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Wetzel families and the continuum

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Abstract

We provide answers to a question brought up by Erdős about the construction of Wetzel families in the absence of the continuum hypothesis: A *Wetzel family* is a family $\mathcal{F}$ of entire functions on the complex plane which pointwise assumes fewer than $|\mathcal{F}|$ values. To be more precise, we show that the existence of a Wetzel family is consistent with all possible values κ of the continuum and, if κ is regular, also with Martin's Axiom. In the particular case of $\kappa = \aleph_2$ this answers the main open question asked by Kumar and Shelah [Fund. Math. **239** (2017) no. 3, 279–288]. In the buildup to this result, we are also solving an open question of Zapletal on strongly almost disjoint functions from Zapletal [Israel J. Math. **97** (1997) no. 1, 101–111]. We also study a strongly related notion of sets exhibiting a universality property via mappings by entire functions and show that these consistently exist while the continuum equals $\aleph_2$.

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1 | INTRODUCTION

This paper is an investigation related to Wetzel's problem which comes from analysis and yet has surprising set-theoretical aspects. While the subjects of analysis and set theory might nowadays be conceived as somewhat distant to each other, there are some examples of topics belonging to them both. Among the most prominent one is the problem of sets of uniqueness, cf. [21], which led Cantor to investigate sets of real numbers and subsequently to the founding of set theory.

One of the greatest contributions of Cantor was the discovery of ordinals and cardinals and the distinction between countable and uncountable sets of reals. This led him to the formulation of the continuum hypothesis, CH, which states that the cardinality of the set of all real numbers is the smallest one conceivable in light of this, the smallest uncountable cardinal $\aleph_1$, cf. [9]. In 1904, Ernst Zermelo axiomatised set theory in a way conforming to mathematician's practise hitherto, cf. [34]. Subsequently, A. Fraenkel added the replacement scheme, cf. [15], thus yielding the system ZFC. Somewhat later, Kurt Gödel showed that the continuum hypothesis cannot be refuted within this system (provided that there is anything which cannot be derived within it), cf. [17]. Gödel conjectured, cf. [18, section 4], that it cannot be proved in it either but only several decades later, Paul Cohen, at the origin an analyst just like Cantor, developed the method of forcing and could prove that this is indeed the case, cf. [12].

Nowadays it is common to subdivide analysis into real and complex analysis. The latter's theorems about its objects of study, holomorphic functions, revealed deep connections between analysis and geometry and found applications in various areas, among them number theory. One striking feature of the family of functions which are holomorphic on some domain of complex numbers opposite the family of those which are merely smooth on an interval of real numbers is the intertwinement of the local and global behaviour of holomorphic functions. Whereas two distinct functions may be both identical and infinitely often differentiable on an interval of real numbers, the situation in the complex domain is quite different, due to the famous 'identity theorem'. According to it, for any two distinct functions holomorphic on some complex domain, the set of points where they agree is discrete (see Proposition 2.1).

As the complex plane is separable, no uncountable set of complex numbers is discrete. Therefore, any two distinct holomorphic functions can only agree on a countable set of points. Subsequently further theorems underscored the difference between the realms of entire functions on the one hand and smooth functions on the real number line on the other. In the middle of the nineteenth century, it emerged from work of Liouville, cf. [19, Chapter 11], that all bounded entire functions are constant. Later, Nevanlinna developed the theory named after him, cf. [10], one upshot of which is that for any two distinct entire functions f and g, there can be at most four complex values a for which the pre-images of $\{a\}$ are equal. Another interesting property that is purely of combinatorial nature and can be stated irrespective of the topological and algebraic structure of $\mathbb{C}$, is Picard's Little Theorem, namely the fact that any non-constant entire function can avoid at most one single value (see e.g. [27, Theorem 16.22]).

The emerging picture of complex analysis in general and of entire functions in particular was one of strong general principles governing their behaviour. Against this backdrop, while writing his dissertation during the sixties of the last century, John E. Wetzel asked (*cum grano salis*) whether any family $\mathcal{F}$ of entire functions such that for each complex number z the set $\{f(z) : f \in \mathcal{F}\}$ is countable, must be countable itself. Dixon showed that, assuming the failure of the continuum hypothesis, this is indeed the case, in fact, this is a corollary of the aforementioned identity theorem, cf. [16]. Shortly thereafter, Erdős proved that not only does the negation of the continuum hypothesis imply this statement, it is equivalent to it, [13]. In fact this result is one among many statements in various areas of mathematics proved to be equivalent to the continuum hypothesis, cf. [6, 31]. Towards the end of his paper, Erdős asked whether the analogue statement resulting from replacing 'countable' by 'fewer than continuum many' can be proved without assuming the continuum hypothesis. Following a suggestion by Martin Goldstern, we subsequently refer to a family of entire functions whose members everywhere assume fewer values than the family has members altogether as a *Wetzel family* (see Definition 3.1). In this terminology, Erdős asked whether the existence of a Wetzel family is provable from ZFC. One might also ask whether the continuum hypothesis is equivalent to the existence of a Wetzel family.

Not long after the development of forcing by Cohen, Solovay and Tennenbaum instigated the theory of iterated forcing and Tony Martin stated what became known as Martin's Axiom, or MA for short, a weakening of the continuum hypothesis. It does not prescribe a particular value for the cardinality of the continuum but it does, for instance, imply that its cardinality is regular. In many cases, when something can be proved for countable sets within ZFC, Martin's Axiom allows us to generalise this to sets with fewer than $2^{\aleph_0}$ elements. Meanwhile Erdős' question remained unanswered.

The threads regarding Wetzel's problem were only picked up again in 2017 by Ashutosh Kumar and Saharon Shelah who answered both questions above in the negative, albeit in a slightly non-satisfactory way, see [22]. They showed that there is no Wetzel family in the side-by-side Cohen model and provided a model with a Wetzel family (and hence a continuum, cf. Lemma 3.2) of cardinality $\aleph_{\omega_1}$. The singularity of $\aleph_{\omega_1}$ is quite crucial in their argument and the result could not be generalised to other cardinals. Moreover, their model necessarily fails to satisfy Martin's Axiom.

It seems that after [22], interest in Wetzel's problem has grown. We are aware of two more papers dealing with it since then, a formalisation of Erdős' proof, [26], and a proof that the continuum hypothesis implies the existence of sparse analytic systems, [11].† But no one yet addressed the open question which Kumar and Shelah ask at the end of [22], of whether the existence of a Wetzel family is consistent with a continuum of cardinality $\aleph_2$. Erdős' proof relied on the fact that any countable dense set of complex numbers is universal for countable sets via entire functions, that is to say that any countable set may be mapped into it via a non-constant entire function (see Definition 3.4 and Proposition 3.5). In fact one finds a few papers from the sixties and seventies of the last century studying similar yet slightly stronger mapping properties, cf. [2, 3, 24, 25, 28]. Kumar and Shelah observed that a Wetzel family would exist in a model of $2^{\aleph_0} = \aleph_2$ in which there is a set of cardinality $\aleph_1$ universal in the sense above for sets of cardinality $\aleph_1$ of complex numbers.

The main result of our paper is Theorem 5.14, that shows that starting from a model of the generalised continuum hypothesis and any cardinal κ of uncountable cofinality, there is a cardinal and cofinality preserving forcing extension with a Wetzel family of size κ. In particular, this completely solves Kumar and Shelah's open problem by showing that Wetzel families put no further restriction on the size of the continuum. Moreover, for regular κ, we can also force Martin's axiom. We also study the notion of universality from above and show that while MA precludes the existence of sufficiently universal sets, they can consistently exist while $2^{\aleph_0} = \aleph_2$.

While some basic knowledge of set theory is needed to understand the main results, a large part of the arguments (with an exception of those in Sections 4 and 7) is more analytic than set theoretic.

The paper is organised as follows. In the following first section, we review some of the preliminaries in complex analysis and forcing that are used in the paper. In the next section, Section 3, we introduce Wetzel families and universal sets and prove some ZFC results about them. Among other things, we show that Wetzel families must have cardinality $2^{\aleph_0}$, we provide a proof of Erdős' result on countable dense sets and we show that universal sets imply the existence of Wetzel families. In Section 4, we show how to force a certain family of strongly almost disjoint functions that serves as a basic (and somewhat necessary) ingredient in the proof of the main result. In fact,

† We have been informed that the authors of [11] have been unaware of [22], and were motivated rather by Erdös' original paper.

it turns out that this solves [33, Question 22]. We also obtain some interesting additional results related to MA that shine some light on one of our open questions. This section can be read completely independently from the rest the paper. Section 5 is the longest and contains the proof the main result, Theorem 5.14. In Section 6, we show that universal sets do not exist under MA + ¬CH. As a corollary, we obtain that the converse of Proposition 3.7, namely the statement that Wetzel families imply the existence of a universal set, does not hold. In the next section, we then show that a universal set, as suggested by Kumar and Shelah, can consistently exist with continuum $\aleph_2$. This uses a proper forcing, based on some of the previous arguments, with pairs of models as side-conditions. We finish the paper with a list of open problems.

2 | PRELIMINARIES

2.1 | Complex analysis

Throughout the paper, $\mathbb{C}$ denotes the set of complex numbers. A function $f : \mathbb{C} \to \mathbb{C}$ is *entire* if it is holomorphic on the domain $\mathbb{C}$, in other words, its complex derivative f' exists in every point $z \in \mathbb{C}$. The set of entire functions will be denoted $\mathcal{H}(\mathbb{C})$.

Proposition 2.1 (see [27, Theorem 10.18]). *Let $f, g \in \mathcal{H}(\mathbb{C})$ and assume that the set $\{z \in \mathbb{C} : f(z) = g(z)\}$ has an accumulation point. Then, $f = g$.*

For $z \in \mathbb{C}$ and δ a positive real number, we let

$$B_\delta(z) = \{z' \in \mathbb{C} : |z - z'| < \delta\}$$

be the ball of radius δ around z. We also define the semi-norms

$$\|f\|_\delta = \sup_{z \in B_\delta(0)} |f(z)|.$$

Recall that a sequence of functions $\bar{f} = \langle f_n : n \in \omega \rangle$ on $\mathbb{C}$ is said to converge uniformly on compact sets, if for every $\delta > 0$, $\bar{f}$ converges uniformly on $B_\delta(0)$, that is, for every $\varepsilon > 0$, there is $N \in \omega$ so that for all $n_0, n_1 > N$,

$$\|f_{n_0} - f_{n_1}\|_\delta < \varepsilon.$$

Among the most useful facts about entire functions that we will use is the following.

Proposition 2.2 (see [27, Theorem 10.28]). *Let $\langle f_n : n \in \omega \rangle$ be a sequence of entire functions that converges uniformly on every compact set. Then, the pointwise limit f of $\langle f_n : n \in \omega \rangle$ is entire.*

Moreover, the sequence of derivatives $\langle f'_n : n \in \omega \rangle$ converges uniformly on every compact set to f'.

2.2 | Forcing

Here, we review a few standard facts about forcing that we will find useful. We use standard forcing notation as used in the reference books [20] or [23].

Lemma 2.3 [23, Lemma V.3.9, V.3.10]. *Let $\mathbb{P} * \dot{\mathbb{Q}}$ be a two-step iteration. Then, $\mathbb{P} * \dot{\mathbb{Q}}$ is ccc if and only if $\mathbb{P}$ is ccc and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is ccc.*

Note of course, that if $\dot{\mathbb{Q}}$ is in fact of the form $\check{\mathbb{Q}}$ for some ground model forcing $\mathbb{Q}$, then also $\mathbb{P} * \dot{\mathbb{Q}}$ is ccc if and only if $\mathbb{P} \times \mathbb{Q}$ is.

Definition 2.4. Let $\mathbb{P}, \mathbb{Q}$ be forcing notions. Then, $\mathbb{P}$ is a sub-forcing of $\mathbb{Q}$ if $\mathbb{P} \subseteq \mathbb{Q}$ and the extension as well as the incompatibility relations agree.

Definition 2.5. Let $M \subseteq V$ be a transitive model of ZF^- (possibly a proper class).† Let $\mathbb{P} \in M$ be a sub-forcing of $\mathbb{Q}$. Then, we write $\mathbb{P} \lessdot_M \mathbb{Q}$ to say that every pre-dense set $E \in M$ of $\mathbb{P}$ is pre-dense in $\mathbb{Q}$. We write $\mathbb{P} \lessdot \mathbb{Q}$ for $\mathbb{P} \lessdot_V \mathbb{Q}$ and say that $\mathbb{P}$ is a complete sub-forcing of $\mathbb{Q}$.

Lemma 2.6 [32, Theorem 6.3]. *The iterative direct limit of ccc forcings is ccc. To be more precise, suppose $\langle \mathbb{P}_\delta : \delta \leqslant \alpha \rangle$ is a sequence of posets, α limit, so that*

(1) *for all $\gamma \leqslant \delta \leqslant \alpha$, $\mathbb{P}_\gamma \lessdot \mathbb{P}_\delta$,*
(2) *for every limit $\delta \leqslant \alpha$, $\bigcup_{\gamma<\delta} \mathbb{P}_\gamma$ is dense in $\mathbb{P}_\delta$,*
(3) *and for all $\delta < \alpha$, $\mathbb{P}_\delta$ is ccc.*

Then also $\mathbb{P}_\alpha$ is ccc.

Lemma 2.7. *Let $\mathbb{P}$ be a complete sub-forcing of $\mathbb{Q}$, $\dot{\mathbb{A}}$ a $\mathbb{P}$-name and $\dot{\mathbb{B}}$ a $\mathbb{Q}$-name for a forcing notion. Then,*

$$\Vdash_{\mathbb{Q}} \dot{\mathbb{A}} \lessdot_{V^{\mathbb{P}}} \dot{\mathbb{B}} \text{ if and only if } \mathbb{P} * \dot{\mathbb{A}} \lessdot \mathbb{Q} * \dot{\mathbb{B}}.$$

Proof. It suffices to notice that a pre-dense subset of $\mathbb{P} * \dot{\mathbb{A}}$ (and, respectively, of $\mathbb{Q} * \dot{\mathbb{B}}$) is precisely the same as a $\mathbb{P}$-name (or $\mathbb{Q}$-name) for a pre-dense subset of $\dot{\mathbb{A}}$ (respectively, $\dot{\mathbb{B}}$). For the direction from left to right, see also, for example, [7, Lemma 13]. □

3 | WETZEL FAMILIES AND UNIVERSAL SETS

Definition 3.1. A family $\mathcal{F} \subseteq \mathcal{H}(\mathbb{C})$ of entire functions is called a *Wetzel family* if for every $z \in \mathbb{C}$, $|\{f(z) : f \in \mathcal{F}\}| < |\mathcal{F}|$.

