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The Local Picard Group of a Ring Extension

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Abstract. Given an integral domain D and a D-algebra R, we introduce the local Picard group $\mathrm{LPic}(R, D)$ as the quotient between the Picard group $\mathrm{Pic}(R)$ and the canonical image of $\mathrm{Pic}(D)$ in $\mathrm{Pic}(R)$, and its subgroup $\mathrm{LPic}_u(R, D)$ generated by the the integral ideals of R that are unitary with respect to D. We show that, when $D \subsetneq R$ is a ring extension that satisfies certain properties (for example, when R is the ring of polynomial $D[X]$ or the ring of integer-valued polynomials $\mathrm{Int}(D)$), it is possible to decompose $\mathrm{LPic}(R, D)$ as the direct sum $\bigoplus \mathrm{LPic}(RT, T)$, where T ranges in a Jaffard family of D. We also study under what hypothesis this isomorphism holds for pre-Jaffard families of D.

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1. Introduction

Let D be an integral domain. The *Picard group* of D, denoted by $\mathrm{Pic}(D)$, is the quotient between the group of invertible (fractional) ideals and the group of principal ideals or, equivalently, the group of isomorphism classes of rank-one projective modules. The Picard group of D is connected, among other topics, to the factorization properties of D: for example, if D is a Dedekind domain, then $\mathrm{Pic}(D)$ is trivial if and only if D is a unique factorization domain.

The Picard group is essentially a global property of a ring, in the sense that it cannot be recovered from the localizations: indeed, the Picard group of a local ring is always trivial. However, in some cases it is possible to understand the structure of the Picard group by choosing carefully a family of localizations: for example, when considering the ring $R = \mathrm{Int}(D)$ of the integer-valued polynomials on a Dedekind domain D, there is an exact sequence

$$0 \longrightarrow \mathrm{Pic}(D) \longrightarrow \mathrm{Pic}(\mathrm{Int}(D)) \longrightarrow \bigoplus_{M \in \mathrm{Max}(D)} \mathrm{Pic}(\mathrm{Int}(D_M)) \longrightarrow 0, \qquad (1)$$

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where $\mathrm{Int}(D_M) = \mathrm{Int}(D)D_M$ is a localization of $\mathrm{Int}(D)$ [1, Theorem VIII.I.9]. Since each D_M is a valuation domain, $\mathrm{Pic}(\mathrm{Int}(D_M))$ is known [1, Theorem VIII.2.8], and thus we can calculate $\mathrm{Pic}(D)$ by localization.

The previous result was generalized in [13] in the context of Jaffard families: a *Jaffard family* of D is a family of flat overrings of D that is complete, independent and locally finite (see Sect. 2.2 for the definitions of these properties). A typical example of a Jaffard family is the family $\{D_M \mid M \in \mathrm{Max}(D)\}$ of localizations of a one-dimensional locally finite domain (for example, a one-dimensional Noetherian domain). It was shown [13, Proposition 4.3] that, if Θ is a Jaffard family of D, then we can construct an exact sequence analogous to (1) for $\mathrm{Pic}(\mathrm{Int}(D))$, and subsequently [13, Theorem 4.7] that there is an isomorphism

$$\mathrm{LPic}(\mathrm{Int}(D), D) \simeq \bigoplus_{T\in\Theta} \mathrm{LPic}(\mathrm{Int}(T), T), \tag{2}$$

where $\mathrm{LPic}(\mathrm{Int}(A), A)$ is the quotient between $\mathrm{Pic}(\mathrm{Int}(A))$ and the canonical image of $\mathrm{Pic}(A)$. (We use the notation we introduce in Sect. 6 of the present paper instead of the notation used in [13].) These results were then extended beyond the Jaffard family context with a derived set-like construction [13, Sections 6 and 7] first introduced in [11].

The purpose of this paper is to extend (1) and (2) beyond the case of integer-valued polynomials to the more general framework of D-algebras. In particular, we want to understand when, given a Jaffard family Θ of D and a D-algebra R, we have an isomorphism

$$\mathrm{LPic}(R, D) \simeq \bigoplus_{T\in\Theta} \mathrm{LPic}(RT, T), \tag{3}$$

fully reducing the study of the local Picard group $\mathrm{LPic}(R, D)$ to localizations; moreover, we want to see whether it is possible to generalize these results to the more general case of pre-Jaffard families. However, the isomorphism (2) does not hold for general D-algebras: thus, the first part of the paper (Sections 3, 4, 5) is devoted to introducing and exploring the additional hypothesis we need to put on R in order for the isomorphism to hold, while the second part (from Sect. 6 onward) adapts the proof of the integer-valued polynomial case to this more general setup. We shall see that several polynomial-like constructions satisfy these conditions: for example, (3) holds not only for $R = \mathrm{Int}(D)$ (which is the topic of [13]), but also for $R = \mathrm{Int}(E, D)$ (the ring of integer-valued polynomials on any $E \subseteq D$), the polynomial ring $R = D[X]$ and the Bhargava ring $R = \mathbb{B}_x(D)$ (see [14]).

More specifically, in Sect. 3 we study D-algebras R that can be endowed with a retract, i.e., with a map $R \longrightarrow D$ that is a D-algebra homomorphism, showing that they are all extensions of D such that $R \cap K = D$ (where K is the quotient field of D; Proposition 3.4). In Sect. 4, we develop the theory of unitary ideals: a fractional ideal I of R is unitary with respect to D if $I \cap K \neq (0)$. We introduce the subgroup $\mathrm{Pic}_u(R, D)$ of $\mathrm{Pic}(R)$ as the subgroup generated by the classes of the unitary integral ideals of R, and we show that it is the kernel of the canonical homomorphism $\mathrm{Pic}(R) \longrightarrow \mathrm{Pic}(RK)$ (Proposition 4.5). In Sect. 5, we introduce pseudo-polynomial algebras over D

as those algebras where every unitary principal ideal is actually generated by an element of D, and we show (Proposition 5.7) that this notion encompasses several constructions contained in the ring of polynomials $K[X]$.

In Sect. 6 we introduce the local Picard group $\mathrm{LPic}(R, D)$ and the unitary local Picard group $\mathrm{LPic}_u(R, D)$ as, respectively, the quotient of $\mathrm{Pic}(R)$ and $\mathrm{Pic}_u(R, D)$ by the canonical image of $\mathrm{Pic}(D)$. We show that these constructions are functorial (Proposition 6.4) and that for retract D-algebras they are actually direct summands of $\mathrm{Pic}(R)$ and $\mathrm{Pic}_u(R, D)$, respectively (Proposition 6.6).

In Sect. 7, we take the proofs of (1) and (2) and show how they can be adapted to the case of D-algebras: in order to hold in the more general context, we need to restrict ourselves to algebras that are retract and pseudo-polynomial and, instead to $\mathrm{LPic}(R, D)$, the best results are obtained when dealing with the group $\mathrm{LPic}_u(R, D)$ induced by unitary integral ideals (Theorems 7.1 and 7.4). We also show several special cases of these theorems. Finally, in Sect. 8, we show under what hypothesis the results about Jaffard families can be generalized to pre-Jaffard families using the derived sequence.

2. Preliminaries

Throughout the paper, D is an integral domain and K is its quotient field. We also suppose that $D \neq K$, i.e., that D is not a field.

A *fractional ideal* of D is a D-submodule I of K such that $dI \subseteq D$ for some $d \in D$, $d \neq 0$. We shall often refer to a fractional ideal simply as an "ideal", while using the term "integral ideal" to refer to fractional ideals contained in D (i.e., to ideals of D in the usual sense).

An *overring* of D is a ring between D and K. If $D \subseteq R$ is a ring extension, with R being an integral domain, then the quotient field of R contains K, and thus it makes sense to consider the intersection $R \cap K$. Moreover, if T is an overring of D, then the set

$$RT := \left\{ \sum_i r_i t_i \mid r_i \in R, t_i \in T \right\}$$

is well-defined, and it is a ring and an overring of R. In particular, if $T = S^{-1}D$ is a localization of D, then $RT = S^{-1}R$ is a localization of R.

We say that two elements $a, b \in D$ are *associated* in D if they generate the same principal ideal, or equivalently if there is a unit u such that $a = ub$.

If E is a subset of K, the ring of *integer-valued polynomials* on E is

$$\mathrm{Int}(E, D) := \{f \in K[X] \mid f(E) \subseteq D\};$$

we also set $\mathrm{Int}(D) := \mathrm{Int}(D, D)$. See [1] for facts about integer-valued polynomials.

2.1. The Picard Group

A fractional ideal I of D is *invertible* if there is a fractional ideal J such that $IJ = D$; in this case, $J = (D : I) = \{x \in K \mid xI \subseteq D\}$. Every invertible ideal is finitely generated. The set $\mathrm{Inv}(D)$ of all the invertible ideals of D is a group under the product of ideals, with identity element D, and contains as a subgroup the set $\mathrm{Princ}(D)$ of all the principal fractional ideals of D. The quotient

$$\mathrm{Pic}(D) := \frac{\mathrm{Inv}(D)}{\mathrm{Princ}(D)}$$

is called the *Picard group* of D; if I is an invertible ideal, we denote by $[I]$ the class of I in $\mathrm{Pic}(D)$. The Picard group can also be constructed as the set of all isomorphism classes of all projective modules of rank 1, with operation given by $[P] \cdot [Q] := [P \otimes Q]$.

The Picard group is a functorial construction, in the sense that a ring homomorphism $\phi : A \longrightarrow B$ between two domains A, B induces a map $\phi^* : \mathrm{Pic}(A) \longrightarrow \mathrm{Pic}(B)$. If ϕ is injective then ϕ^* coincides with the extension of ideals: that is, if I is an invertible ideal of A, then $\phi^*([I]) = [IB]$.

2.2. Jaffard and Pre-Jaffard Families

A *flat overring* of D is an overring that is also flat as a D-module. Let Θ be a family of flat overrings of D: we say that Θ is:

- *complete* if $I = \bigcap\{IT \mid T \in \Theta\}$ for every ideal I of D;
- *independent* if $TT' = K$ for every $T \neq T'$ in Θ;[1]
- *locally finite* if for every $x \in D$, $x \neq 0$, there are only finitely many $T \in \Theta$ such that x is not a unit in D.

