

Global multiplicity results in a Moore–Nehari type problem with a spectral parameter [☆]

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Abstract

This paper analyzes the structure of the set of positive solutions of (1.1), where $a \equiv a_h$ is the piecewise constant function defined in (1.3) for some $h \in (0, 1)$. In our analysis, λ is regarded as a bifurcation parameter, whereas h is viewed as a deformation parameter between the autonomous case when $a = 1$ and the linear case when $a = 0$. In this paper, besides establishing some of the multiplicity results suggested by the numerical experiments of [2], we have analyzed the asymptotic behavior of the positive solutions of (1.1) as $h \uparrow 1$, when the shadow system of (1.1) is the linear equation $-u'' = \pi^2 u$. This is the first paper where such a problem has been addressed. Numerics is of no help in analyzing this singular perturbation problem because the positive solutions blow-up point-wise in $(0, 1)$ as $h \uparrow 1$ if $\lambda < \pi^2$.

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1. Introduction

This paper studies the one-dimensional second-order superlinear problem

$$\begin{cases} -u'' = \lambda u + a(x)u^p, & x \in (0, 1), \\ u(0) = u(1) = 0, \end{cases} \quad (1.1)$$

where $\lambda \in \mathbb{R}$ is regarded as a bifurcation parameter, $p > 1$ is fixed, and $a : [0, 1] \rightarrow [0, \infty)$ is a non-zero piece-wise constant function. Thus, $a \gneq 0$ in the sense that $a \geq 0$, but $a \neq 0$. The main goal of this paper is to study the global structure of the set of positive solutions of (1.1) in terms of the parameter λ and of the nature of the weight function $a(x)$. As $p > 1$, by the uniqueness of solution for the Cauchy problem associated to the differential equation of (1.1), any positive solution, u , of (1.1) must be strongly positive, in the sense that

$$u(x) > 0 \text{ for all } x \in (0, 1), \quad u'(0) > 0 \text{ and } u'(1) < 0. \quad (1.2)$$

In the special case when $\lambda = 0$ and p is an odd integer, the problem (1.1) has been already analyzed by Moore and Nehari [16] for a symmetric piecewise weight function $a(x)$ defined, after a spatial scaling, as

$$a(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1-h}{2}] \cup [\frac{1+h}{2}, 1], \\ 0 & \text{if } x \in (\frac{1-h}{2}, \frac{1+h}{2}), \end{cases} \quad (1.3)$$

where $h \in (0, 1)$ is fixed. The main result of [16] shows that, for some values of $h \in (0, 1)$, the problem (1.1) possesses, at least, three positive solutions: a first solution which is symmetric (even) with respect to $x = 0.5$, and two additional asymmetric solutions, reflected from each other about $x = 0.5$. In recent years, also in the special case when $\lambda = 0$, some additional work has been done by Kajikiya et al. [9] by considering h as a bifurcation parameter and establishing the existence of a symmetry-breaking from the symmetric solution to a continuum of asymmetric solutions at some critical value $h_* \in (0, 1)$ (see also [7] for nodal solutions if $p \in (0, 1)$, and [8] for nodal solutions if $p > 1$). Some related problems where a symmetry-breaking occurs leading to multiplicity of positive solutions can be found in [5,6,17,18].

The main goal of this paper is to ascertain the global topological structure of the set of positive solutions of (1.1), (λ, u) , regarding $\lambda \in \mathbb{R}$ as the primary bifurcation parameter. Then, we will ascertain how changes the underlying global bifurcation diagram in λ as h varies in the interval $(0, 1)$. Thus, h is regarded as the secondary parameter of (1.1) for the special choice (1.3). Throughout this paper, to emphasize the dependence of $a(x)$ on $h \in (0, 1)$, we will eventually set

$$a \equiv a_h.$$

According to, e.g., Amann and López-Gómez [1], under these assumptions, for every $h \in (0, 1)$, the set of non-trivial non-negative solutions, (λ, u) , of problem (1.1) has a component, $\mathcal{C}_h^+$, such that $(\pi^2, 0) \in \mathcal{C}_h^+$ and

$$\mathcal{P}_\lambda(\mathcal{C}_h^+) = (-\infty, \pi^2],$$

where $\mathcal{P}_\lambda$ stands for the λ -projection operator,

$$\mathcal{P}_\lambda(\lambda, u) = \lambda \text{ for all } (\lambda, u) \in \mathbb{R} \times \mathcal{C}([0, 1]; \mathbb{R}).$$

In particular, (1.1) admits a positive solution if, and only if, $\lambda < \pi^2$. Therefore, the condition $\lambda < \pi^2$ will be imposed throughout this paper. Moreover, according to the numerical experiments of Cubillos et al. [2], for every $h \in (0, 1)$, there is a value of the parameter λ , denoted by $\lambda_{h,b}$, which is a subcritical pitchfork bifurcation value for $\mathcal{C}_h^+$. These numerical experiments also suggest that (1.1) has three positive solutions for every $\lambda < \lambda_{h,b}$, one symmetric and two asymmetric, and that

$$\lim_{h \downarrow 0} \lambda_{h,b} = -\infty \quad \text{and} \quad \lim_{h \uparrow 1} \lambda_{h,b} = \pi^2. \quad (1.4)$$

Although the first limit of (1.4) is rather natural, because the limiting shadow problem as $h \downarrow 0$,

$$\begin{cases} -u'' = \lambda u + u^p & \text{in } (0, 1), \\ u(0) = u(1) = 0, \end{cases} \quad (1.5)$$

has a unique positive solution for each $\lambda \in (-\infty, \pi^2)$, which follows by a direct analysis of the phase portrait of the autonomous differential equation

$$-u'' = \lambda u + u^p,$$

the second limit of (1.4) is far from expected, because it entails that, for every $\varepsilon > 0$ and sufficiently small $1 - h$, (1.1) has three positive solutions for every $\lambda \in (-\infty, \pi^2 - \varepsilon]$, while the limiting shadow system of (1.1) as $h \uparrow 1$ is the linear problem

$$\begin{cases} -u'' = \lambda u & \text{in } (0, 1), \\ u(0) = u(1) = 0, \end{cases} \quad (1.6)$$

which has a positive solution if, and only if, $\lambda = \pi^2$. Thus, the global multiplicity result as $h \uparrow 1$, for every $\lambda \in (-\infty, \pi^2)$, is attributable to the nonlinear character of $a_h u^p = u^p$ in $[0, \frac{1-h}{2}) \cup (\frac{1+h}{2}, 1]$, which is a very tiny set as $h \uparrow 1$. Another multiplicity result in a similar vein, caused by spatial heterogeneities, has been documented by López-Gómez and Rabinowitz in [14].

The main goal of this paper is to deliver some analytical proofs of some of these features and analyze the point-wise behavior of the three positive solutions of (1.1) as $h \uparrow 1$. Precisely, our main findings are the following ones:

(F1) For every $\lambda < \pi^2$, (1.1) has a unique positive symmetric solution, u_λ , and

$$\mathcal{C}_{h,s}^+ = \{(\lambda, u_\lambda) : \lambda \in (-\infty, \pi^2]\}$$

is a continuous curve such that $\mathcal{C}_{h,s}^+ \subset \mathcal{C}_h^+$.

(F2) For every $h \in (0, 1)$, there exists $\tilde{\lambda} = \tilde{\lambda}(h) < \pi^2$ such that, for every $\lambda \leq \tilde{\lambda}(h)$, (1.1) has, at least, three positive solutions.

(F3) For any given $\lambda \in (\pi^2/4, \pi^2)$, there exists $R_\lambda > 0$ such that, for every $R > R_\lambda$, the problem (1.1) has, at least, two asymmetric positive solutions for a suitable $h = h(R) \in (0, 1)$. Moreover,

$$\lim_{R \uparrow +\infty} h(R) = 1.$$

(F4) For every $\lambda < \pi^2$ and $h \in (0, 1)$, let u_h be any positive solution of (1.1). Then,

$$\lim_{h \uparrow 1} u_h(x) = +\infty \text{ for all } x \in (0, 1). \quad (1.7)$$

In particular, the components $\mathcal{C}_{h,s}^+$ and $\mathcal{C}_h^+$ blow-up, point-wise in $(0, 1)$, as $h \uparrow 1$.

(F5) For every $\alpha > 0$, there are two increasing sequences, $\{h_n\}_{n \geq 1}$ in $(0, 1)$ and $\{\lambda_n\}_{n \geq 1}$ in $(0, \pi^2)$, such that

$$\lim_{n \rightarrow +\infty} h_n = 1, \quad \lim_{n \rightarrow +\infty} \lambda_n = \pi^2, \quad \text{and } \|u_{\lambda_n}\|_\infty = \alpha \text{ for all } n \geq 1,$$

where u_{λ_n} is the unique symmetric positive solution of (1.1) for $(\lambda, h) = (\lambda_n, h_n)$, $n \geq 1$. Moreover,

$$\lim_{n \rightarrow +\infty} u_{\lambda_n} = \alpha \sin(\pi \cdot) \text{ in } C^1([0, 1]; \mathbb{R}).$$

In other words, $(\pi^2, \alpha \sin(\pi \cdot)) \in \mathcal{C}_1^+$ perturbs into the sequence

$$(\lambda_n, u_{\lambda_n}) \in \mathcal{C}_{h_n, s}^+, \quad n \geq 1.$$

The existence and the uniqueness of the symmetric solution established in (F1) will be proven in Section 2. It combines a phase plane analysis with some sharp comparison results. The multiplicity result established in (F2) will be proven in Section 4 through some phase plane techniques based on a technical device of [15]. The multiplicity result established in (F3) will be proven in Section 5 combining some sophisticated phase plane techniques with the Mountain Climbing Theorem of [4, 19, 10]. Finally, the results stated in (F4) and (F5) will be proven in Section 3.

Although, according to [9], it is already known that

$$\lim_{h \uparrow 1} \|u_h\|_\infty = +\infty \text{ if } \lambda = 0,$$

the main novelty of our Theorem 3.1 establishes that

$$\lim_{h \uparrow 1} u_h(x) = +\infty \text{ for all } \lambda \in (-\infty, \pi^2) \text{ and } x \in (0, 1),$$

which is an extremely delicate issue for $\lambda < 0$. According to (F4) and (F5), the positive solutions of (1.1) blow-up as $h \uparrow 1$, point-wise in $(0, 1)$, if $\lambda < \pi^2$, while, simultaneously, any solution $(\lambda, u) = (\pi^2, \alpha \sin(\pi x))$ of (1.6) perturbs into solutions of (1.1) with $\lambda \sim \pi^2$ and $h \sim 1$. Thus, adopting the terminology of [12], all the positive solutions of (1.1) approximate a metasolution of $-u'' = \pi^2 u$ as $h \uparrow 1$. This is the first occasion that such a phenomenology has been described

in the literature. As a byproduct, it becomes apparent how the pioneering result of Moore and Nehari [16] is actually global in the parameter λ .

2. Symmetric positive solutions of (1.1)

In this section, we will construct the symmetric positive solutions of (1.1) for the choice (1.3) for all $h \in (0, 1)$. To this purpose, we consider the associated planar system

$$\begin{cases} u' = v, \\ v' = -\lambda u - a(x)u^p, \end{cases} \quad (2.1)$$

in the closed half-plane

$$\mathcal{H}^+ := \{(u, v) \in \mathbb{R}^2 : u \geq 0\}.$$

So, depending on whether $a = 1$ or $a = 0$, we are led to analyze one of the following autonomous planar systems

$$(\mathcal{N}) \quad \begin{cases} u' = v, \\ v' = -\lambda u - u^p, \end{cases} \quad \text{or} \quad (\mathcal{L}) \quad \begin{cases} u' = v, \\ v' = -\lambda u, \end{cases} \quad (2.2)$$

respectively. Note that we are throughout assuming that $\lambda < \pi^2$, and that the dynamics of both systems, $(\mathcal{N})$ and $(\mathcal{L})$, change when considering $\lambda < 0$ or $\lambda \in [0, \pi^2)$.

2.1. Symmetric solutions for $\lambda < 0$

Suppose $\lambda < 0$. Then, the set of equilibria of $(\mathcal{N})$ consists of the origin, $(0, 0)$, plus $(\omega, 0)$, where $\omega := (-\lambda)^{\frac{1}{p-1}}$, and the total energy associated to $(\mathcal{N})$ is

$$\mathcal{E}_{\mathcal{N}}(u, v) := \frac{1}{2}v^2 + \frac{\lambda}{2}u^2 + \frac{1}{p+1}u^{p+1}.$$

Thus, $(0, 0)$ is a half-right saddle point whose stable and unstable manifolds provide us with an homoclinic connection of $(0, 0)$ corresponding to the level set $\mathcal{E}_{\mathcal{N}}(u, v) = 0$, surrounding the equilibrium $(\omega, 0)$, which is a local center. Note that the level set $\mathcal{E}_{\mathcal{N}}(u, v) = 0$ crosses the u -axis at the point $(u_{\text{ho}}, 0)$, where

$$u_{\text{ho}} := \left(\frac{-\lambda(p+1)}{2} \right)^{\frac{1}{p-1}}.$$

Thus, $(u_{\text{ho}}, 0)$ is the crossing point of the homoclinic connection of the origin with the positive u -axis. The phase portrait of $(\mathcal{N})$ has been sketched in the left figure of Fig. 1.

By simply looking at the left picture of Fig. 1, it becomes apparent that, for every $u_0 > u_{\text{ho}}$, there exists

$$T := T_{\mathcal{N}}(\lambda, u_0) > 0$$

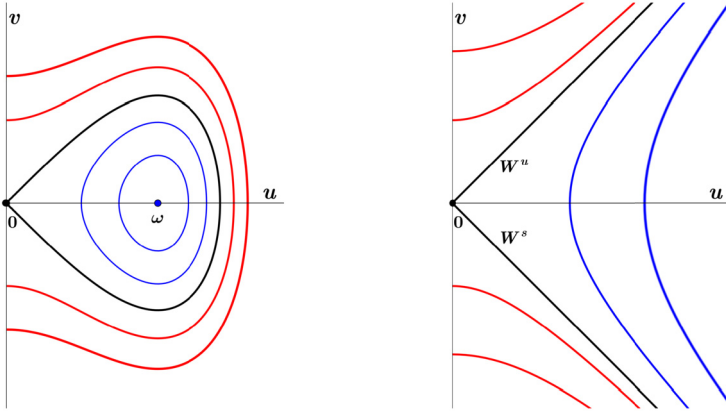


Fig. 1. Phase plane diagrams of $(\mathcal{N})$ (left) and $(\mathcal{L})$ (right) when $\lambda < 0$. Integral curves with positive energy are plotted in red, while those with zero energy are plotted in black and those with negative energy are colored in blue. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

such that the unique solution of $(\mathcal{N})$, $(u(t), v(t))$, satisfying

$$(u(0), v(0)) = (u_0, 0),$$

is defined in $[-T, T]$ and it satisfies

$$\mathcal{E}_{\mathcal{N}}(u(t), v(t)) = \frac{1}{2}v^2(t) + \frac{\lambda}{2}u^2(t) + \frac{1}{p+1}u^{p+1}(t) = \frac{\lambda}{2}u_0^2 + \frac{1}{p+1}u_0^{p+1} \quad (2.3)$$

for all $t \in [-T, T]$. Moreover,

$$(u(\pm T), v(\pm T)) = (0, \mp v_0), \quad v_0 := \sqrt{\lambda u_0^2 + \frac{2}{p+1}u_0^{p+1}},$$

and a standard calculation from (2.3) yields

$$T_{\mathcal{N}}(\lambda, u_0) := \int_0^1 \frac{ds}{\sqrt{\lambda(1-s^2) + \frac{2}{p+1}u_0^{p-1}(1-s^{p+1})}}. \quad (2.4)$$

