European Journal of Mathematics (2026) 12:33
https://doi.org/10.1007/s40879-026-00903-7

RESEARCH ARTICLE

Residual functions and divisorial ideals

Dario Spirito[1]

Received: 25 September 2024 / Revised: 7 April 2026 / Accepted: 29 April 2026

Abstract
We define a *residual function* on a topological space X as a function $f\colon X \longrightarrow \mathbb{Z}$ such that $f^{-1}(0)$ contains an open dense set, and we use this notion to study the freeness of the group of divisorial ideals on a Prüfer domain.

Keywords Prüfer domains · Divisorial ideals · Free group · Residual function · Nowhere dense subsets

Mathematics Subject Classification 13F05 · 13C20 · 54C99

1 Introduction

Let X be a topological space. We say that a function $f\colon X \longrightarrow \mathbb{Z}$ is *residual* if $f^{-1}(0)$ contains an open dense set; thus, residual functions are in a sense functions that are zero almost everywhere in a topological sense. We show that residual functions form a group $\mathcal{H}(X)$ and that the subgroup $\mathcal{H}_b(X)$ of bounded residual functions is a direct summand of the (free) group of all bounded functions $X \longrightarrow \mathbb{Z}$.

As an application, we give a new interpretation of some results in [11] about the group $\mathrm{Div}(D)$ of v-invertible divisorial ideals of a Prüfer domain D. Divisorial ideals are a class of fractional ideals of an integral domain whose study is rooted in the work of Krull [7], and are closely linked to *star operations*, a class of closure operations strongly linked to the multiplicative structure of the ideals of D (see [5, Chapter 32]).

In the case of Prüfer completely integrally closed domains, the set of divisorial ideals form a group $\mathrm{Div}(D)$ that is the completion (in the sense of lattice-ordered groups) of the group $\mathrm{Inv}(D)$ of invertible ideals [6, Proposition 3.1]. When D is an SP-domain (i.e., a domain where every ideal can be written as a product of radical ideals) a mixture of topological and lattice-theoretic considerations can be used to

✉ Dario Spirito
dario.spirito@uniud.it

1 Dipartimento di Scienze Matematiche, Informatiche e Fisiche, Università degli Studi di Udine, Udine, Italy

Published online: 09 July 2026

prove that the groups $\mathrm{Inv}(D)$, $\mathrm{Div}(D)$ and $\mathrm{Div}(D)/\mathrm{Inv}(D)$ are all free [10, Theorem 6.5]; in [11], this case was generalized to the case of the subgroups $\mathrm{Inv}_X(D)$ and $\mathrm{Div}_X(D)$ of ideals whose support is contained in some $X \subseteq \mathrm{Max}(D)$, under the assumption that D is a strongly discrete Prüfer domain and that X contains no critical maximal ideals. The proofs of these results employ the existence and the properties of the Gleason cover of a topological space, following [6, Theorems 5.1 and 5.3].

In [3], the authors showed that, in the case of the ring of integer-valued polynomials over a discrete valuation domain V, the divisoriality of radical unitary ideals can be checked at the topological level by considering the subspace of the completion $\widehat{V}$ associated to the ideal. We generalize their idea by showing that the difference between the ideal functions of an ideal I and of its divisorial closure I^v is residual: in this way, we can prove several results of [11] in a more elementary way, without using the language of lattice-ordered groups nor the Gleason cover construction. For example, we show that if every ideal function is bounded (so that there is an injection from $\mathrm{Inv}(D)$ to the group $\mathcal{F}_b(X, \mathbb{Z})$ of bounded functions $X \longrightarrow \mathbb{Z}$) then we have an injection from $\mathrm{Div}(D)$ to $\mathcal{F}_b(X, \mathbb{Z})/\mathcal{H}_b(X)$.

2 Preliminaries

2.1 Topology

Let X be a topological space. If $Y \subseteq X$, we denote by $\mathrm{cl}(Y)$ the closure of Y.

A subset $Y \subseteq X$ is *dense* if $\mathrm{cl}(Y) = X$, and is *nowhere dense* if $\mathrm{cl}(Y)$ has empty interior. In particular, Y is nowhere dense if and only if $X \setminus Y$ contains an open dense set.

We denote by $\mathcal{C}(X, \mathbb{Z})$ the group of continuous functions from X to $\mathbb{Z}$ (where $\mathbb{Z}$ is endowed with the discrete topology and the operation is the sum), by $\mathcal{C}_b(X, \mathbb{Z})$ the subgroup of bounded continuous functions and by $\mathcal{C}_c(X, \mathbb{Z})$ the subgroup of continuous functions with compact support, where the *support* of f is the closure of $f^{-1}(\mathbb{Z} \setminus \{0\})$.

