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Noncooperative oligopoly in markets with a continuum of traders and a strongly connected set of commodities: A limit theorem

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We consider a mixed version of the Shapley window model, where large traders are represented as atoms and small traders are represented by an atomless part. In the original spirit of Cournot (1838), we partially replicate the mixed exchange economy by increasing the number of atoms, without affecting the atomless part, and ensuring that the measure space of agents remains finite. Our main theorem shows that any sequence of Cournot-Nash allocations of the strategic market games associated with the partial replications of the exchange economy has a limit point for each trader and that the assignment determined by these limit points is a Walras allocation of the original economy. Instead of relying on restrictive assumptions on the characteristics of atoms, as in Busetto et al. (2017), our limit theorem relies on the characteristics of agents in the atomless part and their endogenous price-taking behavior.

1. Introduction

Busetto et al. (2011) proved the existence of a Cournot-Nash equilibrium for the Shapley window model in mixed exchange economies *à la* Shitovitz, where large traders are represented as atoms and small traders are represented by an atomless part (see Shitovitz 1973). The Shapley window model belongs to a very fruitful line of research on noncooperative market games, initiated by Lloyd S. Shapley and Martin Shubik (for a survey of this literature, see Giraud (2003)). The model was informally introduced by Lloyd S. Shapley and subsequently formalized by Sahi and Yao (1989) in the case of exchange economies with a finite number of traders. For this case, the authors proved the existence of a Cournot-Nash equilibrium. The proof provided by Busetto et al. (2011) for the mixed market case is based on the same assumptions used by Sahi and Yao (1989) for the finite case. In particular, it is required that there are at least two atoms with strictly positive endowments, continuously differentiable utility functions, and indifference curves contained in the strict interior of the commodity space. These restrictions are stated by Busetto et al. (2011) in their Assumption 4.

Busetto et al. (2017) analyzed the asymptotic behavior of the Cournot-Nash equilibria of the mixed version of the Shapley window model. They introduced a concept of replication, which they called *à la* Cournot, since it extends to a general equilibrium context the idea of replication in the original spirit of Cournot (1838): it consists in partially replicating the economy by increasing only the number of atoms, this way making them asymptotically negligible, without affecting the atomless part. Under the same assumptions

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of the existence theorem proved by Busetto et al. (2011), these authors proved a theorem establishing that any sequence of Cournot-Nash allocations of the strategic market games associated with the partial replications of the exchange economy has a limit point for each trader and that the assignment determined by these limit points is a Walrasian allocation of the original economy.

Busetto et al. (2018) proved a new existence theorem for the mixed version of the Shapley window model, differing from the one proposed by Busetto et al. (2011) in that it is essentially based on restrictions on endowments and preferences of the atomless part of the economy rather than of atoms. In particular, Busetto et al. (2018) removed Assumption 4 in Busetto et al. (2011) and used the fact – proved by Codognato and Ghosal (2000) – that traders belonging to the atomless part have an endogenous "Walrasian" behavior. In Busetto et al. (2018), this property was exploited to show that, under the assumptions that each commodity is held, in the aggregate, by the atomless part and that traders' utility functions are continuous, strongly monotone, quasi-concave, and measurable, any sequence of prices corresponding to a sequence of Cournot-Nash equilibria has a subsequence which converges to a strictly positive price vector. The authors used this price convergence result to prove their existence theorem, under the assumption that the set of commodities is strongly connected through traders' characteristics, which imposes a joint restriction on the endowments and preferences of the atomless part and is a variant of a hypothesis first proposed by Codognato and Ghosal (2000). This assumption, combined with the continuity properties of the Walrasian correspondence generated by the atomless part's behavior, in turn guarantees that the aggregate matrix of the bids obtained as the limit of a sequence of perturbed Cournot-Nash equilibria is irreducible.

In this paper, we consider the mixed version of the Shapley window model in the formulation proposed by Busetto et al. (2018), with the aim of establishing the asymptotic properties of its equilibria. Sahi and Yao (1989) also proved a limit theorem, in the tradition of cooperative and noncooperative market games, using a symmetric replication *à la* Edgeworth, i.e., they replicated their finite exchange economy replacing each trader by identical copies with the same endowments and utility functions (see also Amir et al. (1990) for an analogous asymptotic result for another kind of strategic market game with complete markets). Here, we consider a new kind of partially symmetric replication *à la* Cournot introduced by Busetto et al. (2017), i.e., we partially replicate the mixed exchange economy replacing each atom by identical copies with the same endowments and utility functions, without affecting the atomless part, and ensuring that the measure space of agents remains finite, to prove a new limit theorem which does not require the restrictions on atoms stated in their Assumption 4.[1] In particular, our main theorem shows that any sequence of Cournot-Nash allocations of the strategic market games associated with the partial replications of the exchange economy has a limit point for each trader and that the assignment determined by these limit points is a Walras allocation of the original economy. The proof of the new limit result rests heavily on the price convergence theorem shown by Busetto et al. (2018). As a consequence, that result turns out to be explained, like those authors' existence theorem, in terms of the characteristics of the atomless part of the economy and the fact that the traders belonging to it have an endogeneous "Walrasian" behavior. Moreover, after our convergence result, we provide an example which illustrates it.

The paper is organized as follows. In Section 2, we introduce the mathematical model. In Section 3, we introduce the replication *à la Cournot*. In Section 4, we restate and prove the price convergence theorem. In Section 5, we prove the existence of an atom-type-symmetric Cournot-Nash equilibrium. In Section 6, we state and prove the limit theorem. In Section 7, we draw some conclusions from our analysis.

2. Mathematical model

We consider an exchange economy, $\mathcal{E}$, with large traders, represented as atoms, and small traders, represented by an atomless part. The space of traders is denoted by the measure space $(T, \mathcal{T}, \mu)$, where T is the set of traders, $\mathcal{T}$ is the σ-algebra of all μ-measurable subsets of T, and μ is a real valued, non-negative, countably additive measure defined on $\mathcal{T}$. We assume that $(T, \mathcal{T}, \mu)$ is finite, i.e., $\mu(T) < +\infty$. This implies that the measure space $(T, \mathcal{T}, \mu)$ contains at most countably many atoms. Let T_1 denote the set of atoms and $T_0 = T \setminus T_1$ the atomless part of T. A null set of traders is a set of measure 0. Null sets of traders are systematically ignored throughout the paper. Thus, a statement asserted for "each" trader in a certain set is to be understood to hold for all such traders except possibly for a null set of traders. A coalition is a nonnull element of $\mathcal{T}$. The word "integrable" is to be understood in the sense of Lebesgue.

In the exchange economy, there are l different commodities. A commodity bundle is a point in R^l_+. An assignment (of commodity bundles to traders) is an integrable function $\mathbf{x}$: $T \to R^l_+$. There is a fixed initial assignment $\mathbf{w}$, satisfying the following assumption.

Assumption 1. $\mathbf{w}(t) > 0$, for each $t \in T$, $\int_{T_0} \mathbf{w}(t)\, d\mu \gg 0$.

An allocation is an assignment $\mathbf{x}$ for which $\int_T \mathbf{x}(t)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$. The preferences of each trader $t \in T$ are described by a utility function $u_t : R^l_+ \to R$, satisfying the following assumptions.

Assumption 2. $u_t : R^l_+ \to R$ is continuous, strongly monotone, and quasi-concave, for each $t \in T$.

Let $\mathcal{B}$ denote the Borel σ-algebra of R^l_+. Moreover, let $\mathcal{T} \bigotimes \mathcal{B}$ denote the σ-algebra generated by all the sets $E \times F$ such that $E \in \mathcal{T}$ and $F \in \mathcal{B}$.

[1] The two kinds of replications mentioned above are both "type symmetric," i.e., atomic traders are replicated symmetrically in terms of their characteristics. Koutsougeras (2009) and Koutsougeras and Meo (2018) proposed to analyze the asymptotic behavior of Cournot-Nash equilibria considering more general sequences of exchange economies with characteristics drawn from compact sets without imposing any kind of type symmetry. Even if type symmetric replication is perhaps a less stringent condition in the framework of partial replication *à la* Cournot than in that of full replication *à la* Edgeworth, we consider that an analysis of the asymptotic properties of the Cournot-Nash equilibria of the mixed version of the Shapley window model based on "asymmetric" sequences should be pursued in further research.

Assumption 3. $u : T \times R^l_+ \to R$, given by $u(t, x) = u_t(x)$, for each $t \in T$ and for each $x \in R^l_+$, is $\mathcal{T} \bigotimes \mathcal{B}$-measurable.

In order to state our last assumption, we need some preliminary definitions. We denote by L the set of commodities $\{1, \dots, l\}$. We say that two commodities $i, j \in L$ stand in relation C if there is coalition T^i in T_0 such that $\mathbf{w}^i(t) > 0$, $\mathbf{w}^r(t) = 0$, for each $r \in L \setminus \{i\}$, $u_t(\cdot)$ is differentiable, additively separable in commodity j, i.e., $u_t(x) = v^j_t(x^j) + v_t(x^1, \dots, x^{j-1}, x^{j+1}, \dots, x^l)$, for each $x \in R^l_+$, and $\frac{dv^j_t(0)}{dx^j} = +\infty$, for each $t \in T^i$.[2] Then, the concept of a set of commodities strongly connected through traders' characteristics can be defined as follows.

Definition 1. The set of commodities L is said to be strongly connected through traders' characteristics if the directed graph $D_L(L, C)$ is strongly connected, i.e., any ordered pair of distinct vertices, i and j, of $D_L(L, C)$ is connected by a path.

We can now state our last assumption.

Assumption 4. The set of commodities L is strongly connected through traders' characteristics.

A price vector is a nonnull vector $p \in R^l_+$. Henceforth, we say that a price vector p is normalized if $p \in \Delta$, where $\Delta = \{p \in R^l_+ : \sum_{i=1}^l p^i = 1\}$. Moreover, we denote by $\partial\Delta$ the boundary of the unit simplex Δ.