Thanks to (2.4), it is apparent that, for every $\lambda < 0$, $T_{\mathcal{N}}(\lambda, u_0)$ is decreasing with respect to u_0 . Moreover,

$$\lim_{u_0 \downarrow u_{\text{ho}}} T_{\mathcal{N}}(\lambda, u_0) = +\infty, \quad \lim_{u_0 \uparrow +\infty} T_{\mathcal{N}}(\lambda, u_0) = 0. \quad (2.5)$$

As far as $(\mathcal{L})$ is concerned, it is a linear system whose unique equilibrium is the origin, $(0, 0)$, which is a saddle point if $\lambda < 0$. The unstable and stable manifolds, W^u and W^s , are the straight lines

$$W^u : v = \sqrt{-\lambda} u, \quad W^s : v = -\sqrt{-\lambda} u,$$

respectively, as sketched in the right picture of Fig. 1. The total energy associated to the linear system $(\mathcal{L})$ is

$$\mathcal{E}_{\mathcal{L}}(u, v) := \frac{1}{2}v^2 + \frac{\lambda}{2}u^2.$$

Thus, the level set $\mathcal{E}_{\mathcal{L}}(u, v) = 0$ consists of $W^u \cup W^s$. Moreover, for every $u_+ > 0$, the necessary time to link $(u_+, 0)$ with an arbitrary point

$$(u_\ell, v_\ell) \in \left\{ (u, v) : \mathcal{E}_{\mathcal{L}}(u, v) = \frac{\lambda}{2}u_+^2 \right\}$$

is given by

$$\begin{aligned} T_{\mathcal{L}}(\lambda, u_+, u_\ell) &:= \int_{u_+}^{u_\ell} \frac{du}{\sqrt{\lambda(u_+^2 - u^2)}} = \frac{1}{\sqrt{-\lambda}} \int_1^{\frac{u_\ell}{u_+}} \frac{ds}{\sqrt{-1 + s^2}} \\ &= \frac{1}{\sqrt{-\lambda}} \ln \left(\frac{u_\ell}{u_+} + \sqrt{\left(\frac{u_\ell}{u_+} \right)^2 - 1} \right), \end{aligned} \quad (2.6)$$

because the function

$$g(s) := \ln(s + \sqrt{s^2 - 1}), \quad s > 1,$$

satisfies $g'(s) = \frac{1}{\sqrt{s^2 - 1}}$ for all $s > 1$.

Consequently, to construct a symmetric positive solution, say u , of (1.1), we should have

$$u'(0.5) = 0, \quad u(0.5) = u_+,$$

for some $u_+ > 0$, and should proceed as sketched next:

Step 1: Since $g'(s) > 0$ for all $s > 1$, the function g is invertible. Thus, we should consider the set of points (u_ℓ, v_ℓ) in the level energy $\mathcal{E}_{\mathcal{L}}(u, v) = \frac{\lambda}{2}u_+^2$ such that

$$\frac{u_\ell}{u_+} = g^{-1} \left(\frac{h}{2} \sqrt{-\lambda} \right) =: C > 1,$$

because, for this choice, (2.6) guarantees that

$$T_{\mathcal{L}}(\lambda, u_+, u_\ell) = \frac{h}{2}.$$

Note that

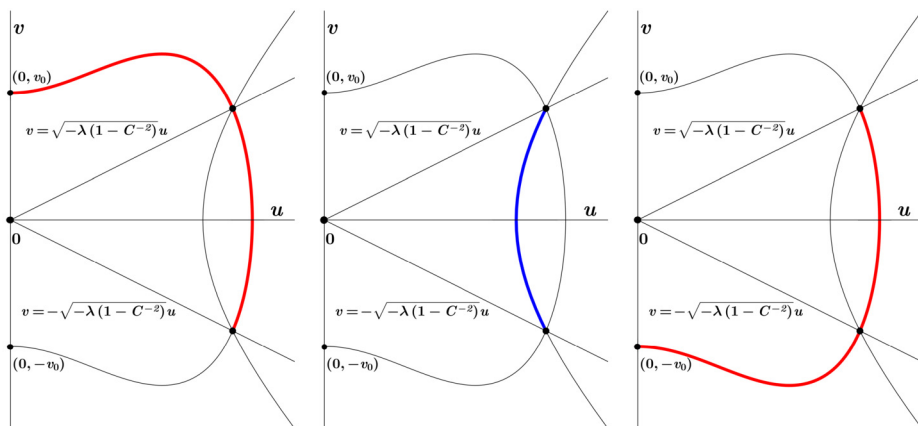


Fig. 2. The phase plane route of a symmetric positive solution of (1.1) in the case $\lambda < 0$: nonlinear interval $[0, \frac{1-h}{2}]$ (left), linear interval $(\frac{1-h}{2}, \frac{1+h}{2})$ (center) and nonlinear interval $[\frac{1+h}{2}, 1]$ (right).

$$\frac{u_\ell}{C} = u_+ = \sqrt{\frac{v_\ell^2}{\lambda} + u_\ell^2}$$

implies

$$v_\ell = \pm \sqrt{-\lambda(1 - C^{-2})} u_\ell.$$

Therefore, the points (u_ℓ, v_ℓ) lying in the straight line

$$v = -\sqrt{-\lambda(1 - C^{-2})} u,$$

need a time $\frac{h}{2}$ to reach $(u_+, 0)$, and, by symmetry, they as well need another $\frac{h}{2}$ unities of time to link $(u_+, 0)$ with

$$(u_\ell, \sqrt{-\lambda(1 - C^{-2})} u_\ell)$$

along the trajectory $\mathcal{E}_{\mathcal{L}}(u, v) = \frac{\lambda}{2} u_+^2$.

Step 2: According to Step 1, any positive solution of (1.1) starting at some point $(0, v_0)$, traveling along the trajectory $\mathcal{E}_{\mathcal{N}} = \frac{v_0^2}{2}$ of the nonlinear system $(\mathcal{N})$ until reaching some point

$$(u_{\frac{1-h}{2}}, v_{\frac{1-h}{2}}) \equiv (u_\ell, -\sqrt{-\lambda(1 - C^{-2})} u_\ell)$$

at $x = \frac{1-h}{2}$ provides us with a symmetric solution of (1.1), as sketched in Fig. 2.

Indeed, such a solution leaves $(0, v_0)$ and follows the trajectory $\mathcal{E}_{\mathcal{N}} = \frac{v_0^2}{2}$ until reaching $(u_{\frac{1-h}{2}}, v_{\frac{1-h}{2}})$ at $x = \frac{1-h}{2}$. Then, it follows the trajectory $\mathcal{E}_{\mathcal{L}}(u, v) = \frac{\lambda}{2} u_+^2$ of the linear system $(\mathcal{L})$ until reaching $(u_{\frac{1-h}{2}}, -v_{\frac{1-h}{2}})$ at $x = \frac{1+h}{2}$. Finally, it travels along the trajectory $\mathcal{E}_{\mathcal{N}} = \frac{v_0^2}{2}$ until reaching $(0, -v_0)$ at $x = 1$, as sketched in the three plots of Fig. 2.

Step 3: Therefore, (1.1) has a symmetric positive solution as soon as there exists some $u_0 > u_{ho}$ such that

$$T_{\mathcal{N}}(\lambda, u_0) + T_{\mathcal{N}}(\lambda, u_0, u_\ell) = \frac{1-h}{2}, \quad (2.7)$$

where $T_{\mathcal{N}}(\lambda, u_0, u_\ell)$ denotes the necessary time to link $(u_0, 0)$ with $(u_\ell, -v_\ell)$ along the trajectory (2.3) of $(\mathcal{N})$. According to (2.3), it is easily seen that

$$T_{\mathcal{N}}(\lambda, u_0, u_\ell) := \int_{\frac{u_\ell}{u_0}}^1 \frac{ds}{\sqrt{\lambda(1-s^2) + \frac{2}{p+1}u_0^{p-1}(1-s^{p+1})}}. \quad (2.8)$$

Thus,

$$\lim_{u_0 \uparrow \infty} T_{\mathcal{N}}(\lambda, u_0, u_\ell) = 0.$$

Consequently, we can infer from (2.5) that there exists $u_0 > u_{ho}$ for which (2.7) holds. So, for every $\lambda < 0$, (1.1) admits, at least, one symmetric positive solution. In Section 2.3, we will prove that, actually, (1.1) has a unique symmetric positive solution. Let denote it by $u \equiv u_\lambda$. According to the previous construction,

$$\|u\|_\infty = u_0$$

is reached at some points $x_0 \in (0, \frac{1-h}{2})$ and $x_1 \in (\frac{1+h}{2}, 1)$ such that $x_0 + x_1 = 1$. Moreover, by Theorem 2.1 of Cubillos et al. [2], we already know that

$$\lim_{\lambda \downarrow -\infty} u_\lambda = 0 \text{ uniformly on compact subsets of } (\frac{1-h}{2}, \frac{1+h}{2}). \quad (2.9)$$

2.2. Symmetric solutions for $\lambda \geq 0$

Throughout this section, we assume that $\lambda \geq 0$. Then, the unique equilibrium of $(\mathcal{N})$ is $(0, 0)$, which is a half-right nonlinear center. Moreover, for every $u_0 > 0$, there exists $T := T_{\mathcal{N}}(\lambda, u_0) > 0$ such that the unique solution of $(\mathcal{N})$, $(u(t), v(t))$, such that $(u(0), v(0)) = (u_0, 0)$, is defined in $[-T, T]$, and it satisfies (2.3) for all $t \in [-T, T]$. As in Section 2.1,

$$(u(\pm T), v(\pm T)) = (0, \mp v_0), \quad v_0 := \sqrt{\lambda u_0^2 + \frac{2}{p+1}u_0^{p+1}},$$

and (2.4) holds. Thus, for every $\lambda \geq 0$, $T_{\mathcal{N}}(\lambda, u_0)$ is decreasing with respect to u_0 , though now, instead of (2.5), we have that

$$\lim_{u_0 \downarrow 0} T_{\mathcal{N}}(\lambda, u_0) = \frac{\pi}{2\sqrt{\lambda}}, \quad \lim_{u_0 \uparrow +\infty} T_{\mathcal{N}}(\lambda, u_0) = 0, \quad (2.10)$$

for all $\lambda > 0$, and

$$\lim_{u_0 \downarrow 0} T_{\mathcal{N}}(0, u_0) = \infty, \quad \lim_{u_0 \uparrow +\infty} T_{\mathcal{N}}(0, u_0) = 0. \quad (2.11)$$

Regarding the linear system ($\mathcal{L}$), we must differentiate the cases $\lambda = 0$ and $\lambda > 0$. Indeed, when $\lambda = 0$ all the energy lines are constant in v and the u -axis of the phase plane consists of equilibria. Thus, in this case, a symmetric positive solution of (1.1) corresponds to a solution starting at some point $(0, v_0)$, with $v_0 > 0$, and running along the trajectory $\mathcal{E}_{\mathcal{N}} = v_0^2/2$ until reaching some point $(u_0, 0)$ at time $x = \frac{1-h}{2}$. Then, this solution should stay at the equilibrium $(u_0, 0)$ for a time h before completing the trajectory $\mathcal{E}_{\mathcal{N}} = v_0^2/2$ and reaching $(0, -v_0)$ at $x = 1$. By the monotonicity of $T_{\mathcal{N}}(0, u_0)$ with respect to u_0 , the existence and the uniqueness of the symmetric positive solution when $\lambda = 0$ follows from (2.11). These solutions are constant in the interval $(\frac{1-h}{2}, \frac{1+h}{2})$.

Next, we assume that $\lambda > 0$. Then, instead of (2.2), it is more appropriate to consider the equivalent systems

$$(\mathcal{N}_{\text{eq}}) \quad \begin{cases} u' = \sqrt{\lambda} v, \\ v' = -\sqrt{\lambda} u - \frac{1}{\sqrt{\lambda}} u^p, \end{cases} \quad (\mathcal{L}_{\text{eq}}) \quad \begin{cases} u' = \sqrt{\lambda} v, \\ v' = -\sqrt{\lambda} u, \end{cases} \quad (2.12)$$

whose associated total energies are

$$\mathcal{E}_{\mathcal{N}_{\text{eq}}}(u, v) := \sqrt{\lambda} \left(\frac{1}{2} v^2 + \frac{1}{2} u^2 + \frac{1}{\lambda(p+1)} u^{p+1} \right)$$

and

$$\mathcal{E}_{\mathcal{L}_{\text{eq}}}(u, v) := \sqrt{\lambda} \left(\frac{1}{2} v^2 + \frac{1}{2} u^2 \right),$$

respectively. For $(\mathcal{L}_{\text{eq}})$ in (2.12), the angular variable in polar coordinates, $\theta(x)$, satisfies

$$\theta'(x) = \frac{v'(x)u(x) - u'(x)v(x)}{u^2(x) + v^2(x)} = -\sqrt{\lambda}$$

and hence, the angle run, clockwise, by a periodic solution in an interval of length h is

$$\theta = \int_0^h \theta'(x) dx = -h\sqrt{\lambda}.$$

Thus, when $\lambda > 0$, a symmetric positive solution of (1.1) should begin at some point $(0, v_0)$ and reach at time $x = (1-h)/2$, through the level set

$$\mathcal{E}_{\mathcal{N}_{\text{eq}}}(u, v) = \sqrt{\lambda} \frac{v_0^2}{2}, \quad (2.13)$$

the (unique) point (u_ℓ, v_ℓ) lying on the line

$$v = \left(\tan \frac{h\sqrt{\lambda}}{2} \right) u. \quad (2.14)$$

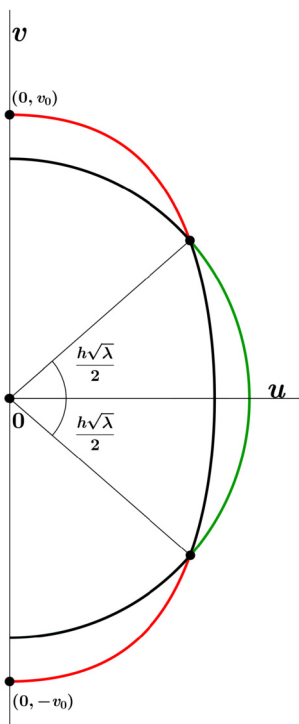


Fig. 3. A symmetric positive solution of (1.1) in the case $\lambda \in (0, \pi^2)$.