A *Jaffard family* of D is a family of flat overrings that is complete, independent and locally finite, and such that $K \notin \Theta$. In particular, if Θ is a Jaffard family, for every nonzero prime ideal P of D there is a unique $T \in \Theta$ such that $PT \neq T$. If T is a flat overring of D, then T is a *Jaffard overring* if T belongs to a Jaffard family of D; this condition can be checked by defining the *orthogonal* of T (with respect to D) as

$$T^\perp := \bigcap\{D_P \mid PT = T\}.$$

Indeed, T is a Jaffard overring if and only if $TT^\perp = K$, and in this case $\{T, T^\perp\}$ is a Jaffard family. See [10] and [5, Section 6.3] for properties of Jaffard families.

The *Zariski topology* on the set $\mathrm{Over}(D)$ of overrings of D is the topology generated by the $\mathcal{B}(x) := \{T \in \mathrm{Over}(D) \mid x \in T\}$, as x ranges in K. A *pre-Jaffard family* of D is a family Θ of flat overrings that is complete, independent, such that $K \notin \Theta$, and compact in the Zariski topology [11]. Any Jaffard family is also a pre-Jaffard family, and a pre-Jaffard family Θ is Jaffard if and only if every $T \in \Theta$ is a Jaffard overring. If Θ is a pre-Jaffard family and there is a $T_\infty \in \Theta$ such that every $T \in \Theta \setminus \{T_\infty\}$ is a Jaffard overring, we say that Θ is a *weak Jaffard family pointed at* T_∞.

Pre-Jaffard families are much more common than Jaffard families, but they do not enjoy the same strong properties of the latter. In order to approximate pre-Jaffard families with Jaffard families, and to extend some results

[1] For general (not necessarily flat) overrings, independence requires that there is no nonzero prime P such that $PT \neq T$ and $PT' \neq T'$ [5, Section 6.2], but this condition reduces to $TT' = K$ for flat overrings [11, Lemma 3.4 and Definition 3.5].

from the latter to the former, we associate to a pre-Jaffard family Θ of D two sequences, one $\{\mathcal{N}^\alpha(\Theta)\}_\alpha$ of subsets of Θ, and another $\{T_\alpha\}_\alpha$ of overrings of D, both indexed by ordinal numbers, in the following way:

- $\mathcal{N}^0(\Theta) := \Theta$, $T_0 := D$;
- if $\alpha = \gamma + 1$ is a successor ordinal, then $\mathcal{N}^\alpha(\Theta)$ is the set of members of $\mathcal{N}^\gamma(\Theta)$ that are not Jaffard overrings of T_γ;
- if α is a limit ordinal, then $\mathcal{N}^\alpha(\Theta) := \bigcap\{\mathcal{N}^\gamma(\Theta) \mid \gamma < \alpha\}$;
- $T_\alpha := \bigcap\{T \mid T \in \mathcal{N}^\alpha(\Theta)\}$.

Then, $\{\mathcal{N}^\alpha(\Theta)\}_\alpha$ is decreasing and $\{T_\alpha\}_\alpha$ is increasing; moreover, $\mathcal{N}^\alpha(\Theta)$ is always a pre-Jaffard family of T_α. A weak Jaffard family is just a pre-Jaffard family such that $\mathcal{N}^1(\Theta)$ is empty or a singleton. We call $\{T_\alpha\}_\alpha$ the *derived sequence* with respect to Θ. If $T_\alpha = K$ for some α, we say that the pre-Jaffard family Θ is *sharp*. See [11] for properties of pre-Jaffard families and of the derived sequence.

3. Retract Algebras

Definition 3.1. Let D be an integral domain, and let R be a D-algebra that is an integral domain. We say that R is a *retract D-algebra* if there is a D-algebra homomorphism $\epsilon : R \longrightarrow D$, which we call a *retract* of R onto D.

A retract is also sometimes called a *Reynolds operator*; its existence implies that D is a direct summand of R, and consequently that D is a pure subring of R. However, since we only need basic facts about retract algebras, we do not use any results about pure extensions.

Example 3.2. (1) Let $R = D[X]$. For any $i \in D$, the evaluation homomorphism $\epsilon_i(f(X)) = f(i)$ is a D-algebra homomorphism, and thus $D[X]$ is a retract D-algebra. The same holds for polynomial rings $D[X_1, \ldots,]$ in any number of indeterminates.

(2) If K is the quotient field of D and $i \in K$, then for every $E \ni i$ the evaluation homomorphisms $\epsilon_i(f(X)) = f(i)$ also makes $\mathrm{Int}(E, D)$ into a retract D-algebra. In particular, taking $i \in D$, we see that $\mathrm{Int}(D)$ is a retract D-algebra.

(3) The power series ring $D[[X]]$ is a retract D-algebra, with retract given by the evaluation at 0, i.e., with the map that associates to the power series $f_0 + f_1 X + \cdots$ the constant term f_0.

Lemma 3.3. *If $\epsilon : R \longrightarrow D$ is a D-algebra homomorphism and $R \neq D$, then ϵ is not injective.*

Proof. If $r \in R \setminus D$ then $r \neq \epsilon(r)$ and $\epsilon(r) = \epsilon(\epsilon(r))$ since $\epsilon(r) \in D$, and thus ϵ is not injective. □

Proposition 3.4. *Let D be an integral domain with quotient field K and let R be a retract D-algebra.*

(a) R is an extension of D (i.e., the canonical D-algebra homomorphism $D \longrightarrow R$ is injective).

(b) If $S \subseteq R$ *is a* D*-algebra, then* S *is a retract* D*-algebra.*
(c) If K *is the quotient field of* D*, then* $R \cap K = D$*.*
(d) The restriction map of spectra

$$\mathrm{Spec}(R) \longrightarrow \mathrm{Spec}(D),$$
$$P \longmapsto P \cap D,$$

is surjective.
(e) If $R \neq D$*, then* $\dim(R) \geq \dim(D) + 1$*.*

Proof. Let ϵ be a retract of R onto D.

(a) Let $i : D \longrightarrow R$ be the canonical map of D-algebras. Then, $\epsilon \circ i$ is the identity on D; therefore, i must be injective, i.e., R is an extension of D.

(b) The restriction of ϵ to S is a D-algebra homomorphism. Hence S is a retract D-algebra.

(c) Let $D \subseteq D' \subseteq K$. Then, any ring homomorphism $\phi : D' \longrightarrow A$ is uniquely determined by the restriction $\phi|_D : D \longrightarrow A$, since any $x \in D'$ can be written as a quotient y/z for $y, z \in D$. (That is, the extension $D \subseteq D'$ is an epimorphism.) In particular, if R is a retract D-algebra and $D' := R \cap D$, then the retract ϵ is a ring homomorphism that is the identity on D; hence, it must be the identity on D'. However, this is impossible if $D' \neq D$. Thus $D' = R \cap K = D$.

(d) The map ϵ induces a map $\epsilon^* : \mathrm{Spec}(D) \longrightarrow \mathrm{Spec}(R)$ given by $\epsilon^*(P) = \epsilon^{-1}(P)$: we claim that $P = \epsilon^{-1}(P) \cap D$. Indeed, since ϵ is the identity on D, we have $P \subseteq \epsilon^{-1}(P) \cap D$; likewise, if $d \in \epsilon^{-1}(P) \cap D$, then $d = \epsilon(d) \in P$ and so $\epsilon^{-1}(P) \cap D \subseteq P$. Hence the restriction map is surjective.

(e) If $\dim(D)$ is infinite the claim is trivial. Suppose that $\dim(D) < \infty$: then, since ϵ is surjective, $\mathrm{Spec}(D)$ is homeomorphic to $V(\ker \epsilon) \subseteq \mathrm{Spec}(R)$. Since D and R are integral domains, the dimensions of $\mathrm{Spec}(R)$ and $\mathrm{Spec}(D)$ can coincide only if $\ker \epsilon = (0)$, i.e., if ϵ is injective. This is impossible by Lemma 3.3. Therefore $\dim(R) \geq \dim(D) + 1$, as claimed. □

Remark 3.5. If $R = D[X]$ or $R = \mathrm{Int}(D)$, there is more than one possible map ϵ. In this case, the prime $\epsilon^{-1}(P)$ may change when changing the map, but its image under the restriction map is always P. For example, if ϵ_d is the map of $D[X]$ defined as the evaluation in d, then $\epsilon_d^{-1}(P) = (P, X - d)$.

Proposition 3.6. *Let* $D \subsetneq R$ *be an integral extension of domains. Then,* R *is not a retract* D*-algebra.*

Proof. Suppose there is a retract $\epsilon : R \longrightarrow D$; since $R \neq D$, ϵ is not injective (Lemma 3.3) and thus $\ker \epsilon \neq (0)$. Let $S := D \setminus \{0\}$: then, $S^{-1}D \subseteq S^{-1}R$ is again an integral extension, and $S^{-1}D = K$ is the quotient field of R. Hence, $S^{-1}R$ must be a field too; however, $\ker \epsilon \cap D = (0)$, and thus $S^{-1} \ker \epsilon \neq S^{-1}R$ is a nonzero prime ideal above (0), a contradiction. Hence R is not a retract D-algebra. □

Remark 3.7. The condition $R \cap K = D$ is not sufficient for R to be a retract D-algebra. For example, if D is integrally closed and $R \neq D$ is an integral extension of D, then $R \cap K = D$, but R is not a retract D-algebra by Proposition 3.6.

The condition is not sufficient even if D is integrally closed in R: for example, let $D = \mathbb{Z}$ and let $R = \mathbb{Z}[\alpha, \beta] \subseteq \overline{\mathbb{Q}(X)}$, where X is an indeterminate over $\mathbb{Z}$ and α, β satisfy

$$\begin{cases} \alpha^2 = X, \\ \beta^2 = 3 - X. \end{cases}$$

Then, $\mathbb{Z}$ is integrally closed in R and $R \cap \mathbb{Q} = \mathbb{Z}$. Suppose there is a retract $\epsilon : R \longrightarrow D$: since $\alpha^2 + \beta^2 = 3$, we also must have

$$\epsilon(\alpha)^2 + \epsilon(\beta)^2 = \epsilon(3) = 3,$$

which is impossible in $\mathbb{Z}$.

We shall often deal with extending a retract D-algebra by a flat overring. In this context, the following result is useful.

