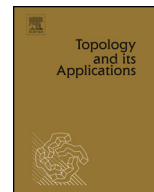


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## Topology and its Applications

journal homepage: [www.elsevier.com/locate/topol](http://www.elsevier.com/locate/topol)An introduction to  $\mathcal{I}$ -characterized subgroups of the circleD. Dikranjan\*, R. Di Santo, A. Giordano Bruno<sup>1</sup>, H. Weber

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## ABSTRACT

A subgroup  $H$  of the circle group  $\mathbb{T}$  is said to be characterized by a sequence  $\mathbf{u} = (u_n)_{n \in \mathbb{N}}$  of integers if  $H = \{x \in \mathbb{T} : u_n x \rightarrow 0\}$ . The introduction of these subgroups was motivated by problems arising from various areas of Mathematics, so they were thoroughly investigated. Recently generalizations of this notion were introduced based on weaker notions of convergence, starting from statistical convergence and ending with  $\mathcal{I}$ -convergence for an ideal  $\mathcal{I}$  of  $\mathbb{N}$ . This survey paper is dedicated to collect the wealth of results and open problems obtained on these new kind of characterized subgroups of  $\mathbb{T}$ .

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## Contents

Introduction . . . . .	2
Acknowledgments . . . . .	4
1. Historical background on characterized subgroups of $\mathbb{T}$ . . . . .	5
1.1. Arithmetic, Hadamard and recursive sequences . . . . .	5
1.2. Theorems A, D, E and Corollary A . . . . .	7
1.3. Arithmetic and recursive characterizing sequences . . . . .	7
2. Statistically characterized subgroups of $\mathbb{T}$ and generalizations . . . . .	10
2.1. Statistically characterized subgroups of $\mathbb{T}$ . . . . .	10
2.2. $\alpha$ -characterized subgroups of $\mathbb{T}$ and weight functions . . . . .	12
2.3. Modulus function . . . . .	13
3. Ideals – generalities . . . . .	15
3.1. The convexity number and the gap number . . . . .	15
3.2. Basic definitions on ideals . . . . .	16
3.3. Maximal ideals vs ultrafilters . . . . .	18
3.4. Analytic $P$ -ideals . . . . .	20
4. Density of ideals and the properties $\text{snt}$ , $(**)$ , $(\dagger)$ and $(*)$ . . . . .	23
4.1. The property $(**)$ and dense ideals . . . . .	23
4.2. $\text{snt}$ vs $(**)$ and $(\dagger)$ . . . . .	25
4.3. The condition $(*)$ . . . . .	27
5. Ideal convergence and $\mathcal{I}$ -characterized subgroups of $\mathbb{T}$ . . . . .	28

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5.1. Ideal convergence . . . . . 28  
 5.2.  $\mathcal{I}$ -characterized subgroups of  $\mathbb{T}$  . . . . . 30  
 5.3. Theorem B: when  $|t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})| = \mathfrak{c}$  . . . . . 32  
 5.4. Theorem C: when  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) \neq t_{\mathbf{u}}(\mathbb{T})$  . . . . . 33  
 6. Eggleston’s dichotomy for  $\mathcal{I}$ -characterized subgroups . . . . . 34  
 6.1. Eggleston’s dichotomy under  $(*)$  . . . . . 35  
 6.2. Eggleston’s dichotomy under  $(\dagger)$  . . . . . 36  
 7. Final comments and open problems . . . . . 38  
 7.1. Ideals . . . . . 38  
 7.2. Continued fractions and recursive sequences . . . . . 39  
 7.3.  $\mathfrak{C}\mathfrak{h}\mathfrak{a}\mathfrak{r}$  vs  $\mathfrak{C}\mathfrak{h}\mathfrak{a}\mathfrak{r}^s$  and  $\mathfrak{C}\mathfrak{h}\mathfrak{a}\mathfrak{r}^{\mathcal{I}}$  . . . . . 39  
 7.4. Questions related to countability or triviality of  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  . . . . . 40  
 7.5. Compact case . . . . . 42  
 References . . . . . 43

**Introduction**

The last survey [31] on characterized subgroups of the circle group  $\mathbb{T}$  was written seven years ago. Its aim was to give a complete picture of this ubiquitous notion, ranging from Algebra (topologically  $\mathbf{u}$ -torsion elements of a topological group) to Topology (precompact topologies on  $\mathbb{Z}$  with or without converging sequences) and Harmonic Analysis (Arbault sets and other classes of ‘thin’ sets). None of these topics will be treated in this survey, the interested reader can address [7,31,34].

In the last few years some new trends appeared in this field, based on weaker notions of convergence: statistical convergence (defined in [22] – see §2),  $\alpha$ -statistical convergence (defined in [10,14] – see §2.2) and  $f^g$ -statistical convergence (defined in [15] – see §2.3). All these specific convergences, including the usual one, are particular cases of the ideal convergence, introduced by Cartan [18] as follows. A non-empty subfamily  $\mathcal{I}$  of  $\mathcal{P}(\mathbb{N})$  is an *ideal of  $\mathbb{N}$*  if  $A \cup B \in \mathcal{I}$  for every  $A, B \in \mathcal{I}$  and if  $A \subseteq B$  in  $\mathcal{P}(\mathbb{N})$  with  $B \in \mathcal{I}$ , then  $A \in \mathcal{I}$ . An easy example of ideal of  $\mathbb{N}$  is the family  $\mathcal{F}in$  of all finite subsets of  $\mathbb{N}$ . A sequence  $(x_n)$  in  $\mathbb{T}$  is said to  $\mathcal{I}$ -converge to a point  $x \in \mathbb{T}$ , denoted by  $x_n \xrightarrow{\mathcal{I}} x$ , if  $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$  for every neighborhood  $U$  of  $x$  in  $\mathbb{T}$  (see Definition 5.1 for a more general definition of  $\mathcal{I}$ -convergence and §5.1 for its properties). The other convergences mentioned above can be obtained as  $\mathcal{I}$ -convergences by means of specific ideals  $\mathcal{I}$  of  $\mathbb{N}$ ; in particular,  $\mathcal{F}in$ -convergence coincides with the usual convergence.

Each of these convergences can be used to define a respective version of characterized subgroup of  $\mathbb{T}$ , of those the following notion of  $\mathcal{I}$ -characterized subgroup of  $\mathbb{T}$  (see Definition 5.5) covers all previous ones: for an ideal  $\mathcal{I}$  of  $\mathbb{N}$  and a sequence  $\mathbf{u} = (u_n)$  in  $\mathbb{Z}$ , a subgroup  $H$  of  $\mathbb{T}$  is  $\mathcal{I}$ -characterized if

$$H = t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) := \{x \in \mathbb{T} : u_n x \xrightarrow{\mathcal{I}} 0\}.$$

In particular,  $t_{\mathbf{u}}^{\mathcal{F}in}(\mathbb{T}) = t_{\mathbf{u}}(\mathbb{T})$  is the subgroup characterized by  $\mathbf{u}$  in the classical sense.

The main aim of this paper is to give an overview on the results obtained on all the new kinds of characterized subgroups of the circle group  $\mathbb{T}$ , with special emphasis on the most general notion of  $\mathcal{I}$ -characterized subgroups. In order to briefly describe the content of the paper, we use the enumeration given below with A, B, C, D, E for the major five results and their corollaries, according to the numbering adopted in [22] and followed afterwards in [14,20,21,23–25,32,44,45]. The precise references to the literature are given in the main text. Some proofs are omitted, especially in the case of published results (unless we propose a different proof). Most of the results of the unpublished preprint [32] are given with a sketch of the proof.

The initial §1 contains background from the classical theory of characterized subgroups of  $\mathbb{T}$ . According to Theorem A,  $t_{\mathbf{u}}(\mathbb{T})$  is an  $F_{\sigma\delta}$  (hence, Borel) subgroup of  $\mathbb{T}$  for any sequence of integers  $\mathbf{u}$ . As a consequence  $t_{\mathbf{u}}(\mathbb{T})$  is Polishable, and so it is either countable or has size  $\mathfrak{c}$  by Corollary A. Moreover, by Theorem D,

$t_{\mathbf{u}}(\mathbb{T}) = \mathbb{T}$  if and only if  $\mathbf{u}$  is eventually 0. On the other hand, Theorem E provides a family of non-characterized  $F_{\sigma}$ -subgroups of  $\mathbb{T}$ .

Another central result in the classical theory of characterized subgroups of  $\mathbb{T}$  is the so-called Eggleston’s dichotomy (see Theorem 1.8): for any sequence  $\mathbf{u}$  in  $\mathbb{N}_+$ ,  $|t_{\mathbf{u}}(\mathbb{T})| = \mathfrak{c}$  if  $u_{n+1}/u_n \rightarrow +\infty$  and  $t_{\mathbf{u}}(\mathbb{T})$  is countable if the sequence  $(u_{n+1}/u_n)$  is bounded. Eggleston’s dichotomy becomes a real dichotomy when the sequence  $\mathbf{u}$  is arithmetic (i.e.,  $u_n \mid u_{n+1}$  for every  $n \in \mathbb{N}$ ) or, more in general, recursive (in the sense of (1.3)).

Finally, we mention that the elements of the characterized subgroups  $t_{\mathbf{u}}(\mathbb{T})$  were completely described in [30] for  $\mathbf{u} \in \mathcal{A}$ , the class of arithmetic sequences (see Theorem 1.7). This result, as well as the previous ones, were later generalized for all kind of statistically characterized subgroups of  $\mathbb{T}$  and for  $\mathcal{I}$ -characterized subgroups of  $\mathbb{T}$ , as we describe in detail in the main part of the paper.

The first step out of the realm of classical characterized subgroups of  $\mathbb{T}$  is described in §2, focused on the statistically characterized subgroups  $t_{\mathbf{u}}^s(\mathbb{T})$  of  $\mathbb{T}$  and further generalizations. In particular, Theorems B and C from [22], specific for statistically characterized subgroups, are recalled in §2.1. They state respectively that for any  $\mathbf{u} \in \mathcal{A}$ ,  $|t_{\mathbf{u}}^s(\mathbb{T})| = \mathfrak{c}$  and  $t_{\mathbf{u}}(\mathbb{T}) \subsetneq t_{\mathbf{u}}^s(\mathbb{T})$  (these two results are actually equivalent to the statement  $|t_{\mathbf{u}}^s(\mathbb{T}) \setminus t_{\mathbf{u}}(\mathbb{T})| = \mathfrak{c}$  in view of Claim 2.1). The counterparts of Theorems B and C for statistically characterized subgroups by recursive sequences as in (1.2) are given as Theorems  $B^R$  and  $C^R$  in §2.1.

In §§2.2, 2.3 we recall, respectively,  $\alpha$ -statistically characterized subgroups of  $\mathbb{T}$  for  $\alpha \in (0, 1]$  and  $f^g$ -statistically characterized subgroups of  $\mathbb{T}$  for  $f$  a modulus function and  $g$  a weight function. The latter notion generalizes the former one, which in turn for  $\alpha = 1$  coincides with that of statistically characterized subgroup. The main known results on this kind of statistically characterized subgroups are mentioned.

It is not surprising that the properties of  $\mathcal{I}$ -characterized subgroups of  $\mathbb{T}$  depend in a strong way on the properties of the ideal  $\mathcal{I}$  of  $\mathbb{N}$ . This is why, before discussing the properties of  $\mathcal{I}$ -characterized subgroups of  $\mathbb{T}$  and related theorems, in §§3, 4 we pay special attention to ideals of  $\mathbb{N}$ , starting from their basic properties and their relation with filters. Beyond some classes of ideals already largely studied in the literature, e.g., maximal ideals, analytic ideals or  $P$ -ideals (that is, closed under countable pseudounions), we pay special attention to ideals  $\mathcal{I}$  of  $\mathbb{N}$  satisfying the four conditions,  $\text{snt}$ ,  $(**)$ ,  $(\dagger)$  and  $(*)$ , recently imposed in [25] in the study of  $\mathcal{I}$ -characterized subgroups.

Our aim is to simplify and unify these four conditions, and most of all to clarify the precise relations among them and with other known properties of ideals  $\mathcal{I}$  of  $\mathbb{N}$ , such as property  $P$ , maximality, translation invariance and density. To this end we avoid the recourse to the related filter  $\mathcal{I}^* := \{\mathbb{N} \setminus A : A \in \mathcal{I}\}$  of complements. In alternative, we make use of two combinatorial invariants  $c(A)$  and  $g(A)$  of infinite subsets  $A$  of  $\mathbb{N}$  (respectively, the convexity number and the gap number) introduced in §3.1. They capture deep intrinsic properties of the ideals and are mainly used in §4, to show that

$$(*) \Rightarrow (\dagger) \Rightarrow \text{snt} \Rightarrow (**)$$

and that the last three properties are equivalent for  $P$ -ideals. Of those four properties,  $\text{snt}$  is defined via (the strong negation of) translation invariance, whereas the remaining three, conveniently described via the convexity number  $c(-)$ , are very similar one to each other and much easier to apply in specific proofs. This is one of the reasons why (when property  $P$  is not explicitly required) we give preference to either  $(\dagger)$  or  $(**)$  instead of  $\text{snt}$ , depending on whether we prefer to have a stronger or weaker condition.

In §5.1 the ideal convergence à la Cartan is recalled, which is used to introduce the main topic of this survey, namely,  $\mathcal{I}$ -characterized subgroups of  $\mathbb{T}$  in §5.2.

In §§5.3, 5.4 respectively, counterparts of Theorems B and C for  $\mathcal{I}$ -characterized subgroups of  $\mathbb{T}$  are studied; namely, sufficient conditions are given ensuring  $|t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})| = \mathfrak{c}$  and  $t_{\mathbf{u}}(\mathbb{T}) \subsetneq t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  for  $\mathbf{u} \in \mathcal{A}$ , respectively. Moreover, the counterparts of Theorems  $B^R$  and  $C^R$  from [25] are recalled, in the case of  $\mathcal{I}$ -characterized subgroups of  $\mathbb{T}$  for analytic  $P$ -ideals  $\mathcal{I}$  which fail to satisfy  $(**)$  (see Theorems 5.19(2) and 5.22(2)).

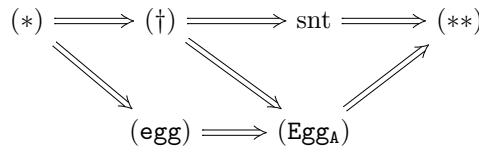
§6 is focused on Eggleston’s dichotomy for  $\mathcal{I}$ -characterized subgroups. Inspired by the classical Eggleston’s dichotomy, Das and Ghosh [25] carried out a remarkable tour de force and found the subtle properties of an ideal  $\mathcal{I}$  of  $\mathbb{N}$  implying that the  $\mathcal{I}$ -characterized subgroups with respect to such an ideal  $\mathcal{I}$  obey Eggleston’s dichotomy. Since we are dealing only with ideals  $\mathcal{I}$  of  $\mathbb{N}$  containing  $\mathcal{F}in$  (so that  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  contains  $t_{\mathbf{u}}(\mathbb{T})$ ), it is only necessary to ensure that  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  is countable provided that the sequence  $(u_{n+1}/u_n)$  is bounded; we briefly say that  $\mathcal{I}$  satisfies (egg) in such a case (see Definition 6.1). The following implications

$$(*) \Rightarrow (\text{egg}) \Rightarrow (**)$$

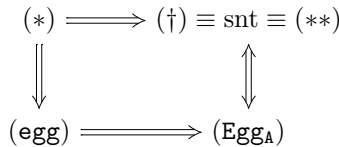
hold and do not need the hypothesis  $P$  imposed on the ideal (for more detail see Theorem 6.5).

Inspired by the fact that for arithmetic sequences the classical Eggleston’s dichotomy becomes a real dichotomy, we say also that an ideal  $\mathcal{I}$  of  $\mathbb{N}$  satisfies  $(\text{Egg}_{\mathcal{A}})$  if for every  $\mathbf{u} \in \mathcal{A}$ , the subgroup  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  is countable if and only if  $(u_{n+1}/u_n)$  is bounded. This property turns out to be weaker than  $(\dagger)$  and stronger than  $(**)$  (see Corollary 6.14). A similar notion and result are given for recursive sequences.

In the next diagram we collect all known implications concerning Eggleston’s dichotomy for an ideal  $\mathcal{I}$  of  $\mathbb{N}$  containing  $\mathcal{F}in$ .



Under the hypothesis that  $\mathcal{I}$  is a  $P$ -ideal, the right triangle collapses:



Our strategy in §§5, 6 is to reduce, as much as possible, the proofs concerning  $\mathcal{I}$ -characterized subgroups to known properties of characterized subgroups (Proposition 1.6 and Lemma 5.7 and their applications in the sketches of proofs of Theorems 5.20, 5.23, 5.25 and 6.5 should give an idea of this line). On the other hand, we relax as much as possible the hypotheses on the ideal  $\mathcal{I}$  in the main results on  $\mathcal{I}$ -characterized subgroups of  $\mathbb{T}$  in §§5, 6, allowing for a wider range of applications. For example, by avoiding the property of being a  $P$ -ideal, we extend the validity of various results to maximal ideals, which in some models of ZFC are never  $P$ -ideals (see Remark 3.16). Moreover, while there are  $2^{\mathfrak{c}}$  many maximal ideals (see Remark 5.21), there are at most  $\mathfrak{c}$  many analytic  $P$ -ideals, and this is the blanket hypothesis imposed in [25,44].

The final §7 collects old and new open problems and related comments motivated by the results recalled in this paper.

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It is a pleasure to thank Prof. Pratulanda Das and his research group (in particular, Dr. Ayan Ghosh) for the fruitful exchange of material and ideas concerning  $\mathcal{I}$ -characterized subgroups of  $\mathbb{T}$ .

We also thank the referee for the useful comments and suggestions.

1. Historical background on characterized subgroups of  $\mathbb{T}$

As usual, denote by  $\mathbb{N}$  the set of natural numbers and  $\mathbb{N}_+ := \{n \in \mathbb{N} : n > 0\}$ , while  $\mathbb{Z}$ ,  $\mathbb{Q}$  and  $\mathbb{R}$  are respectively the group of integers, rationals and reals. For  $m \in \mathbb{N}_+$ ,  $\mathbb{Z}(m) \cong \mathbb{Z}/m\mathbb{Z}$  denotes the cyclic group of order  $m$ , and for  $p$  a prime let  $\mathbb{Z}(p^\infty)$  denote the Prüfer group.

Moreover, let  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$  be the circle group denoted additively, and let  $\varphi: \mathbb{R} \rightarrow \mathbb{T}$  be the canonical projection. The restriction  $\varphi|_{[0,1)}: [0, 1) \rightarrow \mathbb{T}$  is a bijection. On  $\mathbb{T}$ , let  $\| - \|: \mathbb{T} \rightarrow [0, 1/2]$  be the norm defined by  $\|x + \mathbb{Z}\| := \min\{|x - n| : n \in \mathbb{N}\}$  for every  $x \in \mathbb{R}$ .

For  $A, B$  subsets of  $\mathbb{N}$ , we write  $A \subseteq^* B$  when  $A \setminus B$  is finite.

1.1. Arithmetic, Hadamard and recursive sequences

In this paper we intensively use sequences of integers  $\mathbf{u} = (u_n)$  that will be supposed to be actually sequences in  $\mathbb{N}$ , since the convergence of a null sequence  $(u_n x)$  in  $\mathbb{T}$  does not depend on the sign (see also Remark 1.3).

For sequences  $\mathbf{u}$  in  $\mathbb{N}_+$ , we always assume without loss of generality that  $u_0 = 1$  and we put  $b_n^{\mathbf{u}} := u_n/u_{n-1}$  for every  $n \in \mathbb{N}_+$ . When it is clear in the context we simply write  $b_n = b_n^{\mathbf{u}}$ . In these terms

$$u_1 = b_1, u_2 = b_1 b_2, \dots, u_{n+1} = b_{n+1} b_n \dots b_1 = b_{n+1} u_n, \dots$$

We call a sequence  $\mathbf{u}$  in  $\mathbb{N}_+$  an *arithmetic sequence* if all ratios  $b_n^{\mathbf{u}}$  are integers, while  $\mathbf{u}$  is called a *Hadamard sequence* if there exists  $q > 1$  such that  $b_n^{\mathbf{u}} \geq q$  for every  $n \in \mathbb{N}_+$  (cf. Remark 3.4).

From now on we consider only subsequences  $\mathbf{v}$  of sequences  $\mathbf{u}$  in  $\mathbb{N}_+$  starting with  $v_0 = u_0 = 1$ . Clearly, a subsequence of an arithmetic (resp., Hadamard) sequence is arithmetic (resp., Hadamard).

Following [36], given a sequence  $\mathbf{u}$  in  $\mathbb{N}_+$ , call an infinite subset  $A$  of  $\mathbb{N}$ :

- *b-bounded* if the sequence  $(b_n^{\mathbf{u}})_{n \in A}$  is bounded;
- *b-divergent* if the sequence  $(b_n^{\mathbf{u}})_{n \in A}$  diverges to infinity.

In particular,  $\mathbf{u}$  is *b-bounded* if  $\mathbb{N}_+$  is *b-bounded*, that is, the sequence  $\mathbf{b}^{\mathbf{u}} = (b_n^{\mathbf{u}})$  is bounded, and  $\mathbf{u}$  is *b-divergent* if  $\mathbb{N}_+$  is *b-divergent*, that is,  $\lim_{n \rightarrow \infty} b_n^{\mathbf{u}} = \infty$ . Finally, we say that  $\mathbf{u}$  is *b-unbounded* if  $\mathbf{b}^{\mathbf{u}}$  is not bounded.

Let  $\mathcal{S}$ ,  $\mathcal{A}$  and  $\mathcal{H}$  denote respectively the family of strictly increasing sequences of positive integers and its subfamilies of arithmetic (resp., Hadamard) sequences. Clearly,

$$\mathcal{A} \subseteq \mathcal{H} \subseteq \mathcal{S}.$$

Let also  $\mathcal{S}_{bd}$  (resp.,  $\mathcal{A}_{bd}$ ) denote the family of all *b-bounded* (arithmetic) sequences in  $\mathcal{S}$ , and  $\mathcal{S}_{div}$  (resp.,  $\mathcal{A}_{div}$ ) the family of all *b-divergent* (arithmetic) sequences in  $\mathcal{S}$ . Then  $\mathcal{S} \setminus \mathcal{S}_{bd}$  is the family of all *b-unbounded* sequences in  $\mathcal{S}$ .

Another important source of sequences in  $\mathbb{N}_+$  comes from the continued fraction expansion

$$\gamma = [a_0; a_1, a_2, \dots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}$$

of an irrational number  $\gamma$ . Let  $\mathbf{a}^\gamma = (a_n)$ . Since we are mainly interested in  $\varphi(\gamma) \in \mathbb{T}$ , it is not restrictive to assume that  $\gamma \in (0, 1)$  and  $\gamma = [0; a_1, a_2, \dots]$ , as  $\gamma + k = [k; a_1, a_2, \dots]$  for every  $k \in \mathbb{Z}$ .

We are mainly interested in the increasing sequence  $\mathbf{q}^\gamma = (q_n)$  in  $\mathbb{N}_+$  of the best approximation denominators of  $\gamma$ , with  $q_0 = 1$ . In other words, putting  $q_{-1} = 0$ , one has

$$[a_0; a_1, \dots, a_n] = \frac{p_n}{q_n} \quad \text{and} \quad q_{n+1} = a_{n+1}q_n + q_{n-1}, \tag{1.1}$$

for every  $n \in \mathbb{N}$ . When no confusion is possible, we simply write  $\mathbf{a}$  and  $\mathbf{q}$  in place of  $\mathbf{a}^\gamma$  and  $\mathbf{q}^\gamma$ .

**Remark 1.1.** In the above notation  $b_n^{\mathbf{q}} = a_n + 1/b_{n-1}^{\mathbf{q}}$  for every  $n \in \mathbb{N}_+$ , hence

$$\mathbf{q} \text{ is } b\text{-bounded if and only if } \mathbf{a} \text{ is bounded.}$$

With  $A = \{n \in \mathbb{N} : a_n = 1\}$ , it is easy to check that  $\mathbf{q}$  is Hadamard if and only if  $\{a_n : n \in A + 1\}$  is bounded.

The sequences  $\mathbf{q}$  in  $\mathbb{N}_+$  with (1.1) are in particular recursive sequences with  $q_{-1} = 0$  and  $q_0 = 1$ . An alternative way to describe them is to impose, beyond  $q_{-1} = 0$  and  $q_0 = 1$ ,  $q_n \mid q_{n+1} - q_{n-1}$  for all  $n \in \mathbb{N}$ . In other words, letting

$$\mathcal{R} := \{\mathbf{q} = (q_n) \in \mathbb{N}^{\mathbb{N}} : q_0 = 1 \text{ and } q_n \mid q_{n+1} - q_{n-1} \text{ for every } n \in \mathbb{N}_+\}, \tag{1.2}$$

there is a bijection between the irrationals  $\gamma = [0; a_1, a_2, \dots] \in (0, 1)$  and the sequences  $\mathbf{q} \in \mathcal{R}$ . In view of Remark 1.1, this bijection takes the irrationals  $\gamma \in (0, 1)$  with  $\mathbf{a}$  bounded exactly to

$$\mathcal{R}_{bd} := \{\mathbf{q} \in \mathcal{R} : \mathbf{q} \text{ is } b\text{-bounded}\}.$$

One can consider also a more general form of recursive sequences than those defined in (1.1) in order to include the arithmetic ones, namely, sequences  $\mathbf{u}$  in  $\mathbb{N}_+$  of the form

$$u_{n+1} = s_{n+1}u_n + t_{n+1}u_{n-1}, \tag{1.3}$$

with assigned values  $u_{-1} = 0$ ,  $u_0 = 1$  and  $s_n \geq t_n$  in  $\mathbb{N}$ . When  $t_{n+1} = 1$  for every  $n \in \mathbb{N}$ , we obtain exactly (1.1) and when  $t_{n+1} = 0$  for every  $n \in \mathbb{N}$ , we obtain  $\mathbf{u} \in \mathcal{A}$ . So, denoting by  $\mathcal{R}^*$  the class of sequences satisfying (1.3), we have  $\mathcal{A} \subseteq \mathcal{R}^* \supseteq \mathcal{R}$ . Clearly,  $\mathbf{u} \in \mathcal{R}_{bd}^* := \mathcal{R} \cap \mathcal{S}_{bd}$  if and only if the sequence  $(s_n)$  is bounded.

**Example 1.2.** A prominent example of a recursive sequence is the classical Fibonacci sequence  $\mathbf{f} = (f_n) \in \mathcal{R}$  defined by  $f_1 = 1$ ,  $f_2 = 1$  and for all  $n \geq 3$ ,  $f_n = f_{n-1} + f_{n-2}$ . As all  $a_n = 1$  in (1.1),  $\mathbf{f} \in \mathcal{R}_{bd}$ .

Recall that a subgroup  $H$  of  $\mathbb{T}$  is *characterized* if there exists a sequence of integers  $\mathbf{u}$  such that  $H = t_{\mathbf{u}}(\mathbb{T}) := \{x \in \mathbb{T} : u_n x \rightarrow 0\}$ . In this case we say also that  $H$  is *characterized by  $\mathbf{u}$*  and that  $\mathbf{u}$  is a *characterizing sequence* for  $H$ .

**Remark 1.3.** As mentioned above, given a characterized subgroup  $H = t_{\mathbf{u}}(\mathbb{T})$  of  $\mathbb{T}$ , the characterizing sequence  $\mathbf{u}$  can be supposed to be actually a sequence in  $\mathbb{N}$ . As  $\mathbf{u}$  is eventually 0 precisely when  $t_{\mathbf{u}}(\mathbb{T}) = \mathbb{T}$  (see Theorem D below), we further assume that  $\mathbf{u}$  is in  $\mathbb{N}_+$ .