Lemma 3.2. *If $\mathcal{F}$ is a Wetzel family, then $|\mathcal{F}| = 2^{\aleph_0}$ and for every $\lambda < 2^{\aleph_0}$ and all but less than $2^{\aleph_0}$-many $z \in \mathbb{C}$, $|\{f(z) : f \in \mathcal{F}\}| \geqslant \lambda$.*

Proof. Suppose towards a contradiction that $|\mathcal{F}| < 2^{\aleph_0}$. For any distinct $f, g \in \mathcal{H}(\mathbb{C})$, $X_{f,g} = \{z \in \mathbb{C} : f(z) = g(z)\}$ does not have an accumulation point by Proposition 2.1 and thus must be countable. Thus, if $X = \bigcup_{f \neq g \in \mathcal{F}} X_{f,g}$, then

$$|X| \leqslant |\mathcal{F} \times \mathcal{F} \times \omega| < 2^{\aleph_0}.$$

† ZF^- is ZF without the Powerset Axiom.

In particular, there is some $z \in \mathbb{C} \setminus X$. But then for any distinct $f, g \in \mathcal{F}$, $f(z) \neq g(z)$ and so $|\{f(z) : f \in \mathcal{F}\}| = |\mathcal{F}|$, contradicting that $\mathcal{F}$ is Wetzel.

Now let $\lambda < 2^{\aleph_0}$ and $\mathcal{G} \subseteq \mathcal{F}$ have size λ. Then, if $X' = \bigcup_{f \neq g \in \mathcal{G}} X_{f,g}$, as before $|X'| < 2^{\aleph_0}$ and for all $z \in \mathbb{C} \setminus X'$, $|\{f(z) : f \in \mathcal{F}\}| \geqslant |\{f(z) : f \in \mathcal{G}\}| \geqslant \lambda$. □

This lemma in fact shows that a Wetzel family induces a quite non-trivial combinatorial object, especially when $2^{\aleph_0} > \aleph_2$. Namely, consider an enumeration $\langle z_\alpha : \alpha < \kappa \rangle$ of $\mathbb{C}$. For each $\alpha < \kappa$, there is a bijection $e_\alpha : \{f(z_\alpha) : f \in \mathcal{F}\} \to \mu_\alpha$ for some cardinal $\mu_\alpha < 2^{\aleph_0}$. In this way, we can think of each $f \in \mathcal{F}$ as the function $\sigma_f \in \prod_{\alpha < \kappa} \mu_\alpha$, where $\sigma_f(\alpha) = e_\alpha(f(z_\alpha))$. At the same time, the elements of $\mathcal{F}$ and thus of $\{\sigma_f : f \in \mathcal{F}\}$ have pairwise countable intersections.

Under CH, this type of almost disjoint family of functions can be obtained quite easily. For instance, for any $\alpha < \omega_1$, simply let σ_α be constantly 0 below α and constantly equal to α above α.

Also this is not particularly hard when $2^{\aleph_0} = \aleph_2$. Whenever we have constructed $\langle \sigma_\beta : \beta < \alpha \rangle$ for some $\alpha < \omega_2$, we can find a single function $\sigma : \alpha \to \omega_1$ that has countable intersection with all σ_β using a standard diagonalisation argument. Then, we simply let σ_α equal σ below α and constantly equal to α above α.

For larger continuum, the existence of such families becomes much less clear, as a consequence of the larger gap between the countable size of the pairwise intersections and the size of the family and their elements. In fact, in Section 5, we will show how to force these types of families, based on a technique by Baumgartner. This will be a key starting point for the construction of a Wetzel family by forcing.

As a small observation of independent interest, let us mention the following:

Lemma 3.3. *A Wetzel family cannot consist only of polynomials.*

Proof. Suppose $\mathcal{F}$ is such a family. Since $|\mathcal{F}| = 2^{\aleph_0}$ and $2^{\aleph_0}$ has uncountable cofinality, we can assume that all polynomials in $\mathcal{F}$ have the same degree n. Then, pick any $n+1$-many points $a_0, \dots, a_n \in \mathbb{C}$. The set $\{f \restriction \{a_0, \dots, a_n\} : f \in \mathcal{F}\}$ has size less than $2^{\aleph_0}$, because $\mathcal{F}$ is Wetzel. But each $f \in \mathcal{F}$ is uniquely determined by $f \restriction \{a_0, \dots, a_n\}$ and so $\mathcal{F}$ has small size as well, contradicting Lemma 3.2. □

Definition 3.4. We call a set $Y \subseteq \mathbb{C}$, where $|Y| < 2^{\aleph_0}$, *universal (for entire functions)* if for any $X \subseteq \mathbb{C}$ with $|X| < 2^{\aleph_0}$, there is a non-constant $f \in \mathcal{H}(\mathbb{C})$, such that $f(X) \subseteq Y$.

Proposition 3.5 (Erdős, [13]). *Assuming* CH*, any countable dense set is universal.*

Let us write the argument for sake of completeness. In a way, the forcing notions we will use later mimic this construction.

Proof. Let $Y \subseteq \mathbb{C}$ be countable dense and $X \subseteq \mathbb{C}$ be arbitrary countable and enumerated as $\langle z_n : n \in \omega \rangle$. We recursively construct a sequence $\langle f_n : n \in \omega \rangle$ of entire functions that converges uniformly on compact sets. Start by simply letting f_0 be constantly 0. Next, if f_n has been defined, consider

$$g_{n,\xi}(z) = \xi \prod_{m<n} (z - z_m).$$

Note that the set of zeros of $g_{n,\xi}$ is exactly $\{z_m : m < n\}$, when $|\xi| > 0$. Then, there is $\delta > 0$, so that

$$\|g_{n,\xi}\|_n < \frac{1}{2^n},$$

for every $\xi \in B_\delta(0)$. Since Y is dense, there is some $\xi \in B_\delta(0)$ so that

$$f_n(z_n) + g_{n,\xi}(z_n) \in Y \setminus f_n[\{z_m : m < n\}].$$

Then, let $f_{n+1} = f_n + g_{n,\xi}$. Finally, let f be the limit of $\langle f_n : n \in \omega\rangle$. By Proposition 2.2, f is entire. Clearly $f(X) \subseteq Y$ since $f_m(z_n)$ remains constant for $m > n$ and equals $f_{n+1}(z_n) \in Y$. Moreover, f is injective on X, so definitely non-constant. □

Proposition 3.6. *Let Y be a universal set. Then, $|Y|^+ = 2^{\aleph_0}$ and in particular the continuum is a successor cardinal.*

Proof. Suppose that there is $X \subseteq \mathbb{C}$ uncountable with $|Y| < |X| < 2^{\aleph_0}$. If $f \in \mathcal{H}(\mathbb{C})$ and $f''X \subseteq Y$, by the pigeonhole principle, there is $y \in Y$ such that $\{z \in X : f(z) = y\}$ is uncountable. But then this set has an accumulation point and by Proposition 2.1, f is constant. □

Proposition 3.7. *If there is a universal set, there is also a Wetzel family.*

Proof. Let Y be universal and let $\langle z_\alpha : \alpha < \kappa\rangle$ enumerate $\mathbb{C}$. For each $\alpha < \kappa$, let $f_\alpha \in \mathcal{H}(\mathbb{C})$ be non-constant such that $f_\alpha(\{z_\beta : \beta < \alpha\}) \subseteq Y$. We claim that $\mathcal{F} = \{f_\alpha : \alpha < \kappa\}$ is a Wetzel family.

First of all, we note that $|\mathcal{F}| = \kappa$. By Proposition 3.6, κ is a successor and in particular regular. Thus, if $|\mathcal{F}| < \kappa$, there is an unbounded subset $S \subseteq \kappa$ so that $f_\alpha = f_\beta$, for all $\alpha, \beta \in S$. But then for any such $\alpha \in S$, $f_\alpha(\mathbb{C}) \subseteq Y$. This would imply that f_α is constant, as in the proof of Proposition 3.6.

Now let $z_\alpha \in \mathbb{C}$ be arbitrary. Then, $\{f(z_\alpha) : f \in \mathcal{F}\} \subseteq \{f_\beta(z_\alpha) : \beta \leqslant \alpha\} \cup Y$ and the right-hand side has size less than κ. □

Corollary 3.8 (Erdős, [13]). *There is a Wetzel family under* CH.

4 | STRONGLY ALMOST DISJOINT FUNCTIONS AND BAUMGARTNER'S THINNING-OUT FORCING

This section can be skipped entirely if one wants to pass directly to the proof of the main result. The main purpose is to prove Proposition 4.1 below which is used in the setup of our main forcing construction. It is an adaptation of Baumgartner's 'thining-out' technique to obtain certain types of almost disjoint families (see [5]). To be more precise, we show how to obtain the type of family of functions necessitated by a Wetzel family, as shown in Section 3, where pairwise intersections are finite. Interestingly, a different type of strongly almost disjoint family was also used in [22] to obtain a Wetzel family with continuum $\aleph_{\omega_1}$. It is unclear to us how this relates to our argument.

Proposition 4.1 (GCH). *Let κ be an infinite cardinal of uncountable cofinality and for every $\alpha < \kappa$, let $\mu_\alpha = \max(|\alpha|, \aleph_0)$. Then, there is a cardinal and cofinality preserving forcing extension of V where $2^{\aleph_0} = \kappa$ and there is $\langle \sigma_\alpha : \alpha < \kappa\rangle$, such that for all $\alpha < \beta < \kappa$,*

(1) $\sigma_\alpha \in \prod_{\xi<\kappa} \mu_\xi$,
(2) $|\sigma_\alpha \cap \sigma_\beta| < \omega$.

If κ is regular, we additionally have that $|H(\kappa)| = \kappa$.

Proof. Let $S = \bigcup_{\xi\in[\omega,\kappa)} \{\xi\} \times \mu_\xi \subseteq [\omega,\kappa) \times \kappa$ and let K be the set of regular cardinals $\leqslant \kappa$. For every $\lambda \in K$, consider the forcing $\mathbb{P}_\lambda$ consisting of partial functions $p : \kappa \to [S]^{<\lambda}$, such that

(1) $|\operatorname{dom} p| < \lambda$;
(2) for all $\alpha, \beta \in \operatorname{dom} p$, $\operatorname{proj} p(\alpha) = \operatorname{proj} p(\beta)$, that is, $p(\alpha)$ and $p(\beta)$ have the same projection to the first coordinate;
(3) and for all $\alpha \in \operatorname{dom} p$ and $\xi \in \operatorname{proj} p(\alpha) \cap \lambda$, $|p(\alpha) \cap (\{\xi\} \times \mu_\xi)| = \mu_\xi$.

We will simply write $\operatorname{proj} p$ to denote $\operatorname{proj} p(\alpha)$, for any $\alpha \in \operatorname{dom} p$, and if $\operatorname{dom} p = \emptyset$, we let $\operatorname{proj} p = \emptyset$.

For any $p, q \in \mathbb{P}_\lambda$, $q \leqslant p$ if and only if $\operatorname{dom} p \subseteq \operatorname{dom} q$ and for any distinct $\alpha, \beta \in \operatorname{dom} p$,

(1) $p(\alpha) \subseteq q(\alpha)$, and
(2) $q(\alpha) \cap q(\beta) \cap ([\lambda^-,\kappa) \times \kappa) = p(\alpha) \cap p(\beta) \cap ([\lambda^-,\kappa) \times \kappa)$,

where λ^- is the predecessor of λ, if λ is a successor cardinal, or $\lambda^- = \lambda$, if λ is a limit cardinal.

Whenever G is $\mathbb{P}_\lambda$-generic, let $S_\alpha = \bigcup_{p\in G} p(\alpha)$ for every $\alpha < \kappa$. Then, it is not hard to check that every vertical section of S_α is non-empty (see more below) and for any $\alpha < \beta$,

$$|S_\alpha \cap S_\beta \cap ([\lambda^-,\kappa) \times \kappa)| < \lambda.$$

In particular, when λ is a successor cardinal,

$$|S_\alpha \cap S_\beta| < \lambda.$$

In fact, for any $\xi \in [\omega, \lambda^-)$, the section with index ξ of S_α equals μ_ξ. The reason why we include these sections is purely notational.

It is clear that $\mathbb{P}_\lambda$ is $<\lambda$-closed. Ideally we would like to force with $\mathbb{P}_\omega$, since if we then choose σ_α, such that $(\omega+\xi, \sigma_\alpha(\xi)) \in S_\alpha$, for every $\xi < \kappa$, (1) and (2) of the proposition are satisfied. But $\mathbb{P}_\omega$ is far from being ccc. To circumvent this, we use Baumgartner's thinning out trick.

Let $\mathbb{P} \subseteq \prod_{\lambda\in K} \mathbb{P}_\lambda$ consist of all $\bar{p} = \langle p_\lambda : \lambda \in K\rangle$ such that for $\lambda' < \lambda$, $\operatorname{dom} p_{\lambda'} \subseteq \operatorname{dom} p_\lambda$ and for any $\alpha \in \operatorname{dom} p_{\lambda'}$, $p_{\lambda'}(\alpha) \subseteq p_\lambda(\alpha)$. When G is $\mathbb{P}$-generic, we obtain the sets $S^\lambda_\alpha = \bigcup_{\bar{p}\in G} p_\lambda(\alpha)$ for every $\lambda \in K$ and it is very easy to see again that

$$|S^\lambda_\alpha \cap S^\lambda_\beta \cap ([\lambda^-,\kappa) \times \kappa)| < \lambda,$$

for $\alpha < \beta < \kappa$. Let us check that all vertical sections of S^λ_α are non-empty. To this end, let $\bar{p} \in \mathbb{P}$ be arbitrary.

Claim 4.2. There is $\bar{q} \leqslant \bar{p}$ so that $\alpha \in \operatorname{dom} q_\lambda$.

