Physica D 496 (2026) 135323

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Discrete-time maximally superintegrable systems and deformed symmetry algebras: The Calogero–Moser case

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ARTICLE INFO

Communicated by Dr L Ling

Keywords:
Integrable systems
Superintegrable systems
Discrete-time systems
Calogero–Moser model
Symmetry algebras
Polynomial algebras
Deformed algebras
Discretization

ABSTRACT

We determine the complete structure of the symmetry algebras associated with the N-body Calogero–Moser system and its maximally superintegrable discretization. We prove that the discretization naturally leads to a nontrivial deformation of the continuous symmetry algebra, with the discretization parameter playing the rôle of a deformation parameter. This phenomenon illustrates how discrete superintegrable systems can be viewed as natural sources of deformed polynomial algebraic structures. As a byproduct of these results, we also reveal a connection between the above-mentioned symmetry algebras and the Bell polynomials, as a consequence of the trace properties.

1. Introduction

Maximally superintegrable systems form a distinguished subclass of finite-dimensional integrable Hamiltonian systems, characterized by the presence of an exceptionally large number of functionally independent integrals of motion. A classical Hamiltonian system $H(\mathbf{q},\mathbf{p})$ with N degrees of freedom is (Liouville) integrable if it admits N functionally independent, analytic, and single-valued constants of motion $\{ F_1,\dots,F_N \}$, one of them being the Hamiltonian itself, which are mutually in involution, that is, $\{F_i,F_j\}=0$. Under these conditions, the bounded invariant manifolds defined by $F_i=c_i=\text{const}$ with $i=1,\dots,N$ are N-dimensional invariant tori, and the corresponding motions on them are conditionally periodic [1]. If, in addition to the N involutive integrals, there exist k further functionally independent constants of motion $\{ G_1,\dots,G_k \}$ such that $1\leq k\leq N-1$, the system is called *superintegrable* [2]. In this case, the invariant tori have dimension $N-k$. The case $k=1$ defines minimally superintegrable systems (mS), whereas the case $k=N-1$ corresponds to maximally superintegrable systems (MS). In the MS situation, a remarkable property holds: every bounded trajectory is closed, and the motion is strictly periodic [3]. If the integrals of motion are polynomial in the canonical momenta, one speaks of *polynomial superintegrability*. Recalling that the *order*, Ord(–), of a dynamical function is its degree in the canonical momenta, we have that a polynomial superintegrable system is said to be nth-order superintegrable if n is the maximal order among a minimal generating set of constants of motion. Classical examples of MS systems include the isotropic harmonic oscillator and the Kepler–Coulomb model (the prototype examples of second-order MS systems), as well as the Calogero–Moser system [4] and the Toda lattice [5], among many others.

A distinctive feature of MS systems is that their symmetry structure is no longer encoded in an abelian algebra of first integrals. While Liouville integrability is characterized by an abelian algebra generated by the commuting integrals, MS systems possess a non-abelian symmetry algebra generated by all functionally independent constants of motion. To obtain a closed finitely generated algebraic structure, it is sometimes necessary to extend the generating set by including additional linearly independent integrals. In the classical setting, the resulting symmetry algebra is typically a *polynomial Poisson algebra*, closed under the Poisson bracket [2,6]. Only in exceptional cases, the algebra of integrals closes linearly, as in the isotropic harmonic oscillator, where the symmetry algebra is $\mathfrak{su}_N$, and in the Kepler–Coulomb system, where the symmetry algebra is $\mathfrak{so}_{N+1}$ after a suitable rescaling of the generators for negative energies (bounded motion). Such polynomial algebras typically admit nontrivial Casimir invariants and provide a complete algebraic characterization of the dynamics, well beyond the abelian structure associated with Liouville integrability [7]. The quadratic algebras arising

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https://doi.org/10.1016/j.physd.2026.135323
Received 17 February 2026; Received in revised form 23 May 2026; Accepted 28 June 2026
Available online 29 June 2026

in second-order superintegrable systems, and their representation theory, have been developed in depth since the early 1990s (see Miller et al. [2], Kalnins et al. [8] and references therein).

These non-abelian algebraic structures remain central in contemporary research on superintegrability. Their study has revealed deep links between superintegrability and the theory of special functions and orthogonal polynomials (see Miller et al. [2] and references therein). Moreover, they are nowadays employed to derive the spectra of quantum systems, often avoiding direct solution of the associated Schrödinger equation (see, for example, Campoamor-Stursberg et al. [9]). The quantum associative algebras can be understood as deformations, with the deformation parameter $\hbar$, of the underlying classical structures that are recovered in the appropriate $\hbar \to 0$ limit.

An interesting aspect of these symmetry structures is that they can also arise as *nonlinear deformations* of an initial Lie algebra $\mathfrak{g}$. A classical example is provided by the Higgs oscillator [10,11], where in the $N = 2$ case the resulting symmetry algebra is a deformation of $\mathfrak{su}_2$, with the deformation parameter $\kappa > 0$ proportional to the scalar curvature of the underlying space.

In this paper, taking as a prototype example the Calogero–Moser system [12–14], we aim to show the existence of a novel type of deformation of polynomial symmetry algebras associated with MS systems, obtained by considering the discrete-time analogue of the model under investigation. In this setting, the rôle of the deformation parameter is played by the lattice time spacing $h > 0$. To be more specific, the discrete-time analog of a Hamiltonian system is the theory of maps/correspondences[1] of the form:

$$\overline{\mathbf{x}} = \mathbf{F}(\mathbf{x},\mathbf{p}), \quad \overline{\mathbf{p}} = \mathbf{G}(\mathbf{x},\mathbf{p}), \quad \overline{\mathbf{x}} := \mathbf{x}(t+h), \ \overline{\mathbf{p}} := \mathbf{p}(t+h), \tag{1.1}$$

for an unknown sequence $\{ (\mathbf{x}(t),\mathbf{p}(t)) \}_{t\in h\mathbb{Z}} \subset \mathbb{R}^{2N}$, and preserving the canonical Poisson bracket, *i.e.*:

$$\{\overline{x}_i,\overline{x}_j\} = \{\overline{p}_i,\overline{p}_j\} = 0, \quad \{\overline{x}_i,\overline{p}_j\} = \delta_{i,j}. \tag{1.2}$$

A map/correspondence with this property is usually called *symplectic*, and alternatively such a condition can be expressed by saying that the transformation $(\mathbf{x},\mathbf{p}) \longrightarrow (\overline{\mathbf{x}},\overline{\mathbf{p}})$ is canonical. This theory was initiated in the work of Maeda [15].

The theory of integrability for symplectic maps, or more generally, correspondences, was developed in Maeda [15], Veselov [16], Bruschi et al. [17], see also Hietarinta et al. [18, Chap. 6] and the thesis Tran [39, Chap. 1] for a comprehensive exposition of the topic. In analogy with the continuous setting, an integral of motion for a symplectic map/correspondence is a scalar function $I = I(\mathbf{x},\mathbf{p})$ such that it is constant along the solutions, *i.e.* $\overline{I}(\mathbf{x},\mathbf{p}) \equiv I(\overline{\mathbf{x}},\overline{\mathbf{p}}) = I(\mathbf{x},\mathbf{p})$. Then, a symplectic map is *Liouville integrable* if it possesses N functionally independent invariants in involution with respect to the associated Poisson bracket. If the map admits more functionally independent first integrals, the map is *superintegrable*, like its continuous counterpart. So, as in the continuous setting, a superintegrable symplectic map will admit the commuting integrals of motion $\{ F_1,\dots,F_N \}$ and the non-commuting ones $\{ G_1,\dots,G_k \}$, $1 \le k \le N-1$. This implies that also to discrete (maximally) superintegrable systems, one can associate a symmetry algebra, as it was discussed, for instance, in our previous work [19].

However, there is a crucial difference between the continuous and the discrete setting. In the continuous setting, there exists a Hamiltonian function governing the dynamics, which is an integral of motion, while in the discrete setting, the dynamical system is obtained by the repeated application of a canonical transformation. Canonical transformations do not have an associated Hamiltonian function, but rather a generating function. In general, the generating function is not an integral of motion of the associated discrete dynamics. This implies that, although one can write a formal identification between the evolution that can be derived from a generating function and Hamilton's equations of motion, in general, an exact equivalent of the Hamiltonian function does not exist in the discrete case. This may explain why discrete integrable systems appear to be much rarer, and their structure is more rigid with respect to their continuous counterparts.

In many instances, in the limit $h \to 0^+$ a symplectic map (1.1) becomes a system of Hamiltonian equations. In such a case, we say that the symplectic map (1.1) is a *discretization* of the underlying system of Hamiltonian equations. In particular, if the Hamiltonian system is (super)integrable, and so its discretization, we will say that the latter is a *(super)integrable discretization*, see for instance [20]. We underline that preserving the superintegrable nature of a continuous Hamiltonian system is a very delicate task. Indeed, a symplectic map can be simply non-integrable, or integrable (but not superintegrable) discretization of a continuous superintegrable system, as highlighted in the case of the rational anisotropic caged oscillator [21] in Gubbiotti and Latini [22]. This is because, in general, the problem of discretization is ill-posed, in the sense that it does not admit a unique solution, and that different discretizations can have different properties, see also Levi et al. [23].

In this paper, we consider the celebrated Calogero–Moser system, proven to be MS in Wojciechowski [4], to investigate the relationship between the symmetry algebra of a MS system and that of one of its possible MS discretizations [24,25]. In particular, we seek to clarify whether the discretization can give rise to *structural deformations* of a polynomial algebra. That is, if the symmetry algebra of a MS discretization can be understood as a deformation of the original polynomial symmetry algebra of the MS system, the latter recovered in the continuum limit $h \to 0^+$. The question is motivated, for instance, by our previous work [19], where we showed that a class of discretizations of the harmonic oscillator, depending on a parameter, preserves superintegrability, but only gives rise to parametric deformations in the symmetry algebra, in the sense that the deformation can be reabsorbed into the parameters of the model. This result provided us with some interesting insights into the discretization itself, adding, for instance, the idea of preserving not just the integrals of motion, but also the isomorphism class of the symmetry algebra. However, this left open the question of whether or not a maximally superintegrable discretization could induce a structural deformation of the symmetry algebra. The results of the present paper show that the answer is affirmative: discretization can indeed give rise to structural deformations of the symmetry algebra of an MS system.

The paper is structured as follows. First, in Section 2, we discuss the continuous Calogero–Moser system and its superintegrability, and we present its symmetry algebra $\mathcal{A}^{(N)}$ for an arbitrary number of bodies N in Proposition 2.5, with the explicit low-number-of-body examples $N = 2, 3$ following. In Section 3, we proceed in a similar fashion with the Nijhoff–Pang discretization of the Calogero–Moser model [24]. We present the corresponding symmetry algebra $\widetilde{\mathcal{A}}^{(N)}$ of the discrete model in Proposition 3.4, thereby showing **the main result of this paper: that the symmetry algebra in the discrete case can be considered as a polynomial algebra deformation of the continuous one**, the latter recovered when the lattice time spacing h goes to zero, which is the content of Corollary 3.4.1. Finally, in Section 4, we draw conclusions, discuss the implications, and provide further research directions.

