Nagoya Mathematical Journal (2026), **261**:e14 1–16
doi:10.1017/nmj.2025.10095

ARTICLE

Curvature and relative volume forms

Luca Rizzi [1] and Francesco Zucconi [2]

[1]Center for Geometry and Physics, Institute for Basic Science (IBS), Pohang, Republic of Korea
[2]Department of Mathematics, Computer Science and Physics, University of Udine, Udine, Italy

Corresponding author: Luca Rizzi; Email: lucarizzi@ibs.re.kr

Received: 8 June 2025; **Revised:** 22 September 2025; **Accepted:** 22 September 2025

Keywords: strong non-isotrivial families; constant families; higher direct images of relative differential forms; curvature; positivity

2020 Mathematics Subject Classification: *Primary* – 14D06; *Secondary* – 32G05, 32L05

Abstract
Using metric techniques introduced by Berndtsson, we show a result on the constancy of families dominated by a constant variety and, on the opposite side, results on the strong non-isotriviality of certain families of surfaces with positive index and, in arbitrary dimension, in terms of the complex conjugate of a suitable representative of the Kodaira–Spencer class. We also give a metric interpretation of the liftability of relative volume forms.

1. Introduction

In this article, we are concerned with the curvature properties of the higher direct images $R^i f_*(\Omega^{n-i}_{X/B}(\log) \otimes L)$ and in particular of $f_*(\omega_{X/B} \otimes L)$, where $f: X \to B$ is a semistable fibration of smooth complex projective varieties of relative dimension n, L is a Hermitian line bundle on X, and $\Omega^\bullet_{X/B}(\log)$ are the bundles of holomorphic relative logarithmic forms (see Section 2 for details).

If $f: X \to B$ is smooth, it is classically known that $f_*\omega_{X/B}$ is Griffiths semi-positive (see [13], [15]). Expanding on this, if f is not necessarily smooth but B is a curve, a result by Fujita shows that $f_*\omega_{X/B}$ has a holomorphic decomposition $f_*\omega_{X/B} = \mathcal{U} \oplus \mathcal{A}$, where $\mathcal{A}$ is an ample vector bundle and $\mathcal{U}$ is a unitary flat vector bundle (see [4], [5], [8], [9]). This result has been generalized in many directions (see, e.g., [6] for the case $\dim B > 1$, and [18], [23] for the direct image of the relative pluricanonical bundles $f_*(\omega^{\otimes m}_{X/B})$).

Berndtsson, in the seminal work [1], shows that $f_*(\omega_{X/B} \otimes L)$ is semi-positive in the sense of Nakano if L is semi-positive and this provides an evidence for the Griffiths conjecture [12]. Later in [2], it is shown that it is possible to relate the curvature of $f_*(\omega_{X/B} \otimes L)$ to the Kodaira–Spencer class. In particular, if $L = \mathcal{O}_X$, these results highlight a close relation between curvature formulas and the classical infinitesimal Torelli problem.

One key property of this kind of metric result is that they can be meaningfully extended to the whole base B using the notion of singular Hermitian metric, see [26], [30] and the references therein for all definitions.

For convenience, we state our results in the case $\dim B = 1$, nevertheless, most of this work can be generalized to the case $\dim B > 1$.

Using a curvature formula from [2], we give conditions for the isotriviality of $f: X \to B$ over a Zariski open subset of B.

Theorem 1.1. *Let $f: X \to B$ be a family of canonically polarized manifolds. Assume that there exists a surjective morphism $\rho: V \times B \to X$, where V is a projective variety, such that $p := f \circ \rho$ is the projection on B. Then, X is Zariski locally trivial.*

See Theorem 3.12. This result must be seen in the light of the “constancy” results that are at the heart of the recent solution, in some cases, of the Bombieri–Lang conjecture: we think that Theorem 1.1 gives an interesting link between metric estimates and the techniques used in the proof of [42, §2.3 and Theorem 2.6].

On the opposite side, we give a new method to study strong non-isotriviality based on metric evaluations on the conjugate of the Kodaira–Spencer class. It arises from the study of the curvature of certain torsion-free subsheaves of $K^i_{\text{prim}} \otimes \omega_B(E)^{\otimes i}$, where K^i_{prim} is the sheaf of primitive classes in the kernel of $R^i f_*\Omega^{n-i}_{X/B}(\log) \to \omega_B(E) \otimes R^{i+1} f_*\Omega^{n-i-1}_{X/B}(\log)$ and E is the divisor on B of the f-singular values.

Following [22], we give the following definition.

Definition 1.2. A semistable fibration $f: X \to B$ is said to be strongly non-isotrivial if the morphism

$$\rho_f : T_B(-E)^{\otimes n} \to R^n f_* \wedge^n T_{X/B}(-\log)$$

given by the iteration of the Kodaira–Spencer map is not trivial.

Actually, we study the behavior of the morphisms

$$\varphi_i: f_*\omega_{X/B} \otimes T_B(-E)^{\otimes i} \to R^i f_*\Omega^{n-i}_{X/B}(\log) \quad i = 1, \dots, n$$

induced by iteration of the contraction with the Kodaira–Spencer class. In particular, if φ_n is not zero, then f is strongly non-isotrivial.

We denote by k_t the representative of the Kodaira–Spencer class $\xi_t \in H^1(X_t, T_{X_t})$ coming from the so-called horizontal lift of the local vector field $\frac{\partial}{\partial t}$ on B (see [3], [37], [38] and Section 2 for details). We denote a local section of $R^i f_*\Omega^{n-i}_{X/B}(\log)$ by $[u]$; it can be seen as a function that maps a general point $t \in B$ to a cohomology class $[u_t]$ of $(n-i, i)$-forms on the corresponding fiber X_t. We denote by $u_{t,h} \in [u_t]$ the harmonic representative of $[u_t]$ and by $c(\psi)$ the curvature of a smooth metric on $\omega_B(E)$ of local weight ψ. We prove the following theorem.

Theorem 1.3. *If for every section $[u]$ of $\varphi^i(\mathcal{A} \otimes T_B(-E)^{\otimes i}) \cap K^i_{prim}$, $i = 1, \dots, n-1$, the following inequality holds over $B \setminus E$:*

$$\|(\bar{k}_t \cup u_t)_h\|^2 \geq ic(\psi)\|[u_t]\|^2$$

then either $f_*\omega_{X/B}$ is unitary flat or $f: X \to B$ is strongly non-isotrivial.

See Theorem 3.4. Theorem 1.3 must be put in correspondence to the semi-negativity results given in [3], [44].

Thanks to this new metric point of view, we obtain a somewhat surprising result on surfaces S of general type with positive index $\tau := \frac{1}{3}(c_1^2(S) - 2c_2(S))$, where $c_i(S)$, $i = 1, 2$, are the Chern classes of S. Algebraic surfaces with $\tau > 0$, even in the non-simply connected case, have been studied; we can quote here [17] for some examples and [24] for interesting geometric properties.

Theorem 1.4. *Let $f: X \to B$ be a semistable fibration and denote by r the rank of K^0_{prim}. Assume that the general fiber X_t is a surface satisfying $K^2_{X_t} > 8\chi(\mathcal{O}_{X_t}) + 2r + 1$, then f is strongly non-isotrivial.*

https://doi.org/10.1017/nmj.2025.10095 Published online by Cambridge University Press

See Theorem 3.10. We thank the referee for pointing out that, in particular, these families are rigid by [22], [41]. This theorem confirms that conditions for strong non-isotriviality are actually related to upper bounds on the dimension of the vector subspace of liftable top forms (see Remark 3.11).

Finally, motivated by the above-mentioned relation between curvature formulas and Torelli problem, we study how to relate the complex analytic approach stemming from [1], [2] to the theory of Massey products in the algebraic geometry context (cf. [35]).

Massey products have been introduced in [7], [29] and have been useful for solutions of Torelli-type problems (see [32]–[34]). More importantly, they have been used in [7], [35] to study the unitary flat summand $\mathcal{U}$ of $f_*\omega_{X/B}$ following an approach somewhat parallel to the metric one used by Berndtsson (see also [3]). The theory of Massey products can also be generalized in the case $\dim B > 1$ (see, e.g., [31]).

Our result concerns a metric interpretation of the liftability of relative volume forms. To the best of our knowledge, this is the first method proposed for studying the liftability of a Massey product that belongs to the ample part $\mathcal{A}$ of the Fujita decomposition.

More precisely, let $\mathcal{W} \subset f_*\omega_{X/B}$ be the standard subbundle generated via the Massey product technique, see Definition 4.1, and take on $Q := f_*\omega_{X/B}/\mathcal{W}$ the quotient metric of the Hermitian metric defined in [15]. In Proposition 4.5, we give a formula computing the Chern curvature Θ_Q on the class $[\alpha]$, where α is a Massey product in $f_*\omega_{X/B}$. From this explicit formula, it becomes easy to show that the cup product of the Kodaira–Spencer class and a Massey product α in the ample part of the Fujita decomposition vanishes if and only if the curvature of Q degenerates along $[\alpha]$.

