Communications in Contemporary Mathematics
Vol. 16, No. 3 (2014) 1350030 (16 pages)

DOI: 10.1142/S0219199713500302

Radially symmetric solutions to the Hénon–Lane–Emden system on the critical hyperbola

Roberta Musina

Dipartimento di Matematica ed Informatica, Università di Udine
via delle Scienze, 206-33100 Udine, Italy
roberta.musina@uniud.it

K. Sreenadh

Department of Mathematics, Indian Institute of Technology Delhi
Hauz Khas, New Delhi 110016, India
sreenadh@maths.iitd.ac.in

Received 19 December 2012
Revised 23 June 2013
Accepted 26 June 2013
Published 26 September 2013

We use variational methods to study the existence of non-trivial and radially symmetric solutions to the Hénon–Lane–Emden system with weights, when the exponents involved lie on the "critical hyperbola". We also discuss qualitative properties of solutions and non-existence results.

Keywords: Weighted Lane–Emden system; critical hyperbola; Rellich inequality; Sobolev inequality; fourth-order ordinary differential equations; Hamiltonian systems.

Mathematics Subject Classification 2010: 35B09, 35B40, 35B33

1. Introduction

In this paper we discuss the existence, non-existence and qualitative properties of non-trivial radially symmetric solutions u, v to the following weakly coupled system on the punctured space $\mathbb{R}^n \setminus \{0\}$:

$$\begin{cases} -\Delta u = |x|^a |v|^{p-2} v, \\ -\Delta v = |x|^b |u|^{q-2} u. \end{cases} \tag{1.1}$$

Here $n \geq 2$, $a, b \in \mathbb{R}$, $p, q > 1$ belong to the *weighted critical hyperbola*

$$\frac{a+n}{p} + \frac{b+n}{q} = n - 2, \tag{1.2}$$

and that satisfy the standard anticoercivity assumption

$$\frac{1}{p}+\frac{1}{q}<1. \tag{1.3}$$

The Hénon–Lane–Emden system (1.1) is a largely studied problem. In the autonomous case $a=b=0$, Lions proved in [26] the existence of a solution $u \in \mathcal{D}^{2,q'}(\mathbb{R}^n)$, $v \in \mathcal{D}^{2,p'}(\mathbb{R}^n)$ to

$$\begin{cases} -\Delta u = |v|^{p-2}v, \\ -\Delta v = |u|^{q-2}u, \end{cases} \tag{1.4}$$

under the assumptions $n > n/p + n/q = n-2 > 0$. We quote also the paper [21] by Hulshof–Van der Vorst, for additional qualitative properties of the pair u, v.

The role of the "*critical hyperbola*" was first pointed out by Mitidieri [27, 28] for the autonomous case $a=b=0$ (see also [35]). It turns out that (1.4) has no positive, radial solutions $u, v \in C^2(\mathbb{R}^n)$ if p, q are below the critical hyperbola. On the other hand, Serrin and Zou used shooting methods in [36] to prove that (1.1) admits infinitely many positive radial solutions u, v which tend to 0 as $|x| \to +\infty$, provided that the pair p, q is on or above the critical hyperbola.

The *Hénon–Lane–Emden conjecture* has been raised in [34, 12] for a more general class of higher-order system. It says in particular that there is no positive solution for system (1.1) if p, q are under the critical hyperbola. Bidaut-Véron and Giacomini have recently shown in [4] that if $n \ge 3$, $a, b > -2$, then the system (1.1) admits a positive classical radial solution u, v with u, v continuous at the origin if and only if (p, q) is above or on the critical hyperbola. We recall that by [3, Proposition 2.1], no solution to (1.1) is continuous at the origin if $a \le -2$ or $b \le -2$.

Remarkable results about the Hénon–Lane–Emden conjecture have been recently obtained also in [8, 17, 18, 32, 33, 35, 37].

Finally, we recall that the weighted critical hyperbola enters in a natural way in the context of the solvability of Hardy–Hénon type elliptic systems in bounded domains, see, for instance, [16, 9] and the recent papers [5, 6].

In the present paper we first use variational methods to extend the Lions existence theorem to the non-autonomous case. Then we discuss non-existence results. We always assume that (1.2) and (1.3) are satisfied. We limit ourselves to state here some of our results, and we postpone more precise statements to Sec. 3.

Existence. If $a \neq -n$ and $b \neq -n$, then (1.1) has a non-trivial radial solution u, v such that

$$\int_{\mathbb{R}^n} |x|^{-\frac{a}{p-1}} |\Delta u|^{p'} dx < \infty, \quad \int_{\mathbb{R}^n} |x|^{-\frac{b}{q-1}} |\Delta v|^{q'} dx < \infty.$$

Moreover, it holds that

$$\lim_{|x|\to\infty} |x|^{\frac{b+n}{q}} u(x) = \lim_{x\to 0} |x|^{\frac{b+n}{q}} u(x) = \lim_{|x|\to\infty} |x|^{\frac{a+n}{p}} v(x) = \lim_{x\to 0} |x|^{\frac{a+n}{p}} v(x) = 0, \tag{1.5}$$

and u, v are both positive if and only if $a > -n$ and $b > -n$.

Non-existence. Let u, v be a solution to (1.1) on $\mathbb{R}^n \setminus \{0\}$ and assume that either

$$\begin{cases} \lim\limits_{|x|\to\infty} |x|^{\frac{b+n}{q}} u(x), \quad \lim\limits_{|x|\to\infty} |x|^{\frac{a+n}{p}} v(x) \quad \text{exist and that are finite, or} \\ \lim\limits_{x\to 0} |x|^{\frac{b+n}{q}} u(x), \quad \lim\limits_{x\to 0} |x|^{\frac{a+n}{p}} v(x) \quad \text{exist and that are finite.} \end{cases} \tag{1.6}$$

If $a \leq -n$ or $b \leq -n$ and if $u \geq 0, v \geq 0$ then $u \equiv v \equiv 0$.