2.2 Topologies on the spectrum

Let R be a ring and let $\mathrm{Spec}(R)$ be the *spectrum* of R, i.e., the set of prime ideals of R. For every ideal I, let $\mathcal{V}(I) := \{P \in \mathrm{Spec}(R) \mid I \subseteq P\}$.

The *Zariski topology* is the topology on $\mathrm{Spec}(R)$ where the closed sets are the $\mathcal{V}(I)$, as I ranges among the ideals of R. The *inverse topology* on $\mathrm{Spec}(R)$ is the topology generated (as a basis of open sets) by the $\mathcal{V}(I)$, as I ranges among the finitely generated ideals of R. The *constructible topology* on $\mathrm{Spec}(R)$ is the coarsest topology that is finer than both the Zariski and the inverse topology. Under all three topologies, $\mathrm{Spec}(R)$ is compact; under the constructible topology, it is also Hausdorff. On the maximal space $\mathrm{Max}(R)$, the inverse and the constructible topology coincide [2, Corollary 4.4.9 (i)].

If $X \subseteq \mathrm{Spec}(D)$, we denote by X^{inv} the set X endowed with the inverse topology.

2.3 Free groups

All groups considered in this paper are abelian; by a *free group* we always mean a free abelian group, i.e., an abelian group with a basis. Every subgroup of a free group is free.

Let X be a set. For a subset $Y \subseteq X$, we denote by χ_Y the characteristic function of Y, i.e., $\chi_Y(a) = 1$ if $a \in Y$ while $\chi_Y(a) = 0$ if $a \notin Y$.

Let $\mathcal{F}_{\mathrm{b}}(X, \mathbb{Z})$ be the group of bounded function from X to $\mathbb{Z}$, where the operation is the componentwise addition. A subgroup $G \subseteq \mathcal{F}_{\mathrm{b}}(X, \mathbb{Z})$ is a *Specker group* if, for every $f \in G$, we have $\chi_{f^{-1}(n)} \in G$ for every $n \in \mathbb{N}$ [8, Section 1]. If $H \subseteq G$ are Specker groups, then H is a direct summand of G, and the quotient G/H is free [8, Satz 2]. In particular, the group $\mathcal{F}_{\mathrm{b}}(X, \mathbb{Z})$ is free.

2.4 Divisorial ideals

Let D be an integral domain with quotient field K. A *fractional ideal* of D is a subset $I \subseteq K$ such that $dI \subseteq D$ for some $d \in K$, $d \neq 0$. Following [11], the *support* of a fractional ideal I is the set

$$\mathrm{supp}(I) := \{P \in \mathrm{Spec}(D) \mid ID_P \neq D_P\};$$

if $I \subseteq D$ is a proper ideal, then $\mathrm{supp}(I) = \mathcal{V}(I)$.

A fractional ideal I is *invertible* if there is a fractional ideal J such that $IJ = D$; in this case, J is equal to $(D : I) := \{d \in K \mid dI \subseteq D\}$. The set of invertible ideals, under product, is a group denoted by $\mathrm{Inv}(D)$.

The *divisorial closure* of a fractional ideal I is $I^v := (D : (D : I))$, and I is *divisorial* if $I = I^v$. An ideal is *v-invertible* if $(I(D : I))^v = D$. The set of divisorial v-invertible ideals is a group under the *v-product* $I \times_v J := (IJ)^v$, denoted by $\mathrm{Div}(D)$. Every invertible ideal is divisorial and v-invertible, and the inclusion map $\mathrm{Inv}(D) \longrightarrow \mathrm{Div}(D)$ is an injective group homomorphism.

2.5 Prüfer domains

A *valuation domain* is an integral domain V such that, for every x in the quotient field K of V, at least one of x and x^{-1} is in V. To every valuation domain we can associate a totally ordered abelian group Γ_v (the *value group* of V) and a surjective map $v \colon K \setminus \{0\} \longrightarrow \Gamma_v$ (the *valuation* relative to V) that satisfies $v(x+y) \geqslant \min\{v(x), v(y)\}$ and $v(xy) = v(x)+v(y)$ for all $x, y \in K$ (and $x+y \neq 0$ for the first property). A valuation domain V is Noetherian if and only if its value group is isomorphic to $\mathbb{Z}$; in this case, V is said to be a *discrete valuation ring* (DVR).

A *Prüfer domain* is an integral domain D that is locally a valuation domain, i.e., such that D_P is a valuation domain for every $P \in \mathrm{Spec}(D)$. Prüfer domains can be characterized in many ways; for example, D is a Prüfer domain if and only if every finitely generated ideal is invertible. See [5, Chapter IV] and [4, Theorem 1.1.1] for more characterizations.