Let us see that one can even choose  $\mathbf{u} \in \mathcal{S}$  when  $H \neq \mathbb{T}$ . In fact, if  $\mathbf{u}$  has a constant subsequence then  $t_{\mathbf{u}}(\mathbb{T})$  is finite and this case is settled in Remark 1.5. If  $\mathbf{u}$  has no constant subsequences, one can suppose that  $\mathbf{u}$  is non-decreasing (as the convergence of a sequence does not depend on the order of its members). Then, the non-decreasing finite-to-one sequence  $n \mapsto u_n$  obtained in this way can be replaced by a one-to-one subsequence  $\mathbf{v} \in \mathcal{S}$  with the same range, so  $t_{\mathbf{v}}(\mathbb{T}) = t_{\mathbf{u}}(\mathbb{T})$ .

When  $\mathcal{X}$  is one of the classes of sequences introduced above (i.e.,  $\mathcal{S}$ ,  $\mathcal{A}$ ,  $\mathcal{S}_{bd}$ ,  $\mathcal{S}_{div}$ ,  $\mathcal{A}_{bd}$ ,  $\mathcal{A}_{div}$ ,  $\mathcal{H}$ ,  $\mathcal{R}$ ,  $\mathcal{R}_{bd}$ ,  $\mathcal{R}^*$ ,  $\mathcal{R}_{bd}^*$ ), we say that a subgroup  $H$  of  $\mathbb{T}$  is  $\mathcal{X}$ -characterized if there exists  $\mathbf{u} \in \mathcal{X}$  such that  $H = t_{\mathbf{u}}(\mathbb{T})$ . We denote by  $\mathcal{Char}$  (resp.,  $\mathcal{X}\text{-Char}$ ) the family of all characterized (resp.,  $\mathcal{X}$ -characterized) subgroups of  $\mathbb{T}$ .

### 1.2. Theorems A, D, E and Corollary A

Here we recall some basic necessary conditions satisfied by characterized subgroups of  $\mathbb{T}$ .

The following theorem (in a more general form for compact abelian groups) is due to Comfort, Trigoso-Arrieta and Wu [19, Lemma 3.10] (although the precise statement can be only found in the proof of that lemma).

**Theorem A.** [19] *For any sequence  $\mathbf{u}$  in  $\mathbb{N}$ ,  $t_{\mathbf{u}}(\mathbb{T})$  is an  $F_{\sigma\delta}$  (hence, Borel) subgroup of  $\mathbb{T}$ .*

Actually, this immediately follows from the equality  $t_{\mathbf{u}}(\mathbb{T}) = \bigcap_{j=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \{x \in \mathbb{T} : \|u_n x\| \leq 1/j\}$ .

**Corollary A.** *For any sequence  $\mathbf{u}$  in  $\mathbb{N}$ :*

- (1)  $t_{\mathbf{u}}(\mathbb{T})$  is either countable or has size  $\mathfrak{c}$ ;
- (2)  $t_{\mathbf{u}}(\mathbb{T})$  is measurable and has Haar measure 0, unless  $t_{\mathbf{u}}(\mathbb{T}) = \mathbb{T}$ ;
- (3) [11] the group  $t_{\mathbf{u}}(\mathbb{T})$  is Polishable (i.e., admits a finer Polish group topology).

Item (1) depends on the fact that in a separable metric space a Borel set is always countable or has cardinality  $\mathfrak{c}$  in view of Alexandroff-Hausdorff's theorem [50, §37, Theorem 3]. As far as item (2) is concerned, by Theorem A  $t_{\mathbf{u}}(\mathbb{T})$  is a Borel set, so it is measurable. Moreover,  $\mathbb{T}$  has Haar measure 1 and  $\mathbb{T}/t_{\mathbf{u}}(\mathbb{T})$  is divisible, thus  $t_{\mathbf{u}}(\mathbb{T})$  has infinite index in  $\mathbb{T}$  when  $t_{\mathbf{u}}(\mathbb{T}) \neq \mathbb{T}$ , and therefore  $t_{\mathbf{u}}(\mathbb{T})$  has Haar measure 0. Item (3), due to Biró [11], is a consequence of a well-known result of Solecki [57] (see also [58]). It can be found in the proof of Theorem E given in [11]. The finer Polish group topology on  $t_{\mathbf{u}}(\mathbb{T})$  coincides with the topology  $\tau_{\mathbf{u}}$  induced on  $t_{\mathbf{u}}(\mathbb{T})$  by the embedding  $t_{\mathbf{u}}(\mathbb{T}) \rightarrow c_0(\mathbb{T})$ ,  $x \mapsto (u_n x)$ , where  $c_0(\mathbb{T})$  is the group of null-sequences in  $\mathbb{T}$  and carries the topology induced by the topology of uniform convergence of  $\mathbb{T}^{\mathbb{N}}$ .

The next theorem can be briefly resumed by the equality  $\mathcal{Char} = \{\mathbb{T}\} \sqcup \mathcal{S}\text{-Char}$ :

**Theorem D.** [6, Example 2.8] *For a sequence  $\mathbf{u}$  in  $\mathbb{N}$ ,  $t_{\mathbf{u}}(\mathbb{T}) = \mathbb{T}$  if and only if  $\mathbf{u}$  is eventually 0. In particular,  $t_{\mathbf{u}}(\mathbb{T}) \neq \mathbb{T}$  for every  $\mathbf{u} \in \mathcal{S}$ .*

As an immediate consequence,  $\mathbb{T} \in \mathcal{Char} \setminus \mathcal{A}\text{-Char}$ .

**Theorem E.** [11] *Let  $K$  be an uncountable Kronecker set in  $\mathbb{T}$ . Then the subgroup  $\langle K \rangle$  is not Polishable. In particular,  $\langle K \rangle$  cannot be characterized.*

Since  $\langle K \rangle$  is obviously a  $\sigma$ -compact set, hence an  $F_{\sigma}$ -set as well, the last assertion of the theorem provides an example of a non-characterized  $F_{\sigma}$ -subgroup of  $\mathbb{T}$ , thereby answering a question raised in [38] (where some special classes of  $F_{\sigma}$ -subgroups of compact abelian groups were shown to be characterized – see Definition 7.25).

### 1.3. Arithmetic and recursive characterizing sequences

Every arithmetic sequence gives rise to a nice representation:

**Theorem 1.4.** [53] *Let  $\mathbf{u} \in \mathcal{A}$ . For every  $x \in [0, 1)$ , there exists a unique sequence  $(c_n^{\mathbf{u}}(x))_{n \in \mathbb{N}_+}$  in  $\mathbb{N}$  such that*

$$x = \sum_{n=1}^{\infty} \frac{c_n^{\mathbf{u}}(x)}{u_n}, \tag{1.4}$$

with  $c_n^{\mathbf{u}}(x) < b_n^{\mathbf{u}}$  for every  $n \in \mathbb{N}_+$ , and  $c_n^{\mathbf{u}}(x) < b_n^{\mathbf{u}} - 1$  for infinitely many  $n \in \mathbb{N}_+$ .

When no confusion is possible, we shall simply write  $c_n^{\mathbf{u}}$  or  $c_n$  in place of  $c_n^{\mathbf{u}}(x)$ . For  $x \in [0, 1)$ , with canonical representation (1.4), let

$$\text{supp}(x) := \{n \in \mathbb{N}_+ : c_n^{\mathbf{u}} \neq 0\} \quad \text{and} \quad \text{supp}_b(x) := \{n \in \mathbb{N}_+ : c_n^{\mathbf{u}} = b_n^{\mathbf{u}} - 1\}.$$

Clearly,  $\text{supp}_b(x) \subseteq \text{supp}(x)$  and  $\text{supp}_b(x)$  cannot be cofinite by definition.

The following remark considers the case opposite to Theorem D, that is, when a characterized subgroup is finite.

**Remark 1.5.** (a) For  $\mathbf{u} \in \mathcal{A}$  the subgroup  $t_{\mathbf{u}}(\mathbb{T})$  always contains the infinite subgroup  $\mathbb{Z}(\mathbf{u}^\infty) := \langle 1/u_n : n \in \mathbb{N} \rangle$  of  $\mathbb{T}$ . In particular,  $t_{\mathbf{u}}(\mathbb{T})$  cannot be finite.

(b) On the other hand, if  $H$  is any finite subgroup of  $\mathbb{T}$ , then  $H \in \mathcal{C}\text{har}$ . In fact,  $H \cong \mathbb{Z}(m)$  for  $m = |H|$ , and so  $H$  is characterized by a sequence  $\mathbf{u} \in \mathcal{S}$  (e.g.,  $\mathbf{u}$  defined by  $u_0 = 1$  and  $u_n = mn$  for  $n \in \mathbb{N}_+$ ). But the characterizing sequence cannot be arithmetic by item (a).

Therefore, the finite subgroups of  $\mathbb{T}$ , as well as  $\mathbb{T}$  itself, belong to  $\mathcal{C}\text{har}$ , but not to  $\mathcal{A}\text{-}\mathcal{C}\text{har}$ .

The following result from [32], allowing to inflate a subgroup  $t_{\mathbf{u}}(\mathbb{T}) \in \mathcal{A}\text{-}\mathcal{C}\text{har}$  up to a larger subgroup  $t_{\mathbf{v}}(\mathbb{T})$  by means of an appropriate subsequence  $\mathbf{v}$  of  $\mathbf{u}$ , will be needed in the sequel.

**Proposition 1.6.** [32] *Let  $\mathbf{u} \in \mathcal{A}$  and  $(k_n)$  be a strictly increasing sequence in  $\mathbb{N}_+$  such that  $k_n + 1 < k_{n+1}$  for every  $n \in \mathbb{N}$ ,  $b_{k_n}^{\mathbf{u}} \rightarrow +\infty$  and  $b_{k_{n+1}}^{\mathbf{u}} \rightarrow +\infty$ . Let  $K = \{k_n : n \in \mathbb{N}\}$  and  $\mathbf{v} \in \mathcal{S}$  with  $\{v_0, v_1, \dots\} = \{u_n : n \in \mathbb{N} \setminus K\}$ . Then  $t_{\mathbf{u}}(\mathbb{T}) \not\subseteq t_{\mathbf{v}}(\mathbb{T})$ .*

The inclusion  $t_{\mathbf{u}}(\mathbb{T}) \subseteq t_{\mathbf{v}}(\mathbb{T})$  is obvious. The proof that this inclusion is proper uses ideas from [5, Proposition 2.3 and Lemma 2.6].

We recall the following element-wise description of the subgroups in  $\mathcal{A}\text{-}\mathcal{C}\text{har}$ :

**Theorem 1.7.** [30,36] *Let  $\mathbf{u} \in \mathcal{A}$  and  $x \in [0, 1)$ . Then  $\varphi(x) \in t_{\mathbf{u}}(\mathbb{T})$  if and only if either  $\text{supp}(x)$  is finite or  $\text{supp}(x)$  is infinite and for all infinite  $A \subseteq \mathbb{N}$  the following conditions hold:*

- (a) if  $A$  is  $b$ -bounded, then:
  - (1) if  $A \subseteq^* \text{supp}(x)$ , then  $A + 1 \subseteq^* \text{supp}(x)$ ,  $A \subseteq^* \text{supp}_b(x)$  and  $\lim_{n \in A} \frac{c_{n+1} + 1}{b_{n+1}} = 1$ . Moreover, if  $A + 1$  is  $b$ -bounded, then  $A + 1 \subseteq^* \text{supp}_b(x)$  as well;
  - (2) if  $A \cap \text{supp}(x)$  is finite, then  $\lim_{n \in A} \frac{c_{n+1}}{b_{n+1}} = 0$ . Moreover, if  $A + 1$  is  $b$ -bounded, then  $(A + 1) \cap \text{supp}(x)$  is finite as well;
- (b) if  $A$  is  $b$ -divergent, then  $\lim_{n \in A} \varphi\left(\frac{c_n}{b_n}\right) = \lim_{n \in A} \varphi\left(\frac{c_{n+1}}{b_{n+1}}\right) = 0$ .

Next we recall the following remarkable result of Eggleston [41], independently proved in [6].

**Theorem 1.8.** (Eggleston’s dichotomy) *Let  $\mathbf{u}$  be a sequence in  $\mathbb{N}_+$ .*

- (1) If  $\mathbf{u}$  is  $b$ -divergent, then  $|t_{\mathbf{u}}(\mathbb{T})| = \mathfrak{c}$ .
- (2) If  $\mathbf{u}$  is  $b$ -bounded, then  $t_{\mathbf{u}}(\mathbb{T})$  is countable.

**Remark 1.9.** There exists a  $b$ -unbounded  $\mathbf{u} \in \mathcal{S}$  with countable  $t_{\mathbf{u}}(\mathbb{T})$  (for an example see [8]), so “Eggleston’s dichotomy” is not a *real* dichotomy.

To avoid the inconvenience pointed out in the above remark, one can restrict the correspondence  $\mathbf{u} \mapsto |t_{\mathbf{u}}(\mathbb{T})|$  to a smaller class of sequences  $\mathcal{X}$ , so that  $t_{\mathbf{u}}(\mathbb{T})$  is countable if and only if  $\mathbf{u} \in \mathcal{X}_{bd}$  (see Definition 6.1). As the following theorem from [30] shows (see also [36, Corollary 3.8] and [37, Theorem E]), the restriction to  $\mathcal{A}$  has indeed this property.

**Theorem 1.10.** [30] *For a sequence  $\mathbf{u} \in \mathcal{A}$  the following conditions are equivalent:*

- (a)  $|t_{\mathbf{u}}(\mathbb{T})| < \mathfrak{c}$ ;
- (b)  $t_{\mathbf{u}}(\mathbb{T})$  is countable;
- (c)  $t_{\mathbf{u}}(\mathbb{T})$  is a subgroup of  $\mathbb{Q}/\mathbb{Z}$ ;
- (d)  $\mathbf{u}$  is  $b$ -bounded.

**Remark 1.11.** The remarkable fact that  $\mathcal{C}\mathfrak{h}\mathfrak{a}\mathfrak{r}$  contains all countable subgroups of  $\mathbb{T}$  was established by B ır o, Deshouillers and S os [12]. More precisely, all countable subgroups of  $\mathbb{T}$  are contained in  $\mathcal{S}_{bd}\text{-}\mathcal{C}\mathfrak{h}\mathfrak{a}\mathfrak{r}$  [9, Theorem 4.1]. On the other hand,  $\mathcal{C}\mathfrak{h}\mathfrak{a}\mathfrak{r}$  does not contain all  $F_\sigma$ -subgroups, according to Theorem E.

In particular, the subgroup  $\mathbb{Q}/\mathbb{Z}$  of  $\mathbb{T}$  is characterized by the  $b$ -bounded sequence

$$1!, 2!, 2 \cdot 2!, 3!, 2 \cdot 3!, 3 \cdot 3!, 4!, \dots, n!, 2 \cdot n!, \dots, n \cdot n!, (n + 1)!, \dots, \tag{1.5}$$

as pointed out later in [38, Proposition 1.3] (a complete proof of this fact is presented in [13]). Nevertheless,  $\mathbb{Q}/\mathbb{Z} \notin \mathcal{A}\text{-}\mathcal{C}\mathfrak{h}\mathfrak{a}\mathfrak{r}$  in view of Theorem 1.10 (see also [30, Example 2.10]), since every sequence  $\mathbf{u} \in \mathcal{A}$  with  $t_{\mathbf{u}}(\mathbb{T}) = \mathbb{Q}/\mathbb{Z}$  must be  $b$ -bounded, which implies that the set of primes that divide  $u_n$  for some  $n \in \mathbb{N}$  is finite; this contradicts the fact that  $t_{\mathbf{u}}(\mathbb{T})$  must contain  $\varphi(1/p)$  for every prime  $p$ .

Let  $\gamma = [a_0; a_1, a_2, \dots]$  be an irrational number with  $\mathbf{q}^\gamma = (q_n)$  as in (1.1). It is well-known that  $|q_n\gamma - p_n| < 1/q_{n+1}$ , so  $q_n\varphi(\gamma) \rightarrow 0$  in  $\mathbb{T}$ . In our terms this means that  $\langle \varphi(\gamma) \rangle \subseteq t_{\mathbf{q}^\gamma}(\mathbb{T})$ ; Larcher proved that in general this containment can be strict (this follows also from Theorem 1.8) and that the equality  $\langle \varphi(\gamma) \rangle = t_{\mathbf{q}^\gamma}(\mathbb{T})$  holds under the assumption that the sequence  $\mathbf{a}$  is bounded [51, Theorem 1]. In particular:

**Example 1.12.** Let  $\phi$  be the Golden ratio, that is,  $\phi = \frac{1+\sqrt{5}}{2}$ . Then  $\phi = [1; 1, 1, \dots]$  and  $q_n = f_n$  for every  $n \in \mathbb{N}_+$ , so  $\langle \varphi(\phi) \rangle = t_{\mathbf{f}}(\mathbb{T})$  [6].

Kraaikamp and Liardet [48] proved that the boundedness of the sequence  $\mathbf{a}$  in Larcher’s theorem is also necessary. We restate their theorem in terms of the sequence  $\mathbf{q} \in \mathcal{R}$ .

**Theorem 1.13.** *Let  $\mathbf{q} \in \mathcal{R}$ .*

- (1) [48, Theorem 3] *If  $\mathbf{q} \notin \mathcal{R}_{bd}$ , then  $|t_{\mathbf{q}}(\mathbb{T})| = \mathfrak{c}$ .*
- (2) [48, Proposition, page 309] *If  $\gamma \in (0, 1)$  is the irrational with  $\mathbf{q}^\gamma = \mathbf{q}$ , then  $t_{\mathbf{q}}(\mathbb{T}) = \langle \varphi(\gamma) \rangle$  if and only if  $\mathbf{q} \in \mathcal{R}_{bd}$ .*

The criterion [48, Theorem 1] was applied by Bíró, Deshouillers and Sós [12] to settle the case when  $\mathbf{a}$  is unbounded, or equivalently  $\mathbf{q}$  is unbounded (see Remark 1.1). This was done by adding elements to the sequence  $\mathbf{q}$  to produce a larger sequence  $\mathbf{v}^{\mathbf{q}}$  defined by

$$\mathbf{v}^{\mathbf{q}} : q_0 \leq q_1 < 2q_1 < \dots < a_2q_1 < q_2 < 2q_2 < \dots < a_3q_2 < q_3 < 2q_3 < \dots \tag{1.6}$$

(see (1.7) for a more rigorous definition). Clearly,  $\mathbf{q}$  is a subsequence of  $\mathbf{v}^{\mathbf{q}}$  and so  $t_{\mathbf{v}^{\mathbf{q}}}(\mathbb{T}) \subseteq t_{\mathbf{q}}(\mathbb{T})$ .

**Theorem 1.14.** [12, Theorem 1\*] *In the above notations,  $\langle \varphi(\gamma) \rangle = t_{\mathbf{v}^{\mathbf{q}}}(\mathbb{T})$ .*

The sequence (1.5) was defined in [38] extending the sequence  $\mathbf{n} = ((n + 1)!) \in \mathcal{A}$ , following the idea of Vera Sós in defining (1.6), starting from  $\mathbf{q}$ . Clearly, one can unify these two instances by extending any  $\mathbf{u} \in \mathcal{S}$  (not necessarily  $\mathbf{u} \in \mathcal{R}$  or  $\mathbf{u} \in \mathcal{A}$  as above) to the sequence

$$\mathbf{v}^{\mathbf{u}} := \{su_n : 1 \leq s \leq \lfloor b_n \rfloor, n \in \mathbb{N}\}. \tag{1.7}$$

Note that the sequence  $\mathbf{v}^{\mathbf{u}}$  is always  $b$ -bounded.

Theorem 1.14 was significantly extended by Marconato in his bachelor thesis [52]:

**Theorem 1.15.** [52, Theorem 4.3.2] *In the above notations, if  $\mathbf{u}$  is a  $b$ -bounded subsequence of  $\mathbf{v}^{\mathbf{q}}$  and a supersequence of  $\mathbf{q}$ , then  $\langle \varphi(\gamma) \rangle = t_{\mathbf{u}}(\mathbb{T})$ .*

Various results on  $\mathcal{R}^*$ -characterized subgroups of  $\mathbb{T}$  partially extending Theorem 1.13(1) can be found in [8]. More precisely, it is shown in [8, Theorem 4.12] that for  $\mathbf{u} \in \mathcal{R}^*$  Eggleston’s dichotomy becomes a real dichotomy:

**Theorem 1.16.** [8, Theorem 4.12] *For  $\mathbf{u} \in \mathcal{R}^*$ ,  $|t_{\mathbf{u}}(\mathbb{T})| = c$  if and only if  $\mathbf{u}$  is not  $b$ -bounded.*

Nevertheless, this property may strongly fail out of the class  $\mathcal{R}^*$  (namely, for  $b$ -unbounded sequences  $\mathbf{u}$  satisfying (1.3), but without the restriction  $s_n \geq t_n$ , one may have even  $|t_{\mathbf{u}}(\mathbb{T})| = 1$ , see [8, Example 4.13]).

## 2. Statistically characterized subgroups of $\mathbb{T}$ and generalizations

### 2.1. Statistically characterized subgroups of $\mathbb{T}$

For  $A \subseteq \mathbb{N}$ , denote

$$A(n) := \{i \in A : i \leq n\} = A \cap [0, n].$$

The lower and the upper natural density of  $A$  are

$$\underline{d}(A) := \liminf_{n \rightarrow \infty} \frac{|A(n)|}{n} \quad \text{and} \quad \overline{d}(A) := \limsup_{n \rightarrow \infty} \frac{|A(n)|}{n}.$$

If  $\underline{d}(A) = \overline{d}(A)$ , we say that the natural density of  $A$  exists and denote it by  $d(A)$ . As  $\overline{d}$  is a submeasure (see Definition 3.17)

$$\mathcal{I}_{\overline{d}} := \{A \subseteq \mathbb{N} : d(A) = 0\},$$

is an ideal of  $\mathbb{N}$  that contains *Fin*.

A sequence  $(x_n)$  in a topological space  $X$  is said to *statistically converge* to  $x \in X$  if for every neighborhood  $U$  of  $x$  in  $X$ ,  $d(\{n \in \mathbb{N} : x_n \notin U\}) = 0$ , i.e.,  $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}_d$ . We denote this by  $x_n \xrightarrow{\mathcal{I}_d} x$  or  $x_n \xrightarrow{s} x$ . (In  $\mathbb{T}$  this is equivalent to say that for every  $\varepsilon > 0$ ,  $d(\{n \in \mathbb{N} : \|x_n - x\| \geq \varepsilon\}) = 0$ , i.e.,  $\{n \in \mathbb{N} : \|x_n - x\| \geq \varepsilon\} \in \mathcal{I}_d$ .) Clearly,  $x_n \xrightarrow{s} x$  whenever  $x_n \rightarrow x$ .

Following [22], for a sequence of integers  $\mathbf{u}$ , let

$$t_{\mathbf{u}}^s(\mathbb{T}) := \{x \in \mathbb{T} : u_n x \xrightarrow{s} 0\}.$$

A subgroup  $H$  of  $\mathbb{T}$  is *statistically characterized* if there exists a sequence of integers  $\mathbf{u}$  such that  $H = t_{\mathbf{u}}^s(\mathbb{T})$ . Clearly,  $t_{\mathbf{u}}(\mathbb{T}) \subseteq t_{\mathbf{u}}^s(\mathbb{T})$ .

If  $\mathcal{X}$  is one of the classes of sequences introduced above (i.e.,  $\mathcal{S}$ ,  $\mathcal{A}$ , etc.), we say that a subgroup  $H$  of  $\mathbb{T}$  is *statistically  $\mathcal{X}$ -characterized* if there exists  $\mathbf{u} \in \mathcal{X}$  such that  $H = t_{\mathbf{u}}^s(\mathbb{T})$  and we denote by  $\mathfrak{Char}^s$  (resp.,  $\mathcal{X}\text{-}\mathfrak{Char}^s$ ) the family of all statistically characterized (resp., statistically  $\mathcal{X}$ -characterized) subgroups of  $\mathbb{T}$ .

Now we discuss the main results from [22]. First, [22, Theorem A] is the counterpart of Theorem A above for  $t_{\mathbf{u}}^s(\mathbb{T})$ . Obviously, this gives also a counterpart of Corollary A (that is, statistically characterized subgroups are Polishable and the proper ones have Haar measure 0). Similarly, Theorem E holds in this case: if  $K$  is an uncountable Kronecker set in  $\mathbb{T}$ , then the subgroup  $\langle K \rangle$  cannot be statistically characterized (see [22, Proposition 2.6]).

The following two theorems are specific for statistically characterized subgroups.

**Theorem B.** [22, Theorem B] *For any  $\mathbf{u} \in \mathcal{A}$ ,  $|t_{\mathbf{u}}^s(\mathbb{T})| = \mathfrak{c}$ .*

**Theorem C.** [22, Theorem C] *For any  $\mathbf{u} \in \mathcal{A}$ ,  $t_{\mathbf{u}}(\mathbb{T}) \subsetneq t_{\mathbf{u}}^s(\mathbb{T})$ .*

In [22, Proposition 3.5] it was proved directly that  $|t_{(2^n)}^s(\mathbb{T}) \setminus t_{(2^n)}(\mathbb{T})| = \mathfrak{c}$ . A general version of this property can easily be deduced from Theorems B and C by means of the following:

**Claim 2.1.** *If  $A$  is a subgroup of an infinite abelian group  $B$ , then  $A \neq B$  is equivalent to  $|B \setminus A| = |B|$ . Therefore,  $|t_{\mathbf{u}}^s(\mathbb{T}) \setminus A| = |t_{\mathbf{u}}^s(\mathbb{T})|$  for every  $\mathbf{u} \in \mathcal{S}$  and for every proper subgroup  $A$  of  $t_{\mathbf{u}}^s(\mathbb{T})$  if  $t_{\mathbf{u}}^s(\mathbb{T})$  is infinite.*

**Proof.** Obviously,  $|B \setminus A| = |B|$  implies  $A \neq B$ . On the other hand, if  $A \neq B$ , then  $|B \setminus A| \geq |A|$ , as  $B \setminus A$  contains at least a coset of  $A$ , which has cardinality  $|A|$ . Hence,  $|B| = \max\{|B \setminus A|, |A|\} = |B \setminus A|$ . The second assertion follows directly from the first one. Observe that it holds for every  $\mathbf{u} \in \mathcal{A}$  as, in view of Remark 1.5,  $t_{\mathbf{u}}(\mathbb{T}) \subseteq t_{\mathbf{u}}^s(\mathbb{T})$  is infinite.  $\square$

The next corollary follows from Theorems B and C and the second assertion of the above claim:

**Corollary 2.2.** *If  $\mathbf{u} \in \mathcal{A}$ , then  $|t_{\mathbf{u}}^s(\mathbb{T}) \setminus t_{\mathbf{u}}(\mathbb{T})| = \mathfrak{c}$ .*

The counterpart of Theorem D (i.e.,  $t_{\mathbf{u}}^s(\mathbb{T}) \neq \mathbb{T}$  for any  $\mathbf{u} \in \mathcal{A}$ ) was announced without proof in [22, Fact 3.1]. A much more general result is proved in [32] in full detail (see Theorem 5.16).

In [20] the authors investigated also the statistically characterized subgroups  $t_{\mathbf{q}}^s(\mathbb{T})$  of  $\mathbb{T}$  for  $\mathbf{q} \in \mathcal{R}$ , answering [22, Question 6.5]:

**Theorem B<sup>R</sup>.** [20, Theorem B] *If  $\mathbf{q} \in \mathcal{R}$ , then  $|t_{\mathbf{q}}^s(\mathbb{T})| = \mathfrak{c}$ .*

As a consequence of the above theorem, one gets a partial answer to [22, Question 6.4] (see also Problem 7.9 there).

**Corollary 2.3.** [20, Corollary 2.5] *If  $f \in \mathcal{R}$  is the Fibonacci sequence, then  $|t_f^s(\mathbb{T})| = c$ .*

**Theorem C<sup>R</sup>.** [20, Theorem C] *For any  $q \in \mathcal{R}$ ,  $|t_q^s(\mathbb{T}) \setminus t_q(\mathbb{T})| = c$ .*

Compare these results also with Theorem 1.13.

