.11}$$

If  $\mu < \bar{\mu} - 1$ , then  $N < (N-2)(1+a)$  and  $(N-2)a > 2$ , thus by (4.6), (4.9) and (4.11), we obtain

$$\begin{aligned}
 0 < \max_{t \geq 0} \mathcal{I}(t\bar{u}_{\varepsilon}) &\leq \left(\frac{1}{2} - \frac{1}{2 \cdot 2_{\alpha}^*}\right) \left(\frac{\int_{\Omega} |\nabla \bar{u}_{\varepsilon}|^2 dx - \mu \int_{\Omega} \frac{|\bar{u}_{\varepsilon}|^2}{|x|^2} dx - \lambda \int_{\Omega} |\bar{u}_{\varepsilon}|^2 dx}{\left(\int_{\Omega} \int_{\Omega} \frac{|\bar{u}_{\varepsilon}(x)|^{2_{\alpha}^*} |\bar{u}_{\varepsilon}(y)|^{2_{\alpha}^*}}{|x-y|^{\alpha}} dx dy\right)^{\frac{1}{2_{\alpha}^*}}}\right)^{\frac{2_{\alpha}^*}{2_{\alpha}^*-1}} \\
 &\leq \frac{N-\alpha+2}{2(2N-\alpha)} \left(\frac{S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} + O(\varepsilon^{(N-2)a}) - \lambda O(\varepsilon^2)}{\left(S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} - O(\varepsilon^{\frac{(N-2)a}{2N-\alpha}})\right)^{\frac{2_{\alpha}^*}{2N-\alpha}}}\right)^{\frac{2_{\alpha}^*}{2_{\alpha}^*-1}} \\
 &\leq \frac{N-\alpha+2}{2(2N-\alpha)} \left(S_{H,\mu} + O(\varepsilon^{(N-2)a}) - \lambda O(\varepsilon^2)\right)^{\frac{2N-\alpha}{N+2-\alpha}} \\
 &< \frac{N-\alpha+2}{2(2N-\alpha)} S_{H,\mu}^{\frac{2N-\alpha}{N+2-\alpha}}.
 \end{aligned}$$

If  $\mu = \bar{\mu} - 1$ , then  $N = (N - 2)(1 + a)$  and  $(N - 2)a = 2$ , by (4.6), (4.9) and (4.11), we get

$$\begin{aligned} 0 < \max_{t \geq 0} \mathcal{I}(t\bar{u}_\varepsilon) &\leq \left(\frac{1}{2} - \frac{1}{2 \cdot 2_\alpha^*}\right) \left( \frac{\int_\Omega |\nabla \bar{u}_\varepsilon|^2 dx - \mu \int_\Omega \frac{|\bar{u}_\varepsilon|^2}{|x|^2} dx - \lambda \int_\Omega |\bar{u}_\varepsilon|^2 dx}{\left(\int_\Omega \int_\Omega \frac{|\bar{u}_\varepsilon(x)|^{2_\alpha^*} |\bar{u}_\varepsilon(y)|^{2_\alpha^*}}{|x-y|^\alpha} dx dy\right)^{\frac{1}{2_\alpha^*}}} \right)^{\frac{2_\alpha^*}{2_\alpha^*-1}} \\ &\leq \frac{N - \alpha + 2}{2(2N - \alpha)} \left( \frac{S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} + O(\varepsilon^2) - \lambda C \varepsilon^2 |\ln \varepsilon|}{\left(S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} - O(\varepsilon)\right)^{\frac{N-2}{2N-\alpha}}} \right)^{\frac{2_\alpha^*}{2_\alpha^*-1}} \\ &\leq \frac{N - \alpha + 2}{2(2N - \alpha)} \left( S_{H,\mu} + O(\varepsilon^2) - \frac{\lambda \varepsilon^2 |\ln \varepsilon|}{\left(S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} - O(\varepsilon)\right)^{\frac{N-2}{2N-\alpha}}} \right)^{\frac{2N-\alpha}{N+2-\alpha}} \\ &< \frac{N - \alpha + 2}{2(2N - \alpha)} S_{H,\mu}^{\frac{2N-\alpha}{N+2-\alpha}}. \end{aligned}$$