Theorem 1.5. *Let α be a Massey product in $\mathcal{A}$. It is infinitesimally liftable if and only if $\langle \Theta_Q[\alpha], [\alpha]\rangle_Q = 0$.*

For more detailed statements, see Corollaries 4.6 and 4.8.

2. Setting

Let X be a smooth complex projective $n+1$-dimensional variety and B a smooth complex projective curve. In this article, we consider semistable fibrations $f\colon X \to B$, that is, f is a proper surjective morphism with connected fibers $X_t := f^{-1}(t)$, $t \in B$, and such that the singular fibers are reduced and normal crossing. We remark that, thanks to the semistable reduction theorem, see, for example, [19], we can always reduce to this case up to a sequence of blow-ups of X and cyclic Galois coverings of B.

We denote by $E \subset B$ the divisor of singular values of f and by S the (support of the) inverse image $f^{-1}(E)$.

In this section, we give a characterization of some of the key concepts of [31], [35], [36] by means of their metric properties.

2.1. *Preliminary definitions*

We begin by recalling the short exact sequence

$$0 \to f^*\omega_B \to \Omega^1_X \to \Omega^1_{X/B} \to 0, \tag{1}$$

which defines the sheaf of holomorphic relative differentials $\Omega^1_{X/B}$. This sheaf, in general, is not locally free, and for this reason, it is often convenient to consider the logarithmic version of (1):

$$0 \to f^*\omega_B(E) \to \Omega^1_X(\log S) \to \Omega^1_{X/B}(\log) \to 0. \tag{2}$$

We briefly recall that if S is locally given by $z_1z_2\dots z_k = 0$ in appropriate local coordinates, the sheaf $\Omega^1_X(\log S)$ of logarithmic differentials is the locally free $\mathcal{O}_X$-module generated by $dz_1/z_1, \dots, dz_k/z_k, dz_{k+1}, \dots, dz_{n+1}$.

https://doi.org/10.1017/nmj.2025.10095 Published online by Cambridge University Press

In the case of a semistable fibration, (2) is an exact sequence of vector bundles and its p-wedge product, for $p = 1, \dots, n$, is

$$0 \to f^*\omega_B(E) \otimes \Omega^{p-1}_{X/B}(\log) \to \Omega^p_X(\log S) \to \Omega^p_{X/B}(\log) \to 0 \tag{3}$$

and the determinant sheaf of $\Omega^1_{X/B}(\log)$ is

$$\det(\Omega^1_{X/B}(\log)) = \det(\Omega^1_X(\log S)) \otimes f^*\omega_B(E)^\vee = \omega_X(S) \otimes f^*\omega_B(E)^\vee = \omega_X \otimes f^*\omega_B^\vee =: \omega_{X/B}$$

the relative dualizing sheaf of $f\colon X \to B$.

The pushforward via f of Sequence (3) is a long exact sequence of locally free sheaves on B (see [39, Theorem 2.11] cf. [21, Lemma 2.11]), and we give the following definition.

Definition 2.1. We call K^i the kernel of the connecting morphism

$$R^i f_*\Omega^{n-i}_{X/B}(\log) \to \omega_B(E) \otimes R^{i+1} f_*\Omega^{n-i-1}_{X/B}(\log).$$

Remark 2.2. At general point $t \in B$, the fiber of K^i is the kernel of the homomorphism

$$H^i(X_t, \Omega^{n-i}_{X_t}) \to T^\vee_{B,t} \otimes H^{i+1}(X_t, \Omega^{n-i-1}_{X_t})$$

given by the cup product with the Kodaira–Spencer class $\xi_t \in H^1(X_t, T_{X_t})$.

Following [35], we also associate to $f\colon X \to B$ a local system of vector spaces on B, that is, a sheaf on B which is locally isomorphic to a constant sheaf. Denote by $\Omega^1_{X,d}$ the sheaf of de Rham closed holomorphic 1-forms on X, hence $\Omega^1_{X,d} \subset \Omega^1_X(\log S)$ and we consider the composition

$$f_*\Omega^1_{X,d} \hookrightarrow f_*\Omega^1_X(\log S) \to f_*\Omega^1_{X/B}(\log).$$

The image sheaf of $f_*\Omega^1_{X,d} \to f_*\Omega^1_{X/B}(\log)$ is a local systems (see [35, Lemma 2.6]).

Definition 2.3. We call $\mathbb{D}^1$ the local system obtained as image of the morphism $f_*\Omega^1_{X,d} \to f_*\Omega^1_{X/B}(\log)$.

Equivalently, $\mathbb{D}^1$ fits into the following short exact sequence:

$$0 \to \omega_B \to f_*\Omega^1_{X,d} \to \mathbb{D}^1 \to 0. \tag{4}$$

Similarly, we denote by $\mathbb{D}^n$ the local system contained in $f_*\Omega^n_{X/B}(\log) = f_*\omega_{X/B}$ and defined by the de Rham closed n-forms. As shown in [35, Theorem 3.7], $\mathbb{D}^n$ is actually the local system of the so-called second Fujita decomposition of $f_*\omega_{X/B}$. In fact, as recalled in the Introduction, there is a decomposition

$$f_*\omega_{X/B} = \mathcal{U} \oplus \mathcal{A}, \tag{5}$$

where $\mathcal{A}$ is an ample vector bundle (or $\mathcal{A} = 0$) and $\mathcal{U}$ is a unitary flat vector bundle; $\mathbb{D}^n$ is such that $\mathcal{U} = \mathbb{D}^n \otimes_{\mathbb{C}} \mathcal{O}_B$.

Finally, we point out that we have the inclusions $\mathbb{D}^n \subset \mathcal{U} \subseteq K^0 \subseteq f_*\omega_{X/B}$. The difference between the rank of $\mathcal{U}$ and the rank of K^0 is of interest (see, e.g., [11]).

2.2. Hermitian metrics on direct images

Consider the open subset $B \setminus E$ corresponding to the smooth fibers. The vector bundle $f_*\omega_{X/B}$ and, more in general, the bundles of primitive cohomology classes $\mathcal{P}^{n-i,i} \subset R^i f_*\Omega^{n-i}_{X/B}(\log)$ are endowed

https://doi.org/10.1017/nmj.2025.10095 Published online by Cambridge University Press

with a natural smooth Hermitian metric coming from the Hodge metric on the fibers (cf. [13], [15]). Actually, one can also consider the twisted cases $f_*(\omega_{X/B} \otimes L)$ and $R^i f_*(\Omega^{n-i}_{X/B}(\log) \otimes L)$, where L is a Hermitian line bundle on X whose curvature satisfies suitable properties. See [1], [2] for the case where L is semi-positive and [3] for the case where L is semi-negative.

From now on, we fix a Kähler form Ω on X. We denote by ω_t the restriction of Ω to X_t.

Following [1], [2], we first briefly recall the definition of the metric on $f_*\omega_{X/B}$ and some of its key properties. Consider u a local section of $f_*\omega_{X/B}$ over a disk $\Delta \subset B \setminus E$. It can be seen as a function that maps a general point $t \in \Delta$ to a global holomorphic $(n, 0)$-form u_t on the corresponding fiber X_t. The Hermitian norm is then defined as

$$\|u_t\|_t^2 = c_n \int_{X_t} u_t \wedge \bar{u}_t, \tag{6}$$

where $c_n = i^{n^2}$ is a unimodular constant.

Key computations on this metric strongly rely on the notion of a good representative of a section. A representative of u, denoted by $\mathbf{u}$, is a smooth $(n, 0)$-form on $f^{-1}(\Delta) \subset X$ which restricts to u_t on the fibers.

The key point is that we have an explicit expression for the Chern connection $D = D' + D''$ and curvature Θ in terms of the representative $\mathbf{u}$. Indeed, we can write

$$\partial \mathbf{u} = \mu \wedge dt,$$

and, since $\bar\partial \mathbf{u}$ is trivial along the fibers, we can write

$$\bar\partial \mathbf{u} = \nu \wedge d\bar{t} + \eta \wedge dt,$$

where ν and μ are relative forms of bidegree $(n, 0)$ and η is of bidegree $(n-1, 1)$. We have that

$$(D''u)_t = \nu_t d\bar{t} \tag{7}$$

and

$$(D'u)_t = P(\mu_t) dt, \tag{8}$$

where P is the orthogonal projection of $(n, 0)$-forms on the space of holomorphic $(n, 0)$-forms. The $(n-1, 1)$-form η_t has a neat interpretation in terms of the cup product with the Kodaira–Spencer class ξ_t: it is a representative of the cohomology class of the cup product $\xi_t \cup u_t \in H^{n-1,1}(X_t)$. See [1] for all the details on these formulas.