Let us briefly describe our approach. It has already been noticed for instance in [3, 4, 8] that radial solutions to (1.1) are in one-to-one correspondence with trajectories g, f of the Hamiltonian system

$$\begin{cases} -g'' + 2Ag' + \Gamma g = |f|^{p-2} f \quad \text{on } \mathbb{R}, \\ -f'' - 2Af' + \Gamma f = |g|^{q-2} g \quad \text{on } \mathbb{R} \end{cases} \tag{1.7}$$

for suitable constants $A, \Gamma \in \mathbb{R}$ depending on the data. Notice that (1.7) includes the Schrödinger equation $-g'' + \Gamma g = |g|^{p-2} g$, whose relevance with the Caffarelli–Kohn–Nirenberg inequality was pointed out by Catrina and Wang in [13]. For $p = 2$ the system (1.7) reduces to the fourth-order ordinary differential equation

$$g'''' - 2(2A^2 + \Gamma) g'' + \Gamma^2 g = |g|^{q-2} g, \tag{1.8}$$

which is naturally related to second-order dilation invariant inequalities of Rellich–Sobolev type, see [2]. Actually the system (1.7) and Eq. (1.8) have independent interest because of their applications. We shall not attempt to give a complete list of references. We cite for instance [7, 14, 15, 19, 22–25] and references therein. In the monograph [31] one can find several applications and a rich bibliography on these topics.

In Sec. 2 we use the results in [30] and variational methods to get the existence of solutions $g \in W^{2,p'}(\mathbb{R}), f \in W^{2,q'}(\mathbb{R})$ to (1.7); cf. Theorem 2.1. Then we discuss sign properties of solutions to (1.7) having certain behavior at $-\infty$ and/or at $+\infty$. In Sec. 3 we obtain our main theorems about (1.1) as corollaries of our results for (1.7).

In the Appendix we indicate a possible non-radial approach to (1.1).

Notation. For any integer $n \geq 2$ we denote by ω_n the $(n-1)$-dimensional measure of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$.

Let $q \in [1, +\infty)$ and let ω be a non-negative measurable function on a domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$. The weighted Lebesgue space $L^q(\Omega; \omega(x)dx)$ is the space of measurable maps u in Ω with finite norm $(\int_\Omega |u|^q \omega(x) dx)^{1/q}$. For $\omega \equiv 1$ we simply write $L^q(\Omega)$. As usual, $\| \cdot \|_\infty$ is the L^∞-norm.

For any function $\varphi : \mathbb{R} \to \mathbb{R}$, the notation $\varphi(\pm\infty) = c$ means that there exists $\lim_{s\to\pm\infty} \varphi(s) = c$.

2. A 2 × 2 System of Ordinary Differential Equations

In this section we provide conditions for the existence of solutions to (1.7) vanishing at $\pm\infty$ and for the non-existence of positive solutions having non-negative limits at $-\infty$ or at ∞. We start with an existence result.

Theorem 2.1. *Let $p, q \in (1, \infty)$ $A, \Gamma \in \mathbb{R}$ be given, such that $A^2+\Gamma \geq 0$ and $\Gamma \neq 0$. Assume that (1.3) is satisfied. Then the system (1.7) has a non-trivial solution g, f such that $g \in W^{2,p'}(\mathbb{R})$ and $f \in W^{2,q'}(\mathbb{R})$.*

Proof. To simplify notations, we set

$$\mathcal{L}_+\varphi := -\varphi'' + 2A\varphi' + \Gamma\varphi, \quad \mathcal{L}_-\varphi = -\varphi'' - 2A\varphi' + \Gamma\varphi.$$

Since $A^2 + \Gamma \geq 0$ and $\Gamma \neq 0$, from [30, Proposition 5.2] we have that the infimum

$$I_{p',q}(A,\Gamma) = \inf_{\substack{g \in W^{2,p'}(\mathbb{R}) \\ g \neq 0}} \frac{\displaystyle\int_{\mathbb{R}} |\mathcal{L}_+ g|^{p'} ds}{\left(\displaystyle\int_{\mathbb{R}} |g|^q dx\right)^{p'/q}}$$

is achieved by some $g \in W^{2,p'}(\mathbb{R})$ that solves

$$\int_{\mathbb{R}} |\mathcal{L}_+ g|^{p'-2} \mathcal{L}_+ g \mathcal{L}_+ \psi ds = \int_{\mathbb{R}} |g|^{q-2} g\psi \, ds \quad \text{for any } \psi \in W^{2,p'}(\mathbb{R}).$$

Thus $g \in W^{2,p'}(\mathbb{R})$ is a weak solution to the following fourth-order ODE:

$$\mathcal{L}_-(|\mathcal{L}_+ g|^{p'-2}\mathcal{L}_+ g) = |g|^{q-2} g \quad \text{on } \mathbb{R},$$

which is equivalent to the system (1.7), by defining $f = -|\mathcal{L}_+ g|^{p'-2}\mathcal{L}_+ g$. Clearly, $g, f \in C^2(\mathbb{R})$. Now we recall that $\|\mathcal{L}_- \cdot\|_{q'}$ is an equivalent norm in $W^{2,q'}(\mathbb{R})$ by [30, Proposition 5.2]. Since $g \in W^{2,p'}(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$, we have $\mathcal{L}_- f = |g|^{q-2} g \in L^{q'}(\mathbb{R})$, and thus $f \in W^{2,q'}(\mathbb{R})$. □

Remark 2.2. One could exchange g and f in the proof of Theorem 2.1 to find a solution $\tilde{g}, \tilde{f}$, such that $\tilde{f} \in W^{2,q'}(\mathbb{R})$ achieves $I_{q',p}(-A,\Gamma)$ and $\tilde{g} = -|\mathcal{L}_- \tilde{f}|^{p'-2}\mathcal{L}_- \tilde{f}$. This argument does not lead to a multiplicity result for (1.7). To simplify notations we set $m = I_{p',q}(A,\Gamma)$ and $\tilde{m} = I_{q',p}(-A,\Gamma)$. Since $|\mathcal{L}_- f|^{q'} = |g|^q$, $|f|^p = |\mathcal{L}_+ g|^{p'}$, and since g achieves m we find

$$\tilde{m} \leq \frac{\displaystyle\int_{\mathbb{R}} |\mathcal{L}_- f|^{q'} ds}{\left(\displaystyle\int_{\mathbb{R}} |f|^p ds\right)^{q'/p}} = \frac{\displaystyle\int_{\mathbb{R}} |g|^q ds}{\left(\displaystyle\int_{\mathbb{R}} |\mathcal{L}_+ g|^p ds\right)^{q'/p}} = m^{\frac{p-q'}{p} \frac{q}{q-p'}},$$

so that $\tilde{m}^{\frac{q-p'}{q}} \leq m^{\frac{p-q'}{p}}$. In a similar way we get the opposite inequality, and in particular $\tilde{m}^{\frac{q-p'}{q}} = m^{\frac{p-q'}{p}}$. Moreover, $\tilde{f}$ achieves $\tilde{m}$ and $\tilde{g}$ achieves m.