Theorem 1.14 implies that for any irrational  $\gamma \in (0, 1)$ ,  $t_q(\mathbb{T}) = t_{vq}(\mathbb{T})$  if and only if  $q \in \mathcal{R}_{bd}$  (so,  $t_q(\mathbb{T}) \supsetneq t_{vq}(\mathbb{T})$  precisely when  $q \in \mathcal{R} \setminus \mathcal{R}_{bd}$ ). This was partially extended in [20] to statistically characterized subgroups as follows.

**Proposition 2.4.** [20, Proposition 3.3] *Let  $q \in \mathcal{R}_{bd}$ . Then  $t_q^s(\mathbb{T}) = t_{vq}^s(\mathbb{T})$ .*

**Proposition 2.5.** [20, Proposition 3.4] *There exists an irrational  $\gamma \in (0, 1)$  with  $q \in \mathcal{R} \setminus \mathcal{R}_{bd}$ , such that  $t_q^s(\mathbb{T}) \supsetneq t_{vq}^s(\mathbb{T})$ ; namely,  $\gamma = [0; a_1, a_2, \dots]$  with  $(a_{n^2}) = 4n^2$  and  $a_n = 1$  otherwise.*

From Theorem 1.13 we deduce that  $|t_q(\mathbb{T})| = c$ , so  $|t_q^s(\mathbb{T})| = c$  as well (this follows also from Theorem B<sup>R</sup>). Now from Claim 2.1 and this proposition one can immediately deduce that  $|t_q^s(\mathbb{T}) \setminus t_{vq}^s(\mathbb{T})| = c$  (this is obtained in [20, Proposition 3.4 and Corollary 3.5] by a specific more complicated argument).

Here we briefly recall that a subset  $X$  of  $\mathbb{T}$  is called an  $A$ -set (resp., *statistically A-set*) if  $X \subseteq t_u(\mathbb{T})$  (resp.,  $X \subseteq t_u^s(\mathbb{T})$ ) for some  $u \in \mathcal{S}$ . This terminology comes from the theory of thin sets, and more specifically of Arbault sets in Harmonic Analysis (see [17] and see [4] for the connection of thin sets with characterized subgroups of  $\mathbb{T}$ ).

We denote by  $\mathfrak{A}$  (resp.,  $\mathfrak{A}^s$ ) the family of all (statistically)  $A$ -sets of  $\mathbb{T}$ . In particular,  $\mathcal{Char} \subseteq \mathfrak{A}$  and  $\mathcal{Char}^s \subseteq \mathfrak{A}^s$ . Obviously,  $\mathfrak{A} \subseteq \mathfrak{A}^s$ . The following impressive result from [26] shows that the converse inclusion fails in a very strong way, as it states more precisely that

$$\mathcal{A}\text{-}\mathcal{Char}^s \cap \mathfrak{A} = \emptyset.$$

**Theorem 2.6.** [26, Theorem 2.7] *For any  $u \in \mathcal{A}$ ,  $t_u^s(\mathbb{T}) \notin \mathfrak{A}$ , that is,  $t_u^s(\mathbb{T})$  is not contained in any characterized subgroup of  $\mathbb{T}$ .*

This theorem implies that  $t_u^s(\mathbb{T})$  is not contained in  $t_u(\mathbb{T})$  whenever  $u \in \mathcal{A}$ , i.e., this gives a new proof of Theorem C.

The following obvious consequence of Theorem 2.6 is formulated as a separate corollary in view of its importance.

**Corollary 2.7.**  $\mathcal{A}\text{-}\mathcal{Char}^s \cap \mathcal{Char} = \emptyset$ , i.e., *statistically  $\mathcal{A}$ -characterized subgroups of  $\mathbb{T}$  are never characterized.*

In particular, this entails the non-inclusion  $\mathcal{Char}^s \not\subseteq \mathcal{Char}$ , which answers question (ii) in [22, §3] on whether this inclusion holds true. We are not aware whether the converse inclusion  $\mathcal{Char} \subseteq \mathcal{Char}^s$  holds true (see Question 7.12).

### 2.2. $\alpha$ -characterized subgroups of $\mathbb{T}$ and weight functions

Following [3], call *weight* a function  $g: \mathbb{N} \rightarrow \mathbb{R}_{>0}$  with  $\lim_{n \rightarrow \infty} g(n) = \infty$ . The *upper density of weight  $g$  of  $A \in \mathcal{P}(\mathbb{N})$*  is

$$\bar{d}_g(A) := \limsup_{n \rightarrow \infty} \frac{|A(n)|}{g(n)}.$$

As  $\bar{d}_g$  is a submeasure (see Definition 3.17),

$$\mathcal{Z}_g := \{A \in \mathcal{P}(\mathbb{N}) : \bar{d}_g(A) = 0\}$$

is an ideal of  $\mathbb{N}$  that contains *Fin*. Since  $\mathcal{P}(\mathbb{N}) = \mathcal{Z}_g$  if and only if  $n/g(n) \rightarrow 0$ , we assume in the sequel that  $n/g(n) \not\rightarrow 0$ .

**Remark 2.8.** The condition  $n/g(n) \not\rightarrow 0$  means that, at least on some infinite subset  $K = \{k_n : n \in \mathbb{N}\}$  of  $\mathbb{N}$ ,  $g$  has “at most” linear growth, i.e.,  $g(k_n) \leq Ck_n$  for every  $n \in \mathbb{N}$  and a constant  $C > 0$ .

By [24, Lemma 1.5], it is not restrictive to assume that a weight function is non-decreasing (in fact,  $\mathcal{Z}_g = \mathcal{Z}_{g'}$ , where  $g' : \mathbb{N} \rightarrow \mathbb{R}_{>0}$  is defined by  $g'(n) = \inf\{g(i) : i \geq n\}$  for every  $n \in \mathbb{N}$ ). So, let  $\mathbb{G}$  denote the family of all non-decreasing weight functions  $g$  with  $n/g(n) \not\rightarrow 0$ .

The above definition of upper density of weight  $g$  is inspired by the one given in [10] (see [14]) with  $g(n) = n^\alpha$  for a fixed  $\alpha \in (0, 1]$ . In this case,  $d_\alpha := \bar{d}_n^\alpha$  is the *natural density of order  $\alpha$* . For  $\alpha = 1$ ,  $d_1 = d$ .

As in [14], a sequence  $(x_n)$  in a topological space  $X$  is said to  $\alpha$ -statistically converge to  $x \in X$  if for every neighborhood  $U$  of  $x$  in  $X$ ,  $d_\alpha(\{n \in \mathbb{N} : x_n \notin U\}) = 0$ , i.e.,  $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{Z}_n^\alpha$ . We denote this by  $x_n \xrightarrow{\mathcal{Z}_n^\alpha} x$  or  $x_n \xrightarrow{s_\alpha} x$ .

**Remark 2.9.** For  $\alpha \in (0, 1)$ , since  $n^\alpha < n$  for every  $n \in \mathbb{N}_+$ , one has  $d_\alpha \geq \bar{d}$ , so

$$\mathcal{Z}_n^\alpha \subseteq \mathcal{Z}_n = \mathcal{I}_d. \tag{2.1}$$

More generally, for  $g \in \mathbb{G}$ , if  $g(n) \leq n$  for every  $n \in \mathbb{N}$  (which may often occur under the current blanket assumption  $n/g(n) \not\rightarrow 0$ ), then  $\mathcal{Z}_g \subseteq \mathcal{I}_d$ .

From the inclusion (2.1), one concludes that  $x_n \xrightarrow{\mathcal{Z}_n^\alpha} x$  implies  $x_n \xrightarrow{\mathcal{I}_d} x$ .

Following [14], for a sequence of integers  $\mathbf{u}$  and  $\alpha \in (0, 1]$ , let

$$t_{\mathbf{u}}^\alpha(\mathbb{T}) := \{x \in \mathbb{T} : u_n x \xrightarrow{s_\alpha} 0\} \subseteq t_{\mathbf{u}}^s(\mathbb{T});$$

clearly,  $t_{\mathbf{u}}^1(\mathbb{T}) = t_{\mathbf{u}}^s(\mathbb{T})$ . A subgroup  $H$  of  $\mathbb{T}$  is  $\alpha$ -statistically characterized if there exists a sequence of integers  $\mathbf{u}$  such that  $H = t_{\mathbf{u}}^\alpha(\mathbb{T})$ . Clearly, the 1-statistically characterized subgroups of  $\mathbb{T}$  are exactly the statistically characterized ones.

Theorems A, B, C and D above were proved for  $\alpha$ -statistically characterized subgroups in [14], which obviously ensure also the conclusion of Theorem E. Moreover, the following relation was established:

**Theorem 2.10.** [14, Proposition 2.12] For  $\alpha \in (0, 1)$  and  $\mathbf{u} \in \mathcal{A}_{bd}$ ,  $t_{\mathbf{u}}^\alpha(\mathbb{T}) \subsetneq t_{\mathbf{u}}^s(\mathbb{T})$ .

For  $\mathbf{u} \in \mathcal{A}$ , a characterization of the elements of  $t_{\mathbf{u}}^\alpha(\mathbb{T})$  (with  $\alpha \in (0, 1]$ ) in terms of their support, as a counterpart of Theorem 1.7, was proposed in [23, Theorem 2.1]. This result answers [22, Problem 6.10] and provides a positive answer in the general case to [14, Problem 2.14], that is (compare with Theorem 2.10), if  $\mathbf{u} \in \mathcal{A}$  and  $\alpha < \beta$  in  $(0, 1)$ , then  $t_{\mathbf{u}}^\alpha(\mathbb{T}) \subsetneq t_{\mathbf{u}}^\beta(\mathbb{T})$ . We do not state here explicitly [23, Theorem 2.1] since it is completely covered by the more general Theorem 5.12 given below.

### 2.3. Modulus function

Following [24], an unbounded function  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is a *modulus function* if:

- (M1) for  $x \in \mathbb{R}_{\geq 0}$ ,  $f(x) = 0$  if and only if  $x = 0$ ;
- (M2) for every  $x, y \in \mathbb{R}_{> 0}$ ,  $f(x + y) \leq f(x) + f(y)$ ;

- (M3)  $f$  is non-decreasing;
- (M4)  $f$  is right continuous at 0.

These conditions imply continuity.

In [15] the approaches with weight functions and with modulus functions were unified and generalized as follows. Let  $g \in \mathbb{G}$ , let  $f$  be a modulus function and define as in [24] the *upper modular simple density* by letting, for every  $A \in \mathcal{P}(\mathbb{N})$ ,

$$\bar{d}_g^f(A) = \limsup_{n \rightarrow \infty} \frac{f(|A(n)|)}{f(g(n))}$$

As  $\bar{d}_g^f$  is a submeasure (see Definition 3.17),

$$\mathcal{Z}_g(f) := \{A \in \mathcal{P}(\mathbb{N}) : \bar{d}_g^f(A) = 0\}$$

is an ideal, called *modular simple density ideal*, with  $\mathcal{F}in \subseteq \mathcal{Z}_g(f)$ . We have added unbounded in the above definition, as  $\mathcal{Z}_g(f) = \{\emptyset\}$  when  $f$  is bounded.

Clearly, with  $f = id$ , one obtains  $d_g^{id} = d_g$  and  $\mathcal{Z}_g(id) = \mathcal{Z}_g$ . So, for  $f = id$  and  $g(n) = n^\alpha$  with  $\alpha \in (0, 1]$ ,  $d_g^{id} = d_\alpha$  and  $\mathcal{Z}_g(id) = \mathcal{Z}_{n^\alpha}$ .

A sequence  $(x_n)$  in a topological space  $X$  is said to  *$f^g$ -statistically converge* to  $x \in X$  if for every neighborhood  $U$  of  $x$  in  $X$ ,  $d_g^f(\{n \in \mathbb{N} : x_n \notin U\}) = 0$ , i.e.,  $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{Z}_g(f)$  [24, Definition 3.1].

We denote this by  $x_n \xrightarrow{\mathcal{Z}_g(f)} x$  or  $x_n \xrightarrow{f^g} x$ .

Following [24, Definition 3.2], for a sequence of integers  $\mathbf{u}$ , let

$$t_{\mathbf{u}}^{f,g}(\mathbb{T}) := \{x \in \mathbb{T} : u_n x \xrightarrow{f^g} 0\}.$$

A subgroup  $H$  of  $\mathbb{T}$  is  *$f^g$ -statistically characterized* if there exists a sequence of integers  $\mathbf{u}$  such that  $H = t_{\mathbf{u}}^{f,g}(\mathbb{T})$ .

**Remark 2.11.** With  $g(n) = n$  for every  $n \in \mathbb{N}$  we recover the *upper density function*  $\bar{d}^f$  defined in [1] as well as  $\mathcal{Z}(f) := \{A \in \mathcal{P}(\mathbb{N}) : \bar{d}^f(A) = 0\}$ . For this specific  $g$  we briefly write  $t_{\mathbf{u}}^f(\mathbb{T})$  in place of  $t_{\mathbf{u}}^{f,g}(\mathbb{T})$ .

Theorems A, B, C and D above were generalized to  $f^g$ -characterized subgroups in [24, Theorems 3.4, 3.12 and 3.13], so also in this setting Theorem E holds, too. Moreover, the following results, that we collect in a single theorem, give useful comparisons among the several notions of characterized subgroups of  $\mathbb{T}$ .

**Theorem 2.12.** [24, Theorems 4.1, 4.2 and 4.3] *Let  $\mathbf{u} \in \mathcal{A}$ .*

- (1) *For any modulus function  $f$ , there exists  $g \in \mathbb{G}$  such that  $t_{\mathbf{u}}^{f,g}(\mathbb{T}) \subsetneq t_{\mathbf{u}}^\alpha(\mathbb{T})$  and  $t_{\mathbf{u}}^{f,g}(\mathbb{T}) \subsetneq t_{\mathbf{u}}^s(\mathbb{T})$ .*
- (2) *There exists a modulus function  $f$  such that for any  $g \in \mathbb{G}$ ,  $t_{\mathbf{u}}^{f,g}(\mathbb{T}) \neq t_{\mathbf{u}}^\alpha(\mathbb{T})$  and  $t_{\mathbf{u}}^{f,g}(\mathbb{T}) \neq t_{\mathbf{u}}^s(\mathbb{T})$ .*
- (3) *For any modulus function  $f$ , there exist  $\mathfrak{c}$  many  $g \in \mathbb{G}$  such that  $t_{\mathbf{u}}^f(\mathbb{T}) \subsetneq t_{\mathbf{u}}^{f,g}(\mathbb{T})$ .*

Moreover, in [45] Ghosh constructed an uncountable tower of  $f^g$ -characterized subgroups of  $\mathbb{T}$ :

**Theorem 2.13.** [45, Theorem 2.8] *For every modulus function  $f$  and for every  $\mathbf{u} \in \mathcal{A}$ , there exists a family  $\{g_\alpha : \alpha \in (0, 1)\} \subseteq \mathbb{G}$  such that*

$$t_{\mathbf{u}}(\mathbb{T}) \subsetneq \bigcap_{\alpha \in (0,1)} t_{\mathbf{u}}^{f,g_\alpha}(\mathbb{T}) \quad \text{and} \quad \bigcup_{\alpha \in (0,1)} t_{\mathbf{u}}^{f,g_\alpha}(\mathbb{T}) \subsetneq t_{\mathbf{u}}^f(\mathbb{T}),$$

with  $t_{\mathbf{u}}^{f,g\alpha}(\mathbb{T}) \subsetneq t_{\mathbf{u}}^{f,g\beta}(\mathbb{T})$  for every  $\alpha < \beta$  in  $(0, 1)$ .

This also gives the solution of [14, Problems 2.14 and 2.15]:

**Corollary 2.14.** For any  $\mathbf{u} \in \mathcal{A}$ ,

$$t_{\mathbf{u}}(\mathbb{T}) \subsetneq \bigcap_{\alpha \in (0,1)} t_{\mathbf{u}}^{\alpha}(\mathbb{T}) \quad \text{and} \quad \bigcup_{\alpha \in (0,1)} t_{\mathbf{u}}^{\alpha}(\mathbb{T}) \subsetneq t_{\mathbf{u}}^s(\mathbb{T}),$$

with

$$t_{\mathbf{u}}^{\alpha}(\mathbb{T}) \subsetneq t_{\mathbf{u}}^{\beta}(\mathbb{T}) \quad \text{whenever } \alpha < \beta \text{ with } \alpha, \beta \in (0, 1). \tag{2.2}$$

In [21, Theorem 1.1] this was proved for  $\mathbf{u} = (2^n)$ .

### 3. Ideals – generalities

In the sequel we write  $A^* := \mathbb{N} \setminus A$  for  $A \in \mathcal{P}(\mathbb{N})$ .

#### 3.1. The convexity number and the gap number

Every infinite, non-cofinite subset  $A$  of  $\mathbb{N}$  can be written as

$$A = \bigcup_{n \in \mathbb{N}} [a_n, b_n], \quad \text{where } a_n, b_n \in \mathbb{N} \text{ and } a_{n+1} > b_n + 1.$$

Here  $A_n = [a_n, b_n]$  is the  $n$ -th maximal interval of  $A$  and  $G_n = (b_n, a_{n+1})$  the  $n$ -th gap of  $A$ .

One has a natural way to measure the  $n$ -th maximal interval and the  $n$ -th gap of  $A$  as follows: for every  $n \in \mathbb{N}$ , let

$$c_n(A) := |[a_n, b_n]| = b_n - a_n + 1 \quad \text{and} \quad g_n(A) := |(b_n, a_{n+1})| = a_{n+1} - b_n - 1;$$

moreover, let  $g_{-1}(A) := |[0, a_0]| = a_0$ . Then let

$$c(A) := \sup_{n \in \mathbb{N}} c_n(A), \quad ac(A) := \limsup_{n \rightarrow \infty} c_n(A) \quad \text{and} \quad g(A) := \sup_{n \in \mathbb{N} \cup \{-1\}} g_n(A), \quad ag(A) := \limsup_{n \rightarrow \infty} g_n(A).$$

For  $A \in \mathcal{P}(\mathbb{N})$ ,  $c(A)$  is the *convexity number* and  $ac(A)$  is the *asymptotic convexity number* of  $A$ . Clearly,

$$ac(A) \leq c(A) \quad \text{and} \quad c(A) = \infty \text{ if and only if } ac(A) = \infty. \tag{3.1}$$

Moreover,  $g(A)$  is the *gap number* and  $ag(A)$  is the *asymptotic gap number* of  $A$ . Clearly,

$$c(A) = g(A^*) \quad \text{and} \quad ac(A) = ag(A^*). \tag{3.2}$$

These two invariants are monotone in the following sense:

**Lemma 3.1.** Let  $A, B$  be infinite non-cofinite subsets of  $\mathbb{N}$ .

- (1) If  $A \subseteq B$ , then  $c(A) \leq c(B)$  and  $g(A) \geq g(B)$ .
- (2) If  $A \subseteq^* B \subseteq \mathbb{N}$ , then  $ac(A) \leq ac(B)$  and  $ag(A) \geq ag(B)$ .

Let  $\{A_n : n \in \mathbb{N}\}$  be a family of subsets of  $\mathbb{N}$ ; a subset  $A$  of  $\mathbb{N}$  is a pseudointersection of  $\{A_n : n \in \mathbb{N}\}$  if  $A \subseteq^* A_n$  for all  $n \in \mathbb{N}$ , and  $B$  is a pseudounion of  $\{A_n : n \in \mathbb{N}\}$  if  $A_n \subseteq^* B$  for all  $n \in \mathbb{N}$ . The pseudointersection and the pseudounion are not uniquely determined.

**Corollary 3.2.** *Let  $\{A_n : n \in \mathbb{N}\}$  be a family of infinite non-cofinite subsets of  $\mathbb{N}$ .*

- (1) *If  $A$  is an infinite non-cofinite subset of  $\mathbb{N}$  that is a pseudointersection of  $\{A_n : n \in \mathbb{N}\}$ , then  $ac(A) \leq \inf_{n \in \mathbb{N}} ac(A_n)$  and  $ag(A) \geq \sup_{n \in \mathbb{N}} ag(A_n)$ .*
- (2) *If  $B$  is an infinite non-cofinite subset of  $\mathbb{N}$  that is a pseudounion of  $\{A_n : n \in \mathbb{N}\}$ , then  $ac(B) \geq \sup_{n \in \mathbb{N}} ac(A_n)$  and  $ag(B) \leq \inf_{n \in \mathbb{N}} ag(A_n)$ .*

**Proof.** (1) For every  $n \in \mathbb{N}$ , as  $A \subseteq^* A_n$ , it follows that  $ac(A) \leq ac(A_n)$  and  $ag(A) \geq ag(A_n)$  by Lemma 3.1(2).

(2) For every  $n \in \mathbb{N}$ , as  $A_n \subseteq^* B$ , it follows that  $ac(B) \geq ac(A_n)$  and  $ag(B) \leq ag(A_n)$  by Lemma 3.1(2).  $\square$

**Example 3.3.** If  $A$  is an infinite non-cofinite subset of  $\mathbb{N}$  and  $g(A) \leq m$  for some  $m \in \mathbb{N}_+$ , then  $\underline{d}(A) \geq 1/(m+1)$ . On the other hand,  $B = \{k^2 - i : k \in \mathbb{N}_+, 0 \leq i < k\}$  is an infinite non-cofinite subset of  $\mathbb{N}$  with  $g(B) = \infty$  and  $\underline{d}(B) > 0$ .

**Remark 3.4.** The following notions can be found in the literature. A subset  $B = \{b_0 < b_1 < \dots < b_n < \dots\}$  of  $\mathbb{N}$  is:

- (i) *large* if there exists  $m \in \mathbb{N}$  such that  $b_n - b_{n-1} \leq m$  for all  $n \in \mathbb{N}_+$ ;
- (ii) *lacunary* if  $b_{n+1} - b_n \rightarrow \infty$ ;
- (iii) a *Hadamard set* if  $(b_n) \in \mathcal{H}$ .

The notion of Hadamard set was introduced by Kunen and Rudin [49], where they named them also simply “lacunary”. This explains our choice of the term, in order to clearly distinguish them from the lacunary sets as given in item (ii). Clearly, every Hadamard set is lacunary but the converse is not true. On the other hand, lacunary sets are non-large, while the converse implication does not hold in general.

Moreover, for an infinite non-cofinite subset  $A$  of  $\mathbb{N}$ :

- (1)  $A$  is large if and only if  $g(A) < \infty$ ;
- (2)  $A$  is lacunary if and only if  $g_n(A) \rightarrow \infty$  and  $ac(A) = 1$ .

### 3.2. Basic definitions on ideals

An ideal  $\mathcal{I}$  of  $\mathbb{N}$  is *non-trivial* or *proper* if  $\mathbb{N} \notin \mathcal{I}$ , and  $\mathcal{I}$  is *free* if  $\{n\} \in \mathcal{I}$  for every  $n \in \mathbb{N}$ . An ideal  $\mathcal{I}$  of  $\mathbb{N}$  is free if and only if  $\mathcal{I}$  contains the ideal  $\mathcal{F}in$  of all finite subsets of  $\mathbb{N}$ .

If  $\mathcal{I}$  and  $\mathcal{I}'$  are ideals of  $\mathbb{N}$ , we denote

$$\mathcal{I} \dot{\cup} \mathcal{I}' := \{A \cup B : A \in \mathcal{I}, B \in \mathcal{I}'\}.$$

Clearly,  $\mathcal{I} \dot{\cup} \mathcal{I}'$  is the smallest ideal of  $\mathbb{N}$  containing both  $\mathcal{I}$  and  $\mathcal{I}'$ .

For a subset  $A$  of  $\mathbb{N}$  the power-set  $\mathcal{P}(A)$  is a *principal* ideal. A principal ideal  $\mathcal{I} = \mathcal{P}(A)$  is free if and only if  $\mathcal{I} = \mathcal{P}(\mathbb{N})$  is improper.

Since we are mainly interested in free ideals, we will often make recourse to ideals of the form

$$\mathcal{I}_A := \mathcal{P}(A) \dot{\cup} \mathcal{F}in = \{B \in \mathcal{P}(\mathbb{N}) : B \subseteq^* A\},$$

called *principal modulo Fin* (generated by  $A$ ).

For a proper ideal  $\mathcal{I}$  of  $\mathbb{N}$ , the family  $\mathcal{I}^* := \{A^* : A \in \mathcal{I}\}$  is a filter of  $\mathbb{N}$ . So, hereinafter we compare ideals and filters of  $\mathbb{N}$  and deduce properties of ideals from well-known properties of filters.

Clearly, an ideal  $\mathcal{I}$  of  $\mathbb{N}$  is principal if and only if  $\mathcal{I}^*$  is a principal filter.

**Example 3.5.** The Fréchet’s filter  $\mathcal{F}_0 := \{A \in \mathcal{P}(\mathbb{N}) : |\mathbb{N} \setminus A| < \infty\}$  of  $\mathbb{N}$  has as members all cofinite sets in  $\mathbb{N}$ . So,  $\mathcal{F}in^* = \mathcal{F}_0$ .

A family  $\mathcal{B}$  of subsets of  $\mathbb{N}$  is a *base* of an ideal  $\mathcal{I}$  of  $\mathbb{N}$  if for every  $J \in \mathcal{I}$  there exists  $A \in \mathcal{B}$  such that  $J \subseteq A$  (equivalently, when  $\mathcal{B}^*$  is a *base* of the filter  $\mathcal{I}^*$ ). An ideal  $\mathcal{I}$  of  $\mathbb{N}$  is *countably based* if  $\mathcal{I}$  admits a countable base. This is equivalent to the existence of a chain  $A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subseteq \dots$  of members of  $\mathcal{I}$  such that  $\{A_n : n \in \mathbb{N}_+\}$  is a base of  $\mathcal{I}$ .

Recall that a filter  $\mathcal{F}$  of  $\mathbb{N}$  is a *P-filter* if for every countable family  $\{B_n : n \in \mathbb{N}\} \subseteq \mathcal{F}$  there exists a pseudointersection  $B \in \mathcal{F}$ , that is,  $B \subseteq^* B_n$  for all  $n \in \mathbb{N}$ . Analogously, an ideal  $\mathcal{I}$  of  $\mathbb{N}$  is a *P-ideal* if for every countable family  $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{I}$  there exists a pseudounion  $A \in \mathcal{I}$ , that is,  $A_n \subseteq^* A$  for all  $n \in \mathbb{N}$ .

Clearly, for an ideal  $\mathcal{I}$  of  $\mathbb{N}$ :

- $\mathcal{I}$  is countably based if and only if  $\mathcal{I}^*$  is countably based;
- $\mathcal{I}$  is a *P-ideal* if and only if  $\mathcal{I}^*$  is a *P-filter*.

**Lemma 3.6.** Let  $\mathcal{I}$  and  $\mathcal{I}'$  be *P-ideals* of  $\mathbb{N}$ .

- (1) If  $\mathcal{I}$  and  $\mathcal{I}'$  are countably based, then so is  $\mathcal{I} \dot{\cup} \mathcal{I}'$ .
- (2) If  $\mathcal{I}$  and  $\mathcal{I}'$  are *P-ideals*, then so is  $\mathcal{I} \dot{\cup} \mathcal{I}'$ .

**Proof.** (1) is clear.

(2) Let  $\{B_n : n \in \mathbb{N}\}$  be a subfamily of  $\mathcal{I} \dot{\cup} \mathcal{I}'$ , and for every  $n \in \mathbb{N}$  write  $B_n = A_n \cup A'_n$  with  $A_n \in \mathcal{I}$  and  $A'_n \in \mathcal{I}'$ . Since  $\mathcal{I}$  and  $\mathcal{I}'$  are *P-ideals*, there exist  $A \in \mathcal{I}$  and  $A' \in \mathcal{I}'$  such that  $A_n \subseteq^* A$  and  $A'_n \subseteq^* A'$  for all  $n \in \mathbb{N}$ , and so  $A_n \cup A'_n \subseteq^* A \cup A'$  for all  $n \in \mathbb{N}$ . This proves that  $\mathcal{I} \dot{\cup} \mathcal{I}'$  is a *P-ideal*.  $\square$

**Proposition 3.7.** For every  $A \in \mathcal{P}(\mathbb{N})$ , the ideal  $\mathcal{I}_A$  is a countably based *P-ideal*. Moreover, a countably based free ideal  $\mathcal{I}$  is a *P-ideal* if and only if it is principal modulo *Fin*.