Let $\mathbf{X}^0 : T_0 \times \Delta \setminus \partial\Delta \to \mathcal{P}(R^l)$ be a correspondence such that, for each $t \in T_0$ and for each $p \in \Delta \setminus \partial\Delta$, $\mathbf{X}^0(t, p) = \operatorname{argmax}\{u_t(x) : x \in R^l_+ \text{ and } px \le p\mathbf{w}(t)\}$. It is well-known that the previous assumptions guarantee that the correspondence $\mathbf{X}^0(t, \cdot)$ is upper hemicontinuous, for each $t \in T_0$. Finally, let $\mathbf{Z}^0 : R^l_{++} \to \mathcal{P}(R^l)$ be a correspondence which associates with each $p \in R^l_{++}$ the Minkowski difference between the set $\int_{T_0} \mathbf{X}^0(t, p)\, d\mu$ and the set $\{\int_{T_0} \mathbf{w}(t)\, d\mu\}$.[3]

A Walras equilibrium of $\mathcal{E}$ is a pair $(p^*, \mathbf{x}^*)$, consisting of a price vector $p^* \in \Delta \setminus \partial\Delta$ and an allocation $\mathbf{x}^*$, such that, for each $t \in T$, $u_t(\mathbf{x}^*(t)) \ge u_t(y)$, for all $y \in \{x \in R^l_+ : p^*x = p^*\mathbf{w}(t)\}$. A Walras allocation of $\mathcal{E}$ is an allocation $\mathbf{x}^*$ for which there exists a price vector $p^* \in \Delta \setminus \partial\Delta$ such that the pair $(p^*, \mathbf{x}^*)$ is a Walras equilibrium of $\mathcal{E}$.

We define now the strategic market game, Γ, associated with $\mathcal{E}$. It is a slightly reformulated version of the Shapley window model for mixed economies proposed by Busetto et al. (2011).

A strategy correspondence is a correspondence $\mathbf{B} : T \to \mathcal{P}(R^{l^2}_+)$ such that, for each $t \in T$, $\mathbf{B}(t) = \{(b_{ij}) \in R^{l^2}_+ : \sum_{j=1}^l b_{ij} \le \mathbf{w}^i(t),\ i = 1, \dots, l\}$. With some abuse of notation, we denote by $b(t) \in \mathbf{B}(t)$ a strategy of trader t, where $b_{ij}(t)$, $i, j = 1, \dots, l$, represents the amount of commodity i that trader t offers in exchange for commodity j. A strategy profile is an integrable function $\mathbf{b} : T \to R^{l^2}_+$, such that, for each $t \in T$, $\mathbf{b}(t) \in \mathbf{B}(t)$. Given a strategy profile $\mathbf{b}$, we define the aggregate matrix $\bar{\mathbf{B}}$ to be the matrix such that $\bar{\mathbf{b}}_{ij} = (\int_T \mathbf{b}_{ij}(t)\, d\mu)$, $i, j = 1, \dots, l$. Moreover, we denote by $\mathbf{b} \setminus b(t)$ the strategy profile obtained from $\mathbf{b}$ by replacing $\mathbf{b}(t)$ with $b(t) \in \mathbf{B}(t)$, and by $\bar{\mathbf{B}} \setminus b(t)$ the corresponding aggregate matrix.

The following definitions are borrowed from Sahi and Yao (1989).

Definition 2. A nonnegative square matrix A is said to be irreducible if, for every pair (i, j), with $i \ne j$, there is a positive integer k such that $a^{(k)}_{ij} > 0$, where $a^{(k)}_{ij}$ denotes the ij-th entry of the k-th power A^k of A.

Definition 3. Given a strategy profile $\mathbf{b}$, a normalized price vector p is said to be market clearing if

$$p \in \Delta \setminus \partial\Delta,\ \sum_{i=1}^l p^i \bar{\mathbf{b}}_{ij} = p^j \left(\sum_{i=1}^l \bar{\mathbf{b}}_{ji} \right),\ j = 1, \dots, l. \tag{1}$$

By Lemma 1 in Sahi and Yao (1989), there is a unique normalized price vector $p \in \Delta \setminus \partial\Delta$ satisfying (1) if and only if $\bar{\mathbf{B}}$ is irreducible. Then, we denote by $p(\mathbf{b})$ a function which associates with each strategy profile $\mathbf{b}$ the unique normalized price vector $p \in \Delta \setminus \partial\Delta$ satisfying (1), if $\bar{\mathbf{B}}$ is irreducible, and is equal to 0, otherwise.

Given a strategy profile $\mathbf{b}$ and a normalized price vector p, consider the assignment determined as follows:

$$\begin{aligned} \mathbf{x}^j(t, \mathbf{b}(t), p) &= \mathbf{w}^j(t) - \textstyle\sum_{i=1}^l \mathbf{b}_{ji}(t) + \sum_{i=1}^l \mathbf{b}_{ij}(t) \frac{p^i}{p^j}, \text{ if } p \in \Delta \setminus \partial\Delta, \\ \mathbf{x}^j(t, \mathbf{b}(t), p) &= \mathbf{w}^j(t), \text{ otherwise,} \end{aligned}$$

$j = 1, \dots, l$, for each $t \in T$.

Given a strategy profile $\mathbf{b}$ and the function $p(\mathbf{b})$, the traders' final holdings are determined according to this rule and consequently expressed by the assignment

$$\mathbf{x}(t) = \mathbf{x}(t, \mathbf{b}(t), p(\mathbf{b})),$$

for each $t \in T$.[4] It is straightforward to show that this assignment is an allocation.

We are now able to introduce a notion of Cournot-Nash equilibrium for this reformulation of the Shapley window model (see Codognato and Ghosal 2000 and Busetto et al. 2011).

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2 In this definition, differentiability is to be understood as continuous differentiability and it includes the case of infinite partial derivatives along the boundary of the consumption set (for a discussion of this case, see, for instance, Kreps (2012), p. 58). Moreover, it can be proved that the separable utility function used in the definition is the representation of separable preferences (see, for instance, Kreps (2012), p. 42).

3 For a discussion of the properties of the correspondences introduced above and their proofs see, for instance, Debreu (1982), Section 4.

4 In order to save in notation, with some abuse we denote by $\mathbf{x}$ both the function $\mathbf{x}(t)$ and the function $\mathbf{x}(t, \mathbf{b}(t), p(\mathbf{b}))$.

Definition 4. A strategy profile $\hat{\mathbf{b}}$ such that $\bar{\hat{\mathbf{B}}}$ is irreducible is a Cournot-Nash equilibrium of Γ if

$$u_t(\mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))) \geq u_t(\mathbf{x}(t, b(t), p(\hat{\mathbf{b}} \setminus b(t)))),$$

for each $b(t) \in \mathbf{B}(t)$ and for each $t \in T$.[5]

A Cournot-Nash allocation of Γ is an allocation $\hat{\mathbf{x}}$ such that $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$, where $\hat{\mathbf{b}}$ is a Cournot-Nash equilibrium of Γ.

3. Replication à la cournot of $\mathcal{E}$

In this section, we focus on the concept of replication introduced by Busetto et al. (2017), in the original spirit of Cournot (1838). We will use this concept to obtain our limit theorem for the Cournot-Nash equilibria of the mixed version of the Shapley window model. By analogy with the replication proposed by Cournot in a partial equilibrium framework, the concept proposed by Busetto et al. (2017) is obtained by replicating only the atoms of $\mathcal{E}$, while making them asymptotically negligible, and without affecting the atomless part.

This replication *à la Cournot* of $\mathcal{E}$ can be formalized as follows. Let $\mathcal{E}^n$ be an exchange economy characterized as in Section 2, where each atom is replicated n times. For each $t \in T_1$, let tr denote the r-th element of the n-fold replication of t. We assume that, for each $t \in T_1$, $\mathbf{w}(tr) = \mathbf{w}(ts) = \mathbf{w}(t)$, $u_{tr}(\cdot) = u_{ts}(\cdot) = u_t(\cdot)$, $r, s = 1, \dots, n$, and $\mu(tr) = \frac{\mu(t)}{n}$, $r = 1, \dots, n$. Clearly, $\mathcal{E}^1$coincides with $\mathcal{E}$.

Then, the strategic market game Γ^n associated with $\mathcal{E}^n$ can be characterized, *mutatis mutandis*, as in Section 2. Clearly, Γ^1 coincides with Γ. A strategy profile $\mathbf{b}$ of Γ^n is said to be atom-type-symmetric if $\mathbf{b}^n(tr) = \mathbf{b}^n(ts)$, $r, s = 1, \dots, n$, for each $t \in T_1$.

We provide now the definition of an atom-type-symmetric Cournot-Nash equilibrium of Γ^n.

Definition 5. A strategy selection $\hat{\mathbf{b}}$ such that $\bar{\hat{\mathbf{B}}}$ is irreducible is an atom-type-symmetric Cournot-Nash equilibrium of Γ^n if $\hat{\mathbf{b}}$ is atom-type-symmetric and if

$$u_{tr}(\mathbf{x}(tr, \hat{\mathbf{b}}(tr), p(\hat{\mathbf{b}}))) \geq u_{tr}(\mathbf{x}(tr, b(tr), p(\hat{\mathbf{b}} \setminus b(tr)))),$$

for each $b(tr) \in \mathbf{B}(tr)$, $r = 1, \dots, n$, and for each $t \in T_1$;

$$u_t(\mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))) \geq u_t(\mathbf{x}(t, b(t), p(\hat{\mathbf{b}} \setminus b(t)))),$$

for each $b(t) \in \mathbf{B}(t)$ and for each $t \in T_0$.

In order to show the existence of an atom-type-symmetric Cournot-Nash equilibrium of Γ^n, we need to define the notion of a perturbation of this strategic market game, denoted by $\Gamma^n(\epsilon)$ (it was already used by Sahi and Yao (1989), Busetto et al. (2011), and Busetto et al. (2018)).

Given $\epsilon > 0$ and a strategy profile $\mathbf{b}$, we define the aggregate matrix $\bar{\mathbf{B}}_\epsilon$ to be the matrix such that $\bar{\mathbf{b}}_{\epsilon ij} = (\bar{\mathbf{b}}_{ij} + \epsilon)$, $i, j = 1, \dots, l$. Clearly, the matrix $\bar{\mathbf{B}}_\epsilon$ is irreducible. The interpretation is that an outside agency places fixed bids of ϵ for each pair of commodities (i, j).