Proposition 3.8. *Let D be an integral domain and let T be an overring of D. If R is a retract D-algebra, then RT is a retract T-algebra.*

Proof. Let $\epsilon : R \longrightarrow D$ be a retract. If $x \in RT$, then $x = r_1t_1 + \cdots + r_nt_n$ for some $r_i \in R$, $t_i \in T$; we define

$$\widetilde{\epsilon}(x) := \epsilon(r_1)t_1 + \cdots + \epsilon(r_n)t_n \in DT = T.$$

We need to show that $\widetilde{\epsilon}$ is well-defined, i.e., that if

$$x = r_1t_1 + \cdots + r_nt_n = s_1t'_1 + \cdots + s_mt'_m$$

for some $r_i, s_i \in R$, $t_i, t'_i \in T$, then

$$\epsilon(r_1)t_1 + \cdots + \epsilon(r_n)t_n = \epsilon(s_1)t'_1 + \cdots + \epsilon(s_n)t'_n.$$

To do so, it is enough to show that if $r_1t_1 + \cdots + r_nt_n = 0$, then $\epsilon(r_1)t_1 + \cdots + \epsilon(r_n)t_n = 0$.

For each i we can write $t_i = y_i/z_i$ with $y_i, z_i \in D$, $z_i \neq 0$; let $z := z_1 \ldots z_n$. Then, $zt_i \in R$ for every i and $zx = 0$. Thus,

$$0 = \epsilon(zx) = \epsilon(z(r_1t_1 + \cdots + r_nt_n)) = \epsilon(r_1)(zt_1) + \cdots + \epsilon(r_n)(zt_n)$$

since each zt_i is an element of D. The equality

$$\epsilon(r_1)(zt_1) + \cdots + \epsilon(r_n)(zt_n) = 0$$

is an equality in K (the quotient field of D); therefore, we can simplify z and obtain $\epsilon(r_1)t_1 + \cdots + \epsilon(r_n)t_n = 0$, which is what we needed to prove.

By construction, $\widetilde{\epsilon}$ is an homomorphism of T-algebras, and thus it is a retract. Hence RT is a retract T-algebra. □

Proposition 3.9. *Let $D \subseteq R \subseteq A$ be integral domains. If R is a retract D-algebra and A is a retract R-algebra, then A is a retract D-algebra.*

Proof. If $\epsilon : R \longrightarrow D$ and $\epsilon' : A \longrightarrow R$ are retracts, then $\epsilon \circ \epsilon' : A \longrightarrow D$ is a retract. □

4. Unitary Ideals

An important feature of the theory of integer-valued polynomials is the concept of unitary ideals, i.e., ideals of $\mathrm{Int}(D)$ that meet D nontrivially. We generalize this definition to any extension in the following way.

Definition 4.1. Let D be an integral domain with quotient field K and R an extension of D that is an integral domain. We say that a fractional ideal I of R is *unitary with respect to* D if $I \cap K \neq (0)$.

When dealing with integral ideals, the defining condition of a unitary ideal can be written in a slightly different way.

Lemma 4.2. *Let D be an integral domain with quotient field K, and let R be an extension of D. Let I be an integral ideal of R. Then, I is unitary if and only if $I \cap D \neq (0)$.*

Proof. If $I \cap D \neq (0)$, then also $I \cap K \neq (0)$. Conversely, if $x \in I \cap K$, $x \neq 0$, then there is a $y \in D$, $y \neq 0$ such that $yx \in D$. Thus $yx \in yI \cap D \subseteq I \cap D$ and $I \cap D \neq (0)$. □

Proposition 4.3. *Let D be an integral domain and R an extension of D that is an integral domain. An integral ideal I of R is unitary if and only if $IK = RK$.*

Proof. The ring RK is the localization of R at $S := D\setminus\{0\}$; the claim now follows from Lemma 4.2. □

In this paper, we are mainly interested in the study of the Picard group of a ring. In general, whenever R is a D-algebra not contained in K, there will be plenty of ideals of R that are not unitary; however, every ideal can be transformed by multiplication into an unitary ideal, in the sense that, if $x \in I$, $x \neq 0$ then $x^{-1}I$ is unitary (it contains 1). In particular, if we denote by $\mathrm{Inv}_0(R, D)$ the set of all invertible ideals that are unitary with respect to D (which is a subgroup of the group $\mathrm{Inv}(R)$ of all invertible ideals) then the quotient
$$\frac{\mathrm{Inv}_0(R, D)}{\mathrm{Inv}_0(R, D) \cap \mathrm{Princ}(R)} = \frac{\mathrm{Inv}_0(R, D)}{\mathrm{Princ}_u(R, D)}$$
(where $\mathrm{Princ}(R)$ and $\mathrm{Princ}_u(R, D)$ are, respectively, the subgroup of principal ideals and of unitary principal ideals of R) is just equal to the Picard group $\mathrm{Pic}(R)$, since for every I the coset $I\cdot\mathrm{Princ}(R)$ contains elements of $\mathrm{Inv}_0(R, D)$.

A more interesting way to consider unitary ideals is the following.

Definition 4.4. Let R be a D-algebra that extends D. We define $\mathrm{Inv}_u(R, D)$ as the subgroup of $\mathrm{Inv}(R)$ generated by the ideals that are both integral and unitary with respect to D.

Furthermore, we define the *unitary Picard group* of R with respect to D as the quotient
$$\mathrm{Pic}_u(R, D) := \frac{\mathrm{Inv}_u(R, D)}{\mathrm{Inv}_u(R, D) \cap \mathrm{Princ}(R)} = \frac{\mathrm{Inv}_u(R, D)}{\mathrm{Princ}_u(R, D)}.$$

Proposition 4.5. *Let R be a D-algebra that is extends D and is an integral domain, and let $\phi : \mathrm{Pic}(R) \longrightarrow \mathrm{Pic}(RK)$, $[I] \mapsto [IK]$ be the canonical map of Picard groups. Then,* $\ker\phi = \mathrm{Pic}_u(R, D)$.

Proof. Since the set of all integral unitary invertible ideals is a monoid, every $[I] \in \mathrm{Pic}_u(R, D)$ can be written as JL^{-1}, where J, L are integral unitary ideals of R; therefore, to show that $\mathrm{Pic}_u(R, D) \subseteq \ker\phi$, it is enough to prove it for integral unitary ideals. Let thus $[I]$ be integral and unitary: then, $IK \subseteq RK$, while $K \subseteq IK$ since $I \cap K \neq (0)$. Therefore, $IK = RK$ and in particular IK is principal in RK, i.e., $[IK] = [RK]$ and $[I] \in \ker\phi$, as claimed.

Conversely, suppose that $[I] \in \ker\phi$: then, there is a $c \in F$ (where F is the quotient field of R) such that $IK = cRK$, i.e., $c^{-1}IK = RK$. Let $S := D\backslash\{0\}$: then, RK is just the localization $S^{-1}R$, and thus the equality $c^{-1}IK = RK$ can be written as $S^{-1}(c^{-1}I) = S^{-1}R$. Since I is invertible, it is finitely generated; hence there is an $s \in S$ such that $sc^{-1}I \subseteq R$. Thus $sc^{-1}I$ is an integral invertible ideal of R such that $sc^{-1}IK = RK$, i.e., $sc^{-1}I \cap K \neq (0)$. Thus $sc^{-1}I$ is unitary, and $[I] = [sc^{-1}I] \in \mathrm{Pic}_u(R, D)$. The claim is proved. □

Corollary 4.6. *Let R be a D-algebra that is an integral domain, and let I be an invertible ideal of R; let K be the quotient field of D. The following are equivalent:*

(i) *IK is principal;*
(ii) $[I] \in \mathrm{Pic}_u(R, D)$*;*
(iii) *there is a $c \in Q(R)$ such that cI is unitary and integral.*

Proof. The equivalence of the the first two conditions follows from Proposition 4.5, while if (iii) holds then $cIK = cRK$ is principal, so (i) holds. The implication (ii) $\Longrightarrow$ (iii) follows from the last part of the proof of Proposition 4.5. □

Corollary 4.7. *If K is a field and R is a K-algebra, then* $\mathrm{Pic}_u(R, K) = (0)$.

Proof. By Corollary 4.6, $[I] \in \mathrm{Pic}_u(R, K)$ if and only if $IK = I$ is principal. Thus $\mathrm{Pic}_u(R, K)$ is trivial. □

Corollary 4.8. *Let D be an integral domain with quotient field K, $\mathbf{X}$ a family of indeterminates. If R is a D-algebra such that $D[\mathbf{X}] \subseteq R \subseteq K[\mathbf{X}]$, then* $\mathrm{Pic}_u(R, D) = \mathrm{Pic}(R)$.

Proof. We have $K[\mathbf{X}] = D[\mathbf{X}]K \subseteq RK \subseteq K[\mathbf{X}]$, and thus $RK = K[\mathbf{X}]$. The ring of polynomials $K[\mathbf{X}]$ is a unique factorization domain and thus its Picard group is trivial; the claim follows from Proposition 4.5. □

5. Pseudo-polynomial Algebras

In order to prove interesting results on the Picard group, we need to further restrict our attention to another class of D-algebras.

Definition 5.1. Let D be an integral domain with quotient field K, and let R be an extension of D. We say that R is *pseudo-polynomial over* D if every principal integral ideal of R that is unitary over D is generated by an element of D.

The previous definition can be rewritten in the following way.

Lemma 5.2. *Let R be a D-algebra. Then, R is pseudo-polynomial over D if and only if, for every $r \in R\backslash D$, either r is associated in R to some $d \in D$ or $rR \cap D = (0)$.*

Proof. Suppose R is pseudo-polynomial and let $r \in R\backslash D$. If $rR \cap D \neq (0)$, then $I = rR$ is unitary, and thus $I = dR$ is generated by a $d \in D$, i.e., r is associated to d. Conversely, if the property in the statement hold and $I = rR$ is unitary over D, then $rR \cap D \neq (0)$ and thus r is associated to a $d \in D$, i.e., $I = rD = dR$ is generated by an element of D. Thus R is pseudo-polynomial. □

Another interpretation of pseudo-polynomiality is the following: a D-algebra R is pseudo-polynomial over D if, for every $a \in D$, all factors of a in R (i.e., all $f \in R$ such that $a \in fR$) are associated to some element of D; that is, modulo units, all factors of a in R are actually in D.

Lemma 5.3. *Let R be a pseudo-polynomial D-algebra such that $R \cap K = D$. If I is a unitary integral principal ideal of R, then $I \cap D$ is principal (over D).*

Proof. Let $I = rR$ be unitary, and let $J := I \cap D$. Since R is pseudo-polynomial, $I = dR$ for some $d \in D$. If $x \in J$, then $dx^{-1} \in R \cap K = D$ and thus $x \in dD$, i.e., d generates J. □

Remark 5.4. There are D-algebras that are not pseudo-polynomial. For example, if $D = \mathbb{Z}$, $R = \mathbb{Z}[X, 2/X]$, then $I = XR$ is a unitary ideal (since $I \cap \mathbb{Z} = 2\mathbb{Z}$) but I is not generated by an element of $\mathbb{Z}$ (since $X/2 \notin R$). Note also that R is a retract $\mathbb{Z}$-algebra when endowed with the evaluation in 1.