In order to study the qualitative properties of solutions to (1.7) we take advantage of its Hamiltonian structure. Indeed, the system (1.7) is conservative, and any solution g, f satisfies

$$E(g,f) := g'f' - \Gamma gf + \frac{1}{q}|g|^q + \frac{1}{p}|f|^p = \text{constant}. \tag{2.1}$$

Remark 2.3. Let $g \in W^{2,p'}(\mathbb{R})$, $f \in W^{2,q'}(\mathbb{R})$ be a solution to (1.7). By the well-known facts about Sobolev spaces, the functions g, g', f and f' are Hölder continuous on $\mathbb{R}$. Thus $g, f \in C^2(\mathbb{R})$. In addition g, g', f and f' vanish at $\pm\infty$ and hence (2.1) implies

$$g'f' - \Gamma gf + \frac{1}{q}|g|^q + \frac{1}{p}|f|^p \equiv 0 \quad \text{on } \mathbb{R}. \tag{2.2}$$

Remark 2.4. Problem (1.7) is equivalent to a (2×2)-dimensional first-order Hamiltonian system. For $X = (x_1, x_2), Y = (y_1, y_2) \in \mathbb{R}^2$ we set

$$H(X,Y) = y_1y_2 + A(x_1y_1 - x_2y_2) - (A^2 + \Gamma)x_1x_2 + \frac{1}{q}|x_1|^q + \frac{1}{p}|x_2|^p.$$

Then a solution g, f solves (1.7) if and only if $X = (g, f), Y = (f' + Af, g' - Ag)$ solve

$$\begin{cases} X' = \partial_Y H(X,Y), \\ Y' = -\partial_X H(X,Y). \end{cases} \tag{2.3}$$

If $\Gamma \neq 0$ and $\delta := pq - (p+q) > 0$, then $\pm(|\Gamma|^{p/\delta}, |\Gamma|^{-1+q/\delta}\Gamma)$ are equilibrium points for (2.3). Notice that a positive equilibrium exists if and only if $\Gamma > 0$.

From (2.1) we first infer the following *a priori* bound on trajectories having null energy.

Proposition 2.5. *Let $g, f \in C^2(\mathbb{R})$ be a solution to (1.7) such that g, g', f and f' vanish at infinity. Then*

$$\|g\|_\infty^{q-p'} \le \frac{q}{p'}|\Gamma|^{p'}, \quad \|f\|_\infty^{p-q'} \le \frac{p}{q'}|\Gamma|^{q'}.$$

In particular, if $\Gamma = 0$ then $g = f \equiv 0$.

Proof. Let $\overline{s} \in \mathbb{R}$ be such that $|g(\overline{s})| = \|g\|_\infty$. Then $g'(\overline{s}) = 0$ and therefore from (2.1) and since $E(g,f) = 0$ we get

$$\frac{1}{q}\|g\|_\infty^q + \frac{1}{p}|f(\overline{s})|^p = \Gamma f(\overline{s})\|g\|_\infty \le \frac{|\Gamma|^{p'}}{p'}\|g\|_\infty^{p'} + \frac{1}{p}|f(\overline{s})|^p$$

by Young's inequality. The desired *a priori* bound on g follows immediately. The estimate on $\|f\|_\infty$ can be obtained in a similar way. □

In the remaining part of this section, we study the sign of solutions g, f to (1.7). We distinguish the case $\Gamma > 0$ from the case when Γ is non-positive.

Theorem 2.6. *Let $g, f \in C^2(\mathbb{R})$ be a solution to* (1.7), *such that g and f vanish at $\pm\infty$ together with their derivatives. If $\Gamma > 0$ then $g \equiv f \equiv 0$ or $gf > 0$ on $\mathbb{R}$.*

Proof. We start by noticing that the solution g, f satisfies (2.2). In a moment we will prove the following.

Claim. $g(s)f(s) \neq 0$ *for any* $s \in \mathbb{R}$.

Assume that the claim is proved. Then both g and f have constant sign. The function g has at least one critical point $\overline{s}$. By (2.2), it holds that

$$-\Gamma g(\overline{s})f(\overline{s}) + \frac{1}{q}|g(\overline{s})|^q + \frac{1}{p}|f(\overline{s})|^p = 0.$$

Thus $g(\overline{s})f(\overline{s}) > 0$, and therefore $gf > 0$ everywhere in $\mathbb{R}$, which concludes the proof of the theorem.

It remains to prove the claim. Notice that from (2.2) the following facts follow:

$$\text{if } g'(\xi)f'(\xi) = 0 \text{ then } f(\xi) = g(\xi) = 0 \text{ or } f(\xi)g(\xi) > 0, \tag{2.4}$$

$$\text{if } g(\xi)f(\xi) = 0 \text{ then } f(\xi) = g(\xi) = 0 = f'(\xi)g'(\xi) \text{ or } f'(\xi)g'(\xi) < 0. \tag{2.5}$$

By contradiction, assume that g vanishes somewhere. Up to a change of sign and/or inversion $s \mapsto -s$, we can assume that g attains its negative minimum at some $s_1 \in \mathbb{R}$ and that g reaches 0 in (s_1, ∞). Let s_2 be the first zero of g in (s_1, ∞). Thus $g < 0$ on $[s_1, s_2)$, $f(s_1) < 0$ by (2.4), and $g'(s_2) \geq 0$. In addition,

$$\text{if } f'(\bar{s}) = 0 \text{ for some } \bar{s} \in [s_1, s_2), \text{ then } f(\bar{s}) < 0, \tag{2.6}$$

because of (2.4). Now we prove that

$$g'(s_2)f'(s_2) = 0, \quad f(s_2) = 0, \quad f < 0 \quad \text{on } [s_1, s_2). \tag{2.7}$$

If $g'(s_2) = 0$ then (2.7) readily follows from (2.4) and (2.6). If $g'(s_2) > 0$ and $f'(s_2) = f(s_2) = 0$ then (2.6) immediately implies (2.7). In view of (2.5), to conclude the proof of (2.7) we only have to exclude that $g'(s_2) > 0 > f'(s_2)$. We argue by contradiction. If $f'(s_2) < 0$ then $f(s_2) < 0$ by (2.6). Since g is increasing in a neighborhood of s_2 and since g decays at infinity, there is a point $s_3 > s_2$ such that $g'(s_3) = 0$ and $g > 0$ on $(s_2, s_3]$. But then $f(s_3) > 0$ by (2.4). Since $f(s_2), f'(s_2)$ are negative, we infer that f has a minimum $s_4 \in (s_2, s_3)$, with $f(s_4) < 0$. But then $g(s_4) < 0$ by (2.4), which is impossible. Thus (2.7) is proved.