Given $\epsilon > 0$, we denote by $p^\epsilon(\mathbf{b})$ the function which associates, with each strategy profile $\mathbf{b}$, the unique normalized price vector which satisfies

$$\sum_{i=1}^{l} p^i(\bar{\mathbf{b}}_{ij} + \epsilon) = p^j\left(\sum_{i=1}^{l}(\bar{\mathbf{b}}_{ji} + \epsilon)\right), \ j = 1, \dots, l.$$

Then, let us introduce the following notion of equilibrium for $\Gamma^n(\epsilon)$.

Definition 6. Given $\epsilon > 0$, a strategy profile $\hat{\mathbf{b}}^\epsilon$ is an atom-type-symmetric ϵ-Cournot-Nash equilibrium of $\Gamma^n(\epsilon)$ if $\hat{\mathbf{b}}^\epsilon$ is atom-type-symmetric and

$$u_{tr}(\mathbf{x}(tr, \hat{\mathbf{b}}^\epsilon(tr), p^\epsilon(\hat{\mathbf{b}}^\epsilon))) \geq u_{tr}(\mathbf{x}(tr, b(tr), p^\epsilon(\hat{\mathbf{b}}^\epsilon \setminus b(tr)))),$$

for each $b(tr) \in \mathbf{B}(tr)$, $r = 1, \dots, n$, and for each $t \in T_1$;

$$u_t(\mathbf{x}(t, \hat{\mathbf{b}}^\epsilon(t), p^\epsilon(\hat{\mathbf{b}}^\epsilon))) \geq u_t(\mathbf{x}(t, b(t), p^\epsilon(\hat{\mathbf{b}}^\epsilon \setminus b(t)))),$$

for each $b(t) \in \mathbf{B}(t)$ and for each $t \in T_0$.

4. Price convergence theorem

By means of Lemma 9 in Sahi and Yao (1989), Busetto et al. (2011) showed that any convergent sequence of normalized prices corresponding to a sequence of Cournot-Nash equilibria has a convergent subsequence whose limit is a strictly positive normalized price vector. Lemma 9 in Sahi and Yao (1989), and consequently the convergence result in Busetto et al. (2011), are essentially based on the assumption that there are at least two atoms with strictly positive endowments, continuously differentiable utility functions,

[5] Let us notice that, as this definition of a Cournot-Nash equilibrium explicitly refers to irreducible matrices, it applies only to active equilibria (on this point, see Sahi and Yao (1989)).

and indifference curves contained in the strict interior of the commodity space. This restriction is stated by Busetto et al. (2011) in their Assumption 4.

Busetto et al. (2017) also used Lemma 9 in Sahi and Yao (1989) to prove their limit theorem under the same assumption.

Busetto et al. (2018) provided a different price convergence theorem, obtained by removing Assumption 4 in Busetto et al. (2011) and focusing on restrictions concerning endowments and preferences of the atomless part of the economy rather than of atoms. More precisely, they exploited the property of small traders, proved by Codognato and Ghosal (2000), of being "Walrasian" at a Cournot-Nash equilibrium. Their price convergence theorem establishes that any sequence of normalized prices corresponding to a sequence of Cournot-Nash equilibria has a convergent subsequence whose limit is a strictly positive normalized price vector. Here, we state and prove it in a general version which we use to show our new limit theorem for mixed exchange economies where Assumption 4 in Busetto et al. (2011) is relaxed.

Theorem 1. *Under Assumptions 1, 2, and 3, let $\{\hat{p}^n\}$ be a sequence of normalized prices such that $\{\hat{p}^n\} = p(\hat{\mathbf{b}}^n)$, where $\hat{\mathbf{b}}^n$ is an atom-type-symmetric Cournot-Nash equilibrium of Γ^n, for each $n = 1, 2, \ldots$. Then, there exists a subsequence $\{\hat{p}^{k_n}\}$ of the sequence $\{\hat{p}^n\}$ which converges to a price vector $\hat{p} \in \Delta \setminus \partial\Delta$.*

Proof. Let $\{\hat{p}^n\}$ be a sequence of normalized prices such that $\{\hat{p}^n\} = p(\hat{\mathbf{b}}^n)$, where $\hat{\mathbf{b}}^n$ is an atom-type-symmetric Cournot-Nash equilibrium of Γ^n, for each $n = 1, 2, \ldots$. Then, there is a subsequence $\{\hat{p}^{k_n}\}$ of the sequence $\{\hat{p}^n\}$ which converges to a price vector $\hat{p} \in \Delta$ as the unit simplex Δ is a compact set. Suppose that $\hat{p} \in \partial\Delta$, where $\partial\Delta$ denotes the boundary of the unit simplex. Following (Debreu, 1982), let $|x| = \sum_{i=1}^{l} |x^i|$, for each $x \in R^l$, and let $d[0, S] = \inf_{x\in S} |x|$, for each $S \subset R^l$. Then, the sequence $\{d[0, \mathbf{Z}^0(\hat{p}^{k_n})]\}$ diverges to $+\infty$ since $\int_{T_0} \mathbf{w}(t)\, d\mu \gg 0$ and $\hat{p} \in \partial\Delta$, as pointed out by Debreu (1982) in Property (iv). Let $\hat{\mathbf{x}}^n(tr) = \mathbf{x}(tr, \hat{\mathbf{b}}^n(tr), p(\hat{\mathbf{b}}^n))$, $r = 1, \ldots, n$, for each $t \in T_1$, and $\hat{\mathbf{x}}^n(t) = \mathbf{x}(t, \hat{\mathbf{b}}^n(t), p(\hat{\mathbf{b}}^n))$, for each $t \in T_0$, for each $n = 1, 2, \ldots$. Then, $\hat{\mathbf{x}}^n(t) \in \mathbf{X}^0(t, \hat{p}^n)$, for each $t \in T_0$, and for each $n = 1, 2, \ldots$, by Theorem 2 in Codognato and Ghosal (2000). But then, $(\int_{T_0} \hat{\mathbf{x}}^n(t)\, d\mu - \int_{T_0} \mathbf{w}(t)\, d\mu) \in \mathbf{Z}^0(\hat{p}^n)$, for each $n = 1, 2, \ldots$. We have that

$$\int_{T_1} \hat{\mathbf{x}}^n(t)\, d\mu + \int_{T_0} \hat{\mathbf{x}}^n(t)\, d\mu = \int_{T_0} \mathbf{w}(t)\, d\mu + \int_{T_1} \mathbf{w}(t)\, d\mu$$

as $\mathbf{x}^n(tr) = \mathbf{x}^n(ts) = \mathbf{x}^n(t)$, $\mathbf{w}(tr) = \mathbf{w}(ts) = \mathbf{w}(t)$, $r, s = 1, \ldots, n$, for each $t \in T_1$, and $\mu(tr) = \frac{\mu(t)}{n}$, $r = 1, \ldots, n$, for each $n = 1, 2, \ldots$. Then, it must be that

$$\int_{T_0} \hat{\mathbf{x}}^n(t)\, d\mu \leq \int_{T_0} \mathbf{w}(t)\, d\mu + \int_{T_1} \mathbf{w}(t)\, d\mu,$$

for each $n = 1, 2, \ldots$. But then, we have that

$$\left| \int_{T_0} \hat{\mathbf{x}}^{in}(t)\, d\mu - \int_{T_0} \mathbf{w}^i(t)\, d\mu \right| \leq \int_{T_0} \mathbf{w}^i(t)\, d\mu + \int_{T_1} \mathbf{w}^i(t)\, d\mu$$

as $-\int_{T_1} \mathbf{w}^i(t)\, d\mu \leq \int_{T_0} \hat{\mathbf{x}}^{in}(t)\, d\mu \leq 2\int_{T_0} \mathbf{w}^i(t)\, d\mu + \int_{T_1} \mathbf{w}^i(t)\, d\mu$, for each $i \in L$, for each $n = 1, 2, \ldots$. Therefore, it must be that

$$\sum_{i=1}^{l} \left| \int_{T_0} \hat{\mathbf{x}}^{in}(t)\, d\mu - \int_{T_0} \mathbf{w}^i(t)\, d\mu \right| \leq \sum_{i=1}^{l} \left(\int_{T_0} \mathbf{w}^i(t)\, d\mu + \int_{T_1} \mathbf{w}^i(t)\, d\mu \right),$$

for each $n = 1, 2, \ldots$. Moreover, there exists an n_0 such that

$$d[0, \mathbf{Z}^0(\hat{p}^{k_n})] > \sum_{i=1}^{l} \left(\int_{T_0} \mathbf{w}^i(t)\, d\mu + \int_{T_1} \mathbf{w}^i(t)\, d\mu \right),$$

for each $n \geq n_0$, as the sequence $\{d[0, \mathbf{Z}^0(\hat{p}^{k_n})]\}$ diverges to $+\infty$. Then,

$$\sum_{i=1}^{l} \left| \int_{T_0} \hat{\mathbf{x}}^{ik_n}(t)\, d\mu - \int_{T_0} \mathbf{w}^i(t)\, d\mu \right| > \sum_{i=1}^{l} \left(\int_{T_0} \mathbf{w}^i(t)\, d\mu + \int_{T_1} \mathbf{w}^i(t)\, d\mu \right)$$

as $\sum_{i=1}^{l} | \int_{T_0} \hat{\mathbf{x}}^{ik_n}(t)\, d\mu - \int_{T_0} \mathbf{w}^i(t)\, d\mu| \geq d[0, \mathbf{Z}^0(\hat{p}^{k_n})]$, for each $n \geq n_0$, a contradiction. Hence, it must be that $\hat{p} \in \Delta \setminus \partial\Delta$. □

5. Existence theorem

The theorem presented in this section establishes the existence of an atom-type-symmetric Cournot-Nash equilibrium of Γ^n. The proof of the theorem differs from that provided by Busetto et al. (2017) in that it replaces their Assumption 4 on endowments and preferences of atoms (the same as Assumption 4 in Busetto et al. (2011)) with the assumption that the set of commodities is strongly connected through traders' characteristics, which imposes restrictions on endowments and preferences of the atomless part. Our existence theorem is based on that proved by Busetto et al. (2018), which rests crucially on the price convergence theorem presented in Section 4.