2. The continuous N-body Calogero–Moser system and its symmetry algebra

In this section, we review the main results on the superintegrability of the continuous Calogero–Moser model obtained in the seminal paper by S. Wojciechowski [4]. To keep the discussion self-contained, we resume all the relevant properties we will need. Then, we present the explicit form of the symmetry algebra in the N-body case. The calculation of the algebra is performed in two different steps. In the first one,

[1] A recurrence is called a map if it is single valued and a correspondence if it is multi-valued.

we compute the Poisson commutation relations at a formal level, since some relations fall outside the set of integrals of motion. In the second step, using some elements of linear algebra, we show how to obtain the missing relations and give the final form of the symmetry algebra. We also present as examples the symmetry algebras for the two- and the three-body cases.

2.1. Review of the continuous model

The Calogero–Moser system is one of the most famous integrable systems, and one of the first to be introduced in modern times. The system describes the dynamics of N unit-mass particles moving on a line and pairwise interacting via a potential proportional to the inverse of the squared distance. Its history actually goes back to Jacobi, who studied the three-body case in the XIX century [26]. More than a century later, Calogero solved first the three-body quantum case [27] and then the N-body case [12,13]. The integrability of the classical counterpart was conjectured by Calogero in Calogero [12] and later proved by Moser in Moser [14] using isospectral deformations. As mentioned above, the Calogero–Moser system was shown to be maximally superintegrable in Wojciechowski [4]. We note that, in modern times, although the concept of superintegrability was known at least since the 1960s, see *e.g.* Friš et al. [28], Makarov et al. [29], the name was coined in Wojciechowski [4].

The Hamiltonian function of the Calogero–Moser model is the following:

$$H^{(N)} = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \nu^2 \sum_{1\le i<j}^{N} \frac{1}{(q_i-q_j)^2}, \tag{2.1}$$

where ν is the coupling constant parameter, which we assume to be real.[2] For the sake of brevity, hereinafter we will refer to the model (2.1) as the CM model. The Hamilton's equations of motion read as:

$$\dot{q}_i = \{q_i, H^{(N)}\} = p_i, \quad \dot{p}_i = \{p_i, H^{(N)}\} = 2\nu^2 \sum_{\substack{k=1\\k\neq i}}^{N} \frac{1}{(q_i-q_k)^3}, \quad i=1,\dots,N, \tag{2.2}$$

where $\{-,-\}$ is the canonical Poisson bracket, $\{q_i, p_j\} = \delta_{i,j}$.

Following the work of Moser [14], we have that the Hamilton Eq. (2.2) arise a compatibility conditions of the following two matrices $L, M \in \mathfrak{gl}_N(\mathbb{R})$, defined by their matrix elements as follows:

$$L_{jk} = \delta_{jk} p_j + (1-\delta_{jk}) \frac{\mathrm{i}\nu}{q_j - q_k}, \tag{2.3a}$$

$$M_{jk} = -\delta_{jk} \sum_{\substack{\ell=1\\ \ell\neq j}}^{N} \frac{\mathrm{i}\nu}{(q_j-q_\ell)^2} + (1-\delta_{jk}) \frac{\mathrm{i}\nu}{(q_j-q_k)^2}, \tag{2.3b}$$

under the following isospectral evolution equation:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = [M, L], \tag{2.4}$$

where $[-,-]$ denotes the standard matrix commutator. The matrices L and M form what is known in the literature as a *Lax pair*.

The isospectrality condition (2.4) provides N integrals of motion from the Lax matrix L:

$$F_k := \operatorname{tr}(L^k), \qquad k=1,\dots,N, \tag{2.5}$$

such that $\operatorname{Ord}(F_k) = k$. Here, we use the convention of Babelon et al. [30]. There exist other conventions, for example, where the integrals of motion are given by $k^{-1}\operatorname{tr}(L^k)$, as in Wojciechowski [4], Bartocci et al. [31]. We chose the involutive integrals as in (2.5) because, in order to present the general formulas for the symmetry algebra, it is more convenient to have also the quantity $F_0 = N = \text{const}$ well-defined.

Let us observe that the integrals of motion F_k are all non-local, *i.e.* they depend on all dynamical variables. Nevertheless, they are known to be functionally independent and commute between themselves. Moreover, the Hamiltonian belongs to the above set of constants, being $F_2 = 2H^{(N)}$, so that they are a *bona fide* set of integrals of motion to apply Liouville–Arnold theorem.

Superintegrability of the model has been proven in Wojciechowski [4], where a set of additional integrals of motion was found. First, it is known that the system (2.2) admits a second associated generalized isospectral problem for the matrix X:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = [M, X] + L. \tag{2.6}$$

With the above choice of the matrix L (2.3a), there exists a simple solution given by $X = \operatorname{diag}(q_1,\dots,q_N)$. Note that with this choice of the matrix X, the symplectic structure is given by:

$$\omega = \operatorname{tr}(\mathrm{d}X \wedge \mathrm{d}L) = \sum_{i=1}^{N} \mathrm{d}q_i \wedge \mathrm{d}p_i, \tag{2.7}$$

corresponding to the canonical Poisson bracket. Then, it is possible to show that the quantities:

$$J_k := \operatorname{tr}(XL^{k-1}), \qquad k = 1,\cdots,N, \tag{2.8}$$

have a linear time evolution such that $\frac{\mathrm{d}J_k}{\mathrm{d}t} = F_k$, and also holds $\operatorname{Ord}(J_k) = k-1$. This allows us to construct a skew-symmetric family of integrals of motion [4]:

$$K_{m,n} := J_m F_n - J_n F_m, \qquad 1 \le n < m \le N, \tag{2.9}$$

such that $\operatorname{Ord}(K_{m,n}) = m+n-1$. Indeed, using the expressions of the F_k (2.5) and J_k (2.8), it is easy to see that the leading order term in the momenta never vanishes.

So, to summarize, the CM model (2.1) admits a total of $N(N+1)/2$ *linearly independent* integrals of motion:

$$S^{(N)} = \{ F_k \}_{k=1}^{N} \cup \{ K_{m,n} \}_{1\le n<m\le N}, \tag{2.10}$$

of which a functionally independent set is obtained by choosing the subset:

$$S^{(N)}_{\text{f.i.}} = \{ F_k \}_{k=1}^{N} \cup \{ G_m \}_{m=1}^{N-1}, \qquad G_m := K_{m+1,1}. \tag{2.11}$$

The following functional relations hold true:

$$F_j K_{n,i} - F_i K_{n,j} + F_n K_{i,j} = 0, \qquad i,j,n \ge 0. \tag{2.12}$$

To obtain the functional relations between the elements of the set $S^{(N)}$, one restricts the indices (i,j,n) to the combinations with repetitions of elements in the set $\{1,\dots,N\}$, up to the skew-symmetry condition. The functional relation (2.12) follows trivially from rearranging the elements in the following identity:

$$\det \begin{pmatrix} F_j & F_i & F_n \\ F_j & F_i & F_n \\ J_j & J_i & J_n \end{pmatrix} = 0. \tag{2.13}$$

In total, the functional relations (2.12) are $C_3^N = N(N-1)(N-2)/6$, but we don't need all of them to specify the functionally dependent elements. With the same notation as in Eq. (2.11), we can fix the index $i=1$ and obtain:

$$F_j G_{n-1} - F_1 K_{n,j} - F_n G_{j-1} = 0, \tag{2.14}$$

which uniquely determines the integrals of motion $K_{n,j}$ with $2 \le j < n \le N$ in terms of the elements of the set $S^{(N)}_{\text{f.i}}$.

[2] Note that in the literature, see *e.g.* Suris [20], there exists another convention where the potential appears with the opposite sign in front of ν^2. One can switch between them using the replacement $\nu \mapsto \mathrm{i}\nu$, corresponding to repulsive and attractive potentials, respectively.

2.2. The symmetry algebra

The aim of this section is to prove that the Poisson commutation relations of the elements of the set $\mathcal{S}^{(N)}$ (2.10) are closed. Let us start by showing the simplest case, *i.e.* the two-body case. This is the content of the following example:

Example 2.1 ($N = 2$)**.** In the two-body case, the Hamiltonian function is given by:

$$H^{(2)} = \frac{p_1^2 + p_2^2}{2} + \frac{\nu^2}{(q_1 - q_2)^2}\,. \tag{2.15}$$

The two-body case is special among all others. As can be seen from the discussion in the above section, it is the only one where we don't have to consider linearly independent integrals of motion, but only functionally independent ones. Indeed, following formulas (2.10) and (2.11), for $N = 2$ we have:

$$\mathcal{S}^{(2)} = \mathcal{S}^{(2)}_{\text{f.i}} = \{\, F_1, F_2, K_{2,1} \,\}, \tag{2.16}$$

where we can write the integrals of motion explicitly as:

$$F_1 = p_1 + p_2, \qquad F_2 = 2H^{(2)}, \qquad K_{2,1} = (\mathbf{q}\cdot\mathbf{p})F_1 - (q_1 + q_2)F_2\,. \tag{2.17}$$

In this case, we can compute the Poisson commutation relations directly, and obtain the following quadratic algebra structure:

$$\{F_1, F_2\} = 0, \qquad \{F_2, K_{2,1}\} = 0, \qquad \{F_1, K_{2,1}\} = 2F_2 - F_1^2. \tag{2.18}$$

We check that, as required by the general theory, the Hamiltonian is a central element.

Remark 2.2. We remark that the distinguishing features of the $N = 2$ model are related to the fact that it is the only separable case. Indeed, note that through the canonical lift of the point transformation:

$$Q_1 = \frac{q_1 - q_2}{\sqrt{2}}, \quad Q_2 = \frac{q_1 + q_2}{\sqrt{2}}, \tag{2.19}$$

the Hamiltonian (2.15) is mapped to:

$$H^{(2)\prime} = \frac{P_1^2 + P_2^2}{2} + \frac{\nu^2}{2Q_1^2}, \tag{2.20}$$

which admits Q_2 as a cyclic coordinate. Moreover, this Hamiltonian is separable with one linear and two quadratic first integrals:

$$F_1' = P_2, \qquad F_2' = P_1^2 + \frac{\nu^2}{Q_1^2}, \qquad G_1' = Q_2 P_1^2 - Q_1 P_1 P_2 + \nu^2 \frac{Q_2}{Q_1^2}. \tag{2.21}$$

These three close the quadratic algebra:

$$\{F_1', F_2'\} = 0, \qquad \{F_1', G_1'\} = -F_2', \qquad \{F_2', G_1'\} = F_1' F_2'. \tag{2.22}$$

To tackle the general N-body case, we will use the following auxiliary Poisson commutation relations of the quantities F_k and J_k which, in our conventions, read as:

$$\{F_m, F_n\} = 0, \tag{2.23a}$$
$$\{J_m, F_n\} = nF_{m+n-2}, \tag{2.23b}$$
$$\{J_m, J_n\} = (n - m)J_{m+n-2}, \tag{2.23c}$$

for $m, n > 0$. These commutation relations were obtained in Bartocci et al. [31] through the application of the so-called necklace bracket formula for the Poisson bracket of traces. Notice that, in our conventions, the relation $\{J_1, F_1\} = F_0 = N$ is now included in (2.23b). Moreover, since $J_0 = \operatorname{tr}(XL^{-1})$ is well-defined and finite, one recovers the trivial relation $\{J_1, J_1\} = 0$ from the right-hand side of (2.23c). We will use the set of relations (2.23) to derive the Poisson commutation relations of the F_k and the $K_{m,n}$.