**Proof.** Clearly,  $\mathcal{F}in$  and  $\mathcal{P}(A)$  are countably based *P-ideals*, so apply Lemma 3.6 to conclude that  $\mathcal{I}_A$  is a countably based *P-ideal*.

Vice versa, assume that  $\mathcal{I}$  is a free and countably based *P-ideal*, and let  $\{A_n : n \in \mathbb{N}\}$  with  $A_0 \subseteq A_1 \subseteq \dots$  be a base of  $\mathcal{I}$ . Then there exists  $A \in \mathcal{I}$  such that  $A_n \subseteq^* A$  for every  $n \in \mathbb{N}$ . We verify that  $\mathcal{I} = \mathcal{I}_A$ . Indeed, clearly  $\mathcal{P}(A) \subseteq \mathcal{I}$  and  $\mathcal{F}in \subseteq \mathcal{I}$ . For the converse inclusion, let  $X \in \mathcal{I}$ . There exists  $n \in \mathbb{N}$  such that  $X \subseteq A_n \subseteq^* A$ . So,  $X \subseteq^* A$ , hence  $X \in \mathcal{I}_A$ .  $\square$

The above proposition is used in Corollary 3.26 below to see that  $\mathcal{I}_d$  does not admit a countable base.

The next example provides a free ideal which is not a *P-ideal* (another example with this property can be found in [47, Example 3.1(g)]).

**Example 3.8.** Let  $\{A_n : n \in \mathbb{N}\}$  be a family of pairwise disjoint infinite subsets of  $\mathbb{N}$  and

$$\mathcal{I} = \left\{ I \subseteq \mathbb{N} : \exists n \in \mathbb{N}, I \subseteq^* \bigcup_{i=0}^n A_i \right\}$$

be the free ideal of  $\mathbb{N}$  generated by  $\{A_n : n \in \mathbb{N}\}$ . We see that  $\mathcal{I}$  is not a  $P$ -ideal (or, equivalently, that  $\mathcal{I}$  is not principal modulo  $\mathcal{F}in$ ). Suppose the contrary and let  $A \in \mathcal{I}$  be any pseudounion of  $\{A_n : n \in \mathbb{N}\}$ . Then for some  $n \in \mathbb{N}$  we have  $A_{n+1} \subseteq^* A \subseteq^* \bigcup_{i=0}^n A_i$ , hence  $A_{n+1} \subseteq^* \bigcup_{i=0}^n A_i$ , a contradiction.

Let  $\mathcal{I}$  be an ideal of  $\mathbb{N}$ . Following [25], call  $A \in \mathcal{P}(\mathbb{N})$   $\mathcal{I}$ -translation invariant if  $(A+n) \cap \mathbb{N} \in \mathcal{I}$  for every  $n \in \mathbb{Z}$ . Call the ideal  $\mathcal{I}$  translation invariant (briefly, *t.i.*) if every  $A \in \mathcal{I}$  is  $\mathcal{I}$ -translation invariant. Finally, call  $\mathcal{I}$  strongly non-translation invariant (briefly, *snt*) if the only  $\mathcal{I}$ -translation invariant elements of  $\mathcal{I}$  are finite.

**Example 3.9.** The ideals  $\mathcal{F}in$  and  $\mathcal{I}_d$  are translation invariant, while  $\mathcal{F}in$  is also vacuously *snt* (due to the fact that  $\mathcal{F}in$  has no infinite members). Further examples of *snt* and non-*snt* ideals can be found in [25].

Obviously, an *snt* ideal of  $\mathbb{N}$  properly containing  $\mathcal{F}in$  is not translation invariant (shortly, non-*t.i.*) in the strongest possible way. The implication “non-*t.i.*  $\Rightarrow$  *snt*” does not hold in general, ([25, Proposition 2.11] provides a non-*snt* analytic  $P$ -ideal that is non-*t.i.*). To realize better this gap, in Example 4.11 we examine the relation between the property non-*t.i.* and the property  $(**)$  weaker than *snt*.

### 3.3. Maximal ideals vs ultrafilters

Recall that an ideal  $\mathcal{I}$  of  $\mathbb{N}$  is maximal if given any ideal  $\mathcal{J}$  of  $\mathbb{N}$  such that  $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathcal{P}(\mathbb{N})$ , then either  $\mathcal{J} = \mathcal{I}$  or  $\mathcal{J} = \mathcal{P}(\mathbb{N})$ .

**Remark 3.10.** An ideal  $\mathcal{I}$  of  $\mathbb{N}$  is maximal if and only if  $\mathcal{I}^*$  is an ultrafilter. This will allow us to deduce properties of maximal ideals from properties of ultrafilters. In particular, there are  $2^c$  maximal ideals of  $\mathbb{N}$  (see Remark 3.16).

We start recalling the following known facts on ultrafilters.

**Fact 3.11.**

- (1) A filter  $\mathcal{U}$  is an ultrafilter if and only if for every  $L \in \mathcal{P}(\mathbb{N})$ ,  $L \notin \mathcal{U}$  if and only if there exists  $U \in \mathcal{U}$  with  $L \cap U = \emptyset$ .
- (2) Every proper filter of  $\mathbb{N}$  is contained in an ultrafilter.

**Corollary 3.12.** Every proper ideal of  $\mathbb{N}$  is contained in a maximal ideal.

**Proof.** Let  $\mathcal{I}$  be a proper ideal of  $\mathbb{N}$ . By Fact 3.11(2),  $\mathcal{I}^*$  is contained in an ultrafilter  $\mathcal{U}$ . Then  $\mathcal{I}$  is contained in the maximal ideal  $\mathcal{U}^*$  of  $\mathbb{N}$ .  $\square$

One can see this from a different point of view:

**Remark 3.13.** With the symmetric difference and the intersection,  $(\mathcal{P}(\mathbb{N}), \Delta, \cap)$  is a Boolean ring isomorphic to  $\mathbb{Z}(2)^{\mathbb{N}}$ .

- (a) A subfamily  $\mathcal{I}$  of  $\mathcal{P}(\mathbb{N})$  is an ideal of  $\mathbb{N}$  in the sense of §3.2 if and only if  $\mathcal{I}$  is an ideal of the ring  $(\mathcal{P}(\mathbb{N}), \Delta, \cap) \cong \mathbb{Z}(2)^{\mathbb{N}}$ .

From Krull's theorem, every ideal of a unitary ring is contained in a maximal one. This provides an alternative proof of Corollary 3.12.

(b) The isomorphism  $(\mathcal{P}(\mathbb{N}), \Delta, \cap) \cong \mathbb{Z}(2)^{\mathbb{N}}$  allows us to consider  $(\mathcal{P}(\mathbb{N}), \Delta, \cap)$  equipped with the product topology as a topological ring with respect to the operations  $\Delta$  and  $\cap$ .

**Proposition 3.14.** *Let  $\mathcal{I}$  be an ideal of  $\mathbb{N}$ .*

- (1) *Then  $\mathcal{I}$  is maximal if and only if  $A \cap B \in \mathcal{I}$  implies  $A \in \mathcal{I}$  or  $B \in \mathcal{I}$  for every pair  $A, B \in \mathcal{P}(\mathbb{N})$ .*
- (2) *If  $\mathcal{I}$  is maximal, then for every pair of disjoint sets  $A, B \in \mathcal{P}(\mathbb{N})$  either  $A \in \mathcal{I}$  or  $B \in \mathcal{I}$ . In particular, for every  $A \in \mathcal{P}(\mathbb{N})$ , exactly one of the following holds true: either  $A \in \mathcal{I}$  or  $A^* \in \mathcal{I}$ .*

**Proof.** (1) follows from Remark 3.10 and the standard properties of ultrafilters; (2) follows from (1).  $\square$

In the ring  $(\mathcal{P}(\mathbb{N}), \Delta, \cap)$  this means that the ideal  $\mathcal{I}$  is maximal if and only if  $|\mathcal{P}(\mathbb{N})/\mathcal{I}| = 2$ , which is a known property for Boolean rings.

As a consequence of Proposition 3.14, we give the following properties of maximal ideals.

**Corollary 3.15.** *Let  $\mathcal{M}$  be a maximal ideal of  $\mathbb{N}$ . Then:*

- (1)  *$\mathcal{M}$  is free if and only if  $\mathcal{M}$  is not principal;*
- (2) *if  $\mathcal{M}$  is free, then  $\mathcal{M}$  cannot have countable base;*
- (3) *[t.i.  $\Rightarrow$  non-maximal]  $\mathcal{M}$  is not translation invariant.*

**Proof.** (1) If  $\mathcal{M}$  is free, then  $\mathcal{M}$  is obviously non-principal. In order to prove the converse implication, assume that  $\mathcal{M}$  is not free, that is,  $\{n\} \notin \mathcal{M}$  for some  $n \in \mathbb{N}$ . Then  $\mathbb{N} \setminus \{n\} \in \mathcal{M}$  by Proposition 3.14, and so  $\mathcal{P}(\mathbb{N} \setminus \{n\}) \subseteq \mathcal{M}$ . Moreover,  $\{n\} \notin \mathcal{M}$  implies that  $n \notin A$  for every  $A \in \mathcal{M}$ , that is,  $\mathcal{M} \subseteq \mathcal{P}(\mathbb{N} \setminus \{n\})$ . Hence,  $\mathcal{M} = \mathcal{P}(\mathbb{N} \setminus \{n\})$  is principal.

(2) In view of Remark 3.10 and item (1), we have to prove that a non-principal ultrafilter  $\mathcal{U}$  cannot have countable base. Let  $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$  be a countable base of  $\mathcal{U}$ , and we can assume without loss of generality that all  $B_n$  are infinite and that  $B_0 \supseteq B_1 \supseteq B_2 \supseteq \dots$ . Then it is possible to construct two sets  $X = \{x_n : n \in \mathbb{N}\}$  and  $Y = \{y_n : n \in \mathbb{N}\}$  such that  $X \cap Y = \emptyset$  and  $x_n, y_n \in B_n$  for every  $n \in \mathbb{N}$ . Since  $\mathcal{B}$  is a base of  $\mathcal{U}$  and  $X \cap B_n \neq \emptyset$  for every  $n \in \mathbb{N}$ , necessarily  $X \in \mathcal{U}$  by Fact 3.11, and analogously  $Y \in \mathcal{U}$ . But an ultrafilter cannot contain two disjoint elements.

(3) As  $\mathcal{M}$  is maximal, exactly one of the sets  $2\mathbb{N}$  or  $2\mathbb{N} + 1$  belongs to  $\mathcal{M}$ , by Proposition 3.14. Since they are related by a translation (by 1), this shows that  $\mathcal{M}$  is not translation invariant.  $\square$

In the next remark we discuss the non-trivial question on whether a maximal ideal  $\mathcal{M}$  can be a  $P$ -ideal.

**Remark 3.16.** Due to Remark 3.10, there is a bijection between ultrafilters and maximal ideals of  $\mathbb{N}$ . Since the ultrafilters of  $\mathbb{N}$  form the points of the Stone-Ćech compactification  $\beta\mathbb{N}$ , this shows that  $\mathbb{N}$  has  $2^c$  many maximal ideals. (Equivalently, one can work with the compact spectrum of maximal ideals of the Boolean ring  $\mathbb{Z}(2)^{\mathbb{N}} = \mathcal{P}(\mathbb{N})$  – see Remark 3.13, which is homeomorphic to  $\beta\mathbb{N}$ .)

(a) Using the space  $\beta\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$  of non-principal ultrafilters, one can see that a non-principal ultrafilter  $\mathcal{U}$  is a  $P$ -filter precisely when it is a  $P$ -point of the space  $\beta\mathbb{N}^*$  (i.e., any intersection of countably many neighborhoods of  $\mathcal{U}$  in  $\beta\mathbb{N}^*$  is still a neighborhood of  $\mathcal{U}$  in  $\beta\mathbb{N}^*$ ). This is the motivation for the term  $P$ -filter.

(b) The question on whether a maximal ideal  $\mathcal{M}$  can be a  $P$ -ideal is equivalent to the well known question of whether a non-principal ultrafilter  $\mathcal{U}$  can be a  $P$ -filter. It was raised by Rudin [55] who was interested in the homogeneity of the space  $\beta\mathbb{N}^*$ . Since one can construct in ZFC ultrafilters that are not  $P$ -filters (witnessing

the failure of the implication “maximal ideal  $\Rightarrow P$ -ideal”), to establish the non-homogeneity of  $\beta\mathbb{N}^*$  it was enough to see that  $\beta\mathbb{N}^*$  has  $P$ -points. Rudin [55] built such points (i.e., non-principal ultrafilters that are  $P$ -filters) under the assumption of CH (actually, there are  $2^c$   $P$ -points in  $\beta\mathbb{N}^*$  under this assumption). On the other hand, in 1982 Shelah found a model of ZFC (with the additional property  $\mathfrak{c} = \aleph_2$ ) without ultrafilters that are  $P$ -filters. In other words, *the existence of maximal ideals that are  $P$ -ideals is independent of ZFC*.

3.4. Analytic  $P$ -ideals

In Remark 3.13 we noticed that  $(\mathcal{P}(\mathbb{N}), \Delta, \cap)$  is a Boolean ring isomorphic to  $\mathbb{Z}(2)^\mathbb{N}$ , so we can identify these two rings and endow  $\mathbb{Z}(2)^\mathbb{N}$  with the product topology of the discrete topology of  $\mathbb{Z}(2)$ , so that it becomes also a compact metric space.

A subset  $Y$  of a metric space  $X$  is *analytic* if it is continuous image of a Borel set. So, we call an ideal  $\mathcal{I}$  of  $\mathbb{N}$  *analytic* if it is an analytic subset of  $\mathbb{Z}(2)^\mathbb{N}$ .

**Definition 3.17.** A *submeasure* on  $\mathbb{N}$  is a function  $\varphi: \mathcal{P}(\mathbb{N}) \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$  such that:

- (S1)  $\varphi(\emptyset) = 0$ ;
- (S2) if  $A \subseteq B \subseteq \mathbb{N}$ , then  $\varphi(A) \leq \varphi(B)$ ;
- (S3) for all  $A, B \in \mathcal{P}(\mathbb{N})$ ,  $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ .

A submeasure  $\varphi$  such that  $\varphi(A)$  is finite for all  $A \in \mathcal{P}(\mathbb{N})$ , will be called a *finite submeasure*.

Recall that a function  $\varphi: M \rightarrow \mathbb{R}$ , where  $M$  is a topological space, is *lower semicontinuous* if  $x_n \rightarrow x$  in  $M$  implies  $\liminf_{n \rightarrow \infty} \varphi(x_n) \geq \varphi(x)$ .

**Remark 3.18.** Let  $\tau_-$  be the lower topology on  $\mathbb{R}$  having as open sets  $\mathbb{R}$  and all open intervals  $(a, +\infty)$ . A function  $\varphi: M \rightarrow \mathbb{R}$ , where  $M$  is a topological space, is lower semicontinuous if and only if  $\varphi: M \rightarrow (\mathbb{R}, \tau_-)$  is continuous.

**Fact 3.19.** [16] Let  $M$  be a topological space and let  $\varphi, \varphi_n: M \rightarrow \mathbb{R} \cup \{+\infty\}$ ,  $n \in \mathbb{N}$ , be lower semicontinuous functions. Then:

- (1) for every  $\lambda \in \mathbb{R}$ , the set  $\{x \in M : \varphi(x) \leq \lambda\}$  is closed;
- (2)  $\varphi_0 + \varphi_1$  is lower semicontinuous;
- (3)  $\sup_{n \in \mathbb{N}} \varphi_n$  is lower semicontinuous.

**Lemma 3.20.** For a submeasure  $\varphi: \mathcal{P}(\mathbb{N}) \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$  the following conditions are equivalent:

- (1)  $\varphi$  is lower semicontinuous;
- (2)  $\varphi(A) = \lim_{n \rightarrow \infty} \varphi(A(n))$  for all  $A \in \mathcal{P}(\mathbb{N})$ ;
- (3)  $\varphi$  is upward  $\sigma$ -continuous, that is,  $A_n \uparrow A$  in  $\mathcal{P}(\mathbb{N})$  implies  $\varphi(A) = \lim_{n \rightarrow \infty} \varphi(A_n)$ .

**Proof.** (1) $\Rightarrow$ (3) Let  $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{P}(\mathbb{N})$  with  $A_n \uparrow A$ . Since  $\varphi$  is lower semicontinuous and monotone,

$$\varphi(A) \leq \liminf_{n \rightarrow \infty} \varphi(A_n) = \lim_{n \rightarrow \infty} \varphi(A_n) \leq \varphi(A),$$

hence  $\lim_{n \rightarrow \infty} \varphi(A_n) = \varphi(A)$ .

(3)⇒(2) is obvious as  $A(n) \uparrow A$ .

(2)⇒(1) Let  $A_n \rightarrow A$  in  $\mathcal{P}(\mathbb{N})$ . Fixed  $k \in \mathbb{N}$ , there exists  $n_k \in \mathbb{N}$  such that  $A_n \cap [0, k] = A \cap [0, k]$  for every  $n \geq n_k$ . Then

$$\liminf_{n \rightarrow \infty} \varphi(A_n) \geq \liminf_{n \rightarrow \infty} \varphi(A_n \cap [0, k]) = \varphi(A \cap [0, k]).$$

By hypothesis,  $\lim_{k \rightarrow \infty} \varphi(A \cap [0, k]) = \varphi(A)$ , hence we conclude that  $\liminf_{n \rightarrow \infty} \varphi(A_n) \geq \varphi(A)$ .  $\square$

**Fact 3.21.** Any upward  $\sigma$ -continuous submeasure  $\varphi$  on an algebra  $\mathfrak{A}$  of sets is  $\sigma$ -subadditive.

**Proof.** Let  $A, A_n \in \mathfrak{A}$  and  $A = \bigcup_{n \in \mathbb{N}} A_n$ . Then

$$\varphi(A) = \lim_{n \rightarrow \infty} \varphi \left( \bigcup_{i \leq n} A_i \right) \leq \lim_{n \rightarrow \infty} \sum_{i \leq n} \varphi(A_i) = \sum_{n \in \mathbb{N}} \varphi(A_n). \quad \square$$

In the sequel we briefly refer to a lower semicontinuous submeasure as a *lscsm*. For any lscsm  $\varphi$ , Solecki [56] defined the *exhaustive ideal*

$$\text{Exh}(\varphi) := \left\{ A \subseteq \mathbb{N} : \lim_{n \rightarrow \infty} \varphi(A \setminus [0, n]) = 0 \right\}.$$

As mentioned by Farah [42], the following result is apparently folklore, nevertheless we reproduce here his proof in detail.

**Lemma 3.22.** [42, Lemma 1.2.2] If  $\varphi: \mathcal{P}(\mathbb{N}) \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$  is a lscsm, then  $\text{Exh}(\varphi)$  is an  $F_{\sigma\delta}$ -set and a  $P$ -ideal.

**Proof.** For  $m, n \in \mathbb{N}_+$ , the sets  $L_{m,n} = \{A \in \mathcal{P}(\mathbb{N}) : \varphi(A \setminus [0, m]) \leq 1/n\}$  are closed. Indeed, for every  $m \in \mathbb{N}$  the function  $f_m: \mathcal{P}(\mathbb{N}) \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$  defined by  $f_m(A) = \varphi(A \setminus [0, m])$  is lower semicontinuous, as  $f_m = \varphi \circ \sigma_m$ , where  $\sigma_m: A \mapsto A \setminus [0, m]$ . In fact,  $\varphi$  is lower semicontinuous by hypothesis, while  $\sigma_m: \mathcal{P}(\mathbb{N}) \rightarrow \mathcal{P}(\mathbb{N})$  is actually continuous as  $\sigma_m$  coincides with  $A \mapsto (A \Delta [0, m]) \cap A$  in the topological ring  $(\mathcal{P}(\mathbb{N}), \Delta, \cap)$  (using the topological isomorphism  $\mathcal{P}(\mathbb{N}) \cong \mathbb{Z}(2)^{\mathbb{N}}$  from Remark 3.13 and the continuity of ring operations). Since  $f_m$  is lower semicontinuous, Fact 3.19(1) implies that  $L_{m,n}$  is closed. As

$$\text{Exh}(\varphi) = \bigcap_{n \in \mathbb{N}_+} \bigcup_{m \in \mathbb{N}_+} L_{m,n},$$

we conclude that  $\text{Exh}(\varphi)$  is  $F_{\sigma\delta}$ .

To see that  $\text{Exh}(\varphi)$  is a  $P$ -ideal, let  $\{A_n : n \in \mathbb{N}_+\}$  be a subfamily of  $\text{Exh}(\varphi)$ ; we may assume that the sets  $A_n$  are pairwise disjoint. For every  $i \in \mathbb{N}_+$ , let  $n_i \in \mathbb{N}$  such that  $\varphi(A_i \setminus [0, n_i]) \leq 1/2^i$ ; then let

$$A = \bigcup_{i \in \mathbb{N}_+} (A_i \setminus [0, n_i]).$$

Clearly,  $A_i \subseteq^* A$  for every  $i \in \mathbb{N}_+$ . So, it remains to prove that  $A \in \text{Exh}(\varphi)$ . In fact, fixed  $n \in \mathbb{N}_+$ , there exists  $m \in \mathbb{N}_+$  such that  $\varphi(\bigcup_{i=1}^m (A_i \setminus [0, m])) \leq 1/2^n$ . Then, using the  $\sigma$ -subadditivity of  $\varphi$  (see Fact 3.21),

$$\varphi(A \setminus [0, m]) \leq \varphi \left( \bigcup_{i \leq n} (A_i \setminus [0, m]) \right) + \varphi \left( \bigcup_{i > n} (A_i \setminus [0, n_i]) \right) \leq \frac{1}{2^n} + \sum_{i > n} \varphi(A_i \setminus [0, n_i]).$$

As  $\sum_{i>n} \varphi(A_i \setminus [0, n_i]) \leq \sum_{i>n} 1/2^i = 1/2^n$ , we conclude that  $\varphi(A \setminus [0, m]) \leq 1/2^{n-1}$ . As  $n \in \mathbb{N}$  was arbitrary, this proves that  $\lim_{m \rightarrow \infty} \varphi(A \setminus [0, m]) = 0$ .  $\square$

Solecki showed that Lemma 3.22 can be inverted in a strong way:

**Theorem 3.23.** [56, Theorem 3.1] *Every analytic  $P$ -ideal of  $\mathbb{N}$  is of the form  $\text{Exh}(\varphi)$  for some finite lscsm  $\varphi$  on  $\mathbb{N}$  (hence, an  $F_{\sigma\delta}$   $P$ -ideal).*

Theorem 3.23 is applied in Theorem 5.8, that is, [25, Theorem 2.3].

**Remark 3.24.** Since  $\mathcal{P}(\mathbb{N})$  is second countable, it has at most  $\mathfrak{c}$  many open sets and, consequently, at most  $\mathfrak{c}$  many closed sets. Therefore,  $\mathcal{P}(\mathbb{N})$  has at most  $\mathfrak{c}$  many  $F_\sigma$ -sets and, consequently, at most  $\mathfrak{c}$  many  $F_{\sigma\delta}$ -sets. Therefore, Theorem 3.23 implies that  $\mathcal{P}(\mathbb{N})$  has at most  $\mathfrak{c}$  many analytic  $P$ -ideals (compared to  $2^{\mathfrak{c}}$  maximal  $P$ -ideals under the assumption of CH).

In the following result and its corollary, we give a new shorter proof of [3, Proposition 1.1] specialized for  $\mathcal{I}_d$ .

**Proposition 3.25.** *The function  $\varphi_d: \mathcal{P}(\mathbb{N}) \rightarrow \mathbb{R}_{\geq 0}$ , defined by*

$$\varphi_d(A) = \sup_{n \in \mathbb{N}_+} \frac{|A(n)|}{n} \quad \text{for every } A \in \mathcal{P}(\mathbb{N}),$$

*is a finite lscsm and  $\mathcal{I}_d = \text{Exh}(\varphi_d)$ .*

**Proof.** For every  $n \in \mathbb{N}_+$ , define  $\varphi_n: \mathcal{P}(\mathbb{N}) \rightarrow \mathbb{R}_{\geq 0}$  by  $\varphi_n(A) = |A(n)|/n$  for every  $A \in \mathcal{P}(\mathbb{N})$ . Obviously, each  $\varphi_n$  is an upward  $\sigma$ -continuous submeasure, namely, a lscsm by Lemma 3.20. As  $\varphi_d = \sup_{n \in \mathbb{N}} \varphi_n$ , we get that also  $\varphi_d$  is a lscsm by Fact 3.19(3).

To prove the last assertion of the proposition, we proceed as in [3, Proposition 1.1] and start by verifying that  $\mathcal{I}_d \subseteq \text{Exh}(\varphi_d)$ . Let  $A \in \mathcal{I}_d$ . For every  $m \in \mathbb{N}$ ,

$$\varphi_d(A \setminus [0, m]) = \sup_{n \in \mathbb{N}_+} \frac{|(A \setminus [0, m]) \cap [0, n]|}{n} = \sup_{n > m} \frac{|(A \setminus [0, m]) \cap [0, n]|}{n} \leq \sup_{n > m} \frac{|A(n)|}{n}.$$

Therefore,

$$\lim_{m \rightarrow \infty} \varphi_d(A \setminus [0, m]) \leq \lim_{m \rightarrow \infty} \sup_{n > m} \frac{|A(n)|}{n} = \bar{d}(A) = 0.$$

Hence,  $\lim_{m \rightarrow \infty} \varphi_d(A \setminus [0, m]) = 0$  and so  $A \in \text{Exh}(\varphi_d)$ .

We are left with the converse inclusion  $\text{Exh}(\varphi_d) \subseteq \mathcal{I}_d$ . So let  $A \in \text{Exh}(\varphi_d)$ , which means that  $\lim_{m \rightarrow \infty} \varphi_d(A \setminus [0, m]) = 0$ . Fix  $\varepsilon > 0$ . There exists  $m_0 \in \mathbb{N}_+$  such that

$$\varphi_d(A \setminus [0, k]) < \frac{\varepsilon}{2} \quad \text{for every } k \geq m_0.$$

Moreover, let  $n_0 \in \mathbb{N}_+$  such that

$$\frac{|A(m_0)|}{n} < \frac{\varepsilon}{2} \quad \text{for every } n \geq n_0.$$

Then, for all  $n \geq n_0$ ,

$$\frac{|A(n)|}{n} = \frac{|A(m_0)|}{n} + \frac{|A(n) \setminus [0, m_0]|}{n} < \frac{\varepsilon}{2} + \varphi_d(A \setminus [0, m_0]) < \varepsilon.$$

Hence,  $\bar{d}(A) = 0$ , which means that  $A \in \mathcal{I}_d$ .  $\square$

**Corollary 3.26.** *The ideal  $\mathcal{I}_d$  is a P-ideal of  $\mathbb{N}$  and an  $F_{\sigma\delta}$ -set. Consequently,  $\mathcal{I}_d$  is not countably based.*

**Proof.** The first assertion follows from Lemma 3.22 and Proposition 3.25.

To prove the second one assume that  $\mathcal{I}_d$  has a countable base. Since  $\mathcal{I}_d$  is a P-ideal, Proposition 3.7 implies that  $\mathcal{I}_d$  is principal modulo  $\mathcal{F}in$ , namely, there exists an infinite subset  $A$  of  $\mathbb{N}$  such that  $\mathcal{I}_d = \mathcal{I}_A$ . The set  $A^* = \mathbb{N} \setminus A$  is infinite and it contains an infinite subset  $B$  with density 0. Then  $B \in \mathcal{I}_d$  and it must be  $A \cap B \neq \emptyset$ , that is a contradiction because  $B \subseteq A^*$ .  $\square$

**Remark 3.27.** Both Proposition 3.25 and the first assertion of Corollary 3.26 were shown to be true for  $\mathcal{Z}_g$  in [3, Proposition 1.1]. So, in analogy with Corollary 3.26 we can conclude that  $\mathcal{Z}_g$  cannot have countable base.