Then, it moves in a time h an angle $-h\sqrt{\lambda}$ through the level set

$$\mathcal{E}_{\mathcal{L}_{\text{eq}}} = \sqrt{\lambda} \left(\frac{1}{2} v_\ell^2 + \frac{1}{2} u_\ell^2 \right)$$

up to reach the point $(u_\ell, -v_\ell)$. Finally, it runs again along (2.13) from $(u_\ell, -v_\ell)$ to $(0, -v_0)$, as it has been sketched in Fig. 3, where the phase plane route in the nonlinear (resp. linear) level set has been highlighted using red (resp. green) color. Therefore, to prove the existence of a symmetric positive solution it suffices to show that $(0, v_0)$ can be linked in time $(1-h)/2$ to a point (u_ℓ, v_ℓ) lying on the line (2.14) through the integral curve

$$v^2 + u^2 + \frac{2}{\lambda(p+1)} u^{p+1} = \left(\tan \frac{h\sqrt{\lambda}}{2} \right)^2 u_\ell^2 + u_\ell^2 + \frac{2}{\lambda(p+1)} u_\ell^{p+1}.$$

Note that the necessary time to link $(0, v_0)$ and (u_ℓ, v_ℓ) along this integral curve is given by

$$T_{\mathcal{N}}(\lambda, 0, u_\ell) := \frac{1}{\sqrt{\lambda}} \int_0^1 \frac{ds}{\sqrt{\tan^2 \frac{h\sqrt{\lambda}}{2} + 1 - s^2 + \frac{2u_\ell^{p-1}}{\lambda(p+1)} (1 - s^{p+1})}}, \quad (2.15)$$

where we have used that $u' = \sqrt{\lambda}v$. Thus, since $T_{\mathcal{N}}(\lambda, 0, u_\ell)$ is decreasing with respect to u_ℓ and

$$\lim_{u_\ell \uparrow +\infty} T_{\mathcal{N}}(\lambda, 0, u_\ell) = 0,$$

the existence, and the uniqueness, of a symmetric positive solution of (1.1) in case $\lambda > 0$ is guaranteed as soon as

$$\lim_{u_\ell \downarrow 0} T_{\mathcal{N}}(\lambda, 0, u_\ell) = \frac{1}{2} \left(\frac{\pi}{\sqrt{\lambda}} - h \right) > \frac{1-h}{2}, \quad (2.16)$$

which holds true if, and only if, $\lambda < \pi^2$. To show the validity of the limit in (2.16), we set

$$\alpha := \tan^2 \frac{h\sqrt{\lambda}}{2} + 1 = \sec^2 \frac{h\sqrt{\lambda}}{2}.$$

Then,

$$\begin{aligned} \lim_{u_\ell \downarrow 0} T_{\mathcal{N}}(\lambda, 0, u_\ell) &= \frac{1}{\sqrt{\lambda}} \int_0^1 \frac{ds}{\sqrt{\tan^2 \frac{h\sqrt{\lambda}}{2} + 1 - s^2}} = \frac{1}{\sqrt{\lambda}} \int_0^{\frac{1}{\sqrt{\alpha}}} \frac{d\theta}{\sqrt{1 - \theta^2}} \\ &= -\frac{1}{\sqrt{\lambda}} \left(\arccos \cos \frac{h\sqrt{\lambda}}{2} - \frac{\pi}{2} \right) = \frac{1}{2} \left(\frac{\pi}{\sqrt{\lambda}} - h \right), \end{aligned}$$

which provides us with (2.16). Summarizing, by the monotonicity of $T_{\mathcal{N}}(\lambda, 0, u_\ell)$ with respect to u_ℓ , for every $\lambda \in (0, \pi^2)$, (1.1) has a unique symmetric positive solution.

Finally, note that condition $\lambda < \pi^2$ is actually necessary for the existence of a positive solution of (1.1). Indeed, if (1.1) has a positive solution, u , then multiplying the differential equation by $\sin(\pi x)$ and integrating by parts in $(0, 1)$ yields

$$\begin{aligned} \pi^2 \int_0^1 \sin(\pi x) u(x) dx &= \lambda \int_0^1 \sin(\pi x) u(x) dx + \int_0^1 a(x) \sin(\pi x) u^p(x) dx \\ &> \lambda \int_0^1 \sin(\pi x) u(x) dx. \end{aligned}$$

Thus, $\lambda < \pi^2$, because $u(x) > 0$ for all $x \in (0, 1)$ and $a \geq 0$.

2.3. Uniqueness of the symmetric positive solution

The next result establishes the uniqueness of the symmetric positive solutions of (1.1). Note that, in case $\lambda \in [0, \pi^2)$, this result has been already established in Section 2.2

Theorem 2.1. *For every $\lambda \in (-\infty, \pi^2)$, the problem (1.1) has a unique symmetric positive solution. Thus, it has a positive solution if, and only if, $\lambda < \pi^2$.*

Proof. It suffices to prove the result for $\lambda < 0$. So, suppose $\lambda < 0$. Arguing by contradiction, assume that u_1 and u_2 are two symmetric positive solutions of (1.1) and set

$$v_1 = u_1', \quad v_2 = u_2'.$$

Since $u_1(0) = u_2(0) = 0$, by the uniqueness of the underlying Cauchy problem, $v_1(0) \neq v_2(0)$. By inter-exchanging their names, if necessary, we can assume that $v_1(0) > v_2(0)$. Then,

$$w := u_1 - u_2$$

satisfies $w(0) = 0$ and $w'(0) > 0$. Thus, there exists $\eta \in (0, 1]$ such that $w(x) > 0$ for all $x \in (0, \eta)$, and $w(\eta) = 0$. We claim that $\eta = 1$, i.e.

$$w(x) > 0 \quad \text{for all } x \in (0, 1). \quad (2.17)$$

Indeed, since (1.1) is autonomous in $[0, \frac{1-h}{2}]$, u_1 and u_2 lie in different energy levels there in and, in particular, $v_1(0) > v_2(0) > 0$. Thus, since

$$\mathcal{E}_{\mathcal{N}}(0, v_1(0)) > \mathcal{E}_{\mathcal{N}}(0, v_2(0)),$$

it becomes apparent that, for every $x \in [0, \frac{1-h}{2}]$,

$$\mathcal{E}_{\mathcal{N}}(u_1(x), v_1(x)) = \mathcal{E}_{\mathcal{N}}(0, v_1(0)) > \mathcal{E}_{\mathcal{N}}(u_2(x), v_2(x)) = \mathcal{E}_{\mathcal{N}}(0, v_2(0)) > 0. \quad (2.18)$$

Moreover, as sketched in Fig. 2, since u_1 and u_2 are symmetric positive solutions, $u_1(\frac{1-h}{2})$ and $u_2(\frac{1-h}{2})$ should lie in the straight line

$$v = -\sqrt{-\lambda(1 - C^{-2})}u.$$

Consequently, substituting in (2.18), we find that

$$\begin{aligned} \mathcal{E}_{\mathcal{N}}(u_1, v_1) &= \frac{-\lambda(1 - C^{-2})}{2} u_1^2\left(\frac{1-h}{2}\right) + \frac{\lambda}{2} u_1^2\left(\frac{1-h}{2}\right) + \frac{1}{p+1} u_1^{p+1}\left(\frac{1-h}{2}\right) \\ &= \frac{\lambda}{2C^2} u_1^2\left(\frac{1-h}{2}\right) + \frac{1}{p+1} u_1^{p+1}\left(\frac{1-h}{2}\right) \\ &> \mathcal{E}_{\mathcal{N}}(u_2, v_2) = \frac{\lambda}{2C^2} u_2^2\left(\frac{1-h}{2}\right) + \frac{1}{p+1} u_2^{p+1}\left(\frac{1-h}{2}\right) > 0. \end{aligned} \quad (2.19)$$

Thus, since the function

$$\varphi(s) := \frac{\lambda s^2}{2C^2} + \frac{s^{p+1}}{p+1}, \quad s > 0,$$

is increasing if $\varphi(s) > 0$, because, in this case, $\varphi'(s) = \frac{\lambda s}{C^2} + s^p > 0$ for all $s > 0$, (2.19) implies that

$$w(\frac{1-h}{2}) = u_1(\frac{1-h}{2}) - u_2(\frac{1-h}{2}) > 0.$$

In particular, $\eta \neq \frac{1-h}{2}$. Suppose that $\eta \in (0, \frac{1-h}{2})$. Then, since $w(x) > 0$ for all $x \in (0, \eta)$ and $w(\eta) = 0$, necessarily, $w'(\eta) < 0$. Moreover, since $w(\frac{1-h}{2}) > 0$, there exists $\delta \in (\eta, \frac{1-h}{2})$ such that

$$w(x) < 0 \text{ for all } x \in (\eta, \delta), \quad w(\delta) = 0, \quad \text{and } w'(\delta) > 0.$$

On the other hand, by (2.18), we already know that, for every $x \in [0, \frac{1-h}{2}]$,

$$\begin{aligned} \mathcal{E}_{\mathcal{N}}(u_1(x), v_1(x)) &= \frac{1}{2}v_1^2(x) + \frac{\lambda}{2}u_1^2(x) + \frac{1}{p+1}u_1^{p+1}(x) \\ &> \mathcal{E}_{\mathcal{N}}(u_2(x), v_2(x)) = \frac{1}{2}v_2^2(x) + \frac{\lambda}{2}u_2^2(x) + \frac{1}{p+1}u_2^{p+1}(x) > 0. \end{aligned} \quad (2.20)$$

Consequently, since

$$w(\eta) = u_1(\eta) - u_2(\eta) = w(\delta) = u_1(\delta) - u_2(\delta) = 0,$$

it follows from (2.20) that

$$v_1^2(\eta) > v_2^2(\eta), \quad v_1^2(\delta) > v_2^2(\delta),$$

or, equivalently,

$$|v_1(\eta)| > |v_2(\eta)|, \quad |v_1(\delta)| > |v_2(\delta)|.$$

Moreover, since $w'(\eta) < 0$, we have that

$$0 > w'(\eta) = v_1(\eta) - v_2(\eta) \quad \text{and} \quad |v_1(\eta)| > |v_2(\eta)|.$$

So, $v_1(\eta) < 0$. By the clockwise rotation sense in the phase plane, this entails that $v_1(x) < 0$ for all $x \in [\eta, \frac{1-h}{2}]$. Similarly, since

$$0 < w'(\delta) = v_1(\delta) - v_2(\delta) \quad \text{and} \quad |v_1(\delta)| > |v_2(\delta)|,$$

we can infer that $v_1(\delta) > 0$, which is a contradiction. Therefore,

$$w(x) = u_1(x) - u_2(x) > 0 \text{ for all } x \in (0, \frac{1-h}{2}].$$

By the symmetry of the solution, we also have that

$$w(x) = u_1(x) - u_2(x) > 0 \text{ for all } x \in [\frac{1+h}{2}, 1).$$

Finally, since $w(\frac{1-h}{2}) > 0$ and $w(\frac{1+h}{2}) > 0$, if w changes sign in $(\frac{1-h}{2}, \frac{1+h}{2})$, then there would exist a subinterval $[\alpha, \beta] \subsetneq (\frac{1-h}{2}, \frac{1+h}{2})$ such that $w(x) < 0$ for all $x \in (\alpha, \beta)$, and

$$\begin{cases} -w'' = \lambda w & \text{in } [\alpha, \beta], \\ w(\alpha) = 0 = w(\beta). \end{cases} \quad (2.21)$$

Consequently, $w|_{[\alpha, \beta]}$ is an eigenfunction with associated eigenvalue

$$\lambda = \pi^2 / (\beta - \alpha)^2 > 0,$$

which contradicts our assumption that $\lambda < 0$. Therefore, (2.17) holds.

Subsequently, we fix $\lambda < 0$ and denote by D^2 the differential operator $\frac{d^2}{dx^2}$. Then, $w := u_1 - u_2$ satisfies

$$\begin{cases} -D^2 w - a(x)(u_1^p - u_2^p) = \lambda w, & x \in [0, 1], \\ w(0) = 0 = w(1). \end{cases} \quad (2.22)$$

Setting

$$\psi(\xi) := (\xi u_1 + (1 - \xi)u_2)^p, \quad \xi \geq 0,$$

it is apparent that

$$\begin{aligned} u_1^p - u_2^p &= \psi(1) - \psi(0) = \int_0^1 \psi'(\xi) d\xi \\ &= p \int_0^1 (\xi u_1 + (1 - \xi)u_2)^{p-1} d\xi (u_1 - u_2). \end{aligned}$$

Thus, the problem (2.22) can be equivalently written as

$$\begin{cases} \left(-D^2 - a(x)p \int_0^1 (\xi u_1 + (1 - \xi)u_2)^{p-1} d\xi \right) w = \lambda w, & x \in (0, 1), \\ w(0) = 0 = w(1). \end{cases} \quad (2.23)$$

Thus, since, thanks to (2.17), $w > 0$ in $(0, 1)$, it becomes apparent that

$$\lambda = \sigma_1 \left[-D^2 - a(x)p \int_0^1 (\xi u_1 + (1 - \xi)u_2)^{p-1} d\xi \right], \quad (2.24)$$

where $\sigma_1[-D^2 + b(x)]$ denotes the principal eigenvalue of $-D^2 + b(x)$ under Dirichlet boundary conditions. Hence, by the monotonicity of the principal eigenvalue with respect to the potential (see, if necessary, [11]), it follows from (2.24) that

$$\lambda < \sigma_1[-D^2 - a(x)u_1^{p-1}] = \lambda,$$

which is impossible. Therefore, $w \equiv 0$, which ends the proof. $\square$

2.4. Structure of the symmetric positive solutions

According to, e.g., Amann and López-Gómez [1], or adapting the arguments of Lemma 2.3 and Theorem 2.4 of the authors in [13], it becomes apparent that the set of non-trivial solutions of the problem

$$\begin{cases} -u'' = \lambda u + a(x)u^p, & x \in (0, \frac{1}{2}), \\ u(0) = u'(\frac{1}{2}) = 0, \end{cases} \quad (2.25)$$

contains a component of positive solutions bifurcating from $\lambda = \pi^2$, denoted by $\mathcal{C}_{0,5}^+$, such that

$$\mathcal{P}_\lambda(\mathcal{C}_{0,5}^+) = (-\infty, \pi^2].$$

As there is a canonical one-one correspondence between the positive solutions of (2.25) and the symmetric positive solutions of (1.1), we can infer that there exists a component of symmetric positive solutions of (1.1), denoted by $\mathcal{C}_{h,s}^+$, such that

$$(\pi^2, 0) \in \mathcal{C}_{h,s}^+ \text{ and } \mathcal{P}_\lambda(\mathcal{C}_{h,s}^+) = (-\infty, \pi^2]. \quad (2.26)$$

Thus, thanks to Theorem 2.1, the next result holds.

Theorem 2.2. *Besides (2.26), the component $\mathcal{C}_{h,s}^+$ satisfies the following properties:*

- (a) *For every $(\lambda, u_\lambda) \in \mathcal{C}_{h,s}^+$, either $\lambda = \pi^2$ and $u_\lambda = 0$, or $\lambda < \pi^2$ and u_λ is the unique symmetric positive solution of (1.1).*
- (b) *Subsequently, for every $\lambda < \pi^2$, we denote by u_λ the unique symmetric positive solution of (1.1). Then, for every $\sigma \in (-\infty, \pi^2]$,*

$$\lim_{\lambda \rightarrow \sigma} \|u_\lambda\|_\infty = 0 \text{ if and only if } \sigma = \pi^2. \quad (2.27)$$

- (c) *For each $\lambda_0 \in (-\infty, \pi^2)$, the set of solutions*

$$\mathcal{D}_0 := \{u_\lambda : \lambda \in [\lambda_0, \pi^2)\}$$

is bounded in $C^1([0, 1]; \mathbb{R})$, though

$$\lim_{\lambda \downarrow -\infty} \|u_\lambda\|_\infty = +\infty. \quad (2.28)$$

- (d) *The map $\lambda \mapsto u_\lambda$ is continuous. Therefore,*

$$\mathcal{C}_{h,s}^+ = \{(\lambda, u_\lambda) : \lambda \in (-\infty, \pi^2)\}$$

is a continuous curve, as illustrated in Fig. 4.

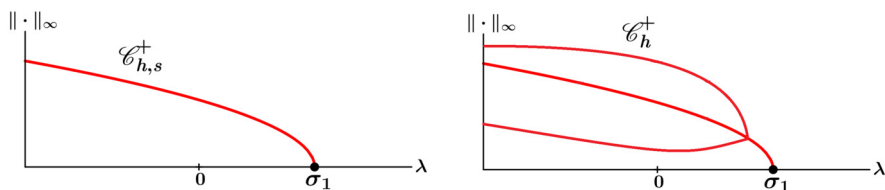


Fig. 4. The component $\mathcal{C}_{h,s}^+$ of (1.1) positive symmetric solutions (left) and a possible component $\mathcal{C}_h^+$ of (1.1) positive solutions (right).