Theorem 2. *Under Assumptions 1, 2, 3, and 4, there exists an atom-type-symmetric Cournot-Nash equilibrium $\hat{\mathbf{b}}$ of Γ^n.*

Proof. We first need to prove the existence of an atom-type-symmetric ϵ-Cournot-Nash equilibrium of $\Gamma^n(\epsilon)$. To do so, we apply, as in Busetto et al. (2011), the Kakutani-Fan-Glicksberg theorem.

We neglect, as usual, the distinction between integrable functions and equivalence classes of such functions and denote by $L_1(\mu, R^{l^2})$ the set of integrable functions taking values in R^{l^2}, by $L_1(\mu, \mathbf{B}(\cdot))$ the set of strategy profiles, and by $L_1(\mu, \mathbf{B}^*(\cdot))$ the set of atom-type-symmetric strategy profiles. Note that the locally convex Hausdorff space we shall be working in is $L_1(\mu, R^{l^2})$, endowed with its weak topology.

The proof of existence of an atom-type-symmetric ϵ-Cournot-Nash equilibrium of $\Gamma^n(\epsilon)$ is articulated in three lemmas.

The first lemma establishes the properties of $L_1(\mu, \mathbf{B}^*(\cdot))$ required to apply the Kakutani-Fan-Glicksberg theorem.

Lemma 1. *Under Assumptions 1, 2, 3, and 4, the set $L_1(\mu, \mathbf{B}^*(\cdot))$ is nonempty, convex and weakly compact.*

Proof. See the proof of Lemma 1 in Busetto et al. (2017). □

Given $\epsilon > 0$, let $\alpha^\epsilon_{tr} : L_1(\mu, \mathbf{B}^*(\cdot)) \to \mathbf{B}(tr)$ be a correspondence such that $\alpha^\epsilon_{tr}(\mathbf{b}) = \operatorname{argmax}\{u_{tr}(\mathbf{x}(t, b(tr), p^\epsilon(\mathbf{b} \setminus b(tr)))) : b(tr) \in \mathbf{B}(tr)\}$, $r = 1, \dots, n$, for each $t \in T_1$ and for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$, and let $\alpha^\epsilon_t : L_1(\mu, \mathbf{B}(\cdot)) \to \mathbf{B}(t)$ be a correspondence such that $\alpha^\epsilon_t(\mathbf{b}) = \operatorname{argmax}\{u_t(\mathbf{x}(t, b(t), p^\epsilon(\mathbf{b} \setminus b(t)))) : b(t) \in \mathbf{B}(t)\}$, for each $t \in T_0$ and for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$. Moreover, let $\alpha^\epsilon : L_1(\mu, \mathbf{B}^*(\cdot)) \to L_1(\mu, \mathbf{B}(\cdot))$ be a correspondence such that $\alpha^\epsilon(\mathbf{b}) = \{\mathbf{b} \in L_1(\mu, \mathbf{B}(\cdot)) : \mathbf{b}(tr) \in \alpha^\epsilon_{tr}(\mathbf{b}),\ r = 1, \dots, n,$ for each $t \in T_1$, and $\mathbf{b}(t) \in \alpha^\epsilon_t(\mathbf{b}),$ for each $t \in T_0\}$, for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$. Finally, let $\alpha^{\epsilon *} : L_1(\mu, \mathbf{B}^*(\cdot)) \to L_1(\mu, \mathbf{B}^*(\cdot))$ be a correspondence such that $\alpha^{\epsilon *}(\mathbf{b}) = \alpha^\epsilon(\mathbf{b}) \cap L_1(\mu, \mathbf{B}^*(\cdot))$, for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$.

The second lemma provides us with the properties of the correspondence $\alpha^{\epsilon *}$.

Lemma 2. *Under Assumptions 1, 2, 3, and 4, given $\epsilon > 0$, the correspondence $\alpha^{\epsilon *}$ is nonempty, convex-valued, and it has a weakly closed graph.*

Proof. Let $\epsilon > 0$ be given. We have that $\alpha^\epsilon_{tr}(\mathbf{b})$ is nonempty, $r = 1, \dots, n$, for each $t \in T_1$ and for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$, by Lemma 4 in Sahi and Yao (1989). Moreover, we have that $\alpha^\epsilon_{tr}(\mathbf{b}) = \alpha^\epsilon_{ts}(\mathbf{b})$ as $u_{tr}(\cdot) = u_{ts}(\cdot)$ and $\mathbf{B}(tr) = \mathbf{B}(ts)$, $r, s = 1, \dots, n$, for each $t \in T_1$ and for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$. Then, there exists a strategy $\tilde{b}(t) \in \mathbf{B}(t)$ such that $\tilde{b}(t) \in \alpha^\epsilon_{tr}(\mathbf{b})$, $r = 1, \dots, n$, for each $t \in T_1$ and for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$. But then, $\alpha^{\epsilon *}(\mathbf{b})$ is nonempty, for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$, as $\alpha^\epsilon(\mathbf{b})$ is nonempty, for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$, by Lemma 2 in Busetto et al. (2011). $\alpha^{\epsilon *}(\mathbf{b})$ is convex, for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$, as $\alpha^\epsilon(\mathbf{b})$ is convex, for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$, by Lemma 2 in Busetto et al. (2011), and $L_1(\mu, \mathbf{B}^*(\cdot))$ is convex, by Lemma 1. α^ϵ has a weakly closed graph, by Lemma 2 in Busetto et al. (2011). Let $\phi : L_1(\mu, \mathbf{B}^*(\cdot)) \to L_1(\mu, \mathbf{B}^*(\cdot))$ be a correspondence such that $\phi(\mathbf{b}) = L_1(\mu, \mathbf{B}^*(\cdot))$, for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$. It is straightforward to verify that ϕ has a weakly closed graph. Then, $\alpha^{\epsilon *}$ has a weakly closed graph as $\alpha^{\epsilon *}(\mathbf{b}) = \alpha^\epsilon(\mathbf{b}) \cap L_1(\mu, \mathbf{B}^*(\cdot))$, for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$, by Theorem 17.25 in Aliprantis and Border (2006). □

Finally, the third lemma proves the existence of an atom-type-symmetric ϵ-Cournot-Nash equilibrium of $\Gamma^n(\epsilon)$.

Lemma 3. *Under Assumptions 1, 2, 3, and 4, given $\epsilon > 0$, there exists an atom-type-symmetric ϵ-Cournot-Nash equilibrium $\hat{\mathbf{b}}^\epsilon$ of $\Gamma^n(\epsilon)$.*

Proof. Let $\epsilon > 0$ be given. The set $L_1(\mu, \mathbf{B}^*(\cdot))$ is nonempty, convex and weakly compact, by Lemma 1. Moreover, the correspondence $\alpha^{\epsilon *}$ is nonempty, convex-valued, and it has a weakly closed graph, by Lemma 2. Then, there exists a fixed point $\hat{\mathbf{b}}^\epsilon$ of the correspondence $\alpha^{\epsilon *}$ by the Kakutani-Fan-Glicksberg Theorem (see Theorem 17.55 in Aliprantis and Border (2006)). Hence, $\hat{\mathbf{b}}^\epsilon$ is an atom-type-symmetric ϵ-Cournot-Nash equilibrium of $\Gamma^n(\epsilon)$. □

To complete the proof of Theorem 2, we have to show that there exists the limit of a sequence of atom-type-symmetric ϵ-Cournot-Nash equilibria of $\Gamma^n(\epsilon)$ and that this limit is an atom-type-symmetric Cournot-Nash equilibrium of Γ^n. Following (Busetto et al., 2011), in this part of the proof we essentially refer to a generalization of the Fatou's lemma in several dimensions provided by Artstein (1979). Let $\epsilon_m = \frac{1}{m}$, $m = 1, 2, \dots$. By Lemma 3, for each $m = 1, 2, \dots$, there is an atom-type-symmetric ϵ-Cournot-Nash equilibrium $\hat{\mathbf{b}}^{\epsilon_m}$ of $\Gamma^n(\epsilon_m)$.

The following lemma provides a version of the price convergence theorem which shows that any sequence of normalized prices corresponding to a sequence of ϵ-Cournot-Nash equilibrium $\hat{\mathbf{b}}^{\epsilon_m}$ of $\Gamma^n(\epsilon_m)$ has a convergent subsequence whose limit is a strictly positive normalized price vector.

Lemma 4. *Under Assumptions 1, 2, and 3, let $\{\hat{p}^{\epsilon_m}\}$ be a sequence of normalized prices such that $\hat{p}^{\epsilon_m} = p^{\epsilon_m}(\hat{\mathbf{b}}^{\epsilon_m})$, where $(\mathbf{b}^{\epsilon_m})$ is an atom-type-symmetric ϵ-Cournot-Nash equilibrium of $\Gamma^n(\epsilon_m)$, for each $m = 1, 2, \dots$. Then, there exists a subsequence $\{\hat{p}^{\epsilon_{k_m}}\}$ of the sequence $\{\hat{p}^{\epsilon_m}\}$ which converges to a price vector $\dot{p} \in \Delta \setminus \partial\Delta$.*