Therefore,

$$\max_{t \geq 0} \mathcal{I}(t\bar{u}_\varepsilon) < \frac{N - \alpha + 2}{2(2N - \alpha)} S_{H,\mu}^{\frac{2N-\alpha}{N+2-\alpha}}.$$

□

*Proof of Theorem 1.2.*: From Lemma 4.1 and the mountain pass theorem (see [41]), there exists a  $(PS)_c$  sequence  $\{u_n\}$  such that

$$\mathcal{I}(u_n) \rightarrow c \text{ and } \mathcal{I}'(u_n) \rightarrow 0 \text{ in } (H_0^1(\Omega))^{-1}$$

at the minimax level

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \mathcal{I}(\gamma(t)),$$

where  $\Gamma$  is defined by

$$\Gamma := \left\{ \gamma \in C([0, 1], H_0^1(\Omega)) : \gamma(0) = 0 \text{ and } \mathcal{I}(\gamma(1)) < 0 \right\}.$$

By the definition of  $c$  and Lemma 4.4, we obtain  $c < \frac{N+2-\alpha}{2(2N-\alpha)} S_{H,\mu}^{\frac{2N-\alpha}{N+2-\alpha}}$ . Together with Lemma 4.2, and Lemma 4.3, we have that the sequence  $\{u_n\}$  contains a strongly convergent subsequence and equation (4.1) has a nontrivial solution, observing that  $\mathcal{I}(u_0) = c > 0$ . □

### 5. Superlinear perturbation

In this section, we consider the existence of solutions for a different perturbation term, that is

$$\begin{cases} -\Delta u(x) - \mu \frac{u}{|x|^2} = \left( \int_\Omega \frac{|u(y)|^{2_\alpha^*}}{|x-y|^\alpha} dy \right) |u|^{2_\alpha^*-1} + \lambda u^q, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{5.1}$$

where  $1 < q < 2^* - 1$ . The energy functional  $\mathcal{I}$  associated to (5.1) is defined as follows:

$$\mathcal{I}(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{\mu}{2} \int_\Omega \frac{|u|^2}{|x|^2} dx - \frac{\lambda}{q+1} \int_\Omega |u|^{q+1} dx - \frac{1}{2 \cdot 2_\alpha^*} \int_\Omega \int_\Omega \frac{|u(x)|^{2_\alpha^*} |u(y)|^{2_\alpha^*}}{|x-y|^\alpha} dx dy.$$

Following the argument for the linear case, with minor modifications we can check that  $\mathcal{I}$  satisfies the Mountain Pass geometry and the  $(PS)$  condition at any level  $c$ , provided  $c < c^* := \frac{N-\alpha+2}{2(2N-\alpha)} S_{H,\mu}^{\frac{2N-\alpha}{N+2-\alpha}}$ . We

omit the details to avoid repetitions. So it suffices to find a path with energy below critical level  $c^*$  and we have the following.

**Lemma 5.1.** *Let  $0 < \mu < \bar{\mu}$ , then there exists  $\varepsilon > 0$  small enough such that*

$$\max_{t \geq 0} \mathcal{I}(t\bar{u}_\varepsilon) \leq c < \frac{N + 2 - \alpha}{2(2N - \alpha)} S_{H,\mu}^{\frac{2N-\alpha}{N+2-\alpha}},$$

provided that

- $N > \frac{2(2\alpha+q+1)}{2\alpha+q-1}$  and  $\lambda > 0$  or
- $N < \frac{2(2\alpha+q+1)}{2\alpha+q-1}$  and  $\lambda > \lambda_0$ , for  $\lambda_0$  large enough.