Finally, by the uniqueness of the component of positive solutions bifurcating from $u = 0$ at $\lambda = \pi^2$, $\mathcal{C}_{h,s}^+$ must be a subcomponent of the component of positive solutions $\mathcal{C}_h^+$ whose existence was established in Section 1.

Proof. Part (a) is a byproduct of the previous analysis. Part (b) is a consequence of the fact that $\mathcal{C}_{0,5}^+$ bifurcates from $u = 0$ at $\lambda = \pi^2$. Thus, by construction, so does it the component $\mathcal{C}_{h,s}^+$. Moreover, it is well known that $\lambda = \pi^2$ is the unique bifurcation value to positive solutions from $u = 0$. Therefore, (2.27) holds.

The fact that, for every $\lambda_0 < \pi^2$, the set $\mathcal{D}_0$ is bounded in $C^1([0, 1]; \mathbb{R})$ is a direct consequence, e.g., of the a priori bounds of [1] and [13]. To show (2.28), suppose that $\lambda < 0$ and let $x_m = x_m(\lambda) \in (0, 1)$ be such that

$$u_\lambda(x_m) = \|u_\lambda\|_\infty.$$

Then, $u'_\lambda(x_m) = 0$ and $u''_\lambda(x_m) \leq 0$. Thus, it follows from (1.1) that

$$\lambda + a(x_m)u_\lambda^{p-1}(x_m) \geq 0$$

and hence, $a(x_m) = 1$. Therefore,

$$u_\lambda(x_m) = \|u_\lambda\|_\infty \geq (-\lambda)^{\frac{1}{p-1}}$$

and, letting $\lambda \downarrow -\infty$ in this estimate, (2.28) holds. This shows Part (c).

The continuity of the map $\lambda \mapsto u_\lambda$ follows, through a rather standard compactness argument, from the uniqueness of the symmetric positive solution. Indeed, necessarily,

$$\lim_{\lambda \rightarrow \mu} \|u_\lambda - u_\mu\|_\infty = 0$$

for all $\mu \in (-\infty, \pi^2]$. Note that Part (c) guarantees the existence of a priori bounds in $[\mu - 1, \pi^2]$. This ends the proof. $\square$

Fig. 4 shows two admissible global bifurcation diagrams of positive solutions of (1.1). In the first bifurcation diagram we are assuming that $\mathcal{C}_h^+$ consists of radially symmetric positive solutions and, hence, $\mathcal{C}_h^+ = \mathcal{C}_{h,s}^+$. In the second one we are representing a case when $\mathcal{C}_{h,s}^+$ is a proper subset of $\mathcal{C}_h^+$. One of the main goals of this paper is ascertaining whether, or not, each of these situations can occur.

3. Limiting behavior of $\mathcal{C}_h^+$ as $h \uparrow 1$

The main goal of this section is analyzing the behavior of the component of positive solutions of (1.1), assuming (1.3), bifurcating from $u = 0$ at $\lambda = \pi^2$, $\mathcal{C}_h^+$, as $h \uparrow 1$, i.e. when the nonlinear system ($\mathcal{N}$) tends towards the linear system ($\mathcal{L}$). In the limiting situation when $h = 1$, (1.1) reduces to ($\mathcal{L}$), which admits some positive solution if, and only if, $\lambda = \pi^2$. Actually, the component of positive solutions bifurcating from $u = 0$ at $\lambda = \pi^2$ for this limiting problem, denoted by $\mathcal{C}_1^+$, is given by

$$\mathcal{C}_1^+ = \{(\pi^2, \alpha \sin(\pi x)) : \alpha \in \mathbb{R}_+ := [0, +\infty)\}.$$

Thus, $\lambda = \pi^2$ is a bifurcation value to a vertical bifurcation from $u = 0$. In particular, $\mathcal{P}_\lambda(\mathcal{C}_1^+) = \{\pi^2\}$. Consequently, the remaining positive solutions of the component $\mathcal{C}_h^+$ for $\lambda < \pi^2$ should be unbounded as $h \uparrow 1$. The next result provides us with the point-wise behavior of the positive solutions of (1.1) as $h \uparrow 1$.

Theorem 3.1. *For any given $\lambda < \pi^2$ and, for every $h \in [0, 1)$, let u_h be a positive solution of (1.1) for the choice (1.3). Then,*

$$\lim_{h \uparrow 1} \|u_h\|_\infty = +\infty. \quad (3.1)$$

Moreover,

$$\lim_{h \uparrow 1} u_h(x) = +\infty \text{ for all } x \in (0, 1). \quad (3.2)$$

In particular, the components $\mathcal{C}_h^+$ blow-up, point-wise in $(0, 1)$, as $h \uparrow 1$.

Proof. As in [3, Lemma 2.1] or [13, Lemma 2.2], we denote by $\mathcal{K} : \mathcal{C}([0, 1]; \mathbb{R}) \rightarrow \mathcal{C}^1([0, 1]; \mathbb{R})$ the linear, continuous and compact operator defined, for every $f \in \mathcal{C}([0, 1]; \mathbb{R})$, by

$$\mathcal{K}f(x) := \int_0^x (y-x)f(y)dy - x \int_0^1 (y-1)f(y)dy,$$

which provides us with the unique solution of the linear problem

$$\begin{cases} -u'' = f & \text{in } [0, 1], \\ u(0) = u(1) = 0. \end{cases}$$

Thus,

$$u_h = \mathcal{K}(\lambda u_h + a_h u_h^p) \text{ for all } h \in [0, 1). \quad (3.3)$$

Arguing by contradiction, suppose that there is a sequence, $\{h_n\}_{n \geq 1}$ in $(0, 1)$ converging to 1, such that, for some constant $C > 0$,

$$\|u_{h_n}\|_\infty \leq C \text{ for all } n \geq 1. \quad (3.4)$$

Then, (3.3) implies that, for every $n \geq 1$,

$$\frac{u_{h_n}}{\|u_{h_n}\|_\infty} = \mathcal{K} \left(\lambda \frac{u_{h_n}}{\|u_{h_n}\|_\infty} + a_{h_n} u_{h_n}^{p-1} \frac{u_{h_n}}{\|u_{h_n}\|_\infty} \right). \quad (3.5)$$

Moreover, thanks to (3.4), the sequence

$$\lambda \frac{u_{h_n}}{\|u_{h_n}\|_\infty} + a_{h_n}(x) u_{h_n}^{p-1} \frac{u_{h_n}}{\|u_{h_n}\|_\infty}, \quad n \geq 1,$$

is bounded in $\mathcal{C}([0, 1]; \mathbb{R})$. Thus, by the compactness of $\mathcal{K}$, there exists a subsequence of $\{u_{h_n}\}_{n \geq 1}$, labeled by h_{n_m} , $m \geq 1$, such that

$$\lim_{m \rightarrow +\infty} \frac{u_{h_{n_m}}}{\|u_{h_{n_m}}\|_\infty} = \psi$$

in $\mathcal{C}^1([0, 1]; \mathbb{R})$, for some $\psi \in \mathcal{C}^1([0, 1]; \mathbb{R})$ such that $\|\psi\|_\infty = 1$ and $\psi \not\equiv 0$. Consequently, particularizing (3.5) at h_{n_m} and letting $m \rightarrow \infty$, shows that

$$\begin{cases} -\psi'' = \lambda \psi & \text{in } (0, 1), \\ \psi(0) = \psi(1) = 0, \end{cases}$$

which is impossible because $\psi \not\equiv 0$ and $\lambda < \pi^2$. Therefore, (3.1) holds.

To prove (3.2) we will use a concavity argument if $\lambda \geq 0$ and a phase plane argument together with the analysis of the linear problem if $\lambda < 0$.

Case 1: Assume $\lambda \geq 0$. Then, for every $x \in [0, 1]$,

$$-u_h''(x) = \lambda u_h(x) + a_h(x) u_h^p(x) \geq 0$$

and, hence, u_h is concave in $[0, 1]$ for all $h \in [0, 1]$. Let $x_{m_h} \in (0, 1)$ be such that

$$u_h(x_{m_h}) = \|u_h\|_\infty.$$

Then, for every $x \in (0, 1)$, we have that

$$x = \begin{cases} \left(1 - \frac{x}{x_{m_h}}\right) 0 + \frac{x}{x_{m_h}} x_{m_h} & \text{if } x \leq x_{m_h}, \\ \left(1 - \frac{1-x}{1-x_{m_h}}\right) 1 + \frac{1-x}{1-x_{m_h}} x_{m_h} & \text{if } x > x_{m_h}, \end{cases}$$

with $\frac{x}{x_{m_h}} \in (x, 1]$ if $x \leq x_{m_h}$, and $\frac{1-x}{1-x_{m_h}} \in (1-x, 1)$ if $x > x_{m_h}$. Consequently, by the concavity of u_h ,

$$u_h(x) \geq \begin{cases} \left(1 - \frac{x}{x_{m_h}}\right) u_h(0) + \frac{x}{x_{m_h}} \|u_h\|_\infty \geq x \|u_h\|_\infty & \text{if } x \leq x_{m_h}, \\ \left(1 - \frac{1-x}{1-x_{m_h}}\right) u_h(1) + \frac{1-x}{1-x_{m_h}} \|u_h\|_\infty > (1-x) \|u_h\|_\infty & \text{if } x > x_{m_h}. \end{cases}$$

Therefore, by (3.1), we find that, for every $x \in (0, 1)$,

$$\lim_{h \uparrow 1} u_h(x) = +\infty.$$

This ends the proof in this case.

Case 2: Assume $\lambda < 0$. We begin by constructing the geometric set of points, $\mathcal{A}_h$, lying in the fourth (resp. first) quadrant and taking time h to reach the negative (resp. positive) v -axis through the linear flow of $(\mathcal{L})$. Note that, for every positive solution u of (1.1), we have that

$$(u(t), v(t)) = (u(t), u'(t)) \notin \mathcal{A}_h \text{ for all } t \in [0, \frac{1-h}{2}], \quad (3.6)$$

because $u(t) > 0$ for all $t \in (0, 1)$. To construct $\mathcal{A}_h$ we adapt an argument of Section 2.1. For every $v_- < 0$, the required time to reach $(0, v_-)$ from a generic point

$$(u_r, v_r) \in \{(u, v) : \mathcal{E}_{\mathcal{L}}(u, v) = v_-^2\}$$

is given by

$$\begin{aligned} T_{\mathcal{L}}(\lambda, u_r, 0) &:= - \int_{u_r}^0 \frac{du}{\sqrt{v_-^2 - \lambda u^2}} = - \frac{1}{\sqrt{|\lambda|}} \int_{\frac{\sqrt{|\lambda|}u_r}{v_-}}^0 \frac{ds}{\sqrt{1+s^2}} \\ &= \frac{1}{\sqrt{|\lambda|}} \ln \left(\frac{\sqrt{|\lambda|}u_r}{v_-} + \sqrt{\left(\frac{\sqrt{|\lambda|}u_r}{v_-}\right)^2 + 1} \right), \end{aligned} \quad (3.7)$$

because $g(s) := \ln(s + \sqrt{s^2 + 1})$ satisfies $g'(s) = \frac{1}{\sqrt{s^2 + 1}}$ for all $s \in \mathbb{R}$. Thus, according to (3.7), the set of points (u_r, v_r) for which

$$T_{\mathcal{L}}(\lambda, u_r, 0) = h$$

can be described through $g(\frac{\sqrt{|\lambda|}u_r}{v_-}) = \sqrt{|\lambda|h}$, or, equivalently,

$$\frac{\sqrt{|\lambda|}u_r}{v_-} = g^{-1}(\sqrt{|\lambda|h}),$$

because $g'(s) > 0$ for all $s \in \mathbb{R}$, and, hence, it is globally invertible. Consequently, since $\mathcal{E}_{\mathcal{L}}(u_r, v_r) = v_-^2$, it becomes apparent that

$$\frac{v_r^2}{|\lambda|u_r^2} - 1 = \frac{v_-^2}{|\lambda|u_r^2} = [g^{-1}(\sqrt{|\lambda|h})]^{-2} \equiv D(h).$$

Therefore,

$$v_r = \pm \sqrt{|\lambda|(D(h) + 1)} u_r.$$

In other words, setting $m_h := \sqrt{|\lambda|(D(h) + 1)}$, we have that

$$\mathcal{A}_h := \{(u_r, v_r) : v_r = \pm m_h u_r\}$$

In particular, the set of points (u_r, v_r) , with $u_r < 0$, reaching the negative v -axis at a time h are those on the straight line

$$v = -m_h u. \quad (3.8)$$

An important feature is that m_h is decreasing with respect to h .

On the other hand, by the proof of Theorem 2.2(c), we already know that, for every $h \in (0, 1)$,

$$x_{m_h} \in (0, \frac{1-h}{2}] \cup [\frac{1+h}{2}, 1) \text{ if } u(x_{m_h}) = \|u_h\|_\infty.$$

Thus, there exists an increasing subsequence, $\{h_n\}_{n \geq 1}$, in $(0, 1)$, such that either

$$\lim_{n \rightarrow \infty} h_n = 1 \text{ and } x_n \equiv x_{m_{h_n}} \in (0, \frac{1-h_n}{2}] \text{ for all } n \geq 1, \quad (3.9)$$

or

$$\lim_{n \rightarrow \infty} h_n = 1 \text{ and } x_n \equiv x_{m_{h_n}} \in [\frac{1+h_n}{2}, 1) \text{ for all } n \geq 1, \quad (3.10)$$

though these alternatives are far from excluding, because the symmetric solutions satisfy both. Assume (3.9) holds. Then, owing to (3.6), we find that

$$\frac{v_{h_n} \left(\frac{1-h_n}{2} \right)}{u_{h_n} \left(\frac{1-h_n}{2} \right)} \geq -m_{h_1}.$$

Note that $v_{h_n} \left(\frac{1-h_n}{2} \right) \leq 0$. Indeed, by (3.9), $v(x_n) = 0$, and, hence, by the clockwise rotation sense in the phase plane, $v(x) \leq 0$ for all $x \in [x_n, \frac{1-h_n}{2}]$. As $m_{h_1} > 0$, the above inequality implies

$$v_{h_n}^2 \left(\frac{1-h_n}{2} \right) \leq m_{h_1}^2 u_{h_n}^2 \left(\frac{1-h_n}{2} \right). \quad (3.11)$$

Thus, as the associated energy to the system $(\mathcal{N})$ is

$$\mathcal{E}_{\mathcal{N}}(u, v) = \frac{1}{2}v^2 + \frac{\lambda}{2}u^2 + \frac{1}{p+1}u^{p+1},$$

it follows from (3.11) that

$$\begin{aligned}
\mathcal{E}_{\mathcal{N}}(u_{h_n}\left(\frac{1-h_n}{2}\right), v_{h_n}\left(\frac{1-h_n}{2}\right)) &= \frac{\lambda}{2} \|u_{h_n}\|_{\infty}^2 + \frac{1}{p+1} \|u_{h_n}\|_{\infty}^{p+1} \\
&= \frac{1}{2} v_{h_n}^2\left(\frac{1-h_n}{2}\right) + \frac{\lambda}{2} u_{h_n}^2\left(\frac{1-h_n}{2}\right) + \frac{1}{p+1} u_{h_n}^{p+1}\left(\frac{1-h_n}{2}\right) \\
&\leq \frac{m_{h_1}^2 + \lambda}{2} u_{h_n}^2\left(\frac{1-h_n}{2}\right) + \frac{1}{p+1} u_{h_n}^{p+1}\left(\frac{1-h_n}{2}\right) \\
&\leq \frac{m_{h_1}^2}{2} u_{h_n}^2\left(\frac{1-h_n}{2}\right) + \frac{1}{p+1} u_{h_n}^{p+1}\left(\frac{1-h_n}{2}\right).
\end{aligned}$$