First of all, we will say that a Poisson commutation relation is *formal* if on the right hand side one can obtain either F_k or a $K_{m,n}$ whose index exceeds N, *i.e. a priori* they contain elements that fall outside the set $\mathcal{S}^{(N)}$ (2.10). Given this definition, we have the following statement:

Proposition 2.3. *The following formal Poisson commutation relations hold:*

$$\{F_i, K_{m,n}\} = i(F_m F_{n+i-2} - F_n F_{m+i-2}), \tag{2.24a}$$
$$\begin{aligned}\{K_{i,j}, K_{m,n}\} &= F_i(mK_{n,j+m-2} + nK_{j+n-2,m}) + F_j(mK_{i+m-2,n} + nK_{m,i+n-2})\\ &\quad + F_m(iK_{i+n-2,j} + jK_{i,j+n-2}) + F_n(iK_{j,i+m-2} + jK_{j+m-2,i}).\end{aligned} \tag{2.24b}$$

Remark 2.4. We remark that putting either $i = j$ or $m = n$ in the above equations, one should obtain an identity of the form $0 = 0$. This is indeed true, taking into account the skew-symmetry of the integrals of motion $K_{m,n}$.

From the technical point of view, the proof of both sets of Poisson commutation relations follows from a judicious application of the Leibniz rule for Poisson brackets.

Proof of Proposition 2.3. Let us prove first Eq. (2.24a). Using the definition of $K_{m,n}$ (2.9), from the Leibniz rule and Eq. (2.23a) we write:

$$\{F_i, K_{m,n}\} = \{F_i, J_m F_n\} - \{F_i, J_n F_m\} = \{F_i, J_m\}F_n - \{F_i, J_n\}F_m. \tag{2.25}$$

Using then the Eq. (2.23b), one arrives at the expression in (2.24a). We remark that the Poisson commutation relations $\{F_i, K_{m,1}\}$ was already obtained with a different method in Wojciechowski [4].

Let us proceed to Eq. (2.24b). Define $R_{m,n} := J_m F_n$, such that:

$$K_{m,n} = R_{m,n} - R_{n,m} =: R_{[m,n]}, \tag{2.26}$$

where the square brackets stand for skew-symmetrization. Note that by construction:

$$R_{i,j} F_k = F_j R_{i,k}, \qquad i, j, k > 0. \tag{2.27}$$

Following Remark 2.4, we can assume, without loss of generality, that $i \neq j$ and $m \neq n$. Introduce a tensor $\mathcal{R}_{i,j,k,l} := \{R_{i,j}, R_{k,l}\}$. We then have by (2.26):

$$\begin{aligned}\{K_{i,j}, K_{m,n}\} &= \mathcal{R}_{i,j,m,n} - \mathcal{R}_{i,j,n,m} - \mathcal{R}_{j,i,m,n} + \mathcal{R}_{j,i,n,m} = \mathcal{R}_{i,j\,[m,n]} - \mathcal{R}_{j,i\,[m,n]}\\ &=: \mathcal{R}_{[i,j]\,[m,n]}.\end{aligned} \tag{2.28}$$

Let us now proceed to the calculation of $\mathcal{R}_{i,j,k,l}$. Using the definition of $R_{i,j}$ and the Leibniz rule, we obtain:

$$\begin{aligned}\mathcal{R}_{i,j,k,l} &= \{J_i F_j, J_k F_l\} = -F_l J_i\{J_k, F_j\} + F_j F_l\{J_i, J_k\} + F_j J_k\{J_i, F_d l\}\\ &= (k - i)F_j F_l J_{i+k-2} + lF_j F_{i+l-2} J_k - jF_l F_{j+k-2} J_i\\ &= -jF_l R_{i,j+k-2} + F_j\big((j - i)R_{i+k-2,l} + lR_{k,i+l-2}\big).\end{aligned} \tag{2.29}$$

Performing an antisymmetrization with respect to i, j and k, l, and doing some algebra, one arrives at the expression (2.24b). □

Now, a crucial point is that the indices in the Poisson commutation relations (2.24) can exceed N. Nevertheless, the symmetry algebra is closed since all the extra terms can be expressed via the elements of the set $\mathcal{S}^{(N)}$ (2.10). Note that the elements $K_{j,i}$ with $j < i$ can always be rewritten using the antisymmetry condition as $K_{j,i} = -K_{i,j}$. Then, at a fixed N, and analyzing the right-hand side of the Poisson commutation relations (2.24), one finds that we need to determine the following additional functionally dependent first integrals:

$$F_{N+1}, \quad F_{N+2}, \quad \ldots, \quad F_{2N-2}, \tag{2.30a}$$
$$K_{1,0}, \quad K_{2,0}, \quad \ldots, \quad K_{N-1,0}, \quad K_{N,0}, \tag{2.30b}$$
$$\begin{array}{lllll} K_{N+1,1}, & K_{N+1,2}, & \ldots, & K_{N+1,N-1}, & K_{N+1,N},\\ K_{N+2,1}, & K_{N+2,2}, & \ldots, & K_{N+2,N-1}, & K_{N+2,N},\\ \vdots & & & & \vdots\\ K_{2N-3,1}, & K_{2N-3,2}, & \ldots, & K_{2N-3,N-1}, & K_{2N-3,N},\\ K_{2N-2,1}, & K_{2N-2,2}, & \ldots, & K_{2N-2,N-1}. & \end{array} \tag{2.30c}$$

Abstractly, the functions F_k, J_k, and $K_{m,n}$ can be defined for arbitrary values of the indices $k, m, n \geq 0$, so the above expressions make perfect sense. First, let us discuss the set $K_{1,0}, \ldots, K_{N,0}$. Analyzing the right-hand side of (2.24b), given $i, j, m, n > 0$ for the functions belonging to $\mathcal{S}^{(N)}$, we note that the terms (2.30b) can appear if at least one of the following equalities holds:

$$j+m-2=0 \implies j=1,\ m=1, \tag{2.31a}$$
$$j+n-2=0 \implies j=1,\ n=1, \tag{2.31b}$$
$$i+m-2=0 \implies i=1,\ m=1, \tag{2.31c}$$
$$i+n-2=0 \implies i=1,\ n=1. \tag{2.31d}$$

Let us observe that, fixing an index $n=0$ in functional relation (2.12), we can write:

$$F_j K_{0,i} - F_i K_{0,j} = N K_{ji}. \tag{2.32}$$

In this way, one is able to express all the products containing $K_{i,0}$.

To proceed with other rows in (2.30), let us note that both F_k and J_k are related to the trace of powers of the Lax matrix. So, following an idea suggested already in Bartocci et al. [31], if $k>N$, we use the Cayley–Hamilton theorem, see *e.g.* Loehr [32], to express them in terms of elements of $\mathcal{S}^{(N)}$ (2.10).

Following the results reported in Appendix A (see formula (A.12)), we can express F_{N+s} through the Bell polynomials evaluated on the power sums, and hence in terms of the traces as:

$$F_{N+s} = -\sum_{k=1}^{N} \frac{1}{k!} B_k\big(-0!F_1, -1!F_2, -2!F_3, \dots, -(k-1)!F_k\big) F_{N+s-k}. \tag{2.33}$$

Note that this is not enough yet, since this last formula expresses F_{N+s} in terms of $F_1,\dots,F_{N+s-1}$. However, it is easy to see that the F_{N+s} can be determined recursively starting from F_{N+1}. Since, as highlighted in formula (2.30a), we need $\{F_{N+s}\}_{s=1}^{N-2}$, it means that to express everything in terms of $F_1,\dots,F_N$, one solves the following triangular system:

$$\begin{cases} F_{N+1} = -\sum_{k=1}^{N} \frac{1}{k!} B_k\big(-0!F_1, -1!F_2, -2!F_3, \dots, -(k-1)!F_k\big) F_{N+1-k}, \\ F_{N+2} = -\sum_{k=1}^{N} \frac{1}{k!} B_k\big(-0!F_1, -1!F_2, -2!F_3, \dots, -(k-1)!F_k\big) F_{N+2-k}, \\ \quad\vdots \\ F_{2N-2} = -\sum_{k=1}^{N} \frac{1}{k!} B_k\big(-0!F_1, -1!F_2, -2!F_3, \dots, -(k-1)!F_k\big) F_{2N-2-k}. \end{cases} \tag{2.34}$$

This expression solves the problem completely. Moreover, we observe that, by induction and since $\deg_{\mathbf{y}} B_k(\mathbf{y}) = k$, it follows that:

$$\deg_{\mathcal{S}^{(N)}} F_{N+s} = N+s, \quad s=1,\dots,N-2. \tag{2.35}$$

We also notice that one can compare formula (2.34) with an analogous one presented for the Toda lattice in Agrotis et al. [5, Section 3].

In the same way, we can multiply on the left the relation (A.10) reported in Appendix A, for $A=L$, by the matrix XL^{s-1} for $s\in\mathbb{N}$. Given that $\operatorname{tr}(XL^{k-1}) = J_k$, taking the traces, we obtain:

$$J_{N+s} = \sum_{i=1}^{N} (-1)^{i+1} e_i(\lambda) J_{N+s-i}. \tag{2.36}$$

Hence, fixing $s\in\mathbb{N}$, we can write:

$$K_{N+s,\ell} = -\sum_{k=1}^{N} \frac{1}{k!} B_k\big(-0!F_1, -1!F_2, -2!F_3, \dots, -(k-1)!F_k\big) K_{N+s-k,\ell}. \tag{2.37}$$

As before, it is not enough yet, since this last formula expresses $K_{N+s,\ell}$ in terms of the elements of $\mathcal{S}^{(N)}$ (2.10). Yet again, it is possible to see that for a fixed ℓ, the function $K_{N+s,\ell}$ can be determined recursively starting from $K_{N+1,\ell}$. Since, as highlighted in formula (2.30c), we need the functions $\{K_{N+s,\ell}\}_{s=1}^{N-2}$, for $\ell\in\{1,\dots,N-1\}$, and the functions $\{K_{N+s,N}\}_{s=1}^{N-3}$. That is, we have to solve the following families of triangular systems:

$$\begin{cases} K_{N+1,\ell} = -\sum_{k=1}^{N} \frac{1}{k!} B_k\big(-0!F_1, -1!F_2, -2!F_3, \dots, -(k-1)!F_k\big) K_{N+1-k,\ell} \\ K_{N+2,\ell} = -\sum_{k=1}^{N} \frac{1}{k!} B_k\big(-0!F_1, -1!F_2, -2!F_3, \dots, -(k-1)!F_k\big) K_{N+2-k,\ell}, \\ \vdots \\ K_{2N-2,\ell} = -\sum_{k=1}^{N} \frac{1}{k!} B_k\big(-0!F_1, -1!F_2, -2!F_3, \dots, -(k-1)!F_k\big) K_{2N-2-k,\ell}, \end{cases} \tag{2.38a}$$

for $\ell\in\{1,\dots,N-1\}$, and

$$\begin{cases} K_{N+1,N} = -\sum_{k=1}^{N} \frac{1}{k!} B_k\big(-0!F_1, -1!F_2, -2!F_3, \dots, -(k-1)!F_k\big) K_{N+1-k,N} \\ K_{N+2,N} = -\sum_{k=1}^{N} \frac{1}{k!} B_k\big(-0!F_1, -1!F_2, -2!F_3, \dots, -(k-1)!F_k\big) K_{N+2-k,N}, \\ \vdots \\ K_{2N-3,N} = -\sum_{k=1}^{N} \frac{1}{k!} B_k\big(-0!F_1, -1!F_2, -2!F_3, \dots, -(k-1)!F_k\big) K_{2N-2-k,N}. \end{cases} \tag{2.38b}$$

These last two expressions solve the problem completely. Moreover, we observe that by induction, taking into account that $\deg_{\mathbf{y}} B_k(\mathbf{y}) = k$, it follows that:

$$\begin{aligned} &\deg_{\mathcal{S}^{(N)}} K_{N+s,1} = N+s-1, && s=1,\dots,N-2, \\ &\deg_{\mathcal{S}^{(N)}} K_{N+s,\ell} = N+s, && s=1,\dots,N-2, \quad \ell=2,\dots,N-1, \\ &\deg_{\mathcal{S}^{(N)}} K_{N+s,N} = N+s, && s=1,\dots,N-3, \end{aligned} \tag{2.39}$$

where we used that for $\ell=1$, since $K_{1,1}=0$, the highest degree Bell polynomial is annihilated.