*Proof.* Case I:  $N > \frac{2(2\alpha+q+1)}{2\alpha+q-1}$ . In this case, it follows from  $1 < q < 2^* - 1$  that we have  $(N - 2)a > N - \frac{N-2}{2}(q + 1)$ . For any  $t > 0$ , we have

$$\begin{aligned} \mathcal{I}(t\bar{u}_\varepsilon) &= \frac{t^2}{2} \int_{\Omega} |\nabla \bar{u}_\varepsilon|^2 dx - \frac{\mu t^2}{2} \int_{\Omega} \frac{|\bar{u}_\varepsilon|^2}{|x|^2} dx - \frac{\lambda t^{q+1}}{q+1} \int_{\Omega} |\bar{u}_\varepsilon|^{q+1} dx - \frac{t^{2 \cdot 2^*_\alpha}}{2 \cdot 2^*_\alpha} \int_{\Omega} \int_{\Omega} \frac{|\bar{u}_\varepsilon(x)|^{2^*_\alpha} |\bar{u}_\varepsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha} dx dy \\ &\leq \frac{t^2}{2} \left( S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} + O(\varepsilon^{(N-2)a}) \right) - \lambda t^{q+1} O(\varepsilon^{N - \frac{N-2}{2}(q+1)}) - \frac{t^{2 \cdot 2^*_\alpha}}{2 \cdot 2^*_\alpha} \left( S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} - O(\varepsilon^{\frac{(N-2)a}{2}}) \right) \\ &=: h(t). \end{aligned}$$

Since  $1 < q < 2^*$ , then it is easy to see that  $h(0) = 0$  and  $\lim_{t \rightarrow +\infty} h(t) = -\infty$ . Thus, there exists  $t_\varepsilon > 0$  such that  $h(t_\varepsilon) = \max_{t > 0} h(t)$  and

$$t_\varepsilon \left( S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} + O(\varepsilon^{(N-2)a}) \right) - \lambda t_\varepsilon^q O(\varepsilon^{N - \frac{N-2}{2}(q+1)}) - t_\varepsilon^{2 \cdot 2^*_\alpha - 1} \left( S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} - O(\varepsilon^{\frac{(N-2)a}{2}}) \right) = 0.$$

For  $\varepsilon >$  sufficiently small,

$$\begin{aligned} t_\varepsilon &= \left( \frac{S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} + O(\varepsilon^{(N-2)a}) - \lambda t_\varepsilon^{q-1} O(\varepsilon^{N - \frac{N-2}{2}(q+1)})}{S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} - O(\varepsilon^{\frac{(N-2)a}{2}})} \right)^{\frac{1}{2 \cdot 2^*_\alpha - 2}} \\ &\leq \left( \frac{S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} + O(\varepsilon^{(N-2)a})}{S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} - O(\varepsilon^{\frac{(N-2)a}{2}})} \right)^{\frac{1}{2 \cdot 2^*_\alpha - 2}} \\ &\leq \left( 1 + O(\varepsilon^{(N-2)a}) \right)^{\frac{1}{2 \cdot 2^*_\alpha - 2}}. \end{aligned}$$

On the other hand, there exists  $t_0 > 0$  independent of  $\varepsilon$  such that for  $\varepsilon >$  small enough,

$$\begin{aligned} t_\varepsilon &= \left( \frac{S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} + O(\varepsilon^{(N-2)a}) - \lambda t_\varepsilon^{q-1} O(\varepsilon^{N - \frac{N-2}{2}(q+1)})}{S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} - O(\varepsilon^{\frac{(N-2)a}{2}})} \right)^{\frac{1}{2 \cdot 2^*_\alpha - 2}} \\ &\geq \left( \frac{S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} + O(\varepsilon^{(N-2)a}) - O(\varepsilon^{N - \frac{N-2}{2}(q+1)})}{S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} - O(\varepsilon^{\frac{(N-2)a}{2}})} \right)^{\frac{1}{2 \cdot 2^*_\alpha - 2}} \\ &> \left( 1 - O(\varepsilon^{N - \frac{N-2}{2}(q+1)}) \right)^{\frac{1}{2 \cdot 2^*_\alpha - 2}} \\ &:= t_0 > 0. \end{aligned}$$