The study of the pseudo-polynomiality of an extension can always be split into two cases.

Proposition 5.5. *Let D be an integral domain with quotient field K, and let R be an integral domain that extends D. Then, R is pseudo-polynomial over D if and only if R is pseudo-polynomial over $R \cap K$ and $R \cap K$ is pseudo-polynomial over D.*

Proof. Let $A := R \cap K$.

Suppose that R is pseudo-polynomial over D, and let $f \in R\backslash A$ be such that fR is unitary over A. Then also $fR \cap D \neq (0)$ (since A and D have the same quotient field) and thus f is associated to some $d \in D$. Since $D \subseteq A$, it follows that R is pseudo-polynomial over A. Moreover, if $f \in A\backslash D$, then fA is unitary over D, and thus $f = ud$ for some $d \in D$ and some unit u of R. However, $u = f^{-1}d \in R \cap K = A$, and likewise $u^{-1} \in A$: hence, f is associated to d also in A. Hence A is pseudo-polynomial over D.

Conversely, suppose that R is pseudo-polynomial over A and A is pseudo-polynomial over D. Let $f \in R\backslash D$ be such that fR is unitary over D: then, either $f \in R\backslash A$ or $f \in A\backslash D$. In the former case, $fR \cap A \neq (0)$, and thus $f = ua$ for some $a \in A$ and some unit $a \in A$; since $aA \cap D \neq (0)$, we have $a = vd$ for some $d \in D$ and some unit v of A. Hence $f = ua = uvd$ and f is associated in R to an element of D. If $f \in A\backslash D$, we have $fR \cap D \neq (0)$ and $fA \cap D \neq (0)$, and thus $f = ud$ for some $d \in D$ and some unit u of A; hence u is a unit of R and f is associated to $d \in D$ also in R. Hence R is pseudo-polynomial over D. □

Example 5.6. When R is an overring of D (i.e., when R is contained in the quotient field of D) then every ideal of R is unitary over D: therefore, R is pseudo-polynomial if and only if every principal integral ideal of R is generated by an element of D. More generally, this criterion holds whenever R is contained in the algebraic closure of the quotient field of D (see Proposition 5.14 below).

For example, every localization of D is pseudo-polynomial over D. On the other hand, consider $D = \mathbb{Z}[X]$, and let R be the valuation domain associated to the valuation v defined by

$$v\left(\sum_{n=0}^{k} a_n X^n\right) = \inf\{2v_{(2)}(a_n) + 3n\},$$

where $v_{(2)}$ is the 2-adic valuation on $\mathbb{Z}$. Then, R is a discrete valuation ring, but no element of D generates the maximal ideal of R, since no element of D has valuation 1. Thus R is not pseudo-polynomial over D.

We give two sufficient conditions for an algebra to be pseudo-polynomial.

Proposition 5.7. *Let D be an integral domain with quotient field K, and let $\mathbf{X}$ be a family of indeterminates over K. If R is a D-algebra contained in $K[\mathbf{X}]$ and $R \cap K$ is pseudo-polynomial over D, then R is pseudo-polynomial over D. In particular, if $R \cap K = D$ then R is pseudo-polynomial over D.*

Proof. By Proposition 5.5, we only need to show that R is pseudo-polynomial over $R \cap K$, i.e., we can suppose without loss of generality that $R \cap K = D$. Let $f \in R\backslash D$: then, f is a non-constant polynomial, and thus $fR \subseteq fK[\mathbf{X}]$ does not contain any constant, i.e., $fR \cap D = (0)$. Thus R is pseudo-polynomial over D, as claimed. □

Corollary 5.8. *Let D be an integral domain. Then, the ring of polynomials $D[X]$ and the ring of integer-valued polynomials* $\mathrm{Int}(D)$ *are pseudo-polynomial D-algebras.*

Proof. Both rings are contained in $K[X]$, and $D[X] \cap K = \mathrm{Int}(D) \cap K = D$. The claim follows from Proposition 5.7. □

Proposition 5.9. *Let D be a unique factorization domain, and let R be an extension of D. If every prime element of D is also a prime element of R, then R is pseudo-polynomial.*

Proof. Let $f \in R \setminus D$, and suppose that $a \in fR \cap D$, with $a \neq 0$. Then, a has a prime factorization $a = p_1 \dots p_n$ in D; since each p_i is also a prime element of R, it follows that $a = p_1 \dots p_n$ is also a prime factorization in R. Since $a \in fR$, f is a divisor of a, and thus f must be associated to a subproduct $p_{i_1} \dots p_{i_k}$ of the factorization of a. Since $p_{i_1} \dots p_{i_k} \in D$, the D-algebra R is pseudo-polynomial. □

Proposition 5.10. *Let D be an integral domain that it is either a unique factorization domain or a Prüfer domain of dimension 1. Then, $D[[X]]$ is a pseudo-polynomial D-algebra.*

Proof. If D is a unique factorization domain, then for every prime element p of D the quotient $D[[X]]/pD[[X]]$ is isomorphic to $(D/pD)[[X]]$, which is an integral domain; thus, p is also a prime element of D and the claim follows from Proposition 5.9.

Suppose that D is a Prüfer domain of dimension 1. Let $f \in D[[X]]$ and suppose $a \in fD[[X]] \cap D$: then, $a = fg$ for some $g \in D[[X]]$. For $h \in Q(D[[X]])$, we denote by $c(h)$ the content of h, i.e., the D-module generated by the coefficients of h. By [9, Corollary 2.9], we have $(c(f)c(g))^2 = c(f)c(g)c(fg) = a \cdot c(f)c(g)$.

Let f_0 and g_0 be, respectively, the constant term of f and g, and let $\tilde{f} := f/f_0$ and $\tilde{g} := g/g_0$. Then, $a = f_0 g_0$, and $c(\tilde{f}) = c(f)/f_0$ and $c(\tilde{g}) = c(g)/g_0$; it follows that

$$(f_0 c(\tilde{f}) g_0 c(\tilde{g}))^2 = a f_0 c(\tilde{f}) g_0 c(\tilde{g}),$$

i.e, $I^2 = I$, where $I := c(\tilde{f})c(\tilde{g})$. Since $1 \in c(\tilde{f})$ and $1 \in c(\tilde{g})$, we have $D \subseteq I$; thus, for every maximal ideal M, we have $D_M \subseteq ID_M$ and $(ID_M)^2 = ID_M$. Since D_M is a one-dimensional valuation domain and I is a fractional ideal of D, the only possibility is $I = D_M$ for every M, and thus $I = D$; hence also $c(\tilde{f}) = c(\tilde{g}) = D$, and $c(f) = (f_0)$. Therefore, f is associated to f_0 in $D[[X]]$, and $D[[X]]$ is pseudo-polynomial over D. □

Example 5.11. The ring $D[[X]]$ of the power series over D is not always pseudo-polynomial. For example, suppose that D is a two-dimensional valuation ring, with prime ideals $(0) \subsetneq P \subsetneq M$. Let $m \in M \setminus P$ and $p \in P \setminus (0)$. Let

$$f := p + pm^{-1}X + pm^{-2}X^2 + \cdots = \sum_{i \geq 0} pm^{-i}X^i = \frac{p}{1 - m^{-1}X}.$$

Then, $f \in D[[X]]$ since $p \in P \subseteq m^k D$ for every k; moreover, $p^2 \in fD[[X]]$ since

$$f \cdot (p - pm^{-1}X) = \frac{p}{1 - m^{-1}X} \cdot p(1 - m^{-1}X) = p^2,$$

and $p - pm^{-1}X \in D[[X]]$. In particular, $fD[[X]]$ is a integral ideal of $D[[X]]$ that is unitary with respect to D.

We claim that $fD[[X]]$ is not generated by any $d \in D$. Indeed, if h is a unit of $D[[X]]$ then its constant term is a unit of D; thus, if $fD[[X]]$

is associated in R to some $d \in D$, then d must be associated in D to the constant term of f, i.e., to p. However,

$$\frac{f}{p} = \frac{p\sum_{i\geq 0} m^{-i}X^i}{p} = \sum_{i\geq 0} m^{-i}X^i \notin D[[X]]$$

since $m^{-1} \notin D$. Thus, $D[[X]]$ is not pseudo-polynomial over D.

Proposition 5.12. *Let $D \subset R \subset A$ be integral domains, and let L be the quotient field of R. If A is pseudo-polynomial over R, R is pseudo-polynomial over D and $A \cap L = R$, then A is pseudo-polynomial over D.*

Proof. Let K be the quotient field of D. Let $a \in A$ be an element such that $aA \cap K \neq (0)$. Then, $aA \cap L \neq (0)$, and thus there is a $r \in R$ such that $aA = rA$. Therefore, if $t \in aA \cap K$ then $tr^{-1} \in A \cap L = R$, and thus $t \in rR$; in particular, $t \in rR \cap K$, and so $rR \cap K \neq (0)$. Since R is pseudo-polynomial over D, there is a $d \in D$ such that $rR = dR$; hence, $aA = rA = rRA = dRA = dA$, and A is pseudo-polynomial over D. □

Example 5.13. The previous proposition does not hold without the hypothesis that $A \cap L = R$. Indeed, let $D = \mathbb{Z}$, $R = \mathbb{Z}[X]$ and let $A = V$ be the valuation overring of R induced by the valuation v defined by

$$v\left(\sum_{n=0}^{k} a_n X^n\right) = \inf\{2v_{(2)}(a_n) + n\}.$$

Then, V is a discrete valuation ring, and its ideals are generated by the powers X^n, which are in R; hence, V is pseudo-polynomial over R. The above part of the section also implies that R is pseudo-polynomial over D.

Let M be the maximal ideal of V. Then, $M \cap \mathbb{Q} \neq (0)$ because $2 \in M$; indeed, $M \cap \mathbb{Q} = 2\mathbb{Z}_{(2)}$. However, M is not generated (over V) by any rational number, since $v(2) = 2$; thus V is not pseudo-polynomial over D.

Integral extensions are often not pseudo-polynomial.

Proposition 5.14. *Let $D \subseteq R$ be integral domains, and let K, L be the quotient fields of D and R (respectively). Suppose that the extension $K \subseteq L$ is algebraic. Then, R is pseudo-polynomial over D if and only if every principal ideal of R is generated by an element of D.*

Proof. It is enough to prove that every ideal of R meets D. Write $D \subseteq A_1 \subseteq A_2 \subseteq R$, where $A_1 := R \cap K$ and A_2 is the integral closure of A_1 in R. Then, both D and A_1 have quotient field K and both A_2 and R have quotient field L; thus, every ideal of R meets A_2 and every ideal of A_1 meets D. The claim will thus be proved if every ideal of A_2 meets A_1.