Of course, the choice of the representative $\mathbf{u}$ is not unique; two such choices differ by a term of the form $v \wedge dt$, but everything mentioned so far is independent from this choice.

Nevertheless, if u is a holomorphic section of $f_*\omega_{X/B}$ and t is a point in Δ, by [1], [2], there is a preferred choice of $\mathbf{u}$ such that μ_t is holomorphic (i.e., $P(\mu_t) = \mu_t$) and η_t is primitive. The key advantage of this choice is that we get $(D'u)_t = \mu_t dt$ and $-c_n \int_{X_t} \eta_t \wedge \bar\eta_t = \|\eta_t\|^2$. This choice of representative allows to compute the Chern curvature formula for the Hermitian metric as follows:

$$\langle \Theta u_t, u_t \rangle_t = \|\eta_t\|^2. \tag{9}$$

See [15] for the original proof. Berndtsson in [2, §3] proves that η_t is the unique harmonic representative of the cohomology class $\xi_t \cup u_t \in H^{n-1,1}(X_t)$, hence we can also write this formula as

$$\langle \Theta u_t, u_t \rangle_t = \|\xi_t \cup u_t\|^2. \tag{10}$$

https://doi.org/10.1017/nmj.2025.10095 Published online by Cambridge University Press

In the case of $\mathcal{P}^{n-i,i}$, the situation is similar, but we have to be a little more careful since a section $[u]$ is a function that maps t to a primitive cohomology class of $(n-i,i)$-forms on the corresponding fiber X_t. For this part, we follow [3].

We denote by $u_{t,h}$ the harmonic representative of this class. By a fundamental result of Kodaira–Spencer, the variation of $u_{t,h}$ with respect to t is smooth. The norm is

$$\|[u_t]\|_t^2 = (-1)^i c_n \int_{X_t} u_{t,h} \wedge \bar{u}_{t,h}. \tag{11}$$

As in the $(n,0)$ case, we consider a representative $\mathbf{u}$ of $[u]$, that is, a smooth $(n-i,i)$-form on $f^{-1}(\Delta)$ which is $\bar{\partial}$-closed on the fibers of f and whose restriction to each X_t belongs to the cohomology class $[u_t]$. In this case, the preferred choice of such a representative is given by the so-called vertical representative.

Definition 2.4. We say that a smooth vector field V on X is horizontal if $df(V)$ is a non-zero $(1,0)$ vector field and $\Omega(V,\bar{U}) = 0$ for every other vector field U with $df(U) = 0$. We say that a form $\mathbf{u}$ is vertical if the contraction $V|\mathbf{u}$ is equal to 0 for any horizontal vector field V.

The vertical representative is unique, see [3, Proposition 5.1]; we consider $\mathbf{u}$ the vertical representative of $[u]$. Similarly as before, we write

$$\bar{\partial}\mathbf{u} = \nu \wedge d\bar{t} + \eta \wedge dt$$

and

$$\partial\mathbf{u} = \zeta \wedge d\bar{t} + \mu \wedge dt.$$

Then, ν and μ are relative forms of bidegree $(n-i,i)$, η is of bidegree $(n-i-1,i+1)$, and ζ is of bidegree $(n-i+1,i-1)$; they are primitive on the fibers (see [3, §7]). As before, we can explicitly compute the Chern connection as

$$(D''[u])_t = [\nu_t]d\bar{t} \tag{12}$$

and

$$(D'[u])_t = [\mu_t]dt. \tag{13}$$

As a representative of the Kodaira–Spencer class ξ_t, we choose the (0,1)-form with coefficients in the tangent space T_X given by $\bar{\partial}V$, where V is the horizontal lift of the $(1,0)$ vector $\partial/\partial t$ on Δ. That is, on X_t, we denote $k_t := \bar{\partial}V_{|X_t}$ and $\xi_t = [k_t] \in H^1(X_t, T_{X_t})$. It turns out that with this choice

$$\eta_t = k_t \cup u_t \tag{14}$$

and

$$\zeta_t = \bar{k}_t \cup u_t \tag{15}$$

if u_t is harmonic.

Remark 2.5. In terms of cohomology classes, this means that

$$[\eta_t] = \xi_t \cup [u_t]$$

and

$$[\zeta_t] = \overline{(\xi_t \cup \overline{[u_t]})}$$

https://doi.org/10.1017/nmj.2025.10095 Published online by Cambridge University Press

since the contraction with $\bar{k}_t$ corresponds to the composition

$$H^{p,q} \cong H^{q,p} \xrightarrow{\xi_t \cup} H^{q-1,p+1} \cong H^{p+1,q-1},$$

where the isomorphisms are given by complex conjugation.

The curvature formula generalizing (9) is

$$\langle \Theta[u_t], [u_t] \rangle_t = \|\eta_{t,h}\|^2 - \|\zeta_{t,h}\|^2, \tag{16}$$

where by $\eta_{t,h}$ and $\zeta_{t,h}$, we mean the harmonic part of η_t and ζ_t, respectively (see [3]).

One of the immediate consequences of the curvature formulas (9) and (16) is that the bundles $K^i_{\text{prim}} := K^i \cap \mathcal{P}^{n-i,i}$ are semi-negatively curved since by definition they are the primitive part of the kernel of the cup product with the Kodaira–Spencer class. For example, if u is a section of $K^0_{\text{prim}} = K^0$, then by (9), we have that $\langle \Theta u_t, u_t \rangle_t = 0$ and for $[u]$, a section of K^i_{prim}, $i \geq 1$, by (16), we have that $\langle \Theta[u_t], [u_t] \rangle_t = -\|\zeta_{t,h}\|^2 \leq 0$. In both cases, since curvature decreases in subbundles, we have that K^i_{prim} is semi-negatively curved on $B \setminus E$ for $i \geq 0$.

Actually, there is an extension across the singularities. Indeed, $\mathcal{P}^{n-i,i}$ and K^i_{prim} are firstly defined on $B \setminus E$, but since they are subsheaves of $R^i f_* \Omega^{n-i}_{X/B}(\log)$, which is defined over the whole base B, they have a natural extension across E. Furthermore, by [3, §9], the chosen smooth Hermitian metric on $B \setminus E$ extends to a singular Hermitian metric. This singular metric is semi-negatively curved on K^i_{prim} in the sense of singular metrics.

2.3. Metric interpretation

The Chern connection D on the vector bundle $f_* \omega_{X/B}$ is compatible with the Gauss–Manin connection. This is shown, for example, in [2, §2.3]. Here, we rephrase it using the objects introduced in the beginning of this section. Using the same notation as above, we denote by Θ the Chern curvature of $f_* \omega_{X/B}$.

Proposition 2.6. *Consider the vector bundle $f_* \omega_{X/B}$ with the Hermitian structure described above. The subsheaves $\mathbb{D}^n \subset \mathcal{U} \subseteq K^0$ of $f_* \omega_{X/B}$ are characterized as follows:*

(i) *$\mathbb{D}^n$ is the kernel of the Chern connection D.*
(ii) *$\mathcal{U}$ is the largest flat subbundle of $f_* \omega_{X/B}$.*
(iii) *K^0 is the kernel of the curvature Θ.*

In particular, K^0 is a (holomorphic) direct summand of $f_* \omega_{X/B}$ if and only if $K^0 = \mathcal{U}$.

Proof. *(i)* First note that $\mathbb{D}^n = \operatorname{Im}(f_* \Omega^n_{X,d} \to f_* \omega_{X/B})$. In particular, if u is a local section of $\mathbb{D}^n$, it follows that, at least locally on the base B, we can choose the representative $\mathbf{u}$ to be a holomorphic de Rham closed n-form. Hence, $\partial \mathbf{u} = \bar{\partial} \mathbf{u} = 0$ and we get that $\mathbb{D}^n$ is contained in the kernel of the connection D by (7) and (8). Vice versa, if u is a holomorphic section in the kernel of D, by (9), we have that $\eta_t = 0$ for all t, so that $\xi_t \cup u_t = 0$. This means that u is in K^0 by Remark 2.2, hence, since K^0 can also be seen as the image of $f_* \Omega^n_X \to f_* \omega_{X/B}$, we can choose $\mathbf{u}$ to be a section of $f_* \Omega^n_X$, so that $\bar{\partial} \mathbf{u} = 0$. Finally, since $D' u = 0$, we also have that $\mu = 0$, that is, $\partial \mathbf{u} = 0$. So we conclude that $\mathbf{u}$ is de Rham closed and u is a local section of $\mathbb{D}^n$.

(ii) By point *(i)*, this follows easily since $\mathcal{U} = \mathbb{D}^n \otimes \mathcal{O}_B$. Indeed, note that $\mathcal{U}$ is contained both in the kernel of the curvature Θ and in the kernel of the second fundamental form of $\mathcal{U}$ in $f_* \omega_{X/B}$.