Therefore, the set of these rules allows us to write explicitly the form of the symmetry algebra of the CM model (2.1). We enclose this result in the following proposition:

Proposition 2.5. *The symmetry algebra $\mathcal{A}^{(N)}$ of the N-body CM model is generated by the elements of the set $\mathcal{S}^{(N)}$ (2.10) with Poisson commutation relations given in Eq. (2.24), provided the additional closure relations given in Eqs. (2.34) and Eq. (2.38). Moreover, $\mathcal{I}^{(N)} := \langle F_i\rangle_{i=1}^{N}$ is an ideal of $\mathcal{A}^{(N)}$. Finally, for $N>2$ the degree of the polynomial symmetry algebra $\mathcal{A}^{(N)}$ is $2N-1$, while $\mathcal{A}^{(2)}$ is quadratic.*

Proof. Everything in the above proposition was proved, except for the last statement about the degree of $\mathcal{A}^{(N)}$ for $N>2$ (the case $N=2$ was discussed in Example 2.1). We have:

$$\deg \mathcal{A}^{(N)} = \max\left\{ \max_{\substack{i=1,\dots,N\\ 1\le n<m\le N}} \deg_{\mathcal{S}^{(N)}}\{F_i, K_{m,n}\},\ \max_{\substack{1\le j<i\le N\\ 1\le n<m\le N}} \deg_{\mathcal{S}^{(N)}}\{K_{i,j}, K_{m,n}\} \right\}. \tag{2.40}$$

We observe that from Eq. (2.24a), taking into account Eqs. (2.35) and (2.39), we have:

$$\begin{aligned} \max_{\substack{i=1,\dots,N\\ 1\le n<m\le N}} \deg_{\mathcal{S}^{(N)}}\{F_i, K_{m,n}\} &= \deg_{\mathcal{S}^{(N)}}\{F_N, K_{N,N-1}\} = \deg_{\mathcal{S}^{(N)}} F_{N-1}F_{2N-2} \\ &= 2N-1. \end{aligned} \tag{2.41}$$

In the same way, from Eq. (2.24b) and taking into account Eqs. (2.35) and (2.39), we have:

$$\begin{aligned} \max_{\substack{1\le j<i\le N\\ 1\le n<m\le N}} \deg_{\mathcal{S}^{(N)}}\{K_{i,j}, K_{m,n}\} &= \deg_{\mathcal{S}^{(N)}}\{K_{N,N-2}, K_{N,N-1}\} \\ &= \deg_{\mathcal{S}^{(N)}} F_{N-2}K_{2N-2,N-2} = 2N-1. \end{aligned} \tag{2.42}$$

Hence, in the end we obtain $\deg \mathcal{A}^{(N)} = 2N-1$, finishing the proof of the proposition. □

We conclude this subsection with a detailed study of the three-body case.

Example 2.6 ($N = 3$)**.** Let us consider the CM model in the three-body case, *i.e.* the Hamiltonian system in Eq. (2.1) with $N = 3$:

$$H^{(3)} = \frac{p_1^2+p_2^2+p_3^2}{2} + \nu^2\left[\frac{1}{(q_1-q_2)^2} + \frac{1}{(q_1-q_3)^2} + \frac{1}{(q_2-q_3)^2}\right]. \tag{2.43}$$

Following formulas (2.10) and (2.11), for $N = 3$, the system is endowed with the following linearly independent integrals of motion:

$$\mathcal{S}^{(3)} = \left\{ F_1, F_2, F_3, K_{2,1}, K_{3,1}, K_{3,2} \right\}. \tag{2.44}$$

From (2.5) and (2.9), these integrals of motion have the following explicit expressions:

$$F_1 = p_1 + p_2 + p_3, \tag{2.45a}$$

$$F_2 = 2H^{(3)}, \tag{2.45b}$$

$$F_3 = \left(p_1^3+p_2^3+p_3^3\right) + 3\nu^2\left[\frac{p_1+p_2}{(q_1-q_2)^2} + \frac{p_1+p_3}{(q_1-q_3)^2} + \frac{p_2+p_3}{(q_2-q_3)^2}\right], \tag{2.45c}$$

and:

$$K_{2,1} = (\mathbf{q}\cdot\mathbf{p})F_1 - (q_1+q_2+q_3)F_2, \tag{2.46a}$$

$$K_{3,1} = (\mathbf{q}\cdot\mathbf{A})F_1 - (q_1+q_2+q_3)F_3, \tag{2.46b}$$

$$K_{3,2} = (\mathbf{q}\cdot\mathbf{A})F_2 - (\mathbf{q}\cdot\mathbf{p})F_3, \tag{2.46c}$$

where the vector $\mathbf{A} := (A_1, A_2, A_3)$ is defined through its components as:

$$A_i := p_i^2 + \sum_{\substack{j=1\\ j\neq i}}^{3} \frac{\nu^2}{(q_i-q_j)^2}, \qquad i = 1,2,3. \tag{2.47}$$

Note that $A_1 + A_2 + A_3 = 2H^{(3)}$.

We can write down the Poisson commutation relations of the algebra $\mathcal{A}^{(3)}$ generated by $\mathcal{S}^{(3)}$. First, there is an abelian part:

$$\{F_2, -\} = 0, \qquad \{F_1, F_3\} = 0. \tag{2.48}$$

Then, there are the following formal Poisson commutation relations from Eq. (2.24a):

$$\begin{aligned} &\{F_1,K_{2,1}\} = -F_1^2 + 3F_2, && \{F_1,K_{3,1}\} = 3F_3 - F_1F_2,\\ &\{F_1,K_{3,2}\} = F_1F_3 - F_2^2, && \{F_3,K_{2,1}\} = 3(F_2^2 - F_1F_3),\\ &\{F_3,K_{3,1}\} = 3(F_2F_3 - F_1F_4), && \{F_3,K_{3,2}\} = 3(F_3^2 - F_2F_4), \end{aligned} \tag{2.49}$$

and the ones from (2.24b):

$$\begin{aligned} &\{K_{2,1},K_{3,1}\} = 2F_1K_{3,1} - F_2(3K_{2,1} + K_{3,0}) + F_3K_{2,0},\\ &\{K_{2,1},K_{3,2}\} = 5F_1K_{3,2} - 4F_2K_{3,1} + 3F_3K_{2,1},\\ &\{K_{3,1},K_{3,2}\} = 3F_1K_{4,2} - F_2(K_{3,2} + 3K_{4,1}) + 2F_3K_{3,1}. \end{aligned} \tag{2.50}$$

Here, by Eq. (2.32), we have $F_3K_{2,0} - F_2K_{3,0} = -3K_{3,2}$. Furthermore, as observed above, to determine the actual Poisson commutation relations, we need to provide the form of F_4, $K_{4,1}$, and $K_{4,2}$. This is readily given using Eqs. (2.33) and (2.37):

$$F_4 = \frac{4}{3}F_1F_3 - F_1^2F_2 + \frac{1}{2}F_2^2 + \frac{1}{6}F_1^4, \tag{2.51a}$$

$$K_{4,1} = F_1K_{3,1} - \frac{1}{2}\left(F_1^2 - F_2\right)K_{2,1}, \tag{2.51b}$$

$$K_{4,2} = F_1K_{3,2} - \frac{1}{6}\left(F_1^3 - 3F_1F_2 + 2F_3\right)K_{2,1}. \tag{2.51c}$$

Observe that, as expected, $\deg_{\mathcal{S}^{(3)}} F_4 = \deg_{\mathcal{S}^{(3)}} K_{4,2} = 4$ and $\deg_{\mathcal{S}^{(3)}} K_{4,1} = 3$. Then, by substituting Eq. (2.51) into formulas (2.49) and (2.50), we obtain the explicit form of the symmetry algebra for the three-body CM system, *i.e.*:

$$\{F_1,K_{2,1}\} = -F_1^2 + 3F_2, \tag{2.52a}$$

$$\{F_1,K_{3,1}\} = -F_1F_2 + 3F_3, \tag{2.52b}$$

$$\{F_1,K_{3,2}\} = F_1F_3 - F_2^2, \tag{2.52c}$$

$$\{F_3,K_{2,1}\} = 3(F_2^2 - F_1F_3), \tag{2.52d}$$

$$\{F_3,K_{3,1}\} = 3F_2F_3 - 4F_1^2F_3 + 3F_1^3F_2 - \frac{3}{2}F_1F_2^2 - \frac{1}{2}F_1^5, \tag{2.52e}$$

$$\{F_3,K_{3,2}\} = 3F_3^2 - 4F_1F_2F_3 + 3F_1^2F_2^2 - \frac{3}{2}F_2^3 - \frac{1}{2}F_1^4F_2, \tag{2.52f}$$

$$\{K_{2,1},K_{3,1}\} = 2F_1K_{3,1} - 3F_2K_{2,1} - 3K_{3,2}, \tag{2.52g}$$

$$\{K_{2,1},K_{3,2}\} = 5F_1K_{3,2} - 4F_2K_{3,1} + 3F_3K_{2,1}, \tag{2.52h}$$

$$\begin{aligned}\{K_{3,1},K_{3,2}\} = {}& 2F_3K_{3,1} + 3F_1^2K_{3,2} - \frac{1}{2}F_1^4K_{2,1} + 3F_1^2F_2K_{2,1}\\ &- F_1F_3K_{2,1} - 3F_1F_2K_{3,1} - \frac{3}{2}F_2^2K_{2,1} - F_2K_{3,2},\end{aligned} \tag{2.52i}$$

together with (2.48). Observe that, as expected, the algebra $\mathcal{A}^{(3)}$ has degree five: the Poisson commutation relations (2.52e), (2.52f), and (2.52i) have all degree five in $\mathcal{S}^{(3)}$. Recall that $K_{3,2}$ is not functionally independent (but linearly independent) on other integrals of motion. Following (2.14), the functional relation linking $K_{3,2}$ with the functionally independent invariants are given by:

$$F_2K_{3,1} - F_1K_{3,2} - F_3K_{2,1} = 0. \tag{2.53}$$

3. The Nijhoff–Pang discretization of the N-body Calogero–Moser system and its symmetry algebra

In this section, we first review the main results of Nijhoff and Pang [24], Ujino et al. [25] concerning the dynamics and integrability of a MS discretization of the CM model. Then, we proceed by computing the corresponding symmetry algebra for the arbitrary N-body case in the same fashion as for the continuous case considered in Section 2. We thereby show the main result of the paper: how discretization leads to a deformation of the polynomial symmetry algebra of the model with respect to the discretization parameter h. We also present, as explicit examples, the deformed symmetry algebra for the two- and the three-body cases.