Let $a \in A_2$: then, a is integral over A_1, and thus it has a minimal polynomial $f(X) = f_0 + f_1X + \cdots + X^n$. Then, $f_0 = -a(f_1 + f_2a + \cdots + a^{n-1})$ belongs to both A_1 and aA_2. and thus $aA_1 \cap A_2 \neq (0)$. The claim is proved. □

Corollary 5.15. *Let R be the integral closure of $\mathbb{Z}$ in a proper extension L of $\mathbb{Q}$. Then, R is not pseudo-polynomial over $\mathbb{Z}$.*

Proof. Since $L \neq \mathbb{Q}$, there is at least one prime p of $\mathbb{Z}$ which splits in R. Thus, the prime ideals over $p\mathbb{Z}$ are not generated by elements of $\mathbb{Z}$. The claim follows from Proposition 5.14. □

Remark 5.16. It is possible for a proper integral extension to be pseudo-polynomial. For example, if V is a valuation domain and W is an extension of V such that the extension of value groups is trivial, then every ideal of W is generated by elements of V.

6. The Local Picard Group

Definition 6.1. Let R be a D-algebra and let $\iota : \mathrm{Pic}(D) \longrightarrow \mathrm{Pic}(R)$ be the canonical map. We define the *local Picard group* of R as a D-algebra as

$$\mathrm{LPic}(R, D) := \frac{\mathrm{Pic}(R)}{\iota(\mathrm{Pic}(D))}.$$

Likewise, if R is an extension of D then the *unitary local Picard group* of $D \subseteq R$ is

$$\mathrm{LPic}_u(R, D) := \frac{\mathrm{Pic}_u(R, D)}{\iota(\mathrm{Pic}(D))}.$$

Remark 6.2. (1) Note that if I is an invertible integral ideal of D then IR is integral and unitary. Thus $\iota(\mathrm{Pic}(D)) \subseteq \mathrm{Pic}_u(R)$ and $\mathrm{LPic}_u(R, D)$ is well-defined.

(2) From the basic properties of groups, we have

$$\frac{\mathrm{LPic}(R, D)}{\mathrm{LPic}_u(R, D)} = \frac{\mathrm{Pic}(R)/\iota(\mathrm{Pic}(D))}{\mathrm{Pic}_u(R, D)/\iota(\mathrm{Pic}(D))} \simeq \frac{\mathrm{Pic}(R)}{\mathrm{Pic}_u(R, D)}.$$

(3) Every ring R can be considered as a $\mathbb{Z}$-algebra; in this case, the map ι is just the zero map. Therefore, the Picard group $\mathrm{Pic}(R)$ can also be seen as the local Picard group $\mathrm{LPic}(R, \mathbb{Z})$. Likewise, if F is a field and R is an F-algebra, $\mathrm{LPic}(R, F)$ is just $\mathrm{Pic}(R)$.

Example 6.3. Let D be an integral domain and $R = D[X]$ the polynomial ring over D. Then, the canonical map $\mathrm{Pic}(D) \longrightarrow \mathrm{Pic}(D[X])$ is surjective if and only if D is seminormal [8, Theorem 1.6], i.e., $\mathrm{LPic}(D[X], D)$ is the trivial group if and only if D is seminormal. More generally, $\mathrm{LPic}(D[X], D)$ is isomorphic to $\mathrm{Pic}(A+XD[X])$, where A is the base ring of D (i.e., $A = \mathbb{Z}$ if D has characteristic 0, $A = \mathbb{F}_p$ is D has characteristic $p > 0$) [3, Theorem 3.8], and a similar result holds with more indeterminates.

Proposition 6.4. *Let D be an integral domain. Then, the assignments $R \mapsto \mathrm{LPic}(R, D)$ and $R \mapsto \mathrm{LPic}_u(R, D)$ give rise to functors from the category of integral D-algebras (where maps are D-algebra homomorphisms) to the category of abelian groups.*

Proof. Let $\phi : R \longrightarrow R'$ be a map of D-algebras. Since Pic is a functor, ϕ induces a map $\phi^* : \mathrm{Pic}(R) \longrightarrow \mathrm{Pic}(R')$ sending $\iota_R(\mathrm{Pic}(D))$ to $\iota_{R'}(\mathrm{Pic}(D))$; hence ϕ^* induces a map $\phi^\sharp : \mathrm{LPic}(R, D) \longrightarrow \mathrm{LPic}(R', D)$. The fact that $\phi \mapsto \phi^\sharp$ respects compositions is seen in the same way.

The proof for the unitary local Picard group is the same. □

Proposition 6.5. *Let $D \subseteq R \subseteq A$ be integral domains. Then,*

$$\mathrm{LPic}(A,R) \simeq \frac{\mathrm{LPic}(A,D)}{\widetilde{\iota}(\mathrm{LPic}(R,D))},$$

and

$$\mathrm{LPic}_u(A,R) \simeq \frac{\mathrm{LPic}_u(A,D)}{\widetilde{\iota_u}(\mathrm{LPic}_u(R,D))},$$

where $\widetilde{\iota} : \mathrm{LPic}(R,D) \longrightarrow \mathrm{LPic}(A,D)$ is the map induced by the inclusion $R \subseteq A$ and $\widetilde{\iota_u}$ is the restriction of $\widetilde{\iota}$ to $\mathrm{LPic}_u(R,D)$.

Proof. Let $\iota_{DR} : \mathrm{Pic}(D) \longrightarrow \mathrm{Pic}(R)$, $\iota_{RA} : \mathrm{Pic}(R) \longrightarrow \mathrm{Pic}(A)$, $\iota_{DA} : \mathrm{Pic}(D) \longrightarrow \mathrm{Pic}(A)$ be the canonical maps. Then, $\iota_{DA} = \iota_{RA} \circ \iota_{DR}$, and thus in particular $\iota_{RA}(\mathrm{Pic}(R)) \supseteq \iota_{DA}(\mathrm{Pic}(D))$. Hence, there is a surjective map

$$\mathrm{LPic}(A,D) = \frac{\mathrm{Pic}(A)}{\iota_{DA}(\mathrm{Pic}(D))} \longrightarrow \frac{\mathrm{Pic}(A)}{\iota_{RA}(\mathrm{Pic}(R))} = \mathrm{LPic}(A,R),$$

whose kernel is

$$\frac{\iota_{RA}(\mathrm{Pic}(R))}{\iota_{DA}(\mathrm{Pic}(D))} = \frac{\iota_{RA}(\mathrm{Pic}(R))}{\iota_{RA} \circ \iota_{DR}(\mathrm{Pic}(D))} = \widetilde{\iota}\left(\frac{\mathrm{Pic}(R)}{\iota_{DR}(\mathrm{Pic}(D)}\right) = \widetilde{\iota}(\mathrm{LPic}(R,D)).$$

The claim for $\mathrm{LPic}(A,D)$ is proved. The case of the unitary Picard group is analogous. □

In Example 6.3, the natural map of $\mathrm{Pic}(D)$ into $\mathrm{Pic}(R)$ is not only injective, but give rise to a direct sum decomposition $\mathrm{Pic}(R) \simeq \mathrm{Pic}(D) \oplus \mathrm{LPic}(R,D)$ [3, Section 2]; this is a more general feature of retract D-algebras, and can be proved essentially in the same way.

Proposition 6.6. *Let D be an integral domain and let R be a retract D-algebra. Then:*

(a) the canonical map $\iota : \mathrm{Pic}(D) \longrightarrow \mathrm{Pic}(R)$ is injective;
(b) $\iota(\mathrm{Pic}(D))$ is a direct summand of $\mathrm{Pic}(R)$ and of $\mathrm{Pic}_u(R,D)$;
(c) $\mathrm{Pic}(R) \simeq \mathrm{Pic}(D) \oplus \mathrm{LPic}(R,D)$;
(d) $\mathrm{Pic}_u(R,D) \simeq \mathrm{Pic}(D) \oplus \mathrm{LPic}_u(R,D)$.

Proof. Let i be the inclusion of D into R. The composition $\epsilon \circ i$ is the identity on D; since $A \mapsto \mathrm{Pic}(A)$ is a functor, it follows that $\epsilon^* \circ i^* = \epsilon^* \circ \iota$ is the identity on $\mathrm{Pic}(D)$. Therefore, ι is injective and the exact sequence

$$0 \longrightarrow \ker(\epsilon^*) \longrightarrow \mathrm{Pic}(R) \xrightarrow{\epsilon^*} \mathrm{Pic}(D) \longrightarrow 0$$

splits. The kernel of ϵ^* is isomorphic to the quotient between $\mathrm{Pic}(R)$ and $\iota(\mathrm{Pic}(D))$, and thus by definition is isomorphic to $\mathrm{LPic}(R,D)$. Hence, $\mathrm{Pic}(R) \simeq \mathrm{Pic}(D) \oplus \mathrm{LPic}(R,D)$.

The reasoning for $\mathrm{LPic}_u(R,D)$ is the same. □

7. Localization of the Local Picard Group

We aim to study the local Picard group through the lens of localization and of extension by Jaffard overrings, as was done for the Picard group of the ring of integer-valued polynomials in [13]. We shall follow the same method of the proofs therein, which are generalizations of the methods given in [1, Chapter VIII].

The following theorem corresponds to [1, Proposition VIII.1.6] and [13, Proposition 4.3].

Theorem 7.1. *Let D be an integral domain and let R be a pseudo-polynomial retract D-algebra. Let Θ be a complete family of flat overrings of D. Then, there are exact sequences*

$$0 \longrightarrow \mathrm{Pic}(D,\Theta) \longrightarrow \mathrm{Pic}(R) \xrightarrow{\pi_\Theta} \prod_{T\in\Theta} \mathrm{Pic}(RT), \tag{4}$$

and

$$0 \longrightarrow \mathrm{Pic}(D,\Theta) \longrightarrow \mathrm{Pic}_u(R,D) \xrightarrow{\pi_\Theta} \prod_{T\in\Theta} \mathrm{Pic}_u(RT,R). \tag{5}$$

Proof. We first show the result for $\mathrm{Pic}(R)$.

