



On a class of abelian groups with sufficiently many T -sequences

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To Dmitri Shakhmatov in honour of his 65th birthday.

Abstract. We describe the one-to-one sequences $\mathbf{u}$ in an abelian group G that have a T -subsequence, and we show that they are the same as those that admit a TB -subsequence. In terms of the Zariski topology, these are precisely the sequences $\mathbf{u}$ such that 0 belongs to the Zariski closure of the supporting set of $\mathbf{u}$. Additionally, we extend this result to $T^\mathcal{I}$ - and $TB^\mathcal{I}$ -subsequences defined using $\mathcal{I}$ -convergence where $\mathcal{I}$ is an ideal of $\mathbb{N}_0$. As a corollary, we describe completely the class $\mathfrak{T}$ (resp., $\mathfrak{T}_B$, $\mathfrak{T}^\mathcal{I}$, $\mathfrak{T}_B^\mathcal{I}$) of abelian groups in which every one-to-one sequence has a T -subsequence (resp., TB -, $T^\mathcal{I}$ -, $TB^\mathcal{I}$ -subsequence). These are precisely the abelian groups that are either almost torsion-free or of prime exponent, in other words, the groups with cofinite Zariski topology [5].

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Notation and terminology. We denote by $\mathbb{N}_0 = \{0, 1, \dots\}$ and $\mathbb{P}$ the sets of natural numbers and prime numbers, respectively, while $\mathbb{N} = \mathbb{N}_0 \setminus \{0\}$. Further, we denote by $\mathbb{Z}$ the integers, by $\mathbb{Q}$ the rationals, by $\mathbb{R}$ the reals, by $\mathbb{T}$ the unit circle group in $\mathbb{C}$, by $\mathbb{Z}(n)$ the cyclic group of order n for $n \in \mathbb{N}$. The cardinality of the continuum 2^ω will also be denoted by $\mathfrak{c}$. For subsets Y, Z of a set X , we briefly write $Y \subseteq^* Z$ to denote that $|Y \setminus Z| < \infty$.

For a sequence $\mathbf{u} = (u_n)$ in an abelian group G and $m \in \mathbb{Z}$, we denote by $m\mathbf{u}$ the sequence (mu_n) . Most often the sequences $\mathbf{u}$ will be one-to-one, but in case G is not torsion-free, $m\mathbf{u}$ need not be one-to-one even if $\mathbf{u}$ is one-to-one. We denote by $|\mathbf{u}|$ the *supporting set* $U = \{u_n : n \in \mathbb{N}_0\}$ regardless of whether

$\mathbf{u}$ is or is not one-to-one. If $\mathbf{v}$ is a subsequence of $\mathbf{u}$, considering $\mathbf{u}$ as a function $\mathbb{N}_0 \rightarrow G$, we write $\mathbf{u}^{-1}(|\mathbf{v}|) = \{n \in \mathbb{N}_0 : u_n \in |\mathbf{v}|\}$.

All groups considered in this paper, unless otherwise stated (e.g., in §2), will be abelian, so additive notation will always be used. For a subset A of an abelian group G , we denote by $\langle A \rangle$ the subgroup of G generated by A . The abelian group G is *divisible* if for every $g \in G$ and positive $n \in \mathbb{N}$, the equation $nx = g$ has a solution in G . Also, we set $nG = \{nx : x \in G\}$ and $G[n] = \{x \in G : nx = 0\}$ for $n \in \mathbb{N}$ and $\text{Soc}(G) = \bigoplus_{p \in \mathbb{P}} G[p]$; moreover, $r(G)$ denotes the free rank of G and $r_p(G)$ its p -rank when $p \in \mathbb{P}$.

All *topological* groups considered in this paper are Hausdorff, and so are Tychonoff spaces. We denote by $\psi(G)$ the pseudocharacter of a topological group G and by $w(G)$ its weight, so if G is finite, one has $w(G) = |G|$. The Pontryagin dual of a topological abelian group G is denoted by $\widehat{G}$, while $\widetilde{G}$ denotes its completion. Then G is *precompact* if $\widetilde{G}$ is compact. For undefined symbols or notions, see [3, 12, 23, 24, 26].

1. Introduction. This paper has two main ingredients: T -sequences and the Zariski topology. The second one is essentially used to resolve a problem concerning the first one (see Question 1.6). We very briefly introduce them below.

1.1. T -sequences and TB -sequences. Following [4, 34], call a sequence $\mathbf{u} = (u_n)$ in a group G a T -sequence (resp., TB -sequence) if there is a Hausdorff (resp., precompact) group topology on G in which $\mathbf{u}$ converges to 0 (here G need not be abelian [34, Definition 3.2.1], but in this paper T -sequences and TB -sequences will appear mainly in the abelian context). For a T - (resp., TB -) sequence $\mathbf{u}$, there is a finest Hausdorff (resp., precompact) group topology $T_{\mathbf{u}}$ (resp., $T_{p\mathbf{u}}$) on G in which $\mathbf{u}$ converges to 0.

Since every precompact group topology is Hausdorff, every TB -sequence is a T -sequence. By [8, Example 1.13(d)], in every infinite abelian group there exists a non-trivial TB -sequence. It was proved earlier that every infinite abelian group admits a non-trivial T -sequence [34, Theorem 2.1.7].

Example 1.1. Here we provide examples and non-examples of T - and TB -sequences.

(a) It is well known that (p^n) and $(n!)$, as well as the Fibonacci sequence (f_n) , are TB -sequences in $\mathbb{Z}$. Moreover, if the ratios u_{n+1}/u_n of a sequence $\mathbf{u} = (u_n)$ in $\mathbb{Z}$ diverge to ∞ , then $\mathbf{u}$ is a TB -sequence [4].

(b) (n^2) and (n^3) are not even T -sequences in $\mathbb{Z}$ [2, Example 5.3.3]. One can show that the sequences $(p^n + 1)$ and $(f_n + 1)$ are not T -sequences either (for this and a more general source of non- T -sequences, see [10]).

(c) As Hans Weber kindly pointed out, the sequence (p_n) of all primes is not a TB -sequence. In fact, (p_n) is not even a T -sequence: by [33, Theorem 4(i)], $\liminf_{n \rightarrow \infty} (p_{n+1} - p_n) \leq 246$; then there exist an integer $0 < d \leq 246$ and a subsequence (p_{n_k}) of (p_n) such that $p_{n_k+1} - p_{n_k} = d$ for all $k \in \mathbb{N}_0$.

(d) In [34] (cf., also [2, Proposition 14.4.18]), an example of a T -sequence which is not a TB -sequence is given. In [9], we find a new class of T -sequences

$\mathbf{u}$ in $\mathbb{Z}$ such that $(\mathbb{Z}, T_{\mathbf{u}})$ is minimally almost periodic (i.e., $(\widehat{\mathbb{Z}}, T_{\mathbf{u}})$ is trivial), so these $\mathbf{u}$ are not TB -sequences.

Recently the concepts of T - and TB -sequence were generalized using ideal convergence in [3] as follows. Recall that an *ideal* $\mathcal{I}$ on $\mathbb{N}_0$ is a subset of the power set of $\mathbb{N}_0$ which is closed under taking subsets and finite unions. The leading example of an ideal on $\mathbb{N}_0$ is the family $\mathcal{Fin}$ of all finite subsets of $\mathbb{N}_0$. Following Cartan [7], for an ideal $\mathcal{I}$ on $\mathbb{N}_0$, a sequence $\mathbf{u} = (u_n)$ in a topological group (G, τ) is said to $\mathcal{I}$ -converge to 0 if $\{n \in \mathbb{N}_0 : u_n \notin U\} \in \mathcal{I}$ for every neighborhood U of 0 in τ . Of course, usual convergence coincides exactly with $\mathcal{Fin}$ -convergence (see [7]).