Thus, we have a lower and upper bound for  $t_\varepsilon$ , independent of  $\varepsilon$ . We can argue as in the proof of Lemma 4.4 to conclude that

$$\begin{aligned}
 0 < \max_{t \geq 0} \mathcal{I}(t\bar{u}_\varepsilon) &\leq \left(\frac{1}{2} - \frac{1}{2 \cdot 2_\alpha^*}\right) \left(\frac{\int_\Omega |\nabla \bar{u}_\varepsilon|^2 dx - \mu \int_\Omega \frac{|\bar{u}_\varepsilon|^2}{|x|^2} dx}{\left(\int_\Omega \int_\Omega \frac{|\bar{u}_\varepsilon(x)|^{2_\alpha^*} |\bar{u}_\varepsilon(y)|^{2_\alpha^*}}{|x-y|^\alpha} dx dy\right)^{\frac{1}{2_\alpha^*}}}\right)^{\frac{2_\alpha^*}{2_\alpha^*-1}} - \lambda C \varepsilon^{N - \frac{N-2}{2}(q+1)} \\
 &\leq \frac{N - \alpha + 2}{2(2N - \alpha)} \left(\frac{S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} + O(\varepsilon^{(N-2)a})}{\left(S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} - O(\varepsilon^{\frac{(N-2)a}{2}})\right)^{\frac{N-2}{2N-\alpha}}}\right)^{\frac{2_\alpha^*}{2_\alpha^*-1}} - \lambda C \varepsilon^{N - \frac{N-2}{2}(q+1)} \\
 &\leq \frac{N - \alpha + 2}{2(2N - \alpha)} \left(S_{H,\mu} + O(\varepsilon^{(N-2)a})\right)^{\frac{2N-\alpha}{N+\alpha-2}} - \lambda C \varepsilon^{N - \frac{N-2}{2}(q+1)} \\
 &\leq \frac{N - \alpha + 2}{2(2N - \alpha)} S_{H,\mu}^{\frac{2N-\alpha}{N+2-\alpha}} + O(\varepsilon^{(N-2)a}) - \lambda C \varepsilon^{N - \frac{N-2}{2}(q+1)} \\
 &< \frac{N - \alpha + 2}{2(2N - \alpha)} S_{H,\mu}^{\frac{2N-\alpha}{N+2-\alpha}}.
 \end{aligned}$$

**Case II:**  $N \leq \frac{2(2a+q+1)}{2a+q-1}$ . In this case, we have  $N \geq \frac{N-2}{2}(1+a)(q+1)$  and  $(N-2)a \leq N - \frac{N-2}{2}(q+1)$ . Similar to Case I, for any fixed  $\varepsilon$ , we can obtain that  $\max_{t > 0} \mathcal{I}(t\bar{u}_\varepsilon)$  can be attained at some  $t_{\varepsilon,\lambda}$  with

$$\|\bar{u}_\varepsilon\|_\mu = \lambda t_{\varepsilon,\lambda}^{q-1} \int_\Omega |\bar{u}_\varepsilon|^{q+1} dx - t_{\varepsilon,\lambda}^{2 \cdot 2_\alpha^* - 2} \int_\Omega \int_\Omega \frac{|\bar{u}_\varepsilon(x)|^{2_\alpha^*} |\bar{u}_\varepsilon(y)|^{2_\alpha^*}}{|x-y|^\alpha} dx dy. \tag{5.2}$$