The map $\mathrm{Pic}(D,\Theta) \longrightarrow \mathrm{Pic}(R)$ is the restriction of the extension map ι, which is injective by Proposition 6.6, and thus it is itself injective. By construction, if $[I] \in \mathrm{Pic}(D,\Theta)$ then IT is principal for every $T \in \Theta$, and thus $IRT = ITR$ is principal; thus, the kernel of π_Θ contains $\iota(\mathrm{Pic}(D,\Theta))$.

Suppose now that $[I] \in \ker \pi_\Theta$; then, IT is principal for every $T \in \Theta$ and thus IK is principal. By the proof of Proposition 4.5, $[I] \in \mathrm{Pic}_u(R,D)$, and thus without loss of generality we can suppose that I is unitary and integral. Then, $(I \cap D)T = T$ for all but finitely many elements of Θ, say $T_1,\dots,T_n$. By Lemma 5.3, for each i there is an $x_i \in IT_i$ such that $IT_i = x_iRT_i$. The ideal $L_i := x_iT_i \cap D$ is finitely generated over D (since T_i is a Jaffard overring [10, Lemma 5.9]) and $L_iT_i = x_iT_i$. Therefore, the ideal $L := L_1 + \cdots + L_n$ is a finitely generated ideal of D; moreover, $LT = T$ if $T \in \Theta\setminus\{T_1,\dots,T_n\}$ and $LT_i = L_iT_i = x_iT_i = IT_i$, and thus L is locally principal. Therefore, L is an invertible ideal such that LT is principal for every $T \in \Theta$ (thus, $L \in \mathrm{Pic}(D,\Theta)$) and $LRT = IRT$ for every $T \in \Theta$. As the family Θ is complete, we have $R = \bigcap\{RT \mid T \in \Theta\}$; therefore, the map $\star : Z \mapsto \bigcap\{ZRT \mid T \in \Theta\}$ is a star operation on R (see for example [7, §32]), and I and LR are invertible ideals of R. Thus

$$I = I^\star = \bigcap_{T\in\Theta} IT = \bigcap_{T\in\Theta} LRT = (LR)^\star = LR,$$

i.e., $[I] = \iota([L])$. Thus $\ker \pi_\Theta \subseteq \iota(\mathrm{Pic}(D,\Theta))$, as claimed.

The result for $\mathrm{Pic}_u(R,D)$ follows by restricting the previous reasoning to unitary ideals and noting that the extension of a unitary integral ideal is still unitary and integral. □

Putting more hypothesis on Θ, we are able to get stronger statements. Lemma 7.3 below is a variant of [10, Lemma 5.9].

Lemma 7.2. *Let D be an integral domain, R a D-algebra with quotient field L and T a flat overring of D. If X_1, X_2 are R-submodules of L, then $(X_1 \cap X_2)T = X_1T \cap X_2T$.*

Proof. If T is a flat overring of D, then TR is a flat overring of R, and since X_1, X_2 are R-modules,

$$(X_1 \cap X_2)T = (X_1 \cap X_2)RT = X_1RT \cap X_2RT = X_1T \cap X_2T,$$

as claimed. □

Lemma 7.3. *Let D be an integral domain and let R be an extension of D; let T be a Jaffard overring of D. Let J be a unitary ideal of RT. If J is finitely generated over RT, then $J \cap R$ is finitely generated over R.*

Proof. Let $T^\perp$ be the orthogonal to T with respect to D, and let $I := J \cap R$. Using Lemma 7.2, we have

$$IT^\perp = (J \cap R)T^\perp = JT^\perp \cap T^\perp = JRTT^\perp \cap T^\perp = JRK \cap T^\perp.$$

Since J is unitary, $J \cap K \neq (0)$; thus, $JRK = (JK)R = KR$ and $IT^\perp = T^\perp$. Hence there is a finitely generated ideal $I_0 \subseteq I$ such that $I_0T^\perp = T^\perp$.

Let $x_1, \dots, x_m$ be the generators of J. Since $IT = (J \cap R)T = JT \cap T = J$, for each i there is a finitely generated ideal $I_i \subseteq I$ such that $x_i \in I_iT$; then, $L := I_0 + I_1 + \cdots + I_m$ is a finitely generated ideal contained in I such that $LT = J = IT$ and $LT^\perp = T^\perp = IT^\perp$. It follows that $L = I$, i.e., $I = J \cap R$ is finitely generated. □

The following is an analogue of Theorems 4.4 and 4.7 of [13].

Theorem 7.4. *Let D be an integral domain and let R be a pseudo-polynomial retract D-algebra. Let Θ be a Jaffard family of D. Then, there is an exact sequence*

$$0 \longrightarrow \mathrm{Pic}(D, \Theta) \longrightarrow \mathrm{Pic}_u(R, D) \xrightarrow{\pi_\Theta} \bigoplus_{T \in \Theta} \mathrm{Pic}_u(RT, T) \longrightarrow 0,$$

and

$$\mathrm{LPic}_u(R, D) \simeq \bigoplus_{T \in \Theta} \mathrm{LPic}_u(RT, T).$$

Proof. By Theorem 7.1, to prove the first claim it is enough to show that the range of π_Θ is the direct sum. Indeed, if $[I] \in \mathrm{Pic}_u(R, D)$, then by Corollary 4.6 we can suppose without loss of generality that I is unitary and integral. Hence, $(I \cap D)T = T$ for all but finitely many $T \in \Theta$, and thus the range of π_Θ is contained in the direct sum.

To prove the converse, we need to show that, given a fixed $T \in \Theta$ and a $[J] \in \mathrm{Pic}_u(RT, T)$, there is an $[I] \in \mathrm{Pic}_u(R, D)$ such that $[IT] = [J]$ and $[IS] = [RS]$ for all $S \in \Theta \setminus \{T\}$. By Corollary 4.6, we can suppose that J is integral and unitary. By Lemma 7.3, $I := J \cap R$ is finitely generated over R. We claim that I satisfies the previous conditions.

Indeed, $IT = (J \cap R)T = JT \cap RT = JT$, while $IS = (J \cap R)S = JS \cap RS = JTS \cap RS = JK \cap RS = RS$ since $J \cap K \neq (0)$ and thus $1 \in JK$. To show that I is invertible, let M be a maximal ideal of R. If

$M \cap D = (0)$, then R_M contains RK and thus RT, for every $T \in \Theta$; hence, IR_M is principal since so is IRT. If $M \cap D = P \neq (0)$, then R_M contains D_P, and thus $R_M \supseteq IS$, where S is the member of Θ such that $PS \neq S$. Thus IR_M is principal since IRS is principal. Therefore, I is locally principal and thus invertible.

Therefore, the direct sum is in the image of π_Θ, and the sequence in the statement is exact.

Consider now the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}(D,\Theta) & \longrightarrow & \mathrm{Pic}(D) & \longrightarrow & \bigoplus_{T\in\Theta} \mathrm{Pic}(T) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \iota_D} & & \downarrow{\scriptstyle \iota_\Theta} & & \\ 0 & \longrightarrow & \mathrm{Pic}(D,\Theta) & \longrightarrow & \mathrm{Pic}_u(R,D) & \longrightarrow & \bigoplus_{T\in\Theta} \mathrm{Pic}_u(RT,T) & \longrightarrow & 0, \end{array}$$

where ι_D and ι_Θ are the natural maps. Since the leftmost vertical map is the equality, the snake lemma gives an isomorphism between the cokernel of ι_D (namely, $\mathrm{LPic}_u(R,D)$) and the cokernel of ι_Θ (namely, the direct sum of the $\mathrm{LPic}_u(RT,T)$). The claim follows. □

Corollary 7.5. *Let D be an integral domain with quotient field K, and let R be a pseudo-polynomial retract D-algebra such that* $\mathrm{Pic}(RK) = (0)$*. Let Θ be a Jaffard family of D. Then, there is an exact sequence*

$$0 \longrightarrow \mathrm{Pic}(D,\Theta) \longrightarrow \mathrm{Pic}(R) \xrightarrow{\pi_\Theta} \bigoplus_{T\in\Theta} \mathrm{Pic}(RT) \longrightarrow 0,$$

and

$$\mathrm{LPic}(R,D) \simeq \bigoplus_{T\in\Theta} \mathrm{LPic}(RT,T).$$

Proof. If $\mathrm{Pic}(RK) = (0)$, then $\mathrm{Pic}_u(R,D) = \mathrm{Pic}(R)$ and $\mathrm{LPic}_u(R,D) = \mathrm{LPic}(R,D)$ (and the same for T, since $\mathrm{Pic}(RTK) = \mathrm{Pic}(RK)$). The claim follows from Theorem 7.4. □

Remark 7.6. Without the hypothesis $\mathrm{Pic}(RK) = (0)$, Corollary 7.5 does not hold. For example, suppose that R is a Prüfer domain: then, the canonical map $\phi : \mathrm{Pic}(R) \longrightarrow \mathrm{Pic}(RK)$ is surjective, and so by Proposition 4.5 we have $\mathrm{Pic}(R)/\mathrm{Pic}_u(R,T) \simeq \mathrm{Pic}(RK)$. If the corollary were true, we would have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}(D,\Theta) & \longrightarrow & \mathrm{Pic}_u(R,D) & \longrightarrow & \bigoplus_{T\in\Theta} \mathrm{Pic}_u(RT,T) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Pic}(D,\Theta) & \longrightarrow & \mathrm{Pic}(R) & \longrightarrow & \bigoplus_{T\in\Theta} \mathrm{Pic}(RT) & \longrightarrow & 0 \end{array}$$

and an application of the snake lemma would give

$$\mathrm{Pic}(RK) \simeq \bigoplus_{T\in\Theta} \mathrm{Pic}(RK).$$

However, if $\mathrm{Pic}(RK) \neq (0)$, then this isomorphism may not hold.

The following corollaries are special cases of Theorem 7.4 and Corollary 7.5.

Corollary 7.7. *Let D be an integral domain and let Θ be a Jaffard family of D. Let $\mathbf{X}$ be a family of independent indeterminates over D. Then,*

$$\mathrm{LPic}(D[\mathbf{X}], D) \simeq \bigoplus_{T\in\Theta} \mathrm{LPic}(T[\mathbf{X}], T).$$

Proof. The polynomial ring $D[\mathbf{X}]$ is a pseudo-polynomial retract D-algebra. Since $D[\mathbf{X}]K = K[\mathbf{X}]$ is a unique factorization domain, we can apply Corollary 7.5. □

Corollary 7.8. *Let D be an integral domain and let Θ be a Jaffard family of D. Let $E \subseteq K$ be a subset such that $dE \subseteq D$ for some $d \neq 0$. Then,*

$$\mathrm{LPic}(\mathrm{Int}(E, D), D) \simeq \bigoplus_{T\in\Theta} \mathrm{LPic}(\mathrm{Int}(E, T), T),$$

and

$$\mathrm{LPic}(\mathrm{Int}(D), D) \simeq \bigoplus_{T\in\Theta} \mathrm{LPic}(\mathrm{Int}(T), T).$$

Proof. Note that $\mathrm{Int}(E, D) \simeq \mathrm{Int}(dE, D)$, and thus we can suppose without loss of generality that $E \subseteq D$.