In order to avoid trivialities, we always suppose an ideal $\mathcal{I}$ to be *proper* (i.e., $\mathcal{I} \neq \mathcal{P}(\mathbb{N}_0)$) and to be *free* (i.e., $\mathcal{Fin} \subseteq \mathcal{I}$). Assuming $\mathcal{Fin} \subseteq \mathcal{I}$, every convergent sequence is also $\mathcal{I}$ -convergent.

Remark 1.2. An ideal $\mathcal{I}$ on $\mathbb{N}_0$ is called a $\mathcal{P}$ -ideal if for any sequence $(I_n)_{n \in \mathbb{N}_0}$ in $\mathcal{I}$, there exists $J \in \mathcal{I}$ such that $I_n \setminus J$ is finite for all $n \in \mathbb{N}_0$. In such a case, a sequence $\mathbf{u}$ in a first-countable topological group (G, τ) $\mathcal{I}$ -converges to 0 if and only if there exists $A \in \mathcal{I}$ such that, with $M = \mathbb{N}_0 \setminus A$, $\lim_{n \in M} u_n = 0$ [29].

Definition 1.3 ([3]). Let $\mathcal{I}$ be an ideal on $\mathbb{N}_0$. A sequence $\mathbf{u}$ in an abelian group G is a $T^{\mathcal{I}}$ -sequence (resp., $TB^{\mathcal{I}}$ -sequence) if there is a Hausdorff (resp., precompact) group topology τ on G such that $u_n \xrightarrow{\tau} 0$ in (G, τ) .

1.2. The Zariski topology and its applications. Let $\mathcal{P}$ be a property of topological groups, we shall say that an abstract group G is *virtually* $\mathcal{P}$ if it admits a Hausdorff group topology τ such that (G, τ) has the property $\mathcal{P}$. A couple of examples will appear below.

The Zariski topology $\mathfrak{Z}_G$ (introduced in detail in §2) has already proved to be a powerful tool in the solution of various problems involving group topologies with some desired property $\mathcal{P}$, i.e., *virtually* $\mathcal{P}$. We briefly recall three of Markov's problems which triggered the birth of the Zariski topology:

- Describe the groups admitting non-discrete Hausdorff group topologies (that is, *virtually non-discrete groups*).
- Describe the abelian groups admitting connected Hausdorff group topologies (that is, *virtually connected groups*).
- Describe the potentially dense subsets of the abelian groups (i.e., those sets that become dense in some Hausdorff group topology).

In all three cases, there is a very simple necessary condition that ensures the required virtual $\mathcal{P}$ property of the group (or its subsets, in the third case), as we explain now.

(a) Let $\mathcal{P}$ be the property “non-discrete”; then the *virtually* $\mathcal{P}$ groups are precisely the *topologizable groups*. Since finite groups cannot be *virtually non-discrete*, Markov [31] asked whether every infinite group is *virtually non-discrete*, i.e., *topologizable*. This Markov problem was solved in the negative only after 40 years with the joint efforts of Ol'shanskij [32] and Shelah [36]

who found various examples of infinite non-topologizable groups (see §2 for the precise connection with the Zariski topology).

(b) If an abelian group G becomes connected in some Hausdorff group topology τ , then $[G : H] \geq \mathfrak{c}$ for every proper τ -closed subgroup H of G . Consequently, every proper Zariski closed subgroup H of G must have index $[G : H] \geq \mathfrak{c}$. So the easy necessary condition for G to be virtually connected becomes:

(M) Every proper Zariski closed subgroup of G must have index $\geq \mathfrak{c}$.

Markov asked if (M) is also a sufficient condition, an affirmative answer was given in [19] in the abelian case.

(c) Since $\mathfrak{Z}_G$ is coarser than any Hausdorff group topology on an abstract group G , every potentially dense subset is also Zariski dense. It is proved in [18] that this condition (i.e., Zariski density) is also sufficient in case $|G| \leq 2^{\mathfrak{c}}$, resolving in this way Markov's problem for groups with $|G| \leq 2^{\mathfrak{c}}$. Further results in the general case can be found in [16].

Item (c) (specialized in the case when all infinite subsets are potentially dense) reveals the relevance of the class $\mathcal{C}$ of (not necessarily abelian) infinite groups with cofinite Zariski topology recently studied in [5] in full detail. The question of whether the class $\mathcal{C}$ may contain non-abelian groups is currently open, actually it is the main conjecture of [5]:

Conjecture 1.4 ([5, Conjecture 8.12]). *Every group in $\mathcal{C}$ is necessarily abelian.*

Remark 1.5. Denote by $\mathcal{G}$ (resp., $\mathcal{Ab}$) the class of all infinite (resp., infinite abelian) groups and put $\mathcal{S} = (\mathcal{G} \setminus \mathcal{C}) \cup (\mathcal{C} \cap \mathcal{Ab}) = \mathcal{G} \setminus (\mathcal{C} \setminus \mathcal{Ab})$, i.e., $\mathcal{S}$ is the class of all groups G such that $G \in \mathcal{C}$ implies $G \in \mathcal{Ab}$, so obviously $\mathcal{Ab} \subseteq \mathcal{S}$. Hence, the answer to Conjecture 1.4 is positive if and only if $\mathcal{S} = \mathcal{G}$.

The results we quote now from [5] witness how large $\mathcal{S}$ is. According to [5, Theorem 2.7], $G \in \mathcal{C}$ if and only if $H \in \mathcal{C}$ for every countable subgroup H of G . By [5, Corollary 5.13], $\mathcal{S}$ contains all locally finite and all locally solvable groups, while it can be deduced from [5, Corollary 5.13] that infinite linear groups and abelian-by-finite groups belong to $\mathcal{S}$, as well as finitely generated, abelian-by-nilpotent-by-finite groups. Furthermore, $\mathcal{S}$ contains also all solvable-by-solvable-by-finite groups by [5, Theorem 5.8(b)].

Summing up, one can observe the following common phenomenon in all three cases (a), (b), and (c) above: for some topological property $\mathcal{P}$, in order to have “virtually $\mathcal{P}$ ” for an abstract group G , one has to ensure that $(G, \mathfrak{Z}_G)$ has some property $\mathcal{P}_3$ very closely related to $\mathcal{P}$. In this paper, we provide new evidence in this line: a fixed one-to-one sequence $\mathbf{u}$ in an abelian group G has a TB -subsequence precisely when it has a Zariski null subsequence (see Remarks 1.8 and 1.13 for more detail).

1.3. Main results. Motivated by the fact that every one-to-one sequence in $\mathbb{Z}$ has a TB -subsequence (pick a very rapidly increasing subsequence and apply Example 1.1(a)), we devote this paper to answer the following question.

Question 1.6. Let G be an abelian group and let $\mathcal{I}$ be an ideal on $\mathbb{N}_0$.

- (a) When does a (fixed) one-to-one sequence $\mathbf{u}$ in G have a T -subsequence (resp., TB -subsequence, $T^{\mathcal{I}}$ -subsequence, $TB^{\mathcal{I}}$ -subsequence)?
 (b) When do *all* one-to-one sequences $\mathbf{u}$ in G have these properties?