Therefore, we can conclude from (3.1) that

$$\lim_{n \rightarrow +\infty} u_{h_n}\left(\frac{1-h_n}{2}\right) = +\infty. \quad (3.12)$$

On the other hand, by construction, we have that, for every $n \geq 1$,

$$-u_{h_n}'' = \lambda u_{h_n} \quad \text{in} \quad \left(\frac{1-h_n}{2}, \frac{1+h_n}{2}\right). \quad (3.13)$$

Since $\lambda < 0$, for every $n \geq 1$, there exist two constants, $A_{h_n}, B_{h_n} \in \mathbb{R}$, such that, for every $x \in \left(\frac{1-h_n}{2}, \frac{1+h_n}{2}\right)$,

$$u_{h_n}(x) = A_{h_n} e^{\sqrt{|\lambda|x}} + B_{h_n} e^{-\sqrt{|\lambda|x}} = e^{-\sqrt{|\lambda|x}} A_{h_n} \left(e^{2\sqrt{|\lambda|x}} + \frac{B_{h_n}}{A_{h_n}} \right).$$

Equivalently, for every $x \in (0, 1)$ and sufficiently large $n \geq 1$,

$$e^{\sqrt{|\lambda|x}} u_{h_n}(x) = A_{h_n} \left(e^{2\sqrt{|\lambda|x}} + \frac{B_{h_n}}{A_{h_n}} \right). \quad (3.14)$$

Note that, for every $x \in (0, 1)$ and sufficiently large n , we have that $x \in \left(\frac{1-h_n}{2}, \frac{1+h_n}{2}\right)$. Thanks to (3.12), we can infer from (3.14) that

$$\begin{aligned}
+\infty &= \lim_{n \rightarrow +\infty} \left(e^{\sqrt{|\lambda|\frac{1-h_n}{2}}} u_{h_n}\left(\frac{1-h_n}{2}\right) \right) \\
&= \lim_{n \rightarrow +\infty} \left(A_{h_n} \left(e^{2\sqrt{|\lambda|\frac{1-h_n}{2}}} + \frac{B_{h_n}}{A_{h_n}} \right) \right).
\end{aligned} \quad (3.15)$$

Thus, along some subsequence, also labeled by n , we find that, either

$$\lim_{n \rightarrow +\infty} A_{h_n} = +\infty, \quad \text{or} \quad \lim_{n \rightarrow +\infty} B_{h_n} = +\infty. \quad (3.16)$$

Suppose that

$$\liminf_{n \rightarrow +\infty} A_{h_n} > -\infty.$$

Then, by (3.14) and (3.15), we find that, for every $x \in (0, 1)$,

$$\begin{aligned}\liminf_{n \rightarrow +\infty} \left(e^{\sqrt{|\lambda|}x} u_{h_n}(x) \right) &= \liminf_{n \rightarrow +\infty} \left(e^{\sqrt{|\lambda|} \frac{1-h_n}{2}} u_{h_n} \left(\frac{1-h_n}{2} \right) \right) \\ &+ \liminf_{n \rightarrow +\infty} \left(A_{h_n} \left(e^{2\sqrt{|\lambda|}x} - e^{2\sqrt{|\lambda|} \frac{1-h_n}{2}} \right) \right) = +\infty.\end{aligned}$$

Consequently,

$$\lim_{n \rightarrow +\infty} u_{h_n}(x) = +\infty.$$

Now, suppose that $\liminf_{n \rightarrow +\infty} A_{h_n} = -\infty$. Then, along some subsequence, labeled again by n , we can assume that

$$\lim_{n \rightarrow +\infty} A_{h_n} = -\infty \quad (3.17)$$

and, thanks to (3.16),

$$\lim_{n \rightarrow +\infty} B_{h_n} = +\infty.$$

In this case, by (3.14) and the positivity of u_{h_n} , we find that, for every $x \in (0, 1)$ and sufficiently large n ,

$$e^{2\sqrt{|\lambda|}x} + \frac{B_{h_n}}{A_{h_n}} < 0.$$

We claim that, for every $x \in (0, 1)$,

$$\limsup_{n \rightarrow +\infty} \left(e^{2\sqrt{|\lambda|}x} + \frac{B_{h_n}}{A_{h_n}} \right) < 0. \quad (3.18)$$

To show (3.18) we will argue by contradiction, assuming that there exist $\tilde{x} \in (0, 1)$ and a subsequence of h_n , relabeled by n , such that

$$\lim_{n \rightarrow +\infty} \left(e^{2\sqrt{|\lambda|}\tilde{x}} + \frac{B_{h_n}}{A_{h_n}} \right) = 0.$$

Then, by (3.17), for every $x > \tilde{x}$, we have that

$$\lim_{n \rightarrow +\infty} \left(e^{\sqrt{|\lambda|}x} u_{h_n}(x) \right) = \lim_{n \rightarrow +\infty} \left(A_{h_n} \left(e^{2\sqrt{|\lambda|}x} + \frac{B_{h_n}}{A_{h_n}} \right) \right) = -\infty,$$

contradicting the positivity of u_{h_n} . Therefore, again by (3.17), it becomes apparent from (3.18) that, for every $x \in (0, 1)$,

$$\lim_{n \rightarrow +\infty} \left(e^{\sqrt{|\lambda|}x} u_{h_n}(x) \right) = \lim_{n \rightarrow +\infty} \left(A_{h_n} \left(e^{2\sqrt{|\lambda|}x} + \frac{B_{h_n}}{A_{h_n}} \right) \right) = +\infty.$$

The case when (3.10) occurs can be proven easily by adapting the proof of the case (3.9). By repetitive, its technical details will be omitted here. As these arguments can be repeated along any increasing sequence of h_n 's converging to 1 as $n \rightarrow +\infty$, the proof is complete. $\square$

The next result establishes that the solutions of the component $\mathcal{C}_1^+$ perturb into solutions of the components $\mathcal{C}_{h,s}^+$ as $h < 1$ perturbs from 1.

Theorem 3.2. *For every $\alpha > 0$, there are two increasing sequences, $\{h_n\}_{n \geq 1}$ in $(0, 1)$ and $\{\lambda_n\}_{n \geq 1}$ in $(0, \pi^2)$, such that*

$$\lim_{n \rightarrow +\infty} h_n = 1, \quad \lim_{n \rightarrow +\infty} \lambda_n = \pi^2, \quad \text{and } \|u_{\lambda_n}\|_\infty = \alpha \text{ for all } n \geq 1, \quad (3.19)$$

where u_{λ_n} is the unique symmetric positive solution of (1.1) for $(\lambda, h) = (\lambda_n, h_n)$, $n \geq 1$. Moreover,

$$\lim_{n \rightarrow +\infty} u_{\lambda_n} = \alpha \sin(\pi \cdot) \quad \text{in } C^1([0, 1]; \mathbb{R}).$$

In other words, $(\pi^2, \alpha \sin(\pi \cdot)) \in \mathcal{C}_1^+$ perturbs into the sequence

$$(\lambda_n, u_{\lambda_n}) \in \mathcal{C}_{h_n, s}^+, \quad n \geq 1.$$

Proof. According to Theorem 2.2, for every $h \in (0, 1)$, (λ, u_λ) bifurcates from $u = 0$ at $\lambda = \pi^2$, $\mathcal{P}_\lambda(\mathcal{C}_{h,s}^+) = (-\infty, \pi^2]$ and (2.28) is satisfied.

Thanks to Theorem 3.1, for every integer $n \geq 1$, we have that

$$\lim_{h \uparrow 1} \|u_{\pi^2 - \frac{1}{n}}\|_\infty = \infty.$$

Thus, there exists $h_n \in (1 - \frac{1}{n}, 1)$ such that

$$\|u_{\pi^2 - \frac{1}{n}}\|_\infty > \alpha \quad \text{for } h = h_n.$$

Hence, thanks to Theorem 2.2, by the continuity of $\mathcal{C}_{h_n, s}^+$, there exists $\lambda_n \in (\pi^2 - \frac{1}{n}, \pi^2)$ such that

$$\|u_{\lambda_n}\|_\infty = \alpha \quad \text{for } h = h_n.$$

By construction, the sequence $(h_n, \lambda_n, u_{\lambda_n})$, $n \geq 1$, satisfies (3.19). Moreover, for every $n \geq 1$,

$$\frac{u_n}{\|u_n\|_\infty} = \mathcal{K} \left(\lambda_n \frac{u_n}{\|u_n\|_\infty} + a_{h_n} u_n^{p-1} \frac{u_n}{\|u_n\|_\infty} \right), \quad (3.20)$$

where $\mathcal{K}$ is the resolvent operator introduced in the proof of Theorem 3.1. Since $\|u_n\|_\infty = \alpha$ for all $n \geq 1$, and $\lim_{n \rightarrow \infty} \lambda_n = \pi^2$, the sequence

$$\lambda_n \frac{u_n}{\|u_n\|_\infty} + a_{h_n} u_n^{p-1} \frac{u_n}{\|u_n\|_\infty}, \quad n \geq 1,$$

is bounded in $[0, 1]$. Thus, by the compactness of $\mathcal{K}$, it follows from (3.20) that, along some subsequence, relabeled by n_m , $m \geq 1$,

$$\lim_{m \rightarrow +\infty} \frac{u_{n_m}}{\|u_{n_m}\|_\infty} = \psi \quad \text{in } C_0^1([0, 1]; \mathbb{R})$$

for some $\psi \in C_0^1([0, 1]; \mathbb{R})$ such that $\|\psi\|_\infty = 1$ and $\psi \gneq 0$. Finally, letting $m \rightarrow \infty$ in (3.20) with $n = n_m$, it becomes apparent that

$$\begin{cases} -\psi'' = \pi^2 \psi & \text{in } (0, 1), \\ \psi(0) = \psi(1) = 0, \end{cases}$$

which implies $\psi(x) = \sin(\pi x)$ for all $x \in [0, 1]$. Therefore,

$$\lim_{m \rightarrow +\infty} \frac{u_{n_m}}{\alpha} = \sin(\pi \cdot) \quad \text{in } C_0^1([0, 1]; \mathbb{R}).$$

As the same argument can be repeated along any subsequence, we find that

$$\lim_{n \rightarrow +\infty} \frac{u_n}{\alpha} = \sin(\pi \cdot) \quad \text{in } C_0^1([0, 1]; \mathbb{R}).$$

This ends the proof. $\square$

4. A multiplicity result for $\lambda < 0$

As already discussed in Section 2, when $\lambda < 0$ in (2.1), the origin of $(\mathcal{N})$ is a local saddle point whose local unstable and stable manifolds conform the homoclinic connection of the level set $\mathcal{E}_{\mathcal{N}}(u, v) = 0$, as illustrated in the left plot of Fig. 1. Moreover, the origin of $(\mathcal{L})$ in (2.1) is a global saddle point whose unstable and stable manifolds are the straight lines

$$W^u : v = \sqrt{-\lambda} u \quad \text{and} \quad W^s : v = -\sqrt{-\lambda} u, \quad (4.1)$$

respectively. Recalling that

$$\mathcal{E}_{\mathcal{N}}(u, v) := \frac{1}{2} (v^2 + \lambda u^2) + \frac{1}{p+1} u^{p+1},$$

the tangent cone of $\mathcal{E}_{\mathcal{N}}(u, v)$ at the origin is given by the zeros of the homogeneous part $v^2 + \lambda u^2$. Denoting it by $V(v^2 + \lambda u^2)$, it is apparent that

$$V(v^2 + \lambda u^2) = \{(u, v) : \mathcal{E}_{\mathcal{L}}(u, v) = 0\} = W^u \cup W^s \cup \{(0, 0)\}. \quad (4.2)$$

Subsequently, we consider the time map $T_{\mathcal{N}}(\lambda, u_0)$ introduced in (2.4) and defined for all $u_0 > u_{\text{ho}}$. Since $T_{\mathcal{N}}(\lambda, u_0)$ is decreasing with respect to $u_0 > u_{\text{ho}}$, it follows from (2.5) that there exists $v_0 > 0$ such that

$$T_{\mathcal{N}}(\lambda, u_0) = \frac{1-h}{4} \quad \text{with} \quad v_0^2 = \lambda u_0^2 + \frac{2}{p+1} u_0^{p+1}, \quad (4.3)$$

i.e. the orbit of $(\mathcal{N})$ through $(0, v_0)$ needs a time $2T_{\mathcal{N}}(\lambda, u_0) = \frac{1-h}{2}$ to reach $(0, -v_0)$. Actually, by the monotonicity of $T_{\mathcal{N}}(\lambda, u)$ with respect to u , for every $v \in (0, v_0]$, the unique solution

of $(\mathcal{N})$ through $(0, v)$ remains in either the first or the fourth quadrants in the time interval $[0, \frac{1-h}{2}]$. Subsequently, we denote by $\Phi_{\mathcal{N}}^+(x, \cdot)$ the flow map of $(\mathcal{N})$ for all $x > 0$, and, setting $\Gamma_0 := \{0\} \times [0, v_0]$, we consider the transformed curve by the flow of the vertical segment Γ_0 after time $\frac{1-h}{2}$, denoted by

$$\Gamma_{0, \frac{1-h}{2}}^+ := \Phi_{\mathcal{N}}^+\left(\frac{1-h}{2}, \Gamma_0\right),$$

which has been plotted using blue light color in the right picture of Fig. 5. By construction, the curve $\Gamma_{0, \frac{1-h}{2}}^+$ lies in the first and fourth quadrants, and

$$\Phi_{\mathcal{N}}^+\left(\frac{1-h}{2}, (0, v_0)\right) = (0, -v_0),$$

as illustrated in the right plot of Fig. 5. Thus, $\Gamma_{0, \frac{1-h}{2}}^+$ links the points $(0, 0)$ and $(0, -v_0)$. Moreover, by the monotonicity of $T_{\mathcal{N}}$, there exists a unique $v_1 \in (0, v_0)$ such that

$$\Phi_{\mathcal{N}}^+\left(\frac{1-h}{2}, (0, v_1)\right) = (u_1, 0) \in \Gamma_{0, \frac{1-h}{2}}^+. \quad (4.4)$$

According to (2.4), (4.3) and (4.4), u_0 and u_1 satisfy

$$\begin{aligned} \frac{1-h}{2} &= 2T_{\mathcal{N}}(\lambda, u_0) = 2 \int_0^1 \frac{ds}{\sqrt{\lambda(1-s^2) + \frac{2}{p+1}u_0^{p-1}(1-s^{p+1})}}, \\ \frac{1-h}{2} &= T_{\mathcal{N}}(\lambda, u_1) = \int_0^1 \frac{ds}{\sqrt{\lambda(1-s^2) + \frac{2}{p+1}u_1^{p-1}(1-s^{p+1})}}. \end{aligned} \quad (4.5)$$

The time map $T_{\mathcal{N}}$ can be further decomposed as follows. For any given $(u_\omega, v_\omega) \in W^u$, with $u_\omega > 0$, by (4.1), we have that $v_\omega = \sqrt{-\lambda u_\omega} > 0$. Next, we consider the orbit of $(\mathcal{N})$

$$\mathcal{E}_{\mathcal{N}}(u, v) = \mathcal{E}_{\mathcal{N}}(u_\omega, v_\omega),$$

departing from Γ_0 at, say, $(0, v_+)$ and crossing the u -axis at, say, $(u_+, 0)$. Then, since

$$v^2 + \lambda u^2 + \frac{2}{p+1}u^{p+1} = v_\omega^2 + \lambda u_\omega^2 + \frac{2}{p+1}u_\omega^{p+1} = \frac{2}{p+1}u_\omega^{p+1},$$

the necessary time to connect $(0, v_+)$ with (u_ω, v_ω) along that orbit, is given by

$$T_{\mathcal{N}}(\lambda, 0, u_\omega) = \int_0^1 \frac{ds}{\sqrt{-\lambda s^2 + \frac{2}{p+1}u_\omega^{p-1}(1-s^{p+1})}}. \quad (4.6)$$