The ring $\mathrm{Int}(E, D)$ is a retract D-algebra since the evaluation in any $d \in E$ is a retract. Moreover, $D[X] \subseteq \mathrm{Int}(E, D)$, and thus $\mathrm{Int}(E, D)K = K[X]$; therefore, $\mathrm{Pic}(\mathrm{Int}(E, D)K) = (0)$, so that $\mathrm{Pic}_u(\mathrm{Int}(E, D), D) = \mathrm{Pic}(\mathrm{Int}(E,D))$ and $\mathrm{LPic}_u(\mathrm{Int}(E, D), D) = \mathrm{LPic}(\mathrm{Int}(E, D), D)$. Finally, $\mathrm{Int}(E, D) \cap K = D$ and thus D is pseudo-polynomial by Proposition 5.7. By Theorem 7.4, we have

$$\mathrm{LPic}(\mathrm{Int}(E, D), D) \simeq \bigoplus_{T\in\Theta} \mathrm{LPic}(\mathrm{Int}(E, D)T, T).$$

The equality $\mathrm{Int}(E, D)T = \mathrm{Int}(E, T)$ follows as in [13, Section 3].

If $E = D$, then $\mathrm{Int}(E, D) = \mathrm{Int}(D)$ and $\mathrm{Int}(D)T = \mathrm{Int}(T)$. The claim is proved. □

Let $x \in D$. The *Bhargava ring* of D with respect to x is [14]

$$\mathbb{B}_x(D) := \{f \in K[X] \mid f(aX + x) \in D[X] \text{ for all } a \in D\}.$$

Corollary 7.9. *Let D be an integral domain and let Θ be a Jaffard family of D. Then,*

$$\mathrm{LPic}(\mathbb{B}_x(D), D) \simeq \bigoplus_{T\in\Theta} \mathrm{LPic}(\mathbb{B}_x(T), T).$$

Proof. The Bhargava ring $\mathbb{B}_x(D)$ satisfies $D[X] \subseteq \mathbb{B}_x(D) \subseteq K[X]$, and $\mathbb{B}_x(D) \cap K = D$; therefore, it is pseudo-polynomial and $\mathrm{Pic}(\mathbb{B}_x(D)K) = (0)$. Moreover, $\mathbb{B}_x(D) \subseteq \mathrm{Int}(x, D)$ and thus $\mathbb{B}_x(D)$ is also a retract D-algebra. By Corollary 7.5, we have $\mathrm{LPic}(\mathbb{B}_x(D), D) \simeq \bigoplus_{T\in\Theta} \mathrm{LPic}(\mathbb{B}_x(D)T, T)$, and we need to show that $\mathbb{B}_x(D)T = \mathbb{B}_x(T)$.

Let $T^\perp$ be the orthogonal of T with respect to D. By [2, Lemma 1.1], since T and $T^\perp$ are sublocalizations and $D = T \cap T^\perp$ we have $\mathbb{B}_x(D) = \mathbb{B}_x(T) \cap \mathbb{B}_x(T^\perp)$; by Lemma 7.2 it follows that

$$\mathbb{B}_x(D)T = (\mathbb{B}_x(T) \cap \mathbb{B}_x(T^\perp))T = \mathbb{B}_x(T)T \cap \mathbb{B}_x(T^\perp)T.$$

Since $\mathbb{B}_x(T)$ is a T-algebra we have $\mathbb{B}_x(T)T = \mathbb{B}_x(T) \subseteq K[X]$. Moreover, $\mathbb{B}_x(T^\perp)T = \mathbb{B}_x(T^\perp)T^\perp T = \mathbb{B}_x(T^\perp)K = K[X]$. Thus $\mathbb{B}_x(T)T \subseteq \mathbb{B}_x(T^\perp)T$ and $\mathbb{B}_x(D)T = \mathbb{B}_x(T)$. The claim is proved. □

Corollary 7.10. *Let D be a locally finite one-dimensional domain, and let R be a pseudo-polynomial retract D-algebra. Then,*

$$\mathrm{LPic}_u(R, D) \simeq \bigoplus_{M\in\mathrm{Max}(D)} \mathrm{Pic}_u(RD_M, D_M),$$

and

$$\mathrm{Pic}_u(R, D) \simeq \mathrm{Pic}(D) \oplus \bigoplus_{M\in\mathrm{Max}(D)} \mathrm{Pic}_u(RD_M, D_M).$$

Proof. The first isomorphism follows from Theorem 7.4 using the family $\Theta := \{D_M \mid M \in \mathrm{Max}(D)\}$ (which is a Jaffard family since D is one-dimensional and locally finite), and the fact that $\mathrm{Pic}(D_M) = (0)$ since each D_M is local. The second isomorphism now follows from Proposition 6.6. □

Corollary 7.11. *Let D be an integral domain and let Θ be a Jaffard family of D; let R be a pseudo-polynomial retract D-algebra. If* $\mathrm{Pic}(D) = (0)$*, then*

$$\mathrm{Pic}_u(R, D) \simeq \bigoplus_{T\in\Theta} \mathrm{Pic}_u(RT, T).$$

Proof. If $\mathrm{Pic}(D) = (0)$, then $\mathrm{Pic}(T) = (0)$ for every Jaffard overring T; thus, $\mathrm{LPic}_u(R, D) = \mathrm{Pic}_u(R, D)$ and $\mathrm{LPic}_u(RT, T) = \mathrm{Pic}_u(RT, T)$. □

Corollary 7.12. *Let D be a locally finite Prüfer domain of dimension* 1*. Then,*

$$\mathrm{Pic}_u(D[[X]], D) \simeq \mathrm{Pic}(D) \oplus \bigoplus_{M\in\mathrm{Max}(D)} \mathrm{Pic}_u(D[[X]]D_M, D_M).$$

Note that $D[[X]]D_M$ is *not* the ring $D_M[[X]]$ of power series over D_M: for example, $\sum_n X^n/3^n$ belongs to $\mathbb{Z}_{(2)}[[X]]$ but not to $\mathbb{Z}[[X]]\mathbb{Z}_{(2)}$.

Proof. The ring $D[[X]]$ is pseudo-polynomial over D by Proposition 5.10 and a retract D-algebra (with ϵ being the evaluation in 0). The claim now follows from Corollary 7.10. □

Remark 7.13. Theorems 7.1 and 7.4 do not hold for general D-algebras. For example, let $D = \mathbb{Z}$ and let R be the integral closure of $\mathbb{Z}$ in a finite extension L of $\mathbb{Q}$. Note that $\mathrm{Pic}_u(R, \mathbb{Z}) = \mathrm{Pic}(R)$ since $RK = L$. Let Θ be the family of localizations of $\mathbb{Z}$ at the maximal ideals. Then, for every $T = \mathbb{Z}_{(p)} \in \Theta$, the ring RT is semilocal (its maximal ideals correspond to the maximal ideals of R over (p), which are finite since $[L : \mathbb{Q}] < \infty$), and thus $\mathrm{Pic}(RT) = (0)$ for every T, and thus also $\mathrm{LPic}(RT, T) = (0)$. On the other hand, $\mathrm{Pic}(D, \Theta) = (0)$, and thus (4) becomes

$$0 \longrightarrow 0 \longrightarrow \mathrm{Pic}(R) \longrightarrow 0.$$

If R does not have unique factorization, $\mathrm{Pic}(R) \neq (0)$ and thus the sequence is not exact. Likewise, $\mathrm{LPic}(R, D) = \mathrm{Pic}(R) \neq (0)$, and thus the isomorphism $\mathrm{LPic}(R, D) \simeq \bigoplus_{T\in\Theta} \mathrm{LPic}(RT, T)$ is not true.

8. Pre-Jaffard Families

The results in the previous section only deal with Jaffard families. As done in [13], under some hypothesis we can extend the results to the more general case of pre-Jaffard families.

We start with an analogue of [13, Proposition 7.2], of which we follow the proof.

Lemma 8.1. *Let D be an integral domain and Θ be a pre-Jaffard family of D; let $\{T_\alpha\}$ be the derived sequence associated to Θ. Let R be a pseudo-polynomial retract D-algebra. Then, the extension map $\mathrm{Pic}_u(R, D) \longrightarrow \mathrm{Pic}_u(RT_\alpha, T_\alpha)$ is surjective.*

Proof. By induction on α. If $\alpha = 1$, let $\mathcal{L}$ be the lattice of Jaffard overrings of D. By the proof of [13, Proposition 6.1], we have $T = \bigcup\{S \mid S \in \mathcal{L}\}$; using the same proof of [13, Lemma 5.2], it follows that $RT = \bigcup\{RS \mid S \in \mathcal{L}\}$. Since for $S \in \Theta\setminus\{T\}$ the map $\mathrm{Pic}_u(R, D) \longrightarrow \mathrm{Pic}_u(RS, S)$ is surjective (Theorem 7.4), the same reasoning of the proof of [13, Lemma 5.1] shows that also $\mathrm{Pic}_u(R, D) \longrightarrow \mathrm{Pic}_u(RT, T)$ is surjective.

If α is a limit ordinal, the claim follows in the same way since $\bigcup_{\gamma<\alpha} T_\gamma = T_\alpha$ [13, Lemma 7.1] and thus $\bigcup_{\gamma<\alpha} RT_\gamma = RT_\alpha$; hence we can apply [13, Lemma 5.1]. If $\alpha = \gamma+1$ is a successor ordinal, then the map $\mathrm{Pic}_u(R, D) \longrightarrow \mathrm{Pic}_u(RT_\alpha, T_\alpha)$ factors as

$$\mathrm{Pic}_u(R, D) \longrightarrow \mathrm{Pic}_u(RT_\gamma, T_\gamma) \longrightarrow \mathrm{Pic}_u(RT_\alpha, T_\alpha);$$

the first map is surjective by inductive hypothesis, while the second one is surjective since we can apply the case $\alpha = 1$ to the T_γ-algebra RT_γ. □

The following result is the analogue of Proposition 6.2 and Theorem 6.4 of [13]. We premise a lemma, that was implicitly used in the proof of [13, Proposition 6.2].

