THE PROBLEM OF TANGENT, MAXIMA AND MINIMA: A HISTORICAL APPROACH TO ITS TEACHING

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The basic notions of infinitesimal calculus are fundamental not only in mathematics but also across almost all scientific disciplines, including physics, engineering, and chemistry (on this vast subject on which a wide literature exists, see the brief, but interesting Usak and Lisko, 2026). Therefore, it is essential that students acquire a deep conceptual understanding of these notions. This Editorial proposes that the history of mathematics can serve as a valuable resource in educational contexts (see, e.g., Barbin et al., 2025; Bussotti, 2021, 2023; Clark et al., 2022. All these works also provide vast references on this subject). More specifically, Pierre de Fermat's (1601-1665) method for solving problems involving tangents, maxima, and minima can provide an effective, intuitive means of introducing students to the foundational concepts of calculus (see Bussotti, 2014, also for references to the literature).

As it is generally taught in upper secondary and technical schools, mathematical analysis often presents fewer initial procedural difficulties for students than Euclidean synthetic geometry. Textbooks typically introduce the fundamental concepts of this branch—function, limit, derivative, and integral—in an intuitive manner, before moving abruptly toward a rigorous formalization in terms of ε-δ language. Furthermore, there is arguably an excessive pedagogical emphasis on certain dry, introductory topics. For instance, a considerable amount of instructional time is spent on determining the domain of a function, a task which essentially reduces to solving systems of equations or inequalities, offering little conceptual richness.

Conversely, an alternative approach introduces the core concepts of analysis by starting directly from the problems that generated them. In this genetic manner, students can naturally comprehend why notions such as function, limit, and derivative became necessary. This approach proceeds from *problems to definitions*, rather than from *definitions to applications*. This historical perspective is coherent with the fact that calculus did not originate from abstract definitions of functions or limits, but from concrete physical and geometric challenges. We do not argue that this historical approach should entirely replace the traditional one; rather, the two should complement each other, organized around three interconnected historical themes: The main themes are three:

1. The problem of tangents and of maxima and minima for a curve.
2. The problem of curvilinear areas.
3. The relationship between these two problems.

In this Editorial, Fermat's method to draw the tangents to a curve and to find maxima and minima in some problems will be explained as well as the way in which it can be exploited in an educational perspective.

Suppose a teacher is working with a final-year class in an upper secondary school. The teacher presents the students with the following problem: given a sufficiently smooth curve, how can one construct the tangent at a given point?

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After an initial brainstorming session, the teacher can guide students toward the intuitive idea that a tangent can be conceptualized as a secant line whose two intersection points with the curve are brought "infinitely close" to one another. This naturally suggests that the notions of infinity and infinitesimals are structurally related to the problem. At this stage, the teacher can introduce the historical context, explaining that this exact question occupied the greatest minds of the seventeenth century.

Let us move, then, to the middle of the seventeenth century, when modern science was emerging. Greek mathematics, especially geometry, is being rediscovered and studied with renewed interest. The leading mathematicians of the age aim not merely to recover the achievements of the ancients but to surpass them. Among the central mathematical challenges of the period were the construction of tangents to arbitrary curves, the determination of maxima and minima of a curve, and the calculation of areas bounded by curved lines. General methods were sought for all these problems.

Johannes Kepler (1571–1630) and Bonaventura Cavalieri (1598–1647) had already developed powerful infinitesimal techniques. Evangelista Torricelli (1608–1647) further advanced Cavalieri's theory of indivisibles. Yet the mathematician who, before Newton and Leibniz, came closest to the fundamental ideas of calculus was Pierre de Fermat in the context of the seventeenth-century France, where figures such as Roberval, Descartes, and Mersenne, besides Fermat himself, were active.

Let us see how Fermat solved the tangent and the maxima and minima problems and how his method can be useful for mathematics education.