Similarly, since

$$v^2 + \lambda u^2 + \frac{2}{p+1} u^{p+1} = \frac{2}{p+1} u_\omega^{p+1} = \lambda u_+^2 + \frac{2}{p+1} u_+^{p+1}, \quad (4.7)$$

the necessary time to connect (u_ω, v_ω) with $(u_+, 0)$ is

$$T_{\mathcal{N}}(\lambda, u_\omega, u_+) = \int_{\frac{u_\omega}{u_+}}^1 \frac{ds}{\sqrt{\lambda(1-s^2) + \frac{2}{p+1} u_+^{p-1} (1-s^{p+1})}}. \quad (4.8)$$

By construction,

$$T_{\mathcal{N}}(\lambda, 0, u_+) = T_{\mathcal{N}}(\lambda, 0, u_\omega) + T_{\mathcal{N}}(\lambda, u_\omega, u_+). \quad (4.9)$$

Since $T_{\mathcal{N}}(\lambda, 0, u_\omega)$ is decreasing with respect to u_ω , the intersection $\Gamma_{0, \frac{1-h}{2}}^+ \cap W^u$ consists of a single point. Moreover, thanks to (4.7), we have that

$$\frac{u_\omega}{u_+} = \left((p+1) \frac{\lambda}{2} u_+^{1-p} + 1 \right)^{\frac{1}{p+1}}.$$

Thus, since $p > 1$ and $\lambda < 0$, it is apparent that $\frac{u_\omega}{u_+}$ is increasing with respect to u_+ . Consequently, $T_{\mathcal{N}}(\lambda, u_\omega, u_+)$ is decreasing with respect to u_+ . Therefore, by the symmetry of $(\mathcal{N})$ and $(\mathcal{L})$ with respect to the u -axis, the curve $\Gamma_{0, \frac{1-h}{2}}^+$ also intersects W^s at a single point, as illustrated by the right plot of Fig. 5.

Similarly, setting $\Gamma_1 := \{0\} \times [-v_0, 0]$, we can shoot backwards in time by the reversed flow $\Phi_{\mathcal{N}}^-(x, \cdot)$ of $(\mathcal{N})$ moving from $x = 1$ to $x = \frac{1+h}{2}$. This provides us with the curve

$$\Gamma_{1, \frac{1+h}{2}}^- := \Phi_{\mathcal{N}}^-\left(\frac{1+h}{2}, \Gamma_1\right),$$

which has been plotted using green color on the right picture of Fig. 5. By construction, the curve $\Gamma_{1, \frac{1+h}{2}}^-$ lies entirely in the first and fourth quadrants. Moreover, by the symmetries of $a(x)$ and the systems $(\mathcal{N})$ and $(\mathcal{L})$, it is apparent that

$$\Gamma_{1, \frac{1+h}{2}}^- = \left\{ (u, v) \in \mathbb{R}^2 : (-u, v) \in \Gamma_{0, \frac{1-h}{2}}^+ \right\}.$$

In particular, it crosses W^u and W^s at two single points, much like $\Gamma_{0, \frac{1-h}{2}}^+$. Fig. 5 sketches the main properties of these curves. Note that

$$\Gamma_{1, \frac{1+h}{2}}^- \cap \Gamma_{0, \frac{1-h}{2}}^+ = \{(u_1, 0)\}.$$

Obviously, to construct positive asymmetric solutions of the problem (1.1), the curves $\Gamma_{0, \frac{1-h}{2}}^+$ and $\Gamma_{1, \frac{1+h}{2}}^-$ should be connected in a time $h \in (0, 1)$ through some orbit of the linear system $(\mathcal{L})$. The next result shows how to do it.

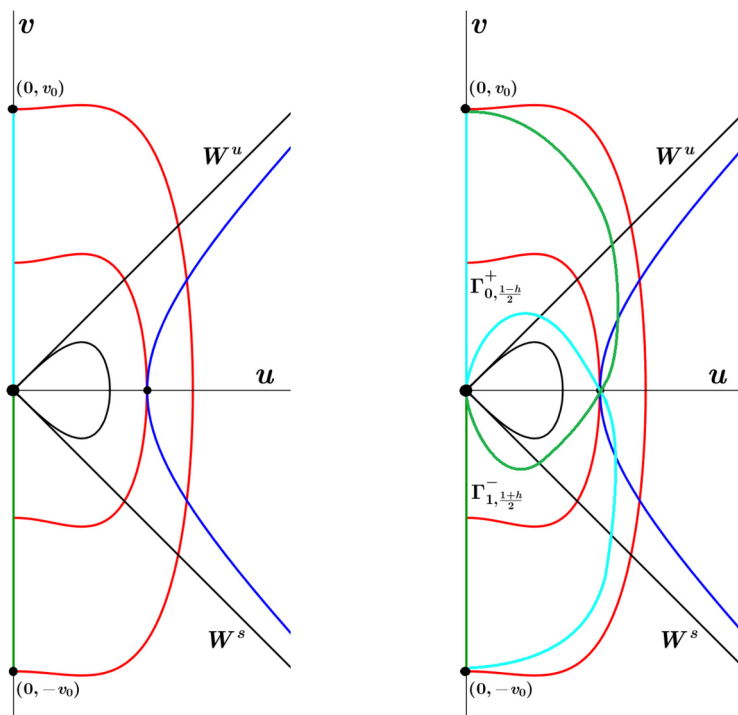


Fig. 5. The sets Γ_0 in blue light and Γ_1 in green (left), as well as the sets $\Gamma_{0, \frac{1-h}{2}}^+$ in blue light and $\Gamma_{1, \frac{1+h}{2}}^-$ in green (right).

Theorem 4.1. For every $h \in (0, 1)$, there exists $\tilde{\lambda}(h) < 0$ such that (1.1) has, at least, two asymmetric positive solutions for each $\lambda \leq \tilde{\lambda}(h)$.

Proof. According the previous analysis, the set

$$\Gamma_{0, \frac{1-h}{2}}^+ \cap \{(u, v) : v \geq \sqrt{-\lambda}u \geq 0\} \quad (4.10)$$

is the piece of $\Gamma_{0, \frac{1-h}{2}}^+$ in the first quadrant connecting $(0, 0)$ with W^u . Similarly,

$$\Gamma_{1, \frac{1+h}{2}}^- \cap \{(u, v) : v \geq \sqrt{-\lambda}u \geq 0\} \quad (4.11)$$

is the piece of $\Gamma_{1, \frac{1+h}{2}}^-$ connecting $(0, v_0)$ with W^u . According to the nature of the flow of $(\mathcal{L})$, there exists $v_* > 0$ such that, for every $\tilde{v} \in (0, v_*)$, the level line of the linear system

$$\mathcal{E}_{\mathcal{L}}(u, v) = \frac{1}{2}v^2 + \frac{\lambda}{2}u^2 = \frac{1}{2}\tilde{v}^2 \quad (4.12)$$

intersects both (4.10) and (4.11). By continuous dependence, the necessary time to connect (4.10) with (4.11) through (4.12) grows to infinity as $\tilde{v} \downarrow 0$.

On the other hand, in the portion of the first quadrant in between the u -axis and W^u , the sets

$$\Gamma_{0, \frac{1-h}{2}}^+ \cap \{(u, v) : 0 \leq v \leq \sqrt{-\lambda u}\} \quad (4.13)$$

and

$$\Gamma_{1, \frac{1+h}{2}}^- \cap \{(u, v) : 0 \leq v \leq \sqrt{-\lambda u}\} \quad (4.14)$$

are the two arcs of curve of $\Gamma_{0, \frac{1-h}{2}}^+$ and $\Gamma_{1, \frac{1+h}{2}}^-$, respectively, connecting the portion of W^u in the first quadrant with the point $(u_1, 0)$ defined in (4.5). Note that

$$(u_1, 0) \in \Gamma_{0, \frac{1-h}{2}}^+ \cap \Gamma_{1, \frac{1+h}{2}}^-.$$

By construction, the interval of times necessary to connect (4.13) to (4.14) with the orbits of $(\mathcal{L})$ passing through some point $(\hat{u}, 0)$ with $\hat{u} \in (0, u_1)$ is bounded. Therefore, by the continuity of the time map associated to $(\mathcal{L})$, the connection times between $\Gamma_{0, \frac{1-h}{2}}^+$ and $\Gamma_{1, \frac{1+h}{2}}^-$ within the first quadrant cover an interval of the form $[\alpha, +\infty)$ for some $\alpha > 0$; α being the minimum connection time between these curves. To complete the proof of the theorem it suffices to show that $\alpha < h$ for sufficiently large $|\lambda| = -\lambda$.

Let $(\hat{u}, \hat{v})$ denote the crossing point of (4.14) with the orbit of $(\mathcal{L})$ through the point $(u_1, 0)$

$$\mathcal{E}_{\mathcal{L}}(u, v) = \frac{1}{2}v^2 + \frac{\lambda}{2}u^2 = \frac{\lambda}{2}u_1^2, \quad (4.15)$$

which is the crossing point between the green and the blue lines on the right picture of Fig. 5. The connection time between $(u_1, 0)$ and $(\hat{u}, \hat{v})$ along (4.15) is given by

$$T_{\mathcal{L}}(\lambda, u_1, \hat{u}) = \int_{u_1}^{\hat{u}} \frac{du}{\sqrt{\lambda(u_1^2 - u^2)}} = \frac{1}{\sqrt{-\lambda}} \int_1^{\frac{\hat{u}}{u_1}} \frac{d\theta}{\sqrt{-1 + \theta^2}},$$

or, equivalently,

$$T_{\mathcal{L}}(\lambda, u_1, \hat{u}) = \frac{1}{\sqrt{-\lambda}} \ln \left(\frac{\hat{u}}{u_1} + \sqrt{\left(\frac{\hat{u}}{u_1} \right)^2 - 1} \right). \quad (4.16)$$

Next, we will show that the quotient $\frac{\hat{u}}{u_1}$ is bounded above. Indeed, coming back to the definition of u_0 and u_1 in (4.5), it is apparent that $u_0 > u_1$. Moreover, since the denominator of the integrands in (4.5),

$$\sqrt{\lambda(1-s^2) + \frac{2}{p+1}u_1^{p-1}(1-s^{p+1})}$$

is defined for all $s \in (0, 1)$, we find that $\lambda + \frac{2}{p+1}u_1^{p-1} \geq 0$ and, hence,

$$u_0 > u_1 \geq \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}} (-\lambda)^{\frac{1}{p-1}}. \quad (4.17)$$

Subsequently, for every $\lambda < 0$, we set

$$\kappa_0(\lambda) := u_0(-\lambda)^{\frac{-1}{p-1}}, \quad \kappa_1(\lambda) := u_1(-\lambda)^{\frac{-1}{p-1}}.$$

Note that, by construction, u_0 and u_1 also vary with λ , as the parameter λ appears in the formulation of (1.1). Then,

$$u_0 = \kappa_0(\lambda)(-\lambda)^{\frac{1}{p-1}}, \quad u_1 = \kappa_1(\lambda)(-\lambda)^{\frac{1}{p-1}},$$

and, thanks to (4.17),

$$\kappa_0(\lambda) > \kappa_1(\lambda) > \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}}.$$

On the other hand, it follows from (4.5) that

$$\begin{aligned} \frac{1-h}{2} \sqrt{-\lambda} &= \int_0^1 \frac{ds}{\sqrt{s^2 - 1 + \frac{2}{p+1} \kappa_1^{p-1}(\lambda)(1-s^{p+1})}} \\ &= 2 \int_0^1 \frac{ds}{\sqrt{s^2 - 1 + \frac{2}{p+1} \kappa_0^{p-1}(\lambda)(1-s^{p+1})}}. \end{aligned}$$

As these integrals are decreasing with respect to $\kappa_0(\lambda)$ and $\kappa_1(\lambda)$, $\frac{1-h}{2} \sqrt{-\lambda}$ decreases with $\lambda < 0$ for fixed h , and

$$\lim_{\theta \uparrow +\infty} \int_0^1 \frac{ds}{\sqrt{s^2 - 1 + \theta(1-s^{p+1})}} = 0,$$

it becomes apparent that, for every $\lambda_* < 0$, there exists a constant $M(\lambda_*) > 0$ such that

$$\left(\frac{p+1}{2}\right)^{\frac{1}{p-1}} < \kappa_1(\lambda) < \kappa_0(\lambda) < M(\lambda_*) \text{ for all } \lambda \in (-\infty, \lambda_*].$$

Thus,

$$\frac{\hat{u}}{u_1} < \frac{u_0}{u_1} = \frac{\kappa_0(\lambda)}{\kappa_1(\lambda)} < M(\lambda_*) \left(\frac{2}{p+1}\right)^{\frac{1}{p-1}} \text{ for all } \lambda \leq \lambda_*. \quad (4.18)$$

Therefore, by (4.18), it follows from (4.16) that there exists $\tilde{\lambda}(h)$ such that

$$T_{\mathcal{L}}(\lambda, u_1, \hat{u}) < h \text{ for all } \lambda \in (-\infty, \tilde{\lambda}(h)].$$

The second asymmetric solution is obtained either by reflection around $x = 0.5$ of the one we have just constructed, or repeating the previous argument in the fourth quadrant. This ends the proof. $\square$

5. A multiplicity result for $\lambda > 0$

In this section, we analyze the existence of, at least, two positive asymmetric solutions of (1.1) when $\lambda > 0$ and h is sufficiently close to 1. As in Section 2.2, to treat the case $\lambda > 0$ it is better to consider, instead of (2.2), the equivalent systems

$$(\mathcal{N}_{\text{eq}}) \begin{cases} u' = \sqrt{\lambda} v, \\ v' = -\sqrt{\lambda} u - \frac{1}{\sqrt{\lambda}} u^p, \end{cases} \quad (\mathcal{L}_{\text{eq}}) \begin{cases} u' = \sqrt{\lambda} v, \\ v' = -\sqrt{\lambda} u. \end{cases} \quad (5.1)$$

If $\lambda > 0$, there is no any restriction on the location of the maxima of the positive solutions of (1.1) in the interval $(0, 1)$. According to Section 2.2, the unique symmetric solution reaches its maximum at $x = 0.5$, where $a(0.5) = 0$ always.