In conclusion, we have that the trajectory g, f solves the system

$$\begin{cases} g'' - 2Ag' - \Gamma g = -|f|^{p-2}f \geq 0 & \text{in } (s_1, s_2), \\ f'' + 2Af' - \Gamma f = -|g|^{q-2}g \geq 0 & \text{in } (s_1, s_2), \\ g, f < 0 & \text{in } (s_1, s_2), \\ g(s_2) = f(s_2) = g'(s_2) = f'(s_2) = 0, \end{cases}$$

which contradicts the Hopf boundary point lemma. The claim and the theorem are completely proved. ∎

The condition $\Gamma > 0$ is also necessary to have the existence of positive solutions vanishing at $\pm\infty$. In view of Remark 2.3, the next proposition applies in particular to solutions $g \in W^{2,p'}(\mathbb{R})$, $f \in W^{2,q'}(\mathbb{R})$.

Proposition 2.7. *Let $g, f \in C^2(\mathbb{R})$ be a solution to* (1.7), *such that g and f vanish at $\pm\infty$ together with their derivatives. If $\Gamma \le 0$ and $gf \ge 0$ on $\mathbb{R}$ then $g \equiv f \equiv 0$.*

Proof. The trajectory g, f has null energy, that is, (2.2) holds. In particular, at any critical point $\bar{s}$ of g one has that $|\Gamma|g(\bar{s})f(\bar{s}) + \frac{1}{q}|g(\bar{s})|^q + \frac{1}{p}|f(\bar{s})|^p = 0$. Thus both g and f vanish at $\bar{s}$. In particular, $\min g = \max g = 0$, and the conclusion follows. □

We conclude this section with two more non-existence results in case $\Gamma \le 0$.

Theorem 2.8. *Assume that the solution $g, f \in C^2(\mathbb{R})$ solves* (1.7) *for some $A \in \mathbb{R}$, $\Gamma \le 0$ and p, q satisfying* (1.3). *In addition, assume that*

$$g(-\infty) = c_g \in [0, \infty), \quad f(-\infty) = c_f \in [0, \infty), \quad g \ge 0 \text{ and } f \ge 0 \quad \text{on } \mathbb{R}.$$

Then $g \equiv f \equiv 0$.

Proof. First of all we notice that g, f cannot be a non-trivial pair of constant functions by Remark 2.4.

The function $h := -f' - 2Af$ is increasing in $\mathbb{R}$, as $h' = g(s)^{q-1} - \Gamma f \ge 0$. Thus it has a limit as $s \to -\infty$. Hence, also f' has a limit as $s \to -\infty$. Clearly

$$f'(-\infty) = 0, \tag{2.8}$$

and therefore from (1.7) we also get

$$-f''(-\infty) = -\Gamma c_f + c_g^{q-1} \ge 0. \tag{2.9}$$

In a similar way we get

$$g'(-\infty) = 0, \quad -g''(\infty) = -\Gamma c_g + c_f^{p-1} \ge 0. \tag{2.10}$$

In particular, from (2.1) and (2.8), (2.10) we infer that

$$g'f' - \Gamma gf + \frac{1}{q}|g|^q + \frac{1}{p}|f|^p = -\Gamma c_g c_f + \frac{1}{q}c_g^q + \frac{1}{p}c_f^p \quad \text{on } \mathbb{R}.$$

Claim 1. *If $c_g = c_f = 0$ then $g \equiv f \equiv 0$.*

To prove the claim, we notice that the trajectory g, f satisfies (2.2). If we assume by contradiction that g or f do not vanish identically, then there exists $s_0 \in \mathbb{R}$ such that $g'(s_0)f'(s_0) < 0$. To fix ideas, assume that $f'(s_0) < 0$. Since $f \ge 0$ and $f(s) \to 0$ as $s \to -\infty$, it means that f must have a positive local maximum $s_1 < s_0$. At the point s_1 the conservation law (2.2) gives $-\Gamma g(s_1)f(s_1) + \frac{1}{q}|g(s_1)|^q + \frac{1}{p}|f(s_1)|^p = 0$, which contradicts $f(s_1) > 0$. The claim is proved.

Claim 2. *If $A \le 0$ then $\Gamma c_f = 0$ and $c_g = 0$.*

By contradiction, assume that $-\Gamma c_f + c_g^{q-1} > 0$. Then the function f is strictly concave and decreasing in a neighborhood of $-\infty$ by (2.9) and (2.8). Thus in particular $c_f > 0$, and therefore form the conservation law we get

$$g'f' - \Gamma gf + \frac{1}{q}|g|^q + \frac{1}{p}|f|^p \geq \frac{1}{p}c_f^p > 0 \quad \text{on } \mathbb{R}. \tag{2.11}$$

Since f is bounded from below, it cannot be strictly concave on $\mathbb{R}$. We claim that f can never be locally convex. Assume that there exists $s_0 \in \mathbb{R}$ such that $f''(s_0) > 0$. Then from (1.7) we have that $-2Af'(s_0) > -\Gamma f(s_0) + g(s_0)^{q-1} \geq 0$. Thus, $A < 0$ and $f'(s_0) > 0$. Since $f'(s) < 0$ for $s \ll 0$, then the function f must have a local minimum $s_1 \in (-\infty, s_0)$. Thus $f'(s_1) = 0$ and $f''(s_1) \geq 0$. But then

$$0 \geq -f''(s_1) = -\Gamma f(s_1) + g(s_1)^{q-1} \geq 0,$$

which implies $\Gamma f(s_1) = g(s_1) = 0$. In particular, $g'(s_1) = 0$, and $g''(s_1) \geq 0$, since s_1 is a minimum for g thanks to the assumption that $g \geq 0$. Thus, (1.7) gives $0 \geq -g''(s_1) = f(s_1)^{p-1} \geq 0$. Thus $f(s_1) = 0$, contradicting (2.11).

We have proved that $f'' \leq 0$ on $\mathbb{R}$. Thus there exists $s_0 \in \mathbb{R}$ such that f is a non-negative constant on $[s_0, \infty)$. But then from (1.7) we infer that $f \equiv g \equiv 0$ on $[s_0, \infty)$, as $\Gamma \leq 0$. We have reached again a contradiction with (2.11), and the claim is proved.

Claim 3. *If $A \geq 0$ then $\Gamma c_g = 0$ and $c_f = 0$.*

It is sufficient to exchange the roles of g and f, and argue as in Claim 2.