A Prüfer domain is *strongly discrete* if no nonzero prime ideal is idempotent (i.e., if $P \neq P^2$ for every prime ideal $P \neq (0)$). Equivalently, a Prüfer domain D is strongly discrete if and only if PD_P is a principal ideal of D_P for every $P \in \mathrm{Spec}(D)$ [4, Sect. 5.3].

3 Residual functions

Definition 3.1 Let X be a topological space. We say that a function $f\colon X \longrightarrow \mathbb{Z}$ is *residual* if $f^{-1}(0)$ contains an open dense subset. We denote by $\mathcal{H}(X)$ the set of residual functions on X, and by $\mathcal{H}_{\mathrm{b}}(X)$ the set of bounded residual functions.

Remark 3.2 (1) Residual functions are "mostly zero", in the sense that they are different from 0 only on a "small" (i.e., nowhere dense) set.

(2) No nonzero continuous function is residual. Indeed, if $f\colon X \longrightarrow \mathbb{Z}$ is continuous and nonzero, there is an $n \neq 0$ such that $f^{-1}(n)$ is nonempty; since f is continuous and $\mathbb{Z}$ is discrete, $f^{-1}(n)$ is clopen, and in particular $f^{-1}(0)$ cannot be dense. Hence f is not residual.

Lemma 3.3 *Let X be a topological space and $f\colon X \longrightarrow \mathbb{Z}$. Then, the following hold:*

(a) *f is residual if and only if $f^{-1}(\mathbb{Z}\setminus\{0\})$ is nowhere dense.*
(b) *If f is bounded, then f is residual if and only if $f^{-1}(n)$ is nowhere dense for every $n \neq 0$.*

Proof (a) If f is residual, then $f^{-1}(0)$ contains an open dense set Ω. In particular, the closure of $Y := f^{-1}(\mathbb{Z}\setminus\{0\})$ is contained in $X\setminus\Omega$, which by the density of Ω cannot contain an open set. Hence Y is nowhere dense.

Conversely, if Y is nowhere dense, then $\Omega := X\setminus \mathrm{cl}(Y)$ is an open set contained in $f^{-1}(0)$, which is dense since $\mathrm{cl}(Y)$ is nowhere dense. The claim is proved.

(b) If f is residual, each $f^{-1}(n)$ is nowhere dense since it is contained in $f^{-1}(\mathbb{Z}\setminus\{0\})$, which is nowhere dense by the previous point.

Conversely, suppose that each $f^{-1}(n)$ is nowhere dense. Since f is bounded, only finitely many such sets are nonempty; hence, $f^{-1}(\mathbb{Z}\setminus\{0\}) := \bigcup_{n\neq 0} f^{-1}(n)$ is a finite union of nowhere dense subsets, and thus it is nowhere dense [13, Problem 25A]. By the previous point, f is residual. □

Proposition 3.4 *The sets $\mathcal{H}(X)$ and $\mathcal{H}_{\mathrm{b}}(X)$ are subgroups of $\mathcal{F}(X,\mathbb{Z})$.*

Proof If $f \in \mathcal{H}(X)$, then $(-f)^{-1}(0) = f^{-1}(0)$ and thus $-f$ is residual. Moreover, if also $g \in \mathcal{H}(X)$, then $(f+g)^{-1}(0) \supseteq f^{-1}(0)\cap g^{-1}(0)$; since the intersection of two dense open sets is dense, $f^{-1}(0)\cap g^{-1}(0)$ contains an open dense set and thus $f+g$ is residual. Hence $\mathcal{H}(X)$ and $\mathcal{H}_{\mathrm{b}}(X) = \mathcal{H}(X)\cap\mathcal{F}_{\mathrm{b}}(X,\mathbb{Z})$ are subgroups of $\mathcal{F}(X,\mathbb{Z})$. □

Theorem 3.5 *Let X be a topological space. Then, the quotient groups*

$$\frac{\mathcal{F}_{\mathrm{b}}(X,\mathbb{Z})}{\mathcal{H}_{\mathrm{b}}(X)} \quad \textit{and} \quad \frac{\mathcal{F}_{\mathrm{b}}(X,\mathbb{Z})}{\mathcal{H}_{\mathrm{b}}(X)\oplus\mathcal{C}_{\mathrm{b}}(X,\mathbb{Z})}$$

are free.

Proof Let $\mathbf{1}$ be the identity function on X, and let $G := \mathcal{H}_b(X) + \mathbf{1}\mathbb{Z}$. We claim that G is a Specker group. Let $f \in G$: then, there are $h \in \mathcal{H}_b(X)$ and $n \in \mathbb{Z}$ such that $f = h + n\mathbf{1}$, and for any $k \in \mathbb{Z}$ we have $f^{-1}(k) = h^{-1}(k-n)$.