However, for asymmetric positive solutions, one can study separately the case where the maximum is reached at $a^{-1}(0)$ and the case where it is attained at $a^{-1}(1)$. In this paper, we restrict ourselves to deal with the special case where, for every positive solution u of (1.1),

$$x_m \in a^{-1}(0) = (\frac{1-h}{2}, \frac{1+h}{2}), \quad \text{where } u(x_m) = \|u\|_{\infty}, \quad (5.2)$$

because this will provide us with two asymmetric solutions for $\lambda < \pi^2$ arbitrarily close to π^2 for h sufficiently close to 1, and this is the first multiplicity result of this nature available in the literature; suggested by the numerical experiments of [2]. In such a case, we should first connect a point in the positive v -semiaxis with a point in the first quadrant through an energy level line of $(\mathcal{N}_{\text{eq}})$ in the time interval $[0, \frac{1-h}{2}]$. For every $\theta_0 \in (0, \pi/2)$, the necessary time to link a point $(0, v_0)$ with a point $(u_{\theta_0}, v_{\theta_0})$ lying in the line $v = (\tan \theta_0)u$ through the orbit of $(\mathcal{E}_{\mathcal{N}_{\text{eq}}})$

$$v^2 + u^2 + \frac{2}{\lambda(p+1)} u^{p+1} = u_{\theta_0}^2 \tan^2 \theta_0 + u_{\theta_0}^2 + \frac{2}{\lambda(p+1)} u_{\theta_0}^{p+1}$$

is given by

$$T_{\mathcal{N}}(\lambda, 0, u_{\theta_0}) := \frac{1}{\sqrt{\lambda}} \int_0^1 \frac{ds}{\sqrt{\tan^2 \theta_0 + 1 - s^2 + \frac{2u_{\theta_0}^{p-1}}{\lambda(p+1)}(1-s^{p+1})}}$$

(see (2.15)). Since this connection should be accomplished in time $\frac{1-h}{2}$, we impose that

$$T_{\mathcal{N}}(\lambda, 0, u_{\theta_0}) = \frac{1-h}{2}. \quad (5.3)$$

Secondly, by (5.2) and for some $\theta_1 \in (0, \pi/2)$, the point $(u_{\theta_0}, v_{\theta_0})$ reached at time $x = \frac{1-h}{2}$ should be connected at time $x = \frac{1+h}{2}$ with a point of the fourth quadrant $(u_{\theta_1}, v_{\theta_1})$ lying in the line $v = -(\tan \theta_1)u$, through the energy line of linear system $(\mathcal{L}_{\text{eq}})$

$$v^2 + u^2 = v_{\theta_0}^2 + u_{\theta_0}^2 = v_{\theta_1}^2 + u_{\theta_1}^2 = \|u\|_\infty^2,$$

which implies

$$\cos \theta = \frac{u_\theta}{\|u\|_\infty} \quad \text{if } \theta \in \{\theta_1, \theta_2\}. \quad (5.4)$$

Since the angular variable component in $(\mathcal{L}_{\text{eq}})$ satisfies $\theta'(x) = -\sqrt{\lambda}$ (see Section 2.2), it follows that

$$-\theta_1 - \theta_0 = \int_{\frac{1-h}{2}}^{\frac{1+h}{2}} \theta'(x) dx = - \int_{\frac{1-h}{2}}^{\frac{1+h}{2}} \sqrt{\lambda} dx = -h\sqrt{\lambda},$$

which is equivalent to ask for

$$\frac{\theta_0 + \theta_1}{\sqrt{\lambda}} = h. \quad (5.5)$$

Finally, by the symmetry of $a_h(x)$, in order to connect $(u_{\theta_1}, v_{\theta_1})$ with the negative v -semiaxis,

$$T_{\mathcal{N}}(\lambda, 0, u_{\theta_1}) = \frac{1}{\sqrt{\lambda}} \int_0^1 \frac{ds}{\sqrt{\tan^2 \theta_1 + 1 - s^2 + \frac{2u_{\theta_1}^{p-1}}{\lambda(p+1)}(1 - s^{p+1})}} = \frac{1-h}{2} \quad (5.6)$$

should hold. By (5.4) and recalling that

$$\tan^2 \theta + 1 = \frac{1}{\cos^2 \theta},$$

integrals $T_{\mathcal{N}}(\lambda, 0, u_{\theta_0})$ and $T_{\mathcal{N}}(\lambda, 0, u_{\theta_1})$ can be rewritten for $\theta \in \{\theta_0, \theta_1\}$ as

$$\begin{aligned} T_{\mathcal{N}}(\lambda, 0, u_\theta) &= \frac{1}{\sqrt{\lambda}} \int_0^1 \frac{ds}{\sqrt{\frac{1}{\cos^2 \theta} - s^2 + \frac{2\|u\|_\infty^{p-1} \cos^{p-1} \theta}{\lambda(p+1)}(1 - s^{p+1})}} \\ &= \frac{\|u\|_\infty^{-\frac{p-1}{2}}}{\sqrt{\lambda}} \int_0^1 \frac{ds}{\sqrt{\frac{1-s^2 \cos^2 \theta}{\|u\|_\infty^{p-1} \cos^2 \theta} + \frac{2 \cos^{p-1} \theta}{\lambda(p+1)}(1 - s^{p+1})}}. \end{aligned} \quad (5.7)$$

Subsequently, we will denote

$$S_{\mathcal{N}}(\lambda, \|u\|_\infty, \theta) := T_{\mathcal{N}}(\lambda, 0, u_\theta) \quad \text{if } \theta \in \{\theta_0, \theta_1\}.$$

According to (5.3), (5.5), (5.6) and (5.7), for any fixed $\lambda \in (0, \pi^2]$, the existence of a positive solution of (1.1) satisfying (5.2) is guaranteed if, and only if,

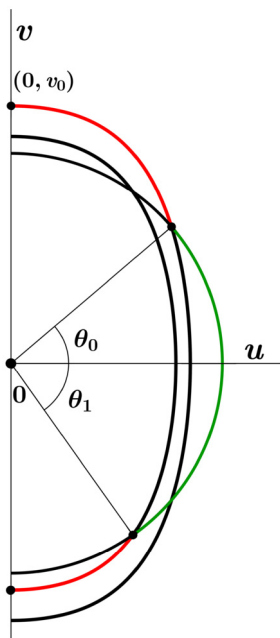


Fig. 6. An asymmetric positive solution of (1.1) ($\theta_0 \neq \theta_1$) satisfying (5.2).

$$\begin{cases} \frac{\theta_0 + \theta_1}{\sqrt{\lambda}} = h, \\ S_{\mathcal{N}}(\lambda, \|u\|_{\infty}, \theta_0) = S_{\mathcal{N}}(\lambda, \|u\|_{\infty}, \theta_1) = \frac{1-h}{2}. \end{cases} \quad (5.8)$$

A solution of (5.8) with $\theta_0 = \theta_1$ corresponds to a symmetric positive solution of (1.1), whose existence and uniqueness has been already established by Theorem 2.1. On the other hand, when (5.8) holds for $\theta_0 = \alpha \neq \beta = \theta_1$, an asymmetric positive solution of (1.1) exists. In such a case, by the symmetry of the problem (1.1), a second asymmetric positive solution arises just by taking $\theta_0 = \beta$ and $\theta_1 = \alpha$. Thus, the asymmetric positive solutions of (1.1) appear always by pairs and these pairs of asymmetric solutions are, by construction, symmetric with respect to $x = 0.5$. Fig. 6 shows the three arcs of trajectory of a positive solution of (1.1) under the assumption (5.2).

In order to determine the asymmetric solutions, we need to study the function $S_{\mathcal{N}}(\lambda, \|u\|_{\infty}, \theta)$ for a fixed $\lambda > 0$ and $\theta \in (0, \pi/2)$. To simplify the notations, for any positive solution u of (1.1), we denote

$$R := \|u\|_{\infty} > 0,$$

and consider the two-variables re-scaled function

$$\begin{aligned} \phi(R, \theta) &:= 2\sqrt{\lambda} S_{\mathcal{N}}(\lambda, R, \theta) \\ &= 2 \int_0^1 \frac{ds}{\sqrt{\frac{1}{\cos^2 \theta} - s^2 + \frac{2R^{p-1} \cos^{p-1} \theta}{\lambda(p+1)} (1-s^{p+1})}} \end{aligned} \quad (5.9)$$

$$= 2R^{-\frac{p-1}{2}} \int_0^1 \frac{ds}{\sqrt{\frac{1-s^2 \cos^2 \theta}{R^{p-1} \cos^2 \theta} + \frac{2 \cos^{p-1} \theta}{\lambda(p+1)} (1-s^{p+1})}}.$$

The next lemma collects some basic properties of $\phi(R, \theta)$.

Lemma 5.1. *The continuous function $\phi : (0, +\infty) \times [0, \pi/2) \rightarrow (0, +\infty)$ defined by (5.9) is of class C^1 in $(0, +\infty) \times (0, \pi/2)$ and it satisfies:*

(i) *it is decreasing with respect to $R > 0$,*

$$\lim_{R \uparrow +\infty} \phi(R, \theta) = 0, \text{ and } \phi(R, \theta) < \lim_{R \downarrow 0} \phi(R, \theta) = \pi - 2\theta,$$

(ii) *it can be extended by continuity to $\pi/2$ by taking $\phi(\cdot, \pi/2) = 0$,*

(iii) *it has a continuous derivative with respect to $\theta \in (0, \pi/2]$, and*

$$\frac{\partial \phi}{\partial \theta}(\cdot, \pi/2) = -2.$$

Proof. First observe that ϕ is also defined for $\theta = 0$ and $R > 0$, with

$$\phi(R, 0) = 2 \int_0^1 \frac{ds}{\sqrt{1-s^2 + \frac{2R^{p-1}}{\lambda(p+1)} (1-s^{p+1})}} < 2 \arcsin 1 = \pi.$$

Moreover, as the integrand is continuous in its domain of definition, ϕ is of class C^1 . The fact that $\phi(R, \theta)$ is positive is obvious by definition. Moreover, since $p > 1$, it follows from (5.9) that, for every $\theta \in [0, \pi/2)$, the mapping $R \mapsto \phi(R, \theta)$ is decreasing and satisfies

$$\lim_{R \uparrow +\infty} \phi(R, \theta) = 0.$$

Also, for every $\theta \in [0, \frac{\pi}{2})$,

$$\lim_{R \downarrow 0} \phi(R, \theta) = 2 \int_0^1 \frac{ds}{\sqrt{\frac{1}{\cos^2 \theta} - s^2}} = 2 \int_0^{\cos \theta} \frac{d\xi}{\sqrt{1-\xi^2}} = \pi - 2\theta.$$

Actually, since $\phi(R, \theta)$ is uniformly bounded above by $\pi - 2\theta$, its asymptotic profile as $R \uparrow +\infty$ is given by

$$\phi(R, \theta) \approx R^{-\frac{p-1}{2}} \left(\frac{2\lambda(p+1)}{\cos^{p-1} \theta} \right)^{\frac{1}{2}} \int_0^1 \frac{ds}{\sqrt{1-s^{p+1}}} = R^{-\frac{p-1}{2}} \left(\frac{2\lambda(p+1)}{\cos^{p-1} \theta} \right)^{\frac{1}{2}} B\left(\frac{1}{p+1}, \frac{1}{2}\right), \quad (5.10)$$

where B is the Euler beta-function (see [13, Section 7.1]). This ends the proof of Part (i). Part (ii) holds because

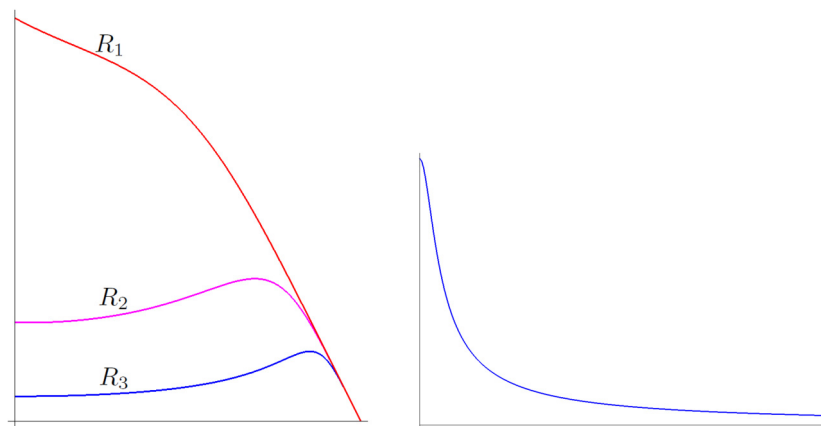


Fig. 7. Graphs of the maps $\theta \mapsto \phi(R_i, \theta)$ for $i \in \{1, 2, 3\}$ (left), and graph of the map $R \mapsto \phi(R, 0)$ on the interval $[0, 80]$ (right).

$$\lim_{\theta \uparrow \frac{\pi}{2}} \phi(\cdot, \theta) = 0.$$

Finally, Part (iii) follows from the fact that, for every $\theta \in (0, \pi/2)$,

$$\frac{\partial \phi}{\partial \theta}(R, \theta) = 2R^{-\frac{p-1}{2}} \sin \theta \int_0^1 \frac{\frac{2(p-1)\cos^{p+1}\theta}{\lambda(p+1)}(1-s^{p+1}) - \frac{1}{R^{p-1}}}{\left(\frac{1-s^2\cos^2\theta}{R^{p-1}} + \frac{2\cos^{p+1}\theta}{\lambda(p+1)}(1-s^{p+1})\right)^{3/2}} ds. \quad (5.11)$$

Indeed, for every $R > 0$, $\frac{\partial \phi}{\partial \theta}(R, \pi/2) = -2$. This ends the proof. $\square$

Fig. 7 left shows the graphs of the map $\theta \mapsto \phi(R, \theta)$ for a series of increasing values of $R > 0$, $R_1 < R_2 < R_3$. For sufficiently small $R > 0$ the map is decreasing, while, for larger values of R , there is a point of maximum in the interval $(0, \pi/2)$. To generate the figure, we have chosen $p = 3$, $\lambda = 6$ and $R_1 = 4$, $R_2 = 20$, $R_3 = 80$, respectively. Note that all the graphs have a common tangent at $\pi/2$ as proved in Lemma 5.1 (iii). The right plot of Fig. 7 shows the graph of the function $\phi(\cdot, 0)$ for $p = 3$ and $\lambda = 6$.

On the other hand, when trying to determine the limit of the derivative as $\theta \downarrow 0$ one should be aware that the integral

$$\int_0^1 \frac{\frac{2(p-1)}{\lambda(p+1)}(1-s^{p+1}) - \frac{1}{R^{p-1}}}{\left(\frac{1-s^2}{R^{p-1}} + \frac{2}{\lambda(p+1)}(1-s^{p+1})\right)^{3/2}} ds$$

is divergent at $s = 1$ and, hence, (5.11) takes an indeterminate form. So, we always assume $\theta > 0$ when analyzing (5.11). For every $R > 0$ and $\theta \in (0, \pi/2]$, it is clear that the sign of (5.11) is the sign of the integrand's numerator.

According to Theorem 3.1, we already know that

$$\lim_{h \uparrow 1} R = \lim_{h \uparrow 1} \|u\|_{\infty} = \infty.$$

Thus, we need a description of the function $\phi(R, \cdot)$ for large R , which is given by the next result.

Lemma 5.2. *Given a, b with $0 < a < b < \pi/2$, there exists $R^* = R^*(a, b)$ such that, for every $R > R^*$, the following holds:*

- (a) *The function $\phi(R, \cdot)$ is increasing in the interval $[a, b]$.*
- (b) *The function $\phi(R, \cdot)$ has a point of maximum at $\theta^* \in (b, \pi/2)$. Moreover, it is the absolute maximum point.*
- (c) *There exists $\theta_* \in [0, a]$ such that*

$$\phi(R, \theta_*) = \min_{\theta \in [0, a]} \phi(R, \theta) > 0 = \phi(R, \pi/2).$$

Proof. First, note that, fixing $\theta \in [a, b]$, the map $R \mapsto \frac{\partial \phi}{\partial \theta}(R, \theta)$ is decreasing and converges to zero as $R \uparrow +\infty$. Indeed, as $R \uparrow +\infty$, we have that

$$\begin{aligned} \frac{\partial \phi}{\partial \theta}(R, \theta) &\approx R^{-\frac{p-1}{2}}(p-1) \sin \theta \left(\frac{2\lambda(p+1)}{\cos^{p+1} \theta} \right)^{\frac{1}{2}} \int_0^1 \frac{ds}{(1-s^{p+1})^{\frac{1}{2}}} \\ &= R^{-\frac{p-1}{2}}(p-1) \sin \theta \left(\frac{2\lambda(p+1)}{\cos^{p+1} \theta} \right)^{\frac{1}{2}} B\left(\frac{1}{p+1}, \frac{1}{2}\right), \end{aligned} \quad (5.12)$$

which provides us with the derivative of (5.10). By (5.12), there exists R^* such that $\frac{\partial}{\partial \theta} \phi(R, \theta) > 0$ for every $R > R^*$ and $\theta \in [a, b]$, proving Part (a).