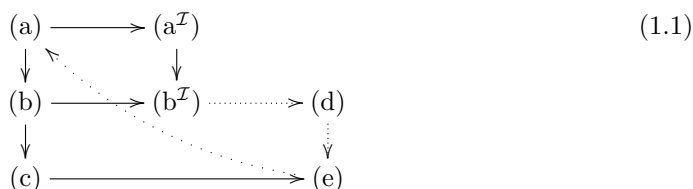
The condition imposed on the sequence $\mathbf{u}$ to be one-to-one is not restrictive since it is necessary. Indeed, if $\mathbf{u}$ has no one-to-one subsequences, then $|\mathbf{u}|$ is finite, so obviously no subsequence of $\mathbf{u}$ can be a T -sequence.

The answer to Question 1.6(a) is given by the following theorem.

Theorem 1.7. *For a one-to-one sequence $\mathbf{u}$ consisting of non-zero elements of an abelian group G and a proper free ideal $\mathcal{I}$ on $\mathbb{N}_0$, the following conditions are equivalent:*

- (a) $\mathbf{u}$ has a TB -subsequence;
 (a $^{\mathcal{I}}$) $\mathbf{u}$ has a $TB^{\mathcal{I}}$ -subsequence;
 (b) $\mathbf{u}$ has a T -subsequence;
 (b $^{\mathcal{I}}$) $\mathbf{u}$ has a $T^{\mathcal{I}}$ -subsequence;
 (c) $\mathbf{u}$ has a subsequence which is a null sequence in $(G, \mathfrak{Z}_G)$;
 (d) $\mathbf{u}$ has a subsequence $\mathbf{u}'$ such that for every finite collection $\mathbf{v}_1, \dots, \mathbf{v}_n$ of subsequences of $\mathbf{u}'$ with:
 (d₁) there exist $m_1, \dots, m_n \in \mathbb{N}$ such that $m_k \mathbf{v}_k$ is a constant sequence for every $k \in \{1, \dots, n\}$,
 (d₂) $|\mathbf{u}'| \subseteq^* |\mathbf{v}_1| \cup \dots \cup |\mathbf{v}_n|$,
 there exists $k \in \{1, \dots, n\}$ such that $m_k \mathbf{v}_k$ is the zero constant sequence;
 (e) $0 \in cl_{\mathfrak{Z}_G}(|\mathbf{u}|)$.

In the following diagram, the solid arrows representing the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (e) as well as the remaining three solid ones forming the rectangle in the upper-left corner of the diagram below, trivially hold.



The implication (e) $\Rightarrow$ (a) is proved in Proposition 4.5, whereas the remaining two implications (b $^{\mathcal{I}}$) $\Rightarrow$ (d) $\Rightarrow$ (e) are proved in §5.

Remark 1.8. (a) As a consequence of the equivalence (a) $\Leftrightarrow$ (c) in the above theorem, we obtain a remarkable connection of $\mathfrak{Z}_G$ with precompact group topologies on an abelian group G , namely: *If a one-to-one sequence $\mathbf{u}$ in G has a null subsequence in $(G, \mathfrak{Z}_G)$, then it has a null subsequence also in some precompact group topology τ on G , i.e., $\mathbf{u}$ has a TB -subsequence.*

(b) Let us discuss now the rectangle in the upper-left corner of the diagram (1.1). Its two horizontal arrows (as well as *all* arrows of the diagram (1.1)) are invertible (equivalences) due to the full force of the theorem. Nevertheless, to obtain (a $^{\mathcal{I}}$) $\Rightarrow$ (a) and (b $^{\mathcal{I}}$) $\Rightarrow$ (b), one needs a “long walk” along the diagram.

We show in Lemma 5.2 that both arrows are invertible (equivalences) without using the full strength of the theorem when $\mathcal{I}$ is a P -ideal. Moreover, the

(precompact) group topology witnessing (b) (resp., (a)) remains *the same* as the topology witnessing (b ^{$\mathcal{I}$}) (resp., (a ^{$\mathcal{I}$})), although the subsequence in (b) (resp., (a)) is not the same as in (b ^{$\mathcal{I}$}) (resp., (a ^{$\mathcal{I}$})) – it is just a subsequence of the respective subsequence.

1.4. The classes $\mathfrak{T}$, $\mathfrak{T}_B$, $\mathfrak{T}^{\mathcal{I}}$, and $\mathfrak{T}_B^{\mathcal{I}}$. The above theorem enables us to describe the following classes of abelian groups:

$\mathfrak{T} = \{G \text{ abelian group} : \text{every one-to-one sequence } \mathbf{u} \text{ in } G \text{ has a } T\text{-subsequence}\},$

$\mathfrak{T}^{\mathcal{I}} = \{G \text{ abelian group} : \text{every one-to-one sequence } \mathbf{u} \text{ in } G \text{ has a } T^{\mathcal{I}}\text{-subsequence}\},$

$\mathfrak{T}_B = \{G \text{ abelian group} : \text{every one-to-one sequence } \mathbf{u} \text{ in } G \text{ has a } TB\text{-subsequence}\},$

$\mathfrak{T}_B^{\mathcal{I}} = \{G \text{ abelian group} : \text{every one-to-one sequence } \mathbf{u} \text{ in } G \text{ has a } TB^{\mathcal{I}}\text{-subsequence}\}.$

Obviously, $G \in \mathfrak{T}_B$ if and only if G satisfies condition (a) of Theorem 1.7 for every one-to-one sequence $\mathbf{u}$ in G ; similarly, $G \in \mathfrak{T}$, $G \in \mathfrak{T}^{\mathcal{I}}$, and $G \in \mathfrak{T}_B^{\mathcal{I}}$ correspond respectively to the conditions (b), (b ^{$\mathcal{I}$}), and (a ^{$\mathcal{I}$}). This explains the labels (A), (B), (A ^{$\mathcal{I}$}), and (B ^{$\mathcal{I}$}) in Theorem 1.12.

In Theorem 1.12, we obtain a complete description of the classes $\mathfrak{T}$, $\mathfrak{T}_B$, $\mathfrak{T}^{\mathcal{I}}$, and $\mathfrak{T}_B^{\mathcal{I}}$ by showing that they all coincide with $\mathcal{C}$. This is why we are left with the description of the class $\mathcal{C}$ (see Fact 1.11) based on the following key concept:

Definition 1.9. An abelian group G is called *almost torsion-free* if $r_p(G)$ is finite for every prime p .

The following remark is intended to give an alternative look at this class:

Remark 1.10. Clearly, an abelian group G is almost torsion-free if and only if $t(G)$ is almost torsion-free. Since the p -rank does not increase when taking the divisible hull $D(G)$ of G , we deduce that G is almost torsion-free if and only if $D(G)$ is almost torsion-free. This occurs precisely when there exist a (possibly infinite) cardinal κ and a sequence $(k_p)_{p \in \mathbb{P}}$ in $\mathbb{N}_0$ such that $D(G) \cong \mathbb{Q}^{(\kappa)} \times \bigoplus_{p \in \mathbb{P}} \mathbb{Z}(p^\infty)^{k_p}$.

One has the following simple description of the class $\mathcal{C}$ unifying two apparently contradictory properties of abelian groups (see [14, 17], [38, Theorem 5.1] and also [2, §4]). For the benefit of the reader, we include a proof of this fact in Lemma 4.1.