3.1. Review of the discrete model

In Nijhoff and Pang [24], Nijhoff and Pang introduced an integrable discretization of the CM system. Later, this discretization was proved to be MS in Ujino et al. [25], building a discrete analog of the G_j (see Eq. (2.11)). Hereinafter, we will address this discrete system as Nijhoff–Pang discretization of the N-body CM system, or in short NPdCM system. The dynamics of the NPdCM system is governed by the following difference equations of motion in canonical form (see Eq. (1.1)):

$$p_k = \frac{c_0}{h}\left[1 - c_0\sum_{j=1}^{N}\frac{1}{\underline{x}_j - x_k + c_0} + c_0\sum_{\substack{j=1\\ j\neq k}}^{N}\frac{1}{x_j - x_k}\right], \tag{3.1a}$$

$$\overline{p}_k = \frac{c_0}{h}\left[1 - c_0\sum_{j=1}^{N}\frac{1}{\overline{x}_k - x_j + c_0} + c_0\sum_{\substack{j=1\\ j\neq k}}^{N}\frac{1}{\overline{x}_k - \overline{x}_j}\right], \tag{3.1b}$$

where $h > 0$ is a discretization parameter, and c_0 is defined such that $c_0^2 := -\mathrm{i}h\nu$. Through the rescaling $x_k(t) = q_k(t) + ht$, the system (3.1) constitutes a discrete version of the continuous equations of motion (2.2).

Similarly to the continuous case, one may find a Lax pair. This can be done by performing a discretization of the isospectral evolution equation (2.4), which can be written as:

$$\overline{L}\widetilde{M} = \widetilde{M}L, \tag{3.2}$$

where the matrices $L, \widetilde{M} \in \mathfrak{gl}_N(\mathbb{R})$ are defined by their matrix elements as follows:

$$L_{jk} = \delta_{jk}p_j + (1-\delta_{jk})\frac{\mathrm{i}\nu}{x_j - x_k}, \tag{3.3a}$$

$$\widetilde{M}_{jk} = \frac{c_0}{\overline{x}_j - x_k + c_0}. \tag{3.3b}$$

We remark that the matrix L can be chosen to be of the same form as in the continuous case, *cf.* (2.3). Therefore, it provides us with the N

integrals of motion, given by the same expression as in the continuous case (2.5).

In turn, a generalization of the Wojciechowski construction (2.9) is nontrivial. First, let us notice that the discrete version of the generalized isospectral problem (2.6) is provided by:

$$\overline{X}\,\widetilde{M} = \widetilde{M}X + h\widetilde{M}L(\mathbb{1} - hc_0^{-1}L)^{-1}, \tag{3.4}$$

and the simplest solution is again $X = \mathrm{diag}\left(x_1, \dots, x_N\right)$. Therefore, the modified integrals of motion for the discrete case, corresponding to (2.9), can be defined as follows:

$$\begin{aligned}\widetilde{K}_{m,n} :&= \mathrm{tr}\left[X\left(\mathbb{1} - hc_0^{-1}L\right)L^{m-1}\right]\mathrm{tr}(L^n) - \mathrm{tr}(L^m)\mathrm{tr}\left[X\left(\mathbb{1} - hc_0^{-1}L\right)L^{n-1}\right],\\ &1 \le n < m \le N.\end{aligned} \tag{3.5}$$

One can observe that in the continuum limit $h \to 0$, we recover the integral of motion $K_{m,n}$ (2.9) from the continuous case. Note that here we have $\mathrm{Ord}(\widetilde{K}_{m,n}) = m + n$. Indeed, from Eq. (3.2) and (3.4), we have:

$$\overline{X}(\mathbb{1} - hc_0^{-1}\overline{L}) = \widetilde{M}\left[X(\mathbb{1} - hc_0^{-1}L) + hL\right]\widetilde{M}^{-1}. \tag{3.6}$$

We use the cyclicity of trace together with a property of similarity transformation:

$$(\overline{L})^n = (\widetilde{M}L\widetilde{M}^{-1})^n = \widetilde{M}L^n\widetilde{M}^{-1} \implies \mathrm{tr}[\overline{L}^n] = \mathrm{tr}[L^n]. \tag{3.7}$$

Thus, using linearity and cyclicity of trace again, one can prove that $\widetilde{K}_{m,n}$ is conserved:[3]

$$\begin{aligned}\overline{\widetilde{K}}_{m,n} &= \mathrm{tr}\left[\overline{X}\left(\mathbb{1} - hc_0^{-1}\overline{L}\right)\overline{L}^{m-1}\right]\mathrm{tr}\left(\overline{L}^n\right) - \mathrm{tr}\left(\overline{L}^m\right)\mathrm{tr}\left[\overline{X}\left(\mathbb{1} - hc_0^{-1}\overline{L}\right)\overline{L}^{n-1}\right]\\ &= \mathrm{tr}\left[XL^{m-1} - hc_0^{-1}XL^m + hL^m\right]\mathrm{tr}(L^n) - \mathrm{tr}(L^m)\mathrm{tr}\left[XL^{n-1}\right.\\ &\quad\left. - hc_0^{-1}XL^n + hL^n\right]\\ &= \mathrm{tr}\left[X(\mathbb{1} - hc_0^{-1}L)L^{m-1}\right]\mathrm{tr}(L^n) - \mathrm{tr}(L^m)\mathrm{tr}\left[X(\mathbb{1} - hc_0^{-1}L)L^{n-1}\right]\\ &= \widetilde{K}_{m,n}.\end{aligned} \tag{3.8}$$

Therefore, the NPdCM model admits a total of $N(N+1)/2$ *linearly independent* integrals of motion:

$$\widetilde{S}^{(N)} = \{\, F_k \,\}_{k=1}^N \cup \{\, \widetilde{K}_{m,n} \,\}_{1 \le n < m \le N}\,, \tag{3.9}$$

of which a functionally independent subset is given by:

$$\widetilde{S}^{(N)}_{\mathrm{f.i.}} = \{\, F_k \,\}_{k=1}^N \cup \{\, \widetilde{G}_m \,\}_{m=1}^{N-1}\,, \qquad \widetilde{G}_m := \widetilde{K}_{m+1,1}, \tag{3.10}$$

cf. Eqs. (2.10) and (2.11).

By construction, upon replacing $K_{i,j} \mapsto \widetilde{K}_{i,j}$, the same functional relations as in the continuous case (see Eq. (2.12)) hold true:

$$F_j\widetilde{K}_{n,i} - F_i\widetilde{K}_{n,j} + F_n\widetilde{K}_{i,j} = 0, \qquad i,j,n \ge 0, \tag{3.11}$$

where the same remark we made in the continuous setting also applies here.

Finally, let us observe that the modified integrals of motion $\widetilde{K}_{m,n}$ can be rewritten as follows:

$$\widetilde{K}_{m,n} = K_{m,n} - hc_0^{-1}K^{(1)}_{m+1,n+1} = K_{m,n} + \alpha\sqrt{h}K^{(1)}_{m+1,n+1}, \qquad \alpha := \frac{1+\mathrm{i}}{\sqrt{2\nu}}, \tag{3.12}$$

where $K_{m,n}$ is an integral of motion in the continuous case (2.9), and $K^{(1)}_{m+1,n+1}$ is the first member of the so-called higher flow:

$$K^{(a)}_{m,n} = J_mF_{a+n-2} - J_nF_{a+m-2}, \tag{3.13}$$

reported in Wojciechowski [4]. Notice that $\mathrm{Ord}(K^{(a)}_{m,n}) = a + n + m - 3$ for $m \ne n$, and $K^{(2)}_{m,n} \equiv K_{m,n}$.

[3] We remark that the proof for the subset of the functionally independent integrals of motion, *i.e.* for $\{\, G_m = K_{m+1,1} \,\}_{m=1}^{N-1}$ was done in Ujino et al. [25].

3.2. The symmetry algebra

The aim of this section is to obtain the expressions defining the symmetry algebra of the NPdCM system. We do it in the same fashion as for the continuous case (see Section 2.2). We also present the main result of the paper, namely the derivation of the symmetry algebra of the discretized model as a deformation of its continuous counterpart. Since this phenomenon can already be observed for a low number-of-bodies case, let us again start by presenting an example.

Example 3.1 ($N = 2$). For the two-body problem, the symmetry algebra is generated by:

$$\widetilde{S}^{(2)} = \widetilde{S}^{(2)}_{\mathrm{f.i}} = \{\, F_1, F_2, \widetilde{K}_{2,1} \,\}\,, \tag{3.14}$$

where F_1 and F_2 remain unchanged under discretization, *cf.* Eq. (2.17), and the modified constant of motion, from (3.12), is given by:

$$\begin{aligned}\widetilde{K}_{2,1} &= K_{2,1} + \alpha\sqrt{h}K^{(1)}_{3,2} = (\mathbf{x}\cdot\mathbf{p})F_1 - (x_1 + x_2)F_2\\ &\quad + \frac{1}{2}\alpha\sqrt{h}(x_1 - x_2)(p_1 - p_2)\left(F_1^2 - F_2\right).\end{aligned} \tag{3.15}$$

As we already observed, the integral of motion $\widetilde{K}_{2,1}$ is of order three. One can now compute the Poisson commutation relations and arrive at the following set of relations:

$$\begin{aligned}&\{F_1, F_2\} = 0, \quad \{F_1, \widetilde{K}_{2,1}\} = 2F_2 - F_1^2,\\ &\{F_2, \widetilde{K}_{2,1}\} = \alpha\sqrt{h}\left(F_1^2 - 2F_2\right)\left(F_1^2 - F_2\right).\end{aligned} \tag{3.16}$$

The first two brackets are unaltered with respect to the continuous case, see (2.18), while the third one is now non-zero. This is in line with the fact that, as was discussed in Section 1, in the discrete-time setting, there is no complete analogue of the Hamiltonian, and being in involution with the "Hamiltonian" no longer implies to be an integral of motion, and *vice versa*. It is easy to see that in the continuum limit we have:

$$\lim_{h\to 0}\{F_2, \widetilde{K}_{2,1}\} = \{F_2, K_{2,1}\} = 0. \tag{3.17}$$

In this sense, the polynomial algebra $\widetilde{\mathcal{A}}^{(2)}$, generated by $\widetilde{S}^{(2)}$ can be seen as a *deformation* of the continuous symmetry algebra $\mathcal{A}^{(2)}$ with respect to the discretization parameter h. We remark that the following nontrivial phenomenon occurs: since the discretization of $K_{m,n}$ increases its order, the associated polynomial symmetry algebra becomes of higher degree in the generators. This will be shown to be true below for an arbitrary N. For $N = 2$, the continuous symmetry algebra was quadratic, while the corresponding deformed symmetry algebra in the discrete case turns out to be quartic.