It follows from (5.2) that  $t_{\varepsilon,\lambda} \rightarrow 0$  as  $\lambda \rightarrow +\infty$ . Thus

$$\max_{t > 0} \mathcal{I}(t\bar{u}_\varepsilon) = \mathcal{I}(t_{\varepsilon,\lambda} \bar{u}_\varepsilon) \leq \frac{t_{\varepsilon,\lambda}^2}{2} \|\bar{u}_\varepsilon\|_\mu - \frac{t_{\varepsilon,\lambda}^{2 \cdot 2_\alpha^*}}{2 \cdot 2_\alpha^*} \int_\Omega \int_\Omega \frac{|\bar{u}_\varepsilon(x)|^{2_\alpha^*} |\bar{u}_\varepsilon(y)|^{2_\alpha^*}}{|x-y|^\alpha} dx dy \rightarrow 0 \text{ as } \lambda \rightarrow +\infty,$$

which yields that there exists  $\lambda_0 > 0$  such that

$$\max_{t > 0} \mathcal{I}(t\bar{u}_\varepsilon) < \frac{N + 2 - \alpha}{2(2N - \alpha)} S_{H,\mu}^{\frac{2N-\alpha}{N+2-\alpha}}, \text{ for all } \lambda > \lambda_0.$$

□

*Proof of Theorem 1.3:* In order to finish the proof of Theorem 1.3, we define

$$\Gamma := \left\{ \gamma \in C([0, 1], H_0^1(\Omega)) : \gamma(0) = 0 \text{ and } \mathcal{I}(\gamma(1)) < 0 \right\}$$

and the minimax level

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \mathcal{I}(\gamma(t)).$$

Using Lemma 5.1, we get that  $c < \frac{N+2-\alpha}{2(2N-\alpha)} S_{H,\mu}^{\frac{2N-\alpha}{N+2-\alpha}}$ , provided that

- $N > \frac{2(2a+q+1)}{2a+q-1}$  and  $\lambda > 0$  or
- $N < \frac{2(2a+q+1)}{2a+q-1}$  and  $\lambda > \lambda_0$ , for  $\lambda_0$  large enough.

Similarly to the proof of Theorem 1.2, the equation (5.1) has a nontrivial solution. □

### 6. Nonlocal superlinear perturbation

In this section, we finally consider the following nonlocal perturbation,

$$\begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = \left( \int_{\Omega} \frac{|u(y)|^{2^*_{\alpha}}}{|x-y|^{\alpha}} dy \right) |u|^{2^*_{\alpha}-1} + \lambda \left( \int_{\Omega} \frac{|u(y)|^p}{|x-y|^{\alpha}} dy \right) |u|^{p-1}, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{6.1}$$

Nontrivial solutions of (6.1) are equivalent to nonzero critical points of the following energy functional

$$\mathcal{J}(u) := \frac{1}{2} \|u\|_{\mu}^2 - \frac{1}{2 \cdot 2^*_{\alpha}} \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2^*_{\alpha}} |u(y)|^{2^*_{\alpha}}}{|x-y|^{\alpha}} dx dy - \frac{\lambda}{2p} \int_{\Omega} \int_{\Omega} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{\alpha}} dx dy.$$

Similarly as in the previous sections, one can check that  $\mathcal{J}$  satisfies the Mountain Pass geometry and the (PS) condition at any level  $c$  such that  $c < \frac{N+2-\alpha}{2(2N-\alpha)} S_{H,\mu}^{\frac{2N-\alpha}{N+2-\alpha}}$ , so we omit the details. Let us just point out that for a bounded sequence  $\{u_n\}$  in  $H_0^1(\Omega)$ ,  $p$  to a subsequence,  $u_n \rightharpoonup u_0$  in  $H_0^1(\Omega)$ , and then, since  $u_n \rightarrow u_0$  in  $L^s(\Omega)$  for  $s \in [1, 2^*_{\alpha})$ , we have that

$$\int_{\Omega} \int_{\Omega} \frac{|u_n(x)|^p |u_n(y)|^p}{|x-y|^{\alpha}} dx dy \rightarrow \int_{\Omega} \int_{\Omega} \frac{|u_0(x)|^p |u_0(y)|^p}{|x-y|^{\alpha}} dx dy, \quad p \in (1, 2^*_{\alpha} - 1).$$

We therefore turn to the following estimate.