Fact 1.11 ([17, Fact 3.9]). *An abelian group G is either almost torsion-free or of prime exponent precisely when $\mathfrak{Z}_G$ is the cofinite topology on G .*

Now Theorem 1.12 answers Question 1.6(b) and shows, among others, that $\mathfrak{T} = \mathfrak{T}^{\mathcal{I}} = \mathfrak{T}_B = \mathfrak{T}_B^{\mathcal{I}} = \mathcal{C}$, namely, imposing the equivalent conditions of Theorem 1.7 to all one-to-one sequences in an abelian group G brings to the very restrictive property $G \in \mathcal{C}$ (i.e., G must be almost torsion-free or of prime exponent):

Theorem 1.12. *The following are equivalent for an abelian group G and a proper free ideal $\mathcal{I}$ on $\mathbb{N}_0$:*

(A) $G \in \mathfrak{T}_B$;

- (A ^{$\mathcal{T}$}) $G \in \mathfrak{T}_B^{\mathcal{T}}$;
- (B) $G \in \mathfrak{T}$;
- (B ^{$\mathcal{T}$}) $G \in \mathfrak{T}^{\mathcal{T}}$;
- (C) $\mathfrak{Z}_G$ is the cofinite topology (i.e., $G \in \mathcal{C}$);
- (C*) G is either almost torsion-free or of prime exponent.

The following diagram and remark illustrate the scheme of the proof of this theorem.

$$\begin{array}{ccc}
 (A) & \longrightarrow & (A^{\mathcal{T}}) \\
 \downarrow & \nearrow & \downarrow \\
 (B) & \longrightarrow & (B^{\mathcal{T}}) \\
 \vdots & \searrow & \vdots \\
 (C) & \longleftrightarrow & (C^*)
 \end{array} \tag{1.2}$$

Remark 1.13. The four solid arrows in the upper rectangle of the diagram are straightforward. The equivalence $(C) \Leftrightarrow (C^*)$ is proved in Lemma 4.1, while $(B^{\mathcal{T}}) \Rightarrow (C^*)$ is proved in Lemma 4.3. The remaining (and hardest) implication $(C^*) \Rightarrow (A)$ is proved in Lemma 4.4 and Corollary 4.6.

To conclude, we give here a direct short proof of $(B) \Rightarrow (C)$ without the assumption that the group G is abelian. We have to check that imposing (B), every countably infinite subset X of G must be $\mathfrak{Z}_G$ -dense. Fix arbitrarily $x \in G$. To prove that $x \in cl_{\mathfrak{Z}_G}(X)$, enumerate arbitrarily X as a one-to-one sequence $(x_n) = \{x_1, x_2, \dots\}$. This provides a new one-to-one sequence $\mathbf{u} = (u_n)$ of G defined by $u_n = x_n x^{-1}$. Then (B) gives a T -subsequence $\mathbf{v} = (u_{n_k})$ of $\mathbf{u}$. As $\mathbf{v}$ converges to e_G in the Hausdorff topology $T_{\mathbf{v}} \geq \mathfrak{Z}_G$, we deduce that $u_{n_k} \rightarrow e_G$ in $(G, \mathfrak{Z}_G)$. As right translations are continuous in $(G, \mathfrak{Z}_G)$, it follows that $x_{n_k} = u_{n_k} x \rightarrow x$ in $(G, \mathfrak{Z}_G)$. So $x \in cl_{\mathfrak{Z}_G}(X)$. Therefore, if G satisfies (B), then $G \in \mathcal{C}$. In the notation of Remark 1.5, this implies that if G satisfies (B) and $G \in \mathcal{P}$, then G is abelian so G satisfies all items of Theorem 1.12.

Question 1.14. Does the implication $(C) \Rightarrow (B)$ hold for any group G ?

Due to Theorem 1.12, the answer to this question is positive modulo a positive answer to Conjecture 1.4.

From item (C) of Theorem 1.12 and Fact 1.11, we deduce that the classes $\mathfrak{T}_B$, etc. are stable under taking subgroups, but not stable under finite direct products or quotients.

The paper is organized as follows. In §2, background on Zariski topology is provided (here we explain why Theorem 1.7 fails in the non-abelian case), while in §3, we recall some useful facts on T -sequences and metrizability. In §4, we propose a proof of Theorem 1.12 independent from Theorem 1.7. Finally, §5 contains the proof of Theorem 1.7.

2. Background on Zariski topology.

2.1. Definition and historical background on Zariski and Markov topologies.

Markov introduced the following four special families of subsets of a group G :

Definition 2.1 ([30]). A subset X of a group G is called:

- (a) *elementary algebraic* if there exist an integer $n > 0$, elements $g_1, \dots, g_n \in G$, and $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ such that $X = \{x \in G : g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \dots g_n x^{\varepsilon_n} = e_G\}$;
- (b) *algebraic* if X is an intersection of finite unions of elementary algebraic subsets of G ;
- (c) *unconditionally closed* if X is closed in *every* Hausdorff group topology on G ;
- (d) *potentially dense* if X is dense in *some* Hausdorff group topology on G .

Clearly, the algebraic sets are unconditionally closed. The topology on a group G having as open sets the complements of algebraic sets is called *Zariski topology* and it is denoted by $\mathfrak{Z}_G$ (it was explicitly introduced by Bryant [6] who called it *verbal topology*). The shifts $x \mapsto xa$, $x \mapsto ax$ and the inversions $x \mapsto x^{-1}$ are continuous in the Zariski topology, so $(G, \mathfrak{Z}_G)$ is a *quasi-topological group* in the sense of [1].

Remark 2.2. Since we are working mainly with abelian groups, we recall that the Zariski topology for an abelian group G looks much simpler, as the elementary algebraic sets have the form $E_{a,n} := \{x \in G : nx = a\}$ for some $n \in \mathbb{N}$ and $a \in G$. Clearly, $E_{a,n} \neq \emptyset$ if and only if $a \in nG$, in such a case $E_{a,n} = b + G[n]$ for any $b \in E_{a,n}$. So, the cosets of the subgroups $G[n]$, $n \in \mathbb{N}$, form a prebase of the $\mathfrak{Z}_G$ -closed sets of G .

Moreover, the algebraic sets are simply the finite unions of elementary algebraic ones [6, 17].

Definition 2.3. Let G be a group. Denote by:

- (a) $\mathfrak{M}_G$ the infimum of all Hausdorff group topologies on G and call it the *Markov topology* of G ;
- (b) $\mathcal{P}_G$ the infimum of all precompact group topologies on G , call it the *precompact Markov topology* of G .

Obviously, $\mathfrak{Z}_G \leq \mathfrak{M}_G \leq \mathcal{P}_G$, as Zariski closed sets are closed in any Hausdorff group topology, as mentioned above. An important fact established in [17] is that these three topologies coincide when G is abelian. The equality $\mathfrak{Z}_G = \mathfrak{M}_G$ in this case was proved also by Sipacheva [37], while Markov [30] proved that $\mathfrak{Z}_G = \mathfrak{M}_G$ for countable groups G . Toller and the first named author [21] showed that the inequality $\mathfrak{M}_G \neq \mathcal{P}_G$ may occur for some linear groups that are nilpotent of class 2. Recently Shakhmatov and Yañez [35] established that $\mathfrak{Z}_G = \mathfrak{M}_G \neq \mathcal{P}_G$ for free groups G . More information can be found in Remark 2.4 and the papers [14, 15, 17–19], as well as in the surveys [14, 20, 22].

Theorem 1.7 fails in the non-abelian context for groups that admit no precompact topology but admit non-discrete metrizable ones (e.g., the countable nilpotent class 2 groups G with discrete $\mathcal{P}_G$ described in [21]). Indeed, such a group G obviously has some T -sequence $\mathbf{u}$, but $\mathbf{u}$ (as well as any other one-to-one sequence in G) does not have $TB^{\mathcal{I}}$ -subsequences. Similar examples can be built in free groups, using the results from [35].