(iii) If u is a holomorphic section of K^0, locally on B, the representative $\mathbf{u}$ can be chosen as a holomorphic form, as in point *(i)*. So $\bar{\partial} \mathbf{u} = 0$ and $\partial \mathbf{u} = \mu \wedge dt$, where μ is a holomorphic section of

https://doi.org/10.1017/nmj.2025.10095 Published online by Cambridge University Press

$f_*\omega_{X/B}$. Hence, $\Theta u = D''D'u = 0$. Vice versa, if u is a holomorphic section of $f_*\omega_{X/B}$ with $\Theta u = 0$, by (10), u_t is in the kernel of the cup product with ξ_t for all t, hence u is a section of K^0, by Remark 2.2. □

Remark 2.7. We have already seen that K^0 is semi-negatively curved with the metric induced from $f_*\omega_{X/B}$. It seems actually interesting to understand when K^0 is maximal inside $f_*\omega_{X/B}$ with this property. For example, in Section 3, we study the kernels of the iterated cup product with the Kodaira–Spencer class.

3. Isotriviality and strong non-isotriviality

In this section, we use the metrics introduced in Section 2 to study possible conditions for the triviality or, on the opposite side, strong non-isotriviality of the fibration $f: X \to B$.

3.1. *Strongly non-isotrivial fibrations*

We give the following definition.

Definition 3.1. We denote by $\phi_i: f_*\omega_{X/B} \to \omega_B(E)^{\otimes i} \otimes R^i f_*\Omega^{n-i}_{X/B}(\log)$ the composition of the connecting morphisms obtained from the direct images of (3):

$$\phi_i: f_*\omega_{X/B} \to \omega_B(E) \otimes R^1 f_*\Omega^{n-1}_{X/B}(\log) \to \cdots \to \omega_B(E)^{\otimes i} \otimes R^i f_*\Omega^{n-i}_{X/B}(\log).$$

Similarly, we denote by $\varphi_i: f_*\omega_{X/B} \otimes T_B(-E)^{\otimes i} \to R^i f_*\Omega^{n-i}_{X/B}(\log)$ the morphism obtained from ϕ_i by tensoring with $T_B(-E)^{\otimes i}$.

In the literature, they are referred to as the Griffiths–Yukawa coupling or as iterated Higgs maps.

These morphisms are given on smooth fibers by iterated cup product with the Kodaira–Spencer class. From the Introduction, we recall that if ϕ_n is not zero (equivalently, φ_n is not zero), then f is strongly non-isotrivial. By the Fujita decomposition (5), $f_*\omega_{X/B} = \mathcal{U} \oplus \mathcal{A}$ and, since $\mathcal{U}$ is contained in $K^0 = \ker \phi_1$, we look at the image of $\mathcal{A}$.

In particular, note that if $\mathcal{A}$ is not contained in K^0 (i.e., $\mathcal{A} \neq 0$), and also $\phi_i(\mathcal{A})$ is not contained in $K^i \otimes \omega_B(E)^{\otimes i}$, for $i = 1, \dots, n-1$, then the image $\phi_n(\mathcal{A})$ is not zero and f is strongly non-isotrivial. Actually, in our setting, we are dealing with primitive forms. In fact, the holomorphic $(n,0)$-forms on the fibers are primitive and hence the images $\phi_i(\mathcal{A})$ are bundles of primitive classes, since the cup product of a primitive form with the Kodaira–Spencer class is again primitive.

3.1.1. *Curvature conditions for strong non-isotriviality*

A possible condition that ensures that $\phi_i(\mathcal{A})$ is not contained in $K^i_{\text{prim}} \otimes \omega_B(E)^{\otimes i}$, $i = 1, \dots, n-1$, since $\mathcal{A}$ is ample, is that $K^i_{\text{prim}} \otimes \omega_B(E)^{\otimes i}$ induces a metric with semi-negative curvature on $\phi_i(\mathcal{A}) \cap (K^i_{\text{prim}} \otimes \omega_B(E)^{\otimes i})$.

We have seen in Section 2 how to define a smooth metric on $\mathcal{P}^{n-i,i}|_{B\setminus E}$. Given a metric on $\omega_B(E)$ of local weight ψ and curvature denoted by $c(\psi)$, we consider the tensor metric on the bundle $\mathcal{P}^{n-i,i} \otimes \omega_B(E)^{\otimes i}|_{B\setminus E}$. The following result gives a curvature formula for this metric. We write an element of $\mathcal{P}^{n-i,i} \otimes \omega_B(E)^{\otimes i}$ at the point t as $[u_t] \otimes l_t$, where l_t is of norm 1 and u_t is harmonic on X_t.

Proposition 3.2. *The curvature Θ_i of $\mathcal{P}^{n-i,i} \otimes \omega_B(E)^{\otimes i}|_{B\setminus E}$ satisfies*

$$\langle \Theta_i([u_t] \otimes l_t), [u_t] \otimes l_t \rangle = \|\eta_{t,h}\|^2 - \|\zeta_{t,h}\|^2 + ic(\psi)\|[u_t]\|^2, \tag{17}$$

where $\eta_{t,h}$ and $\zeta_{t,h}$ are defined as in (16).

https://doi.org/10.1017/nmj.2025.10095 Published online by Cambridge University Press

Proof. This is the formula for the curvature of a tensor product. Call for simplicity $F = \mathcal{P}^{n-i,i}$ and $L = \omega_B(E)^{\otimes i}$. Then, $\Theta_{F\otimes L} = \Theta_F \otimes I_L + I_F \otimes \Theta_L$ and

$$\langle \Theta_{F\otimes L}[u_t] \otimes l_t, [u_t] \otimes l_t\rangle = \langle \Theta_F[u_t], [u_t]\rangle \|l_t\|^2 + \langle \Theta_L l_t, l_t\rangle \|[u_t]\|^2$$

and we are done using that $\|l_t\| = 1$ and Equation (16). □

We prove the following proposition.

Proposition 3.3. *Assume that there exists a smooth metric on $\omega_B(E)$ of local weight ψ such that on $B \setminus E$, we have $\|(\bar{k}_t \cup u_t)_h\|^2 \geq ic(\psi)\|[u_t]\|^2$ for every section $[u]$ of $\varphi^i(\mathcal{A} \otimes T_B(-E)^{\otimes i}) \cap K^i_{prim}$. Then, $\phi_i(\mathcal{A}) \cap (K^i_{prim} \otimes \omega_B(E)^{\otimes i})$ admits a semi-negative singular Hermitian metric.*

Proof. As recalled in Section 2, by [3], K^i_{prim} has a semi-negatively curved singular Hermitian metric. We consider on $K^i_{\text{prim}} \otimes \omega_B(E)^{\otimes i}$ the singular metric given by the tensor product and on $\phi_i(\mathcal{A}) \cap (K^i_{\text{prim}} \otimes \omega_B(E)^{\otimes i})$ its restriction.

On $B \setminus E$, these metrics are smooth and coincide with the restriction of the metric on $\mathcal{P}^{n-i,i} \otimes \omega_B(E)^{\otimes i}|_{B\setminus E}$ discussed above.

For the sections of $K^i_{\text{prim}} \otimes \omega_B(E)^{\otimes i}$, we have by (15) and (17),

$$\langle \Theta_i([u_t] \otimes l_t), [u_t] \otimes l_t\rangle = -\|\zeta_{t,h}\|^2 + ic(\psi)\|[u_t]\|^2 = -\|(\bar{k}_t \cup u_t)_h\|^2 + ic(\psi)\|[u_t]\|^2. \tag{18}$$

By our hypothesis, the metric on $\phi_i(\mathcal{A}) \cap (K^i_{\text{prim}} \otimes \omega_B(E)^{\otimes i})$ is then semi-negatively curved on $B \setminus E$, since curvature decreases in subbundles.

By the results in [3], it is not difficult to see that it is semi-negative everywhere in the sense of singular metrics. □

So, under the hypotheses of Proposition 3.3, the image $\phi_i(\mathcal{A})$ is not contained in $K^i_{\text{prim}} \otimes \omega_B(E)^{\otimes i}$, $i = 1, \ldots, n-1$, since $\mathcal{A}$ is ample. Hence, we conclude the following theorem.