Proof. Let $\{\hat{p}^{\epsilon_m}\}$ be a sequence of normalized prices such that $\hat{p}^{\epsilon_m} = p^{\epsilon_m}(\hat{\mathbf{b}}^{\epsilon_m})$, where $(\mathbf{b}^{\epsilon_m})$ is an atom-type-symmetric ϵ-Cournot-Nash equilibrium of $\Gamma^n(\epsilon_m)$, for each $m = 1, 2, \dots$. Then, there is a subsequence $\{\hat{p}^{\epsilon_{k_m}}\}$ of the sequence $\{\hat{p}^{\epsilon_m}\}$ which converges to a price vector $\dot{p} \in \Delta$ as the unit simplex Δ is a compact set. Suppose that $\dot{p} \in \partial\Delta$, where $\partial\Delta$ denotes the boundary of the unit simplex. Then, the sequence $\{d[0, \mathbf{Z}^0(\hat{p}^{\epsilon_{k_m}})]\}$ diverges to $+\infty$ since $\int_{T_0} \mathbf{w}(t)\, d\mu \gg 0$ and $\dot{p} \in \partial\Delta$, as pointed out by Debreu (1982) in Property (iv). Let $\hat{\mathbf{x}}^{\epsilon_m}(tr) = \mathbf{x}(t, \hat{\mathbf{b}}^{\epsilon_m}(tr), p^{\epsilon_m}(\hat{\mathbf{b}}^{\epsilon_m}))$, $r = 1, \dots, n$, for each $t \in T_1$, and $\hat{\mathbf{x}}^{\epsilon_m}(t) = \mathbf{x}(t, \hat{\mathbf{b}}^{\epsilon_m}(t), p^{\epsilon_m}(\hat{\mathbf{b}}^{\epsilon_m}))$ for each $t \in T_0$, for each $m = 1, 2, \dots$. Then, $\hat{\mathbf{x}}^{\epsilon_m}(t) \in \mathbf{X}^0(t, \hat{p}^{\epsilon_m})$, for each $t \in T_0$, and for each $m = 1, 2, \dots$, by Theorem 2 in Codognato and Ghosal (2000). But then, $(\int_{T_0} \hat{\mathbf{x}}^{\epsilon_m}(t)\, d\mu - \int_{T_0} \mathbf{w}(t)\, d\mu) \in \mathbf{Z}^0(\hat{p}^{\epsilon_m})$, for each $m = 1, 2, \dots$. We have that

$$\int_{T_1} \hat{\mathbf{x}}^{\epsilon_m}(t)\, d\mu + \int_{T_0} \hat{\mathbf{x}}^{\epsilon_m}(t)\, d\mu = \int_{T_0} \mathbf{w}(t)\, d\mu + \int_{T_1} \mathbf{w}(t)\, d\mu + l\epsilon_m e,$$

where e is a vector such that $e_i = 1$, for each $i \in L$, as $\mathbf{x}^{\epsilon_m}(tr) = \mathbf{x}^{\epsilon_m}(ts) = \mathbf{x}^n(t)$, $\mathbf{w}(tr) = \mathbf{w}(ts) = \mathbf{w}(t)$, $r, s = 1, \dots, n$, for each $t \in T_1$, and $\mu(tr) = \frac{\mu(t)}{n}$, $r = 1, \dots, n$, for each $m = 1, 2, \dots$. Then, it must be that

$$\int_{T_0} \hat{\mathbf{x}}^{\epsilon_m}(t)\, d\mu \leq \int_{T_0} \mathbf{w}(t)\, d\mu + \int_{T_1} \mathbf{w}(t)\, d\mu + l\epsilon_m e,$$

for each $m = 1, 2, \dots$. But then, we have that

$$\left| \int_{T_0} \hat{\mathbf{x}}^{i\epsilon_m}(t)\, d\mu - \int_{T_0} \mathbf{w}^i(t)\, d\mu \right| \leq \int_{T_0} \mathbf{w}^i(t)\, d\mu + \int_{T_1} \mathbf{w}^i(t)\, d\mu + l\epsilon_m$$

as $-\int_{T_1} \mathbf{w}^i(t)\, d\mu - l\epsilon \leq \int_{T_0} \hat{\mathbf{x}}^{i\epsilon_m}(t)\, d\mu \leq 2\int_{T_0} \mathbf{w}^i(t)\, d\mu + \int_{T_1} \mathbf{w}^i(t)\, d\mu + l\epsilon_m$, for each $i \in L$, for each $m = 1, 2, \dots$. Therefore, it must be that

$$\sum_{i=1}^{l} \left| \int_{T_0} \hat{\mathbf{x}}^{i\epsilon_m}(t)\, d\mu - \int_{T_0} \mathbf{w}^i(t)\, d\mu \right| \leq \sum_{i=1}^{l} \left(\int_{T_0} \mathbf{w}^i(t)\, d\mu + \int_{T_1} \mathbf{w}^i(t)\, d\mu + l\epsilon_m \right),$$

for each $m = 1, 2, \dots$. Moreover, there exists an m_0 such that

$$d[0, \mathbf{Z}^0(\hat{p}^{\epsilon_{k_m}})] > \sum_{i=1}^{l} \left(\int_{T_0} \mathbf{w}^i(t)\, d\mu + \int_{T_1} \mathbf{w}^i(t)\, d\mu + l\epsilon_m \right),$$

for each $m \geq m_0$, as the sequence $\{d[0, \mathbf{Z}^0(\hat{p}^{k_m})]\}$ diverges to $+\infty$ and the sequence $\{\epsilon_m\}$ converges to 0. Then,

$$\sum_{i=1}^{l} \left| \int_{T_0} \hat{\mathbf{x}}^{i\epsilon_{k_m}}(t)\, d\mu - \int_{T_0} \mathbf{w}^i(t)\, d\mu \right| > \sum_{i=1}^{l} \left(\int_{T_0} \mathbf{w}^i(t)\, d\mu + \int_{T_1} \mathbf{w}^i(t)\, d\mu + l\epsilon_m \right)$$

as $\sum_{i=1}^{l} |\int_{T_0} \hat{\mathbf{x}}^{i\epsilon_{k_m}}(t)\, d\mu - \int_{T_0} \mathbf{w}^i(t)\, d\mu| \geq d[0, \mathbf{Z}^0(\hat{p}^{\epsilon_{k_m}})]$, for each $m \geq m_0$, a contradiction. Hence, it must be that $\dot{p} \in \Delta \setminus \partial\Delta$. □

The fact that the sequence $\{\bar{\hat{\mathbf{B}}}^{\epsilon_m}\}$ belongs to the compact set $\{(b_{ij}) \in R_+^{l^2} : b_{ij} \leq n\int_{T_1} \mathbf{w}^i(t)\, d\mu + \int_{T_0} \mathbf{w}^i(t)\, d\mu\ i, j = 1, \dots, l\}$, the sequence $\{\hat{\mathbf{b}}^{\epsilon_m}(tr)\}$ belongs to the compact set $\mathbf{B}(tr)$, $r = 1, \dots, n$, for each $t \in T_1$, and the sequence $\{\hat{p}^{\epsilon_m}\}$, where $\hat{p}^{\epsilon_m} = p^{\epsilon_m}(\hat{\mathbf{b}}^{\epsilon_m})$, for each $m = 1, 2, \dots$, belongs to the unit simplex Δ, implies that there is a subsequence $\{\bar{\hat{\mathbf{B}}}^{\epsilon_{k_m}}\}$ of the sequence $\{\bar{\hat{\mathbf{B}}}^{\epsilon_m}\}$ which converges to an element of the set $\{(b_{ij}) \in R_+^{l^2} : b_{ij} \leq n\int_{T_1} \mathbf{w}^i(t)\, d\mu + \int_{T_0} \mathbf{w}^i(t)\, d\mu,\ i, j = 1, \dots, l\}$, a subsequence $\{\hat{\mathbf{b}}^{\epsilon_{k_m}}(tr)\}$ of the sequence $\{\hat{\mathbf{b}}^{\epsilon_m}(tr)\}$ which converges to an element of the set $\mathbf{B}(tr)$, $r = 1, \dots, n$, for each $t \in T_1$, and a subsequence $\{\hat{p}^{\epsilon_{k_m}}\}$ of the sequence $\{\hat{p}^{\epsilon_m}\}$ which converges to a price vector $\dot{p} \in \Delta \setminus \partial\Delta$, by Lemma 4. Since the sequence $\{\hat{\mathbf{b}}^{\epsilon_{k_m}}\}$ satisfies the assumptions of Theorem A in Artstein (1979), there is a function $\hat{\mathbf{b}}$ such that $\hat{\mathbf{b}}(tr)$ is the limit of the sequence $\{\hat{\mathbf{b}}^{\epsilon_{k_m}}(tr)\}$, $r = 1, \dots, n$, for each $t \in T_1$, $\hat{\mathbf{b}}(t)$ is a limit point of the sequence $\{\hat{\mathbf{b}}^{\epsilon_{k_m}}(t)\}$, for each $t \in T_0$, and such that the sequence $\{\bar{\hat{\mathbf{B}}}^{\epsilon_{k_m}}\}$ converges to $\bar{\hat{\mathbf{B}}}$. Then, $\hat{\mathbf{b}}(tr) = \hat{\mathbf{b}}(ts)$ as $\{\hat{\mathbf{b}}^{\epsilon_{k_m}}(tr)\} = \{\hat{\mathbf{b}}^{\epsilon_{k_m}}(ts)\}$, $r, s = 1, \dots, n$, for each $t \in T_1$, and $\hat{\mathbf{b}}(tr)$ is the limit of the sequence $\{\hat{\mathbf{b}}^{\epsilon_{k_m}}(tr)\}$, $r = 1, \dots, n$, for each $t \in T_1$. Hence, it can be proved, by the same argument used by Busetto et al. (2018) to show their existence theorem, that $\hat{\mathbf{b}}$ is an atom-type-symmetric Cournot-Nash equilibrium of Γ^n. □

6. Limit theorem

In this section, we state and prove our limit theorem. It establishes that, given a sequence of atom-type-symmetric Cournot-Nash equilibria of Γ^n, for $n = 1, 2, \dots$, there exists a Walras allocation of $\mathcal{E}$ with the following property: for each trader $t \in T$, the value of this Walras allocation at t is a limit point of the sequence of final holdings of t associated with the sequence of atom-type-symmetric Cournot-Nash equilibria of Γ^n, for $n = 1, 2, \dots$.