Remark 2.4. (a) Markov did not introduce any of the above recalled three topologies, he formulated his problems by only using the terms “algebraic set” and “unconditionally closed set”. The Markov topology (introduced in [13–15]) is very useful for a short formulation of his topologization problem: *a group G is topologizable if and only if $\mathfrak{M}_G$ is non-discrete*. A necessary condition for this is the non-discreteness of $\mathfrak{Z}_G$, easily described in algebraic terms; this explains the utility of $\mathfrak{Z}_G$, especially in groups with $\mathfrak{M}_G = \mathfrak{Z}_G$.

(b) Clearly a group possessing a one-to-one T -sequence is topologizable. Using this simple fact, it was proved in [34, Theorem 3.2.4] that a countable group G with non-discrete $\mathfrak{Z}_G$ is topologizable (this follows also by the more general fact that $\mathfrak{Z}_G = \mathfrak{M}_G$ in this case). The topology $\mathfrak{Z}_G$ does not explicitly appear in [34].

(c) In the above terms, the result proved in [38, Theorem 5.1] is actually much stronger than that formulated in Fact 1.11: *An abelian group G is either almost torsion-free or of prime exponent precisely when every proper unconditionally closed subset of G is finite* (i.e., when $\mathfrak{M}_G$ is the cofinite topology on G).

Example 2.5. We add here another relevant example of a virtual property related to Zariski–Markov topologies.

The *compact Markov topology* $\mathfrak{C}_G$ of a group G was defined in [13] as the infimum of all compact group topologies on G . Obviously, an infinite group G admits compact Hausdorff group topologies (i.e., G is virtually compact) precisely when $\mathfrak{C}_G$ is non-discrete. In 1944, Halmos proved that the group of the reals $\mathbb{R}$ is virtually compact, by producing a compact group topology on $\mathbb{R}$ via the algebraic isomorphism between $\mathbb{R}$ and the Pontryagin dual of the discrete group $\mathbb{Q}$. He asked for a description of the virtually compact abelian groups. Kaplansky gave a partial (yet relevant) solution of Halmos’ problem, which was resolved in the late fifties of the last century by Harrison [25] and Hulanicki [27, 28].

2.2. Further background on the Zariski topology on abelian groups.

Remark 2.6. The fact that $\mathfrak{Z}_G = \mathfrak{M}_G = \mathcal{P}_G$ for an abelian group G implies that $T_{\mathbf{u}} \geq T_{p\mathbf{u}} \geq \mathfrak{Z}_G$ for every T -sequence $\mathbf{u}$ in G , hence $u_n \rightarrow 0$ with respect to $\mathfrak{Z}_G$. This explains the necessity of the condition $u_n \rightarrow 0$ with respect to $\mathfrak{Z}_G$ for a sequence $\mathbf{u}$ which is a T -sequence or a TB -sequence.

In the proofs below, we apply some useful facts, whose proofs can be found in [17]. The first one is somewhat folklore (although, not completely trivial):

Fact 2.7 ([17, Theorem 10.2]). *If H is a subgroup of an abelian group G , then every precompact group topology on H can be “extended” to a precompact group topology on G .*

If we need to build a TB -subsequence of some sequence $\mathbf{u}$ in G , we can assume that $\mathbf{u}$ generates G since by Fact 2.7 any precompact group topology on the subgroup H generated by $|\mathbf{u}|$ witnessing that some subsequence $\mathbf{v}$ of $\mathbf{u}$ is a TB -sequence in H can be “extended” to a precompact group topology of G witnessing that $\mathbf{v}$ is a TB -sequence in G as well.

The next theorem (actually, it is only a small part of [17, Theorem C]) shows that the Zariski closure of a single subset X of an abelian group G can be “realized” by a precompact metrizable topology on G :

Theorem 2.8 ([17, Theorem C]). *If G is an abelian group with $|G| \leq \mathfrak{c}$, then for every $X \subseteq G$, there exists a precompact metrizable group topology τ on G such that $cl_\tau(X) = cl_{\mathfrak{Z}_G}(X)$.*

Remark 2.9. Following [17, §1.5], call a countably infinite subset A of an abelian group G a $\mathfrak{Z}_G$ -atom if A has the cofinite topology (that is, the coarsest T_1 topology) as a subspace of $(G, \mathfrak{Z}_G)$. In these terms, $G \in \mathcal{C}$ if and only if every countably infinite subset of G is a $\mathfrak{Z}_G$ -atom.

The $\mathfrak{Z}_G$ -atoms determine the topological space $(G, \mathfrak{Z}_G)$ precisely as convergent sequences determine the topology of a Fréchet–Urysohn space. Indeed, $x \in cl_{\mathfrak{Z}_G}(X)$ for some $X \subseteq G$ precisely when either $x \in X$ or there exists a $\mathfrak{Z}_G$ -atom $A = \{a_n : n \in \mathbb{N}\} \subseteq X$ such that $\{x\} \cup A$ is still a $\mathfrak{Z}_G$ -atom (i.e., $a_n \rightarrow x$ with respect to $\mathfrak{Z}_G$). So, the quasi-topological group $(G, \mathfrak{Z}_G)$ is a Fréchet–Urysohn space, in particular, it is sequential ([17, Corollary 8.9]). The proof of Theorem 2.8 is heavily based on this property of $\mathfrak{Z}_G$ -atoms.

If $|G| \leq \mathfrak{c}$, this follows also from the above Theorem 2.8 since for a subset $X \subseteq G$ and $x \in cl_{\mathfrak{Z}_G}(X)$, there is a precompact metrizable topology τ on G such that $cl_\tau(X) = cl_{\mathfrak{Z}_G}(X)$. So, in particular, there exists a sequence (x_n) in X such that $x_n \rightarrow x$ in (G, τ) . Since $\mathfrak{Z}_G$ is coarser than τ , $x_n \rightarrow x$ also in $(G, \mathfrak{Z}_G)$.

3. T -sequences and metrizability. It is well known that the topology $T_{\mathbf{u}}$ is not metrizable if $\mathbf{u}$ is a T -sequence in an abelian group G [34]. However, in Proposition 3.2, we see that there always exists a (strictly) coarser metrizable group topology in which $\mathbf{u}$ converges to 0.

Lemma 3.1. *Let $(G, \mathcal{T})$ be an abelian topological group having an open countable subgroup G_0 . Then there exists a metrizable group topology τ on G coarser than $\mathcal{T}$.*

Proof. Since the countable subgroup G_0 is open, we deduce that $\psi(G) = \psi(G_0) = \omega$. According to a folklore fact (see for example [2, Corollary 5.2.20]), there exists a metrizable group topology on G coarser than $\mathcal{T}$. $\square$

Proposition 3.2. *If $\mathbf{u}$ is a T -sequence in an abelian group G , then there is a metrizable group topology τ on G strictly coarser than $T_{\mathbf{u}}$ such that $u_n \rightarrow 0$ in (G, τ) .*

Proof. Since $\langle u_n : n \in \mathbb{N}_0 \rangle$ is open in $(G, T_{\mathbf{u}})$, by Lemma 3.1, there exists a metrizable group topology τ on G coarser than $T_{\mathbf{u}}$. Since $u_n \rightarrow 0$ in $T_{\mathbf{u}}$, it also converges to 0 in the coarser topology τ . $\square$

Remark 3.3. The above proposition remains true (with a similar proof) also for $T^{\mathcal{I}}$ -sequences: if $\mathbf{u}$ is a $T^{\mathcal{I}}$ -sequence in a group G , then there is a metrizable group topology τ on G such that $u_n \xrightarrow{\mathcal{I}} 0$ in (G, τ) .