Theorem 3.4. *Let $f : X \to B$ be a semistable fibration and assume that there exists a smooth metric on $\omega_B(E)$ of local weight ψ such that on $B \setminus E$, we have $\|(\bar{k}_t \cup u_t)_h\|^2 \geq ic(\psi)\|[u_t]\|^2$ for every section $[u]$ of $\varphi^i(\mathcal{A} \otimes T_B(-E)^{\otimes i}) \cap K^i_{prim}$, $i = 1, \ldots, n-1$. Then, either $f_*\omega_{X/B}$ is unitary flat or the fibration is strongly non-isotrivial.*

Note that a parallel approach is given by the study of the semi-positivity of $\mathcal{A} \otimes T_B(-E)^{\otimes i}$ for $i = 1, \ldots, n-1$. Similarly as before, this condition shows that $\varphi_i(\mathcal{A} \otimes T_B(-E)^{\otimes i})$ is not contained in K^i_{prim} for $i = 1, \ldots, n-1$ and hence φ_n is not zero and f is strongly non-isotrivial (or $\mathcal{A}$ is zero). We skip the computations since they are similar to the previous ones. We note that we only need the semi-positivity of $\mathcal{A} \otimes T_B(-E)^{\otimes n-1}$ since it implies the semi-positivity of $\mathcal{A} \otimes T_B(-E)^{\otimes i}$, $i = 1, \ldots, n-2$. This semi-positivity is given on $B \setminus E$ by the inequality $\|(k_t \cup u_t)_h\|^2 \geq (n-1)c(\psi)\|u_t\|^2$.

Theorem 3.5. *Let $f : X \to B$ be a semistable fibration on a smooth projective curve B and assume that there exists a smooth metric on $\omega_B(E)$ of local weight ψ such that on $B \setminus E$, we have $\|(k_t \cup u_t)_h\|^2 \geq (n-1)c(\psi)\|u_t\|^2$ for all u in $\mathcal{A}$. Then, either $f_*\omega_{X/B}$ is unitary flat or the fibration is strongly non-isotrivial.*

Remark 3.6. In principle, there is no correlation between the inequalities of Theorems 3.4 and 3.5, except when $n = 2$. In fact, in this case, note that the condition $\|(k_t \cup u_t)_h\|^2 \geq c(\psi)\|u_t\|^2$ of Theorem 3.5 together with the inequality $\|(\bar{k}_t \cup (k_t \cup u_t)_h)_h\|^2 \leq \|(k_t \cup u_t)_h\|\|u_t\|$ coming from Cauchy formula together with the fact that k_t and $\bar{k}_t$ are adjoint (see Proposition 3.7 below), give the condition $\|(\bar{k}_t \cup (k_t \cup u_t)_h)_h\|^2 \geq c(\psi)\|(k_t \cup u_t)_h\|^2$. This is the condition required in Theorem 3.4 since $k_t \cup u_t$ gives, while u_t varies in $\mathcal{A}$, all the elements in $\varphi^i(\mathcal{A} \otimes T_B(-E)^{\otimes i}) \cap K^i_{\text{prim}}$.

https://doi.org/10.1017/nmj.2025.10095 Published online by Cambridge University Press

As a final note, everything in this section can actually be generalized for fibrations over a higher-dimensional base. In fact, even when B is not a curve, by [6], the direct image $f_*\omega_{X/B}$ has a decomposition $f_*\omega_{X/B} = \mathcal{U} \oplus \hat{\mathcal{A}}$, where $\mathcal{U}$ is unitary flat and $\hat{\mathcal{A}}$ is generically ample, that is, $\hat{\mathcal{A}}$ restricts to an ample vector bundle on the general complete intersection smooth curve in B. Hence, it is possible to consider the generalization of the morphism ϕ_i and φ_i as $\phi_i\colon f_*\omega_{X/B} \to \mathrm{Sym}^i\Omega^1_B(\log E) \otimes R^i f_*\Omega^{n-i}_{X/B}(\log)$ and $\varphi_i\colon f_*\omega_{X/B} \otimes \mathrm{Sym}^i T_B(-\log E) \to R^i f_*\Omega^{n-i}_{X/B}(\log)$ and argue similarly to obtain similar conditions involving the curvature of the symmetric bundles.

3.1.2. A case on surfaces

Now, consider again the composition ϕ_n

$$\phi_n\colon f_*\omega_{X/B} \to \omega_B(E) \otimes R^1 f_*\Omega^{n-1}_{X/B}(\log) \to \cdots \to \omega_B(E)^{\otimes n} \otimes R^n f_*\mathcal{O}_X$$

and restrict it to the general smooth fiber X_t as

$$\phi_{n,t}\colon H^{n,0}(X_t) \to T^\vee_{B,t} \otimes H^{n-1,1}(X_t) \to \cdots \to (T^\vee_{B,t})^{\otimes n} \otimes H^{0,n}(X_t). \tag{19}$$

Since $\dim B = 1$, choosing a local coordinate around t, we identify $T^\vee_{B,t} \cong \mathbb{C}$ and, with a little abuse of notation, we denote by $\xi_t\cup$ every map

$$\xi_t\cup\colon H^{p,q}(X_t) \to H^{p-1,q+1}(X_t)$$

since they are all defined by the cup product with the Kodaira–Spencer class ξ_t.

Proposition 3.7. *Let $[u] \in H^{p,q}_{prim}(X_t)$ and $[v] \in H^{p-1,q+1}_{prim}(X_t)$, $p+q=n$. Then,*

$$\langle \xi_t \cup [u], [v]\rangle = \langle [u], \overline{\xi_t \cup \overline{[v]}}\rangle.$$

Proof. We choose harmonic representatives u_h and v_h of the classes $[u]$ and $[v]$, respectively. Also, we represent the class ξ_t by its representative k_t as above.

We have

$$\begin{aligned}\langle \xi_t \cup [u], [v]\rangle &= (-1)^{q+1} c_n \int_{X_t} (\xi_t \cup [u])_h \wedge \bar{v}_h \\ &= (-1)^{q+1} c_n \int_{X_t} (k_t \cup u_h)_h \wedge \bar{v}_h = (-1)^{q+1} c_n \int_{X_t} (k_t \cup u_h) \wedge \bar{v}_h\end{aligned}$$

and

$$\begin{aligned}\langle [u], \overline{\xi_t \cup \overline{[v]}}\rangle &= (-1)^{q} c_n \int_{X_t} u_h \wedge (\xi_t \cup \overline{[v]})_h \\ &= (-1)^{q} c_n \int_{X_t} u_h \wedge (k_t \cup \bar{v}_h)_h = (-1)^{q} c_n \int_{X_t} u_h \wedge (k_t \cup \bar{v}_h).\end{aligned}$$

It is now easy to see, for example, writing locally u_h, v_h, and k_t, that $(-1)^{q+1}c_n \int_{X_t}(k_t \cup u_h) \wedge \bar{v}_h = (-1)^q c_n \int_{X_t} u_h \wedge (k_t \cup \bar{v}_h)$. This can, for example, be verified on decomposable forms, since everything is linear. □

Remark 3.8. From this formula, we easily get the condition $K^q_{\mathrm{prim},t} \subset \overline{\xi_t(H^{q+1,p-1}_{\mathrm{prim}}(X_t))}^{\perp}$.

Denote by ξ_t^q the composition of the cup product with ξ_t taken q times,

$$\xi_t^q\colon H^{n,0}(X_t) \to H^{n-q,q}(X_t).$$

https://doi.org/10.1017/nmj.2025.10095 Published online by Cambridge University Press

Hence, we have the following condition for strong non-isotriviality.

Proposition 3.9. *If* $\xi_t^{n-1}(H^{n,0}(X_t)) \cap \overline{\xi_t(H^{n,0}(X_t))} \neq \{0\}$*, then the fibration is strongly non-isotrivial.*

Proof. This comes from the fact that all the elements of $K^{n-1}_{\text{prim},t}$ are contained in $\overline{\xi_t(H^{n,0}_{\text{prim}}(X_t))}^{\perp}$ by the previous remark. Hence, $\phi_{n,t} = \xi_t^n \neq 0$. □

In the case of $n = 2$, this condition suggests an interesting numerical bound concerning families of surfaces with positive index.

Theorem 3.10. *Let* $f: X \to B$ *be a semistable fibration and denote by r the rank of* K^0*. Assume that the general fiber* X_t *is a surface satisfying* $K^2_{X_t} > 8\chi(\mathcal{O}_{X_t}) + 2r + 1$*, then f is strongly non-isotrivial.*

Proof. By Proposition 3.9, it is enough to show that the intersection $\xi_t(H^{2,0}(X_t)) \cap \overline{\xi_t(H^{2,0}(X_t))}$ is nontrivial in $H^{1,1}_{\text{prim}}(X_t)$. This is true if $\dim \xi_t(H^{2,0}(X_t)) > h^{1,1}_{\text{prim}}/2$, that is, $p_g - r > h^{1,1}_{\text{prim}}/2$. Now since $c_2 = 2 - 4q + 2p_g + h^{1,1}$ and $h^{1,1}_{\text{prim}} \leq h^{1,1} - 1$, by Noether's formula $12\chi(\mathcal{O}_{X_t}) = K^2_{X_t} + c_2$, the above condition is implied by

$$2p_g - 2r > -2p_g + 4q - 2 + 12\chi(\mathcal{O}_{X_t}) - K^2_{X_t} - 1,$$

which we can write as

$$K^2_{X_t} > 8\chi(\mathcal{O}_{X_t}) + 2r + 1.$$

□

Remark 3.11. Note that since $9\chi(\mathcal{O}_{X_t}) \leq K^2_{X_t}$ by the Miyaoka–Yau inequality [25], [43], our assumption can hold only if $r \leq \frac{p_g - q}{2}$.