Proof. If for all $\nu \in K$, $\alpha \notin \operatorname{dom} p_\nu$, we let $\operatorname{dom} q_\nu = \operatorname{dom} p_\nu \cup \{\alpha\}$, $q_\nu \restriction \operatorname{dom} p_\nu = p_\nu \restriction \operatorname{dom} p_\nu$ and

$$q_\nu(\alpha) = \{(\xi, 0) : \xi \in \operatorname{proj} p_\nu \setminus \nu\} \cup \bigcup_{\xi\in\operatorname{proj} p_\nu \cap \nu} \{\xi\} \times \mu_\xi,$$

for every $\nu \in K$. Otherwise, there is a minimal $\mu \in K$ so that $\alpha \in \operatorname{dom} p_\mu$. In this case, pick the least a_ξ so that $(\xi, a_\xi) \in p_\mu(\alpha)$, for every $\xi \in \operatorname{proj} p_\mu$. Then, for each $\nu \in K$, we let q_ν extend p_ν as before, but with

$$q_\nu(\alpha) = p_\nu(\alpha) \cup \{(\xi, a_\xi) : \xi \in \operatorname{proj} p_\nu \setminus \nu\} \cup \bigcup_{\xi \in \operatorname{proj} p_\nu \cap \nu} \{\xi\} \times \mu_\xi.$$

□

Now let $\xi \in [\omega, \kappa)$ be arbitrary.

Claim 4.3. There is $\bar{q} \leqslant \bar{p}$ so that $\xi \in \operatorname{proj} q_\lambda$.

Proof. If $\xi < \lambda^-$, simply extend every single $p_\nu(\beta)$, for $\nu \geqslant \lambda$ and $\beta \in \operatorname{dom} p_\nu$, by adding $\{\xi\} \times \mu_\xi$. This works since pairwise intersections occurring below $\nu^- \geqslant \lambda^-$ do not matter when extending p_ν.

If $\xi \in [\lambda^-, \lambda)$, λ is a successor and $\mu_\xi = \lambda^-$. Then, we can assume already that for every $\nu \geqslant \lambda^+$ and $\beta \in \operatorname{dom} p_\nu$, $\{\xi\} \times \mu_\xi \subseteq p_\nu(\beta)$, using the same procedure as before. Since $|\operatorname{dom} p_\lambda| < \lambda$ and thus $|\operatorname{dom} p_\lambda| \leqslant \lambda^-$, it is easy to find pairwise disjoint sets $X_\beta \subseteq \mu_\xi = \lambda^-$ of size $\mu_\xi = \lambda^-$, for all $\beta \in \operatorname{dom} p_\lambda$. Simply extend each $p_\lambda(\beta)$ by adding $\{\xi\} \times X_\beta$.

Finally, if $\xi \geqslant \lambda$, we can assume already that $\xi \in \operatorname{proj} p_\nu$, for $\nu = |\xi|^+$, using what we have just shown. Then, we can easily find pairwise distinct a_β, so that $(\xi, a_\beta) \in p_\nu(\beta)$, for all $\beta \in \operatorname{dom} p_\nu$, since $|\operatorname{dom} p_\nu| \leqslant \nu^- = |\xi| = \mu_\xi$ and $|p_\nu(\beta) \cap (\{\xi\} \times \mu_\xi)| = \mu_\xi$, for every $\beta \in \operatorname{dom} p_\nu$. Now simply extend all $p_{\nu'}(\beta)$, for $\nu' \in K \cap [\lambda, \nu)$ and $\beta \in \operatorname{dom} p_{\nu'}$, by adding in (ξ, a_β). □

The σ_α as defined above, for $S_\alpha = S_\alpha^\omega$, are then as required.

Claim 4.4. $\mathbb{P}$ preserves all regular cardinals.

Proof. Let $\lambda \in K$. We will show how to factor $\mathbb{P}$ into a two-step iteration of a $< \lambda^+$-closed and a λ^+-cc forcing. Let $\mathbb{P}_0 = \{\bar{p} \upharpoonright [\lambda^+, \kappa] : \bar{p} \in \mathbb{P}\}$ and note that $\mathbb{P}_0$ is $< \lambda^+$-closed and thus does not add sequences of length λ.[†] Let G_0 be $\mathbb{P}_0$-generic over V. Let $S_\alpha = S_\alpha^{\lambda^+}$, for $\alpha < \kappa$, be defined as before, that is,

$$S_\alpha = \bigcup_{\bar{p} \in G_0} p_{\lambda^+}(\alpha).$$

Let $\mathbb{P}_1$ consist of all $\bar{p} \upharpoonright \lambda^+$, for $\bar{p} \in \mathbb{P}$, where

$$p_\lambda(\alpha) \subseteq S_\alpha,$$

for each $\alpha \in \operatorname{dom} p_\lambda$. Then, it is easy to verify that if G_1 is $\mathbb{P}_1$-generic over $V[G_0]$, $V[G_0][G_1]$ is a $\mathbb{P}$-generic extension of V. Work in $V[G_0]$ and suppose that $\langle \bar{p}^i : i < \lambda^+ \rangle$ is an anti-chain in $\mathbb{P}_1$.

Note that for any $i < \lambda^+$, there is $\delta < \lambda$ so that $\operatorname{dom} p^i_{\lambda'}$ as well as $p^i_{\lambda'}(\alpha)$ are the same for all $\lambda' \in [\delta, \lambda) \cap K$ and fixed $\alpha \in \operatorname{dom} p^i_\lambda$, since these sets grow and are subsets of $\operatorname{dom} p^i_\lambda$ and $p^i_\lambda(\alpha)$,

[†] When $\lambda = \kappa$, then $\mathbb{P}_0$ is simply the trivial forcing.

respectively, which have size strictly less than λ.[†] We may assume that this δ is the same for all $i < \lambda^+$ and let

$$D^i = \operatorname{dom} p^i_{\lambda'},$$

for any, equivalently all $\lambda' \in [\delta, \lambda) \cap K$, if this exists.[‡]

Since $\lambda^{<\lambda} = \lambda$, a Δ-system argument lets us assume that there are sets D and E such that $D = D^i \cap D^j$ and $\operatorname{dom} p^i_\lambda \cap \operatorname{dom} p^j_\lambda = E$, for all $i < j < \lambda^+$. Note that we can also do the same for all $\lambda' \in \delta \cap K$ simultaneously, as $\delta < \lambda$. To be more precise, we can assume $D_{\lambda'} = \operatorname{dom} p^i_{\lambda'} \cap \operatorname{dom} p^j_{\lambda'}$, for all $\lambda' \in \delta \cap K$ and $i < j < \lambda^+$. Moreover, using another Δ-system argument, we can assume $R^i_\lambda \cap R^j_\lambda = R$, for some fixed R and all $i < j < \lambda^+$, where $R^i_\lambda = \bigcup_{\alpha \in \operatorname{dom} p^i_\lambda} p^i_\lambda(\alpha)$.

Now comes the heart of the thinning out argument. Let $i < \lambda^+$ be arbitrary, then note that for any distinct α, β and any λ', $p^i_{\lambda'}(\alpha) \cap S(\beta)$ is a $< \lambda$ sized subset of $S(\alpha) \cap S(\beta)$, which has size at most λ. Also R has size $< \lambda$. Thus, we find $i < j < \lambda^+$ such that

$$p^i_{\lambda'}(\alpha) \cap S(\beta) = p^j_{\lambda'}(\alpha) \cap S(\beta)$$

and

$$p^i_{\lambda'}(\alpha) \cap R = p^j_{\lambda'}(\alpha) \cap R,$$

for all $\lambda' \in \delta \cap K$ and distinct $\alpha, \beta \in D_{\lambda'}$, also for any, equivalently all, $\lambda' \in [\delta, \lambda) \cap K$ and distinct $\alpha, \beta \in D$, and for $\lambda' = \lambda$ and distinct $\alpha, \beta \in E$. It is straightforward to check that $\bar{p}^i$ and $\bar{p}^j$ are compatible.

Now let us check that all regular cardinals θ are preserved. Suppose that in $V^{\mathbb{P}}$, $\operatorname{cf}(\theta) = \mu < \theta$. If $\mu \geqslant \kappa$, then either κ is regular and we have shown above that $\mathbb{P}$ is κ^+-cc, so in particular θ-cc. Or κ is singular, $\mu \geqslant \kappa^+$ and $|\mathbb{P}| \leqslant \kappa^+$, so $\mathbb{P}$ is κ^{++}-cc and also θ-cc. If $\mu < \kappa$, $\mathbb{P}$ is the iteration of a μ^+-closed and a μ^+-cc forcing and $\theta \geqslant \mu^+$. □

It is clear that in $V^{\mathbb{P}}$, $2^{\aleph_0} \geqslant \kappa$ since $\{\sigma_\alpha \restriction \omega : \alpha < \kappa\}$ is an almost disjoint set of functions from ω to ω. In the other direction, note that $2^\lambda \leqslant \kappa$ for any regular $\lambda < \operatorname{cf}(\kappa)$. Namely $\mathbb{P}$ is $\mathbb{P}_0 * \dot{\mathbb{P}}_1$, where $\mathbb{P}_0$ does not add any new sequences of length λ and $\mathbb{P}_1$ has size at most κ and is λ^+-cc. A counting of names argument then shows $2^\lambda \leqslant \kappa$. In particular, $2^{\aleph_0} = \kappa$ and $|H(\kappa)| = 2^{<\kappa} = \kappa$, when κ is regular. □

Corollary 4.5. *The answer to [33, Question 22] is positive. Namely, assuming* GCH*, it is possible to add arbitrarily large strongly almost disjoint families of functions in* $\aleph_\omega^{\aleph_{\omega+1}}$ *without collapsing cardinals.*

The following is of further interest, especially for Question 8.1 at the end of the paper. In the second paragraph after the proof of Lemma 3.2, we have shown the following using a simple construction:

[†] Of course, if λ is a successor cardinal, this is trivial as we may choose δ such that $K \cap [\delta, \lambda)$ is empty. Otherwise, recall that λ is regular.

[‡] If $K \cap [\delta, \lambda)$ is empty, D^i is left undefined as it will be irrelevant.

Proposition 4.6. $2^{\aleph_0} = \aleph_2$ *implies that there is* $\langle \sigma_\alpha : \alpha < \omega_2 \rangle$ *so that for any* $\alpha < \beta < \omega_2$, $\sigma_\alpha \in {}^{\omega_2}\omega_1$ *and* $|\sigma_\alpha \cap \sigma_\beta| < \omega_1$.

In fact, $\mathsf{MA} + 2^{\aleph_0} = \aleph_2$ (in particular PFA) implies that we can even assume finite intersections, and thus the conclusion of Proposition 4.1 holds.

Proposition 4.7. $\mathsf{MA} + 2^{\aleph_0} = \aleph_2$ *implies that there is* $\langle \sigma_\alpha : \alpha < \omega_2 \rangle$ *so that for any* $\alpha < \beta < \omega_2$, $\sigma_\alpha \in {}^{\omega_2}\omega_1$ *and* $|\sigma_\alpha \cap \sigma_\beta| < \omega$.

Proof. We recursively construct a sequence as above, but where each σ_α is an element of $\prod_{\beta<\omega_2} \max(\omega_1, \beta + 1)$. This clearly makes no difference. We also ensure that σ_α always constantly maps to α on $[\alpha, \omega_2)$. Suppose that $\langle \sigma_\beta : \beta < \alpha \rangle$ has been constructed. Using a simple diagonalisation argument, we find a function $S : \alpha \to [\omega_1]^\omega$, so that for any $\beta < \alpha$, $\{\xi < \alpha : \sigma_\beta(\xi) \in S(\xi)\}$ is countable. Now let $\mathbb{P}$ be the natural poset with finite conditions adding a function $\sigma \in \prod_{\xi<\alpha} S(\xi)$ that has finite intersection with each σ_β, $\beta < \alpha$. To be more precise, $\mathbb{P}$ consists of pairs (s, w) where s is a finite partial function $s \in \prod_{\xi\in\mathrm{dom}(s)} S(\xi)$ and $w \in [\alpha]^{<\omega}$. A condition (t, u) extends (s, w) if $s \subseteq t$, $w \subseteq u$ and for every $\beta \in w$, $(t \setminus s) \cap \sigma_\beta = \emptyset$. Once we have shown that $\mathbb{P}$ is ccc, we can apply MA to find σ. Then simply let $\sigma_\alpha \restriction \alpha = \sigma$ and $\sigma_\alpha \restriction [\alpha, \omega_2)$ constantly equal α.

So suppose that $\langle (s_\delta, w_\delta) : \delta < \omega_1 \rangle$ is an uncountable anti-chain in $\mathbb{P}$. Without loss of generality, we can assume that $\langle \mathrm{dom}(s_\delta) : \delta < \omega_1 \rangle$ forms a Δ-system with root r and that $s_\delta \restriction r = s$ and $|\mathrm{dom}(s_\delta)| = n$, for all δ and some fixed s and n. Also we may assume that $\langle w_\delta : \delta < \omega_2 \rangle$ is a Δ-system with root w. Since $\langle \mathrm{dom}(s_\delta) \setminus r : \delta < \omega_1 \rangle$ is pairwise disjoint and for each σ_β, $\sigma_\beta \cap \bigcup_{\xi<\alpha} (\{\xi\} \times S(\xi))$ is countable, there is a large enough $\gamma \in [\omega, \omega_1)$, so that for every $\delta \in [\gamma, \omega_1)$,

$$s_\delta \restriction (\mathrm{dom}(s_\delta) \setminus r) \cap \sigma_\beta = \emptyset,$$

for all $\beta \in \bigcup_{i\in\omega} w_i$. Note that this means that $(s_i \cup s_\delta, w_i \cup w_\delta) \leqslant (s_i, w_i)$, for every $i \in \omega$. Thus, the only way in which (s_i, w_i) and (s_δ, w_δ) can be incompatible, is if there is $\beta \in w_\delta \setminus w$ and $\xi \in \mathrm{dom}(s_i) \setminus r$, so that

$$\sigma_\beta(\xi) = s_i(\xi).$$

For any $\delta \geqslant \gamma$, let us define functions ξ_δ and β_δ with domain ω so that $\xi_\delta(i) \in \mathrm{dom}(s_i) \setminus r$ is a witness ξ as above for $\beta_\delta(i) \in w_\delta \setminus w$. Since for all $i \in \omega$, $\mathrm{dom}(s_i) \setminus r$ has finite size $\leqslant n$, there must be $\delta_0 < \delta_1$ and an infinite $x \subseteq \omega$ so that

$$\xi_{\delta_0} \restriction x = \xi_{\delta_1} \restriction x.$$

Since $w_{\delta_0} \setminus w$ and $w_{\delta_1} \setminus w$ are finite, we find an infinite $y \subseteq x$ so that both $\beta_{\delta_0} \restriction y$ and $\beta_{\delta_1} \restriction y$ are constant, say with values β^0 and β^1. Now $\beta^0 \neq \beta^1$, since $w_{\delta_0} \setminus w$ and $w_{\delta_1} \setminus w$ are disjoint. But then $\sigma_{\beta^0} \cap \sigma_{\beta^1}$ is infinite, yielding a contradiction. □

5 | WETZEL FAMILIES WITH ARBITRARY CONTINUUM AND MA

In this section, we prove our main result that Wetzel families can coexist with arbitrary values of the continuum and in combination with Martin's Axiom.