An analogue of Proposition 3.2 does not hold for TB -sequences due to the following natural restriction:

Lemma 3.4. *An abelian group G admits a precompact metrizable group topology if and only if $|G| \leq \mathfrak{c}$.*

Proof. If G admits a precompact group topology, then its completion $\tilde{G}$ is compact and metrizable. Since algebraically $\widehat{\tilde{G}} = \tilde{\widehat{G}}$, $\widehat{G}$ is countable. The map $G \rightarrow \mathbb{T}^{G^\wedge}$, $x \mapsto (\chi(x))$, is injective, hence $|G| \leq |\mathbb{T}^{G^\wedge}| = \mathfrak{c}$.

Let now G satisfy $|G| \leq \mathfrak{c}$. We see below that there exists an embedding $j: G \rightarrow \mathbb{T}^\mathbb{N}$. The group topology induced on G under the embedding j is precompact and metrizable.

The group $D = \mathbb{T}^\omega$ is divisible, with $r(D) = \mathfrak{c} = r_p(D)$ for every prime p . As $r(G) \leq \mathfrak{c} = r(D)$ and $r_p(G) \leq \mathfrak{c} = r_p(D)$ for every prime p , it suffices to check that there exists an embedding $G \rightarrow D \cong \left(\bigoplus_{\mathfrak{c}} \mathbb{Q} \right) \oplus \left(\bigoplus_{p \in \mathbb{P}} \bigoplus_{\mathfrak{c}} \mathbb{Z}(p^\infty) \right)$. Indeed, fix a free subgroup F of G such that G/F is torsion, then $r(F) = r(G)$. Obviously $F \cap \text{Soc}(G) = \{0\}$. The direct sum $H = F \oplus \text{Soc}(G)$ can obviously be embedded in D . Since D is divisible, this embedding extends to G . This extension is necessarily a monomorphism since H is essential in G (i.e., whenever $0 \neq x \in G$, there exists $m \in \mathbb{N}$ such that $0 \neq mx \in H$). $\square$

Corollary 3.5. *If $\mathbf{u}$ is a TB -sequence in an abelian group G of cardinality $\leq \mathfrak{c}$, then there is a precompact metrizable group topology τ on G such that $u_n \rightarrow 0$ in (G, τ) .*

Proof. Assume that $\mathbf{u}$ is a TB -sequence and $|G| \leq \mathfrak{c}$. By assumption, $T_{p\mathbf{u}}$ is a precompact group topology on G such that $u_n \rightarrow 0$ in $(G, T_{p\mathbf{u}})$. By Lemma 3.1 applied to $H := \langle u_n : n \in \mathbb{N}_0 \rangle$, there is a metrizable group topology τ_0 on H coarser than $T_{p\mathbf{u}}|_H$. In particular, τ_0 is precompact and $u_n \rightarrow 0$ in (H, τ_0) . Since $D_0 := (H, \tau_0)^\wedge$ is countable, there exists a countable subgroup $D_1 \leq \widehat{G}$ such that $D_0 = \{\chi|_H : \chi \in D_1\}$. By Lemma 3.4, there is a countable subgroup D_2 of $\widehat{G/H}$ separating the points of G/H . Let $D_3 := \{\psi \circ \pi : \psi \in D_2\}$, where $\pi: G \rightarrow G/H$ is the canonical projection. Then the subgroup $D := D_1 + D_3$ of $\widehat{G}$ is countable and separates the points of G . Hence, the topology τ on G generated by the characters of D is precompact and satisfies $\tau|_H = \tau_0$, so $u_n \rightarrow 0$ in (G, τ) . $\square$

The following result witnesses the relevance of TB -sequences $\mathbf{u}$ and the respective topologies $T_{p\mathbf{u}}$.

Theorem 3.6 ([8]). *If G is a countable abelian group, then every precompact metrizable group topology τ on G has the form $T_{p\mathbf{u}}$ for some TB -sequence $\mathbf{u}$ in G .*

Proof. Let τ be a precompact metrizable group topology on G . The compact dual group K of the discrete abelian group G is metrizable and the dual H of (G, τ) is a countable subgroup of K since τ metrizable. Then, by [11], H is characterizable, i.e., there exists a sequence $\mathbf{u}$ in $K^\wedge = G$ such that

$H = \{h \in K : h(u_n) \rightarrow 0 \text{ in } \mathbb{T}\}$. Since τ is the initial topology of the characters in H , the fact that $h(u_n) \rightarrow 0$ in $\mathbb{T}$ for all $h \in H$ implies that $u_n \rightarrow 0$ in (G, τ) . Therefore, $\mathbf{u}$ is a TB -sequence in G . $\square$

4. Existence of T -subsequences and proof of Theorem 1.12. Here we discuss an *independent proof* of Theorem 1.12, that means not making use of Theorem 1.7. Clearly $\mathfrak{T}_B \subseteq \mathfrak{T} \subseteq \mathfrak{T}^I$ and $\mathfrak{T}_B \subseteq \mathfrak{T}_B^I \subseteq \mathfrak{T}^I$, so the following implications hold:

$$(A) \Rightarrow (A^I) \Rightarrow (B^I) \quad \text{and} \quad (A) \Rightarrow (B) \Rightarrow (B^I), \quad (4.1)$$

in view of the solid implications in the left upper corner of diagram (1.1) applied for every sequence $\mathbf{u}$.

Even if the next result is known (see Fact 1.11), we include a proof for the reader's convenience. We use the obvious fact that (C^*) for an abelian group G is equivalent to the property

$$\text{for every prime } p, \text{ the subgroup } G[p] \text{ is either finite or } G[p] = G. \quad (4.2)$$

Lemma 4.1. *For an abelian group G , $(C) \Leftrightarrow (C^*)$.*

Proof. If (C) holds, then $\mathfrak{Z}_G$ is the cofinite topology of G , so (4.2) (that is, (C^*)) obviously holds since each subgroup $G[p]$ is $\mathfrak{Z}_G$ -closed.

Now assuming (C^*) , we prove (C) . If G has prime exponent p , then the only proper elementary algebraically closed subsets of G are finite, so $\mathfrak{Z}_G$ is the cofinite topology of G and (C) holds.

Assume now that G is almost torsion-free. To see that $\mathfrak{Z}_G$ is the cofinite topology of G , it is enough to check that $|G[n]| < \infty$ for every $n \in \mathbb{N}$ by Remark 2.2. All subgroups $G[p]$, with p prime, are finite by (4.2); so $G[p^k]$ is finite for every $k \in \mathbb{N}$, as $G[p]$ is essential in $G[p^k]$. This implies that all subgroups $G[n]$, with $n \in \mathbb{N}$, are finite too since $G[n]$ is a finite direct sum of all $G[p^k]$ with $p^k \mid n$. $\square$

Our next and last objective is to prove the implications

$$(B^I) \Rightarrow (C^*) \Rightarrow (A). \quad (4.3)$$

Along with (4.1) and Lemma 4.1, this pair of implications will complete the proof of Theorem 1.12.

Fact 4.2. *Let G be an abelian group, let $\mathcal{I}$ be a free ideal on $\mathbb{N}_0$ and let $\mathbf{u}$ be a T^I -sequence (resp., TB^I -sequence) in G . Then:*

- (a) $m\mathbf{u} = (mu_n)$ is a T^I -sequence (resp., TB^I -sequence) in G for every $m \in \mathbb{N}$;
- (b) $\mathbf{u}^{-1}(|\mathbf{v}|) \in \mathcal{I}$ for every subsequence $\mathbf{v}$ of $\mathbf{u}$ which is a non-zero constant.