If $k \neq n$, then $h^{-1}(k-n)$ is nowhere dense, and thus $\chi_{f^{-1}(k)}$ is residual, and in particular it belongs to G. If $k = n$, then $h^{-1}(0)$ contains an open dense subset, and thus $X \setminus h^{-1}(0)$ is nowhere dense. Thus, its characteristic function is residual, and so

$$\chi_{f^{-1}(n)} = \mathbf{1} - \chi_{X \setminus h^{-1}(0)} \in G.$$

Therefore, every $\chi_{f^{-1}(k)}$ belongs to G, and thus G is Specker. It follows that $\mathcal{F}_b(X, \mathbb{Z}) = G \oplus H$ for some free group H. Moreover, $\mathcal{H}_b(X) \cap \mathbf{1}\mathbb{Z} = (0)$, and thus G is the direct sum of $\mathcal{H}_b(X)$ and $\mathbf{1}\mathbb{Z}$: hence, $\mathcal{F}_b(X, \mathbb{Z}) = \mathcal{H}_b(X) \oplus \mathbf{1}\mathbb{Z} \oplus H$, and so $\mathcal{F}_b(X, \mathbb{Z})/\mathcal{H}_b(X)$ is isomorphic to the free group $\mathbf{1}\mathbb{Z} \oplus H$.

We now show that $G' := \mathcal{H}_b(X) \oplus \mathcal{C}_b(X, \mathbb{Z})$ is Specker. Let $f \in G'$: then, $f = h + g$ with $h \in \mathcal{H}_b(X)$ and $g \in \mathcal{C}_b(X, \mathbb{Z})$. Fix $n \in \mathbb{Z}$, and, for $a \in \mathbb{Z}$, let $Y_a := h^{-1}(a) \cap g^{-1}(n-a)$. Then,

$$f^{-1}(n) = \bigcup_{a \in \mathbb{Z}} Y_a.$$

Moreover, the union above is disjoint (as $h^{-1}(a)$ and $h^{-1}(a')$ are disjoint if $a \neq a'$) and only finitely many of the intersections are nonempty (as h, g are bounded). Hence,

$$\chi_{f^{-1}(n)} = \sum_{a \in \mathbb{Z}} \chi_{Y_a},$$

and it is enough to show that each χ_{Y_a} belongs to G'. If $a \neq 0$, the set Y_a is nowhere dense since it is contained in the nowhere dense set $h^{-1}(a)$, and thus its characteristic function is residual (i.e., in $\mathcal{H}_b(X)$). If $a = 0$, then

$$Y_0 = g^{-1}(n) \setminus (g^{-1}(n) \setminus h^{-1}(0)) = g^{-1}(n) \setminus (g^{-1}(n) \cap h^{-1}(\mathbb{Z} \setminus \{0\})).$$

Let $Y' := g^{-1}(n) \cap h^{-1}(\mathbb{Z} \setminus \{0\})$. Then, $Y' \subseteq g^{-1}(n)$, and thus

$$\chi_{Y_0} = \chi_{g^{-1}(n)} - \chi_{Y'}.$$

Since g is continuous, $g^{-1}(n)$ is clopen and $\chi_{g^{-1}(n)}$ is continuous; on the other hand, $Y' \subseteq h^{-1}(\mathbb{Z} \setminus \{0\})$ is nowhere dense, and thus its characteristic function is residual. Hence also $\chi_{Y_0} \in G'$, and so $\chi_{f^{-1}(n)} \in G'$. Therefore, G' is Specker and $\mathcal{F}_b(X, \mathbb{Z})/G'$ is free. □

4 Algebraic applications

Throughout the section, let D be a Prüfer domain. As done in [11] for invertible ideals, we want to associate to certain subsets $X \subseteq \mathrm{Max}(D)$ a subgroup $\mathrm{Div}_X(D)$ of $\mathrm{Div}(D)$, and to each element of $\mathrm{Div}_X(D)$ a function $X \longrightarrow \mathbb{Z}$.

Definition 4.1 Let D be an integral domain and let $X \subseteq \mathrm{Spec}(D)$. We define $\mathrm{Div}_X(D)$ as the subgroup of $\mathrm{Div}(X)$ of the ideals $I \in \mathrm{Div}(D)$ such that $\mathrm{supp}(I)$, $\mathrm{supp}((D:I)) \subseteq X$.