5.1 Adding entire functions

Definition 5.1. The poset $\mathbb{Q}$ consists of all conditions

$$p = (a_p, f_p, \varepsilon_p, m_p),$$

where $a_p \in [\mathbb{C}]^{<\omega}$, $f_p \in \mathcal{H}(\mathbb{C})$, ε_p is a positive rational number and $m_p \in \omega$. A condition q extends p if and only if $a_p \subseteq a_q$, $f_q \upharpoonright a_p = f_p \upharpoonright a_p$, $\varepsilon_q \leqslant \varepsilon_p$, $m_p \leqslant m_q$ and $\|f_q - f_p\|_{m_p} \leqslant \varepsilon_p - \varepsilon_q$.

Definition 5.2. Let $H : X \to \mathcal{P}(\mathbb{C})$, for some $X \subseteq \mathbb{C}$. Then, we define

$$\mathbb{Q}(H) = \{p \in \mathbb{Q} : a_p \subseteq X \wedge \forall z \in a_p(f_p(z) \in H(z))\}.$$

Note that the notion of incompatibility of conditions $p, q \in \mathbb{Q}(H)$ is not dependent on H. Namely, if r extends p and q in $\mathbb{Q}$, then $(a_p \cup a_q, f_r, \varepsilon_r, m_r) \in \mathbb{Q}(H)$ and also extends p and q. In other words, $\mathbb{Q}(H)$ is a sub-forcing of $\mathbb{Q}$ (see Definition 2.4). For most considerations, it is also not relevant in which transitive model of set theory M, we evaluate the definition of $\mathbb{Q}(H)$, as long as $H \in M$.

Lemma 5.3. *Let $M \subseteq V$ be a transitive model of* ZF^-*(possibly a proper class) and $H \in M$ be a partial function from $\mathbb{C}$ to $\mathcal{P}(\mathbb{C})$. Then, $\mathbb{Q}(H)^M$ is a dense sub-forcing of $\mathbb{Q}(H)^V$.*

Proof. Note that conditions p such that f_p is a polynomial in coefficients in the field generated by $\operatorname{dom} H \cup \bigcup_{z \in \operatorname{dom} H} H(z) \subseteq M$ form a dense sub-poset of $\mathbb{Q}(H)^V$. Namely, if $\|f - g\|_m = \delta < \varepsilon$, and $f \upharpoonright a = g \upharpoonright a$, $(a, g, \varepsilon - \delta, m) \leqslant (a, f, \varepsilon, m)$. □

In particular, any iteration of the form $\mathbb{Q}(H_0) * \dot{\mathbb{Q}}(H_1)$, where H_0, H_1 are in the ground model, is equivalent to the product $\mathbb{Q}(H_0) \times \mathbb{Q}(H_1)$ and the ccc of the iteration is equivalent to that of the product. This will be used at least implicitly in several arguments.

Lemma 5.4. *Suppose that $H(z)$ is dense in $\mathbb{C}$, for every $z \in \operatorname{dom} H$. Then, $\mathbb{Q}(H)$ generically adds an entire function f such that $f(z) \in H(z)$ for every $z \in \operatorname{dom} H$.*

Proof. Let G be $\mathbb{Q}(H)$-generic over V. For any $n \in \omega$, the set $D_n = \{p \in \mathbb{Q}(H) : \varepsilon_p < \frac{1}{n} \wedge m_p > n\}$ is clearly dense open. Moreover, for any $\xi \in \mathbb{C}$ and any $p \in \mathbb{Q}(H)$, consider

$$f_\xi(z) = f_p(z) + \xi \prod_{y \in a_p} (z - y).$$

Note that $f_\xi(y) = f_p(y)$, for every $y \in a_p$. Let $z \in \operatorname{dom} H \setminus a_p$ be arbitrary. Since $H(z)$ is dense, we can easily find a small enough ξ so that

$$\delta := \|f_p - f_\xi\|_{m_p} < \varepsilon_p$$

and $f_\xi(z) \in H(z)$. Then, $(a_p \cup \{z\}, f_\xi, \varepsilon_p - \delta, m_p) \leqslant p$. This shows $E_z = \{q \in \mathbb{Q}(H) : z \in a_q\}$ is dense open. We claim that for any sequences $\langle p_n : n \in \omega \rangle$ and $\langle q_n : n \in \omega \rangle$, with $p_n, q_n \in D_n \cap$

G, $\langle f_{p_n} : n \in \omega\rangle$ and $\langle f_{q_n} : n \in \omega\rangle$ converge uniformly on compact sets to the same function $f \in \mathcal{H}(\mathbb{C})$. To see this, use Proposition 2.2 and notice that when $p, q \in D_n \cap G$ are arbitrary, there is $r \leqslant p, q$ and thus

$$\|f_p - f_q\|_n \leqslant \|f_p - f_r\|_n + \|f_r - f_q\|_n < \frac{1}{n} + \frac{1}{n}.$$

For any $z \in \operatorname{dom} H$, we can find a decreasing sequence $\langle p_n : n \in \omega\rangle$ such that $p_n \in D_n \cap E_z \cap G$, for every n. Then, $f(z) = \lim_{n\to\infty} f_{p_n}(z) = f_{p_0}(z) \in H(z)$. □

Let us make the following interesting observation that will somewhat elucidate the necessity of the approach taken in the proof of the main result.

Lemma 5.5. *Let* $\operatorname{dom} H$ *be uncountable and suppose that there is an entire* f *such that* $f(z) \in H(z)$*, for every* $z \in \operatorname{dom} H$*. Then,* $\mathbb{Q}(H)$ *is not ccc.*

Proof. Let $n \in \omega$ be such that $B_n(0) \cap \operatorname{dom} H$ is uncountable and let $\varepsilon > 0$ be so that $\|\operatorname{Re}(f')\|_n < \varepsilon$. For any $z_0 \in B_n(0) \cap \operatorname{dom} H$, define

$$f_{z_0}(z) = 2\varepsilon(z - z_0) + f(z_0).$$

Note that there is $\delta > 0$ and $m \geqslant n$ so that whenever $\|g - f_{z_0}\|_m < \delta$, then $\|g' - 2\varepsilon\|_n < \varepsilon$.[†] Let $p_{z_0} = (\{z_0\}, f_{z_0}, \delta, m)$. We claim that $\{p_{z_0} : z_0 \in B_n(0) \cap \operatorname{dom} H\}$ is an anti-chain. Namely, suppose $z_0, z_1 \in B_n(0) \cap \operatorname{dom} H$ are arbitrary and that $r \leqslant p_{z_0}, p_{z_1}$. Then, $\|f_r - f_{z_0}\|_m < \delta$ and so $\|f_r' - 2\varepsilon\|_n < \varepsilon$. In particular, for any $z \in B_n(0)$, $\operatorname{Re}(f_r'(z)) > \varepsilon$. At the same time, by the complex mean value theorem (see e.g. [14, Theorem 2.2]),

$$\varepsilon < \operatorname{Re}\left(\frac{f_r(z_0) - f_r(z_1)}{z_0 - z_1}\right) = \operatorname{Re}\left(\frac{f(z_0) - f(z_1)}{z_0 - z_1}\right) < \varepsilon,$$

which poses a contradiction. □

It would be interesting to obtain some sort of converse to Lemma 5.5. For instance, suppose that H only maps to countable sets. Does the non-ccc of $\mathbb{Q}(H)$ imply at least that there is a Borel function f, with $f(z) \in H(z)$ for uncountably many $z \in \operatorname{dom} H(z)$?

Corollary 5.6. *Let* $\operatorname{dom} H$ *be uncountable and suppose that* $H(z)$ *is dense in* $\mathbb{C}$*, for every* $z \in \operatorname{dom} H$*. Then,* $\mathbb{Q}(H) \times \mathbb{Q}(H)$ *is not ccc.*

Proof. Either $\mathbb{Q}(H)$ is already not ccc, or $\operatorname{dom} H$ is preserved to be uncountable and by Lemmas 5.3 and 5.4, we are in the situation of Lemma 5.5 after forcing with $\mathbb{Q}(H)$ once. Thus, $\Vdash_{\mathbb{Q}(H)}$ '$\mathbb{Q}(H)$ is not ccc' and by Lemma 2.3, $\mathbb{Q}(H) * \dot{\mathbb{Q}}(H) \cong \mathbb{Q}(H) \times \mathbb{Q}(H)$ is not ccc. □

Thus, the forcings $\mathbb{Q}(H)$ can generally not be recycled in a ccc construction. If one wants to add another entire function, one has to pass to a new H.

[†] For instance, this follows easily from Proposition 2.2. This part of the argument strongly depends on the special geometry of holomorphic functions. The statement is clearly not true for functions that are merely infinitely often differentiable.

5.2 | Tools for the successor step

Lemma 5.7. *Let $l, m \in \omega$ and $K \subseteq \mathbb{C}^{l+1}$ be compact, such that every element of K is one-to-one. Then, there is $L > 0$ such that for any $\bar{z} \in K$, there is $g \in \mathcal{H}(\mathbb{C})$ with $\|g\|_m < L$, $g(z_i) = 1$ for every $i < l$ and $g(z_l) = 0$.*

Proof. Consider an interpolation formula such as

$$g(\bar{z}, z) = \sum_{i<k} \frac{(z - z_k)}{(z_i - z_k)} \prod_{\substack{j<k \\ j \neq i}} \frac{(z - z_j)}{(z_i - z_j)},$$

and simply note that $\bar{z} \mapsto g(\bar{z}, \cdot)$ is a continuous map from K to $\mathcal{H}(\mathbb{C})$ in the norm $\|\cdot\|_m$. The claim follows from the compactness of K. □

Lemma 5.8. *Let $H_0, \dots, H_n$ be such that $\mathbb{Q}(H_0) \times \cdots \times \mathbb{Q}(H_n)$ is ccc. Let $z \in \mathbb{C}$ be arbitrary and $H_0' \supseteq H_0$, where $\operatorname{dom} H_0' = \operatorname{dom} H_0 \cup \{z\}$ and $H_0'(z)$ is countable. Then, $\mathbb{Q}(H_0') \times \mathbb{Q}(H_1) \times \cdots \times \mathbb{Q}(H_n)$ is ccc.*

Proof. Suppose towards a contradiction that $\langle \bar{p}_\alpha : \alpha < \omega_1 \rangle$ is an anti-chain in $\mathbb{Q}(H_0') \times \mathbb{Q}(H_1) \times \cdots \times \mathbb{Q}(H_n)$. Then, we may assume without loss of generality that for every $\alpha < \omega_1$, $z \in a_{p_\alpha(0)}$ and $f_{p_\alpha(0)}(z) = y$, for some fixed $y \in H_0'(z)$. Otherwise, we find an uncountable anti-chain in $\mathbb{Q}(H_0) \times \cdots \times \mathbb{Q}(H_n)$. Furthermore, we may assume $\varepsilon_{p_\alpha(0)} = \varepsilon$, $m_{p_\alpha(0)} = m$, $|a_{p_\alpha(0)}| = l + 1$ and $a_{p_\alpha(0)} \setminus \{z\}$ is enumerated by $\bar{z}_\alpha = \langle z_{\alpha,i} : i < l \rangle$, for every α and some fixed ε, m and l. Even more, we can assume $\|f_{p_\alpha(0)} - f_{p_\beta(0)}\|_m < \frac{\varepsilon}{2}$, for every $\alpha, \beta < \omega_1$.

Then, there is some $\beta < \omega_1$, such that $\bar{z}_\beta$ is an ω_1-accumulation point of $\{\bar{z}_\alpha : \alpha < \omega_1\}$ in $\mathbb{C}^l$, in the sense that for any open neighbourhood of $\bar{z}_\beta$, there are uncountably many α with $\bar{z}_\alpha$ in said neighbourhood.† Let $O \ni \bar{z}_\beta$ be a compact neighbourhood of $\bar{z}_\beta$ so that every element of O is one-to-one and does not have z in any coordinate. This is easily possible as $\bar{z}_\beta$ is one-to-one and $z \notin \{z_{\beta,i} : i < l\} = a_{p_\beta(0)} \setminus \{z\}$. According to Lemma 5.7, and considering $K = O \times \{z\}$, there is $L > 0$ such that for any $\bar{z} \in O$, there is $g \in \mathcal{H}(\mathbb{C})$ such that $g(z) = 0$, $g(z_i) = 1$, for all $i < l$, and $\|g\|_m < L$. Now let $\varepsilon' < \frac{\varepsilon}{2L}$ and for each $\alpha < \omega_1$, let $\bar{p}_\alpha'$ be such that $p_\alpha'(i) = p_\alpha(i)$, for $i > 0$ and

$$p_\alpha'(0) = (a_{p_\alpha(0)} \setminus \{z\}, f_{p_\alpha(0)}, \varepsilon', m).$$

Then, note that $\bar{p}_\alpha' \in \mathbb{Q}(H_0) \times \cdots \times \mathbb{Q}(H_n)$, for every α. Thus, there are $\gamma < \delta < \omega_1$ such that $\bar{z}_\gamma, \bar{z}_\delta \in O$ and $\bar{q}' \leqslant \bar{p}_\gamma', \bar{p}_\delta'$, for some $\bar{q}' \in \mathbb{Q}(H_0) \times \cdots \times \mathbb{Q}(H_n)$. Let $g \in \mathcal{H}(\mathbb{C})$ be such that $\|g\|_m < L$, $g(z_{\delta,i}) = 1$ for every $i < l$ and $g(z) = 0$. Let $k = f_{q'(0)} - f_{p_\gamma(0)}$ and consider $f = f_{p_\gamma(0)} + g \cdot k$. Note that

$$\|f - f_{p_\gamma(0)}\|_m = \|g \cdot k\|_m \leqslant \|g\|_m \cdot \|k\|_m$$
$$< L \cdot \frac{\varepsilon}{2L} = \frac{\varepsilon}{2}.$$

† When $l = 0$, then $\mathbb{C}^l$ contains one element, namely the empty sequence, which all $\bar{z}_\alpha$ then equal to.