**Lemma 6.1.** *Let  $0 < \mu < \bar{\mu}$ , then there exists  $\varepsilon > 0$  small enough such that*

$$\max_{t \geq 0} \mathcal{J}(t\bar{u}_{\varepsilon}) < c < \frac{N+2-\alpha}{2(2N-\alpha)} S_{H,\mu}^{\frac{2N-\alpha}{N+2-\alpha}},$$

provided that

- $N > \frac{2a-\alpha+2p}{a+p-2}$  and  $\lambda > 0$ , or
- $N \leq \frac{2a-\alpha+2p}{a+p-2}$  and  $\lambda$  is sufficiently large.

*Proof.* By the definition of  $\bar{u}_{\varepsilon}$ , we have

$$\begin{aligned} \mathbb{D} &= \int_{\Omega} \int_{\Omega} \frac{|\bar{u}_{\varepsilon}(x)|^p |\bar{u}_{\varepsilon}(y)|^p}{|x-y|^{\alpha}} dx dy \\ &\geq \int_{B_{\delta}} \int_{B_{\delta}} \frac{|\bar{u}_{\varepsilon}(x)|^p |\bar{u}_{\varepsilon}(y)|^p}{|x-y|^{\alpha}} dx dy \\ &= \int_{B_{\delta}} \int_{B_{\delta}} \frac{|u_{\mu,\varepsilon}(x)|^p |u_{\mu,\varepsilon}(y)|^p}{|x-y|^{\alpha}} dx dy \\ &\geq \int_{B_{\delta}} \int_{B_{\delta}} \frac{C_{\varepsilon}^{(2-N)p}}{\left( \left| \frac{x}{\varepsilon} \right|^{1-a} + \left| \frac{y}{\varepsilon} \right|^{1+a} \right)^{\frac{(N-2)p}{2}} |x-y|^{\alpha} \left( \left| \frac{y}{\varepsilon} \right|^{1-a} + \left| \frac{x}{\varepsilon} \right|^{1+a} \right)^{\frac{(N-2)p}{2}}} dx dy \\ &= \int_{B_{\delta}} \int_{B_{\delta}} \frac{C_{\varepsilon}^{2N-\alpha-(N-2)p}}{\left( \left| \frac{x}{\varepsilon} \right|^{1-a} + \left| \frac{y}{\varepsilon} \right|^{1+a} \right)^{\frac{(N-2)p}{2}} \left| \frac{x}{\varepsilon} - \frac{y}{\varepsilon} \right|^{\alpha} \left( \left| \frac{y}{\varepsilon} \right|^{1-a} + \left| \frac{x}{\varepsilon} \right|^{1+a} \right)^{\frac{(N-2)p}{2}}} d\frac{x}{\varepsilon} d\frac{y}{\varepsilon} \\ &= C_{\varepsilon}^{2N-\alpha-(N-2)p} \int_{B_{\delta}} \int_{B_{\delta}} \frac{1}{\left( |x|^{1-a} + |y|^{1+a} \right)^{\frac{(N-2)p}{2}} |x-y|^{\alpha} \left( |y|^{1-a} + |x|^{1+a} \right)^{\frac{(N-2)p}{2}}} dx dy \\ &= O(\varepsilon^{2N-\alpha-(N-2)p}). \end{aligned}$$