**4. Density of ideals and the properties snt, (\*\*), (†) and (\*)**

In this section we focus on five properties of ideals: snt, (\*\*), (†), (\*) and density. Of those the first three will play a key role in §§5, 6. What unifies the properties (\*\*), (†), (\*) and, up to a point, density (see Lemma 4.5), is the cardinal invariant  $c(-)$  defined in §3.1. From this point of view the property snt, defined strictly in terms of translations, apparently remains isolated. It is, of course, related to the weaker property non-t.i. which is similarly related exclusively to translations (see Examples 3.9 and 4.11 for the connection between non-t.i., snt and (\*\*), respectively). Our aim in this section will be to clarify the close connections among all these five properties (see the diagrams at the end of the section).

*4.1. The property (\*\*) and dense ideals*

The next property of a proper free ideal  $\mathcal{I}$  of  $\mathbb{N}$  was formulated in the proof of [25, Proposition 2.12]. We prefer to give it here in the following simpler equivalent form using the convexity number:

(\*\*)  $c(A) < \infty$  for every infinite  $A \in \mathcal{I}$ .

We shall often use also  $\neg(**)$ , which means that the ideal  $\mathcal{I}$  in question contains an infinite member  $A$  with  $c(A) = \infty$ , that is, with arbitrarily large intervals. In particular,  $\mathcal{P}(\mathbb{N})$  satisfies  $\neg(**)$ , which justifies the exclusion of  $\mathcal{P}(\mathbb{N})$  in the above definition.

**Lemma 4.1.** *Let  $\mathcal{I}$  be a proper free ideal of  $\mathbb{N}$ . The following conditions are equivalent:*

- (1)  $\neg(**)$  holds;
- (2) for every sequence  $(k_n)$  in  $\mathbb{N}_+$  there exists a strictly increasing sequence  $(c_n)$  in  $\mathbb{N}$  such that  $c_{n+1} > c_n + k_n$  for every  $n \in \mathbb{N}$  and  $A = \bigcup_{n \in \mathbb{N}} [c_n, c_n + k_n] \in \mathcal{I}$ ;
- (3) the filter  $\mathcal{I}^*$  contains a (necessarily infinite) set  $A^*$  with  $g(A^*) = \infty$ .

**Proof.** The equivalence of (1) and (3) follows from (3.2) and (2) $\Rightarrow$ (1) is clear.

(1) $\Rightarrow$ (2) Fix a sequence  $(k_n)$  in  $\mathbb{N}_+$ . Since  $\mathcal{I}$  fails to satisfy (\*\*), after possibly passing to a subset, we can assume that  $\mathcal{I}$  contains a set  $A_0 = \bigcup_{n \in \mathbb{N}} [a_n, b_n]$ , where  $a_n, b_n \in \mathbb{N}$ ,  $b_n + 1 < a_{n+1}$  and  $\lim_{n \rightarrow \infty} (b_n - a_n) = \infty$ . Hence there exists a strictly increasing sequence  $(m_n)$  such that  $k_n \leq b_{m_n} - a_{m_n}$  for every  $n \in \mathbb{N}$ . Then

$A = \bigcup_{n \in \mathbb{N}} [a_{m_n}, a_{m_n} + k_n] \subseteq A_0$  since  $[a_{m_n}, a_{m_n} + k_n] \subseteq [a_{m_n}, b_{m_n}]$  for every  $n \in \mathbb{N}$ . This implies that  $A \in \mathcal{I}$  and  $a_{m_n} + k_n \leq b_{m_n} < a_{m_{n+1}}$  for every  $n \in \mathbb{N}$ , so  $(c_n) = (a_{m_n})_{n \in \mathbb{N}}$  is the desired sequence.  $\square$

**Definition 4.2.** [42, p.11] An ideal  $\mathcal{I}$  of  $\mathbb{N}$  is called *dense* if every infinite subset of  $\mathbb{N}$  contains an infinite subset belonging to  $\mathcal{I}$ .

Maximal ideals are dense and fail to have the property (\*\*):

**Lemma 4.3.** [Maximal  $\Rightarrow \neg(**)$  & dense] *Every free maximal ideal  $\mathcal{M}$  of  $\mathbb{N}$  is dense and fails to satisfy (\*\*).*

**Proof.** Let  $\mathcal{M}$  be a maximal ideal of  $\mathbb{N}$  and let  $A$  be a subset of  $\mathbb{N}$  with  $c(A) = g(A) = c(A^*) = \infty$  (e.g., the set  $A$  from Example 4.7). Since  $A$  and  $A^*$  are disjoint, at least one of them belongs to  $\mathcal{M}$  by Proposition 3.14, hence  $\mathcal{M}$  fails to satisfy (\*\*).

Let  $A$  be an infinite subset of  $\mathbb{N}$ . Then  $A$  contains two disjoint infinite subsets  $C$  and  $D$ . In view of Proposition 3.14, either  $C$  or  $D$  belongs to  $\mathcal{M}$ . Hence,  $\mathcal{M}$  is dense.  $\square$

This lemma, combined with Remark 3.16, shows that “ $\neg(**) \not\Rightarrow P$ ”, as witnessed by any maximal ideal that is not a  $P$ -ideal.

Now we see that the ideal  $\mathcal{Z}_g$  fails to satisfy (\*\*) for any  $g \in \mathbb{G}$ . Even if this follows from Proposition 4.6 (as  $\mathcal{Z}_g$  is a  $P$ -ideal by Remark 3.27 and  $\mathcal{Z}_g$  is dense by [24, Proposition 2.2]), we prefer to give a direct proof. In particular,  $\mathcal{I}_d$  turns out to be a dense ideal with  $\neg(**)$  which is non-maximal in view of Proposition 3.14 as both  $2\mathbb{N}$  and  $2\mathbb{N} + 1$  do not belong to  $\mathcal{I}_d$ .

**Proposition 4.4.** *For any  $g \in \mathbb{G}$ ,  $\mathcal{Z}_g$  satisfies  $\neg(**)$ . In particular,  $\mathcal{I}_d$  does not satisfy (\*\*).*

**Proof.** As  $\lim_{n \rightarrow \infty} g(n) = \infty$ , there exists a sequence  $(a_n)$  in  $\mathbb{N}$  such that

$$g(a_n) > n(a_{n-1} + 2n + 1) \quad \text{and} \quad a_{n+1} > a_n + n \quad \text{for every } n \in \mathbb{N}. \tag{4.1}$$

Now  $A = \bigcup_{n \in \mathbb{N}} [a_n, a_n + n]$  has  $c(A) = \infty$  and we prove that  $\lim_{k \rightarrow \infty} \frac{|A(k)|}{g(k)} = 0$ , so  $A \in \mathcal{Z}_g$ . Hence,  $\mathcal{Z}_g$  does not satisfy (\*\*). It is enough to check that for every  $k \in \mathbb{N}$  and the unique  $n$  with  $a_n \leq k < a_{n+1}$  one has

$$\frac{|A(k)|}{g(k)} \leq \frac{1}{n}, \tag{4.2}$$

as  $n \rightarrow \infty$  when  $k \rightarrow \infty$ . First we check (4.2) when  $a_n \leq k \leq a_n + n$ . The monotonicity of  $g$  and (4.1) give

$$\frac{|A(k)|}{g(k)} \leq \frac{|[0, a_{n-1} + n - 1]| + |[a_n, a_n + n]|}{g(k)} = \frac{a_{n-1} + n + n + 1}{g(k)} \leq \frac{a_{n-1} + 2n + 1}{g(a_n)} < \frac{1}{n}. \tag{4.3}$$

Now assume that  $a_n + n < k < a_{n+1}$ . This assumption implies the equality  $A(k) = A(a_n + n)$ . Hence,

$$\frac{|A(k)|}{g(k)} = \frac{|A(a_n + n)|}{g(k)} \leq \frac{|A(a_n + n)|}{g(a_n + n)} \leq \frac{1}{n}, \tag{4.4}$$

where the first inequality is due to the monotonicity of  $g$ , whereas the second one is obtained from (4.3) with  $k^* = a_n + n \in [a_n, a_n + n]$ . Now (4.3) and (4.4) conclude the proof of (4.2) for all  $k \in \mathbb{N}$ .  $\square$

In Example 4.7 we show that a dense ideal need not satisfy  $\neg(**)$ , while in Proposition 4.6 we verify that dense  $P$ -ideals never satisfy (\*\*).

**Lemma 4.5.** Let  $\mathcal{I}$  be a dense ideal of  $\mathbb{N}$  and let  $A = \{a_0 < a_1 < \dots < a_n < \dots\} \in \mathcal{I}$ . Then, for every  $k \in \mathbb{N}$ , there exists an infinite subset  $J_k$  of  $\mathbb{N}$  such that  $A_k := \bigcup_{n \in J_k} [a_n, a_n + k] \in \mathcal{I}$ , so  $c(A_k) = ac(A_k) \geq k + 1$ .

**Proof.** We argue by induction. For  $k = 0$  choose  $J_0 = \mathbb{N}$  and note that  $A_0 = A$ . Assume that the property holds for some  $k \in \mathbb{N}$ . Since  $\mathcal{I}$  is dense, the infinite set  $\{a_n + k + 1 : n \in J_k\}$  contains an infinite subset  $B = \{a_n + k + 1 : n \in J_{k+1}\} \in \mathcal{I}$ . Clearly,  $J_{k+1} \subseteq J_k$  is infinite. Since

$$A_{k+1} := \bigcup_{n \in J_{k+1}} [a_n, a_n + k + 1] \subseteq \bigcup_{n \in J_k} [a_n, a_n + k] \cup B = A_k \cup B \in \mathcal{I},$$

it follows that  $A_{k+1} \in \mathcal{I}$ .  $\square$

**Proposition 4.6.** If a  $P$ -ideal  $\mathcal{I} \not\subseteq Fin$  is either dense or translation invariant, then  $\mathcal{I}$  does not satisfy (\*\*).

**Proof.** In both cases for any  $k \in \mathbb{N}$  there exists  $A_k \in \mathcal{I}$  containing infinitely many intervals of length  $k$ . When  $\mathcal{I}$  is dense this is ensured by Lemma 4.5. In the remaining case pick any infinite subset  $A \in \mathcal{I}$  and note that  $A + n \in \mathcal{I}$  for every  $n \in \mathbb{N}$  as  $\mathcal{I}$  is translation invariant. Now  $A_k := \bigcup_{n=0}^k (A + n) \in \mathcal{I}$  and does the job.

Since  $\mathcal{I}$  is a  $P$ -ideal, there exists  $B \in \mathcal{I}$  such that  $A_k \subseteq^* B$  for any  $k \in \mathbb{N}$ . Then  $B$  contains arbitrary large intervals, i.e.,  $\mathcal{I}$  fails to satisfy (\*\*).  $\square$

In item (a) of the next example we show that the implication dense  $\Rightarrow \neg(**)$  in the above proposition may fail if  $\mathcal{I}$  is not a  $P$ -ideal.

**Example 4.7.** (a) [dense  $\not\Rightarrow \neg(**)$ ] Consider the family  $\mathcal{B}$  of all infinite non-cofinite sets  $B \subseteq \mathbb{N}$  such that  $c(B) < \infty$  and  $g_n(B) \rightarrow \infty$ . As  $\mathcal{B}$  is closed under finite unions and passage to infinite subsets,  $\mathcal{I} = \mathcal{B} \cup Fin$  is an ideal properly containing  $Fin$ . Moreover,  $\mathcal{I}$  is obviously dense and satisfies (\*\*).

Clearly,  $\mathcal{I}$  cannot be a  $P$ -ideal in view of Proposition 4.6, since it is a dense ideal satisfying (\*\*). Nevertheless, one can check this directly. Indeed, for every  $k \in \mathbb{N}$  put  $A_k = \bigcup_{m=k}^\infty [m^3, m^3 + k]$ , then  $c(A_k) = k + 1$  and  $g_n(A_k) = \lfloor (k + n)^3 + k, (k + n + 1)^3 \rfloor \rightarrow \infty$ , so  $A_k \in \mathcal{I}$ . By Corollary 3.2, every pseudounion  $A$  of  $\{A_k : k \in \mathbb{N}\}$  has  $c(A) = \infty$ , hence  $A \notin \mathcal{I}$ .

(b) [ $\neg(**)$   $\not\Rightarrow$  dense] Let  $A \subseteq \mathbb{N}$  such that  $c(A) = \infty$  and  $|A^*| = \infty$  (e.g.,  $A = \bigcup_{n \in \mathbb{N}} [n^2, n^2 + n]$ ). The ideal  $\mathcal{I}_A$  is a free  $P$ -ideal that fails to satisfy (\*\*), but it is not dense.

#### 4.2. Snt vs (\*\*) and (†)

According to [25, Proposition 2.12], an analytic  $P$ -ideal  $\mathcal{I}$  of  $\mathbb{N}$  is snt if and only if (in our terminology) for every  $A^* \in \mathcal{I}^*$  there exists an  $m \in \mathbb{N}_+$  such that  $ag(A^*) \leq m - 1$ . It is easy to see (by choosing  $m$  a bit bigger and using (3.2)) that this is equivalent to (\*\*). Anyway the proof of [25, Proposition 2.12] establishes, for analytic  $P$ -ideals, the equivalence of snt not with (\*\*), but with the following stronger condition for a proper free ideal  $\mathcal{I}$  of  $\mathbb{N}$ :

(†) there exists  $m \in \mathbb{N}_+$  such that  $ac(A) < m$  for every infinite  $A \in \mathcal{I}$

(here, again, (†) is formulated in our terminology). Hereinafter, we aim to explain the precise connection between (\*\*) and (†) when the ideal in question need not be a  $P$ -ideal. In fact, for any ideal  $\mathcal{I}$  of  $\mathbb{N}$ , (†)  $\Rightarrow$  (\*\*), as (†) is a stronger (“uniform”) version of (\*\*), since for every  $A \in \mathcal{I}$ ,  $c(A) < \infty$  if and only if  $ac(A) < \infty$ , by (3.1). The converse implication (\*\*)  $\Rightarrow$  (†) does not hold in general as we see in the next example, which shows moreover that “snt & (\*\*)  $\not\Rightarrow$  (†)  $\not\Rightarrow P$ ”.

**Example 4.8.** (a) [snt & (\*\*)  $\not\Rightarrow$  ( $\dagger$ )] Let  $\mathcal{I}$  be the ideal of  $\mathbb{N}$  generated by  $\{\mathbb{N} \setminus 2^n\mathbb{N} : n \in \mathbb{N}_+\}$ . Since every  $A \in \mathcal{I}$  is contained in  $\mathbb{N} \setminus 2^n\mathbb{N}$  for some  $n \in \mathbb{N}_+$ , we get that  $c(A) \leq 2^n - 1 = c(\mathbb{N} \setminus 2^n\mathbb{N})$ ; hence,  $\mathcal{I}$  satisfies (\*\*) and obviously,  $\mathcal{I}$  is snt. On the other hand, for an arbitrarily fixed  $m \in \mathbb{N}_+$ ,  $c(\mathbb{N} \setminus 2^m\mathbb{N}) = ac(\mathbb{N} \setminus 2^m\mathbb{N}) > m$ , that is,  $\mathcal{I}$  does not satisfy ( $\dagger$ ).

Moreover,  $\mathcal{I} \not\subseteq \mathcal{I}_d$ , actually this fails spectacularly, since  $\mathcal{I}$  has members which have density arbitrarily close to 1. Roughly speaking,  $\mathcal{I}$  is an snt ideal with arbitrarily big (with respect to density) sets, even if  $\mathcal{I}$  satisfies (\*\*), so it has no member containing arbitrarily big intervals.

(b) [( $\dagger$ )  $\not\Rightarrow$   $P$ ] Let  $\mathcal{I}$  be the ideal of  $\mathbb{N}$  defined in Example 3.8, which is not a  $P$ -ideal. On the other hand,  $\mathcal{I}$  satisfies ( $\dagger$ ) since  $\mathcal{I} \subseteq \mathcal{I}_{2\mathbb{N}}$ .

The precise relations among these properties, namely ( $\dagger$ )  $\Rightarrow$  snt  $\Rightarrow$  (\*\*)  $\xrightarrow{P}$  ( $\dagger$ ), are described in the next theorem. Clearly, the ideal *Fin* has all these properties.

**Theorem 4.9.** *The implications ( $\dagger$ )  $\Rightarrow$  snt  $\Rightarrow$  (\*\*) hold for every free ideal  $\mathcal{I}$  of  $\mathbb{N}$ . If  $\mathcal{I}$  is also a  $P$ -ideal, then (\*\*)  $\Rightarrow$  ( $\dagger$ ), i.e., all three properties are equivalent for  $\mathcal{I}$ .*

**Proof.** Let  $\mathcal{I}$  be a free ideal of  $\mathbb{N}$ . To check the implication ( $\dagger$ )  $\Rightarrow$  snt, assume that  $\mathcal{I}$  is not snt, so there exists an infinite  $A' \in \mathcal{I}$  which is  $\mathcal{I}$ -translation invariant. Let  $A = \{a_1 < a_2 < \dots\}$  be a lacunary subset of  $A'$ ; clearly,  $A \in \mathcal{I}$  and  $A$  is  $\mathcal{I}$ -translation invariant. To verify the failure of ( $\dagger$ ), fix an arbitrary  $m \in \mathbb{N}_+$  and let  $B_m = \mathbb{N} \cap \bigcup_{k=0}^m (A - k)$ . As  $A$  is  $\mathcal{I}$ -translation invariant,  $(A - k) \cap \mathbb{N} \in \mathcal{I}$  for every  $k \in \mathbb{Z}$ , so  $B_m \in \mathcal{I}$ . Use the lacunarity of  $A$  and pick an  $i_m \in \mathbb{N}_+$  such that for every  $i \geq i_m$ ,  $a_{i+1} - a_i > m$ . Then for  $i \geq i_m$  the intervals  $[a_i - m, a_i]$  are pairwise disjoint and witness  $ac(B_m) > m$ .

To check the implication snt  $\Rightarrow$  (\*\*), assume that  $\mathcal{I}$  is a free ideal satisfying  $\neg(**)$ . Then there exists a strictly increasing sequence  $(a_n)$  in  $\mathbb{N}$  such that  $A = \bigcup_{n \in \mathbb{N}} (a_n, a_n + n] \in \mathcal{I}$ , with  $a_{n+1} > a_n + n$  for every  $n \in \mathbb{N}$ . Let  $B = \{a_n + \lfloor n/2 \rfloor : n \in \mathbb{N}\}$ . Then  $B \subseteq A$ , and so  $B \in \mathcal{I}$ . For  $k \in \mathbb{Z}$ ,  $(B + k) \cap \mathbb{N} \subseteq^* A$ , hence  $B + k \in \mathcal{I}$  as  $\mathcal{I}$  is free. This proves that  $B$  is  $\mathcal{I}$ -translation invariant, therefore  $\mathcal{I}$  cannot be snt.

To prove the missing implication assume that  $\mathcal{I}$  is a free  $P$ -ideal with  $\neg(\dagger)$ , that is, for every  $m \in \mathbb{N}_+$  there exists an infinite  $A_m \in \mathcal{I}$  with  $ac(A_m) > m$ . Since  $\mathcal{I}$  is a  $P$ -ideal, there exists  $A \in \mathcal{I}$  such that  $A_m \subseteq^* A$  for every  $m \in \mathbb{N}_+$ . Hence,  $ac(A_m) > m$  for every  $m \in \mathbb{N}_+$ , implies that  $ac(A) = \infty$ , by Corollary 3.2(2). Therefore,  $\mathcal{I}$  does not satisfy (\*\*).  $\square$

**Corollary 4.10.** [snt  $\Rightarrow$  non-maximal] *Every maximal free ideal of  $\mathbb{N}$  is non-snt.*

**Proof.** Every maximal free ideal of  $\mathbb{N}$  satisfies  $\neg(**)$ , by Lemma 4.3. It remains to add that  $\neg(**)$  implies non-snt by Theorem 4.9.  $\square$

In Example 3.9 we pointed out the gap between the properties snt and non-t.i. In the next example we compare non-t.i. with the weaker property (\*\*). Both non-implications non-t.i.  $\not\Rightarrow$  (\*\*) and (\*\*)  $\not\Rightarrow$  non-t.i. should be compared with the obvious implication snt  $\Rightarrow$  non-t.i., witnessing once again the gap between snt and (\*\*) (see Theorem 4.9).

**Example 4.11.** (a) [(\*\*)  $\not\Rightarrow$  non-t.i.] and [t.i.  $\not\Rightarrow$   $\neg(**)$ ] To simultaneously witness these two non-implications we need to display an ideal that is both t.i. and (\*\*).

Let  $\emptyset \neq A \subseteq \mathbb{N}$  and  $A_k = \mathbb{N} \cap (\bigcup_{i=-k}^k (A + i))$  for  $k \in \mathbb{N}$ . Then

$$\mathcal{I}(A) := \{I \subseteq \mathbb{N} : \exists k \in \mathbb{N}, I \subseteq A_k\}$$

is a t.i. free ideal of  $\mathbb{N}$ . Indeed,  $\mathcal{I}(A)$  is free since if  $a \in A$ , then for every sufficiently large  $k$  (say,  $k \geq a$ ),  $A_k$  contains the initial segment  $[0, a + k]$  of  $\mathbb{N}$ . This implies also that  $\mathcal{I}(A)$  is t.i., since  $(A + k) \cap \mathbb{N} \in \mathcal{I}(A)$  for every  $k \in \mathbb{Z}$ .

Since  $\mathcal{I}(A) = \mathcal{F}in$  is snt when  $A$  is finite, we assume from now on that  $A$  is *infinite*, so we can write  $A = \{a_n : n \in \mathbb{N}\}$ , where  $(a_n)$  is an increasing sequence, and assume in addition that  $(a_{n+1} - a_n)$  is strictly increasing. Since  $\mathcal{I}(A)$  is t.i., this means that  $\mathcal{I}(A)$  is non-snt.

To check that  $\mathcal{I}(A)$  satisfies (\*\*), for every  $k \in \mathbb{N}$  pick an  $n_k \in \mathbb{N}$  such that  $a_{n+1} - a_n > 2k$  for every  $n \geq n_k$ . This entails  $(a_i + [-k, k]) \cap (a_j + [-k, k]) = \emptyset$  for distinct  $i, j > n_k$ . Therefore,  $c(A_k) \leq a_{n_k+1}$ . Since the family of sets  $A_k$  generates  $\mathcal{I}(A)$ , this implies  $c(I) < \infty$  for all  $I \in \mathcal{I}(A)$ , i.e.,  $\mathcal{I}(A)$  is a t.i. ideal satisfying (\*\*). In particular,  $\mathcal{I}(A)$  is non-maximal by Lemma 4.3.

(b) [non-t.i.  $\not\Rightarrow$  (\*\*)] Let  $\mathcal{M}$  be a maximal ideal. Then  $\mathcal{M}$  is non-t.i. by Corollary 3.15 and fails to satisfy (\*\*) by Lemma 4.3.

4.3. The condition (\*)

Here we recall a last condition for a proper free ideal  $\mathcal{I}$  of  $\mathbb{N}$ , stronger than snt, (\*\*) and ( $\dagger$ ), that was introduced in [25, Proposition 2.13]. We prefer to reformulate it equivalently in our terms:

(\*) *there is an infinite set  $A \subseteq \mathbb{N}$  such that  $c(A) < \infty$  and  $\mathcal{I} \subseteq \mathcal{I}_A$ .*

Clearly,  $\mathcal{F}in$  satisfies (\*), while (\*) implies non-dense for any ideal  $\mathcal{I}$  of  $\mathbb{N}$ . On the other hand, (\*)  $\not\Rightarrow P$ , by Example 4.8 (since the ideal  $\mathcal{I}$  produced there satisfies  $\mathcal{I} \subseteq \mathcal{I}_{2\mathbb{N}}$ ).

**Lemma 4.12.** [(\*)  $\Rightarrow$  ( $\dagger$ )] *For any ideal  $\mathcal{I}$  of  $\mathbb{N}$ , (\*) implies ( $\dagger$ ).*

**Proof.** By (\*), there exists  $A \in \mathcal{P}(\mathbb{N})$  with  $c(A) = m \in \mathbb{N}_+$  such that  $\mathcal{I} \subseteq \mathcal{I}_A$ . Then, for any  $C \in \mathcal{I}$ , as  $C \subseteq^* A$ ,  $ac(C) \leq ac(A) \leq m$  by Lemma 3.1(2). This proves that  $\mathcal{I}$  satisfies ( $\dagger$ ).  $\square$

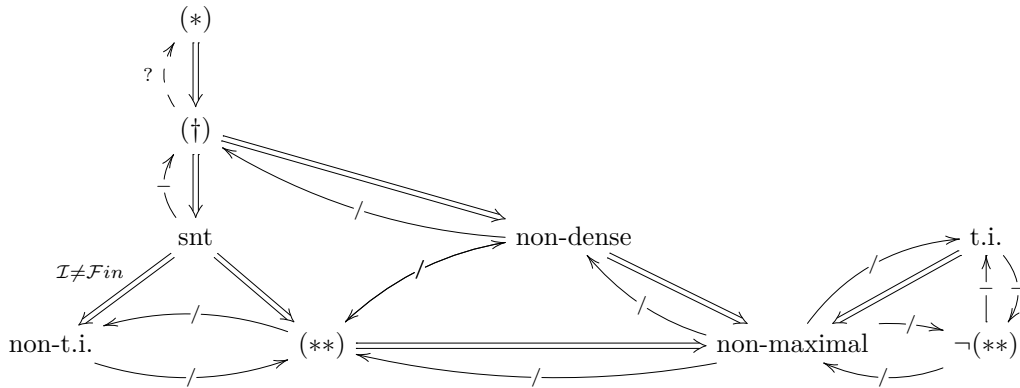
The implication ( $\dagger$ )  $\Rightarrow$  (\*) is claimed in [25, Proposition 2.13], but the argument used in its proof contains flaws. As the proof of [25, Theorem 4.6] makes essential use of [25, Proposition 2.13], that argument works only under the stronger assumption (\*) (see Remark 6.6 below).

**Lemma 4.13.** [( $\dagger$ )  $\Rightarrow$  non-dense] *If an ideal  $\mathcal{I}$  of  $\mathbb{N}$  satisfies ( $\dagger$ ), then  $\mathcal{I}$  is not dense.*

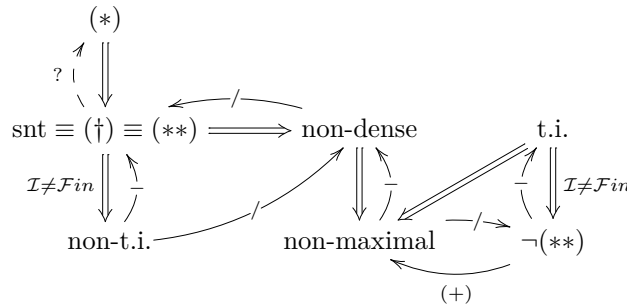
**Proof.** Assume that  $\mathcal{I}$  is dense. Then for every  $k \in \mathbb{N}_+$  there exists  $A_k \in \mathcal{I}$  with  $ac(A_k) > k$ , by Lemma 4.5. This means that  $\mathcal{I}$  fails to satisfy ( $\dagger$ ).  $\square$

**Remark 4.14.** [non-dense  $\stackrel{P}{\not\Rightarrow}$  ( $\dagger$ )] The implication in the above lemma cannot be inverted. Actually, that an ideal  $\mathcal{I}$  of  $\mathbb{N}$  is not dense does not imply ( $\dagger$ ) even when  $\mathcal{I}$  is a  $P$ -ideal. In fact, Example 4.7 produces a  $P$ -ideal  $\mathcal{I}$  that is neither dense nor satisfies (\*\*). So it remains to recall that the latter property is equivalent to ( $\dagger$ ) for  $P$ -ideals.

In the following two diagrams we collect the implications and the non-implications that we have proved for free ideals (the non-implications are supported by counterexamples that are either given above or are immediate, e.g.,  $\mathcal{I}_{2\mathbb{N}}$  is non-maximal and non-translation invariant, etc.). The dashed arrows with the symbol “?” mean that we do not know whether the implication is available (see Question 7.1). We omit the obsolete arrows representing non-implications (i.e., those that trivially follow from this diagram).