It is necessary to emphasize that the following derivations are a modern reconstruction of Fermat's reasoning in modern algebraic notation. For, the aim of this Editorial is not a precise historical reconstruction of Fermat's methods, but an analysis of the way in which his concepts can be exploited in modern mathematics education.

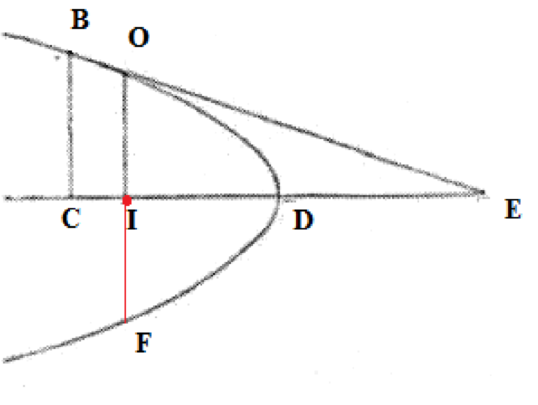
Fermat's Method

The parabola. The ancients had solved the problem of constructing tangents for several types of curves, most notably the conic sections and certain special curves such as the Archimedean spiral. What was still lacking, however, was a general method.

In his *Methodus ad disquirendam maximam et minimam* [*Method for determining maxima and minima*] (1637. See Fermat, 1891, pp. 133-180), Fermat approached the problem in the following way. Let us consider the case of a parabola and use an analytic argument.

The goal is to determine the tangent to a parabola at a point *B* (see Fig. 1). In what follows, the method of Fermat is transcribed into modern mathematical symbols because, as already highlighted, the aim of this Editorial is not historical research, but the use of notions taken from history of mathematics in an educative context.

Figure 1
The adaptation of the diagram used by Fermat to draw the tangent to a parabola



Note. Retrieved from Fermat, 1891, p. 135.

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- *CE* is the axis of the parabola.
- *C* is the orthogonal projection of *B* onto the axis.
- *O* is a point lying on the tangent, distinct from *B*. It is assumed to be "infinitely close" to *B*.
- *I* is the orthogonal projection of *O* onto the axis. Fermat sets *IC=e*.
- *D* is the vertex of the parabola.
- *E* is the point where the tangent intersects the axis.

Let us now see the steps through which Fermat solved the problem:

A) Let *F* (not shown in Fermat's original figure. I have added it) be the point where the segment *OI* intersects the parabola. By the definition of a parabola, we have:

$$\frac{CI}{ID} = \frac{BC^2}{FI^2}.$$

Since O does not belong to the parabola, it is

$$1) \quad \frac{CI}{ID} > \frac{BC^2}{OI^2}.$$

B) The triangles *BCE* and *OIE* are similar. Therefore, it is

$$2) \quad BC^2 : OI^2 = CE^2 : IE^2.$$

C) From 1) and 2), it follows

$$3) \quad \frac{CI}{ID} > \frac{CE^2}{IE^2}.$$

D) Now Fermat sets *CD=d*; *CE=a, CI=e*. The inequality 3) is so transformed into

$$4) \quad \frac{d}{d-e} > \frac{a^2}{a^2+e^2-2ae}.$$

Thus:

$$5) \quad d(a^2 + e^2 - 2ae) > a^2(d - e).$$

Ergo:

$$6) \quad de^2 - 2aed > -a^2e.$$

E) Since *e* is assumed to be vanishingly small, inequality 6) is not really an inequality. Nor is it, strictly speaking, an equation. Fermat uses the term *adaequatio* (here indicated through the symbol $\approx$). We, therefore, have:

$$7) \quad de^2 - 2aed \approx -a^2e.$$

F) As *e* is very close to zero, but not exactly zero, it is possible to divide the *adaequatio* by *e*, thus obtaining:

$$8) \quad de - 2ad \approx -a^2.$$

G) Once again, since *e* is assumed to be vanishingly small, and since all the other terms in the *adaequatio*, apart from those containing *e*, involve only finite quantities, the finite quantities multiplied by *e* may be deleted. In this way, the *adaequatio* is transformed into a genuine equation, which in the present case is

$$9) \quad a=2d.$$

Recalling that *CD* = *d* and *CE* = *a*, one obtains the correct relation for the tangent to the parabola. This relation had already been proved by Apollonius (262-190 B.C.) in his *Conics*, although by a completely different method.