Consider now a non-idempotent prime ideal P of D. Then, $Q := \bigcap_{n\geqslant 1} P^n$ is a prime ideal of D, and it is the largest prime properly contained in P; therefore, the quotient D_P/QD_P is a one-dimensional valuation domain whose maximal ideal PD_P/QD_P is not idempotent. Hence, D_P/QD_P is a discrete valuation ring, and so it carries a discrete valuation v'_P. In particular, if I is a fractional ideal of D such that $QD_P \subsetneq ID_P \subseteq D_Q$, we can define a value

$$w_P(I) := v'_P(ID_P/QD_P) \in \mathbb{Z}.$$

Definition 4.2 Preserve the notation above. We say that I is *evaluable* on X if $w_P(I)$ is well defined for every $P \in X$.

In particular, if $X \subseteq \mathrm{Max}(D)$ is open (with respect to the inverse topology) and $I \in \mathrm{Div}_X(D)$ then I is evaluable on X [11, Lemma 5.3], and thus also every $I \in \mathrm{Inv}_X(D)$ is evaluable on X.

Suppose now that I is evaluable on $X \subseteq \mathrm{Spec}(D)$. For every $P \in X$, we have a well-defined quantity $w_P(I) \in \mathbb{Z}$; therefore, we have a well-defined *ideal function*

$$\begin{aligned} \nu_I\colon X &\longrightarrow \mathbb{Z}, \\ P &\longmapsto w_P(I). \end{aligned}$$

We study the properties of these maps.

Lemma 4.3 *Let $X \subseteq \mathrm{Max}(D)$, and let I, J be fractional ideals that are evaluable on X. Then, the following hold:*

(a) *IJ is evaluable on X, and $\nu_{IJ} = \nu_I + \nu_J$.*
(b) *If I, J are proper ideals with $\mathcal{V}(I), \mathcal{V}(J) \subseteq X$ and $\nu_I = \nu_J$, then $I = J$.*

Proof See [11, Lemmas 5.2 and 5.4 (a)]. □

Suppose now that $X \subseteq \mathrm{Max}(D)$ is open, with respect to the inverse topology. Every $I \in \mathrm{Div}_X(D)$ is evaluable on X, and thus to each such I we can associate an ideal function ν_I; therefore, we can define a map

$$\begin{aligned} \Psi_0\colon \mathrm{Div}_X(D) &\longrightarrow \mathcal{F}(X, \mathbb{Z}), \\ I &\longmapsto \nu_I. \end{aligned}$$

This map, however, is *not* a group homomorphism. Indeed, if $I, J \in \mathrm{Div}_X(D)$, by Lemma 4.3 we have $\nu_{IJ} = \nu_I + \nu_J$; however, the product $I \times_v J$ is not equal to IJ but rather to $(IJ)^v$, and $\nu_{IJ} \neq \nu_{(IJ)^v}$ (at least if $I, J \subseteq D$, by Lemma 4.3).

To understand how far ν_I and ν_{I^v} are, we define the following set.

Definition 4.4 Let D be a strongly discrete Prüfer domain and let $X \subseteq \mathrm{Max}(D)$ be open with respect to the inverse topology. Let I be a fractional ideal that is evaluable on X. We set

$$\lambda_X(I) := \{M \in X \mid \nu_{I^v}(M) \neq \nu_I(M)\}.$$

Definition 4.5 ([11, Definition 5.7]) Let D be a Prüfer domain, and let $X \subseteq \mathrm{Max}(D)$ be open with respect to the inverse topology. We say that $P \in X$ is:

- *critical* (with respect to X) if there is no radical ideal $I \in \mathrm{Inv}_X(D)$ such that $I \subseteq P$;
- *bounded-critical* (with respect to X) if there is no ideal $I \in \mathrm{Inv}_X(D)$ such that $I \subseteq P$ and ν_I is bounded.

We denote by $\mathrm{Crit}_X(D)$ and ${}^{\omega}\mathrm{Crit}_X(D)$, respectively, the set of critical ideals and bounded-critical ideals with respect to X.

Lemma 4.6 *Let D be a strongly discrete Prüfer domain and let $X \subseteq \mathrm{Max}(D)$ be open with respect to the inverse topology. Let I, J be fractional ideals of D that are evaluable on X. Then, $\lambda_X(I) \subseteq \lambda_X(IJ)$. If J is invertible, then $\lambda_X(I) = \lambda_X(IJ)$.*

Proof Let $P \in \lambda_X(I)$. Since $I^v J \subseteq (IJ)^v = (IJ)^v$ [5, Proposition 32.2 (c) and Theorem 34.1 (2)], we have

$$\nu_{(IJ)^v}(P) \leqslant \nu_{I^v J}(P) = \nu_{I^v}(P) + \nu_J(P) < \nu_I(P) + \nu_J(P) = \nu_{IJ}(P).$$

Hence $P \in \lambda_X(IJ)$. If J is invertible, then

$$\lambda_X(I) \subseteq \lambda_X(IJ) \subseteq \lambda_X(IJ(D:J)) = \lambda_X(I)$$

since $J(D:J) = D$. Hence $\lambda_X(I) = \lambda_X(IJ)$. $\square$

Lemma 4.7 *Let D be a strongly discrete Prüfer domain and let $X \subseteq \mathrm{Max}(D)$ be open with respect to the inverse topology. Let $I \subseteq D$ be an ideal with $\mathcal{V}(I) \subseteq X$. Then, $I^v = D$ if and only if ν_I is residual, with respect to the inverse topology.*

Proof We prove that $I^v \neq D$ if and only if ν_I is not residual.