The next diagram describes the situation under the blanket assumption that the ideal is a  $P$ -ideal.



The arrow  $(+)$  depends on additional set-theoretic assumption. More precisely, this implication does not hold when there exists a maximal ideal that is also a  $P$ -ideal (as every maximal ideal satisfies  $\neg(**)$ ), i.e., it is equivalent to Rudin’s problem on the existence of a maximal  $P$ -ideal (see Remark 3.16). Hence, under CH,  $(+)$  is a non-implication, whereas the implication  $\neg(**) \Rightarrow \text{non-maximal}$  vacuously holds for  $P$ -ideals in Shelah’s model where all  $P$ -ideals are non-maximal.

5. Ideal convergence and  $\mathcal{I}$ -characterized subgroups of  $\mathbb{T}$

5.1. Ideal convergence

We start recalling the following notion due to Cartan.

**Definition 5.1.** (Cartan [18]) Let  $\mathcal{I}$  be an ideal of  $\mathbb{N}$ . In a topological space  $X$ , a sequence  $(x_n)$  is:

- (1)  $\mathcal{I}$ -convergent to  $x \in X$  if for every neighborhood  $U$  of  $x$  in  $X$ ,  $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$ , and we briefly write  $x_n \xrightarrow{\mathcal{I}} x$  or  $\mathcal{I}\text{-}\lim_{n \rightarrow \infty} x_n = x$ ;
- (2)  $\mathcal{I}^*$ -convergent to  $x \in X$  if there exists  $M \in \mathcal{I}^*$  such that  $x_n \xrightarrow[n \in M]{} x$ , and we briefly write  $x_n \xrightarrow{\mathcal{I}^*} x$  or  $\mathcal{I}^*\text{-}\lim_{n \rightarrow \infty} x_n = x$ .

Both  $\text{Fin}$ -convergence and  $\text{Fin}^*$ -convergence coincide with the usual convergence, while  $\mathcal{I}_d$ -convergence obviously coincides with statistical convergence [47, Example 3.1(b)].

Clearly, if  $\mathcal{I} \subseteq \mathcal{J}$  are ideals of  $\mathbb{N}$ , then  $x_n \xrightarrow{\mathcal{I}} x$  implies  $x_n \xrightarrow{\mathcal{J}} x$ .

Since  $\mathcal{I}^*$ -convergence is much easier to understand and deal with, the next lemma demonstrates the utility of  $P$ -ideals:

**Lemma 5.2.** [47, Proposition 3.2 and Theorem 3.2(ii)] *Let  $\mathcal{I}$  be a free  $\mathcal{P}$ -ideal of  $\mathbb{N}$ ,  $X$  a first countable topological space,  $x \in X$  and  $(x_n)$  a sequence in  $X$ . Then  $x_n \xrightarrow{\mathcal{I}} x$  if and only if  $x_n \xrightarrow{\mathcal{I}^*} x$ .*

The next remark motivates the fact that we consider most often free proper ideals in the framework of ideal convergence.

**Remark 5.3.** Let  $X$  be a  $T_1$  topological space,  $\mathbf{x} = (x_n)$  a sequence in  $X$  and  $x \in X$ . Let  $A$  be an infinite subset of  $\mathbb{N}$  and consider  $\mathcal{I}$ -convergence for  $\mathcal{I} = \mathcal{P}(A)$ . Then

$$\mathcal{I}\text{-}\lim_{n \rightarrow \infty} x_n = x \text{ if and only if } x_n = x \text{ for } n \in \mathbb{N} \setminus A.$$

Indeed, assume that  $\mathcal{I}\text{-}\lim_{n \rightarrow \infty} x_n = x$ . If  $x_n \neq x$  for some  $n \in \mathbb{N}$ , pick a neighborhood  $U$  of  $x$  such that  $x_n \notin U$ . Then

$$n \in \mathbf{x}(U) := \{k \in \mathbb{N} : x_k \notin U\},$$

so  $\mathcal{I}\text{-}\lim_{n \rightarrow \infty} x_n = x$  yields  $\mathbf{x}(U) \in \mathcal{I}$ , i.e.,  $n \in \mathbf{x}(U) \subseteq A$ . Therefore,  $n \in \mathbb{N} \setminus A$  implies  $x_n = x$ , as required.

In particular:

- (1)  $\mathcal{I}\text{-}\lim_{n \rightarrow \infty} x_n = x$  is always true for  $\mathcal{I} = \mathcal{P}(\mathbb{N})$ , and this justifies the use of proper ideals;
- (2) if  $A = \mathbb{N} \setminus \{n_1, \dots, n_k\}$  is a cofinite subset of  $\mathbb{N}$ , then for  $\mathcal{I} = \mathcal{P}(A)$ ,  $\mathcal{I}\text{-}\lim_{n \rightarrow \infty} x_n = x$  if and only if  $x_{n_i} = x$  for  $i = 1, \dots, k$ .

The phenomenon pointed out in (2) is quite unusual (counterintuitive), since one expects that the convergence of a sequence should not depend on a finite part of the sequence. To avoid this problem and to ensure the quite natural requirement that the usual convergence implies  $\mathcal{I}$ -convergence, we impose the mild restraint of freeness on  $\mathcal{I}$  (i.e.,  $\mathcal{F}in \subseteq \mathcal{I}$ ).

We add here two more items to [47, Proposition 3.1].

**Proposition 5.4.** *Let  $\mathcal{I}$  be a free ideal, let  $X$  be a Hausdorff topological group with at least two points and let  $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$  and  $\mathbf{y} = (y_n)_{n \in \mathbb{N}}$  be sequences in  $X$ .*

- (1) *Every constant sequence  $(x)_{n \in \mathbb{N}}$  in  $X$   $\mathcal{I}$ -converges to  $x \in X$ .*
- (2) *The limit of any  $\mathcal{I}$ -convergent sequence is unique.*
- (3) *If each subsequence of  $\mathbf{x}$  has a subsequence that  $\mathcal{I}$ -converges to  $x \in X$ , then  $\mathbf{x}$   $\mathcal{I}$ -converges to  $x$ .*
- (4) *In case  $\mathcal{I}$  contains an infinite set, then the following property does not hold: if  $\mathbf{x}$   $\mathcal{I}$ -converges to  $x \in X$ , then every subsequence of  $\mathbf{x}$   $\mathcal{I}$ -converges to  $x$ .*
- (5) *If  $\mathbf{x}$   $\mathcal{I}$ -converges to  $x \in X$  and  $\mathbf{y}$   $\mathcal{I}$ -converges to  $y \in X$ , then  $(x_n y_n)_{n \in \mathbb{N}}$   $\mathcal{I}$ -converges to  $xy \in X$  and  $(x_n^{-1})_{n \in \mathbb{N}}$   $\mathcal{I}$ -converges to  $x^{-1} \in X$ .*
- (6) *If  $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$   $\mathcal{I}$ -converges to  $x \in X$  and  $\mathcal{I}$  is translation invariant, then  $(x_{n+1})_{n \in \mathbb{N}}$   $\mathcal{I}$ -converges to  $x \in X$ .*

**Proof.** Items (1)–(4) come from [47, Proposition 3.1].

(5) First we note that a sequence  $(z_n)$  in  $X$   $\mathcal{I}$ -converges to  $z \in X$  if and only if  $(z_n z^{-1})$   $\mathcal{I}$ -converges to  $1 \in X$ , if and only if  $(z z_n^{-1})$   $\mathcal{I}$ -converges to  $1 \in X$ . Hence, it suffices to check that  $(x_n y_n^{-1})$   $\mathcal{I}$ -converges to 1 whenever both  $\mathbf{x}$  and  $\mathbf{y}$   $\mathcal{I}$ -converge to 1.

Let  $U$  be a neighborhood of 1 in  $X$  and  $V$  be a neighborhood of 1 in  $X$  such that  $VV^{-1} \subseteq U$ . As  $\mathbf{x}$   $\mathcal{I}$ -converges to 1 in  $X$  and  $\mathbf{y}$   $\mathcal{I}$ -converges to 1 in  $X$ , one has  $\mathbf{x}(V) \in \mathcal{I}$  and  $\mathbf{y}(V) \in \mathcal{I}$ . Since

$$\{n \in \mathbb{N} : x_n y_n^{-1} \notin U\} \subseteq \mathbf{x}(V) \cup \mathbf{y}(V) \in \mathcal{I},$$

we proved in this way that  $(x_n y_n^{-1})$   $\mathcal{I}$ -converges to 1 in  $X$ .  $\square$

5.2.  $\mathcal{I}$ -characterized subgroups of  $\mathbb{T}$

**Definition 5.5.** [25] Let  $\mathcal{I}$  be an ideal of  $\mathbb{N}$ . For a sequence of integers  $\mathbf{u}$ , let

$$t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) := \{x \in \mathbb{T} : u_n x \xrightarrow{\mathcal{I}} 0\}.$$

A subgroup  $H$  of  $\mathbb{T}$  is  $\mathcal{I}$ -characterized if  $H = t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  for some sequence  $\mathbf{u}$  in  $\mathbb{Z}$ .

If  $\mathcal{X}$  is one of the classes of sequences introduced above (i.e.,  $\mathcal{S}$ ,  $\mathcal{A}$ , etc.), we say that a subgroup  $H$  of  $\mathbb{T}$  is  $\mathcal{X}$ - $\mathcal{I}$ -characterized if there exists  $\mathbf{u} \in \mathcal{X}$  such that  $H = t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$ .

We denote by  $\mathcal{C}\mathfrak{h}\mathfrak{a}\mathfrak{r}^{\mathcal{I}}$  (resp.,  $\mathcal{X}\text{-}\mathcal{C}\mathfrak{h}\mathfrak{a}\mathfrak{r}^{\mathcal{I}}$ ) the family of all  $\mathcal{I}$ -characterized (resp.,  $\mathcal{X}$ - $\mathcal{I}$ -characterized) subgroups of  $\mathbb{T}$ .

Clearly, for any ideal  $\mathcal{I}$  of  $\mathbb{N}$  and any sequence  $\mathbf{u}$  of integers,  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  is a subgroup of  $\mathbb{T}$ . In the study of  $\mathcal{I}$ -characterized subgroups  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  one can assume without loss of generality that  $\mathbf{u}$  is in  $\mathbb{N}$  by the definition. Moreover, if  $\mathcal{I}$  is free and  $\mathbf{u}$  is eventually 0, then  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) = \mathbb{T}$ . And clearly, if  $\mathcal{I} \subseteq \mathcal{J}$  are ideals of  $\mathbb{N}$ , then  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) \subseteq t_{\mathbf{u}}^{\mathcal{J}}(\mathbb{T})$ .

In the sequel the ideals are often assumed to be proper and free, and the next example shows why these two conditions are important.

**Example 5.6.** If  $\mathcal{I}$  is a free ideal of  $\mathbb{N}$ , then  $t_{\mathbf{u}}(\mathbb{T}) \subseteq t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  for every sequence  $\mathbf{u}$  in  $\mathbb{N}$ .

(a) For  $\mathcal{I} = \mathcal{P}(\mathbb{N} \setminus \{0\})$  and  $\mathbf{u} \in \mathcal{A}$  with  $u_0 = 1$  and  $u_n = n!$  for every  $n \in \mathbb{N}_+$ , we have  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) = \{0\}$ , by Remark 5.3(2). On the other hand,  $t_{\mathbf{u}}(\mathbb{T})$  is uncountable by Theorem 1.8(1), thus the inclusion  $t_{\mathbf{u}}(\mathbb{T}) \subseteq t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  strongly fails.

(b) For  $\mathcal{I} = \mathcal{P}(\mathbb{N})$  we have  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) = \mathbb{T}$  for every sequence  $\mathbf{u}$  in  $\mathbb{N}$ , by Remark 5.3(1). In particular for  $\mathbf{u} \in \mathcal{S}$  with  $u_n = n$  for every  $n \in \mathbb{N}$ , we have  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) = \mathbb{T}$  while  $t_{\mathbf{u}}(\mathbb{T}) = \{0\}$ .

Example 5.6(b) shows why the ideal  $\mathcal{P}(\mathbb{N})$  is of little interest. Anyway, the results in this section (with exception of Theorems 5.16 and 5.17) hold also for  $\mathcal{I} = \mathcal{P}(\mathbb{N})$ . This is why we do not assume explicitly the ideal to be proper in the statements. On the other hand, in the whole §6 the ideal  $\mathcal{P}(\mathbb{N})$  is automatically ruled out by the conditions (\*\*), (†) and (\*).

The following result is needed in the sequel.

**Lemma 5.7.** Let  $A \subseteq \mathbb{N}$  such that  $\mathbb{N} \setminus A$  is infinite and let  $\mathbf{u} \in \mathcal{S}$ . Let  $(k_n)$  be a strictly increasing sequence in  $\mathbb{N}$  with  $\{k_n : n \in \mathbb{N}\} = \mathbb{N} \setminus A$ , and let  $v_n = u_{k_n}$  for every  $n \in \mathbb{N}$ . Then for every free ideal  $\mathcal{I}$  of  $\mathbb{N}$  containing  $A$ , one has  $t_{\mathbf{u}}(\mathbb{T}) \subseteq t_{\mathbf{v}}(\mathbb{T}) = t_{\mathbf{u}}^{\mathcal{I}^A}(\mathbb{T}) \subseteq t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$ .

**Proof.** Both inclusions in the above chain are obvious, so it only remains to prove the equality  $t_{\mathbf{v}}(\mathbb{T}) = t_{\mathbf{u}}^{\mathcal{I}^A}(\mathbb{T})$ . Assume that  $x \in t_{\mathbf{v}}(\mathbb{T})$ , so  $u_{k_n} x \rightarrow 0$ . Pick  $\varepsilon > 0$ , then  $F := \{k_n : \|u_{k_n} x\| \geq \varepsilon\} \in \mathcal{F}in \subseteq \mathcal{I}$ . Therefore,

$$\{n \in \mathbb{N} : \|u_n x\| \geq \varepsilon\} \subseteq F \cup A \in \mathcal{I}_A,$$

i.e.,  $u_n x \xrightarrow{\mathcal{I}_A} 0$ , that is,  $x \in t_{\mathbf{u}}^{\mathcal{I}_A}(\mathbb{T})$ . Vice versa, if  $x \in t_{\mathbf{u}}^{\mathcal{I}_A}(\mathbb{T})$ , then  $u_n x \xrightarrow{\mathcal{I}_A} 0$ . Then for every  $\varepsilon > 0$  one has  $B := \{n \in \mathbb{N} : \|u_n x\| \geq \varepsilon\} \subseteq^* A$ . Therefore,  $\{k_n : \|u_{k_n} x\| \geq \varepsilon\} = B \setminus A$  is finite, i.e.,  $u_{k_n} x \rightarrow 0$ . Hence,  $x \in t_{\mathbf{v}}(\mathbb{T})$ .  $\square$

Here and in the following section we recall the main known results on  $\mathcal{I}$ -characterized subgroups of  $\mathbb{T}$ . The conditions (\*), (\*\*) and (†) described in detail in the previous sections are heavily used here. By Theorem 4.9, free  $P$ -ideals are snt precisely when they satisfy (\*\*), therefore hereinafter we use only (\*\*) when we deal with snt  $P$ -ideals.

First, we recall that Theorem A holds in the present general setting under suitable hypotheses on  $\mathcal{I}$ , and so also Corollary A; we formulate them for the sake of completeness:

**Theorem 5.8.** [25, Theorem 2.3] *Let  $\mathcal{I}$  be a free analytic  $P$ -ideal of  $\mathbb{N}$  and  $\mathbf{u} \in \mathcal{S}$ . Then  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  is an  $F_{\sigma\delta}$ -subgroup of  $\mathbb{T}$  containing  $t_{\mathbf{u}}(\mathbb{T})$ .*

**Corollary 5.9.** [45] *Let  $\mathcal{I}$  be a free analytic  $P$ -ideal of  $\mathbb{N}$  and  $\mathbf{u} \in \mathcal{S}$ . Then  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  is Polishable.*

So, Theorem E holds also in this setting, that is, it gives  $F_{\sigma}$ -subgroups of  $\mathbb{T}$  non- $\mathcal{I}$ -characterized for any free analytic  $P$ -ideal  $\mathcal{I}$  of  $\mathbb{N}$ .

We recall the following fundamental claim from [25].

**Lemma 5.10.** [25, Lemma 3.2] *Let  $\mathcal{I}$  be a free analytic  $P$ -ideal of  $\mathbb{N}$  and  $\mathbf{u} \in \mathcal{A}$ . If  $x \in [0, 1)$  has  $\text{supp}(x) \in \mathcal{I}$  and  $\text{supp}(x)$  is  $\mathcal{I}$ -translation invariant, then  $\varphi(x) \in t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$ .*

As direct consequence one obtains:

**Corollary 5.11.** [44, Lemma 2.3, Corollary 2.4] *Let  $\mathcal{I}$  be a free translation invariant analytic  $P$ -ideal of  $\mathbb{N}$  and  $\mathbf{u} \in \mathcal{A}$ . If  $x \in [0, 1)$  has  $\text{supp}(x) \in \mathcal{I}$ , then  $\varphi(x) \in t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$ . Hence,  $|t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})| = \mathfrak{c}$  whenever  $\mathcal{I} \neq \mathcal{F}in$ .*

The next theorem generalize Theorems 1.7 and [23, Theorem 2.1]. For two subsets  $A, B$  of  $\mathbb{N}$  and an ideal  $\mathcal{I}$  of  $\mathbb{N}$ ,  $A \subseteq^{\mathcal{I}} B$  means that  $A \setminus B \in \mathcal{I}$ , and  $A \subseteq_{\mathcal{I}} B$  that  $A \subseteq B$  and  $B \setminus A \in \mathcal{I}$ . The proof given in [44] has some gaps, eliminated recently in [33], where the hypothesis “analytic  $P$ -ideal” is relaxed and the conditions (a) and (b) are simplified.

**Theorem 5.12.** [44, Theorem 2.9] *Let  $x \in [0, 1)$ , let  $\mathcal{I}$  be a free translation invariant analytic  $P$ -ideal of  $\mathbb{N}$  and  $\mathbf{u} \in \mathcal{A}$ . Then  $\varphi(x) \in t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  if and only if either  $\text{supp}(x) \in \mathcal{I}$  or if  $\text{supp}(x) \notin \mathcal{I}$ , then for all  $A \subseteq \mathbb{N}$  with  $A \notin \mathcal{I}$  the following conditions hold:*

- (a) if  $A$  is  $b$ -bounded then:
  - (a1) if  $A \subseteq^{\mathcal{I}} \text{supp}(x)$ , then  $A + 1 \subseteq^{\mathcal{I}} \text{supp}(x)$ ,  $A \subseteq^{\mathcal{I}} \text{supp}_b(x)$  and there exists  $A' \subseteq_{\mathcal{I}} A$  such that  $\lim_{n \in A'} \frac{c_{n+1+1}}{b_{n+1}} = 1$ . Moreover, if  $A + 1$  is  $b$ -bounded, then  $A + 1 \subseteq^{\mathcal{I}} \text{supp}_b(x)$ .
  - (a2) If  $A \cap \text{supp}(x) \in \mathcal{I}$ , then there exists  $A' \subseteq_{\mathcal{I}} A$  such that  $\lim_{n \in A'} \frac{c_{n+1}}{b_{n+1}} = 0$ . Moreover, if  $A + 1$  is  $b$ -bounded, then  $(A + 1) \cap \text{supp}(x) \in \mathcal{I}$  as well.
- (b) if  $A$  is  $b$ -divergent, then  $\lim_{n \in B} \varphi\left(\frac{c_n}{b_n}\right) = 0$  for some  $B \subseteq_{\mathcal{I}} A$ .

The following general result provides a more precise version of [44, Corollary 2.15] (see Corollary 5.14 below). The proof is similar to that of [44, Corollary 2.15], but we give it here for the sake of completeness.

**Theorem 5.13.** *For any two free translation invariant  $P$ -ideals  $\mathcal{I}_1, \mathcal{I}_2$  of  $\mathbb{N}$  and  $\mathbf{u} \in \mathcal{A}$ , if  $\mathcal{I}_2 \not\subseteq \mathcal{I}_1$  then  $t_{\mathbf{u}}^{\mathcal{I}_2}(\mathbb{T}) \not\subseteq t_{\mathbf{u}}^{\mathcal{I}_1}(\mathbb{T})$ .*

**Proof.** Let  $B \in \mathcal{I}_2 \setminus \mathcal{I}_1$ . As  $B \notin \mathcal{I}_1$ , by [44, Corollary 2.14] there exists  $x \in [0, 1)$  with  $\text{supp}(x) \subseteq B$  such that  $\varphi(x) \notin t_{\mathbf{u}}^{\mathcal{I}_1}(\mathbb{T})$ . But  $\mathcal{I}_2$  is an ideal, hence  $\text{supp}(x) \in \mathcal{I}_2$ . Therefore, by Lemma 5.10,  $\varphi(x) \in t_{\mathbf{u}}^{\mathcal{I}_2}(\mathbb{T})$  and then  $t_{\mathbf{u}}^{\mathcal{I}_2}(\mathbb{T}) \not\subseteq t_{\mathbf{u}}^{\mathcal{I}_1}(\mathbb{T})$ .  $\square$

The next result gives the answer to [14, Problem 2.14] in terms of ideals (see (2.2) of Corollary 2.14).

**Corollary 5.14.** [44, Corollary 2.15] *For any two free translation invariant analytic  $P$ -ideals  $\mathcal{I}_1, \mathcal{I}_2$  of  $\mathbb{N}$  and  $\mathbf{u} \in \mathcal{A}$ , if  $\mathcal{I}_1 \subsetneq \mathcal{I}_2$  then  $t_{\mathbf{u}}^{\mathcal{I}_1}(\mathbb{T}) \subsetneq t_{\mathbf{u}}^{\mathcal{I}_2}(\mathbb{T})$ .*

Applying this theorem with  $\mathcal{I}_1 \neq \mathcal{F}in$  one obtains also the following immediate corollary:

**Corollary 5.15.** *For every  $\mathbf{u} \in \mathcal{A}$ , one has  $|t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) \setminus t_{\mathbf{u}}(\mathbb{T})| = \mathfrak{c}$  for any free translation invariant analytic  $P$ -ideal  $\mathcal{I} \neq \mathcal{F}in$  of  $\mathbb{N}$ .*

**Proof.** Since  $\mathcal{F}in$  is a free translation invariant analytic  $P$ -ideal of  $\mathbb{N}$ , from Corollary 5.14 one can deduce that  $t_{\mathbf{u}}(\mathbb{T}) \subsetneq t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$ . Since  $|t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})| = \mathfrak{c}$  by Corollary 5.11, we deduce, with Claim 2.1, that  $|t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) \setminus t_{\mathbf{u}}(\mathbb{T})| = \mathfrak{c}$ .  $\square$

By Theorem 4.9, free translation invariant analytic  $P$ -ideals satisfy  $\neg(**)$ , hence the above corollary follows also from Theorem 5.23; the above proof has the advantage to be much simpler.

In the next two subsections, we discuss, in the line of Theorems B and C above, when  $|t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})| = \mathfrak{c}$  and when  $t_{\mathbf{u}}(\mathbb{T}) \subsetneq t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$ . Moreover, in Section 6 we consider the case when  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  is countable. Before that, it seems natural to analyze when  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  is non-trivial, that is, a proper infinite subgroup of  $\mathbb{T}$ . The following two results from [32] go in this direction.

Since, fixed a sequence  $\mathbf{u}$  of integers, the correspondence  $\mathcal{I} \mapsto t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  is monotone increasing, one may intuitively expect that  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) = \mathbb{T}$  may (often) occur when the ideal  $\mathcal{I}$  is maximal. The next result, extending Theorem D, shows that this is not the case under the mild condition that  $\mathbf{u}$  is a Hadamard sequence.

**Theorem 5.16.** [32] *Let  $\mathcal{I}$  be a proper ideal of  $\mathbb{N}$  and  $\mathbf{u} \in \mathcal{H}$ . Then  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) \neq \mathbb{T}$ .*

The following theorem goes in the opposite direction.

**Theorem 5.17.** [32] *Let  $\mathcal{I}$  be a translation invariant proper ideal of  $\mathbb{N}$  and let  $\mathbf{u}$  be a sequence of integers such that  $u_n = P(n)$  for every  $n \in \mathbb{N}$ , where  $P(x) \in \mathbb{Z}[x]$ . Then  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  is finite.*

For  $\mathbf{u} \in \mathcal{S}$  the set  $W_{\mathbf{u}} := \{x \in \mathbb{T} : (u_n x) \text{ is uniformly distributed in } \mathbb{T}\}$  was introduced and studied by Weyl [59], who proved that  $W_{\mathbf{u}} = \mathbb{T} \setminus \mathbb{Q}/\mathbb{Z}$  in case  $\mathbf{u}$  is as in the above theorem. The next corollary of Theorem 5.17 should be compared to [22, Theorem 6.2] where under the weaker hypothesis  $W_{\mathbf{u}} = \mathbb{T} \setminus \mathbb{Q}/\mathbb{Z}$  on  $\mathbf{u}$  the much weaker conclusion  $t_{\mathbf{u}}^s(\mathbb{T}) \subseteq \mathbb{Q}/\mathbb{Z}$  is obtained.

**Corollary 5.18.** *Let  $\mathbf{u}$  be a sequence such that  $u_n = P(n)$  for every  $n \in \mathbb{N}$ , where  $P(x) \in \mathbb{Z}[x]$ . Then  $t_{\mathbf{u}}^s(\mathbb{T})$  is finite.*

5.3. Theorem B: when  $|t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})| = \mathfrak{c}$

As translation invariant implies  $\neg(**)$ , item (1) of the next theorem covers [44, Corollary 2.4] and for  $\mathcal{I} = \mathcal{I}_d$  is exactly Theorem B. Furthermore, one can apply Corollary 5.15 and add to the conclusion also  $|t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) \setminus t_{\mathbf{u}}(\mathbb{T})| = \mathfrak{c}$  (see Theorem 5.22(1)). Item (2) is a generalization of Theorem B<sup>R</sup>.

**Theorem 5.19.** *Let  $\mathcal{I}$  be a free analytic  $P$ -ideal  $\mathcal{I}$  of  $\mathbb{N}$  satisfying  $\neg(**)$ .*

- (1) [25, Theorem 3.5] If  $\mathbf{u} \in \mathcal{A}$ ,  $|t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})| = \mathfrak{c}$ .  
 (2) [25, Theorem 4.3] If  $\mathbf{q} \in \mathcal{R}$ , then  $|t_{\mathbf{q}}^{\mathcal{I}}(\mathbb{T})| = \mathfrak{c}$ .

In Theorem 5.19(1), which follows from Lemma 5.10, one can relax the hypothesis that the ideal is an analytic  $P$ -ideal:

**Theorem 5.20.** [32] *Let  $\mathcal{I}$  be a free ideal of  $\mathbb{N}$  with  $\neg(**)$  and let  $\mathbf{u} \in \mathcal{A}$ . Then  $|t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})| = \mathfrak{c}$ . In particular,  $|t_{\mathbf{u}}^{\mathcal{M}}(\mathbb{T})| = \mathfrak{c}$  for every proper maximal ideal  $\mathcal{M}$  of  $\mathbb{N}$ .*

The proof of this theorem is short and simple. To check the first assertion one exploits the property  $\neg(**)$  to produce an  $A \in \mathcal{I}$  with  $c(A) = \infty$ . Then a subsequence  $\mathbf{w}$  of  $\mathbf{u}$  can be chosen with support set  $\mathbb{N} \setminus A$ . As a result  $\mathbf{w} \in \mathcal{A} \setminus \mathcal{A}_{bd}$ , and so  $|t_{\mathbf{w}}(\mathbb{T})| = \mathfrak{c}$  by Theorem 1.10. As  $t_{\mathbf{w}}(\mathbb{T}) \subseteq t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$ , by Lemma 5.7, one immediately gets  $|t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})| = \mathfrak{c}$ . The second assertion of the theorem follows from the first one and the fact that every proper maximal ideal satisfies  $\neg(**)$  by Lemma 4.3.