Now we are in position to conclude the proof. By Claim 1, we only have to show that $c_g = c_f = 0$. Thus we are done if $A = 0$, thanks to Claims 2 and 3. We have to study the case

$$A < 0, \quad \Gamma = c_g = 0 \tag{2.12}$$

and the case $A > 0$, $\Gamma = 0 = c_f$ that can be handled in a similar way. Assume that (2.12) holds. Since g solves $-g'' + 2Ag' = f^{p-1} \geq 0$, then the function $-g' + 2Ag$ is non-decreasing on $\mathbb{R}$. Hence $-g' + 2Ag \geq 0$ by (2.10) and since $c_g = 0$. Thus $g' \leq 2Ag \leq 0$ on $\mathbb{R}$, that is, $g \equiv 0$, because it is non-increasing and non-negative. The proof is complete. □

Since the system (1.7) is invariant with respect to inversion $s \mapsto -s$, then clearly the next result holds as well.

Theorem 2.9. *Assume that the pair $g, f \in C^2(\mathbb{R})$ solves (1.7) for some $A \in \mathbb{R}$, $\Gamma \leq 0$ and p, q satisfying (1.3). In addition, assume that*

$$g(\infty) = c_g \in [0, \infty), \quad f(\infty) = c_f \in [0, \infty), \quad g \geq 0 \text{ and } f \geq 0 \quad \text{on } \mathbb{R}.$$

Then $g \equiv f \equiv 0$.

We conclude this section with a result that holds in case $p = 2 < q$.

Theorem 2.10. *Let $q \in (2, \infty)$ and assume that $A^2 + \Gamma \geq 0$, $\Gamma \neq 0$. Up to translations in $\mathbb{R}$, composition with the inversion $s \mapsto -s$ and change of sign, the system*

$$\begin{cases} -g'' + 2Ag' + \Gamma g = f & \text{on } \mathbb{R}, \\ -f'' - 2Af' + \Gamma f = |g|^{q-2}g & \text{on } \mathbb{R} \end{cases}$$

has a unique non-trivial solution (g, f) such that $g \in H^2(\mathbb{R})$ and $f \in W^{2,q'}(\mathbb{R})$. Moreover, g is even, positive and strictly decreasing on $(0, \infty)$, and f is positive if and only if $\Gamma > 0$.

Proof. Existence is given by Theorem 2.1. Notice that g is smooth and solves

$$g'''' - 2(2A^2 + \Gamma)g'' + \Gamma^2 g = |g|^{q-2}g. \tag{2.13}$$

On the other hand, since $(2A^2 + \Gamma)^2 \geq \Gamma^2$, then Theorem 2.2 in [2] implies that (2.13) has a unique solution g (up to the above transforms), which can be taken to be positive, even and strictly decreasing on $(0, \infty)$. The uniqueness of f is immediate. The last statement concerning the sign of f follows by Theorem 2.6 and Proposition 2.7. □

Remark 2.11. Clearly, f is even if and only if $A = 0$.

3. The Hénon–Lane–Emden System

In this section we provide conditions for the existence of solutions to (1.1) in suitable energy spaces and for the non-existence of positive solutions having certain behavior at 0 or at ∞.

We start by introducing some weighted Sobolev spaces. Let $\theta \in (1, \infty)$ and $\alpha \in \mathbb{R}$ be given, such that $\alpha \notin \{2\theta - n, np - n\}$. Then we can use the results in [30] to define the Banach space $\mathcal{D}^{2,\theta}_{\mathrm{r}}(\mathbb{R}^n; |x|^\alpha dx)$ as the completion of radial functions in $C^2_c(\mathbb{R}^n \backslash \{0\})$ with respect to the norm

$$\|u\|_\alpha = \left(\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^\theta \, dx \right)^{1/\theta}.$$

To any pair of radial functions $u, v \in C^2(\mathbb{R}^n \backslash \{0\})$, we associate the pair $g, f \in C^2(\mathbb{R})$ defined by

$$u(x) = |x|^{-\lambda_1} g(-\log|x|), \quad v(x) = |x|^{-\lambda_2} f(-\log|x|), \tag{3.1}$$

where

$$\lambda_1 = \frac{b + n}{q}, \quad \lambda_2 = \frac{a + n}{p}.$$

We will always assume that (p, q) belongs to the *critical hyperbola* in (1.2), that is, $\lambda_1 + \lambda_2 = n - 2$.

We introduce also the constants

$$\Gamma = \frac{n+a}{p}\,\frac{n+b}{q} = \lambda_1\lambda_2, \quad A = \frac{n-2}{2} - \lambda_1 = -\frac{n-2}{2} + \lambda_2. \tag{3.2}$$

Notice that

$$A^2 + \Gamma = \left(\frac{n-2}{2}\right)^2 \geq 0.$$

A direct computation shows that a radial pair u, v solves (1.1) on $\mathbb{R}^n \backslash \{0\}$ if and only if the trajectory g, f solves (1.7) with Γ, A given by (3.2). Thanks to the results in previous section we first get the next existence theorem.

Theorem 3.1. *Let $n \geq 2$, $a, b \in \mathbb{R}\backslash\{-n\}$ and $p, q > 1$. Assume that* (1.3) *and* (1.2) *are satisfied. Then the Hénon–Lane–Emden system* (1.1) *has a radially symmetric solution*

$$u \in \mathcal{D}^{2,p'}_{\mathrm{r}}(\mathbb{R}^n; |x|^{-\frac{a}{p-1}}dx), \quad v \in \mathcal{D}^{2,q'}_{\mathrm{r}}(\mathbb{R}^n; |x|^{-\frac{b}{q-1}}dx). \tag{3.3}$$

Moreover, u, v satisfies (1.5).

Proof. Define Γ, A as in (3.2), and notice that $\Gamma \neq 0$, and $A^2 + \Gamma \geq 0$. By Theorem 2.1, we see that there exist $f \in W^{2,p'}(\mathbb{R})$ and $g \in W^{2,q'}(\mathbb{R})$ satisfying (1.7). Now using the Emden–Fowler transformation in (3.1) and the results in [30], we get a pair u, v that satisfies (3.3) and solves (1.1). The conclusion readily follows from $|x|^{\frac{b+n}{q}}u(x) = g(-\log|x|)$, $|x|^{\frac{a+n}{p}}v(x) = f(-\log|x|)$ and the fact that g and f vanish at $\pm\infty$. □

Theorem 3.2. *Let $n \geq 2$, $a, b \in \mathbb{R}$ and $p, q > 1$. Assume that* (1.3) *and* (1.2) *are satisfied. Let $u, v \in C^2(\mathbb{R}^n\backslash\{0\})$ be a radially symmetric solution to* (1.1) *on $\mathbb{R}^n\backslash\{0\}$.*

(i) *If $a > -n$, $b > -n$ and if a solution u, v satisfies* (1.5), *then $u \equiv v \equiv 0$ or $uv > 0$ on $\mathbb{R}$.*
(ii) *Assume that* (1.6) *holds. If $a \leq -n$ or $b \leq -n$ and if $u \geq 0, v \geq 0$ then $u \equiv v \equiv 0$.*

Proof. Define A, Γ and use the Emden–Fowler transform $(u, v) \mapsto (g, f)$ as before. Notice that $\Gamma > 0$ in case (i) and $\Gamma \leq 0$ in case (ii). Then apply Theorem 2.8 and Theorem 2.6. □

In the next corollary we emphasize the impact of Theorem 3.2 in case $n = 2$, when Theorem 3.1 gives existence on the critical hyperbola whenever $a, b \neq -2$.