3.2. Conditions for triviality of a fibration

In this section, we look at the opposite problem and give conditions for the triviality of $f: X \to B$ over a Zariski open subset of B. So far, we have considered the direct image of the relative dualizing sheaf $f_*\omega_{X/B}$, here, we will work on the direct image of the relative pluricanonical bundles $f_*(\omega^{\otimes m}_{X/B})$. Hence, we briefly recall how to construct a Hermitian metric on the vector bundle associated with $f_*(\omega_{X/B} \otimes L)$, where L is a line bundle on X endowed with a singular metric assumed to be smooth when restricted to the general fiber. We denote by ψ the local weight of this metric and assume that L is semi-positively curved.

Without covering all the details, we just highlight the differences with the untwisted case of Section 2. In particular, on an appropriate open subset of B, the metric in (6) is replaced by

$$\|u_t\|^2_t = \int_{X_t} c_n u_t \wedge \bar{u}_t e^{-\psi} \tag{20}$$

and the curvature in (9) is replaced by

$$\langle \Theta u_t, u_t \rangle_t = f_*(c_n i\partial\bar{\partial}\psi \wedge \mathbf{u} \wedge \bar{\mathbf{u}} e^{-\psi})/dV_t + \|\eta_t\|^2, \tag{21}$$

where $dV_t = idt \wedge d\bar{t}$.

For our purposes, we take $L = \omega^{\otimes m-1}_{X/B}$ and we prove the following theorem which complements the constancy result proved in [42].

Theorem 3.12. *Let* $f: X \to B$ *be a family of canonically polarized manifolds. Assume that there exists a surjective morphism* $\rho: V \times B \to X$*, where V is a projective variety, such that* $p := f \circ \rho$ *is the projection on B. Then, X is Zariski locally trivial.*

https://doi.org/10.1017/nmj.2025.10095 Published online by Cambridge University Press

Proof. By [18], [23], the direct image $f_*(\omega_{X/B}^{\otimes m})$ has a so-called Catanese–Fujita–Kawamata decomposition

$$f_*(\omega_{X/B}^{\otimes m}) = \mathcal{U} \oplus \mathcal{A},$$

where $\mathcal{U}$ is unitary flat and $\mathcal{A}$ is ample (or zero). On the other hand, we also get an injective morphism $f_*(\omega_{X/B}^{\otimes m}) \hookrightarrow p_*((\Omega^n_{V\times B/B})^{\otimes m}) = H^0(V, (\Omega^n_V)^{\otimes m}) \otimes \mathcal{O}_B$. From this, it is not difficult to see that $\mathcal{A} = 0$ and $\mathcal{U} = \mathcal{O}_B^l$ is trivial.

Now, we write $f_*(\omega_{X/B}^{\otimes m}) = f_*(\omega_{X/B} \otimes \omega_{X/B}^{\otimes m-1})$ and take $L = \omega_{X/B}^{\otimes m-1}$. L admits a singular metric which is smooth and with strictly positive curvature when restricted to the smooth fibers (cf. [37]). By [27, Theorem 3.3.5], the vector bundle $f_*(\omega_{X/B}^{\otimes m})$ admits a singular Hermitian metric with semi-positive curvature. But by the fact that $f_*(\omega_{X/B}^{\otimes m})$ is trivial, the curvature of this metric cannot be strictly positive definite; actually, it is zero, see, for example, [16, Lemma 13.2]. On the open subset where f is smooth, this metric coincides with the one recalled above in (20) and by the explicit curvature formula given in [2, Theorem 1.2] and following remarks, the Kodaira–Spencer class vanishes at every point of this open subset.

Moreover, by the argument in [2, §4.1, p. 1216], the Kodaira–Spencer class ξ_t is represented by the form $\bar{\partial} V_{\psi\,|X_t}$, where V_ψ is a (in principle only smooth) vector field which turns out to be holomorphic since the curvature of $f_*(\omega_{X/B}^{\otimes m})$ is not strictly positive. The fibration f is then analytically locally trivial.

To show that f is trivial over a Zariski open subset U of B, we can proceed then as in [42, p. 22]. □

Remark 3.13. Theorem 3.12 relies on the existence of an ample part in $f_*(\omega_{X/B}^{\otimes m})$. For the study of this ample part using algebraic techniques, see [40].

4. Massey product and curvature formula

Using curvature formulas, we can give conditions on the liftability of Massey products. First, we briefly recall this notion, which is one of the main constructions of [28], [35]. If a Massey product α_t is Massey trivial, see Definition 4.4 below, then its cup product with the Kodaira–Spencer class ξ_t is zero, that is, α_t is liftable. The consequences of this are well studied in relation to Torelli-type problems. On the other hand, the case where α_t is not Massey trivial, but still liftable, is not well understood, especially when the Massey product is a section of the ample part $\mathcal{A}$ of the Fujita decomposition (5). Hence, below we give a metric interpretation of this condition. To our best knowledge, it is the first time that such an interpretation is studied.

From the direct image of Sequence (1), we get a short exact sequence on B defining the vector bundle K_∂:

$$0 \to \omega_B \to f_*\Omega^1_X \to K_\partial \to 0. \tag{22}$$

Intuitively, we can think of $K_\partial \subseteq f_*\Omega^1_{X/B}$ as the sheaf of holomorphic one forms on the fibers of f which are liftable to a tubular neighborhood in X of such fibers. For a more in depth study of K_∂, see [10], [11], [28] for the case $n = 2$ and [35] for the general case.

Note that if in Sequence (22) we consider, instead of $f_*\Omega^1_X$, the sheaf $f_*\Omega^1_{X,d}$, we obtain the exact sequence (4) and recover the local system $\mathbb{D}^1$ as introduced in Section 2.

By [28, Lemma 3.5] or [35, Lemma 2.2], Sequence (22) splits, and from now on, we choose and fix one such splitting. Now, consider $n+1$ linearly independent sections $\eta_1, \dots, \eta_{n+1} \in \Gamma(\Delta, K_\partial)$ on an open subset $\Delta \subset B$ of coordinate t and denote by $s_1, \dots, s_{n+1} \in \Gamma(\Delta, f_*\Omega^1_X)$ the liftings of $\eta_1, \dots, \eta_{n+1}$ according to the chosen splitting of Sequence (22).

Definition 4.1. We call $\alpha_i \in \Gamma(\Delta, f_*\omega_{X/B})$, $i = 1, \dots, n+1$, the relative forms defined by $\eta_1 \wedge \cdots \wedge \hat{\eta}_i \wedge \cdots \wedge \eta_{n+1}$ and $\mathcal{W}$ the (local) $\mathcal{O}_B$-submodule of $f_*\omega_{X/B}$ generated by the α_i.

https://doi.org/10.1017/nmj.2025.10095 Published online by Cambridge University Press

Note that by $\hat{\eta}_i$, we mean that η_i is excluded from the wedge product.

Definition 4.2. The Massey product of $\eta_1, \dots, \eta_{n+1}$ is the section $\alpha \in \Gamma(\Delta, f_*\omega_{X/B})$ defined by $s_1 \wedge \cdots \wedge s_{n+1} = \alpha \wedge dt$. We say that the sections $\eta_1, \dots, \eta_{n+1}$ are Massey trivial if α is a section of the submodule $\mathcal{W}$.

Remark 4.3. The crucial result is that if $\eta_1, \dots, \eta_{n+1}$ are Massey trivial, we can find liftings $\tilde{s}_i$ such that

$$\tilde{s}_1 \wedge \cdots \wedge \tilde{s}_{n+1} = 0; \tag{23}$$

in particular, we can take α to be zero. See, for example, [35, Proposition 4.10].

Since $\mathbb{D}^1$ is a subsheaf of K_∂, it makes sense to construct Massey products starting from sections of $\mathbb{D}^1$, that is, $\eta_i \in \Gamma(\Delta, \mathbb{D}^1)$. One of the key points in [28], [35] is exactly to consider this setting.

The construction of Massey products can also be done pointwise, that is, for a fixed regular value $t \in B$. This is the original approach of [7], [29], [34]. In this case, one considers $\eta_{1,t}, \dots, \eta_{n+1,t} \in K_{\partial,t} \leq H^0(X_t, \Omega^1_{X_t})$ the restriction of the η_i to the fiber X_t. Similarly, $\mathcal{W}_t = \langle \alpha_{1,t}, \dots, \alpha_{n+1,t} \rangle \leq H^0(X_t, \omega_{X_t})$ and $\alpha_t \in H^0(X_t, \omega_{X_t})$ are restriction on X_t of $\mathcal{W}$ and of α, respectively. We have a pointwise version of Definition 4.2

Definition 4.4. We say that $\eta_{1,t}, \dots, \eta_{n+1,t}$ are Massey trivial (at t) if $\alpha_t \in \mathcal{W}_t$.