**Remark 5.21.** As the hypothesis of Theorem 5.20 is much weaker than in Theorem 5.19(1), it has a wider range of applications. In particular, the maximal ideals  $\mathcal{M}$  of  $\mathbb{N}$  are  $2^{\mathfrak{c}}$ , while the analytic  $P$ -ideals  $\mathcal{I}$  of  $\mathbb{N}$  are at most  $\mathfrak{c}$  (see Remark 3.24).

Moreover, in certain models of ZFC (see Remark 3.16(b)), maximal ideals are never  $P$ -ideals, so Theorem 5.19(1) cannot be applied to maximal ideals at all in such a case.

#### 5.4. Theorem C: when $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) \neq t_{\mathbf{u}}(\mathbb{T})$

The following general versions of Theorems C and  $C^R$  for  $\mathcal{I}$ -characterized subgroups were obtained in [25]. Clearly, the conclusion holds true also when  $\mathcal{I} = \mathcal{P}(\mathbb{N})$  as in this case  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) = \mathbb{T}$  as noted in Example 5.6(b) while  $t_{\mathbf{u}}(\mathbb{T})$  is a proper subgroup of  $\mathbb{T}$  by Theorem D, so Claim 2.1 applies. The same idea works also for the subsequent results.

**Theorem 5.22.** *Let  $\mathcal{I}$  be a free analytic  $P$ -ideal of  $\mathbb{N}$  satisfying  $\neg(**)$ .*

- (1) [25, Theorems 3.5 and 3.6] If  $\mathbf{u} \in \mathcal{A}$ , then  $|t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) \setminus t_{\mathbf{u}}(\mathbb{T})| = \mathfrak{c}$ .  
 (2) [25, Theorem 4.5] If  $\mathbf{q} \in \mathcal{R}$ , then  $|t_{\mathbf{q}}^{\mathcal{I}}(\mathbb{T}) \setminus t_{\mathbf{q}}(\mathbb{T})| = \mathfrak{c}$ .

*In particular, in both cases  $t_{\mathbf{u}}(\mathbb{T}) \subsetneq t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$ .*

As translation invariant implies  $\neg(**)$ , item (1) of the above theorem covers [44, Corollary 2.5]. Again, Lemma 5.10 is fundamental in the proof of Theorem 5.22(1).

The next theorem implies Theorem 5.22(1), and moreover it provides a simultaneous proof of Theorems B and C for  $\mathcal{A}$ - $\mathcal{I}$ -characterized subgroups when  $\mathcal{I}$  satisfies  $\neg(**)$ :

**Theorem 5.23.** [32] *Let  $\mathbf{u} \in \mathcal{A}$  and  $\mathcal{I}$  be a free ideal of  $\mathbb{N}$  with  $\neg(**)$ . Then  $\mathbf{u}$  has a  $b$ -unbounded subsequence  $\mathbf{w}$  (so  $|t_{\mathbf{w}}(\mathbb{T})| = \mathfrak{c}$ ) such that  $t_{\mathbf{u}}(\mathbb{T}) \subsetneq t_{\mathbf{w}}(\mathbb{T}) \subseteq t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$ . Hence,  $|t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) \setminus t_{\mathbf{u}}(\mathbb{T})| = \mathfrak{c}$ .*

Let us briefly outline the idea of the proof of the theorem. As in the sketch of the proof of Theorem 5.20, property  $\neg(**)$  produces  $A \in \mathcal{I}$  with  $c(A) = \infty$ , hence a subsequence  $\mathbf{w}$  of  $\mathbf{u}$  with support the set  $\mathbb{N} \setminus A$ . Then  $|t_{\mathbf{w}}(\mathbb{T})| = \mathfrak{c}$  (by Theorem 1.10), as  $\mathbf{w}$  is  $b$ -unbounded. The inclusion  $t_{\mathbf{w}}(\mathbb{T}) \subseteq t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  follows again from Lemma 5.7. The crucial inequality  $t_{\mathbf{u}}(\mathbb{T}) \subsetneq t_{\mathbf{w}}(\mathbb{T})$  is the hardest part of this proof, its proof relies on Proposition 1.6.

The hypothesis  $\neg(**)$  cannot be relaxed in Theorem 5.23. Indeed, in view of Theorem 6.5,  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  is countable, whenever  $\mathcal{I}$  satisfies  $(*)$  and  $\mathbf{u} \in \mathcal{S}_{bd}$ .

Since the ideal  $\mathcal{I}_d$  fails to satisfy  $(**)$  by Proposition 4.4, Theorem 5.23 can be applied to obtain the following result that simultaneously covers Theorems B and C for statistically characterized subgroups.

**Corollary 5.24.** *Let  $\mathbf{u} \in \mathcal{A}$ . Then  $\mathbf{u}$  has a  $b$ -unbounded subsequence  $\mathbf{w}$  (so that  $|t_{\mathbf{w}}(\mathbb{T})| = \mathfrak{c}$ ) and  $t_{\mathbf{u}}(\mathbb{T}) \subsetneq t_{\mathbf{w}}(\mathbb{T}) \subseteq t_{\mathbf{u}}^s(\mathbb{T})$ , so  $t_{\mathbf{u}}(\mathbb{T}) \subsetneq t_{\mathbf{u}}^s(\mathbb{T})$  and  $|t_{\mathbf{u}}^s(\mathbb{T})| = \mathfrak{c}$ , hence  $|t_{\mathbf{u}}^s(\mathbb{T}) \setminus t_{\mathbf{u}}(\mathbb{T})| = \mathfrak{c}$ .*

The main conclusion of Theorem 5.23 is  $|t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) \setminus t_{\mathbf{u}}(\mathbb{T})| = \mathfrak{c}$  for  $\mathcal{I}$  an ideal of  $\mathbb{N}$  with  $\neg(**)$  and  $\mathbf{u} \in \mathcal{A}$ . This is obtained also in the next theorem for  $\mathbf{u} \in \mathcal{A}_{div}$  and for any free ideal  $\mathcal{I}$  of  $\mathbb{N}$  containing at least one infinite subset of  $\mathbb{N}$ .

**Theorem 5.25.** *Let  $\mathbf{u} \in \mathcal{A}_{div}$ . Then  $|t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) \setminus t_{\mathbf{u}}(\mathbb{T})| = \mathfrak{c}$  for every free ideal  $\mathcal{I} \neq Fin$  of  $\mathbb{N}$ .*

**Proof.** We first give a brief sketch of the proof of the proper inclusion  $t_{\mathbf{u}}(\mathbb{T}) \subsetneq t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  established in [32]. Pick an infinite set  $M$  in  $\mathcal{I}$  with  $0 \notin M$ . Choose  $\mathbf{v} = (v_n)$  such that  $\{v_0, v_1, \dots\} = \{u_n : n \in \mathbb{N} \setminus M\}$ . Then  $t_{\mathbf{u}}(\mathbb{T}) \subsetneq t_{\mathbf{v}}(\mathbb{T}) \subseteq t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$ . The second inclusion follows from Lemma 5.7. The argument of the first (proper) inclusion, making use of  $b_n^{\mathbf{u}} \rightarrow \infty$ , can be deduced from Proposition 1.6 and the hypothesis  $\mathbf{u} \in \mathcal{A}_{div}$ . In particular,  $t_{\mathbf{u}}(\mathbb{T}) \subsetneq t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$ .

Our hypothesis  $\mathbf{u} \in \mathcal{A}_{div}$  implies  $|t_{\mathbf{u}}(\mathbb{T})| = \mathfrak{c}$ , in view of Theorem 1.8. This gives, combined with the above proper inclusion,  $|t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) \setminus t_{\mathbf{u}}(\mathbb{T})| = \mathfrak{c}$ , according to Claim 2.1 applied to  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$ .  $\square$

Theorem 5.25 provides a negative answer to the following question of Ghosh.

**Problem 5.26.** [46, Problem 1] *Let  $\mathbf{u} \in \mathcal{A}$  and  $\mathcal{I}$  be a free analytic  $P$ -ideal satisfying  $(**)$ . Is  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) = t_{\mathbf{u}}(\mathbb{T})$ ?*

### 6. Eggleston’s dichotomy for $\mathcal{I}$ -characterized subgroups

As pointed out in [25], the counterpart of Eggleston’s dichotomy (Theorem 1.8) for  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  fails when  $\mathcal{I}$  satisfies  $\neg(**)$  (since  $|t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})| = \mathfrak{c}$  in that case for all  $\mathbf{u} \in \mathcal{A} \cup \mathcal{R}$  regardless of whether  $\mathbf{u}$  is  $b$ -bounded or  $b$ -divergent, see Theorem 5.20). This motivated them to introduce the  $snt$  property for ideals and establish that for this kind of ideals the natural counterpart of Eggleston’s dichotomy holds true. In order to conveniently discuss these issues we propose the following notion describing Eggleston’s dichotomy for  $\mathcal{I}$ -characterized subgroups.

**Definition 6.1.** For a free ideal  $\mathcal{I}$  of  $\mathbb{N}$  and a class  $\mathcal{X}$  of sequences of integers, we say that:

- (a)  $\mathcal{I}$  is an *Eggleston ideal* (and briefly denote this property by **(egg)**) if  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  is countable for every  $\mathbf{u} \in \mathcal{S}_{bd}$ ;
- (b)  $\mathcal{I}$  is a *perfect Eggleston ideal with respect to  $\mathcal{X}$*  (and briefly denote this property by **(Egg $_{\mathcal{X}}$ )**) if for every  $\mathbf{u} \in \mathcal{X}$  the subgroup  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  is countable if and only if  $\mathbf{u} \in \mathcal{X}_{bd}$ .

We do not add “and  $|t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})| = \mathfrak{c}$  whenever  $\mathbf{u} \in \mathcal{S}_{div}$  (resp.,  $\mathbf{u} \in \mathcal{X}_{div}$ )” in **(egg)** (resp., **(Egg $_{\mathcal{X}}$ )**) as the reader may expect, since this equality obviously holds true from the inclusion  $t_{\mathbf{u}}(\mathbb{T}) \subseteq t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  and the equality  $|t_{\mathbf{u}}(\mathbb{T})| = \mathfrak{c}$ , by way of the standard Eggleston’s dichotomy (Theorem 1.8).

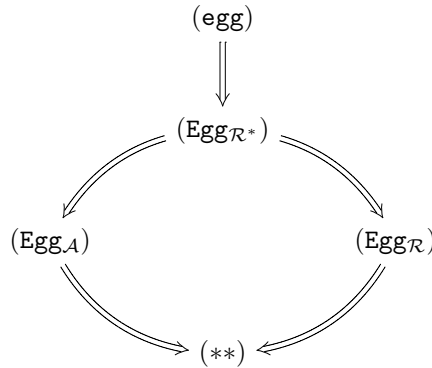
**Example 6.2.** The ideal  $Fin$  satisfies **(egg)**. Moreover,  $Fin$  satisfies **(Egg $_{\mathcal{R}^*}$ )** by Theorem 1.16, and so also **(Egg $_{\mathcal{A}}$ )** and **(Egg $_{\mathcal{R}}$ )**. On the other hand,  $Fin$  does not satisfy **(Egg $_{\mathcal{S}}$ )**, in view of Remark 1.9.

Clearly, **(Egg $_{\mathcal{S}}$ )** implies **(egg)**, although we are not aware whether there exists a free ideal  $\mathcal{I}$  of  $\mathbb{N}$  satisfying **(Egg $_{\mathcal{S}}$ )** (see Question 7.3).

**Lemma 6.3.** *Let  $\mathcal{X}$  be a class of sequences of integers. If  $\mathcal{J} \subseteq \mathcal{I}$  are ideals of  $\mathbb{N}$ ,  $\mathcal{J}$  satisfies  $(\text{Egg}_{\mathcal{X}})$  and  $\mathcal{I}$  satisfies  $(\text{egg})$ , then  $\mathcal{I}$  satisfies also  $(\text{Egg}_{\mathcal{X}})$ .*

**Proof.** Assume that  $\mathbf{u} \in \mathcal{X}_{bd}$ . Then  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  is countable, by the hypothesis that  $\mathcal{I}$  satisfies  $(\text{egg})$ . On the other hand,  $t_{\mathbf{u}}^{\mathcal{J}}(\mathbb{T}) \subseteq t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  is countable, as  $t_{\mathbf{u}}^{\mathcal{J}}(\mathbb{T})$  is countable. Therefore,  $\mathbf{u} \in \mathcal{X}_{bd}$  as  $\mathcal{J}$  satisfies  $(\text{Egg}_{\mathcal{X}})$ .  $\square$

**Proposition 6.4.** *For a free ideal  $\mathcal{I}$  of  $\mathbb{N}$ , the following implications hold.*



**Proof.** The implications  $(\text{egg}) \Rightarrow (\text{Egg}_{\mathcal{R}^*}) \Rightarrow (\text{Egg}_{\mathcal{R}})$  follow from Lemma 6.3 and Example 6.2. The implication  $(\text{Egg}_{\mathcal{R}^*}) \Rightarrow (\text{Egg}_{\mathcal{A}})$  is trivial since  $\mathcal{A} \subseteq \mathcal{R}^*$ .

To prove the implications  $(\text{Egg}_{\mathcal{A}}) \Rightarrow (**)$  and  $(\text{Egg}_{\mathcal{R}}) \Rightarrow (**)$ , note that the hypothesis that  $\mathcal{I}$  satisfies  $(\text{Egg}_{\mathcal{A}})$  (resp.,  $(\text{Egg}_{\mathcal{R}})$ ) yields that for every  $\mathbf{u} \in \mathcal{A}_{bd}$  (resp.,  $\mathbf{u} \in \mathcal{R}_{bd}$ ) the subgroup  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  is countable, which means that  $\mathcal{I}$  satisfies  $(**)$ , in view of Theorem 5.20.  $\square$

6.1. Eggleston’s dichotomy under  $(*)$

Here we show that  $(*)$  ensures  $(\text{egg})$  (although it cannot ensure  $(\text{Egg}_{\mathcal{X}})$ , as witnessed by  $\mathcal{F}in$ ). The impact of the weaker condition  $(\dagger)$  on Eggleston’s dichotomy will be examined in §6.2.

**Theorem 6.5.** *Let  $\mathcal{I}$  be an ideal of  $\mathbb{N}$ .*

- (1) *If  $\mathcal{I}$  satisfies  $(*)$ , then  $\mathcal{I}$  is an Eggleston ideal.*
- (2) *For  $P$ -ideals of  $\mathbb{N}$  the implications  $(*) \Rightarrow (\text{egg}) \Rightarrow (\dagger)$  hold.*

**Proof.** Item (1) was proved in [32] as follows. Making use of Lemma 5.7,  $(*)$  and  $\mathbf{u} \in \mathcal{S}_{bd}$ , a  $b$ -bounded subsequence  $\mathbf{v}$  of  $\mathbf{u}$  was built with  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) \subseteq t_{\mathbf{v}}(\mathbb{T})$ , so an application of Theorem 1.8(2) yields that  $t_{\mathbf{v}}(\mathbb{T})$  (hence, also  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$ ) is countable.

The first implication in (2) follows from (1) without any recourse to property  $P$ . The second implication follows from Proposition 6.4 and Theorem 4.9.  $\square$

**Remark 6.6.** Item (1) of Theorem 6.5 was obtained in [25, Theorem 4.6] for  $\text{snt}$  ideals. In the statement of [25, Theorem 4.6] the ideal is not explicitly required to be an analytic  $P$ -ideal, but in the proof of the theorem the criterion [25, Proposition 2.13] is applied, and in that criterion “analytic  $P$ -ideal” is necessary along with the even stronger hypothesis  $(*)$  (see the comment after Lemma 4.12). Theorem 6.5 shows that  $(*)$  is enough.

While the proof of [25, Theorem 4.6] uses ideas very close to those from the proof of Theorem 1.8(2) given in [6, Theorem 3.3], the proof of Theorem 6.5(1) is quite simple as it directly applies Theorem 1.8(2) (see the sketch given above).

In view of Theorem 6.5, if the properties (†) and (\*) were equivalent for  $P$ -ideals (see Question 7.1), then for  $P$ -ideals all three properties (egg), (†) and (\*) would coincide, so Eggleston’s dichotomy for ideal convergence would hold precisely for ideals satisfying (\*). Since in [25] a positive answer to Question 7.1 is silently assumed, it is safe to say that this remarkable fact (modulo the missing answer to Question 7.1) was established as a main result in [25].

**Corollary 6.7.** *For a free ideal  $\mathcal{I}$  of  $\mathbb{N}$  satisfying (\*) and  $\mathbf{q} \in \mathcal{R}^*$ , the subgroup  $t_{\mathbf{q}}^{\mathcal{I}}(\mathbb{T})$  is countable if and only if  $\mathbf{q} \in \mathcal{R}_{bd}^*$ .*

**Proof.** Shortly, we have to check that (\*)  $\Rightarrow$  (Egg $\mathcal{R}^*$ ) for ideals satisfying (\*). This follows from Theorem 6.5(1) combined with Proposition 6.4.  $\square$

This result was obtained in [25, Corollary 4.7] under the more stringent hypothesis that  $\mathcal{I}$  is a free snt analytic  $P$ -ideal of  $\mathbb{N}$ . Since it is deduced there from [25, Theorem 4.6], it is necessary to impose (\*) rather than snt (see Remark 6.6).

**Corollary 6.8.** *For a free ideal  $\mathcal{I}$  of  $\mathbb{N}$  the implications (\*)  $\Rightarrow$  (egg)  $\Rightarrow$  (Egg $\mathcal{R}^*$ )  $\Rightarrow$  (Egg $\mathcal{R}$ )  $\Rightarrow$  (\*\*) hold.*

Theorem 6.5 shows that  $\mathcal{S}_{bd}\text{-}\mathcal{C}\text{har}^{\mathcal{I}} \subseteq \mathcal{S}_{bd}\text{-}\mathcal{C}\text{har} \subseteq \mathcal{C}\text{har}$  when the ideal  $\mathcal{I}$  satisfies (\*), as  $\mathcal{S}_{bd}\text{-}\mathcal{C}\text{har}$  contains all countable subgroups of  $\mathbb{T}$  – see Remark 1.11).

6.2. Eggleston’s dichotomy under (†)

Here we show that to ensure Eggleston’s dichotomy for  $|t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})|$  when only arithmetic sequences  $\mathbf{u}$  are involved, one can relax the condition (\*) to (†).

Let us note that under the hypothesis  $\mathbf{u} \in \mathcal{A}_{bd}$ , the equality  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) = t_{\mathbf{u}}(\mathbb{T})$  in the conclusion of the next theorem implies that  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  is countable (this should be compared to the fact that  $\mathcal{I}$  satisfying (Egg $\mathcal{A}$ ) yields that  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  is countable whenever  $\mathbf{u} \in \mathcal{A}_{bd}$ ). According to Theorem 4.9, (†)  $\Rightarrow$  snt  $\Rightarrow$  (\*\*)  $\stackrel{P}{\Rightarrow}$  (†), so when dealing with  $P$ -ideals, we prefer to use the equivalent form (†) instead of snt.

**Theorem 6.9.** [25, Theorem 3.8] *Let  $\mathcal{I}$  be a proper free analytic  $P$ -ideal of  $\mathbb{N}$  and let  $\mathbf{u} \in \mathcal{A}_{bd}$ . Then  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) = t_{\mathbf{u}}(\mathbb{T})$  if and only if  $\mathcal{I}$  satisfies (†).*

An independent and shorter proof of the following stronger version of Theorem 6.9 is given in [32]; more precisely, the ideal is not required to be analytic. We include a brief sketch of the proof for reader’s convenience.

**Theorem 6.10.** [32] *Let  $\mathcal{I}$  be any proper ideal of  $\mathbb{N}$  and  $\mathbf{u} \in \mathcal{A}_{bd}$ . Then*

$$\mathcal{I} \text{ satisfies (†)} \Rightarrow t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) = t_{\mathbf{u}}(\mathbb{T}) \Rightarrow \mathcal{I} \text{ satisfies (**)} \tag{6.1}$$

*If  $\mathcal{I}$  is a  $P$ -ideal, then all three conditions in (6.1) are equivalent.*

**Proof.** The first implication in (6.1) is proved in [32]. For the second one observe that  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) = t_{\mathbf{u}}(\mathbb{T})$  and the hypothesis  $\mathbf{u} \in \mathcal{A}_{bd}$  yield that  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  is countable by Eggleston’s dichotomy (Theorem 1.8), which means that  $\mathcal{I}$  satisfies (\*\*), in view of Theorem 5.20. The last assertion follows from Theorem 4.9.  $\square$

The following example shows that the first implication in (6.1) fails when  $\mathbf{u}$  is not arithmetic:

**Example 6.11.** For the set  $A = 2\mathbb{N} + 1$  with  $c(A) = 1 < \infty$ , the  $P$ -ideal  $\mathcal{I}_A$  satisfies  $(*)$  (hence,  $(\dagger)$  as well by Lemma 4.12). Let  $\mathbf{u}$  be the sequence with  $u_0 = 1$  and  $u_n = n$  for every  $n \in \mathbb{N}_+$  and  $\mathbf{v}$  the sequence with  $v_n = u_{2n}$  for every  $n \in \mathbb{N}$ . Then  $\mathbf{u} \in \mathcal{S}_{bd}$  and  $t_{\mathbf{u}}(\mathbb{T}) = \{0\}$ . On the other hand, Lemma 5.7 gives  $t_{\mathbf{u}^A}(\mathbb{T}) = t_{\mathbf{v}}(\mathbb{T}) = \langle \varphi(1/2) \rangle$ . In particular,  $t_{\mathbf{u}}(\mathbb{T}) \neq t_{\mathbf{u}^A}(\mathbb{T})$ .

The following consequence of the above theorem provides a counterpart for  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  of Theorem 1.10 (covering the case  $\mathcal{I} = Fin$ ) for ideals  $\mathcal{I}$  satisfying  $(\dagger)$ .

**Corollary 6.12.** *For a free ideal  $\mathcal{I}$  of  $\mathbb{N}$  satisfying  $(\dagger)$  and  $\mathbf{u} \in \mathcal{A}$ , the following are equivalent:*

- (1)  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  is countable;
- (2)  $\mathbf{u} \in \mathcal{A}_{bd}$ ;
- (3)  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) = t_{\mathbf{u}}(\mathbb{T}) \subseteq \mathbb{Q}/\mathbb{Z}$ ;
- (4)  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) \subseteq \mathbb{Q}/\mathbb{Z}$  (so  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  is torsion and countable);
- (5)  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) = t_{\mathbf{u}}(\mathbb{T})$  is countable.

**Proof.** The implications  $(3) \Rightarrow (4) \Rightarrow (1)$  and  $(3) \Rightarrow (5) \Rightarrow (1)$  are trivial. The implication  $(1) \Rightarrow (2)$  follows from the inclusion  $t_{\mathbf{u}}(\mathbb{T}) \subseteq t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  and Theorem 1.10.

$(2) \Rightarrow (3)$  The equality  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) = t_{\mathbf{u}}(\mathbb{T})$  follows from Theorem 6.10. The inclusion follows from the hypothesis  $\mathbf{u} \in \mathcal{A}_{bd}$  and Theorem 1.10. This is the only implication where we use the assumption that  $\mathcal{I}$  satisfies  $(\dagger)$ .  $\square$

**Remark 6.13.** (a) Corollary 6.12 is false without the assumption  $\mathbf{u} \in \mathcal{A}$ : take the classical analytic  $P$ -ideal  $\mathcal{I} = Fin$  satisfying  $(\dagger)$  (see Remark 1.9).

(b) We show in Example 6.16(a), that “countable” cannot be relaxed in item (5). In other words,  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) = t_{\mathbf{u}}(\mathbb{T})$  with  $\mathbf{u} \in \mathcal{A}$  does not imply that  $\mathbf{u}$  is  $b$ -bounded.

(c) Theorem 5.25 provides a nice “reinforced” dichotomy (restricted to the class  $\mathcal{A}$ ) for every fixed free ideal  $\mathcal{I} \neq Fin$  of  $\mathbb{N}$  satisfying  $(\dagger)$ :

$$\text{if } \mathbf{u} \in \mathcal{A}_{bd} \text{ then } t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) \text{ is countable, while } \mathbf{u} \in \mathcal{A}_{div} \text{ entails } |t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) \setminus t_{\mathbf{u}}(\mathbb{T})| = \mathfrak{c}.$$

The equivalence of (1) and (2) in Corollary 6.12 shows that the ideals of  $\mathbb{N}$  satisfying  $(\dagger)$  satisfy also  $(Egg_{\mathcal{A}})$ . This is a part of the first assertion of the next corollary. The second assertion can be attributed to [25], although it cannot be found there formulated in this way. The corollary follows from Proposition 6.4 and Theorem 4.9.

**Corollary 6.14.**  $(\dagger) \Rightarrow (Egg_{\mathcal{A}}) \Rightarrow (**)$  for free ideals of  $\mathbb{N}$ . In particular,  $(Egg_{\mathcal{A}})$  is equivalent to  $(\dagger)$  for  $P$ -ideals.

Corollary 6.14 provides a correct and reinforced version of [25, Corollary 3.9] (see also [46, Theorem 6]). Without the assumption  $\mathbf{u} \in \mathcal{A}$  this corollary fails as explained in Remark 6.13(b).

**Remark 6.15.** It follows from the above corollary and Corollary 6.8 that for a free  $P$ -ideal  $\mathcal{I}$  of  $\mathbb{N}$  the implications  $(*) \Rightarrow (egg) \Rightarrow (Egg_{\mathcal{R}^*}) \Rightarrow (Egg_{\mathcal{R}}) \Rightarrow (\dagger) \Leftrightarrow (Egg_{\mathcal{A}})$  hold. Therefore, a positive answer to Question 7.1 (granting the equivalence of  $(\dagger)$  and  $(*)$  for  $P$ -ideals) would give the equivalence of all these properties for  $P$ -ideals.

We have seen above that, under suitable hypotheses on the ideal  $\mathcal{I} \neq Fin$  of  $\mathbb{N}$ ,  $\mathbf{u} \in \mathcal{A}_{bd}$  implies  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) = t_{\mathbf{u}}(\mathbb{T})$  and  $\mathbf{u} \in \mathcal{A}_{div}$  implies  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) \neq t_{\mathbf{u}}(\mathbb{T})$ . The next example answers the question of whether  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) = t_{\mathbf{u}}(\mathbb{T})$  or not when  $\mathbf{u} \in \mathcal{A} \setminus (\mathcal{A}_{bd} \cup \mathcal{A}_{div})$  showing that both outcomes are possible.

**Example 6.16.** Let  $\mathcal{I} = \mathcal{I}_{2\mathbb{N}_+}$ . We show that there are  $\mathbf{u}, \mathbf{v} \in \mathcal{A} \setminus (\mathcal{A}_{bd} \cup \mathcal{A}_{div})$  such that  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) = t_{\mathbf{u}}(\mathbb{T})$  and  $t_{\mathbf{v}}^{\mathcal{I}}(\mathbb{T}) \neq t_{\mathbf{v}}(\mathbb{T})$ .

Let  $(a_n)$  be a sequence in  $\mathbb{N}$  with

$$1 < a_{n+1} - a_n \rightarrow +\infty. \tag{6.2}$$

(a) Let  $u_{2n-1} = 2^{a_n}$  and  $u_{2n} = 2^{a_n+1} = 2u_{2n-1}$  for  $n \in \mathbb{N}_+$  and  $u_0 = 1$ . Then  $\mathbf{u} \in \mathcal{A} \setminus (\mathcal{A}_{bd} \cup \mathcal{A}_{div})$ ; in fact,  $\mathbf{u} \notin \mathcal{A}_{bd}$  as  $b_{2n+1}^{\mathbf{u}} \rightarrow +\infty$  in view of (6.2) and  $\mathbf{u} \notin \mathcal{A}_{div}$  as  $b_{2n}^{\mathbf{u}} = 2$  for every  $n \in \mathbb{N}_+$ . Let  $\mathbf{w}$  be the sequence defined by  $w_0 = 1$  and  $w_n = u_{2n-1}$  for every  $n \in \mathbb{N}_+$ . Then, by Lemma 5.7, we have  $t_{\mathbf{u}}(\mathbb{T}) \subseteq t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) = t_{\mathbf{w}}(\mathbb{T})$ .