Proof. Let τ be a Hausdorff (resp., precompact) topology on G witnessing the fact that $\mathbf{u}$ is a T^I -sequence (resp., TB^I -sequence).

(a) Use the fact that the multiplication by m is a continuous homomorphism and continuous homomorphisms preserve the property of being a T^I -sequence (resp., TB^I -sequence).

(b) Assume that $\mathbf{v}$ is a non-zero constant subsequence of $\mathbf{u}$. Pick a neighborhood U of 0 in τ such that U avoids the constant sequence $\mathbf{v}$. Then $\mathbf{u} \xrightarrow{I} 0$ implies that $\mathbf{u}^{-1}(|\mathbf{v}|) \subseteq \{n \in \mathbb{N} : u_n \notin U\} \in \mathcal{I}$, so $\mathbf{u}^{-1}(|\mathbf{v}|) \in \mathcal{I}$. $\square$

The next lemma shows the first implication in (4.3), and in particular that the class $\mathfrak{T}^{\mathcal{I}}$ is quite narrow.

Lemma 4.3. *Let G be an abelian group and let $\mathcal{I}$ be a proper free ideal on $\mathbb{N}_0$. If $G \in \mathfrak{T}^{\mathcal{I}}$, then either G is almost torsion-free or G has prime exponent.*

In particular, if $G \in \mathfrak{T}^{\mathcal{I}}$ for an infinite torsion abelian group G , then G is of prime exponent.

Proof. Assume that G is neither almost torsion-free nor has prime exponent, i.e., G fails to satisfy (C*). Then by the equivalence between (C*) and (4.2), there exists a prime p such that the subgroup $G[p]$ is proper and infinite. We aim to prove that this yields $G \notin \mathfrak{T}^{\mathcal{I}}$.

Since $G[p]$ is infinite, we can pick a one-to-one sequence $\mathbf{a} = (a_n)$ in $G[p]$. Since $G \neq G[p]$, there exists $b \in G \setminus G[p]$. Then the sequence $\mathbf{u} = (u_n)$, with $u_n = a_n + b$ for every $n \in \mathbb{N}$, is a one-to-one sequence that is not a $T^{\mathcal{I}}$ -sequence by Fact 4.2 since $p\mathbf{u}$ is the constant non-null sequence (pb) . Similarly, no subsequence of $\mathbf{u}$ can be a $T^{\mathcal{I}}$ -sequence. Therefore, $\mathbf{u}$ witnesses that $G \notin \mathfrak{T}^{\mathcal{I}}$.

The last assertion trivially follows from the first one. $\square$

We are left with the last missing implication (C*) $\Rightarrow$ (A) in (4.3).

The next lemma follows from Lemma 4.1 and Proposition 4.5, so it is formally obsolete. Nevertheless, we keep its proof based on a simple but useful idea, whereas Proposition 4.5 uses the full power of Theorem 2.8.

Lemma 4.4. *If G is any abelian group of prime exponent, then $G \in \mathfrak{T}_B$.*

Proof. If G is finite, then $G \in \mathfrak{T}_B$. Assume that G is infinite and let the prime p be the exponent of G . Then G is an infinite-dimensional vector space over $\mathbb{Z}/p\mathbb{Z}$.

Let $\mathbf{u}$ be a one-to-one sequence in G . We build inductively a subsequence $\mathbf{v} = (v_n)$ of $\mathbf{u}$ such that $\{v_n : n \in \mathbb{N}_0\}$ is linearly independent. Put $v_0 = u_0$ if $u_0 \neq 0$, otherwise put $v_0 = u_1 \neq 0$ (recall that $\mathbf{u}$ is one-to-one). Assume that for some $k \in \mathbb{N}_0$, all $v_0, \dots, v_k$ have been already chosen such that $\{v_0, \dots, v_k\}$ is linearly independent. Let V be the subspace of G generated by $v_0, \dots, v_k$. As $\dim G > \dim V$, there exists $m > k$ such that $u_m \notin V$, hence $\{v_0, \dots, v_k, u_m\}$ is linearly independent. Now we can choose $v_{k+1} := u_m$.

By Fact 2.7, it is enough to verify that $\mathbf{v}$ is a TB -sequence in the subgroup H of G generated by $|\mathbf{v}|$. Since $H = \bigoplus_{n \in \mathbb{N}_0} \langle v_n \rangle \cong \bigoplus_{n \in \mathbb{N}_0} \mathbb{Z}(p)$, consider the injective homomorphism $\sigma: H \rightarrow \mathbb{Z}(p)^{\mathbb{N}_0}$, where $\mathbb{Z}(p)^{\mathbb{N}_0}$ carries the compact product topology. The topology τ' induced by σ on H is precompact and $v_n \rightarrow 0$ in (H, τ') . $\square$

This lemma is not true for abelian groups G of arbitrary finite exponent since $G \in \mathfrak{T}_B$ for an infinite torsion abelian group G implies that G has prime exponent by Lemma 4.3.

Lemma 4.4 covered (C*) $\Rightarrow$ (A) for abelian groups of prime exponent. The missing part of the implication (C*) $\Rightarrow$ (A) as well as the implication (e) $\Rightarrow$ (a) of Theorem 1.7 are covered by the next proposition.

Proposition 4.5 ((e) $\Rightarrow$ (a) of Theorem 1.7). *Let G be an abelian group and $\mathbf{u}$ be a one-to-one sequence in G consisting of non-zero elements. If $0 \in cl_{\mathfrak{Z}_G}(|\mathbf{u}|)$, then $\mathbf{u}$ has a TB -subsequence.*

Proof. Let $H = \langle |\mathbf{u}| \rangle$. According to [15], the Zariski topology of H coincides with the topology induced by $\mathfrak{Z}_G$ on H , hence $0 \in cl_{\mathfrak{Z}_H}(|\mathbf{u}|)$. Therefore, by Fact 2.7, we can assume that $|\mathbf{u}|$ generates G ; so that G is countable (and in particular $|G| \leq \mathfrak{c}$). By Theorem 2.8, there exists a precompact metrizable group topology τ on G such that $cl_{\mathfrak{Z}_G}(|\mathbf{u}|) = cl_\tau(|\mathbf{u}|)$. Then $0 \in cl_\tau(|\mathbf{u}|)$ according to our hypothesis $0 \in cl_{\mathfrak{Z}_G}(|\mathbf{u}|)$. Since (G, τ) is metrizable, there exists a subsequence $\mathbf{v}$ of $\mathbf{u}$ with $v_n \rightarrow 0$ in (G, τ) ; then $\mathbf{v}$ is a TB -sequence as τ is precompact. $\square$

Corollary 4.6. *If G is any almost torsion-free abelian group, then $G \in \mathfrak{T}_B$.*

Proof. By Lemma 4.1, $\mathfrak{Z}_G$ is the cofinite topology. Fix a one-to-one sequence $\mathbf{u}$ in G . Since the set $|\mathbf{u}|$ is infinite, it is $\mathfrak{Z}_G$ -dense. Hence, $0 \in cl_{\mathfrak{Z}_G}(|\mathbf{u}|)$. By Proposition 4.5, $\mathbf{u}$ has a TB -subsequence. $\square$

This concludes the proof of Theorem 1.12. The realization Theorem 2.8 is applied only in the proof of Proposition 4.5, i.e., only when G is almost torsion-free as Proposition 4.5 applies only in Corollary 4.6.