Corollary 3.3. *Let $n = 2$ and $p, q > 1$. Assume that* (1.3) *and* (1.2) *are satisfied, and in addition assume that $a, b \neq -2$. Let $u, v \in C^2(\mathbb{R}^2\backslash\{0\})$ be a radially symmetric and non-negative solution to* (1.1) *satisfying* (1.6). *Then $u \equiv v \equiv 0$.*

In Theorem 3.2 we saw that the sign of Γ affects the sign of the product uv. However, at least in case $p = 2$, the function u never changes sign, also in case $\Gamma < 0$. The next result for problem

$$\begin{cases} -\Delta u = |x|^a v, \\ -\Delta v = |x|^b |u|^{q-2} u \end{cases} \tag{3.4}$$

is an immediate consequence to Theorem 2.10.

Theorem 3.4. *Let $n \geq 2$, $a, b \in \mathbb{R}$ and $q > 1$. Assume that $a, b \neq -n$, and*

$$\frac{a+n}{2} + \frac{b+n}{q} = n - 2$$

is satisfied. Up to dilations, compositions with the Kelvin transform and change of sign, problem (3.4) has a unique non-trivial radial solution $u \in \mathcal{D}^{2,2}_{\mathrm{r}}(\mathbb{R}^n; |x|^{-a} dx)$, $v \in \mathcal{D}^{2,q'}_{\mathrm{r}}(\mathbb{R}^n; |x|^{-\frac{b}{q-1}} dx)$. Moreover, u is positive, and v is positive if and only if $a, b > -n$.

Appendix. A Non-Radial Approach

Following [39, 9], we notice that (1.1) is formally equivalent to the fourth-order equation

$$\Delta(|x|^\alpha |\Delta u|^{\theta-2} \Delta u) = |x|^b |u|^{q-2} u, \tag{A.1}$$

where $\theta = p' = \frac{p}{p-1}$ and $\alpha = -\frac{a}{p-1}$. Equation (A.1) is variational. In particular, its non-trivial solutions can be found as critical points for the functional

$$u \mapsto \frac{\displaystyle\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^\theta dx}{\left(\displaystyle\int_{\mathbb{R}^n} |x|^b |u|^q dx\right)^{\theta/q}}$$

on a suitable function space. Let us introduce the weighted Rellich constant

$$\mu_\theta(\alpha) := \inf_{\substack{u \in C^2_c(\mathbb{R}^n \setminus \{0\}) \\ u = u(|x|), u \neq 0}} \frac{\displaystyle\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^\theta dx}{\displaystyle\int_{\mathbb{R}^n} |x|^{\alpha - 2\theta} |u|^\theta dx}. \tag{A.2}$$

The best constant $\mu_\theta(\alpha)$ is explicitly known in few cases. We define Γ as in (3.2) and we notice that

$$\Gamma = \left(\frac{n+\alpha}{\theta} - 2\right)\left(n - \frac{n+\alpha}{\theta}\right) \tag{A.3}$$

if (1.2) is satisfied. The value of $\mu_2(\alpha)$ (case $\theta = 2$) is known from [20, 11]:

$$\mu_2(\alpha) = \min_{k \in \mathbb{N} \cup \{0\}} |\Gamma + k(n - 2 + k)|^2.$$

For general $\theta > 1$, Mitidieri proved in [29] that $\mu_\theta(\alpha) = |\Gamma|^\theta$, provided that $\Gamma \geq 0$.

From now on, we assume that $\mu_\theta(\alpha) > 0$. Then we can define the space $\mathcal{D}^{2,\theta}(\mathbb{R}^n; |x|^\alpha dx)$ as the closure of functions in $C^2_c(\mathbb{R}^n \backslash \{0\})$ with respect to the norm

$$\|u\|^\theta = \int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^\theta dx.$$

Lemma A.1. *Let $\theta > 1$, $\alpha \in \mathbb{R}$ be given, such that $\mu_\theta(\alpha) > 0$. Let $q \geq \theta$ and assume that $q \leq \theta^{**} := \frac{\theta n}{n-2\theta}$ if $n > 2\theta$. Then there exists a constant $c > 0$ such that*

$$\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^\theta \, dx$$
$$\geq c \left(\int_{\mathbb{R}^n} |x|^{-n+q\frac{n-2\theta+\alpha}{\theta}} |u|^q dx \right)^{\theta/q} \quad \text{for any } u \in \mathcal{D}^{2,\theta}(\mathbb{R}^n; |x|^\alpha dx).$$

Proof. If $n > 2\theta$ the conclusion readily follows via interpolation with the Sobolev inequality. For a proof in lower dimensions, we use the Emden–Fowler transform $T : C^2_c(\mathbb{R}^n \backslash \{0\}) \to \in C^2_c(\mathbb{R} \times \mathbb{S}^{n-1})$, $T : u \mapsto g$ defined via

$$u(x) = |x|^{\frac{2\theta-n-\alpha}{\theta}} g\left(-\log|x|, \frac{x}{|x|} \right).$$

We denote by Δ_σ the Laplace–Beltrami operator on $\mathbb{S}^{n-1}$ and by g'', g' the derivatives of $g = g(s, \sigma)$ with respect to $s \in \mathbb{R}$. By direct computation one has that

$$\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^\theta dx = \int_{\mathbb{R}} \int_{\mathbb{S}^{n-1}} |\Delta_\sigma g + g'' - 2Ag' - \Gamma g|^\theta ds d\sigma,$$

$$\int_{\mathbb{R}^n} |x|^{\alpha-2\theta} |u|^\theta dx = \int_{\mathbb{R}} \int_{\mathbb{S}^{n-1}} |g|^\theta ds d\sigma,$$

where Γ is given by (A.3) and $A = \frac{2(\theta-\alpha)+n(\theta-2)}{2\theta}$. Thus, using the assumption $\mu_\theta(\alpha) > 0$, one proves that

$$\|g\|^\theta := \int_{\mathbb{R}} \int_{\mathbb{S}^{n-1}} |\Delta_\sigma g + g'' - 2Ag' - \Gamma g|^\theta ds d\sigma$$

is an equivalent norm on $W^{2,\theta}(\mathbb{R} \times \mathbb{S}^{n-1})$. Therefore, T can be regarded as an isometry between Banach spaces, and the conclusion readily follows by using the Sobolev embedding $W^{2,\theta}(\mathbb{R} \times \mathbb{S}^{n-1}) \hookrightarrow L^q(\mathbb{R} \times \mathbb{S}^{n-1})$. □