As $\|f_{p_\gamma(0)} - f_{p_\delta(0)}\| < \frac{\varepsilon}{2}$, we immediately find

$$\|f - f_{p_\delta(0)}\|_m < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Also, note that since $f_{q'(0)}(z_{\gamma,i}) = f_{p_\gamma(0)}(z_{\gamma,i})$, $k(z_{\gamma,i}) = 0$ for every $i < l$. Then, by choice of g and the fact that $\bar{q}' \leqslant \bar{p}'_\gamma, \bar{p}'_\delta$, it is easy to compute $f_{p_\gamma(0)} \restriction a_{p_\gamma(0)} \subseteq f$ and $f_{p_\delta(0)} \restriction a_{p_\delta(0)} \subseteq f$. Thus, if we let $\bar{q}$ be such that $q(i) = q'(i)$ for $i > 0$, and

$$q(0) = (a_{p_\gamma(0)} \cup a_{p_\delta(0)}, f, \varepsilon^*, m),$$

for some small enough $\varepsilon^* < \varepsilon$, we have $\bar{q} \leqslant \bar{p}_\gamma, \bar{p}_\delta$, contradicting our initial assumption. □

Note that by a simple inductive argument, the previous lemma implies that we can extend simultaneously each H_i in countably many arbitrary points with arbitrary countable sets of values and preserve the ccc:

Proposition 5.9. *Let $H_0, \dots, H_n$ be such that $\mathbb{Q}(H_0) \times \cdots \times \mathbb{Q}(H_n)$ is ccc. Let $H'_0 \supseteq H_0, \dots, H'_n \supseteq H_n$ be such that for every $i \leqslant n$,*

1. *$\operatorname{dom} H'_i \setminus \operatorname{dom} H_i$ is countable,*
2. *and for any $z \in \operatorname{dom} H'_i \setminus \operatorname{dom} H_i$, $H'_i(z)$ is countable.*

Then, $\mathbb{Q}(H'_0) \times \cdots \times \mathbb{Q}(H'_n)$ is ccc.

5.3 | Tools for the limit step

Lemma 5.10. *Let $M \subseteq V$ be a transitive model of ZF^- (possibly a proper class). Let $f \in \mathcal{H}(\mathbb{C})$, $a \in [\mathbb{C}]^{<\omega} \cap M$ such that $f \restriction a \in M$, $K \in [\mathbb{C} \setminus a]^{<\omega} \cap M$ such that $f \restriction K$ is constant, $\varepsilon > 0$ and $m \in \omega$. Moreover, for any $\xi \in \mathbb{C}$, let*

$$g_\xi(z) = \xi \sum_{x \in K} \prod_{y \in A \setminus \{x\}} \frac{(z-y)}{(x-y)},$$

where $A = a \cup K$. Then, there is $\tilde{f} \in \mathcal{H}(\mathbb{C}) \cap M$ and $\delta > 0$ such that $f \restriction a \subseteq \tilde{f}$, $\tilde{f} \restriction K$ is constant, $\|\tilde{f} - f\|_m < \varepsilon$ and

1. *$\forall \xi \in B_\delta(0)(\|\tilde{f} + g_\xi - f\|_m < \varepsilon)$,*
2. *$\exists \xi \in B_\delta(0) \forall z \in K(\tilde{f}(z) + g_\xi(z) = f(z))$.*

Whenever $f \in M$, we can assume $\tilde{f} = f$.

Before we continue to the proof, let us note that $g_\xi \in \mathcal{H}(\mathbb{C})$ is simply a function such that $g_\xi \restriction K$ is constantly ξ and $g_\xi(z) = 0$, for $z \in a$. Also, the smaller the absolute value of ξ is, the smaller $\|g_\xi\|_m$ gets.

Proof. We may assume, without losing generality, that m is large enough such that $A \subseteq B_m(0)$. Let $\delta < \frac{\varepsilon}{2}$ be small enough so that for any $\xi \in B_\delta(0)$, $\|g_\xi\|_m < \frac{\varepsilon}{2}$. Since $f \restriction a \in M$, we can easily find a function $\tilde{f} \in \mathcal{H}(\mathbb{C}) \cap M$ such that $f \restriction a \subseteq \tilde{f}$, $\tilde{f}$ is constant on K and $\|\tilde{f} - f\|_m < \delta < \varepsilon$. If $f \in M$ already, we may simply use $\tilde{f} = f$. For (1),

$$\begin{aligned}\|\tilde{f} + g_\xi - f\|_m &\leqslant \|\tilde{f} - f\|_m + \|g_\xi\|_m \\ &< \delta + \delta < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

For (2), define $\xi = f(z) - \tilde{f}(z)$, for any, equivalently every, $z \in K$. Then, $\xi \in B_\delta(0)$ and for any $z \in K$, $\tilde{f}(z) + g_\xi(z) = f(z)$. □

For a set F of finite partial functions on $\mathbb{C}$, let

$$\mathbb{Q}_0(F) := \{p \in \mathbb{Q} : f_p \restriction a_p \in F\}.$$

Forcings $\mathbb{Q}(H)$ are a particular type of forcings of the form $\mathbb{Q}_0(F)$. As with $\mathbb{Q}(H)$, the interpretation of $\mathbb{Q}_0(F)$ in any transitive model that contains F is easily seen to be a dense subset of $\mathbb{Q}_0(F)$ as interpreted in V. The following lemma will be formulated for this more general type of forcing, since that will be needed in Section 7. The proof of this core lemma originates in ideas fleshed out in Burke's [8]. For instance, Claim 2.8 in the aforementioned paper corresponds roughly to Claim 5.12 below. Burke remarks that this is a version of an argument by Shelah from [29].

Lemma 5.11. *Let $M \subseteq V$ be as in Lemma 5.10, $F \in M$, $z \in \mathbb{C} \cap (M \setminus \bigcup_{h \in F} \operatorname{dom} h)$ and c a Cohen real over M.*† *Furthermore, let $F' = F \cup \{h \cup \{(z, c)\} : h \in F\}$ and $\mathbb{P} \in M$ be a forcing notion that is dense in a forcing $\mathbb{P}' \in V$. Then,*

$$\mathbb{Q}_0(F) \times \mathbb{P} \lessdot_M \mathbb{Q}_0(F') \times \mathbb{P}'.$$

Proof. It is easy to see that $\mathbb{Q}_0(F) \times \mathbb{P}$ is a sub-forcing of $\mathbb{Q}_0(F') \times \mathbb{P}'$ (the incompatibility relation is preserved). Now let $E \in M$, $E \subseteq \mathbb{Q}_0(F) \times \mathbb{P}$ be pre-dense in $\mathbb{Q}_0(F) \times \mathbb{P}$ and suppose towards a contradiction there exists $\bar{p} = (p_0, p_1) \in \mathbb{Q}_0(F') \times \mathbb{P}'$, such that $\bar{p} \perp E$, where $z \in a_{p_0}$ and so $f_{p_0}(z) = c$. By extending $\bar{p}$, we may assume without loss of generality that $p_1 \in \mathbb{P}$ and that $a_{p_0} \subseteq B_{m_{p_0}}(0)$.

Let $a := a_{p_0} \setminus \{z\}$, $K := \{z\}$, $f := f_{p_0}$, $\varepsilon := \frac{\varepsilon_{p_0}}{4}$, $m := m_{p_0}$ and apply Lemma 5.10 to find $\tilde{f} \in M$ and $\delta > 0$ as in the conclusion of the lemma. Let $\tilde{p}_0 := (a, \tilde{f}, \frac{\varepsilon_{p_0}}{2}, m)$. Then, $(\tilde{p}_0, p_1) \in (\mathbb{Q}_0(F) \times \mathbb{P}) \cap M$. Since $c \in B_\delta(\tilde{f}(z))$, we may find a basic open set $O \subseteq B_\delta(\tilde{f}(z))$ such that $c \in O$.

In the following, for a condition p and a subset E of a poset, we write $p \leqslant E$ to mean that p extends some element of E.

Claim 5.12. There is a dense open set $U \subseteq O$ coded in M so that for every $d \in U$, there is $\bar{q} \in \mathbb{Q}_0(F) \times \mathbb{P}$ with $\bar{q} \leqslant E, (\tilde{p}_0, p_1)$ and $f_{q_0}(z) = d$.

Once we prove the claim we are done. Namely, as $c \in O$ is Cohen generic over M, $c \in U$. Then, according to the claim, there is $\bar{q} \leqslant E, (\tilde{p}_0, p_1)$ such that $f_{q_0}(z) = c$. Letting $r_0 :=$

† That is, c is in any open dense subset of $\mathbb{C}$ coded in M.

$(a_{r_0}, f_{q_0}, \varepsilon_{q_0}, m_{q_0})$, where

$$a_{r_0} := a_{q_0} \cup \{z\},$$

we clearly have $\bar{r} = (r_0, q_1) \leqslant \bar{q}$ and $\bar{r} \in \mathbb{Q}_0(F') \times \mathbb{P}'$. Moreover, we have $r_0 \leqslant p_0$ and thus $\bar{r} \leqslant \bar{p}, E$:

$$\begin{aligned}\|f_{p_0} - f_{r_0}\|_{m_{p_0}} &= \|f_{p_0} - f_{q_0}\|_m \leqslant \|f_{p_0} - \tilde{f}\|_m + \|\tilde{f} - f_{q_0}\|_m \\ &< \frac{\varepsilon_{p_0}}{4} + \left(\frac{\varepsilon_{p_0}}{2} - \varepsilon_{q_0}\right) = \frac{3}{4}\varepsilon_{p_0} - \varepsilon_{r_0} \\ &< \varepsilon_{p_0} - \varepsilon_{r_0}.\end{aligned}$$

This contradicts the assumption that $\bar{p} \perp E$.

Proof of Claim. Work in M. Let $O_0 \subseteq O$ be an arbitrary non-empty open set. We will find a non-empty open set $O_1 \subseteq O_0$ that will be included in U. Let $e \in O_0$ be an arbitrary rational complex number. Then, we find $\xi \in B_\delta(0)$ such that $f_\xi(z) = e$, where

$$f_\xi = \tilde{f} + g_\xi$$

and g_ξ is the function from the statement of Lemma 5.10. Then, by (1) of the lemma,

$$\begin{aligned}\varepsilon^* := \|f_\xi - \tilde{f}\|_{m_{p_0}} &\leqslant \|f_\xi - f_{p_0}\|_m + \|f_{p_0} - \tilde{f}\|_m \\ &< \frac{\varepsilon_{p_0}}{4} + \frac{\varepsilon_{p_0}}{4} = \varepsilon_{\tilde{p}_0}.\end{aligned}$$

Consider a condition $q_0' = (a, f_\xi, \varepsilon_{q_0'}, m) \in \mathbb{Q}_0(F)$, where $\varepsilon_{q_0'} < \varepsilon_{\tilde{p}_0} - \varepsilon^*$ and $\varepsilon_{q_0'}$ is small enough so that $B_{2\varepsilon_{q_0'}}(e) \subseteq O_0$. Then, $q_0' \leqslant \tilde{p}_0$ as

$$\|f_\xi - \tilde{f}\|_{m_{\tilde{p}_0}} = \varepsilon^* = \varepsilon_{\tilde{p}_0} - (\varepsilon_{\tilde{p}_0} - \varepsilon^*) < \varepsilon_{\tilde{p}_0} - \varepsilon_{q_0'}.$$

In particular, $(q_0', p_1) \leqslant (\tilde{p}_0, p_1)$. Now let $\bar{q}'' \in \mathbb{Q}_0(F) \times \mathbb{P}$ be such that $\bar{q}'' = (q_0'', q_1'') \leqslant E, (q_0', p_1)$. Let $\gamma \in (0, \varepsilon_{q_0'})$ be small enough so that for any $v \in B_\gamma(0)$ there is an entire function h_v with $\|h_v\|_{m_{q_0''}} < \varepsilon_{q_0''}$, $h_v(z) = v$ and $h_v \restriction a_{q_0''}$ constantly equals 0 (e.g. using a similar formula as in Lemma 5.10).

As $\|f_{q_0''} - f_\xi\|_{m_{q_0'}} < \varepsilon_{q_0'}$, we have that for any $v \in B_\gamma(0)$ and h_v as above,

$$\begin{aligned}\|(f_{q_0''} + h_v) - f_\xi\|_{m_{q_0'}} &\leqslant \|f_{q_0''} - f_\xi\|_{m_{q_0'}} + \|h_v\|_{m_{q_0'}} \\ &< \varepsilon_{q_0'} + \varepsilon_{q_0''} \leqslant 2\varepsilon_{q_0'}.\end{aligned}$$

Now we let $O_1 := B_\gamma(f_{q_0''}(z)) \subseteq B_{2\varepsilon_{q_0'}}(e) \subseteq O_0$. The inclusion follows, since for any $d \in B_\gamma(f_{q_0''}(z))$, $|d - e| = |f_{q_0''}(z) + h_v(z) - f_\xi(z)| < 2\varepsilon_{q_0'}$ for some $v \in B_\gamma(0)$. U is constructed in M as the union of all sets O_1 that we obtain in this way.

Let us check that this works. So working in V, let $d \in O_1$ be arbitrary. Then, there is $v \in B_\gamma(0)$ and h_v as before so that $f_{q_0''}(z) + h_v(z) = d$.

Let $q_0 = (a_{q_0}, f_{q_0}, \varepsilon_{q_0}, m_{q_0})$, where $a_{q_0} = a_{q_0''}$, $f_{q_0} = f_{q_0''} + h_v$,

$$\varepsilon_{q_0} < \varepsilon_{q_0''} - \|f_{q_0} - f_{q_0''}\|_{m_{q_0''}}$$

and $m_{q_0} = m_{q_0''}$. Then, $f_{q_0''} \restriction a_{q_0''} = f_{q_0} \restriction a_{q_0}$ as $h_v \restriction a_{q_0''}$ is constantly 0. So if we let $\bar{q} = (q_0, q_1'')$, then $\bar{q} \in \mathbb{Q}_0(F) \times \mathbb{P}$ and $\bar{q} \leqslant \bar{q}'' \leqslant E, (\tilde{p}_0, p_1)$:

$$\begin{aligned}\|f_{q_0} - f_{q_0''}\|_{m_{q_0''}} &= \varepsilon_{q_0''} - (\varepsilon_{q_0''} - \|f_{q_0} - f_{q_0''}\|_{m_{q_0''}}) \\ &< \varepsilon_{q_0''} - \varepsilon_{q_0}.\end{aligned}$$

Moreover, $f_{q_0}(z) = d$ as required. This finishes the proof of the claim. □

Proposition 5.13. *Let $M \subseteq V$ be as in Lemma 5.10, $H_0, \dots, H_n \in M$ and $H_0' \supseteq H_0, \dots, H_n' \supseteq H_n$ be partial functions from $\mathbb{C}$ to $\mathcal{P}(\mathbb{C})$ such that*

1. *$\mathrm{dom}(H_i') \setminus \mathrm{dom}(H_i) \subseteq M$, for all $i \leqslant n$;*
2. *for any pairwise distinct pairs (i_j, z_j), $j \leqslant l$, where $i_j \leqslant n$, $z_j \in \mathrm{dom}(H_{i_j}') \setminus \mathrm{dom}(H_{i_j})$, and any sequence $\langle c_j : j \leqslant l\rangle$, $c_j \in H_{i_j}'(z_j)$, we have that $\langle c_j : j \leqslant l\rangle$ is mutually Cohen generic over M.*[†]

Then, $\mathbb{Q}(H_0) \times \dots \times \mathbb{Q}(H_n) \lessdot_M \mathbb{Q}(H_0') \times \dots \times \mathbb{Q}(H_n')$.