Suppose that ν_I is not residual. The set $Z := \nu_I^{-1}(\mathbb{Z} \setminus \{0\})$ is equal to the closed set $\mathcal{V}(I)$ of the Zariski topology associated to I; hence, Z is closed in the constructible topology, and since the inverse and the constructible topology agree on $\mathrm{Max}(D)$ then Z is closed in X, with respect to the inverse topology. Therefore, if ν_I is not residual there is subset $\Omega \subseteq Z$ that is open in the inverse topology; since a basis of the inverse topology is given by the $\mathcal{V}(J)$, as J ranges among the finitely generated ideals of D, we can suppose without loss of generality that $\Omega = \mathcal{V}(J)$ for some finitely generated ideal J. By [11, Theorem 6.10], $X \setminus \mathrm{Crit}_X(D)$ is dense in X, and thus there is a $P \in \mathcal{V}(J)$ that is not critical; by definition, we can find a radical ideal $J' \in \mathrm{Inv}_X(D)$ contained in P. Let $L := J + J'$: then, L is finitely generated, it is a radical ideal (as it contains J', which is a radical ideal contained only in maximal ideals) and contains J, and thus $\mathcal{V}(L) \subseteq \mathcal{V}(J) \subseteq \mathcal{V}(I)$. Hence $I \subseteq L$ and thus $I^v \subseteq L^v = L$. In particular, $I^v \neq D$.

Conversely, if $I^v \neq D$, then $I \subseteq L$ for some finitely generated ideal $L \subsetneq D$; hence, $\mathcal{V}(L)$ is nonempty and open in the inverse topology, and

$$\mathcal{V}(L) \subseteq \mathcal{V}(I) = \nu_I^{-1}(\mathbb{Z} \setminus \{0\}).$$

In particular, ν_I cannot be residual. □

Proposition 4.8 *Let D be a strongly discrete Prüfer domain and let $X \subseteq \mathrm{Max}(D)$ be open with respect to the inverse topology. Let I be a fractional ideal that is evaluable on X. Then, $\lambda_X(I)$ is nowhere dense in X^{inv}.*

Proof If $I^v = D$, then ν_I is residual by Lemma 4.7, and thus $\lambda_X(I) = \nu_I^{-1}(\mathbb{Z} \setminus \{0\})$ is nowhere dense.

If I is arbitrary, then $\lambda_X(I) \subseteq \lambda_X(I(D:I))$, and the latter is nowhere dense by the previous part of the proof since I is v-invertible and thus $(I(D:I))^v = D$. Hence also $\lambda_X(I)$ is nowhere dense. □

Proposition 4.9 *Let D be a strongly discrete Prüfer domain and let $X \subseteq \mathrm{Max}(D)$ be open with respect to the inverse topology. Then, the map*

$$\begin{aligned} \Psi \colon \mathrm{Div}_X(D) &\longrightarrow \mathcal{F}(X, \mathbb{Z})/\mathcal{H}(X^{\mathrm{inv}}), \\ I &\longmapsto \nu_I + \mathcal{H}(X^{\mathrm{inv}}) \end{aligned}$$

is an injective group homomorphism.

Proof Let $I, J \in \mathrm{Div}_X(D)$: to show that $\Psi(I \times_v J) = \Psi(I) + \Psi(J)$, we need to show that $\nu_{I \times_v J} - (\nu_I + \nu_J)$ is residual. However, $I \times_v J = (IJ)^v$ and $\nu_I + \nu_J = \nu_{IJ}$; hence,

$$h := \nu_{I \times_v J} - (\nu_I + \nu_J) = \nu_{(IJ)^v} - \nu_{IJ}$$

is a function such that $h^{-1}(\mathbb{Z} \setminus \{0\}) = \lambda_X(IJ)$, which is nowhere dense by Proposition 4.8. Thus h is residual, as claimed, and so Ψ is a group homomorphism.