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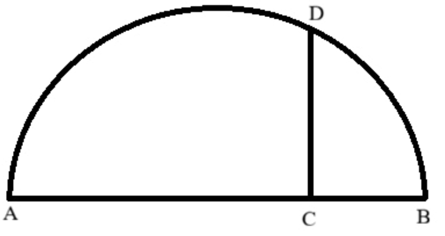
A problem of maximum and minimum. Before adding some educational comments on the usefulness of Fermat's methods, it is worth presenting a further application concerning a maximum problem. This will help to illustrate the generality of this method, which shows that, from a mathematical standpoint, drawing a tangent to a curve and finding a maximum or a minimum are closely related problems. This is the problem:

Given a semicircle with diameter AB, let DC be perpendicular to the diameter. We seek the maximum value of the sum $AC + CD$ (see Fig. 2).

Figure 2
The diagram used by Fermat to determine the maximum in the proposed problem



Note. Retrieved from Fermat 1891, p. 154.

Let *b* be the diameter and set $AC=a$, then it is $CD = \sqrt{ba - a^2}$. It is a matter of finding the maximum of the quantity $a + \sqrt{ba - a^2}$. Fermat writes that, if in this case the rule is applied without any special care, one arrives at expressions of too high a degree. First, he denotes the unknown maximum by O. Thus, it holds:

$$a + \sqrt{ba - a^2} = o \Rightarrow o - a = \sqrt{ba - a^2} \Rightarrow o^2 + a^2 - 2oa = ba - a^2.$$

Now Fermat isolates the terms holding the highest power of *o*, thus obtaining:

$$o^2 = ba - 2a^2 + 2oa.$$

Since o is the maximum, its square will also be a maximum. Therefore, the expression $ba - 2a^2 + 2oa$ also is. No radical appears, so it is possible to proceed with the *adaequatio*, replacing $AC = a$ with $AC + e = a + e$. The expression gets, hence, the form

$$ba - 2a^2 + 2oa \approx ba + ba - 2a^2 - 2e^2 - 4ae + 2oa + 2oe.$$

By eliminating the common terms and dividing by e, one has

$$b + 2o \approx 2e + 4a.$$

If the quantity 2*e* is deleted, the expression becomes

$$o = 2a - \frac{b}{2}.$$

Thus, finally one has the equation

$$2a - \frac{b}{2} = a + \sqrt{ba - a^2}.$$

After very easy passages, one finds the searched value of *a*.

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Educational Observations

When evaluating Fermat's method for classroom implementation, four key conceptual nodes emerge:

1) The concept of a limit seems to be present *in nuce*, since it can be interpreted as the tendency of one point to approach another. Indeed, while dealing with *e*, two points of what nowadays we call abscissa axis tend to converge and correspondingly two points of the parabola do. However, Fermat's reasoning is much more pragmatic: he essentially treats *e* as an element of his symbolic framework implicitly endowed with an infinitesimal character. He then discards this "amphibious" entity, poised between the actual and the potential, at the first convenient opportunity. This is what makes his way of reasoning so remarkable.
2) The symbol *e* may be interpreted as an infinitesimal increment of the independent variable. In this sense, it may be related to our concept of a differential. From a historical point of view, this interpretation can be accused of presentism, but from an educational standpoint, it is useful and can be exploited.
3) Fermat's method is, to use a modern expression, theoretically applicable to any differentiable function. In practice, however, it can be applied only to a class of functions, not to all of them. The fundamental difficulty of the method is that the division of *f*(*x*+*e*)-*f*(*x*) by *e* must be algebraically possible. For example, a simple function $f(x) = 2^x$. In this case, it is $\frac{f(x+e)-f(x)}{e} = \frac{2^{x+e}-2^x}{e} = \frac{2^x.2^e-2^x}{e} = \frac{2^x(2^e-1)}{e}$. Fermat's approach requires to simplify algebraically the fraction $\frac{2^e-1}{e}$. However, this operation is impossible through elementary algebra. Expansion in Taylor series would be necessary, but this concept was unavailable to Fermat.
4) The concept itself of function cannot be attributed to Fermat because a precise relation between an independent and a dependent variable is not yet present, although an intuitive idea between the two variables is. This is sufficient for the functioning of Fermat's method.