4.1. A curvature formula on Massey products

In this section, we consider a disk $\Delta \subset B \setminus E$. Using the same notation as above, take sections $\eta_1, \dots, \eta_{n+1} \in \Gamma(\Delta, K_\partial)$ and α their Massey product. The notion of Massey triviality, see Definition 4.2, means that α is a section of the O_B-submodule $\mathcal{W} = \langle \alpha_1, \dots, \alpha_{n+1} \rangle \subseteq f_*\omega_{X/B}{}_{|\Delta}$, hence it is natural to study the quotient $Q := f_*\omega_{X/B}{}_{|\Delta}/\mathcal{W}$. Indeed, the class $[\alpha]$ does not depend on the choice of liftings s_i of η_i, while the section α does. Following this idea, we assume that $\mathcal{W}$ is a non-trivial subbundle of $f_*\omega_{X/B}{}_{|\Delta}$ and we compute the curvature of Q on the section $[\alpha]$ and relate it to the cup product with the Kodaira–Spencer class and the pointwise liftability of α. The result is a version of Formula (10), the difference is that working on the quotient Q makes the key role of the local system $\mathbb{D}^1$ more clear.

In this section, let $\mathcal{E} := f_*\omega_{X/B}{}_{|\Delta}$ and, to avoid confusion, here we denote by $\langle , \rangle_{\mathcal{E}}$ its metric introduced in Section 2 and by $\langle , \rangle_Q$ the quotient metric on Q.

By the above discussion, the case to be studied is when α_t is not contained in the vector space $\mathcal{W}_t = \langle \alpha_{1,t}, \dots, \alpha_{n+1,t} \rangle$. Without loss of generality, we assume that, by a standard orthogonalization process, $\alpha_{1,t}, \dots, \alpha_{m,t}, \alpha_t$, $m \leq n+1$, are a unitary basis of $\langle \mathcal{W}_t, \alpha_t \rangle$ (note that $\alpha_{1,t}, \dots, \alpha_{n+1,t}$ are not necessarily linearly independent). We can extend to a C^∞ unitary frame of $\mathcal{E}$, denoted by $\alpha'1, \dots, \alpha'm, \alpha', \tau_1, \dots, \tau_k$, such that $\alpha'1, \dots, \alpha'm$ is a smooth frame for $\mathcal{W}$, $\alpha', \tau_1, \dots, \tau_k$ is a smooth frame for $\mathcal{W}^\perp \cong Q$, and $\alpha'1, \dots, \alpha'm, \alpha'$ restrict to $\alpha_{1,t}, \dots, \alpha_{m,t}, \alpha_t$ on X_t.

We compute $\langle \Theta_Q[\alpha_t], [\alpha_t] \rangle_Q$ by means of the second fundamental form $\mathfrak{S}$ of Q in $\mathcal{E}$. Following the notation of [14], the second fundamental form $\mathfrak{S} \colon \mathcal{A}^0(Q) \to \mathcal{A}^1(\mathcal{W})$ is defined as the composition of the connection D on $\mathcal{E}$ restricted to $Q \cong \mathcal{W}^\perp$

$$D_{|Q} \colon \mathcal{A}^0(Q) \to \mathcal{A}^1(\mathcal{E})$$

and the projection to $\mathcal{W}$

$$\mathcal{A}^1(\mathcal{E}) \to \mathcal{A}^1(\mathcal{W}).$$

https://doi.org/10.1017/nmj.2025.10095 Published online by Cambridge University Press

We have

$$\langle \Theta_Q[\alpha_t], [\alpha_t] \rangle_Q = \langle \Theta_{\mathcal{E}} \alpha_t, \alpha_t \rangle_{\mathcal{E}} - \langle \mathfrak{S}\alpha_t, \mathfrak{S}\alpha_t \rangle_{\mathcal{E}}.$$

The first of these summands is given by (10), that is,

$$\langle \Theta_{\mathcal{E}} \alpha_t, \alpha_t \rangle_{\mathcal{E}} = \|\xi_t \cup \alpha_t\|^2$$

hence we are left with the second summand.

In our chosen unitary frame, we write $\mathfrak{S}\alpha_t = \lambda_1 \alpha_{1,t} + \cdots + \lambda_m \alpha_{m,t}$, for certain $(0, 1)$-forms λ_i, [20, Proposition 1.6.6]. Explicitly, since $\mathcal{W}$ is a holomorphic subbundle of $\mathcal{E}$,

$$\lambda_i = \langle (D\alpha')_t, \alpha_{i,t} \rangle_{\mathcal{E}} = -\langle \alpha_t, (D\alpha' i)_t \rangle_{\mathcal{E}} = -\left(\int_{X_b} c_n \alpha_t \wedge \overline{P(\mu' i)} \right) d\bar{t},$$

where we use the same notation as Section 2, Equation (8).

Proposition 4.5. *The following formula computes the curvature of Q on the class $[\alpha_t]$:*

$$\langle \Theta_Q[\alpha_t], [\alpha_t] \rangle_Q = \|\xi_t \cup \alpha_t\|^2 + \sum_i \left| \int_{X_b} c_n \alpha_t \wedge \overline{P(\mu' i)} \right|^2. \tag{24}$$

As pointed out above, the case where the sections η_i are in the local system $\mathbb{D}^1$ is of particular interest. The following corollary indeed specifies what happens in this case.

Corollary 4.6. *Let $\eta_i \in \Gamma(\Delta, \mathbb{D}^1)$, then, for every $t \in \Delta$, α_t is liftable iff $\langle \Theta_Q[\alpha_t], [\alpha_t] \rangle_Q = 0$.*

Proof. Note that if the η_i are in the local system $\mathbb{D}^1$, then the α_i are in $\mathbb{D}^n$, hence we can assume that the α_i' are holomorphic and by Proposition 2.6, the summation in the formula of Proposition 4.5 is zero. □

Remark 4.7. In particular, α is a section of K^0 if and only if $\langle \Theta_Q[\alpha_t], [\alpha_t] \rangle_Q = 0$ for every $t \in \Delta$. Of course by Formula (10), α is a section of K^0 if and only if $\langle \Theta_{\mathcal{E}} \alpha_t, \alpha_t \rangle_{\mathcal{E}} = 0$ for every $t \in \Delta$. The condition on the curvature of Q is, in general, stronger in the sense that it points to directions where the curvature of the quotient bundle is degenerate.

Another interesting point not fully analyzed in the literature is when the Massey product is a section of $\mathcal{A}$, the ample part of the Fujita decomposition. In this case, we can relax the assumption on the η_i and obtain the same conclusion.

Corollary 4.8. *Assume that $\eta_i \in \Gamma(\Delta, \mathbb{D}^1 \otimes \mathcal{O}_B)$ and that their Massey product α is a holomorphic section of $\mathcal{A}$, then α_t is liftable iff $\langle \Theta_Q[\alpha_t], [\alpha_t] \rangle_Q = 0$.*

Proof. In this case, the α_i are in $\mathbb{D}^n \otimes \mathcal{O}_B = \mathcal{U}$, the unitary flat part of the Fujita decomposition. Since the decomposition $f_* \omega_{X/B} = \mathcal{U} \oplus \mathcal{A}$ is orthogonal, we get that $D\alpha' i$ is not necessarily zero, but is orthogonal to α. Hence, once again the summation in the formula of Proposition 4.5 is zero. □

The above corollaries give a metric interpretation of the liftability of Massey products. Based on our experience concerning the case of trivial Massey products, we expect these results to be useful in the future.

Funding statement. The first author has been supported by the Institute for Basic Science (IBS-R003-D1) and by European Union funds, NextGenerationEU. The second author has been supported by the grant DIMA Geometry PRIDZUCC, by PRIN 2017 Prot. 2017JTLHJR "Geometric, algebraic and analytic methods in arithmetics" and by the project DM737 RIC COLLAB ZUCCONI "Birational geometry, Fano manifolds and torus actions" CUP G25F21003390007, European Union funds, NextGenerationEU.

The authors are members of INdAM-GNSAGA.