Under the assumptions in Lemma A.1, we have that the infimum

$$S_{\theta,q}(\alpha) := \inf_{\substack{u \in \mathcal{D}^{2,\theta}(\mathbb{R}^n; |x|^\alpha dx) \\ u \neq 0}} \frac{\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^\theta dx}{\left(\int_{\mathbb{R}^n} |x|^{-n+q\frac{n-2\theta+\alpha}{\theta}} |u|^q dx \right)^{\theta/q}}$$

is positive. Notice that for $n > 2\theta$, $\alpha = 0$ and $q = \theta^{**}$ we have that

$$S_{\theta,\theta^{**}}(0) = S^{**}(\theta) := \inf_{\substack{u\in\mathcal{D}^{2,\theta}(\mathbb{R}^n)\\ u\neq 0}} \frac{\displaystyle\int_{\mathbb{R}^n} |\Delta u|^\theta dx}{\left(\displaystyle\int_{\mathbb{R}^n} |u|^{\theta^{**}} dx\right)^{\theta/\theta^{**}}},$$

which is the best constant in the Sobolev embedding $\mathcal{D}^{2,\theta}(\mathbb{R}^n) \hookrightarrow L^{\theta^{**}}(\mathbb{R}^n)$, see [1,38]. The next existence results can be proved, for instance, by using the techniques in [11] (proof of Theorem 1.2). We omit the details.

Theorem A.2. *Let $\theta > 1$, $\alpha \in \mathbb{R}$ be given, in such a way that the infimum in* (A.2) *is positive. Let $q > \theta$.*

(i) *Assume that $n \geq 3$ and $q < \theta^{**}$ if $n > 2\theta$. Then $S_{\theta,q}(\alpha)$ is achieved.*
(ii) *If $n > 2\theta$ and $S_{\theta,\theta^{**}}(\alpha) < S^{**}(\theta)$ then $S_{\theta,\theta^{**}}(\alpha)$ is achieved.*

Remark A.3. Thanks to the results in [29] we know that $\mu_\theta(\alpha) = |\Gamma|^\theta > 0$, whenever $\Gamma > 0$. We suspect that in this case the infimum $S_{\theta,q}(\alpha)$ is always achieved by radial functions. We leave this as an open problem.

From Theorem A.2 one can easily infer sufficient conditions for the existence of (minimal energy) solutions to the Hénon–Lane–Emden system (1.1), whenever $\mu_{p'}(\alpha) > 0$. More can be said when $p = 2$. From [20, 11] we know that $\mu_2(-a) > 0$ if and only if $-\Gamma$ is not an eigenvalue of the Laplace–Beltrami operator on the sphere, where now

$$\Gamma = \left(\frac{n+a}{2}\right)\left(\frac{n+b}{q}\right), \quad \frac{a+n}{2} + \frac{b+n}{q} = n-2.$$

From now on we assume that

$$-\left(\frac{n+a}{2}\right)\left(\frac{n+b}{q}\right) \neq k(n-2+k) \text{ for any integer } k \geq 0. \tag{A.4}$$

By *ground state solutions* to (3.4) we mean solutions u, v such that u achieves the infimum

$$\inf_{\substack{u\in\mathcal{D}^{2,2}(\mathbb{R}^n;|x|^{-a}dx)\\ u\neq 0}} \frac{\displaystyle\int_{\mathbb{R}^n} |x|^{-a}|\Delta u|^2 dx}{\left(\displaystyle\int_{\mathbb{R}^n} |x|^b|u|^q ds\right)^{2/q}}.$$

For convenience of the reader we summarize here the main results for (3.4) that can be obtained as immediate corollaries of the results in [10].

Theorem A.4. *Let $q > 2$ and assume that $q < 2^{**} := \frac{2n}{n-4}$ if $n \geq 5$.*

(i) *If* (A.4) *holds, then* (3.4) *has a ground state solution $\overline{u}, \overline{v}$.*

(ii) *For every $q > 2$ and for every integer $k \geq 1$ there exists $\delta > 0$ such that if*

$$0 < |\Gamma + k(n - 2 + k)| < \delta$$

then $\overline{u}$ is not radially symmetric. Thus problem (1.1) *has at least two distinct weak solutions.*

(iii) *If $|\Gamma| > \frac{n-1}{q-2}(1 + \sqrt{q-1})$ then $\overline{u}$ is not radially symmetric. Thus problem* (1.1) *has at least two distinct weak solutions.*

(iv) *Assume that $-\Gamma > \frac{n-1}{2}$. Then there exists $q_\alpha > 2$ such that no ground state solution to* (3.4) *can be positive.*

In the limiting case $n \geq 5$ and $q = 2^{**}$ the problem is more difficult. We limit ourselves to point out some corollaries to the results in [10] in case $n \geq 6$.

Theorem A.5. *Assume $n \geq 6$ and that* (A.4) *is satisfied. If in addition $|a+2| > 2$, then the problem*

$$\begin{cases} -\Delta u = |x|^a v & \text{on } \mathbb{R}^n \\ -\Delta v = |x|^b |u|^{\frac{8}{n-4}} u & \text{on } \mathbb{R}^n \end{cases} \tag{A.5}$$

has a ground state solution $\overline{u}, \overline{v}$. Moreover, the conclusions (ii)–(iv) *in Theorem* A.4 *still hold. In particular,* (A.5) *has a radial and a non-radial weak solutions.*

References

[1] T. Aubin, Problèmes isopérimétriques et espaces de Sobolev, *J. Differential Geom.* **11** (1976) 573–598.

[2] M. Bhakta and R. Musina, Entire solutions for a class of variational problems involving the biharmonic operator and Rellich potentials, *Nonlinear Anal.* **75** (2012) 3836–3848.

[3] M. F. Bidaut-Véron, Local and global behavior of solutions of quasilinear equations of Emden–Fowler type, *Arch. Ration. Mech. Anal.* **107** (1989) 293–324.