Proof. This is an inductive argument using Lemma 5.11. Let $E \in M$ be pre-dense in $\mathbb{Q}(H_0) \times \dots \times \mathbb{Q}(H_n)$. Towards a contradiction, suppose that l is minimal such that there is $\bar{p} \in \mathbb{Q}(H_0') \times \dots \times \mathbb{Q}(H_n')$, $\bar{p} \perp E$ and $\{(i, z) : i \leqslant n, z \in a_{p_i} \setminus \mathrm{dom}(H_i)\}$ is enumerated by a sequence $\langle (i_j, z_j) : j \leqslant l\rangle$. Then, according to (2), $\langle c_j : j \leqslant l\rangle = \langle f_{p_{i_j}}(z_j) : j < l\rangle$ is mutually Cohen generic over M. In $M[c_0, \dots, c_{l-1}]$, consider

$$H_i'' = H_i \cup \{(z_j, \{c_j\}) : j < l, i_j = i\},$$

for every $i \leqslant n$. Then, by the minimality of l, E is still pre-dense in $\mathbb{Q}(H_0'') \times \dots \times \mathbb{Q}(H_n'')$. Now apply Lemma 5.11 once to accommodate the full condition $\bar{p}$ and reach a contradiction. □

5.4 | The main theorem

Theorem 5.14 (GCH). *Let κ be an infinite cardinal of uncountable cofinality. Then, there is a cofinality and cardinal preserving forcing extension in which*

1. *$2^{\aleph_0} = \kappa$,*
2. *there is a Wetzel family,*
3. *if κ is regular,* MA *holds.*

† That is, $\langle c_j : j \leqslant l\rangle$ is in any open dense subset of $\mathbb{C}^{l+1}$ coded in M.

Proof. Start with the model obtained in Proposition 4.1, where there is an almost disjoint sequence $\langle \sigma_\alpha : \alpha < \kappa \rangle$ in $\prod_{\xi<\kappa} \mu_\xi$, $\mu_\xi = \max(|\xi|, \aleph_0)$. This is our ground model V now. Fix a bookkeeping function B with domain κ. The details of B are going to be discussed at the end. We are going to recursively define a ccc finite support iteration $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \kappa \rangle$. Additionally, there will be the following objects for each $\alpha < \kappa$:

(1) $\mathbb{P}_{\alpha+1}$-names $\dot{C}_{\alpha,\xi}$, $\xi < \mu_\alpha$, for pairwise disjoint countable dense sets of complex numbers, such that any $\bar{c} \in (\bigcup_{\xi<\mu_\alpha} C_{\alpha,\xi})^{<\omega}$ is mutually Cohen generic over $V^{\mathbb{P}_\alpha}$,
(2) a countable set $X_\alpha \subseteq \kappa$,
(3) a $\mathbb{P}_\alpha$-name $\dot{z}_\alpha$ for a complex number,
(4) a $\mathbb{P}_{\alpha+1}$-name $\dot{f}_\alpha$ for an entire function, such that $\Vdash_{\mathbb{P}_{\alpha+1}} \dot{f}_\alpha(\dot{z}_\delta) \in \bigcup_{\xi<\mu_\delta} \dot{C}_{\delta,\xi}$, for all $\delta < \alpha$.

For any $\sigma \in \prod_{\xi<\alpha} \mu_\xi$, we then let $\dot{H}_\sigma$ be a $\mathbb{P}_\alpha$-name for

$$\{(z_\delta, C_{\delta,\sigma(\delta)}) : \delta < \alpha\}.$$

We will inductively prove that for any $n \in \omega$ and $\xi_0 < \cdots < \xi_n \in \kappa \setminus \bigcup_{\delta<\alpha} X_\delta$,

$$\Vdash_{\mathbb{P}_\alpha} \mathbb{Q}(\dot{H}_{\sigma_{\xi_0}\restriction\alpha}) \times \cdots \times \mathbb{Q}(\dot{H}_{\sigma_{\xi_n}\restriction\alpha}) \text{ is ccc.} \qquad (*)$$

Start with $\mathbb{P}_0 = \{\mathbb{1}\}$. Clearly $(*)$ above is satisfied when $\alpha = 0$.[†] Suppose we have constructed $\mathbb{P}_\alpha$ and we showed that $(*)$ holds. Then, we first define a forcing notion $\mathbb{P}_\alpha^+$ extending $\mathbb{P}_\alpha$.

Suppose $B(\alpha)$ is a $\mathbb{P}_\alpha$-name $\dot{\mathbb{A}}$ for a ccc poset of size $< \kappa$. In not, we simply let $\mathbb{P}_\alpha^+ = \mathbb{P}_\alpha$. In $V^{\mathbb{P}_\alpha}$, there may be $\xi_0 < \cdots < \xi_n \in \kappa \setminus \bigcup_{\delta<\alpha} X_\delta$ such that

$$\mathbb{A} \times \mathbb{Q}(H_{\sigma_{\xi_0}\restriction\alpha}) \times \cdots \times \mathbb{Q}(H_{\sigma_{\xi_n}\restriction\alpha}) \text{ is not ccc.}$$

Note then, that $\mathbb{Q}(H_{\sigma_{\xi_0}\restriction\alpha}) \times \cdots \times \mathbb{Q}(H_{\sigma_{\xi_n}\restriction\alpha})$ forces that $\mathbb{A}$ is not ccc and this remains the case in any further ccc extension, by Lemma 2.3. Also, this forces that for any $\xi'_0 < \cdots < \xi'_m \in \kappa \setminus \bigcup_{\delta<\alpha} X_\delta \cup \{\xi_0 < \cdots < \xi_n\}$,

$$\mathbb{Q}(H_{\sigma_{\xi'_0}\restriction\alpha}) \times \cdots \times \mathbb{Q}(H_{\sigma_{\xi'_m}\restriction\alpha}) \text{ is still ccc.}$$

If no such $\xi_0 < \cdots < \xi_n$ exist, then $\mathbb{A}$ preserves that all such products are ccc, again by Lemma 2.3. We let $\mathbb{P}_\alpha^+$ be $\mathbb{P}_\alpha * \dot{\mathbb{B}}$, where $\dot{\mathbb{B}}$ is a $\mathbb{P}_\alpha$-name for $\mathbb{Q}(H_{\sigma_{\xi_0}\restriction\alpha}) \times \cdots \times \mathbb{Q}(H_{\sigma_{\xi_n}\restriction\alpha})$ or $\mathbb{A}$, depending on which of the cases occur. In V, using the ccc of $\mathbb{P}_\alpha$, we let X_α^- be a countable set that contains any such $\xi_0, \ldots, \xi_n$ that we might choose in the former case. Then, $(*)$ still holds if we replace $\mathbb{P}_\alpha$ by $\mathbb{P}_\alpha^+$ and if $\xi_0 < \cdots < \xi_n$ are in $\kappa \setminus (X_\alpha^- \cup \bigcup_{\delta<\alpha} X_\delta)$, by what we have already noted.

Next, we let $\dot{z}_\alpha = B(\gamma)$, where γ is least such that $B(\gamma)$ is a $\mathbb{P}_\alpha$-name for a complex number distinct from any $\dot{z}_\delta$, $\delta < \alpha$. Let $\mathbb{C}_{\mu_\alpha}$ be the forcing for adding mutually generic Cohen reals $\langle c_{\alpha,\xi,i} : \xi < \mu_\alpha, i \in \omega \rangle$. Let $\mathbb{P}_\alpha^{++} = \mathbb{P}_\alpha^+ \times \mathbb{C}_{\mu_\alpha}$ and $\dot{C}_{\alpha,\xi}$ be a $\mathbb{P}_\alpha^{++}$-name for $\{c_{\alpha,\xi,i} : i \in \omega\}$. For any $\xi_0 < \cdots < \xi_n \in \kappa \setminus (X_\alpha^- \cup \bigcup_{\delta<\alpha} X_\delta)$, we still have

$$\Vdash_{\mathbb{P}_\alpha^{++}} \mathbb{Q}(\dot{H}_{\sigma_{\xi_0}\restriction\alpha}) \times \cdots \times \mathbb{Q}(\dot{H}_{\sigma_{\xi_n}\restriction\alpha}) \text{ is ccc,}$$

[†] $\mathbb{Q}(\emptyset)$ has a countable dense subset, for example, consisting of those conditions p, where f_p is a polynomial in rational coefficients.

since Cohen forcing preserves the ccc of any poset. Let $\eta_\alpha \in \kappa \setminus (X_\alpha^- \cup \bigcup_{\delta<\alpha} X_\delta)$ be arbitrary, let $X_\alpha = X_\alpha^- \cup \{\eta_\alpha\}$,

$$\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha^{++} * \mathbb{Q}(\dot{H}_{\sigma_{\eta_\alpha}\restriction\alpha}),$$

and $\dot{f}_\alpha$ be a name for the entire function added by $\mathbb{Q}(H_{\sigma_{\eta_\alpha}\restriction\alpha})$, as described in Lemma 5.4.

Now, for any $\xi_0 < \cdots < \xi_n \in \kappa \setminus (\bigcup_{\delta<\alpha+1} X_\delta)$,

$$\Vdash_{\mathbb{P}_{\alpha+1}} \mathbb{Q}(\dot{H}_{\sigma_{\xi_0}\restriction\alpha}) \times \cdots \times \mathbb{Q}(\dot{H}_{\sigma_{\xi_n}\restriction\alpha}) \text{ is ccc}$$

and by Proposition 5.9,

$$\Vdash_{\mathbb{P}_{\alpha+1}} \mathbb{Q}(\dot{H}_{\sigma_{\xi_0}\restriction\alpha+1}) \times \cdots \times \mathbb{Q}(\dot{H}_{\sigma_{\xi_n}\restriction\alpha+1}) \text{ is ccc.}$$

So $(*)$ holds at $\alpha+1$. It remains to show that $(*)$ is preserved in limits α. So once again, let $\xi_0 < \cdots < \xi_n \in \kappa \setminus (\bigcup_{\delta<\alpha} X_\delta)$. Then, there is $\beta < \alpha$ so that $\sigma_{\xi_i}(\delta) \neq \sigma_{\xi_j}(\delta)$ and hence following, $C_{\delta,\sigma_{\xi_i}(\delta)} \cap C_{\delta,\sigma_{\xi_j}(\delta)} = \emptyset$, for any $i < j \leqslant n$ and $\delta \in [\beta, \alpha)$. Thus, by Proposition 5.13,

$$\Vdash_{\delta+1} \mathbb{Q}(\dot{H}_{\sigma_{\xi_0}\restriction\delta}) \times \cdots \times \mathbb{Q}(\dot{H}_{\sigma_{\xi_n}\restriction\delta}) \lessdot_{V^{\mathbb{P}_\delta}} \mathbb{Q}(\dot{H}_{\sigma_{\xi_0}\restriction\delta+1}) \times \cdots \times \mathbb{Q}(\dot{H}_{\sigma_{\xi_n}\restriction\delta+1}),$$

which, according to Lemma 2.7, is equivalent to

$$\mathbb{P}_\delta * \mathbb{Q}(\dot{H}_{\sigma_{\xi_0}\restriction\delta}) \times \cdots \times \mathbb{Q}(\dot{H}_{\sigma_{\xi_n}\restriction\delta}) \lessdot \mathbb{P}_{\delta+1} * \mathbb{Q}(\dot{H}_{\sigma_{\xi_0}\restriction\delta+1}) \times \cdots \times \mathbb{Q}(\dot{H}_{\sigma_{\xi_n}\restriction\delta+1}).$$

By induction, it is then easy to see that for any limit $\delta \in [\beta, \alpha]$, $\mathbb{P}_\delta * \mathbb{Q}(\dot{H}_{\sigma_{\xi_0}\restriction\delta}) \times \cdots \times \mathbb{Q}(\dot{H}_{\sigma_{\xi_n}\restriction\delta})$ is the direct limit of the forcings $\mathbb{P}_{\delta'} * \mathbb{Q}(\dot{H}_{\sigma_{\xi_0}\restriction\delta'}) \times \cdots \times \mathbb{Q}(\dot{H}_{\sigma_{\xi_n}\restriction\delta'})$, for $\delta' \in [\beta, \delta)$ and for $\delta < \alpha$, we know already by $(*)$ that[†]

$$\mathbb{P}_\delta * \mathbb{Q}(\dot{H}_{\sigma_{\xi_0}\restriction\delta}) \times \cdots \times \mathbb{Q}(\dot{H}_{\sigma_{\xi_n}\restriction\delta}) \text{ is ccc.}$$

Thus, by Lemma 2.6, $\mathbb{P}_\alpha * \mathbb{Q}(\dot{H}_{\sigma_{\xi_0}\restriction\alpha}) \times \cdots \times \mathbb{Q}(\dot{H}_{\sigma_{\xi_n}\restriction\alpha})$ is itself ccc. In particular, $(*)$ now follows, by Lemma 2.3.

This finishes the construction. The bookkeeping function B is supposed to enumerate all $\mathbb{P}$-names for complex numbers, and in case κ is regular, also all $\mathbb{P}$-names for ccc forcings on ordinals $< \kappa$ unboundedly often. This is a standard argument. When κ is regular, it suffices to let B enumerate all elements of $H(\kappa)$ unboundedly often. A standard argument then shows that MA holds after forcing with $\mathbb{P}$. When κ is not regular, let B enumerate all elements of $\mathcal{P}_\omega^\kappa(\kappa)$, where $\mathcal{P}_\omega^0(\kappa) = \kappa$, $\mathcal{P}_\omega^{\alpha+1}(\kappa) = \mathcal{P}_\omega^\alpha(\kappa) \cup [\mathcal{P}_\omega^\alpha(\kappa)]^{\leqslant\omega}$, and we take unions at limits. Since $2^{\aleph_0} = \kappa$, $|\mathcal{P}_\omega^\kappa(\kappa)| = \kappa$ as well. It is then standard to see that $\mathbb{P} \subseteq \mathcal{P}_\omega^\kappa(\kappa)$, using the ccc and choosing appropriate names for $\dot{\mathbb{Q}}_\alpha$, $\alpha < \kappa$.