References

[1] B. Berndtsson, *Curvature of vector bundles associated to holomorphic fibrations*, Ann. Math. 169 (2009), no. 2, 531–560.
[2] B. Berndtsson, *Strict and nonstrict positivity of direct image bundles*, Math. Z. 269 (2011), 1201–1218.
[3] B. Berndtsson, M. Păun and X. Wang, *Algebraic fiber spaces and curvature of higher direct images*, J. Inst. Math. Jussieu 21 (2022), no. 3, 973–1028.
[4] F. Catanese and M. Dettweiler, *The direct image of the relative dualizing sheaf needs not be semi-ample*, C. R. Math. Acad. Sci. Paris 352 (2014), no. 3, 241–244.
[5] F. Catanese and M. Dettweiler, "Answer to a question by Fujita on variation of Hodge structures" in *Higher Dimensional Algebraic Geometry: In Honour of Professor Yujiro Kawamata's Sixtieth Birthday*, edited by K. Oguiso, C. Birkar, S. Ishii, S. Takayama, Mathematical Society of Japan, Tokyo, 2017, pp. 73–102.
[6] F. Catanese and Y. Kawamata, *Fujita decomposition over higher dimensional base*, Eur. J. Math. 5 (2019), no. 3, 720–728.
[7] A. Collino and G. P. Pirola, *The Griffiths infinitesimal invariant for a curve in its Jacobian*, Duke Math. J 78 (1995), no. 1, 59–88.
[8] T. Fujita, *On Kähler fiber spaces over curves*, J. Math. Soc. Japan 30 (1978), no. 4, 779–794.
[9] T. Fujita, *The sheaf of relative canonical forms of a Kähler fiber space over a curve*, Proc. Japan Acad. Ser. A Math. Sci. 54 (1978), no. 7, 183–184.
[10] V. González-Alonso, L. Stoppino and S. Torelli, *On the rank of the flat unitary summand of the Hodge bundle*, Trans. Am. Math. Soc. 372 (2019), no. 12, 8663–8677.
[11] V. González-Alonso and S. Torelli, *Families of curves with Higgs field of arbitrarily large kernel*, Bull. London Math. Soc 53 (2021), no. 2, 493–506.
[12] P. Griffiths, "Hermitian differential geometry, Chern classes, and positive vector bundles" in *Global Analysis (Papers in Honor of K. Kodaira)*, edited by D. C. Spencer and S. Iyanaga, University Tokyo Press, Tokyo, 1969, pp. 185–251.
[13] P. Griffiths, *Periods of integrals on algebraic manifolds. III. Some global differential-geometric properties of the period mapping*, Inst. Hautes Études Sci. Publ. Math. 38 (1970), 125–180.
[14] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics, Wiley-Interscience, New York, 1978.
[15] P. Griffiths and L. Tu, "Curvature properties of the Hodge bundles" in *Topics in Transcendental Algebraic Geometry (Princeton, N.J., 1981/1982)*, Annals of Mathematics Studies, 106, Princeton University Press, Princeton, NJ, 1984, pp. 29–49.
[16] C. Hacon, M. Popa, and C. Schnell, "Algebraic fiber spaces over abelian varieties: Around a recent theorem by Cao and Păun" in *Local and Global Methods in Algebraic Geometry*, edited by N. Budur, T. de Fernex, R. Docampo, K. Tucker, Contemporary Mathematics, 712, American Mathematical Society, Providence, RI, 2018, pp. 143–195.
[17] M. Inoue, *Some new surfaces of general type*, Tokyo J. Math 17 (1994), no. 2, 295–319.
[18] M. Iwai, *Almost nef regular foliations and Fujita's decomposition of reflexive sheaves*, Ann. Sc. Norm. Super. Pisa Cl. Sci 23 (2022), no. 5, 719–743.
[19] G. Kempf, F. F. Knudsen, D. Mumford and B. Saint-Donat, *Toroidal embeddings. I*, Lecture Notes in Mathematics, 339. Springer-Verlag, Berlin, 1973.
[20] S. Kobayashi, *Differential geometry of complex vector bundles*, Publications of the Mathematical Society of Japan, 15, Kanô Memorial Lectures, 5, Princeton University Press, Princeton, NJ, 1987.
[21] J. Kollár, *Higher direct images of dualizing sheaves. II*, Ann. Math 124 (1986), no. 1, 171–202.
[22] S. Kovács, "Strong non-isotriviality and rigidity" in *Recent Progress in Arithmetic and Algebraic Geometry*, edited by Y. Kachi, S. B. Mulay, P. Tzermias, Contemporary Mathematics, 386, American Mathematical Society, Providence, RI, 2005, pp. 47–55.
[23] L. Lombardi and C. Schnell, *Singular Hermitian metrics and the decomposition theorem of Catanese, Fujita, and Kawamata*, Proc. Am. Math. Soc. 152 (2024), no. 1, 137–146.
[24] S. S. Lu and S. T. Yau, *Holomorphic curves in surfaces of general type*, Proc. Natl. Acad. Sci. USA 87 (1990), 80–82.
[25] Y. Miyaoka, *On the Chern numbers of surfaces of general type*, Invent. Math. 42 (1977), 225–237.
[26] M. Păun, "Singular Hermitian metrics and positivity of direct images of pluricanonical bundles" in *Algebraic geometry: Salt Lake City 2015*, edited by T. de Fernex, B. Hassett, M. Mustaţă, M. Olsson, M. Popa, R. Thomas, Proceedings of Symposia in Pure Mathematics, 97.1, American Mathematical Society, Providence, RI, 2018, pp. 519–553.
[27] M. Păun and S. Takayama, *Positivity of twisted relative pluricanonical bundles and their direct images*, J. Algebr. Geom. 27 (2018), no. 2, 211–272.
[28] G. P. Pirola and S. Torelli, *Massey products and Fujita decompositions on fibrations of curves*, Collect. Math. 71 (2020), no. 1, 39–61.
[29] G. P. Pirola and F. Zucconi, *Variations of the Albanese morphisms*, J. Algeb. Geom. 12 (2003), no. 3, 535–572.
[30] H. Raufi, *Singular Hermitian metrics on holomorphic vector bundles*, Ark. Mat. 53 (2015), no. 2, 359–382.
[31] L. Rizzi, *Massey products and Fujita decomposition over higher dimensional base*, Collect. Math. 75 (2024), no. 2, 511–534.
[32] L. Rizzi and F. Zucconi, *Generalized adjoint forms on algebraic varieties*, Ann. Mat. Pura Appl. 196 (2017), no. 3, 819–836.
[33] L. Rizzi and F. Zucconi, *On Green's proof of infinitesimal Torelli theorem for hypersurfaces*, Atti Accad. Naz. Lincei Rend. Lincei Cl. Sci. Fis. Mat. Natur 29 (2018), no. 4, 689–709.

[34] L. Rizzi and F. Zucconi, *Differential forms and quadrics of the canonical image*, Ann. Mat. Pura Appl. 199 (2020), no. 6, 2341–2356.

[35] L. Rizzi and F. Zucconi, *Fujita decomposition and Massey product for fibered varieties*, Nagoya Math. J. 247 (2022), 624–652.

[36] L. Rizzi and F. Zucconi, *Local systems of Castelnuovo-type and factorization of semistable families*, J. Geom. Phys. 211 (2025), 1–13.

[37] G. Schumacher, *Positivity of relative canonical bundles and applications*, Invent. Math. 190 (2012), no. 1, 1–56.

[38] Y. T. Siu, "Curvature of the Weil-Petersson metric in the moduli space of compact Kähler-Einstein manifolds of negative first Chern class" in *Contributions to Several Complex Variables*, edited by A. Howard, P. M. Wong, Aspects of Mathematics, E9, Friedrich Vieweg, Braunschweig, 1986, pp. 261–298.

[39] J. H. M. Steenbrink, "Mixed Hodge structure on the vanishing cohomology" in *Real and complex singularities (Proceedings of Ninth Nordic Summer School/NAVF Symposium in Mathematics, Oslo, 1976)*, edited by P. Holm, Sijthoff and Noordhoff, Alphen aan den Rijn, 1977, pp. 525–563.

[40] E. Viehweg, *Weak positivity and the stability of certain Hilbert points*, Invent. Math. 96 (1989), no. 3, 639–667.

[41] E. Viehweg and K. Zuo, "Discreteness of minimal models of Kodaira dimension zero and subvarieties of moduli stacks" in *Surveys in Differential Geometry, vol. VIII (Boston, MA, 2002)*, edited by S. T. Yau, Int. Press, Somerville, MA, 2003, pp. 337–356.

[42] J. Xie and X. Yuan, *Partial heights and the geometric Bombieri-Lang conjecture*, preprint, arXiv.2305.14789.

[43] S. T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*, Commun. Pure Appl. Math. 31 (1978), no. 3, 339–411.

[44] K. Zuo, *On the negativity of kernels of Kodaira-Spencer maps on Hodge bundles and applications*, Kodaira's issue. Asian J. Math. (1) 4 (2000), 279–301.

Cite this article: L. Rizzi and F. Zucconi, Curvature and relative volume forms. *Nagoya Mathematical Journal* (2026);261, 1–16. https://doi.org/10.1017/nmj.2025.10095