Let us now proceed to analyze the structure of the symmetry algebra for an arbitrary N-body NPdCM model. Taking into account that in the discrete case, the integral of motion $\widetilde{K}_{m,n}$ (3.5) can be rewritten via the constant of motion $K_{m,n}$ of the continuous model and the term $K^{(1)}_{m+1,n+1}$ (see Eq. (3.12)), we can use the results of Section 2.

First, directly from Proposition 2.3 and taking into account the presence of the additional term, one can obtain the following result:

Proposition 3.2. *The following formal expressions for the Poisson brackets hold true in the discrete case:*

$$\{F_i, \widetilde{K}_{m,n}\} = \mathrm{i}(F_mF_{i+n-2} - F_nF_{i+m-2}) + \mathrm{i}\alpha\sqrt{h}\,\left(F_mF_{i+n-1} - F_nF_{i+m-1}\right), \tag{3.18a}$$

$$\begin{aligned}\{\widetilde{K}_{i,j}, \widetilde{K}_{m,n}\} = \;& F_i(m\widetilde{K}_{n,j+m-2} + n\widetilde{K}_{j+n-2,m}) + F_j(m\widetilde{K}_{i+m-2,n} + n\widetilde{K}_{m,i+n-2})\\ &+ F_m(i\widetilde{K}_{i+n-2,j} + j\widetilde{K}_{i,j+n-2}) + F_n(i\widetilde{K}_{j,i+m-2} + j\widetilde{K}_{j+m-2,i})\\ &+ \alpha\sqrt{h}\left[F_i(m\widetilde{K}_{n,j+m-1} + n\widetilde{K}_{j+n-1,m}) + F_j(m\widetilde{K}_{i+m-1,n}\right.\\ &+ n\widetilde{K}_{m,i+n-1}) + F_m(i\widetilde{K}_{i+n-1,j} + j\widetilde{K}_{i,j+n-1}) + F_n(i\widetilde{K}_{j,i+m-1}\\ &\left.+ j\widetilde{K}_{j+m-1,i})\right].\end{aligned} \tag{3.18b}$$

Remark 3.3. It is easy to see that in the continuum limit $h \to 0$, the relations (3.18) reduce exactly to their continuous counterparts (2.24), see Proposition 2.3. From this point of view, the symmetry algebra of the discrete model can be considered as a *deformed polynomial algebra* with respect to the discretization parameter h, which plays the rôle of deformation parameter. The demonstration of this non-trivial phenomenon constitutes the main result of the paper.

Proof of Proposition 3.2. By Eq. (3.12), we have that: $\{F_i, \widetilde{K}_{m,n}\} = \{F_i, K_{m,n}\} + \alpha\sqrt{h}\{F_i, K^{(1)}_{m+1,n+1}\}$. The explicit form of the bracket $\{F_i, K_{m,n}\}$ is known (see Eq. (2.24a)). The bracket $\{F_i, K^{(1)}_{m+1,n+1}\}$, in turn, can be computed as follows:

$$\{F_i, K^{(1)}_{m+1,n+1}\} = i(F_m F_{i+n-1} - F_n F_{i+m-1}). \tag{3.19}$$

Combining these expressions, one arrives at (3.18a).

Using again (3.12), one can rewrite the Poisson bracket (3.18b) as follows:

$$\begin{aligned}\{\widetilde{K}_{i,j}, \widetilde{K}_{m,n}\} =& \{K_{i,j} + \alpha\sqrt{h}K^{(1)}_{i+1,j+1}, K_{m,n} + \alpha\sqrt{h}K^{(1)}_{m+1,n+1}\} \\ =& \{K_{i,j}, K_{m,n}\} + \alpha\sqrt{h}\{K_{i,j}, K^{(1)}_{m+1,n+1}\} + \alpha\sqrt{h}\{K^{(1)}_{i+1,j+1}, K_{m,n}\} \\ &+ \alpha^2\, h\{K^{(1)}_{i+1,j+1}, K^{(1)}_{m+1,n+1}\},\end{aligned} \tag{3.20}$$

where we recall that $K_{i,j} \equiv K^{(2)}_{i,j}$ is an integral of motion for the continuous CM model. Hence, the bracket $\{K_{i,j}, K_{m,n}\}$ was already calculated in the continuous case (see Eq. (2.24b)). The remaining brackets can be computed analogously, using (3.13) and (2.23).

Combining these terms once computed, performing some algebra, and reconstructing the terms $\widetilde{K}_{i,j}$ using again Eq. (3.12), one arrives at (3.18b). □

Proceeding as in the continuous setting, we see that for a fixed N to make the formal Poisson commutation relations (3.18) the actual ones given in terms of elements in the set $\widetilde{S}^{(N)}$, we need to express the following additional functionally dependent first integrals:

$$F_{N+1}, \quad F_{N+2}, \quad \dots, \quad F_{2N-2}, \quad F_{2N-1}, \tag{3.21a}$$

$$\widetilde{K}_{1,0}, \quad \widetilde{K}_{2,0}, \quad \dots, \quad \widetilde{K}_{N-1,0}, \quad \widetilde{K}_{N,0}, \tag{3.21b}$$

$$\begin{matrix} \widetilde{K}_{N+1,1}, & \widetilde{K}_{N+1,2}, & \dots, & \widetilde{K}_{N+1,N-1}, & \widetilde{K}_{N+1,N}, \\ \widetilde{K}_{N+2,1}, & \widetilde{K}_{N+2,2}, & \dots, & \widetilde{K}_{N+2,N-1}, & \widetilde{K}_{N+2,N}, \\ \vdots & & & & \vdots \\ \widetilde{K}_{2N-2,1}, & \widetilde{K}_{2N-2,2}, & \dots, & \widetilde{K}_{2N-2,N-1}, & \widetilde{K}_{2N-2,N}, \\ \widetilde{K}_{2N-1,1}, & \widetilde{K}_{2N-1,2}, & \dots, & \widetilde{K}_{2N-1,N-1}. & \end{matrix} \tag{3.21c}$$

Notice that we need one more row with respect to continuous case (2.30).

Then, adopting the same procedure shown in the continuous case (see Section 2.2), one can express the relations, presented in Proposition 3.2, in a closed form for a given N. Since the integrals of motion $\{F_i\}_{i=1}^N$ remain unchanged under discretization, the same iterative formulas (2.34) are applicable. Moreover, by construction the same Eq. (2.37) holds true upon the replacement $K_{i,j} \mapsto \widetilde{K}_{i,j}$, *i.e.*:

$$\widetilde{K}_{N+s,\ell} = -\sum_{k=1}^{N} \frac{1}{k!} B_k\left(-0!F_1, -1!F_2, -2!F_3, \dots, -(k-1)!F_k\right)\widetilde{K}_{N+s-k,\ell}. \tag{3.22}$$

This allows us to write the explicit form of the symmetry algebra not only for the continuous CM model, but also for its Nijhoff–Pang discretization. We conclude by extending Proposition 2.5 to the discrete setting:

Proposition 3.4. *The symmetry algebra $\widetilde{\mathcal{A}}^{(N)}$ of the N-body NPdCM model is generated by the elements of the set $\widetilde{S}^{(N)}$ (3.10) with Poisson commutation relations given in Eq. (3.18), provided the additional closure relations given in Eqs. (2.34) and (3.22). Moreover, $\mathcal{I}^{(N)} := \langle F_i\rangle_{i=1}^N$ is an ideal of $\widetilde{\mathcal{A}}^{(N)}$. Finally, for any $N \geq 2$, the degree of the polynomial symmetry algebra $\widetilde{\mathcal{A}}^{(N)}$ is $2N$.*

Proof. As for the continuous case, everything in the previous proposition was already proved except for the degree of the symmetry algebra. To determine the degree of the symmetry algebra in the discrete case, we can use the same argument as in the proof of Proposition 2.5, *mutatis mutandis*. Hence, taking into account the new terms proportional to the discretization parameter in Eq. (3.18), which have a higher degree, we obtain the thesis. □

We stress that the discrete symmetry algebra $\widetilde{\mathcal{A}}^{(N)}$ reduces to the continuous one in the continuum limit, *i.e.* $h \to 0$, without the necessity of rescaling other parameters. That is, the following result holds:

Corollary 3.4.1. *The symmetry algebra $\widetilde{\mathcal{A}}^{(N)}$ of the N-body NPdCM model is a deformation of the symmetry algebra $\mathcal{A}^{(N)}$ N-body CM model, that is the Poisson commutation relations of $\widetilde{\mathcal{A}}^{(N)}$ reduce to those of $\mathcal{A}^{(N)}$ by letting the discretization parameter $h \to 0$.*

The above result relates the discrete and continuous CM systems on an algebraic level and constitutes a remarkable property of the model.

Remark 3.5. We remark that the discretization of the CM model not only deforms the symmetry algebra with respect to the deformation parameter h, but also increases its degree, provided by the contribution of the new additional terms, proportional to h. For $N > 2$, we have $\deg \widetilde{\mathcal{A}}^{(N)} = \deg \mathcal{A}^{(N)} + 1$, *i.e.* the degree of the polynomial algebra of the CM model is increased by one, while for $N = 2$ we have $\deg \widetilde{\mathcal{A}}^{(2)} = \deg \mathcal{A}^{(2)} + 2$, *i.e.* the degree is raised by two.

We conclude this section with a detailed study of the discrete three-body case.