What can be done in the classroom, then, is the following: one introduces the idea that certain types of problems, such as drawing the tangent to a curve at a given point, cannot, in general, be solved by means of the methods learned up to that stage (namely, geometric and synthetic methods, together with algebraic manipulations and trigonometry). It becomes necessary to introduce new entities, somehow connected with the infinitesimal. Fermat was among the first mathematicians to provide a systematic procedure for addressing such problems in a manner closely related to later developments of infinitesimal calculus.

One possible objection to using the parabola as an example is that, in the case of the parabola, both synthetic and algebraic methods exist for constructing the tangent at a point. It should therefore be shown that a method such as Fermat's also works for transcendental curves (this item will not be addressed here, but Fermat dealt with curves such as cycloid), for which synthetic or algebraic methods may not exist.

At this point, it is important to emphasize that Fermat's method has definite limitations:

1. It may require algebraic manipulations that cannot actually be carried out.
2. It is based on the symbol *e*, whose nature is not clearly defined. Thus, in a teaching sequence, a necessary step after presenting Fermat's method is to provide a more precise characterization of the nature of *e*.
3. Before doing so, however, it is first appropriate to show that Fermat's method also applies to problems of maxima and minima which, from a conceptual point of view, are almost identical to the problem of constructing tangents.

In retrospect, what is lacking is the idea of representing a curve as a function of a single variable. Also absent, strictly speaking, is the concept of a neighbourhood of a point, which is likewise connected with the possibility of expressing a curve as a function of a single variable. Newton and Leibniz would make significant progress in this direction.

Nevertheless, what Fermat achieved is remarkable, and in many respects he deserves, together with Newton and Leibniz, to be regarded as one of the true founders of infinitesimal calculus.

Accordingly, in a teaching sequence, the next step should be to clarify both the relationships among variables and the nature of the infinitesimal element, so as to obtain, if possible, a general method free from exceptions with regard to the problems of constructing tangents and determining maxima and minima of curves.

The educational value of Fermat's method lies not only in its historical significance but also in its capacity to support the development of fundamental calculus concepts. Through the analysis of tangent and optimisation

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problems, students may progressively construct an intuitive understanding of variation, infinitesimal change, and limiting processes before encountering formal definitions.

In connection with such item, the teacher may encourage discussion around the status of the quantity e. Students frequently perceive it as neither an ordinary number nor exactly zero. Such discussions naturally reproduce some of the conceptual difficulties historically encountered in the development of calculus.

Students often interpret differentiation as a purely computational procedure. Fermat's method helps reverse this perspective by placing the original mathematical problem before the formal algorithm used to solve it.

Once students recognise both the power and the limitations of Fermat's procedure, they become more receptive to the introduction of derivative and limit concepts. Formalisation no longer appears as an arbitrary requirement imposed by mathematicians, but as a response to difficulties already encountered in an intuitive framework.

From this perspective, the history of mathematics is not merely a source of cultural enrichment. It becomes a cognitive resource that allows students to experience, in a simplified form, the same conceptual tensions that historically led to the creation of infinitesimal calculus.

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