If $I \in \ker \Psi$, then ν_I is residual. The proper ideal $I \cap D$ is divisorial, it satisfies $\mathcal{V}(I \cap D) \subseteq X$ and is v-invertible [11, Lemma 4.3]. Since $\nu_{I \cap D}^{-1}(\mathbb{Z} \setminus \{0\}) \subseteq \nu_I^{-1}(\mathbb{Z} \setminus \{0\})$, the function $\nu_{I \cap D}$ is residual and thus by Lemma 4.7, $I \cap D = (I \cap D)^v = D$. Thus $D \subseteq I$; it follows that $(D:I) \subseteq D$. Since Ψ is a group homomorphism, $\nu_{(D:I)} \in -\nu_I + \mathcal{H}(X^{\mathrm{inv}}) = \mathcal{H}(X^{\mathrm{inv}})$, i.e., $(D:I)$ is residual; it follows that $(D:I) = (D:I)^v = D$. Thus $I = D$, as claimed. □

The following theorem was proved with different methods in [11, Propositions 7.1 and 7.2].

Theorem 4.10 *Let D be a strongly discrete Prüfer domain and let $X \subseteq \mathrm{Max}(D)$ be open with respect to the inverse topology.*

(a) *If* ${}^{\omega}\mathrm{Crit}_X(D) = \emptyset$*, then* $\mathrm{Div}_X(D)$ *is free.*
(b) *If* $\mathrm{Crit}_X(D) = \emptyset$*, then* $\mathrm{Div}_X(D)/\mathrm{Inv}_X(D)$ *is free.*

Proof (a) Since ${}^{\omega}\mathrm{Crit}_X(D)$ is empty, the ideal function ν_I is bounded for every $I \in \mathrm{Div}_X(D)$ [11, Lemma 4.4 and Corollary 5.9], and thus the image of the group homomorphism $\Psi\colon \mathrm{Div}_X(D) \longrightarrow \mathcal{F}(X,\mathbb{Z})/\mathcal{H}(X^{\mathrm{inv}})$ of Proposition 4.9 is contained in

$$\frac{\mathcal{F}_{\mathrm{b}}(X,\mathbb{Z})}{\mathcal{H}(X^{\mathrm{inv}}) \cap \mathcal{F}_{\mathrm{b}}(X,\mathbb{Z})} = \frac{\mathcal{F}_{\mathrm{b}}(X,\mathbb{Z})}{\mathcal{H}_{\mathrm{b}}(X^{\mathrm{inv}})}.$$

Therefore $\mathrm{Div}_X(D)$ is isomorphic to a subgroup of $\mathcal{F}_{\mathrm{b}}(X,\mathbb{Z})/\mathcal{H}_{\mathrm{b}}(X^{\mathrm{inv}})$; as the latter is free by Theorem 3.5, also $\mathrm{Div}_X(D)$ is free.

(b) Suppose now that $\mathrm{Crit}_X(D) = \emptyset$. Let $G := \mathcal{H}_{\mathrm{b}}(X^{\mathrm{inv}}) + \mathcal{C}_{\mathrm{c}}(X^{\mathrm{inv}},\mathbb{Z})$, and consider the map

$$\begin{aligned} \Psi'\colon \mathrm{Div}(D) &\longrightarrow \mathcal{F}_{\mathrm{b}}(X,\mathbb{Z})/G, \\ I &\longmapsto \nu_I + G. \end{aligned}$$

We claim that $\ker\Psi' = \mathrm{Inv}_X(D)$. Indeed, if $I \in \mathrm{Inv}_X(D)$ then ν_I is continuous by [11, Proposition 5.8], and thus $\nu_I \in G$. Conversely, suppose that $I \in \ker\Psi'$. Then, $\nu_I = h + g$ with $h \in \mathcal{H}_{\mathrm{b}}(X^{\mathrm{inv}})$ and $g \in \mathcal{C}_{\mathrm{c}}(X^{\mathrm{inv}},\mathbb{Z})$. By [11, Lemma 6.4], there is an invertible ideal J such that $\nu_J = g$; set $J^{-1} := (D : J)$. Then, IJ^{-1} is a divisorial ideal, which is in $\ker\Psi'$ as $\mathrm{Inv}_X(D) \subseteq \ker\Psi'$. Moreover, $\nu_{IJ^{-1}} \in \mathcal{H}_{\mathrm{b}}(X^{\mathrm{inv}})$; by the previous part of the proof, $IJ^{-1} = D$, i.e., $I = J$. Hence $I \in \mathrm{Inv}_X(D)$ and $\ker\Psi' = \mathrm{Inv}_X(D)$.

Therefore, Ψ' induces an injective map

$$\frac{\mathrm{Div}_X(D)}{\mathrm{Inv}_X(D)} \longrightarrow \frac{\mathcal{F}_{\mathrm{b}}(X,\mathbb{Z})}{G}.$$

Since the group on the right hand side is free (Theorem 3.5), $\mathrm{Div}_X(D)/\mathrm{Inv}_X(D)$ is free too. □

We single out two special cases.