[4] M. F. Bidaut-Véron and H. Giacomini, A new dynamical approach of Emden–Fowler equations and systems, *Adv. Differential Equations* **15** (2010) 1033–1082.

[5] D. Bonheure, E. Moreira dos Santos and M. Ramos, Ground state and non-ground state solutions of some strongly coupled elliptic systems, *Trans. Amer. Math. Soc.* **364** (2012) 447–491.

[6] ———, Symmetry and symmetry breaking for ground state solutions of some strongly coupled elliptic systems, *J. Funct. Anal.* **264** (2013) 62–96.

[7] B. Buffoni, Periodic and homoclinic orbits for Lorentz–Lagrangian systems via variational methods, *Nonlinear Anal.* **26** (1996) 443–462.

[8] J. Busca and R. Manasevich, A Liouville-type theorem for Emden system, *Indiana Univ. Math. J.* **51** (2002) 37–51.

[9] M. Calanchi and B. Ruf, Radial and non-radial solutions for Hénon–Hardy type elliptic systems, *Calc. Var. Partial Differential Equations* **38** (2010) 111–133.

[10] P. Caldiroli and R. Musina, Caffarelli–Kohn–Nirenberg type inequalities for the weighted biharmonic operator in cones, *Milan J. Math.* **79** (2011) 657–687.

[11] ———, Rellich inequalities with weights, *Calc. Var. Partial Differential Equations* **45** (2012) 147–164.
[12] G. Caristi, L. D'Ambrosio and E. Mitidieri, Representation formulae for solutions to some classes of higher order systems and related Liouville theorems, *Milan J. Math.* **76** (2008) 27–67.
[13] F. Catrina and Z.-Q. Wang, On the Caffarelli–Kohn–Nirenberg inequalities: Sharp constants, existence (and nonexistence), and symmetry of extremal functions, *Comm. Pure Appl. Math.* **54** (2001) 229–258.
[14] A. R. Champneys and J. F. Toland, Bifurcation of a plethora of multi-modal homoclinic orbits for autonomous Hamiltonian systems, *Nonlinearity* **6** (1993) 665–721.
[15] Y. Chen and P. J. McKenna, Travelling waves in a nonlinearly suspended beam: Some computational results and four open questions, *Philos. Trans. R. Soc. Lond. Ser. A* **355** (1997) 2175–2184.
[16] D. G. de Figueiredo, I. Peral and J. D. Rossi, The critical hyperbola for a Hamiltonian system with weights, *Ann. Mat. Pura Appl. (4)* **187** (2008) 531–545.
[17] M. Fazly, Liouville type theorems for stable solutions of certain elliptic systems, *Adv. Nonlinear Stud.* **12** (2012) 1–17.
[18] M. Fazly and N. Ghoussoub, On the Hénon–Lane–Emden conjecture, preprint (2011); arXiv:1107.5611v2.
[19] F. Gazzola and H. C. Grunau, Radial entire solutions for supercritical biharmonic equations, *Math. Ann.* **334** (2006) 905–936.
[20] N. Ghoussoub and A. Moradifam, Bessel pairs and optimal Hardy and Hardy–Rellich inequalities, *Math. Ann.* **349** (2011) 1–57.
[21] J. Hulshof and R. C. A. M. Van der Vorst, Asymptotic behavior of ground states, *Proc. Amer. Math. Soc.* **124** (1996) 2423–2431.
[22] G. W. Hunt, H. M. Bolt and J. M. T. Thompson, Structural localization phenomena and the dynamical phase-space analogy, *Proc. R. Soc. London A* **425** (1989) 245–267.
[23] G. W. Hunt and E. Lucerna Neto, Localized buckling in long axially-loaded cylindrical shells, *J. Mech. Phys. Solids* **39** (1991) 881–894.
[24] G. W. Hunt and M. K. Wadee, Comparative Lagrangian formulations for localized buckling, *Proc. R. Soc. London A* **434** (1991) 485–502.
[25] M. Lazzo and P. G. Schmidt, Oscillatory radial solutions for subcritical biharmonic equations, *J. Differential Equations* **247** (2009) 1479–1504.
[26] P. L. Lions, The concentration-compactness principle in the calculus of variations. The limit case, Part 1, *Rev. Mat. Iberoam.* **1** (1985) 145–201.
[27] E. Mitidieri, A Rellich type identity and applications, *Comm. Partial Differential Equations* **18** (1993) 125–151.
[28] ———, Nonexistence of positive solutions of semilinear elliptic systems in $\mathbf{R}^N$, *Differential Integral Equations* **9** (1996) 465–479.
[29] ———, A simple approach to Hardy inequalities, *Math. Notes* **67** (2000) 479–486; translation from *Mat. Zametki* **67** (2000) 563–572.
[30] R. Musina, Weighted Sobolev spaces of radially symmetric functions, to appear in *Ann. Mat. Pura Appl.* (2013), doi:10.1007/s10231-013-0348-4.
[31] L. A. Peletier and W. C. Troy, *Spatial Patterns*, Progress in Nonlinear Differential Equations and their Applications, Vol. 45 (Birkhäuser, Boston, MA, 2001).
[32] Q. H. Phan, Liouville-type theorems and bounds of solutions for Hardy–Hénon elliptic systems, *Adv. Differential Equations* **17** (2012) 605–634.
[33] P. Poláĉik, P. Quittner and Ph. Souplet, Singularity and decay estimates in superlinear problems via Liouville-type theorems, Part I: Elliptic systems, *Duke Math. J.* **139** (2007) 555–579.

[34] J. Serrin and H. Zou, Non-existence of positive solutions of semilinear elliptic systems, in *A Tribute to Ilya Bakelman*, Discourses in Mathematics and Its Applications, No. 3 (Texas A & M University Press, College Station, TX, 1993), pp. 55–68.

[35] ———, Non-existence of positive solutions of Hénon–Lane–Emden systems, *Differential Integral Equations* **9** (1996) 635–653.

[36] ———, Existence of positive solutions of the Hénon–Lane–Emden system, *Atti Semin. Mat. Fis. Univ. Modena* **46** (1998) 369–380.

[37] P. Souplet, The proof of the Hénon–Lane–Emden conjecture in four space dimensions, *Adv. Math.* **221** (2009) 1409–1427.

[38] G. Talenti, Best constants in Sobolev inequality, *Ann. Mat. Pura Appl.* **110** (1976) 353–372.

[39] X. J. Wang, Sharp constant in a Sobolev inequality, *Nonlinear Anal.* **20** (1993) 261–268.