Finally, after forcing with $\mathbb{P}$, we have that $\langle z_\alpha : \alpha < \kappa \rangle$ enumerates the complex numbers and for every $\delta, \alpha < \kappa$,

$$f_\alpha(z_\delta) \in \{f_\beta(z_\delta) : \beta \leqslant \delta\} \cup \bigcup_{\xi < \mu_\delta} C_{\delta,\xi},$$

which has size $\leqslant |\delta| + \mu_\delta \cdot \aleph_0 = \mu_\delta < \kappa = 2^{\aleph_0}$. Thus, $\mathcal{F} = \{f_\alpha : \alpha < \kappa\}$ is a Wetzel family. □

[†] To be slightly more accurate, the direct limit is a dense sub-forcing of $\mathbb{P}_\delta * \mathbb{Q}(\dot{H}_{\sigma_{\xi_0}\restriction\delta}) \times \cdots \times \mathbb{Q}(\dot{H}_{\sigma_{\xi_n}\restriction\delta})$.

Let us note that it is not very important that the σ_α's had finite pairwise intersections and we could easily get by with assuming only countable intersections. In that case, we would just have to split up the limit stages into countable and uncountable cofinalities. The proof for uncountable cofinality stays the same and for countable cofinality the ccc follows easily from the previous steps.

Also, some interesting modifications can be made to the forcing construction above. For example, instead of taking care of all complex numbers along the iteration, we can leave out some values. For example, we may leave out exactly 0. The resulting family $\mathcal{F}$ is then a Wetzel family on the modified domain $\mathbb{C} \setminus \{0\}$, while all values $f(0)$, for $f \in \mathcal{F}$, are pairwise distinct. To see this, note that if $z \notin \operatorname{dom} H$, then the function added by $\mathbb{Q}(H)$ maps to a generic complex number at z.

More generally, for any given infinite $\Omega \subseteq \mathbb{C}$, we can construct a family of entire functions that is Wetzel on Ω, while attaining $2^{\aleph_0}$-many values at any point outside of Ω. For κ regular, we can force $\diamondsuit_\kappa$ over the model from Proposition 4.1 without collapsing cardinals or changing cofinalities, by adding a κ-Cohen real in the standard way. This does not affect the result of Proposition 4.1. Then, using a standard guessing argument in the iteration of Theorem 5.14, it should not be hard to modify the construction in order to obtain a model where such families exist for every infinite subset $\Omega \subseteq \mathbb{C}$. Whenever $\Omega_\alpha \subseteq \langle z_\beta : \beta < \alpha \rangle$ is guessed at step α, we may force with $\mathbb{Q}(H_{\sigma_{\eta_\alpha} \restriction \alpha} \restriction \Omega_\alpha)$ instead. Here note that if $\mathbb{Q}(H)$ is ccc, then for any restriction $H' \subseteq H$, $\mathbb{Q}(H')$ is a sub-forcing (not necessarily complete) of $\mathbb{Q}(H)$ and thus also ccc.

6 | MA AND UNIVERSAL SETS

In this section, we show that under $\mathsf{MA} + \neg\mathsf{CH}$ there is no universal set. Recall that MA is saying that for any ccc partial order $\mathbb{P}$ and any family $\mathcal{D}$ of less than $2^{\aleph_0}$-many dense subsets of $\mathbb{P}$, there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$, for every $D \in \mathcal{D}$.

We begin by introducing the ccc poset that we will use. The forcing $\mathbb{S}$ shall consist of conditions of the form $p = (w, s) = (w_p, s_p)$, where $w \in [\mathbb{C}]^{<\omega}$ and s is a finite sequence of open intervals with rational endpoints in $(0, 1) \subseteq \mathbb{R}$ such that

(a) $\forall i < j < |s| \forall x \in s(i) \forall y \in s(j)(y < x^8)$,
(b) $\forall z_0, z_1 \in w(|z_0 - z_1| \in \bigcup_{i<|s|} s(i) \cup \{0\})$.

A condition q extends p if and only if $w_p \subseteq w_q$ and $s_p \subseteq s_q$.

Lemma 6.1. *$\mathbb{S}$ is ccc.*

Proof. Suppose $p_\alpha \in \mathbb{S}$, for $\alpha < \omega_1$ are pairwise incompatible. Then, we may assume without loss of generality that $s_{p_\alpha} = s$ and $|w_{p_\alpha}| = n$ is the same for all $\alpha < \omega_1$. Let us write $w_{p_\alpha} = \{z_i^\alpha : i < n\}$, for every α and consider the function $d : \mathbb{C}^n \times \mathbb{C}^n \to [0, \infty)^{n \times n}$, where

$$d(\langle z_i : i < n \rangle, \langle z_i' : i < n \rangle) = \langle |z_i - z_j'| : (i, j) \in n \times n \rangle.$$

d is clearly continuous. Moreover, since $\{\langle z_i^\alpha : i < n \rangle : \alpha < \omega_1\} \subseteq \mathbb{C}^n$ is uncountable, there is some $\alpha < \omega_1$ such that $\langle z_i^\alpha : i < n \rangle$ is an accumulation point of that set. Thus, let $\alpha_k \neq \alpha$, $k \in \omega$, be such that $\langle z_i^{\alpha_k} : i < n \rangle \to \langle z_i^\alpha : i < n \rangle$ as $k \to \infty$. Let c be the left endpoint of $s(|s| - 1)$, that is, c is the the infimum of $\bigcup_{i<|s|} s(i)$. Now note that

$$d(\langle z_i^{\alpha} : i < n\rangle, \langle z_i^{\alpha} : i < n\rangle) \in \left(\bigcup_{i<|s|} s(i) \cup [0, c^8)\right)^{n\times n}.$$

Thus, by continuity, there is k large enough so that

$$d(\langle z_i^{\alpha_k} : i < n\rangle, \langle z_i^{\alpha} : i < n\rangle) \in \left(\bigcup_{i<|s|} s(i) \cup [0, c^8)\right)^{n\times n}.$$

In other words, for all $z_0, z_1 \in w_{p_\alpha} \cup w_{p_{\alpha_k}}$, $|z_0 - z_1| \in \bigcup_{i<|s|} s(i) \cup [0, c^8)$. Let $0 < b < c^8$, where b is strictly bigger than the maximal distance between points in $w_{p_\alpha} \cup w_{p_{\alpha_k}}$ that lies in $(0, c^8)$. Similarly, let $0 < a < b$, where a is strictly smaller than the minimal such distance. Letting I be the interval (a, b), $(w_{p_\alpha} \cup w_{p_{\alpha_k}}, s^\frown I)$ is a condition extending both p_α and p_{α_k}, while $\alpha \neq \alpha_k$. This is a contradiction to the assumption that $p_\alpha \perp p_{\alpha_k}$. □

In the following, Q is the set of rational numbers.

Lemma 6.2. *For every $z \in \mathbb{C}$ and every $n \in \omega$, the sets $D_z = \{q \in \mathbb{S} : z \in w_q + (Q + iQ)\}$ and $E_n = \{q \in \mathbb{S} : |s_q| \geqslant n\}$ are dense in $\mathbb{S}$.*

Proof. Let $p \in \mathbb{S}$ be arbitrary. If $w_p = \emptyset$, then clearly $q = (\{z\}, s_p)$ extends p and lies in D_z. Otherwise, there is $z_0 \in w_p$ and a small open neighbourhood O of z_0 so that for any $z_1 \in O$, there is an extension q of p with $z_1 \in w_q$. This is similar to the argument in the proof of Lemma 6.1. O clearly contains a rational translate of z. The case of E_n is obvious. □

Lemma 6.3. *Let $O \subseteq (0, \infty)$ be open containing arbitrarily small values. Then, there is an uncountable set $X \subseteq \mathbb{R}$ such that $\{|z_0 - z_1| : z_0, z_1 \in X, z_0 \neq z_1\} \subseteq O$.*

Proof. By recursion construct a Cantor scheme, that is, a map φ from $2^{<\omega}$ to non-empty open intervals of $\mathbb{R}$, such that for every $t \subseteq t'$, $\overline{\varphi(t')} \subseteq \varphi(t)$, $\operatorname{diam}(\varphi(t)) \leqslant \frac{1}{|t|+1}$ and $\varphi(t^\frown 0) \cap \varphi(t^\frown 1) = \emptyset$. Start with $\varphi(\emptyset) = (0, 1)$ and given $\varphi(t)$, find $\varphi(t^\frown 0)$ and $\varphi(t^\frown 1)$ such that for every $x \in \varphi(t^\frown 0)$, $y \in \varphi(t^\frown 1)$, $|x - y| \in O$. This is possible since O is open and contains arbitrarily small numbers > 0. Clearly, $X = \bigcap_{n\in\omega} \bigcup_{t\in 2^n} \varphi(t)$ works. □

Lemma 6.4 (essentially [1, Proposition 9.4]). *Let $X, Y \subseteq \mathbb{C}$, X uncountable, and assume that for every $x \in \{|z_0 - z_1| : z_0, z_1 \in X, z_0 \neq z_1\}$, $y \in \{|z_0 - z_1| : z_0, z_1 \in Y, z_0 \neq z_1\}$,*

$$\min(x, y) < \max(x, y)^2.$$

Then, there is no non-constant entire function f such that $f''X \subseteq Y$.

Proof. Suppose there is such f. Since f is non-constant, we can find an accumulation point $x \in X$ of X such that $f'(x) \neq 0$. Let $x_n \to x$, where $x_n \in X$ for every $n \in \omega$. Then there is $\langle n_k : k \in \omega\rangle$

such that for all k,

$$|x_{n_k} - x| < |f(x_{n_k}) - f(x)|^2$$

or for all k,

$$|f(x_{n_k}) - f(x)| < |x_{n_k} - x|^2.$$

The former is impossible since then

$$\frac{|f(x_{n_k}) - f(x)|}{|x_{n_k} - x|} \geqslant \frac{|f(x_{n_k}) - f(x)|}{|f(x_{n_k}) - f(x)|^2} = \frac{1}{|f(x_{n_k}) - f(x)|}$$

which does not converge, and from the latter, we follow that

$$\frac{|f(x_{n_k}) - f(x)|}{|x_{n_k} - x|} \leqslant \frac{|x_{n_k} - x|^2}{|x_{n_k} - x|} = |x_{n_k} - x|$$

which converges to 0. This contradicts that $f'(x) \neq 0$. □

Theorem 6.5. MA + ¬CH *implies that there is no universal set.*

Proof. Let $Y \subseteq \mathbb{C}$, $|Y| < 2^{\aleph_0}$. Using MA, find a filter $G \subseteq \mathbb{S}$ intersecting all sets D_z for $z \in Y$ and E_n for $n \in \omega$ from Lemma 6.2. Let $Z = \bigcup_{p \in G} w_p$, $s = \bigcup_{p \in P} s_p$ and $U = \bigcup_{n \in \omega} s(n)$. Then, $Y \subseteq Z + (Q + iQ)$ and $\{|z_0 - z_1| : z_0, z_1 \in Z, z_0 \neq z_1\} \subseteq U$. Let a_n be the left endpoint of $s(n)$, for every $n \in \omega$ and consider $O = \bigcup_{n \in \omega}(a_n^4, a_n^2)$. Then, note that for every $x \in U$, $y \in O$,

$$\min(x, y) < \max(x, y)^2.$$

Finally apply Lemma 6.3 to find a set $X \subseteq \mathbb{C}$ of size $\aleph_1$ such that $\{|z_0 - z_1| : z_0, z_1 \in X, z_0 \neq z_1\} \subseteq O$. We claim that there is no entire f such that $f''X \subseteq Y$. Otherwise, as $Y \subseteq Z + (Q + iQ)$, there is an uncountable $X' \subseteq X$ and there are rationals r_0, r_1, such that

$$f''X \subseteq Z + (r_0 + ir_1).$$

Clearly, $\{|z_0 - z_1| : z_0, z_1 \in Z + (r_0 + ir_1), z_0 \neq z_1\} = \{|z_0 - z_1| : z_0, z_1 \in Z, z_0 \neq z_1\} \subseteq U$. This contradicts Lemma 6.4 □

Corollary 6.6. *The existence of a Wetzel family does not imply the existence of a universal set.*

Proof. Taking $\kappa = \aleph_2$, this follows from Theorem 6.5 and Theorem 5.14. Note that if we are allowed to assume the existence of a weakly inaccessible cardinal, this already follows from the main result in combination with Proposition 3.6. □

7 | A UNIVERSAL SET WITH $2^{\aleph_0} = \aleph_2$

The construction in the following section will use a countable support iteration of proper forcing notions. For more information, we refer the reader to [20, Chapter 31].

Theorem 7.1 (CH). *There is a proper forcing extension of V preserving all cardinals and cofinalities in which $\mathbb{C}^V$ is universal and $2^{\aleph_0} = \aleph_2$. In particular, the existence of a universal set is consistent with $2^{\aleph_0} = \aleph_2$.*

We will construct a forcing notion that uses models as side conditions. For now let us fix an arbitrary set Y of complex numbers. We then say that a pair (M, N) is a node, if $(M, \in, Y \cap M), (N, \in, Y \cap N)$ are countable elementary submodels of $(H(\omega_1), \in, Y)$ and $(M, \in, Y \cap M) \in N$. In particular, M and N are transitive. A side condition is a finite set $s = \{(M_i, N_i) : i \leqslant n\}$ of nodes, where $(M_i, N_i) \in M_{i+1}$ for every $i < n$.

Recall the poset $\mathbb{Q}$ from Definition 5.1. The forcing $\mathbb{P}(Y)$ then consists of pairs (w, s) where $s = \{(M_i, N_i) : i \leqslant n\}$ is a side condition, $w = (a, f, \varepsilon, m) \in M_n \cap \mathbb{Q}$, $a \subseteq M_{n-1}$ and for every $i < n$, $\langle f(z) : z \in a \cap (M_i \setminus \bigcup_{j<i} M_j)\rangle \subseteq M_{i+1} \cap Y$ is mutually Cohen generic over N_i. In other words, the elements of a that appear in a model M_i, but not before, are mapped to mutually generic-over-N_i complex numbers that lie in $Y \cap M_{i+1}$. A condition (v, t) extends (w, s) if v extends w in $\mathbb{Q}$ and $s \subseteq t$.