Theorem 3. *Under Assumptions 1, 2, 3, and 4, let $\{\hat{\mathbf{b}}^{\Gamma^n}\}$ be a sequence of atom-type-symmetric Cournot-Nash equilibria of Γ^n, $\{\hat{\mathbf{b}}^n\}$ be a sequence of strategy profiles of Γ, and $\{\hat{p}^n\}$ be a sequence of prices such that $\hat{\mathbf{b}}^n(t) = \hat{\mathbf{b}}^{\Gamma^n}(tr)$, $r = 1, \dots, n$, for each $t \in T_1$, $\hat{\mathbf{b}}^n(t) = \hat{\mathbf{b}}^{\Gamma^n}(t)$, for each $t \in T_0$, and $\hat{p}^n = p(\hat{\mathbf{b}}^{\Gamma^n})$, for each $n = 1, 2, \dots$. Then,*
(i) there exists a subsequence $\{\hat{\mathbf{b}}^{k_n}\}$ of the sequence $\{\hat{\mathbf{b}}^n\}$, a subsequence $\{\hat{p}^{k_n}\}$ of the sequence $\{\hat{p}^n\}$, a strategy profile $\hat{\mathbf{b}}$ of Γ, and a price vector $\hat{p} \in \Delta \setminus \partial\Delta$, such that $\hat{\mathbf{b}}(t)$ is the limit of the sequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$, for each $t \in T_1$, $\hat{\mathbf{b}}(t)$ is a limit point of the sequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$, for each $t \in T_0$, the sequence $\{\bar{\hat{\mathbf{B}}}^{k_n}\}$ converges to $\bar{\hat{\mathbf{B}}}$, and the sequence $\{\hat{p}^{k_n}\}$ converges to $\hat{p}$;
(ii) $\hat{\mathbf{x}}(t)$ is the limit of the sequence $\{\hat{\mathbf{x}}^{k_n}(t)\}$, for each $t \in T_1$, and $\hat{\mathbf{x}}(t)$ is a limit point of the sequence $\{\hat{\mathbf{x}}^{k_n}(t)\}$, for each $t \in T_0$, where $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), \hat{p})$ for each $t \in T$, and $\hat{\mathbf{x}}^{k_n}(t) = \mathbf{x}(t, \hat{\mathbf{b}}^{k_n}(t), \hat{p}^{k_n})$, for each $t \in T$, and for each $n = 1, 2, \dots$;
(iii) the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium of $\mathcal{E}$.

Proof. (i) The fact that the sequence $\{\bar{\hat{\mathbf{B}}}^n\}$ belongs to the compact set $\{(b_{ij}) \in R_+^{l^2} : b_{ij} \leq \int_T \mathbf{w}^i(t)\, d\mu,\ i, j = 1, \dots, l\}$, the sequence $\{\hat{\mathbf{b}}^n(t)\}$ belongs to the compact set $\mathbf{B}(t)$, for each $t \in T_1$, and the sequence $\{\hat{p}^n\}$, belongs to the unit simplex Δ, implies that there is a subsequence $\{\bar{\hat{\mathbf{B}}}^{k_n}\}$ of the sequence $\{\bar{\hat{\mathbf{B}}}^n\}$ which converges to an element of the set $\{(b_{ij}) \in R_+^{l^2} : b_{ij} \leq \int_T \mathbf{w}^i(t)\, d\mu,\ i, j = 1, \dots, l\}$, a subsequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$ of the sequence $\{\hat{\mathbf{b}}^n(t)\}$ which converges to an element of the set $\mathbf{B}(t)$, for each $t \in T_1$, and a subsequence $\{\hat{p}^{k_n}\}$ of the sequence $\{\hat{p}^n\}$ which converges to a price vector $\hat{p} \in \Delta \setminus \partial\Delta$, by Theorem 1. Since the sequence $\{\hat{\mathbf{b}}^{k_n}\}$ satisfies the assumptions of Theorem A in Artstein (1979), there is a function $\hat{\mathbf{b}}$ such that $\hat{\mathbf{b}}(t)$ is the limit of the sequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$, for each $t \in T_1$, $\hat{\mathbf{b}}(t)$ is a limit point of the sequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$, for each $t \in T_0$, and such that the sequence $\{\bar{\hat{\mathbf{B}}}^{k_n}\}$ converges to $\bar{\hat{\mathbf{B}}}$.

(ii) $\hat{\mathbf{x}}(t)$ is the limit of the sequence $\{\hat{\mathbf{x}}^{k_n}(t)\}$, for each $t \in T_1$, as $\hat{\mathbf{b}}(t)$ is the limit of the sequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$, for each $t \in T_1$, and the sequence $\{\hat{p}^{k_n}\}$ converges to $\hat{p}$, $\hat{\mathbf{x}}(t)$ is a limit point of the sequence $\{\hat{\mathbf{x}}^{k_n}(t)\}$, for each $t \in T_0$, as $\hat{\mathbf{b}}(t)$ is a limit point of the sequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$, for each $t \in T_0$, and the sequence $\{\hat{p}^{k_n}\}$ converges to $\hat{p}$.
(iii) $\bar{\hat{\mathbf{B}}}^{\Gamma^n} = \bar{\hat{\mathbf{B}}}^n$ as $\bar{\hat{\mathbf{b}}}_{ij}^{\Gamma^n} = \sum_{t \in T_1} \sum_{r=1}^n \hat{\mathbf{b}}_{ij}^{\Gamma^n}(tr)\mu(tr) + \int_{t \in T_0} \hat{\mathbf{b}}_{ij}^{\Gamma^n}(t)\, d\mu = \sum_{t \in T_1} n\hat{\mathbf{b}}_{ij}^n(t)\frac{\mu(t)}{n} + \int_{t \in T_0} \hat{\mathbf{b}}_{ij}^n(t)\, d\mu = \sum_{t \in T_1} \hat{\mathbf{b}}_{ij}^n(t)\mu(t) + \int_{t \in T_0} \hat{\mathbf{b}}_{ij}^n(t)\, d\mu = \bar{\hat{\mathbf{b}}}_{\mathbf{ij}}^{\mathbf{n}}$, $i, j = 1, \dots, l$, for $n = 1, 2, \dots$. Then, $\hat{p}^n = p(\hat{\mathbf{b}}^n)$ as $\hat{p}^n$ and $\{\hat{\mathbf{b}}^{\Gamma^n}\}$ satisfy equation (1) and $\bar{\hat{\mathbf{B}}}^{\Gamma^n} = \bar{\hat{\mathbf{B}}}^n$, for $n = 1, 2, \dots$. But then, by continuity, $\hat{p}$ and $\hat{\mathbf{b}}$ must also satisfy equation (1). We now show that, if two commodities $i, j \in L$ stand in the relation C, then $\bar{\hat{\mathbf{b}}}_{ij} > 0$. Suppose that $\bar{\hat{\mathbf{b}}}_{ij} = 0$. Let O be the subset of T^i such that Assumption 4 does not hold, for each $t \in O$, and let V be the subset of T_0 such that $\hat{\mathbf{b}}(t)$ is a not limit point of the sequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$, for each $t \in V$. It must be that $\mu(O) = \mu(V) = 0$ as we ignore null sets. Then, the set $\bar{T}^i = T^i \setminus (O \cup V)$ is measurable, as the sets T^i and $O \cup V$ are measurable and a $\mathcal{T}$ is a σ-algebra, and $\mu(\bar{T}^i) > 0$ as $\mu(T^i) > 0$ and $\mu(O \cup V) = 0$. But then, we have that $\int_{\bar{T}^i} \hat{\mathbf{b}}_{ij}(t)\, d\mu = 0$ as $\bar{\hat{\mathbf{b}}}_{ij} = 0$ and $\mu(\bar{T}^i) > 0$. Consider a trader $\tau \in \bar{T}^i$ for whom $\hat{\mathbf{b}}_{ij}(\tau) = 0$. $\hat{\mathbf{b}}(\tau)$ is a limit point of the sequence $\{\hat{\mathbf{b}}^{k_n}(\tau)\}$ as $\tau \notin V$. Then, there is a subsequence $\{\hat{\mathbf{b}}^{h_{k_n}}(\tau)\}$ of this sequence which converges to $\hat{\mathbf{b}}(\tau)$. But then, the subsequence $\{\hat{\mathbf{x}}^{h_{k_n}}(\tau)\}$ of the sequence $\{\hat{\mathbf{x}}^{k_n}(\tau)\}$ converges to $\hat{\mathbf{x}}(\tau)$ as the sequence $\{\hat{\mathbf{b}}^{h_{k_n}}(\tau)\}$ converges to $\hat{\mathbf{b}}(\tau)$ and the sequence $\{\hat{p}^{h_{k_n}}\}$ converges to $\hat{p}$. Therefore, we have that $\hat{\mathbf{x}}^j(\tau) = 0$ as $\hat{\mathbf{b}}_{ij}(\tau) = 0$ and $\hat{\mathbf{x}}(\tau) \in \mathbf{X}^0(\tau, \hat{p})$ as $\hat{\mathbf{x}}^{h_{k_n}}(\tau) \in \mathbf{X}^0(\tau, \hat{p}^{h_{k_n}})$, for each $n = 1, 2, \dots$, by Theorem 2 in Codognato and Ghosal (2000), and the correspondence $\mathbf{X}^0(\tau, \cdot)$ is upper hemicontinuous. Therefore, we have that $\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^j} = +\infty$ as $i, j \in L$ stand in the relation C and $\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^j} \leq \nu \hat{p}^j$, where ν is a Lagrange multiplier, by the necessary conditions of the Kuhn-Tucker Theorem. Moreover, there must be a commodity h such that $\hat{\mathbf{x}}^h(\tau) > 0$ as $u_\tau(\cdot)$ is strongly monotone, by Assumption 2, and $\hat{p}\mathbf{w}(\tau) > 0$. Then, $\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^h} = \nu \hat{p}^h$, by the necessary conditions of the Kuhn-Tucker Theorem. But then, $\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^h} = +\infty$ as $\nu = +\infty$, contradicting the assumption that $u_\tau(\cdot)$ is continuously differentiable. Therefore, if two commodities $i, j \in L$ stand in the relation C, then $\bar{\hat{\mathbf{b}}}_{ij} > 0$. This implies that the matrix $\bar{\hat{\mathbf{B}}}$ is irreducible by our Assumption 4 and by the argument used by Codognato and Ghosal (2000) in the proof of their Theorem 2. Consider the pair $(\hat{p}, \hat{\mathbf{x}})$. It is straightforward to show that the assignment $\hat{\mathbf{x}}$ is an allocation as $\hat{p}$ and $\hat{\mathbf{b}}$ satisfy equation (1) and that $\hat{\mathbf{x}}(t) \in \{x \in R_+^l : \hat{p}x = \hat{p}\mathbf{w}(t)\}$, for each $t \in T$. Suppose that $(\hat{p}, \hat{\mathbf{x}})$ is not a Walras equilibrium of $\mathcal{E}$. Then, there exists a trader $\tau \in T$ and a commodity bundle $\tilde{x} \in \{x \in R_+^l : \hat{p}x = \hat{p}\mathbf{w}(\tau)\}$ such that $u_\tau(\tilde{x}) > u_\tau(\hat{\mathbf{x}}(\tau))$. By Lemma 5 in Codognato and Ghosal (2000), there exist real numbers $\tilde{\lambda}^j \geq 0$, with $\sum_{j=1}^l \tilde{\lambda}^j = 1$, such that

$$\tilde{x}^j = \tilde{\lambda}^j \frac{\sum_{i=1}^l \hat{p}^i \mathbf{w}^i(\tau)}{\hat{p}^j}, \; j = 1, \dots, l.$$

Let $\tilde{b}_{ij} = \mathbf{w}^i(\tau)\tilde{\lambda}^j$, $i, j = 1, \dots, l$. Then, it is straightforward to verify that

$$\tilde{x}^j = \mathbf{w}^j(\tau) - \sum_{i=1}^l \tilde{b}_{ji} + \sum_{i=1}^l \tilde{b}_{ij} \frac{\hat{p}^i}{\hat{p}^j},$$

for each $j = 1, \dots, l$. Consider the following cases.