Lemma 8.2. *Let Θ be a weak Jaffard family of D pointed at T_∞. Let I be a finitely generated ideal of D such that IT_∞ is principal. Then, there are only finitely many $T \in \Theta$ such that IT is not principal.*

Proof. Without loss of generality we can suppose that $I \subseteq D$. Let Λ be the set of all $T \in \Theta$ such that IT is not principal.

Let $I = (x_1, \dots, x_n)$. Suppose that $IT_\infty = fT_\infty$: then, there are $t_1, \dots, t_n \in T_\infty$ such that $f = x_1t_1 + \cdots + x_nt_n$. Consider the set $\Omega := \mathcal{B}(t_1, \dots, t_n, fx_1^{-1}, \dots, fx_n^{-1}) \subseteq \Theta$ of all elements of Θ that contain each t_i and each fx_i^{-1}: then, Ω is an open set with respect to the Zariski topology, and $T \in \Omega$ if and only if $IT = fT$. In particular, $T_\infty \in \Omega$; thus, $\Lambda_0 := \Theta \setminus \Omega$ is a closed set of Θ not containing T_∞, and $\Lambda \subseteq \Lambda_0$.

Since Θ is compact in the Zariski topology and Λ_0 is closed, Λ_0 is compact. Let $A := \bigcap\{T \mid T \in \Lambda_0\}$: then, each T is a flat overring of A, and it is also a Jaffard overring of A since each such T is a Jaffard overring of D. If P is a prime ideal of D such that $PT = T$ for every $T \in \Lambda_0$ then

$$AD_P = \left(\bigcap_{T \in \Lambda_0} T\right) D_P = \bigcap_{T \in \Lambda_0} TD_P = K,$$

using [4, Corollary 5]. Thus, Λ_0 is a Jaffard family of A, and in particular it is locally finite. Consider IA and the extensions IAT for $T \in \Lambda$: then, $IAT = IT$ is not principal for such T, and thus $IAT \neq T$. By local finiteness, Λ_0 must be finite, as claimed. $\square$

Proposition 8.3. *Let D be an integral domain and let R be a pseudo-polynomial retract D-algebra. Let Θ be a weak Jaffard family of D pointed at T_∞. Let $\pi_\Theta : \mathrm{Pic}_u(R, D) \longrightarrow \prod\{\mathrm{Pic}_u(RT, T) \mid T \in \Theta\}$ be the extension map and let Δ be its cokernel. Then, there are exact sequences*

$$0 \longrightarrow \bigoplus_{T \in \Theta\setminus\{T_\infty\}} \mathrm{Pic}_u(RT, T) \longrightarrow \Delta \longrightarrow \mathrm{Pic}_u(RT_\infty, T_\infty) \longrightarrow 0,$$

and

$$0 \longrightarrow \bigoplus_{T \in \Theta\setminus\{T_\infty\}} \mathrm{LPic}_u(RT, T) \longrightarrow \mathrm{LPic}_u(R, D) \longrightarrow \mathrm{LPic}_u(RT_\infty, T_\infty) \longrightarrow 0.$$

Proof. The inclusion of Δ into the direct sum $\prod_{T \in \Theta} \mathrm{Pic}_u(RT, T)$ induces a projection map $\pi' : \Delta \longrightarrow \mathrm{Pic}_u(RT_\infty, T_\infty)$, which is surjective since it factorizes the surjective extension map $\mathrm{Pic}_u(R, D) \longrightarrow \mathrm{Pic}_u(RT_\infty, T_\infty)$.

The kernel of π' contains exactly the extensions of the classes $[I] \in \mathrm{Pic}_u(R, D)$ such that I becomes principal in each RT, for $T \in \Theta \setminus \{T_\infty\}$. Using Theorem 7.4, we obtain that the direct sum $\bigoplus\{\mathrm{Pic}_u(RT, T) \mid T \in \Lambda\}$ is contained in the kernel; conversely, if $[I] \in \ker \pi'$ then $IT_\infty = fT_\infty$ for some f and thus I is not principal for only finitely many $T \in \Theta$ (Lemma 8.2); thus the kernel is contained in the direct sum. Therefore, $\ker \pi' = \bigoplus\{\mathrm{Pic}_u(RT, T) \mid T \in \Lambda\}$, and the first sequence in the statement is exact.

The diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \displaystyle\bigoplus_{T \in \Theta\setminus\{T_\infty\}} \mathrm{Pic}(T) & \longrightarrow & \dfrac{\mathrm{Pic}(D)}{\mathrm{Pic}(D, \Theta)} & \longrightarrow & \mathrm{Pic}(T_\infty) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \displaystyle\bigoplus_{T \in \Theta\setminus\{T_\infty\}} \mathrm{Pic}_u(RT, T) & \longrightarrow & \Delta & \longrightarrow & \mathrm{Pic}_u(RT_\infty, T_\infty) & \longrightarrow & 0
\end{array}$$

commutes, the vertical maps are injective, and the horizontal rows are exact (the top one by [13, Lemma 6.3], the bottom one by the previous part of the proof). Applying the snake lemma, we obtain an exact sequence

$$0 \longrightarrow \bigoplus_{T\in\Theta\setminus\{T_\infty\}} \mathrm{LPic}_u(RT,T) \longrightarrow \frac{\Delta}{\mathrm{Pic}(D)/\mathrm{Pic}(D,\Theta)} \longrightarrow \mathrm{LPic}_u(RT_\infty,T_\infty) \longrightarrow 0,$$

that becomes the one in the statement by noting that

$$\frac{\Delta}{\mathrm{Pic}(D)/\mathrm{Pic}(D,\Theta)} = \frac{\mathrm{Pic}_u(R,D)/\mathrm{Pic}(D,\Theta)}{\mathrm{Pic}(D)/\mathrm{Pic}(D,\Theta)} \simeq \frac{\mathrm{Pic}_u(R,D)}{\mathrm{Pic}(D)} = \mathrm{LPic}_u(R,D)$$

by definition. □

The following is analogous to Theorem 7.3 of [13].

Theorem 8.4. *Let D be an integral domain and let R be a pseudo-polynomial retract D-algebra. Let Θ be a pre-Jaffard family of D, and let $\{T_\alpha\}$ be the derived series of D. Fix an ordinal α and suppose that $\mathrm{LPic}_u(RT,T)$ is a free group for each $T\in\Theta\setminus\mathcal{N}^\alpha(\Theta)$. Then, there is an exact sequence*

$$0 \longrightarrow \bigoplus_{T\in\Theta\setminus\mathcal{N}^\alpha(\Theta)} \mathrm{LPic}_u(RT,T) \longrightarrow \mathrm{LPic}_u(R,D) \longrightarrow \mathrm{LPic}_u(RT_\alpha,T_\alpha) \longrightarrow 0.$$

Proof. By induction on α: if $\alpha=1$ then Θ is a weak Jaffard family and the claim follows from Proposition 8.3.

If $\alpha=\gamma+1$ is a successor ordinal, then we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{T\in\Theta\setminus\mathcal{N}^\gamma(\Theta)} \mathrm{LPic}_u(RT,T) & \longrightarrow & \mathrm{LPic}_u(R,D) & \longrightarrow & \mathrm{LPic}_u(RT_\gamma,T_\gamma) & \longrightarrow & 0\\
 & & \downarrow f & & \| & & \downarrow g & & \\
0 & \longrightarrow & L & \longrightarrow & \mathrm{LPic}_u(R,D) & \longrightarrow & \mathrm{LPic}_u(RT_\alpha,T_\alpha) & \longrightarrow & 0,
\end{array} \tag{6}$$

where L is the kernel of $\mathrm{LPic}_u(R,D)\longrightarrow\mathrm{LPic}_u(RT_\alpha,T_\alpha)$. The two rows are exact: the first one by induction, the second one by definition and by Lemma 8.1. The snake lemma and the fact that the middle vertical arrow is the identity give an exact sequence

$$0 \longrightarrow \bigoplus_{T\in\Theta\setminus\mathcal{N}^\gamma(\Theta)} \mathrm{LPic}_u(RT,T) \longrightarrow L \longrightarrow \bigoplus_{T\in\mathcal{N}^\gamma(\Theta)\setminus\mathcal{N}^\alpha(\Theta)} \mathrm{LPic}_u(RT,T) \longrightarrow 0,$$

which splits since $\mathrm{LPic}_u(RT,T)$ is free for $T\in\mathcal{N}^\gamma(\Theta)\setminus\mathcal{N}^\alpha(\Theta)$, by hypothesis. The claim follows reading the second row of (6).

If α is a limit ordinal, and L_γ is the kernel of the surjective map $\mathrm{LPic}_u(R,D)\longrightarrow\mathrm{LPic}_u(RT_\gamma,T_\gamma)$, then $\{L_\gamma\}_{\gamma<\alpha}$ is a chain of free subgroups such that every element is a direct summand of the following ones and whose union is L_α. Therefore, by [12, Lemma 5.6] (or [6, Chapter 3, Lemma 7.3]) L_α is the direct sum of the quotients

$$\frac{L_{\gamma+1}}{L_\gamma} \simeq \frac{\mathrm{LPic}_u(RT_{\gamma+1},T_{\gamma+1})}{\mathrm{LPic}_u(RT_\gamma+,T_\gamma)} \simeq \bigoplus_{T\in\mathcal{N}^{\gamma+1}(\Theta)\setminus\mathcal{N}^\gamma(\Theta)} \mathrm{LPic}_u(RT,T).$$

Hence, $L_\alpha\simeq\bigoplus\{\mathrm{LPic}_u(RT,T)\mid T\in\Theta\setminus\mathcal{N}^\alpha(\Theta)\}$. The claim is proved. □

Corollary 8.5. *Let D be an integral domain and let R be a pseudo-polynomial retract D-algebra. Let Θ be a sharp pre-Jaffard family of D, and suppose that $\mathrm{LPic}_u(RT, T)$ is a free group for each $T \in \Theta$. Then,*

$$\mathrm{LPic}_u(R, D) \simeq \bigoplus_{T \in \Theta} \mathrm{LPic}_u(RT, T).$$

Proof. If Θ is sharp, by definition there is an ordinal α such that $T_\alpha = K$. By the previous theorem, the cokernel of $\bigoplus_{T \in \Theta} \mathrm{LPic}_u(RT, T) \longrightarrow \mathrm{LPic}_u(R, D)$ is $\mathrm{LPic}_u(RK, K)$. By Corollary 4.7, the latter is a quotient of the trivial group $\mathrm{Pic}_u(RK, K)$, and thus it is itself trivial. The claim is proved. □

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Declarations

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