Lemma 7.2. *$\mathbb{P}(Y)$ is proper.*

Proof. Let $K \preccurlyeq H(\theta)$ be countable with $Y \in K$, for some large θ. Furthermore, let $K \in K^+ \preccurlyeq H(\theta)$ be another countable model. Consider $M = K \cap H(\omega_1)$ and $N = K^+ \cap H(\omega_1)$. Then, (M, N) is a node, since both M and $Y \cap M$ are hereditarily countable in K^+ and thus elements of N. Elementarity of $(M, \in, Y \cap M)$ and $(N, \in, Y \cap N)$ follows easily from the definability of $H(\omega_1)$ within K, K^+ and the elementarity of K, K^+. Any subset of M that lies in K^+ is an element of N, since M is countable in K^+. In particular, $\mathbb{P}(Y) \cap K = \mathbb{P}(Y) \cap M \in N$. Moreover, for any subset $A \in K$ of $\mathbb{P}(Y)$, $A \cap K = A \cap M \in N$.

In the following, $r \parallel A$ means that r is compatible with some element of A.

Claim 7.3. Let $A \in N$ be any pre-dense subset of $\mathbb{P}(Y) \cap M$ and r be any condition of the form $r = (w, s \cup \{(M, N)\} \cup t)$, where $s \in M$ and $t \cap M = \emptyset$. In other words, r is any condition that contains (M, N) in its side condition. Then, $r \parallel A$.

Proof. We proceed by induction on the length of t. If t is empty, then $(w, s) \in \mathbb{P}(Y) \cap M$. By assumption, $(w, s) \parallel A$ since A is pre-dense in $\mathbb{P}(Y) \cap M$. Thus, there is $(w', s') \leqslant (w, s)$ in M, extending an element of A. Then, $(w', s' \cup \{(M, N\}))$ is a condition extending $(w, s \cup \{(M, N)\})$ and that same element of A.

Now suppose $t = t_0 \cup \{(M_1, N_1)\}$, where $t_0 \in M_1$ and let (M_0, N_0) be the last node of t_0, or (M, N) if $t_0 = \emptyset$. Let

$$\mathbb{Q}_0 = \begin{cases} \{w' \in \mathbb{Q} : \exists s' \in M, s \subseteq s'((w', s') \in \mathbb{P}(Y))\} & \text{if } t_0 = \emptyset \\ \{w' \in \mathbb{Q} : \exists s' \in M, s \subseteq s'(w', s' \cup \{(M, N)\} \cup t_0) \in \mathbb{P}(Y)\} & \text{otherwise.} \end{cases}$$

Then, note that $\mathbb{Q}_0 \in N_0$ is a forcing of the form $\mathbb{Q}_0(F)$ for $F \in N_0$.[†] Furthermore, let $E = \{w' \in \mathbb{Q} : \exists s'((w', s') \in A)\}$. Then, $E \subseteq \mathbb{Q}_0$, $E \in N_0$ and by the inductive hypothesis, it is pre-dense in

[†] To see that $\mathbb{Q}_0, F \in N_0$, note that the subset of M_0 consisting of countable elementary sub-models of $(H(\omega_1), \in, Y)$ can be defined in N_0 as the elements of M_0 that are elementary sub-models of $(M_0, \in, Y \cap M_0)$.

$\mathbb{Q}_0$. Now use Lemma 5.11, for $\mathbb{P}, \mathbb{P}'$ trivial forcings, and a similar argument as in Proposition 5.13 to finish the inductive step. □

Finally, if $(w, s) \in K$ is an arbitrary condition, then we have shown that $(w, s \cup \{(M, N)\})$ is a master condition over K. This proves the lemma. □

Lemma 7.4. *Let $p, Y \in K \preccurlyeq H(\theta)$, K countable, for large θ, and let c be a Cohen real over K. Then, there is a master condition $q \leqslant p$ over K so that $q \Vdash c$ is Cohen over $K[\dot{G}]$.*

This is known as 'almost preserving $\sqsubseteq^{\text{Cohen}}$' in [4, section 6.3.C] and implies the preservation of non-meager sets in countable support iterations.

Proof. Let $M = K \cap H(\omega_1)$ and H be a $\text{Coll}(\omega, \lambda)$-generic over $K[c]$, where $\lambda = |H(\omega_1)|^K$. Then, c is still a Cohen real over $K[H]$ and $K[H] \models |M| = \omega$. Moreover, $\mathbb{P}(Y) \cap K = \mathbb{P}(Y) \cap M \in K[H]$, since $\mathbb{P}(Y) \cap M$ is definable from $M, Y \cap M \in K[H]$. Now let $K^+ \preccurlyeq H(\theta)$ be countable with $K, H, c \in K^+$. Let $p = (w, s)$ and $N = K^+ \cap H(\omega_1)$. Then, $q = (w, s \cup \{(M, N)\})$ is a master condition over K, as in the proof of Lemma 7.2. Let $G \ni q$ be $\mathbb{P}(Y)$-generic over V. According to Claim 7.3, for any pre-dense subset $A \in K^+$ of $\mathbb{P}(Y) \cap K$, $A \cap G \neq \emptyset$. Thus, $G \cap K$ is $\mathbb{P}(Y) \cap K$-generic over K^+ and in particular over $K[H][c] \subseteq K^+$. But $\mathbb{P}(Y) \cap K$ is a countable forcing in $K[H]$ and thus equivalent to Cohen forcing (or, in the simplest case, a trivial forcing) witnessed through an isomorphism in $K[H]$. So $K[H][c][G \cap K]$ is a Cohen extension of $K[H][c]$, and c is still Cohen generic over $K[H][G \cap K]$. In particular, c is still Cohen generic over $K[G] = K[G \cap K] \subseteq K[H][G \cap K]$. □

Lemma 7.5. *Let $Y \subseteq \mathbb{C}$ be everywhere non-meager. Then, $\mathbb{P}(Y)$ generically adds a non-constant entire function f such that $f(\mathbb{C}^V) \subseteq Y$.*

Proof. Let $(w, s) \in \mathbb{P}(Y)$ and $z \in \mathbb{C}$. Extending (w, s) further, we can assume $z \in M_0$ for some $(M_0, N_0) \in s$, where M_0 is minimal with this property and there is a successor $(M_1, N_1) \in s$ of (M_0, N_0). Since Y is everywhere non-meager, M_1 knows this and since N_0 is countable in M_1, there is $c \in Y \cap M_1$ that is arbitrarily close to $f_w(z)$ and Cohen generic over $N_0[c_0, \dots, c_n]$, where $c_0, \dots, c_n$ enumerates the mutual Cohen generics $f_w(x)$, for $x \in a_w$ that first appear in M_0. The rest then follows as in the proof of Lemma 5.4. □

Proof of Theorem 7.1. Let $Y = \mathbb{C}^V$ and iterate $\mathbb{P}(Y)$ in a countable support iteration of length ω_2. By Lemma 7.4 and [4, Lemma 6.3.17, 6.3.20], Y stays everywhere non-meager along the iteration. Everything else follows from standard counting of names arguments and the fact that $\mathbb{P}(Y)$ has size $\aleph_1$ under CH. □

It seems that it would also suffice to consider nodes where M, N are merely countable transitive models of ZF^-, $(M, \in, Y \cap M)$ is countable in N and $Y \cap M$ is non-meager in M. This has the slight advantage that $\mathbb{P}(Y) \cap M$ is already a definable subclass of M and not just definable in N. Also in the proof of Lemma 7.4 we could directly let $K^+ = K[H][c]$.

There is a also a ccc way to do this starting from a model of $\diamondsuit$. This is a modification of the construction presented in [8], from which our result draws its main inspiration. Instead of generically adding an $\in$-increasing sequence of nodes using side conditions, we start already with a given sequence $\langle (M_\alpha, N_\alpha) : \alpha < \omega_1 \rangle$, where $\langle N_\alpha : \alpha < \omega_1 \rangle$ is an 'oracle' (see [30]). It can then be shown that the resulting forcing, consisting of those $w \in \mathbb{Q}$ such that $(w, s) \in \mathbb{P}(Y)$ for every $s \subseteq \langle (M_\alpha, N_\alpha) : \alpha < \omega_1 \rangle$, is 'oracle-cc'. In fact, our proper forcing is built directly from this construction. The advantage is that it has a much easier setup and does not depend on a particular chosen sequence of nodes. Also there might be a bigger potential of generalising it to continuum higher than $\aleph_2$, although this is not very clear to us.

8 | OPEN QUESTIONS

Question 8.1. Does MA or PFA imply that there is a Wetzel family? Is $\mathsf{MA} + 2^{\aleph_0} = \aleph_2$ sufficient?

Recall that $\mathrm{non}(\mathcal{M})$ is the least size of a non-meager set.

Question 8.2. Is every universal set non-meager under $\neg\mathsf{CH}$? In particular, can we replace MA with $\mathrm{non}(\mathcal{M}) = 2^{\aleph_0}$ in Theorem 6.5?

Question 8.3. Is the existence of a universal set consistent with $2^{\aleph_0} = \aleph_3$? With $2^{\aleph_0} = \kappa$ for arbitrary successor cardinal κ?

Recall that a domain $\Omega \subseteq \mathbb{C}$ is any open connected subset of $\mathbb{C}$. We may then define the analogous notion of Wetzel families on Ω for functions that are holomorphic on Ω.

Question 8.4. Let $\Omega \subset \mathbb{C}$ be any domain and suppose that there is a Wetzel family on Ω. Does there exist a Wetzel family on the whole of $\mathbb{C}$? What about $\Omega = \mathbb{C} \setminus \{0\}$?

Question 8.5. Can we characterise when $\mathbb{Q}(H)$ is ccc?

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REFERENCES

1. U. Abraham, M. Rubin, and S. Shelah, *On the consistency of some partition theorems for continuous colorings, and the structure of $\aleph_1$-dense real order types*, Ann. Pure Appl. Logic **29** (1985), no. 2, 123–206.
2. K. F. Barth and W. J. Schneider, *Entire functions mapping countable dense subsets of the reals onto each other monotonically*, J. Lond. Math. Soc. (2) **2** (1970), 620–626.
3. K. F. Barth and W. J. Schneider, *Entire functions mapping arbitrary countable dense sets and their complements onto each other*, J. Lond. Math. Soc. (2) **4** (1972), 482–488.
4. T. Bartoszyński and H. Judah, *Set theory*, A K Peters, Ltd., Wellesley, MA, 1995. On the structure of the real line.
5. J. E. Baumgartner, *Almost-disjoint sets the dense set problem and the partition calculus*, Ann. of Math. Logic **9** (1976), no. 4, 401–439.
6. B. L. Bohl, *Propositions equivalent to the continuum hypothesis*, Report, Utah State University, 1968.
7. J. Brendle and V. Fischer, *Mad families, splitting families and large continuum*, J. Symb. Log. **76** (2011), no. 1, 198–208.
8. M. R. Burke, *Entire functions mapping uncountable dense sets of reals onto each other monotonically*, Trans. Amer. Math. Soc. **361** (2009), no. 6, 2871–2911.
9. G. Cantor, *Ein Beitrag zur Mannigfaltigkeitslehre*, J. Reine Angew. Math. **84** (1877), 242–258.
10. W. Cherry and Z. Ye, *Nevanlinna's theory of value distribution*, Springer, Berlin, Heidelberg, 2001.
11. B. Cody, S. Cox, and K. Lee, *Sparse analytic systems*, Forum Math. Sigma **11** (2023), e58.
12. P. Cohen, *The independence of the continuum hypothesis*, Proc. Natl. Acad. Sci. USA **50** (1963), 1143–1148.
13. P. Erdös, *An interpolation problem associated with the continuum hypothesis*, Michigan Math. J. **11** (1964), no. 1, 9–10.
14. J.-Cl. Evard and F. Jafari, *A complex Rolle's theorem*, Amer. Math. Monthly **99** (1992), no. 9, 858–861.
15. A. H. Fraenkel, *Zu den Grundlagen der Cantor-Zermeloschen Mengenlehre*, Math. Ann. **86** (1922), no. 3–4, 230–237.
16. S. R. Garcia and A. L. Shoemaker, *Wetzel's problem, Paul Erdös and the continuum hypothesis: a mathematical mystery*, Notices Amer. Math. Soc. **62** (2015), 243–247.
17. K. Gödel, *The consistency of the continuum hypothesis*, Annals of Mathematics Studies, no. 3. Princeton University Press, Princeton, NJ, 1940.
18. K. Gödel, *What is Cantor's continuum problem?* Amer. Math. Monthly **54** (1947), no. 9, 515–525.
19. J. Gray, *The real and the complex: a history of analysis in the 19th century*, Springer International Publishing, Cham, 2015.
20. T. Jech, *Set theory*, Springer, Berlin, Heidelberg, 2003.
21. A. S. Kechris and A. Louveau, *Descriptive set theory and the structure of sets of uniqueness*, London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1987.
22. A. Kumar and S. Shelah, *On a question about families of entire functions*, Fund. Math. **239** (2017), no. 3, 279–288.
23. K. Kunen, *Set theory*, Studies in Logic: Mathematical, College Publications, London, 2011.
24. W. D. Maurer, *Conformal equivalence of countable dense sets*, Proc. Amer. Math. Soc. **18** (1967), 269–270.
25. J. W. Nienhuys and J. Thiemann, *On the existence of entire functions mapping countable dense sets onto each other*, Indag. Math **38** (1976), 331–334.
26. L. C. Paulson, *Wetzel: Formalisation of an undecidable problem linked to the continuum hypothesis*. Intelligent Computer Mathematics, 2022, pp. 92–106, DOI 10.1007/978-3-031-16681-5_6
27. W. Rudin, *Real and complex analysis*, 3rd ed., McGraw-Hill Book Co., New York, 1987.
28. D. Sato and S. Rankin, *Entire functions mapping countable dense subsets of the reals onto each other monotonically*, Bull. Aust. Math. Soc. **10** (1974), no. 1, 67–70.
29. S. Shelah, *Independence results*, J. Symb. Log. **45** (1980), no. 3, 563–573.
30. S. Shelah, *Proper and improper forcing*, Perspectives in Mathematical Logic, 2nd ed., Springer, Berlin, 1998.
31. W. Sierpiński, *Hypothèse du continu*, 2nd edition, Chelsea Publishing Co., New York, 1956.
32. R. M. Solovay and S. Tennenbaum, *Iterated Cohen extensions and Souslin's problem*, Ann. of Math. **94** (1971), no. 2, 201–245.
33. J. Zapletal, *Strongly almost disjoint functions*, Israel J. Math. **97** (1997), no. 1, 101–111.
34. E. Zermelo, *Beweis, daßjede Menge wohlgeordnet werden kann*, Math. Ann. **59** (1904), no. 4, 514–516.