Let us see that  $t_{\mathbf{w}}(\mathbb{T}) = t_{\mathbf{u}}(\mathbb{T})$ . Indeed, if  $x \in t_{\mathbf{w}}(\mathbb{T})$  one has  $u_{2n-1}x = w_n x \rightarrow 0$ , hence  $u_{2n}x = 2u_{2n-1}x \rightarrow 0$  as well, thus  $x \in t_{\mathbf{u}}(\mathbb{T})$ . This proves that  $t_{\mathbf{w}}(\mathbb{T}) \subseteq t_{\mathbf{u}}(\mathbb{T})$ , and so  $t_{\mathbf{u}}(\mathbb{T}) = t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$ .

(b) By Theorem 1.10,  $t_{(2^{a_n})}(\mathbb{T})$  is non-torsion, as  $(2^{a_n}) \in \mathcal{A}_{div}$ . Pick a non-torsion  $x \in t_{(2^{a_n})}(\mathbb{T})$  and let  $C = \{c \in \mathbb{N} : \|2^c x\| > 1/4\}$ . Assume for a contradiction that  $C$  is finite and let  $c_0 = \max C$ . Then  $\|2^k x\| \leq 1/4$  for all  $k > c_0$ . Then  $z = 2^{c_0} x$  has the property  $\|2^t z\| \leq 1/4$  for all  $t \in \mathbb{N}$ . It is a folklore fact that  $\|2^t z\| \leq 1/4$  for all  $t \in \mathbb{N}$  yields  $2^t z \rightarrow 0$ , so  $z \in t_{(2^n)}(\mathbb{T}) = \mathbb{Z}(2^\infty)$ , and hence also  $x \in \mathbb{Z}(2^\infty)$ , a contradiction. This proves that  $C$  is infinite, so we can write it as a strictly increasing sequence  $C = (c_n)$ .

Passing to subsequences of  $(a_n)$  and  $(c_n)$  if necessary, we can choose a strictly increasing sequence  $(k_n)$  in  $\mathbb{N}$  with  $a_{k_n} < c_{k_n} < a_{k_{n+1}}$  and define the sequence  $(d_n)$  as follows:  $d_{2n+1} = a_{k_n}$  for all  $n \in \mathbb{N}$ , while  $d_{2n+2} = a_{k_n} + 1$  if  $n$  is even, and  $d_{2n+2} = c_{k_n}$  if  $n$  is odd;  $d_0 = 1$ . So  $(d_n)$  is the sequence

$$d_0 = 1, d_1 = a_{k_0}, d_2 = a_{k_0} + 1, d_3 = a_{k_1}, d_4 = c_{k_1}, d_5 = a_{k_2}, d_6 = a_{k_2} + 1, d_7 = a_{k_3}, d_8 = c_{k_3}, d_9 = a_{k_4}, \dots$$

Let  $v_0 = w_0 = 1$  and, for every  $n \in \mathbb{N}_+$ , let  $v_n = 2^{d_n}$  and  $w_n = v_{2n+1} = 2^{d_{2n+1}} = 2^{a_{k_n}}$ . Then  $\mathbf{w} \in \mathcal{A}_{div}$  in view of (6.2) and so  $\mathbf{v} \notin \mathcal{A}_{bd}$ ; moreover,  $\mathbf{v} \notin \mathcal{A}_{div}$  as  $b_{2n}^{\mathbf{v}} = 2$  for every odd  $n \in \mathbb{N}_+$ , and so  $\mathbf{v} \in \mathcal{A} \setminus (\mathcal{A}_{bd} \cup \mathcal{A}_{div})$ . By Lemma 5.7,  $t_{\mathbf{v}}^{\mathcal{I}}(\mathbb{T}) = t_{\mathbf{w}}(\mathbb{T})$ . This implies  $x \in t_{\mathbf{w}}(\mathbb{T}) = t_{\mathbf{v}}^{\mathcal{I}}(\mathbb{T})$ , as  $\mathbf{w}$  is a subsequence of  $(2^{a_n})$  and  $x \in t_{(2^{a_n})}(\mathbb{T})$ . On the other hand,  $x \notin t_{\mathbf{v}}(\mathbb{T})$  as  $v_n x \not\rightarrow 0$  is witnessed by the subsequence  $(v_{2n+2} x)$  with  $n$  odd, that is,  $(2^{c_{k_n}} x)$ , and the defining property of  $C = (c_n)$ . Therefore,  $x \in t_{\mathbf{v}}^{\mathcal{I}}(\mathbb{T}) \setminus t_{\mathbf{v}}(\mathbb{T})$ .

**7. Final comments and open problems**

*7.1. Ideals*

We start with the following question on ideals inspired by Lemma 4.12.

**Question 7.1.** Does  $(\dagger)$  imply  $(*)$  for any  $(P)$ -ideal of  $\mathbb{N}$ ?

The next question (asking whether  $(\dagger)$  implies (egg)) would have a vacuously positive answer in case the question has a positive answer:

**Question 7.2.** Can  $(*)$  in Theorem 6.5 and Corollaries 6.7 and 6.8 be relaxed to  $(\dagger)$ ?

**Question 7.3.** Does there exist a free ideal  $\mathcal{I}$  satisfying (Egg<sub>S</sub>)?

It is natural to ask whether it is possible to relax “translation invariant” in Theorem 5.13:

**Question 7.4.** Given any two free  $P$ -ideals  $\mathcal{I}_1, \mathcal{I}_2$  of  $\mathbb{N}$  and  $\mathbf{u} \in \mathcal{A}$ , does  $\mathcal{I}_2 \not\subseteq \mathcal{I}_1$  imply  $t_{\mathbf{u}}^{\mathcal{I}_2}(\mathbb{T}) \not\subseteq t_{\mathbf{u}}^{\mathcal{I}_1}(\mathbb{T})$ ?

The following question posed by Lydia Außenhofer is related to Theorem 5.13 as well as Question 7.4:

**Question 7.5.** Given two distinct maximal ideals  $\mathcal{I}$  and  $\mathcal{J}$ , does there always exist  $\mathbf{u} \in \mathcal{A}$  such that  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) \neq t_{\mathbf{u}}^{\mathcal{J}}(\mathbb{T})$ ?

Indeed, if Question 7.4 has a positive answer, then we obtain an immediate positive answer to Question 7.5, for maximal  $P$ -ideals. Unfortunately, maximal ideals are not translation invariant, so Theorem 5.13 cannot be applied to get an immediate positive answer to Question 7.4.

7.2. Continued fractions and recursive sequences

Among other open problems left in [12], we mention the following one due to Maharam and Stone:

**Problem 7.6.** [12, Problem 3] Given an irrational  $\gamma \in (0, 1)$ , find all sequences  $\mathbf{u}$  in  $\mathbb{N}_+$  such that  $t_{\mathbf{u}}(\mathbb{T}) = \langle \varphi(\gamma) \rangle$ .

Theorem 1.15 provides a partial answer to this problem, so we recall the following more specific question related to that theorem that was left open in [52, Problem 1]:

**Question 7.7.** [52] Given an irrational  $\gamma \in (0, 1)$ , if  $\mathbf{u}$  is a subsequence of  $\mathbf{v}^{\mathbf{q}}$  and a supersequence of  $\mathbf{q}$  such that  $\langle \varphi(\gamma) \rangle = t_{\mathbf{u}}(\mathbb{T})$ , must  $\mathbf{u}$  be  $b$ -bounded?

The next conjecture related to Problem 7.6 is an extension of [25, Conjecture 4.8] and is motivated by Theorem 6.5 as well as Theorem 1.13.

**Conjecture 7.8.** [25, Conjecture 4.8] Let  $\mathcal{I}$  be an ideal of  $\mathbb{N}$  satisfying (\*). If  $\mathbf{q} \in \mathcal{R}_{bd}$  and  $\gamma$  is the irrational in  $(0, 1)$  with  $\mathbf{q}^\gamma = \mathbf{q}$ , then  $t_{\mathbf{q}}^{\mathcal{I}}(\mathbb{T}) = \langle \varphi(\gamma) \rangle$ .

Even the following particular case of this conjecture, which is the (most delicate) part of [22, Question 6.4], still remains open:

**Problem 7.9.** Compute  $t_{\mathbf{f}}^s(\mathbb{T})$ , where  $\mathbf{f} \in \mathcal{R}$  is the Fibonacci sequence.

The rest of [22, Question 6.4], namely,  $|t_{\mathbf{f}}^s(\mathbb{T})| = \mathfrak{c}$  (so  $t_{\mathbf{f}}^s(\mathbb{T})$  is distinct from  $t_{\mathbf{f}}(\mathbb{T})$ ), was already answered in [20, Corollary 2.5] (see Corollary 2.3). Note that after trading the ideal  $\mathcal{I}_d$  for an ideal  $\mathcal{I}$  satisfying (\*),  $t_{\mathbf{f}}^{\mathcal{I}}(\mathbb{T})$  becomes countable, by Theorem 6.5.

While [20, Proposition 3.4] (see Proposition 2.5) produces an irrational  $\gamma$  such that  $\mathbf{q}^\gamma$  is  $b$ -unbounded and  $t_{\mathbf{v}^{\mathbf{q}}}^s(\mathbb{T}) \neq t_{\mathbf{q}^\gamma}^s(\mathbb{T})$ , [20, Problem 3.6] asks whether  $t_{\mathbf{v}^{\mathbf{q}}}^s(\mathbb{T})$  is countably infinite in that situation.

**Question 7.10.** [20, Problem 3.7] Does the equality  $t_{\mathbf{v}^{\mathbf{q}}}^s(\mathbb{T}) = t_{\mathbf{q}}^s(\mathbb{T})$  in Proposition 2.4 fail for every  $\mathbf{q} \in \mathcal{R} \setminus \mathcal{R}_{bd}$ ?

Problem 3.8 of the same paper asks whether, in case of a positive answer to Question 7.10, it is true that for any  $\gamma$  as in that question  $|t_{\mathbf{q}}^s(\mathbb{T}) \setminus t_{\mathbf{v}^{\mathbf{q}}}^s(\mathbb{T})| = \mathfrak{c}$ . The answer to this quest is “Yes”, since  $|t_{\mathbf{q}}^s(\mathbb{T})| = \mathfrak{c}$ , by Theorem B<sup>R</sup>. So  $t_{\mathbf{v}^{\mathbf{q}}}^s(\mathbb{T}) \neq t_{\mathbf{q}}^s(\mathbb{T})$  implies  $|t_{\mathbf{q}}^s(\mathbb{T}) \setminus t_{\mathbf{v}^{\mathbf{q}}}^s(\mathbb{T})| = \mathfrak{c}$  by Claim 2.1.

The following question, related to Theorem B<sup>R</sup>, is inspired by [22, Question 6.5].

**Question 7.11.** Is  $|t_{\mathbf{q}}(\mathbb{T})| = \mathfrak{c}$  for any  $\mathbf{q} \in \mathcal{R}^*$ ?

7.3.  $\mathcal{C}\text{har}$  vs  $\mathcal{C}\text{har}^s$  and  $\mathcal{C}\text{har}^{\mathcal{I}}$

The next general question is motivated by Corollary 2.7.

**Question 7.12.** Is  $\mathcal{C}\text{har}$  contained in  $\mathcal{C}\text{har}^s$ ? Namely, are all characterized subgroups of  $\mathbb{T}$  also statistically characterized?

Since all countable subgroups of  $\mathbb{T}$  are characterized, one can specify more:

**Question 7.13.** Are all countable subgroups of  $\mathbb{T}$  statistically characterized? More specifically:

- (a) does  $\mathcal{C}\text{har}^s$  contain  $\mathbb{Q}/\mathbb{Z}$ ?
- (b) does  $\mathcal{C}\text{har}^s$  contain all infinite cyclic subgroups of  $\mathbb{T}$ ?

The above question is related to Remark 1.11 and the fact that  $\mathbb{Q}/\mathbb{Z} \in \mathcal{C}\text{har} \setminus \mathcal{A}\text{-}\mathcal{C}\text{har}$ :

**Question 7.14.** [22, Question 6.6] Let  $\mathbf{u}$  be the sequence in (1.5) from Remark 1.11. Is  $t_{\mathbf{u}}^s(\mathbb{T}) = \mathbb{Q}/\mathbb{Z}$ ? Is it countable or  $|t_{\mathbf{u}}^s(\mathbb{T})| = \mathfrak{c}$ ?

One can formulate a variant of Question 7.13 for a free ideal  $\mathcal{I} \not\subseteq \mathcal{F}in$ , not necessarily  $\mathcal{I} = \mathcal{I}_d$ :

**Question 7.15.** Let  $\mathcal{I} \not\subseteq \mathcal{F}in$  be a free ideal of  $\mathbb{N}$ .

- (a) Does the inclusion  $\mathcal{C}\text{har} \subseteq \mathcal{C}\text{har}^{\mathcal{I}}$  hold true?
- (b) Does  $\mathcal{C}\text{har}^{\mathcal{I}}$  contain all countable subgroups of  $\mathbb{T}$ ? What about  $\mathbb{Q}/\mathbb{Z}$  or the infinite cyclic subgroups of  $\mathbb{T}$ ?

Theorem 6.5 provides an answer to Question 7.15 in the case of subgroups  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  with  $\mathbf{u} \in \mathcal{S}_{bd}$  and  $\mathcal{I}$  with (\*). Indeed, by item (1) of that theorem  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  is countable, hence characterized.

On other hand, one can formulate a variant of the problem raised by Corollary 2.7:

**Question 7.16.** For which ideals  $\mathcal{I}$  of  $\mathbb{N}$ ,  $\mathcal{A}\text{-}\mathcal{C}\text{har}^{\mathcal{I}} \cap \mathcal{C}\text{har} = \emptyset$ ?

According to Corollary 2.7, this holds true for  $\mathcal{I} = \mathcal{I}_d$ , while Theorem 6.5(1) entails that, when  $\mathcal{I}$  satisfies (\*) the subgroups from  $\mathcal{S}_{bd}\text{-}\mathcal{C}\text{har}^{\mathcal{I}}$  are countable (see Remark 1.11), so  $\mathcal{S}_{bd}\text{-}\mathcal{C}\text{har}^{\mathcal{I}} \subseteq \mathcal{S}_{bd}\text{-}\mathcal{C}\text{har} \subseteq \mathcal{C}\text{har}$ . This gives a negative answer to Question 7.16 for  $\mathcal{I}$  satisfying (\*), as under this condition  $\mathcal{A}_{bd}\text{-}\mathcal{C}\text{har} \subseteq \mathcal{A}\text{-}\mathcal{C}\text{har}^{\mathcal{I}} \cap \mathcal{C}\text{har}$ .

The (negation of the) equality  $\mathcal{A}\text{-}\mathcal{C}\text{har}^{\mathcal{I}} \cap \mathcal{C}\text{har} = \emptyset$  from Question 7.16 can also be presented in the following alternative form:

**Question 7.17.** For which ideals  $\mathcal{I}$  of  $\mathbb{N}$ ,  $\mathcal{X}\text{-}\mathcal{C}\text{har}^{\mathcal{I}} \subseteq \mathcal{C}\text{har}$  for some class of sequences  $\mathcal{X}$ ?

7.4. Questions related to countability or triviality of  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$

**Question 7.18.** Is it possible to relax the restraint  $\mathbf{u} \in \mathcal{A}$  in Theorem 5.20 to  $\mathbf{u} \in \mathcal{H}$ ?

To see that in general Theorems B and C does not hold true for statistically characterized subgroups and for any sequence in  $\mathcal{S}$ , recall that Corollary 5.18 provided a sufficient condition for *finiteness* of  $t_{\mathbf{u}}^s(\mathbb{T})$  for  $\mathbf{u} \in \mathcal{S}$ . Namely, if there exists a polynomial  $P(x) \in \mathbb{Z}[x]$  with  $u_n = P(n)$  for almost all  $n \in \mathbb{N}$ , then  $t_{\mathbf{u}}^s(\mathbb{T})$  is finite (under the same hypothesis, it was shown in [22, Example 6.1] that  $t_{\mathbf{u}}^s(\mathbb{T}) \subseteq \mathbb{Q}/\mathbb{Z}$ ). This leaves the following question open:

**Question 7.19.** [22, Question 6.7] Are there other countable subgroups in  $\mathcal{C}\text{har}^s$  beyond these ones?

In the second half of [20, Problem 3.6] the authors posed the following challenging question which is obviously very closely related to the above one.

**Question 7.20.** [20, Problem 3.6] Does there exist  $\mathbf{u} \in \mathcal{S}$  such that  $t_{\mathbf{u}}^s(\mathbb{T})$  is countably infinite?

Following the line of [22, Theorem 6.2], we see that  $W_{\mathbf{u}} \cap t_{\mathbf{u}}^s(\mathbb{T}) = \emptyset$  for any  $\mathbf{u} \in \mathcal{S}$ . Indeed, assume that there exists  $x \in W_{\mathbf{u}} \cap t_{\mathbf{u}}^s(\mathbb{T})$ . As  $x \in t_{\mathbf{u}}^s(\mathbb{T})$ , the complement of the set  $C := \{n \in \mathbb{N} : u_n x \in \varphi((-1/4, 1/4))\}$  has density 0, so  $\bar{d}(C) = 1$ . This contradicts the fact that  $\bar{d}(C) = 1/2$  since  $(u_n x)$  is uniformly distributed in  $\mathbb{T}$ , as  $x \in W_{\mathbf{u}}$ .

Therefore, imposing  $W_{\mathbf{u}} = \mathbb{T} \setminus \mathbb{Q}/\mathbb{Z}$  on the sequence  $\mathbf{u} \in \mathcal{S}$ , we immediately get  $t_{\mathbf{u}}^s(\mathbb{T}) \subseteq \mathbb{Q}/\mathbb{Z}$  (by Weyl’s uniform equidistribution theorem this condition is weaker than the hypothesis of Corollary 5.18). In particular,  $t_{\mathbf{u}}^s(\mathbb{T})$  is countable whenever  $W_{\mathbf{u}} = \mathbb{T} \setminus \mathbb{Q}/\mathbb{Z}$ .

Of course, the obvious counterpart for  $\mathcal{Ch}\alpha\tau^{\mathcal{I}}$  of the above questions can be formulated also for ideals  $\mathcal{I}$  distinct from  $\mathcal{F}in$  and  $\mathcal{I}_d$ :

**Question 7.21.** For which  $\mathbf{u} \in \mathcal{S}$  and ideals  $\mathcal{I} \notin \{\mathcal{F}in, \mathcal{I}_d\}$  the subgroup  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  is countably infinite?

Some partial answers were already given above. Indeed, by Theorems 5.20 and 5.19(2), respectively, neither  $\mathcal{I}$  can have the property  $\neg(**)$  if  $\mathbf{u} \in \mathcal{A}$  nor  $\mathcal{I}$  can be a free non-snt analytic  $P$ -ideal if  $\mathbf{u} \in \mathcal{R}$  (as  $|t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})| = \mathfrak{c}$  in these two cases). On the other hand, positive evidence is provided by Theorem 6.5(1), since  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  is countable whenever  $\mathbf{u} \in \mathcal{S}_{bd}$  and  $\mathcal{I}$  is an ideal of  $\mathbb{N}$  satisfying  $(*)$ .

The following questions are inspired by Theorems 5.16 and 5.17.

**Question 7.22.** [32] Is  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) \neq \mathbb{T}$  for every proper ideal  $\mathcal{I}$  of  $\mathbb{N}$  and every  $\mathbf{u} \in \mathcal{S}$ ?

In order to positively answer this question it is enough to consider maximal ideals of  $\mathbb{N}$ , which have some additional properties (for example, dense or  $\neg(*)$ ).

**Question 7.23.** [32] For which ideals  $\mathcal{I}$  of  $\mathbb{N}$  and which sequences  $\mathbf{u}$  of integers,  $t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  is finite?

Since a finite subgroup  $F$  of  $\mathbb{T}$  has the form  $F = \langle \varphi(1/q) \rangle \cong \mathbb{Z}(q)$  for some  $q \in \mathbb{N}_+$ , one can show that  $t_{\mathbf{u}}(\mathbb{T}) = F$  for some  $\mathbf{u} \in \mathcal{S}$  precisely when there is a sequence  $\mathbf{v}$  in  $\mathbb{N}_+$  such that  $u_n = qv_n$  eventually and  $t_{\mathbf{v}}(\mathbb{T}) = \{0\}$  [32]. In this way the above problem can be reduced, in the case  $\mathcal{I} = \mathcal{F}in$ , to the problem of determining which sequences  $\mathbf{u} \in \mathbb{N}_+$  give  $t_{\mathbf{u}}(\mathbb{T}) = \{0\}$ . This leaves open the following problem:

**Problem 7.24.** Which sequences  $\mathbf{u}$  of integers give  $t_{\mathbf{u}}(\mathbb{T}) = \{0\}$ ?

A sequence  $\mathbf{u} = (u_n)$  of integers is called a  $T$ -sequence if there exists a Hausdorff group topology  $\tau$  on  $\mathbb{Z}$  such that  $u_n \rightarrow 0$  in  $\tau$ . This notion was introduced Protasov and Zelenyuk [54] in the framework of arbitrary groups. Every  $T$ -sequence  $\mathbf{u}$  in  $\mathbb{Z}$  gives rise to the finest Hausdorff group topology  $T_{\mathbf{u}}$  on  $\mathbb{Z}$  which makes  $\mathbf{u}$  converge to 0. Since the continuous characters  $\chi: (\mathbb{Z}, T_{\mathbf{u}}) \rightarrow \mathbb{T}$  can be identified, via the correspondence  $\chi \mapsto \chi(1)$ , with the elements of the subgroup  $t_{\mathbf{u}}(\mathbb{T})$ , one can give an answer to the above question when  $\mathbf{u}$  is a  $T$ -sequence. Namely, for a  $T$ -sequence  $\mathbf{u}$  one has  $t_{\mathbf{u}}(\mathbb{T}) = \{0\}$  if and only if the topological group  $(\mathbb{Z}, T_{\mathbf{u}})$  is minimally almost periodic, i.e., has no non-trivial continuous characters. Examples of such sequences were provided in [54].

7.5. Compact case

For a compact abelian group  $G$ , let  $\widehat{G}$  denote the Pontryagin dual of  $G$ , i.e., the group of all continuous homomorphisms (characters)  $G \rightarrow \mathbb{T}$  equipped with the compact-open topology (e.g., see [2,40]). The remarkable fact that  $(\widehat{\mathbb{Z}}, \widehat{T_{\mathbf{u}}}) = (t_{\mathbf{u}}(\mathbb{T}), \tau_{\mathbf{u}})$  for every  $T$ -sequence  $\mathbf{u}$  in  $\mathbb{Z}$  was first observed by Gabrielyan [43].

**Definition 7.25.** For an ideal  $\mathcal{I}$  of  $\mathbb{N}$ , a subgroup  $H$  of a compact abelian group  $G$  is  $\mathcal{I}$ -characterized by a sequence  $\mathbf{u} = (u_n)$  in  $\widehat{G}$  if

$$H = s_{\mathbf{u}}^{\mathcal{I}}(G) := \{x \in G : u_n(x) \xrightarrow{\mathcal{I}} 0\}.$$

Clearly,  $s_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T}) = t_{\mathbf{u}}^{\mathcal{I}}(\mathbb{T})$  for any sequence  $\mathbf{u}$  in  $\widehat{\mathbb{T}} = \mathbb{Z}$ .

When  $\mathcal{I} = \mathcal{F}$ in one speaks of *characterized subgroups* of  $G$ , introduced in [34,39] and further studied in [9,35,38,39,43]. Analogously, when  $\mathcal{I} = \mathcal{I}_d$  one speaks of *statistically characterized subgroups* of  $G$ .

Due to the general form of Theorem A proved in [19], the characterized subgroups of the compact abelian groups are  $F_{\sigma\delta}$ -sets.

We recall here the following very general problem from [22]:

**Problem 7.26.** [22, Problem 6.11] *Study the statistically characterized subgroups of the compact abelian groups. Are they Borel sets?*

Clearly, the problem can be extended to  $\mathcal{I}$ -characterized subgroups:

**Problem 7.27.** *Study the  $\mathcal{I}$ -characterized subgroups of the compact abelian groups. Are they Borel sets in case  $\mathcal{I}$  is an analytic  $P$ -ideal?*

NOTE ADDED IN PROOFS. At the last stages of preparation of the revised version of this survey we came across three papers of Das and Ghosh [27–29] which provide some new and relevant results on the subgroups  $t_{\mathbf{v}^{\mathbf{u}}}(\mathbb{T})$  and  $t_{\mathbf{v}^{\mathbf{u}}}^s(\mathbb{T})$  for  $\mathbf{u} \in \mathcal{A}$  (see (1.7) for the definition of  $\mathbf{v}^{\mathbf{u}}$ ). Here is the main result of [27]:

**Theorem 7.28.** [27, Theorem 2.3] *If  $\mathbf{u} \in \mathcal{A}$  and  $x \in [0, 1)$ , then  $\varphi(x) \in t_{\mathbf{v}^{\mathbf{u}}}(\mathbb{T})$  if and only if  $\text{supp}(x)$  is finite.*

From this theorem the authors deduce the nice equality  $t_{\mathbf{v}^{\mathbf{u}}}(\mathbb{T}) = t_{\mathbf{u}}(\mathbb{T}) \cap \mathbb{Q}/\mathbb{Z}$  for  $\mathbf{u} \in \mathcal{A}$  ([27, Theorem 2.7]), which largely generalizes (with  $\mathbf{u} = \mathbf{n}$ ) [38, Proposition 1.3] described in Remark 1.11.

Continuing the work started in [27], the paper [28] concerns the statistically characterized subgroups  $t_{\mathbf{v}^{\mathbf{u}}}^s(\mathbb{T})$  for  $\mathbf{u} \in \mathcal{A}$ . If  $\mathbf{u}$  is  $b$ -bounded, then  $t_{\mathbf{v}^{\mathbf{u}}}^s(\mathbb{T}) = t_{\mathbf{u}}^s(\mathbb{T})$  [28, Theorem 2.5], and if  $\mathbf{u}$  is  $b$ -divergent, then  $t_{\mathbf{u}}(\mathbb{T}) \not\subseteq t_{\mathbf{v}^{\mathbf{u}}}^s(\mathbb{T})$  [28, Theorem 2.10] and so also  $t_{\mathbf{u}}^s(\mathbb{T}) \not\subseteq t_{\mathbf{v}^{\mathbf{u}}}^s(\mathbb{T})$  [28, Corollary 2.9]. Moreover, there exist  $b$ -unbounded sequences  $\mathbf{u} \in \mathcal{A}$  such that  $t_{\mathbf{u}}^s(\mathbb{T}) = t_{\mathbf{v}^{\mathbf{u}}}^s(\mathbb{T})$  [28, Proposition 2.7], but also  $b$ -unbounded sequences  $\mathbf{u} \in \mathcal{A}$  such that  $t_{\mathbf{v}^{\mathbf{u}}}^s(\mathbb{T}) \subsetneq t_{\mathbf{u}}^s(\mathbb{T})$  [28, Proposition 2.8]. Finally, an answer to [22, Question 6.6] (so Question 7.14 as well) is given by showing that  $t_{\mathbf{v}^{\mathbf{n}}}^s(\mathbb{T})$  has cardinality  $\mathfrak{c}$ , so it strictly contains  $t_{\mathbf{v}^{\mathbf{n}}}(\mathbb{T}) = \mathbb{Q}/\mathbb{Z}$ .

Pushing the investigation in [27] in another direction, in [29] Das, Ghosh and Aziz study the characterized subgroups  $t_{\mathbf{e}}(\mathbb{T})$  where  $\mathbf{u} \in \mathcal{A}$  and  $\mathbf{e}$  is a sequence of integers such that  $\mathbf{u} \subseteq \mathbf{e} \subseteq \mathbf{v}^{\mathbf{u}}$ . In particular, they completely describe the elements in  $t_{\mathbf{e}}(\mathbb{T})$  extending Theorem 1.7 to this class of sequences.

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