Example 3.6 ($N = 3$). In the same fashion as we have done for the continuous CM model (*cf.* Example 2.6), let us now demonstrate an application of the general formulas (3.18) to the three-body NPdCM system. From (3.9), we have that the symmetry algebra is generated by the set:

$$\widetilde{S}^{(3)} = \left\{ F_1, F_2, F_3, \widetilde{K}_{2,1}, \widetilde{K}_{3,1}, \widetilde{K}_{3,2} \right\}. \tag{3.23}$$

Since the integrals of motion $\{F_i\}_{i=1}^N$ remain unchanged under discretization, their explicit expression is given by the same formulas (2.45). The modified additional constants of motion, in turn, can be written as follows:

$$\widetilde{K}_{2,1} = K_{2,1} + \alpha\sqrt{h}\,\left[(\mathbf{x}\cdot\mathbf{A})F_1 - (\mathbf{x}\cdot\mathbf{p})F_2\right], \tag{3.24a}$$

$$\widetilde{K}_{3,1} = K_{3,1} + \alpha\sqrt{h}\,\left[\left(2\sum_{i=1}^{3} x_i p_i A_i - \sum_{i=1}^{3} x_i p_i^3 + \nu^2\left(Y_{1,2} + Y_{1,3} + Y_{2,3}\right)\right)F_1 - (\mathbf{x}\cdot\mathbf{p})F_3\right], \tag{3.24b}$$

$$\widetilde{K}_{3,2} = K_{3,2} + \alpha\sqrt{h}\,\left[\left(2\sum_{i=1}^{3} x_i p_i A_i - \sum_{i=1}^{3} x_i p_i^3 + \nu^2\left(Y_{1,2} + Y_{1,3} + Y_{2,3}\right)\right)F_2 - (\mathbf{x}\cdot\mathbf{A})F_3\right], \tag{3.24c}$$

where $K_{i,j}$ are integrals of motion for the continuous system (see Eq. (2.46)), $\mathbf{A}$ is defined again by Eq. (2.47), and we also introduced the following auxiliary quantity:

$$Y_{i,j} := \frac{p_i x_j + p_j x_i}{(x_i - x_j)^2}. \tag{3.25}$$

Let us proceed to compute the Poisson commutation relations of the symmetry algebra $\widetilde{\mathcal{A}}^{(3)}$ for the discrete model, generated by the elements in the set $\widetilde{S}^{(3)}$. The abelian part of the algebra, generated by $\{F_1, F_2, F_3\}$, is preserved. Then, on a formal level, we can compute the relations, using formula (3.18a):

$$\{F_1, \widetilde{K}_{2,1}\} = -F_1^2 + 3F_2, \tag{3.26a}$$

$$\{F_1, \widetilde{K}_{3,1}\} = -F_1 F_2 + 3F_3, \tag{3.26b}$$

$$\{F_1,\widetilde{K}_{3,2}\}=F_1F_3-F_2^2, \tag{3.26c}$$

$$\{F_2,\widetilde{K}_{2,1}\}=2\alpha\sqrt{h}\left(F_2^2-F_1F_3\right), \tag{3.26d}$$

$$\{F_2,\widetilde{K}_{3,1}\}=2\alpha\sqrt{h}\left(F_2F_3-F_1F_4\right), \tag{3.26e}$$

$$\{F_2,\widetilde{K}_{3,2}\}=2\alpha\sqrt{h}\left(F_3^2-F_2F_4\right), \tag{3.26f}$$

$$\{F_3,\widetilde{K}_{2,1}\}=3(F_2^2-F_1F_3)+3\alpha\sqrt{h}(F_2F_3-F_1F_4), \tag{3.26g}$$

$$\{F_3,\widetilde{K}_{3,1}\}=3(F_2F_3-F_1F_4)+3\alpha\sqrt{h}(F_3^2-F_1F_5), \tag{3.26h}$$

$$\{F_3,\widetilde{K}_{3,2}\}=3(F_3^2-F_2F_4)+3\alpha\sqrt{h}(F_3F_4-F_2F_5), \tag{3.26i}$$

and from (3.18b):

$$\begin{aligned}\{\widetilde{K}_{2,1},\widetilde{K}_{3,1}\}&=2F_1\widetilde{K}_{3,1}-F_2(3\widetilde{K}_{2,1}+\widetilde{K}_{3,0})+F_3\widetilde{K}_{2,0}\\&\quad+\alpha\sqrt{h}\Big(3F_3\widetilde{K}_{2,1}-4F_2\widetilde{K}_{3,1}+F_1(2\widetilde{K}_{3,2}+\widetilde{K}_{4,1})\Big),\end{aligned} \tag{3.26j}$$

$$\begin{aligned}\{\widetilde{K}_{2,1},\widetilde{K}_{3,2}\}&=5F_1\widetilde{K}_{3,2}-4F_2\widetilde{K}_{3,1}+3F_3\widetilde{K}_{2,1}\\&\quad+2\alpha\sqrt{h}\Big(\frac{3}{2}F_1\widetilde{K}_{4,2}+F_3\widetilde{K}_{3,1}-F_2\widetilde{K}_{4,1}-2F_2\widetilde{K}_{3,2}\Big),\end{aligned} \tag{3.26k}$$

$$\begin{aligned}\{\widetilde{K}_{3,1},\widetilde{K}_{3,2}\}&=3F_1\widetilde{K}_{4,2}-F_2(\widetilde{K}_{3,2}+3\widetilde{K}_{4,1})+2F_3\widetilde{K}_{3,1}\\&\quad+\alpha\sqrt{h}(3F_1\widetilde{K}_{5,2}-3F_2\widetilde{K}_{5,1}+3F_3\widetilde{K}_{4,1}-2F_1\widetilde{K}_{4,3}-4F_3\widetilde{K}_{3,2}).\end{aligned} \tag{3.26l}$$

Following the same algorithm as in the continuous case, we now have to express the elements $F_4,F_5,\widetilde{K}_{2,0},\widetilde{K}_{3,0},\widetilde{K}_{4,1},\widetilde{K}_{4,2},\widetilde{K}_{4,3},\widetilde{K}_{5,1},\widetilde{K}_{5,2}$ in terms of the generators (3.23). We have, similarly to the continuous case (see (3.11) and (2.32)) that $F_3\widetilde{K}_{2,0}-F_2\widetilde{K}_{3,0}=-3\widetilde{K}_{3,2}$. Then, from (2.33) and (3.22), we obtain:

$$F_4=\frac{4}{3}F_1F_3-F_1^2F_2+\frac{1}{2}F_2^2+\frac{1}{6}F_1^4, \tag{3.27a}$$

$$F_5=\frac{5}{6}(F_1^2F_3-F_1^3F_2+F_2F_3)+\frac{1}{6}F_1^5, \tag{3.27b}$$

$$\widetilde{K}_{4,1}=F_1\widetilde{K}_{3,1}-\frac{1}{2}\left(F_1^2-F_2\right)\widetilde{K}_{2,1}, \tag{3.27c}$$

$$\widetilde{K}_{4,2}=F_1\widetilde{K}_{3,2}-\frac{1}{6}\left(F_1^3-3F_1F_2+2F_3\right)\widetilde{K}_{2,1}, \tag{3.27d}$$

$$\widetilde{K}_{4,3}=-\frac{1}{6}\left(F_1^3-3F_1F_2+2F_3\right)\widetilde{K}_{3,1}+\frac{1}{2}\left(F_1^2-F_2\right)\widetilde{K}_{3,2}, \tag{3.27e}$$

$$\widetilde{K}_{5,1}=\frac{1}{3}\left(F_3-F_1^3\right)\widetilde{K}_{2,1}+\frac{1}{2}\left(F_1^2+F_2\right)\widetilde{K}_{3,1}, \tag{3.27f}$$

$$\widetilde{K}_{5,2}=\left(\frac{1}{2}F_1F_2-\frac{1}{3}F_3-\frac{1}{6}F_1^3\right)F_1\widetilde{K}_{2,1}+\frac{1}{2}\left(F_1^2+F_2\right)\widetilde{K}_{3,2}. \tag{3.27g}$$

Combining (3.27) with (3.26), we finally arrive at the closed-form expressions for the Poisson commutation relations, completely defining the structure of the symmetry algebra $\widetilde{\mathcal{A}}^{(3)}$:

$$\{F_1,\widetilde{K}_{2,1}\}=-F_1^2+3F_2, \tag{3.28a}$$

$$\{F_1,\widetilde{K}_{3,1}\}=-F_1F_2+3F_3, \tag{3.28b}$$

$$\{F_1,\widetilde{K}_{3,2}\}=F_1F_3-F_2^2, \tag{3.28c}$$

$$\{F_2,\widetilde{K}_{2,1}\}=2\alpha\sqrt{h}\left(F_2^2-F_1F_3\right), \tag{3.28d}$$

$$\{F_2,\widetilde{K}_{3,1}\}=\frac{1}{3}\alpha\sqrt{h}\left[6F_2F_3-F_1\left(8F_1F_3-6F_1^2F_2+3F_2^2+F_1^4\right)\right], \tag{3.28e}$$

$$\{F_2,\widetilde{K}_{3,2}\}=\frac{1}{3}\alpha\sqrt{h}\left[6F_3^2-F_2\left(8F_1F_3-6F_1^2F_2+3F_2^2+F_1^4\right)\right], \tag{3.28f}$$

$$\begin{aligned}\{F_3,\widetilde{K}_{2,1}\}&=3(F_2^2-F_1F_3)+\frac{1}{2}\alpha\sqrt{h}\big[6F_2F_3-F_1\big(8F_1F_3-6F_1^2F_2\\&\quad+3F_2^2+F_1^4\big)\big],\end{aligned} \tag{3.28g}$$

$$\begin{aligned}\{F_3,\widetilde{K}_{3,1}\}&=3F_2F_3-4F_1^2F_3+3F_1^3F_2-\frac{3}{2}F_1F_2^2-\frac{1}{2}F_1^5\\&\quad+\frac{1}{2}\alpha\sqrt{h}\left(F_3-F_1^3\right)\left(F_1^3-5F_1F_2+6F_3\right),\end{aligned} \tag{3.28h}$$

$$\begin{aligned}\{F_3,\widetilde{K}_{3,2}\}&=3F_3^2-4F_1F_2F_3+3F_1^2F_2^2-\frac{3}{2}F_2^3-\frac{1}{2}F_1^4F_2\\&\quad+\frac{1}{2}\alpha\sqrt{h}\big(F_1^4F_3-F_1^5F_2+5F_1^3F_2^2-11F_1^2F_2F_3\\&\quad+8F_1F_3^2-2F_2^2F_3\big),\end{aligned} \tag{3.28i}$$

$$\begin{aligned}\{\widetilde{K}_{2,1},\widetilde{K}_{3,1}\}&=2F_1\widetilde{K}_{3,1}-3F_2\widetilde{K}_{2,1}-3\widetilde{K}_{3,2}\\&\quad+\frac{1}{2}\alpha\sqrt{h}\Big[\left(F_1F_2+6F_3-F_1^3\right)\widetilde{K}_{2,1}+2(F_1^2-4F_2)\widetilde{K}_{3,1}\\&\quad+4F_1\widetilde{K}_{3,2}\Big],\end{aligned} \tag{3.28j}$$

$$\begin{aligned}\{\widetilde{K}_{2,1},\widetilde{K}_{3,2}\}&=5F_1\widetilde{K}_{3,2}-4F_2\widetilde{K}_{3,1}+3F_3\widetilde{K}_{2,1}\\&\quad+\frac{1}{2}\alpha\sqrt{h}\Big[\left(5F_1^2F_2-2F_1F_3-2F_2^2-F_1^4\right)\widetilde{K}_{2,1}\\&\quad+4(F_3-F_1F_2)\widetilde{K}_{3,1}+2(3F_1^2-4F_2)\widetilde{K}_{3,2}\Big],\end{aligned} \tag{3.28k}$$

$$\begin{aligned}\{\widetilde{K}_{3,1},\widetilde{K}_{3,2}\}&=2F_3\widetilde{K}_{3,1}+3F_1^2\widetilde{K}_{3,2}-\frac{1}{2}F_1^4\widetilde{K}_{2,1}+3F_1^2F_2\widetilde{K}_{2,1}-F_1F_3\widetilde{K}_{2,1}\\&\quad-3F_1F_2\widetilde{K}_{3,1}-\frac{3}{2}F_2^2\widetilde{K}_{2,1}-F_2\widetilde{K}_{3,2}\\&\quad+\frac{1}{2}\alpha\sqrt{h}\Big[\left(5F_1^3F_2+F_2F_3-5F_1^2F_3-F_1^5\right)\widetilde{K}_{2,1}\\&\quad+\Big(\frac{2}{3}F_1^4-5F_1^2F_2+\frac{22}{3}F_1F_3-3F_2^2\Big)\widetilde{K}_{3,1}\\&\quad+\left(F_1^3+5F_1F_2-8F_3\right)\widetilde{K}_{3,2}\Big].\end{aligned} \tag{3.28l}$$

The functional relation is given by:

$$F_2\widetilde{K}_{3,1}-F_1\widetilde{K}_{3,2}-F_3\widetilde{K}_{2,1}=0 \tag{3.29}$$

Let us compare this algebra with the continuous case (see Example 2.6). Note that the relations (3.28a)–(3.28c) are preserved, *i.e.* they coincide with the continuous counterparts (2.52a)–(2.52c).