5. Proof of Theorem 1.7. We start this last section by proving our first main theorem stated in the introduction.

Proof of Theorem 1.7. As explained in (1.1), we only need to prove the implications (b $^{\mathcal{I}}$) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (a). Since the last one follows from Proposition 4.5, we are left with only the proof of the first two.

(b $^{\mathcal{I}}$) $\Rightarrow$ (d) Assume for a contradiction that $\mathbf{u}$ has a $T^{\mathcal{I}}$ -subsequence $\mathbf{v}$, but fails to satisfy (d), i.e., for every subsequence $\mathbf{u}'$ of $\mathbf{u}$ there exist $m_1, \dots, m_n \in \mathbb{N}$ and subsequences $\mathbf{v}_1, \dots, \mathbf{v}_n$ of $\mathbf{u}'$ such that $|\mathbf{u}'| \subseteq^* |\mathbf{v}_1| \cup \dots \cup |\mathbf{v}_n|$ and $m_k \mathbf{v}_k$ is a non-zero constant sequence for every $k \in \{1, \dots, n\}$. Apply this assumption to $\mathbf{u}' = \mathbf{v}$, so $|\mathbf{v}| \subseteq^* |\mathbf{v}_1| \cup \dots \cup |\mathbf{v}_n|$, i.e., the subset $F := |\mathbf{v}| \setminus \bigcup_{k=1}^n |\mathbf{v}_k|$ of G is finite. Denote by g_k the non-zero element of G such that $m_k \mathbf{v}_k$ is the constant sequence $(g_k)_{n \in \mathbb{N}}$. For every $k \in \{1, \dots, n\}$, let

$$S_k := \mathbf{v}^{-1}(|\mathbf{v}_k|) = \{n \in \mathbb{N} : v_n \in |\mathbf{v}_k|\} \subseteq \{n \in \mathbb{N} : m_k v_n \in |m_k \mathbf{v}_k|\} \\ = \{n \in \mathbb{N} : m_k v_n = g_k\}.$$

The sequence $m_k \mathbf{v}$ is a $T^{\mathcal{I}}$ -sequence, by Fact 4.2(a), and $m_k \mathbf{v}_k = (g_k)_{n \in \mathbb{N}}$ is its non-zero constant subsequence so $S_k \subseteq \{n \in \mathbb{N} : m_k v_n = g_k\} = (m_k \mathbf{v})^{-1}(|m_k \mathbf{v}_k|) \in \mathcal{I}$, by Fact 4.2(b). Therefore, $S_k \in \mathcal{I}$. From $|\mathbf{v}| \subseteq^* |\mathbf{v}_1| \cup \dots \cup |\mathbf{v}_n|$, we conclude that

$$\mathbb{N} = \mathbf{v}^{-1}(|\mathbf{v}|) \subseteq \mathbf{v}^{-1} \left(\bigcup_{k=1}^n |\mathbf{v}_k| \right) \cup \bigcup_{g \in F} \mathbf{v}^{-1}(g) = \bigcup_{k=1}^n S_k \cup \bigcup_{g \in F} \mathbf{v}^{-1}(g),$$

where $|\bigcup_{g \in F} \mathbf{v}^{-1}(g)| < \infty$ as $\mathbf{v}$ is one-to-one. This entails $\mathbb{N} \in \mathcal{I}$ since $\bigcup_{k=1}^n S_k \in \mathcal{I}$, a contradiction.

(d) $\Rightarrow$ (e) Let $\mathbf{u}'$ be a subsequence of $\mathbf{u}$ as in (d) and assume for a contradiction that $0 \notin cl_{\mathfrak{Z}_G}(|\mathbf{u}'|)$. Then $0 \notin cl_{\mathfrak{Z}_G}(|\mathbf{u}'|)$ as well. In view of Remark 2.2, $cl_{\mathfrak{Z}_G}(|\mathbf{u}'|) = \bigcup_{i=1}^n (a_i + G[m_i])$ for appropriate $m_1, \dots, m_n \in \mathbb{N}$ and $a_1, \dots, a_n \in G$. Our assumption yields $0 \notin a_k + G[m_k]$, so $a_k \notin G[m_k]$, i.e., $b_k := m_k a_k \neq 0$ for every $k \in \{1, \dots, n\}$. Let I be the non-empty set of all those $k \in \{1, \dots, n\}$ such that $|\mathbf{u}'| \cap (a_k + G[m_k])$ is infinite. For $k \in I$, take the one-to-one subsequence $\mathbf{v}_k$ of $\mathbf{u}'$ supported by the countably infinite set $|\mathbf{u}'| \cap (a_k + G[m_k])$. Then $|\mathbf{u}'| \subseteq^* \bigcup_{k \in I} |\mathbf{v}_k|$ since $|\mathbf{u}'| \cap (a_k + G[m_k])$ is finite whenever $k \notin I$. Moreover, for every $k \in I$, the sequence $m_k \mathbf{v}_k$ is the constant non-zero sequence (b_k) . This contradicts the property of the subsequence $\mathbf{u}'$ granted by (d). $\square$

Remark 5.1. The implication $(B^{\mathcal{I}}) \Rightarrow (C^*)$ of Theorem 1.12 can be obtained alternatively also from the just proved implication $(b^{\mathcal{I}}) \Rightarrow (e)$ of Theorem 1.7 applied to all one-to-one sequences $\mathbf{u}$. Indeed, the conclusion “ $0 \in cl_{\mathfrak{Z}_G}(|\mathbf{u}|)$ for all $\mathbf{u}$ ” obtained in this way shows that $cl_{\mathfrak{Z}_G}(|\mathbf{u}|)$ is dense in G , as translations are $\mathfrak{Z}_G$ -continuous. This shows that all countably infinite subsets of G are $\mathfrak{Z}_G$ -dense, hence $\mathfrak{Z}_G$ is the cofinite topology.

Now we directly prove the equivalences $(a^{\mathcal{I}}) \Leftrightarrow (a)$ and $(b^{\mathcal{I}}) \Leftrightarrow (b)$ from Theorem 1.7 when $\mathcal{I}$ is a P -ideal.

Lemma 5.2. *Let $\mathbf{u}$ be a one-to-one sequence in an abelian group G and let $\mathcal{I}$ be a proper free P -ideal $\mathcal{I}$ on $\mathbb{N}_0$. If $\mathbf{u}$ has a $T^{\mathcal{I}}$ -subsequence (resp., $TB^{\mathcal{I}}$ -subsequence), then $\mathbf{u}$ has a T -subsequence (resp., TB -subsequence).*

Proof. We prove the statement for T -sequences, the proof of its counterpart for TB -sequence is similar.

We can assume without loss of generality that $\mathbf{u}$ itself is a $T^{\mathcal{I}}$ -sequence. According to Proposition 3.2 and Remark 3.3, there is a metrizable group topology τ on G such that $u_n \xrightarrow{\tau} 0$ in (G, τ) . By Remark 1.2, there exists a subsequence $\mathbf{v}$ of $\mathbf{u}$ such that $v_n \rightarrow 0$ in (G, τ) . Hence, $\mathbf{v}$ is the desired T -subsequence of $\mathbf{u}$. $\square$

The above proof shows that while Theorem 1.7 proves the equivalence $(b^{\mathcal{I}}) \Leftrightarrow (b)$ without claiming anything about the topology witnessing (b), one can achieve to have the topology witnessing (b) the same as the topology witnessing $(b^{\mathcal{I}})$, under the additional assumption that $\mathcal{I}$ is a P -ideal. Similarly, for $(a^{\mathcal{I}}) \Leftrightarrow (a)$.

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