It follows from  $p < 2^*_\alpha$  that  $2N - \alpha - (N - 2)p > 0$ . For any  $t > 0$ , we have

$$\begin{aligned} \mathcal{J}(t\bar{u}_\varepsilon) &= \frac{t^2}{2} |\bar{u}_\varepsilon|_\mu^2 - \frac{t^{2 \cdot 2^*_\alpha}}{2 \cdot 2^*_\alpha} \int_\Omega \int_\Omega \frac{|\bar{u}_\varepsilon(x)|^{2^*_\alpha} |\bar{u}_\varepsilon(y)|^{2^*_\alpha}}{|x - y|^\alpha} dx dy - \frac{\lambda t^{2p}}{2p} \int_\Omega \int_\Omega \frac{|\bar{u}_\varepsilon(x)|^p |\bar{u}_\varepsilon(y)|^p}{|x - y|^\alpha} dx dy \\ &\leq \frac{t^2}{2} \left( S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} + O(\varepsilon^{(N-2)a}) \right) - \frac{t^{2 \cdot 2^*_\alpha}}{2 \cdot 2^*_\alpha} \left( S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} - O(\varepsilon^{\frac{(N-2)a}{2}}) \right) - \lambda t^{2p} O(\varepsilon^{2N-\alpha-(N-2)p}) \\ &=: \tilde{h}(t). \end{aligned}$$

Since  $1 < p < 2^*_\alpha$ , then it is easy to see that  $h(0) = 0$  and  $\lim_{t \rightarrow +\infty} h(t) = -\infty$ . Thus, there exists  $t_\varepsilon > 0$  such that  $h(t_\varepsilon) = \max_{t>0} f(t)$  and

$$t_\varepsilon \left( S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} + O(\varepsilon^{(N-2)a}) \right) - t^{2 \cdot 2^*_\alpha - 1} \left( S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} - O(\varepsilon^{\frac{(N-2)a}{2}}) \right) - \lambda t^{2p-1} O(\varepsilon^{2N-\alpha-(N-2)p}) = 0.$$

For  $\varepsilon >$  sufficiently small,

$$\begin{aligned} t_\varepsilon &= \left( \frac{S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} + O(\varepsilon^{(N-2)a}) - \lambda t^{2p-1} O(\varepsilon^{2N-\alpha-(N-2)p})}{S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} - O(\varepsilon^{\frac{(N-2)a}{2}})} \right)^{\frac{1}{2 \cdot 2^*_\alpha - 2}} \\ &\leq \left( \frac{S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} + O(\varepsilon^{(N-2)a})}{S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} - O(\varepsilon^{\frac{(N-2)a}{2}})} \right)^{\frac{1}{2 \cdot 2^*_\alpha - 2}} \\ &\leq \left( 1 + O(\varepsilon^{(N-2)a}) \right)^{\frac{1}{2 \cdot 2^*_\alpha - 2}} := t_1. \end{aligned}$$

On the other hand, there exists  $t_1 > 0$  independent of  $\varepsilon$  such that, for  $\varepsilon >$  small enough,

$$\begin{aligned} t_\varepsilon &\geq \left( \frac{S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} + O(\varepsilon^{(N-2)a}) - O(\varepsilon^{2N-\alpha-(N-2)p})}{S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} - O(\varepsilon^{\frac{(N-2)a}{2}})} \right)^{\frac{1}{2 \cdot 2^*_\alpha - 2}} \\ &> \left( 1 - O(\varepsilon^{2N-\alpha-(N-2)p}) \right)^{\frac{1}{2 \cdot 2^*_\alpha - 2}} := t_0 > 0. \end{aligned}$$