Case 1. $\tau \in T_1$. Let $\rho = k_1$ and let $\bar{\hat{\mathbf{B}}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho)$ be the aggregate matrix corresponding to the strategy profile $\hat{\mathbf{b}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho)$, where $\tilde{b}(\tau\rho) = \tilde{b}$, for $n = 1, 2, \dots$. Let $\Delta\bar{\hat{\mathbf{B}}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho)$ and $\Delta\bar{\hat{\mathbf{B}}}^{k_n}$ denote the diagonal matrices of row sums of, respectively, $\bar{\hat{\mathbf{B}}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho)$ and $\bar{\hat{\mathbf{B}}}^{k_n}$, for each $n = 1, 2, \dots$. Moreover, let $q_{\tau\rho}^{\Gamma^{k_n}}$, and q^{k_n} denote the vectors of the cofactors of the first column of, respectively, $\Delta\bar{\hat{\mathbf{B}}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho) - \bar{\hat{\mathbf{B}}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho)$ and $\Delta\bar{\hat{\mathbf{B}}}^{k_n} - \bar{\hat{\mathbf{B}}}^{k_n}$, for each $n = 1, 2, \dots$. Clearly, $q^{\Gamma^{k_n}} = q^{k_n}$ as $\bar{\hat{\mathbf{B}}}^{\Gamma^{k_n}} = \bar{\hat{\mathbf{B}}}^{k_n}$, for $n = 1, 2, \dots$. Let $\Delta\bar{\hat{\mathbf{B}}}$ be the diagonal matrix of row sums of $\bar{\hat{\mathbf{B}}}$ and q be the cofactors of the first column of $\Delta\bar{\hat{\mathbf{B}}} - \bar{\hat{\mathbf{B}}}$. The sequence $\{q^{k_n}\}$ converges to q as the sequence $\bar{\hat{\mathbf{B}}}^{k_n}$ converges to $\bar{\hat{\mathbf{B}}}$. Let $\bar{w} = \max\{\mathbf{w}^1(\tau), \dots, \mathbf{w}^l(\tau)\}$. Consider the matrix $\bar{\hat{\mathbf{B}}}^{\Gamma^{k_n}} - \bar{\hat{\mathbf{B}}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho)$, for each $n = 1, 2, \dots$. Then, we have that $\bar{\hat{\mathbf{b}}}_{ij}^{\Gamma^{k_n}} - \bar{\hat{\mathbf{b}}}_{ij}^{\Gamma^{k_n}} \setminus \tilde{b}_{ij}(\tau\rho) = \left(\frac{1}{k_n}\hat{\mathbf{b}}_{ij}^{\Gamma^{k_n}}(\tau\rho) - \frac{1}{k_n}\tilde{b}_{ij}(\tau\rho)\right)$, $i, j = 1, \dots, l$, for each $n = 1, 2, \dots$. But then, the sequence of Euclidean distances $\left\{\left\|\bar{\hat{\mathbf{B}}}^{\Gamma^{k_n}} - \bar{\hat{\mathbf{B}}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho)\right\|\right\}$ converges to 0 as $\left|\frac{1}{k_n}\hat{\mathbf{b}}_{ij}^{\Gamma^{k_n}}(\tau\rho) - \frac{1}{k_n}\tilde{b}_{ij}(\tau\rho)\right| = \frac{1}{k_n}\left|\hat{\mathbf{b}}_{ij}^{\Gamma^{k_n}}(\tau\rho) - \tilde{b}_{ij}(\tau\rho)\right| \leq \frac{1}{k_n}\bar{w}$, $i, j = 1, \dots, l$, for each $n = 1, 2, \dots$. The sequence $\left\{\bar{\hat{\mathbf{B}}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho)\right\}$ converges to $\bar{\hat{\mathbf{B}}}$ as, by the triangle inequality,

$$\begin{aligned} \left\|\bar{\hat{\mathbf{B}}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho) - \bar{\hat{\mathbf{B}}}\right\| &\leq \left\|\bar{\hat{\mathbf{B}}}^{\Gamma^{k_n}} - \bar{\hat{\mathbf{B}}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho)\right\| + \left\|\bar{\hat{\mathbf{B}}}^{\Gamma^{k_n}} - \bar{\hat{\mathbf{B}}}\right\| \\ &= \left\|\bar{\hat{\mathbf{B}}}^{\Gamma^{k_n}} - \bar{\hat{\mathbf{B}}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho)\right\| + \|\bar{\hat{\mathbf{B}}}^{k_n} - \bar{\hat{\mathbf{B}}}\|, \end{aligned}$$

for each $n = 1, 2, \dots$, and the sequences $\left\{\left\|\bar{\hat{\mathbf{B}}}^{\Gamma^{k_n}} - \bar{\hat{\mathbf{B}}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho)\right\|\right\}$ and $\left\{\left\|\bar{\hat{\mathbf{B}}}^{k_n} - \bar{\hat{\mathbf{B}}}\right\|\right\}$ converge to 0. Then, the sequence $\{q_{\tau\rho}^{\Gamma^{k_n}}\}$ converges to q. We have that $u_{\tau\rho}(\mathbf{x}(\tau\rho, \hat{\mathbf{b}}^{\Gamma^{k_n}}(\tau\rho), p(\hat{\mathbf{b}}^{\Gamma^{k_n}}))) \geq u_{\tau\rho}(\mathbf{x}(\tau\rho, \tilde{b}(\tau\rho), p(\hat{\mathbf{b}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho))))$ as $\hat{\mathbf{b}}^{\Gamma^{k_n}}$ is an atom-type-symmetric Cournot-Nash equilibrium of Γ^{k_n}, for each $n = 1, 2, \dots$. Then, we have that $u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}^{k_n}(\tau), p^{k_n})) \geq u_\tau(\mathbf{x}(\tau, \tilde{b}(\tau\rho), q_{\tau\rho}^{\Gamma^{k_n}}))$ as $u_{\tau\rho}(\cdot) = u_\tau(\cdot)$, $\hat{\mathbf{b}}^{\Gamma^{k_n}}(\tau\rho) = \hat{\mathbf{b}}^{k_n}(\tau)$, $p(\mathbf{b}^{\Gamma^{k_n}}) = \hat{p}^{k_n}$, $p(\hat{\mathbf{b}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho)) = \beta_{k_n} q_{\tau\rho}^{\Gamma^{k_n}}$, with $\beta_{k_n} > 0$, by Lemma 2 in Sahi and Yao, for each $n = 1, 2, \dots$. But then, it must be that

$$u_\tau(\hat{\mathbf{x}}(\tau)) = u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}(\tau), \hat{p})) \geq u_\tau(\mathbf{x}(\tau, \tilde{b}(\tau\rho), q)) = u_\tau(\tilde{x}),$$

as the sequence $\{\hat{\mathbf{b}}^{k_n}(\tau)\}$ converges to $\hat{\mathbf{b}}(\tau)$, the sequence $\{\hat{p}^{k_n}\}$ converges to $\hat{p}$, the sequence $\{q_{\tau\rho}^{\Gamma^{k_n}}\}$ converges to q, $\tilde{b}(\tau\rho) = \tilde{b}$, $\hat{p} = \theta q$, with $\theta > 0$, by Lemma 2 in Sahi and Yao, and $u_\tau(\cdot)$ is continuous, by Assumption 2. Since we have assumed that $u_\tau(\tilde{x}) > u_\tau(\hat{\mathbf{x}}(\tau))$, this generates a contradiction.

Case 2. $\tau \in T_0$. Let $\{\hat{\mathbf{b}}^{h_{k_n}}(\tau)\}$ be a subsequence of the sequence $\{\hat{\mathbf{b}}^{k_n}(\tau)\}$ which converges to $\hat{\mathbf{b}}(\tau)$. Moreover, let $\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}} \setminus \tilde{b}(\tau)$ be a strategy profile obtained by replacing $\hat{\mathbf{b}}^{h_{k_n}}(\tau)$ in $\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}}$ with $\tilde{b}$, for each $n = 1, 2, \dots$. We have that $u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}^{\Gamma^{h_{k_n}}}(\tau), p(\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}}))) \geq$

$u_\tau(\mathbf{x}(\tau, \tilde{b}(\tau), p(\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}} \setminus \tilde{b}(\tau))))$ as $\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}}$ is an atom-type-symmetric Cournot-Nash equilibrium of $\Gamma^{h_{k_n}}$, for each $n = 1, 2, \ldots$. Then, we have that $u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}^{h_{k_n}}(\tau), \hat{p}^{h_{k_n}})) \geq u_\tau(\mathbf{x}(\tau, \tilde{b}(\tau), \hat{p}^{h_{k_n}}))$ as $\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}}(\tau) = \hat{\mathbf{b}}^{h_{k_n}}(\tau)$, $p(\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}}) = \hat{p}^{h_{k_n}}$, and $p(\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}} \setminus \tilde{b}(\tau)) = p(\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}}) = \hat{p}^{h_{k_n}}$, by Lemma 1 in Codognato and Ghosal (2000). But then, it must be that

$$u_\tau(\hat{\mathbf{x}}(\tau)) = u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}(\tau), \hat{p})) \geq u_\tau(\mathbf{x}(\tau, \tilde{b}(\tau), \hat{p})) = u_\tau(\tilde{x}),$$

as the sequence $\{\hat{\mathbf{b}}^{h_{k_n}}(\tau)\}$ converges to $\hat{\mathbf{b}}(\tau)$, the sequence $\{p^{h_{k_n}}\}$ converges to $\hat{p}$, $\tilde{b}(\tau) = \tilde{b}$, and $u_\tau(\cdot)$ is continuous, by Assumption 2. Since we have assumed that $u_\tau(\tilde{x}) > u_\tau(\hat{\mathbf{x}}(\tau))$, this generates a contradiction. Hence, the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium of $\mathcal{E}$. □

The following example illustrates the limit result provided by Theorem 3.