Now, by Part (a) and recalling that $\phi(R, \pi/2) = 0$, the existence of a point of maximum at $(b, \pi/2)$ holds, which proves Part (b). Moreover, by (5.10) and (5.12), it becomes apparent that, for sufficiently large R , θ^* is a point of absolute maximum. Finally, Part (c) follows by the continuity of $\phi(R, \theta)$. This ends the proof. $\square$

Thanks to Lemma 5.2, for every $\lambda > 0$ and $\varepsilon > 0$, there exists $R^* > 0$ such that, for every $R > R^*$, $\phi(R, \cdot)$ is increasing in $[\varepsilon, \frac{\pi}{2} - \varepsilon]$ and all the critical points of $\phi(R, \cdot)$ lie on $[0, \varepsilon] \cup (\frac{\pi}{2} - \varepsilon, \frac{\pi}{2})$. For such R 's, we set

$$\phi_M = \phi_M(R) := \max_{\theta \in [0, \frac{\pi}{2}]} \phi(R, \theta).$$

By Lemma 5.2 (b), such a value is achieved at some $\theta \in [\pi/2 - \varepsilon, \pi/2]$. Accordingly, we set

$$\theta_M = \theta_M(R, \varepsilon) := \min\{\theta \in [\pi/2 - \varepsilon, \pi/2] : \phi(R, \theta) = \phi_M\}.$$

We also denote by $\theta_m = \theta_m(R, \varepsilon) \in [0, \varepsilon]$ the maximum value of θ for which $\phi(R, \theta)$ has a minimum in $[0, \varepsilon]$, taking $\theta_m = 0$ if there are no critical points in $[0, \varepsilon]$. Moreover, we set

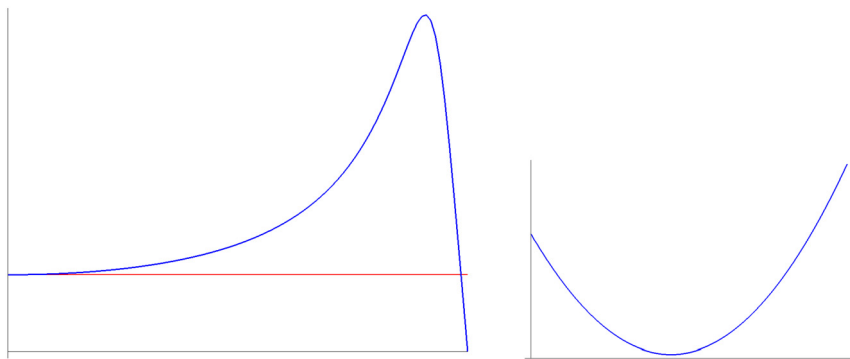


Fig. 8. Graph of $\theta \mapsto \phi(R, \cdot)$ for large R (left), and graph of $\theta \mapsto \phi(R, \theta)$ for large R in a right neighborhood of $\theta = 0$ (right).

$$\phi_m = \phi_m(R) := \phi(R, \theta_m) > 0, \quad \bar{\theta} := \min\{\theta \in [\pi/2 - \varepsilon, \pi/2) : \phi(R, \theta) = \phi_m\}.$$

With these notations, by Lemma 5.2, the next result holds.

Proposition 5.1. *For every $R > R^*$, the restriction of the function $\phi(R, \cdot)$ to the interval $[\theta_m, \bar{\theta}]$ satisfies the following conditions:*

- (a) $\phi(R, \theta_m) = \phi(R, \bar{\theta}) = \phi_m > 0$, and $\phi(R, \theta) > \phi_m$ for all $\theta \in (\theta_m, \bar{\theta})$.
- (b) $\phi(R, \cdot)$ is increasing in $[\theta_m, \frac{\pi}{2} - \varepsilon]$.
- (c) $\theta_M \in [\frac{\pi}{2} - \varepsilon, \frac{\pi}{2})$ with

$$\phi(R, \theta_M) = \max_{\theta \in [\theta_m, \theta_M]} \phi(R, \theta) \geq \phi(R, \frac{\pi}{2} - \varepsilon) > \phi_m.$$

There is numerical evidence that, for large $R > 0$, there is a unique point of maximum for $\phi(R, \cdot)$, θ_M , and that $\phi(R, \cdot)$ is decreasing in $[\theta_M, \pi/2]$, i.e., θ_M is the unique $\theta \in [\pi/2 - \varepsilon, \pi/2)$ such that $\phi(R, \theta) = \phi_m$. However, since this assertion has not been proved in the analysis of the function $\phi(R, \cdot)$, we are forced to make the definitions of θ_M and $\bar{\theta}$ a bit more intricate.

Fig. 8 shows the main features of Lemma 5.2 and Proposition 5.1. The left plot of Fig. 8 provides a genuine graph of $\phi(R, \cdot)$ for large R . Apparently, the figure shows the presence of a unique (maximum) critical point in the interval $(0, \pi/2)$; the red horizontal line crossing the graph at the points (θ_m, ϕ_m) and $(\bar{\theta}, \phi_m)$ at the level $y = \phi_m$. As this figure might suggest that $\phi(R, \cdot)$ is increasing in the whole interval $[0, \theta_M]$, the plot on the right shows a zoom of the graph of $\phi(R, \cdot)$ in a neighborhood of $\theta = 0$, with a minimum, θ_m , close to $\theta = 0$. These graphs have been computed for $p = 3$ and $\lambda = 6$ with $R = 300$ and $\theta \in [0, 0.05]$. In such case, $\phi(300, \cdot) \in [0.15116, 0.15122]$.

As, for every $h \in (1 - \frac{\phi_M}{\sqrt{\lambda}}, 1 - \frac{\phi_m}{\sqrt{\lambda}})$, the equation

$$\phi(R, \theta) = (1 - h)\sqrt{\lambda}, \quad \theta \in (0, \pi/2),$$

has, at least, two solutions: one with $\theta \in (\theta_m, \theta_M)$ and a second one with $\theta \in (\theta_M, \bar{\theta})$, we can continuously parameterize these solutions as follows.

Lemma 5.3. *There exists a continuous map $\gamma = (\gamma_1, \gamma_2) : [0, 1] \rightarrow \mathbb{R}^2$ such that*

- (a) $\gamma_1(s) \in (\theta_m, \theta_M)$ and $\gamma_2(s) \in (\theta_M, \bar{\theta})$ for all $s \in (0, 1)$,
- (b) $\gamma_1(0) = \theta_m$, $\gamma_2(0) = \bar{\theta}$, and $\gamma_1(1) = \theta_M = \gamma_2(1)$,
- (c) $\phi(R, \gamma_1(s)) = \phi(R, \gamma_2(s))$ for all $s \in [0, 1]$.

Lemma 5.3 is just a special version of the Mountain Climbing Theorem, applied to the function $\phi(R, \cdot)$. There are different versions of this theorem which either require that the function under consideration is piecewise monotone, as in [19], or that it has no intervals where it is constant, as in [4, 10]. The function $\phi(R, \cdot)$ is real analytic in a closed sub-interval J of $(0, \pi/2)$ with $[\theta_m, \bar{\theta}] \subset \text{int} J$. Therefore, $\phi(R, \cdot)$ must have finitely many critical points in $[\theta_m, \bar{\theta}]$ and the Mountain Climbing Theorem can be applied.

The geometric interpretation of Lemma 5.3 goes as follows. There are two climbers, $(\gamma_1(s), \phi(R, \gamma_1(s)))$ and $(\gamma_2(s), \phi(R, \gamma_2(s)))$, aiming to attain the highest pick of a mountain at $\theta = \theta_M$, climbing continuously on the two sides

$$\{(\theta, \phi(R, \theta)) : \theta \in [\theta_m, \theta_M)\} \quad \text{and} \quad \{(\theta, \phi(R, \theta)) : \theta \in (\theta_M, \bar{\theta}]\}$$

of the graph of $\phi(R, \cdot)$. The two climbers start at the time $s = 0$ at the same level, ϕ_m , though at opposite sides of the peak, and both meet at the summit, (θ_M, ϕ_M) , at the time $s = 1$. While climbing, they maintain the same altitude at each time. It is apparent that if $\theta_M \in (\theta_m, \pi/2)$ is the unique point of maximum of $\phi(R, \cdot)$ for R large (see Fig. 8 and the discussion after Proposition 5.1), then, one can take the functions

$$\eta_1 := (\phi(R, \cdot)|_{[\theta_m, \theta_M]})^{-1}, \quad \eta_2 := (\phi(R, \cdot)|_{[\theta_M, \bar{\theta}]})^{-1},$$

defined on $[\phi(R, 0), \phi_M]$, and re-parameterize them to $[0, 1]$, by setting

$$\gamma_i(s) := \eta_i(\phi_m + s(\phi_M - \phi_m)), \quad s \in [0, 1], \quad (i = 1, 2),$$

without invoking the Mountain Climbing Theorem. Using this theorem allows us to avoid a further analysis of $\frac{\partial \phi}{\partial \theta}(R, \theta)$ in $[\pi - \varepsilon, \frac{\pi}{2})$ in order to prove rigorously that θ_M is the unique local maximum of $\phi(R, \cdot)$ for sufficiently large $R > 0$. As it can be proved that, for some $\theta(R) \in (\theta_M, \pi/2)$,

$$\lim_{R \uparrow +\infty} \frac{\partial \phi}{\partial \theta}(R, \theta(R)) = +\infty = \lim_{R \uparrow +\infty} \frac{\partial^2 \phi}{\partial \theta^2}(R, \theta(R))$$

the underlying analysis can become quite hard due to the lack of information on the sign of the derivatives near $\pi/2$.

Now, we are ready to prove the existence of asymmetric positive solutions for (1.1).

Theorem 5.1. *For any given $\lambda \in (\pi^2/4, \pi^2)$, there exists $R_\lambda > 0$ such that, for every $R > R_\lambda$, the problem (1.1) has, at least, two asymmetric positive solutions for a suitable $h \in (0, 1)$. Moreover, $\lim_{R \uparrow +\infty} h = 1$.*

Proof. As we have previously remarked, the asymmetric positive solutions come in pairs. Thus, it suffices to show the existence of, at least, one asymmetric solution. In terms of

$$\phi(R, \theta) \equiv 2\sqrt{\lambda} S_{\mathcal{N}}(\lambda, R, \theta),$$

(5.8) can be, equivalently expressed, as

$$\begin{cases} \theta_0 + \theta_1 = h\sqrt{\lambda}, \\ \phi(R, \theta_0) = (1-h)\sqrt{\lambda} = \phi(R, \theta_1). \end{cases} \quad (5.13)$$

By Lemma 5.3 with $\theta_0 = \gamma_1(s)$ and $\theta_1 = \gamma_2(s)$, we have that, for every $s \in (0, 1)$,

$$\phi(R, \gamma_1(s)) = \phi(R, \gamma_2(s)) = (1-h(s))\sqrt{\lambda} \in (\phi_m, \phi_M). \quad (5.14)$$

Thus, we need only to solve the first equation of (5.13), which reads as

$$\gamma_1(s) + \gamma_2(s) + \phi(R, \gamma_1(s)) = \sqrt{\lambda} \quad (5.15)$$

by (5.14). Equivalently, $g(s) = \sqrt{\lambda}$, where we are setting

$$g(s) := \gamma_1(s) + \gamma_2(s) + \phi(R, \gamma_1(s)), \quad s \in [0, 1].$$

Since $\phi(R, \theta) < \pi - 2\theta$ for every $R > 0$, we find from Lemma 5.3 that

$$\begin{aligned} \lim_{s \downarrow 0} g(s) &= \theta_m + \bar{\theta} + \phi(R, \theta_m) = \theta_m + \bar{\theta} + \phi(R, \bar{\theta}) \\ &< \theta_m + \bar{\theta} + \pi - 2\bar{\theta} = \theta_m + \pi - \bar{\theta}. \end{aligned}$$

Thus, recalling that $\theta_m \leq \varepsilon$ and $\bar{\theta} \geq \pi/2 - \varepsilon$, it becomes apparent that

$$\lim_{s \downarrow 0} g(s) < \pi/2 + 2\varepsilon,$$

provided $R > R^*(\varepsilon) \equiv R^*(\varepsilon, \pi/2 - \varepsilon)$, where $R^*(\varepsilon, \pi/2 - \varepsilon)$ is the value of R given by Lemma 5.2. As we are assuming that $\sqrt{\lambda} > \pi/2$, we can make the choice

$$2\varepsilon \in (0, \sqrt{\lambda} - \pi/2).$$

This ensures that, for every $R > R^*(\varepsilon)$,

$$\lim_{s \downarrow 0} g(s) < \sqrt{\lambda}. \quad (5.16)$$

Thanks again to Lemma 5.3,

$$\lim_{s \uparrow 1} g(s) = 2\theta_M + \phi(R, \theta_M) = 2\theta_M + \phi_M.$$

Since $\sqrt{\lambda} < \pi$ and $2\theta_M \geq \pi - 2\varepsilon$, if $\varepsilon > 0$ is chosen so that

$$\sqrt{\lambda} < \pi - 2\varepsilon,$$

then, for every $R > R^*(\varepsilon)$, we find that

$$\lim_{s \uparrow 1} g(s) \geq \pi - 2\varepsilon > \sqrt{\lambda},$$

besides (5.16). Note that ε has been chosen so that

$$\frac{\pi}{2} + 2\varepsilon < \sqrt{\lambda} < \pi - 2\varepsilon.$$

Therefore, by the continuity of $g(s)$, there is $\hat{s} \in (0, 1)$ such that $g(\hat{s}) = \sqrt{\lambda}$. By construction, $\gamma_1(\hat{s}) < \theta_M < \gamma_1(\hat{s})$, so that we can take $\theta_0 = \gamma_1(\hat{s})$ and $\theta_1 = \gamma_2(\hat{s})$, or $\theta_0 = \gamma_2(\hat{s})$ and $\theta_1 = \gamma_1(\hat{s})$, to get the desired asymmetric solutions. Finally, from (5.14), we find that

$$\phi(R, \gamma_1(\hat{s})) = \phi(R, \gamma_2(\hat{s})) = (1 - h(\hat{s}))\sqrt{\lambda} \in (\phi_m, \phi_M), \quad (5.17)$$

providing us with the precise value of h for which we can guarantee the existence of two asymmetric solutions,

$$h(\hat{s}) = 1 - \frac{\phi(R, \gamma_1(\hat{s}))}{\sqrt{\lambda}}. \quad (5.18)$$

Observe that, thanks to (5.17),

$$\lim_{R \uparrow +\infty} (1 - h(\hat{s}))\sqrt{\lambda} \leq \lim_{R \uparrow +\infty} \phi_M := \lim_{R \uparrow +\infty} \phi(R, \theta_M) = 0.$$

Consequently,

$$\lim_{R \uparrow +\infty} h(\hat{s}) = 1. \quad (5.19)$$

This ends the proof. $\square$

Note that, recalling that $R = \|u\|_\infty$, (5.19) is a counterpart of (3.1).

Data availability

No data was used for the research described in the article.

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