The three-body example reveals the phenomena in more detail, compared to Example 3.1. For instance, here not only the relations $\{F_2,-\}$ become non-zero (as it was already discussed before), but also the relations (3.28g)–(3.28l), being already nontrivial in the continuous case (*cf.* (2.52d)–(2.52i)), are now deformed with respect to h as well. This clearly follows from the general formulas (3.18).

4. Discussion and outlook

In this paper, we computed the explicit symmetry algebras for the celebrated N-body Calogero–Moser model and its discretization, introduced by Nijhoff and Pang, for an arbitrary number of bodies N. The structure of the symmetry algebra is given by its Poisson commutation relations and closure relations in the form of recursive formulas (see Propositions 2.5 and 3.4 for the continuous and discrete models, respectively). The final formulas, obtained through a judicious application of Cayley–Hamilton theorem, contain a peculiar class of special polynomials: the Bell polynomials. We underline that Bell polynomials should appear in every case, either continuous or discrete, when the integrals of motion are given by traces and superintegrability is obtained through an additional isospectral problem like (2.6). Finally, we illustrated the results by presenting the low-number-of-bodies cases $N=2,3$.

Comparing the obtained symmetry algebras, we demonstrated a remarkable property: the symmetry algebra of the discrete case is a non-trivial deformation of the one obtained in the continuous case with respect to the discretization parameter h. Moreover, this deformation does not preserve the degree of the continuous symmetry algebra: in the discrete setting, higher degree terms emerge, increasing the degree of the polynomial algebra. To be more precise, for $N>2$, the degree increases by one, from $2N-1$ to $2N$, whereas for $N=2$ it increases by two, from two to four. In the case of the CM symmetry algebra, the contraction from the deformed to the non-deformed algebra is simply given by taking the continuous limit $h\to 0$. To our knowledge, this is the first example of such a phenomenon. Therefore, our central result is that maximally superintegrable discretizations can naturally yield deformed polynomial Poisson algebras. We observe that the deformation of the symmetry algebra we exhibit in this paper is possible because of the presence of modified integrals of motion of higher order for the discrete system. This is in contrast with the much simpler example of the harmonic oscillator we showed in Drozdov et al. [19], where the discretized invariants are modified, but of the same order as the continuous ones, and the same holds true for the corresponding symmetry algebras. For this reason, in this case, a condition for a non-trivial deformation to happen is that the discrete invariants are modified in a nontrivial way, *e.g.* with their order being increased. We observe that, in the example we considered in this paper, and as well in the one we considered earlier in Drozdov et al. [19], the deformation in the discretization, whether trivial or non-trivial, appears in the integrals of motion, while the Poisson bracket is kept canonical. Following the celebrated Ge–Marsden theorem [33], different discretizations can arise by keeping the integrals of motion as in the continuous counterpart and deforming the Poisson bracket. Clearly, a final possibility would be to discretize deforming *both* the Poisson bracket and the integrals of motion.

Our findings potentially suggest an inverse problem: can one find a new discretization of a given continuous model, preserving the superintegrable properties, by deforming a known symmetry algebra? This points a new direction of research and expands the potential for applying the algebraic approach in search of new discretizations. We reserve to address the problem of carrying out a systematic search for discrete MS models whose invariants and/or Poisson bracket are non-trivially modified for future research. Here, we limit ourselves to comment that, since a similar result is known for the continuous Toda lattice [5], a natural system to check will be its discretization, see *e.g.* Suris [20, Section 3.10]. Another natural candidate to consider is the CM system with an external harmonic potential or Kepler–Coulomb potential (see, for example, Correa et al. [34] and references therein), whose discretization was considered in Ujino et al. [35]. To our knowledge, the symmetry algebra of the (discrete) CM model with an external potential has not been determined yet, and due to the differences between this model and the "free" one, in our opinion, it poses already a challenging problem. Finally, another possible direction could be an application of the obtained results in the context of modern representation theory and Calogero–Moser spaces [36].

CRediT authorship contribution statement

Pavel Drozdov: Writing – review & editing, Writing – original draft, Validation, Software, Project administration, Methodology, Investigation, Formal analysis; **Giorgio Gubbiotti:** Writing – review & editing, Writing – original draft, Validation, Supervision, Software, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization; **Danilo Latini:** Writing – review & editing, Writing – original draft, Validation, Supervision, Software, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

PD, GG, and DL acknowledge the support of the research project Mathematical Methods in NonLinear Physics (MMNLP), Gruppo 4-Fisica Teorica of INFN. This work has been partially supported by the National Group of Mathematical Physics (GNFM) of the Italian Institute for High Mathematics (INdAM).

PD acknowledges the support of the Università degli Studi di Milano (for partially funding the research visit during which significant progress on this work was made), and of the Ph.D. program of the Università degli Studi di Udine.

DL has been partially funded by MUR - Dipartimento di Eccellenza 2023–2027, codice CUP G43C22004580005 - codice progetto DECC23_012_DIP.

Data availability

No data was used for the research described in the article.

Appendix A. Symmetric polynomials and Newton's identity

In this appendix, for the convenience of the reader, we report some results on the symmetric polynomials and Newton's identity that we need in the text. For a more general presentation of the topic, we refer to Macdonald [37] and references therein.

Let $\mathbf{x} := (x_1, \dots, x_N) \in \mathbb{R}^N$. Then, the *$k$-th power sum* is:

$$f_k(\mathbf{x}) := \sum_{i=1}^{N} x_i^k = x_1^k + \cdots + x_N^k. \tag{A.1}$$

Moreover, the *elementary symmetric polynomial* $e_k(\mathbf{x})$ is defined to be the sum of all distinct products of $1 \le k \le N$ variables $\{x_1, \dots, x_N\}$:

$$e_k(\mathbf{x}) := \sum_{i_1<\dots<i_k} x_{i_1} \cdots x_{i_k}. \tag{A.2}$$

Symmetric polynomials appear naturally in the factorization of a single-variable monic polynomial of degree N with roots $x_1, \dots, x_N$ [38]:

$$\begin{aligned}\prod_{i=1}^{N}(X - x_i) =& X^N - e_1(\mathbf{x})X^{N-1} + e_2(\mathbf{x})X^{N-2} - e_3(\mathbf{x})X^{N-3} \\ &+ \cdots + (-1)^N e_N(\mathbf{x}) = 0.\end{aligned} \tag{A.3}$$

The relation between elementary symmetric polynomials and power sums is given by the following result:

Newton's identity. The following identity holds true:

$$e_k(\mathbf{x}) = \frac{1}{k}\sum_{i=1}^{k}(-1)^{i-1} e_{k-i}(\mathbf{x}) f_i(\mathbf{x}). \tag{A.4}$$

From Newton's identity (A.4), it is possible to express the elementary symmetric polynomials in terms of the power sums:

$$e_k(\mathbf{x}) = \frac{1}{k!}\det\begin{bmatrix} f_1(\mathbf{x}) & 1 & 0 & \cdots & 0 \\ f_2(\mathbf{x}) & f_1(\mathbf{x}) & 2 & & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ f_{k-1}(\mathbf{x}) & f_{k-2}(\mathbf{x}) & & \ddots & k-1 \\ f_k(\mathbf{x}) & f_{k-1}(\mathbf{x}) & \cdots & \cdots & f_1(\mathbf{x}) \end{bmatrix}. \tag{A.5}$$

Lastly, we observe that Eq. (A.5) can be expressed through the so-called *complete exponential Bell polynomials*, an infinite family of inhomogeneous polynomials of total degree k in k variables, explicitly given by Comtet [40, Section 3.3]:

$$B_k(y_1, \dots, y_k) := k! \sum_{1j_1+\dots+kj_k=k} \prod_{i=1}^{k} \frac{y_i^{j_i}}{(i!)^{j_i} j_i!}. \tag{A.6}$$

Indeed, through the properties of determinants, it is possible to see that we can rewrite formula (A.5) through (A.6) as:

$$e_k(\mathbf{x}) = \frac{(-1)^k}{k!} B_k\big(-f_1(\mathbf{x}),\ -1!f_2(\mathbf{x}),\ -2!f_3(\mathbf{x}),\ \dots,\ -(k-1)!f_k(\mathbf{x})\big). \tag{A.7}$$

Let us now recall that given a $N \times N$ matrix A, its trace is a sum of its eigenvalues:

$$\operatorname{tr} A = \lambda_1 + \cdots + \lambda_N. \tag{A.8}$$

This readily implies that, for all $k \in \mathbb{N}$, we have the identity:

$$\operatorname{tr}(A^k) = \sum_{i=1}^{N} \lambda_i^k = f_k(\lambda), \tag{A.9}$$

where we used the power sum notation (A.1).

Next, let us apply the Cayley–Hamilton theorem Loehr [32] to the matrix A. This famous result states that a matrix solves its own characteristic equation:

$$A^N - e_1(\lambda)A^{N-1} + e_2(\lambda)A^{N-2} + \cdots + (-1)^{N-1}e_{N-1}(\lambda)A + (-1)^N e_N(\lambda)\mathbb{1}_N = \mathbb{0}_N, \tag{A.10}$$

where $\mathbb{0}_N$ and $\mathbb{1}_N$ are zero and identity matrices of size $N \times N$ respectively, and we used (A.3) to express the coefficient of the characteristic polynomial in terms of its roots, namely the eigenvalues $\lambda = (\lambda_1, \dots, \lambda_N)$. Now, multiplying the relation (A.10) by the matrix A^s for $s \in \mathbb{N}$ and taking the traces, using (A.9), we obtain:

$$f_{N+s} = \sum_{i=1}^{N} (-1)^{i+1} e_i(\lambda) f_{N+s-i}. \tag{A.11}$$

Therefore, using (A.7), we arrive at:

$$f_{N+s} = -\sum_{k=1}^{N} \frac{1}{k!} B_k\left(-0!f_1, -1!f_2, -2!f_3, \dots, -(k-1)!f_k\right) f_{N+s-k}, \tag{A.12}$$

for $s > 0$. The formula (A.12) allows us to express f_{N+s} in terms of $f_1, \dots, f_{N+s-1}$. Thus, by applying this formula recursively, one can express f_{N+s} in terms of $\{ f_1, \dots, f_N \}$, see also Lavoie [41].

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