Example 1. Consider the following specification of the exchange economy $\mathcal{E}$ satisfying Assumptions 1, 2, 3, and 4. $l = 2$, $T_1 = \{2, 3, \ldots, m\}$, with $m > 2$, $T_0 = [0, 1]$, $\mu(t) = \frac{1}{m}$, for each $t \in T_1$, T_0 is taken with Lebesgue measure, $\mathbf{w}(t) = (4, 0)$, $u_t(x) = x^1 + \ln x^2$, for each $t \in T_1$, $\mathbf{w}(t) = (0, 2)$, $u_t(x) = \ln x^1 + x^2$, for each $t \in [0, \frac{1}{2}]$, $\mathbf{w}(t) = (4, 0)$, $u_t(x) = x^1 + \ln x^2$, for each $t \in [\frac{1}{2}, 1]$. Then, there is a unique sequence $\{\hat{\mathbf{b}}^{\Gamma^n}\}$ of atom-type-symmetric Cournot-Nash equilibria of Γ^n, a sequence of strategy profiles $\{\hat{\mathbf{b}}^n\}$ of Γ such that $\hat{\mathbf{b}}^n(t) = \hat{\mathbf{b}}^{\Gamma^n}(tr)$, $r = 1, \ldots, n$, for each $t \in T_1$, $\hat{\mathbf{b}}^n(t) = \hat{\mathbf{b}}^{\Gamma^n}(t)$, for each $t \in T_0$, for each $n = 1, 2, \ldots$, which converges pointwise to a strategy profile $\hat{\mathbf{b}}$ of Γ, a sequence $\{\hat{p}^n\}$ of prices such that $\hat{p}^n = p(\hat{\mathbf{b}}^{\Gamma^n})$, for each $n = 1, 2, \ldots$, which converges to a price vector $\hat{p}$, and a sequence $\{\hat{\mathbf{x}}^n(t)\}$ such that $\hat{\mathbf{x}}^n(t) = \mathbf{x}(t, \hat{\mathbf{b}}^n(t), \hat{p}^n)$, for each $t \in T$, which converges pointwise to an allocation $\hat{\mathbf{x}}(t)$ such that $\mathbf{x}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), \hat{p})$, for each $t \in T$, where the pair $(\hat{p}, \hat{\mathbf{x}})$ is the unique Walras equilibrium of $\mathcal{E}$.

Proof. There is a unique sequence $\{\hat{\mathbf{b}}^{\Gamma^n}\}$ of atom-type-symmetric Cournot-Nash equilibria of Γ^n, a sequence of strategy profiles $\{\hat{\mathbf{b}}^n\}$ of Γ such that

$$\begin{aligned}\hat{\mathbf{b}}^n(t) &= \hat{\mathbf{b}}^{\Gamma^n}(tr)\\ &= \left(0, \frac{mn - 4 + \sqrt{9m^2n^2 - 16mn + 16}}{4mn - 4}, 0, 0\right),\end{aligned}$$

$r = 1, \ldots, n$, for each $t \in T_1$, $\hat{\mathbf{b}}^n(t) = \hat{\mathbf{b}}^{\Gamma^n}(t) = (0, 0, 1, 0)$, for each $t \in [0, \frac{1}{2}]$, $\hat{\mathbf{b}}^n(t) = \hat{\mathbf{b}}^{\Gamma^n}(t) = (0, 1, 0, 0)$, for each $t \in [\frac{1}{2}, 1]$, for each $n = 1, 2, \ldots$, which converges pointwise to a strategy profile $\hat{\mathbf{b}}$ of Γ such that $\hat{\mathbf{b}}(t) = (0, 1, 0, 0)$, for each $t \in T_1$, $\hat{\mathbf{b}}(t) = (0, 0, 1, 0)$, for each $t \in [0, \frac{1}{2}]$, $\hat{\mathbf{b}}(t) = (0, 1, 0, 0)$, for each $t \in [\frac{1}{2}, 1]$, a sequence $\{\hat{p}^n\}$ of prices such that

$$\begin{aligned}\hat{p}^n &= p(\hat{\mathbf{b}}^{\Gamma^n})\\ &= \left(\frac{2mn - 2}{5mn + \sqrt{9m^2n^2 - 16mn + 16} - 8}, \frac{3mn + \sqrt{9m^2n^2 - 16mn + 16} - 6}{5mn + \sqrt{9m^2n^2 - 16mn + 16} - 8}\right),\end{aligned}$$

for each $n = 1, 2, \ldots$, which converges to a price vector $\hat{p} = (\frac{1}{4}, \frac{3}{4})$, and a sequence $\{\hat{\mathbf{x}}^n(t)\}$ such that

$$\begin{aligned}\hat{\mathbf{x}}^n(t) &= \mathbf{x}(t, \hat{\mathbf{b}}^n(t), \hat{p}^n)\\ &= \left(4 - \frac{mn - 4 + \sqrt{9m^2n^2 - 16mn + 16}}{4mn - 4}, \frac{mn - 4 + \sqrt{9m^2n^2 - 16mn + 16}}{6mn + 2\sqrt{9m^2n^2 - 16mn + 16} - 12}\right),\end{aligned}$$

for each $t \in T_1$,

$$\begin{aligned}\mathbf{x}^n(t) &= \mathbf{x}(t, \hat{\mathbf{b}}^n(t), \hat{p}^n)\\ &= \left(\frac{3mn + \sqrt{9m^2n^2 - 16mn + 16} - 6}{2mn - 2}, 1\right),\end{aligned}$$

for each $t \in [0, \frac{1}{2}]$,

$$\begin{aligned}\mathbf{x}^n(t) &= \mathbf{x}(t, \hat{\mathbf{b}}^n(t), \hat{p}^n)\\ &= \left(3, \frac{2mn - 2}{3mn + \sqrt{9m^2n^2 - 16mn + 16} - 6}\right),\end{aligned}$$

for each $t \in [\frac{1}{2}, 1]$, which converges pointwise to an allocation $\hat{\mathbf{x}}(t)$ such that $\mathbf{x}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), \hat{p}) = (3, \frac{1}{3})$, for each $t \in T_1$, $\mathbf{x}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), \hat{p}) = (3, 1)$, for each $t \in [0, \frac{1}{2}]$, $\mathbf{x}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), \hat{p}) = (3, \frac{1}{3})$, for each $t \in [\frac{1}{2}, 1]$, where the pair $(\hat{p}, \hat{\mathbf{x}})$ is the unique Walras equilibrium of $\mathcal{E}$. □

7. Conclusion

The main theorem of this paper – Theorem 3 – is a limit result for the mixed version of the Shapley window model proposed by Busetto et al. (2018). It is innovative with respect to previous results in the same line in that it is crucially based on the Walrasian properties of atomless part's behavior, and can be applied to economic structures left uncovered by the limit theorem proved by Busetto et al. (2017). In our theorem, all traders may indeed have corner endowments, and indifference curves which touch the boundary of the consumption set. In particular, it covers the case of bilateral oligopoly with a competitive fringe for each commodity. We leave for further research the problem of proving a limit theorem for a bilateral oligopoly configuration without a competitive fringe, a case which violates Assumption 1 of our Theorem 3.

Mas-Colell (1982) provided a framework to deal with the asymptotic Cournotian foundations of Walrasian equilibrium theory. He considered both the case of symmetric Cournotian exchange, i.e., strategic market games, and the case of a production economy where firms have a Cournotian behavior and consumers are price-takers. The case we have considered in this paper, the mixed version of the Shapley window model, is a kind of new "hybrid" of which we have started to study the asymptotic foundational properties in terms of the new concept of symmetric partial replication *à la* Cournot. In particular, we have shown that, under our four assumptions, any sequence of Cournot-Nash allocations of the strategic market games associated with the partial replications of the exchange economy "converges" to a Walras allocation of the original economy. According to the foundational paradigm proposed by Mas-Colell (1982) we should show that the converse of this result also holds: namely, we should show that any, possibly regular, Walras allocation is the limit of a sequence of Cournot-Nash allocations of the strategic market games associated with the partial replications of our exchange economy. We leave this further challenging foundational task for future research.

As we have reminded, Codognato and Ghosal (2000) proved an equivalence result *à la* Aumann (1964) for a version of the Shapley window model with an atomless continuum of traders, under the same assumptions of our Theorem 3. García-Cutrin and Hervés-Beloso (1993) and Hüsseinov (1994) proposed to reconcile, by means of the notion of core, the asymptotic results, obtained by replicating an exchange economy with a finite number of traders, and the equivalence results, obtained for limit exchange economies with a continuum of traders. They showed that, when only a finite number of traders' characteristics can be distinguished, atoms can be split into a continuum of identical traders generating an equivalence between asymptotic results and results in the continuum (see also Hervés-Beloso and Moreno-García (2020) for a sketch of an application of these results to strategic market games). We leave for further research an investigation of an application of this methodology to the mixed version of the Shapley window model with the goal of reconciling our Theorem 3 with the equivalence theorem provided by Codognato and Ghosal (2000).

Data availability

No data was used for the research described in the article.

Declaration of competing interest

none

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