Thus,  $t_\varepsilon$  has a lower bound and an upper bound independent of  $\varepsilon$ . Now, by the elementary inequality  $(m + n)^q < m^q + \varrho(m + n)^{q-1}n$  for  $m, n > 0$  and  $\varrho \geq 1$ , we obtain

$$\begin{aligned} 0 < \max_{t \geq 0} \mathcal{J}(t\bar{u}_\varepsilon) &\leq \left( \frac{1}{2} - \frac{1}{2 \cdot 2^*_\alpha} \right) \left( \frac{\int_\Omega |\nabla \bar{u}_\varepsilon|^2 dx - \mu \int_\Omega \frac{|\bar{u}_\varepsilon|^2}{|x|^2} dx}{\left( \int_\Omega \int_\Omega \frac{|\bar{u}_\varepsilon(x)|^{2^*_\alpha} |\bar{u}_\varepsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha} dx dy \right)^{\frac{1}{2^*_\alpha}}} \right)^{\frac{2^*_\alpha}{2^*_\alpha - 1}} - \lambda O(\varepsilon^{2N-\alpha-(N-2)p}) \\ &\leq \frac{N - \alpha + 2}{2(2N - \alpha)} \left( \frac{S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} + O(\varepsilon^{(N-2)a})}{\left( S_{H,\mu}^{\frac{2N-\alpha}{N-\alpha+2}} - O(\varepsilon^{\frac{(N-2)a}{2}}) \right)^{\frac{N-2}{2N-\alpha}}} \right)^{\frac{2^*_\alpha}{2^*_\alpha - 1}} - \lambda O(\varepsilon^{2N-\alpha-(N-2)p}) \tag{6.2} \\ &= \frac{N - \alpha + 2}{2(2N - \alpha)} \left( S_{H,\mu} + O(\varepsilon^{(N-2)a}) \right)^{\frac{2N-\alpha}{N+2-\alpha}} - \lambda O(\varepsilon^{2N-\alpha-(N-2)p}) \\ &\leq \frac{N - \alpha + 2}{2(2N - \alpha)} S_{H,\mu}^{\frac{2N-\alpha}{N+2-\alpha}} + O(\varepsilon^{(N-2)a}) - \lambda O(\varepsilon^{2N-\alpha-(N-2)p}). \end{aligned}$$

Now we distinguish the following cases:

(i) In the case  $N > \frac{2a-\alpha+2p}{a+p-2}$ , by (6.2) we get

$$\max_{t \geq 0} \mathcal{J}(t\bar{u}_\varepsilon) \leq \frac{N-\alpha+2}{2(2N-\alpha)} S_{H,\mu}^{\frac{2N-\alpha}{N+2-\alpha}} + O(\varepsilon^{(N-2)a}) - O(\varepsilon^{2N-\alpha-(N-2)p}).$$

In view of  $(N-2)a > 2N-\alpha-(N-2)p > 0$ , we get the conclusion for  $\varepsilon$  small enough.

(ii) In the case  $N \leq \frac{2a-\alpha+2p}{a+p-2}$ , by (6.2) we have

$$\max_{t \geq 0} \mathcal{J}(t\bar{u}_\varepsilon) \leq \frac{N-\alpha+2}{2(2N-\alpha)} S_{H,\mu}^{\frac{2N-\alpha}{N+2-\alpha}} + O(\varepsilon^{(N-2)a}) - \lambda O(\varepsilon^{2N-\alpha-(N-2)p}).$$

for  $\lambda = \varepsilon^{-\theta}$  with  $\theta > 2N-\alpha-(N-2)p-(N-2)a$ , we also get the conclusion.  $\square$

*Proof of Theorem 1.4.*: We know that there exists a  $(PS)_c$  sequence  $\{u_n\}$  of  $\mathcal{J}$  with  $c < \frac{N+2-\alpha}{2(2N-\alpha)} S_{H,\mu}^{\frac{2N-\alpha}{N+2-\alpha}}$  provided that

- $N > \frac{2a-\alpha+2p}{a+p-2}$  and  $\lambda > 0$ , or
- $N \leq \frac{2a-\alpha+2p}{a+p-2}$  and  $\lambda$  is sufficiently large.

Similarly as in the previous sections, this yields a nontrivial solution for equation (6.1).  $\square$

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## Declarations

**Conflict of interest** The authors declare no Conflict of interest.

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