The first is about almost Dedekind domains, i.e., integral domains that are locally discrete valuation domains. In this case, for $X = \mathrm{Max}(D)$ the ideal function ν_I is defined for all fractional ideals I, and $\nu_I(P)$ coincides with the valuation of ID_P. An almost Dedekind domain is *bounded* if every ideal function is bounded. An *SP-domain* is an integral domain such that every ideal can be written as a product of radical ideals; every Dedekind domain is an SP-domain, and every SP-domain is an almost Dedekind domain.

As consequences of Theorem 4.10, the results of the following corollary have already been proved with different methods: point (a) using completion properties of lattice-ordered groups [12], point (b) by using the Gleason cover of a topological space [10, Theorem 6.5].

Corollary 4.11 *Let D be an almost Dedekind domain. Then, the following hold:*

(a) *If D is bounded,* $\mathrm{Div}(D)$ *is free.*

(b) *If D is an SP-domain, then* $\mathrm{Div}(D)/\mathrm{Inv}(D)$ *is free.*

Proof Let $X := \mathrm{Max}(D)$. Since D is one-dimensional, $\mathrm{Div}_X(D) = \mathrm{Div}(D)$ and $\mathrm{Inv}_X(D) = \mathrm{Inv}(D)$. If D is bounded, then ${}^{\omega}\mathrm{Crit}_X(D) = \varnothing$ and $\mathrm{Div}(D)$ is free by Theorem 4.10. If D is an SP-domain, $\mathrm{Crit}_X(D) = \mathrm{Crit}(D) = \varnothing$ [9, Theorem 2.1], the claim follows again from Theorem 4.10. □

The second case deals with integer-valued polynomials. If D is an integral domain with quotient field K, the ring $\mathrm{Int}(D)$ of integer-valued polynomials is the ring of all $f \in K[T]$ such that $f(D) \subseteq D$. A proper ideal I of $\mathrm{Int}(D)$ is *unitary* if $I \cap D \neq (0)$, and a fractional ideal I is *almost-unitary* if $dI \subseteq \mathrm{Int}(D)$ for some $d \in D$. We denote by $\mathrm{Div}^u(\mathrm{Int}(D))$ and $\mathrm{Inv}^u(\mathrm{Int}(D))$ the groups of almost-unitary divisorial and invertible ideals, respectively. We say that $\mathrm{Int}(D)$ *behaves well under localization* if $\mathrm{Int}(D)_P = \mathrm{Int}(D_P)$ for all prime ideals P of D.

Corollary 4.12 *Let D be an SP-domain such that every residue field is finite and* $\mathrm{Int}(D)$ *behaves well under localization. Then, the groups* $\mathrm{Div}^u(\mathrm{Int}(D))$ *and* $\mathrm{Div}^u(\mathrm{Int}(D))/\mathrm{Inv}^u(\mathrm{Int}(D))$ *are free.*

Proof Since D has finite residue fields and $\mathrm{Int}(D)$ behaves well under localization, it is a Prüfer domain [1, Proposition VI.1.6]. Let X be the set of all unitary maximal ideals. Then, $\mathrm{Crit}_X(D) = \varnothing$: indeed, if $M \in X$, then $M \cap D$ contains a finitely generated radical ideal I, and thus $I \cdot \mathrm{Int}(D)$ is a finitely generated radical ideal contained in M. Thus $v_I(P) \leqslant 1$ for all $P \in X$, and so M is not critical. Since $\mathrm{Div}^u(D) = \mathrm{Div}_X(D)$ and $\mathrm{Inv}^u(D) = \mathrm{Inv}_X(D)$, the claim follows from Theorem 4.10. □

Funding Open access funding provided by Università degli Studi di Udine within the CRUI-CARE Agreement.

Declarations

Conflict of interest The author declares no conflict of interest.

References

1. Cahen, P.-J., Chabert, J.-L.: Integer-Valued Polynomials. Mathematical Surveys and Monographs, vol. 48. American Mathematical Society, Providence (1997)
2. Dickmann, M., Schwartz, N., Tressl, M.: Spectral Spaces. New Mathematical Monographs, vol. 35. Cambridge University Press, Cambridge (2019)
3. Finocchiaro, C.A., Loper, K.A.: Topological aspects of the ideal theory in rings of integer-valued polynomials. Proc. Amer. Math. Soc. **152**(5), 1809–1819 (2024)

4. Fontana, M., Huckaba, J.A., Papick, I.J.: Prüfer Domains. Monographs and Textbooks in Pure and Applied Mathematics, vol. 203. Marcel Dekker, New York (1997)
5. Gilmer, R.: Multiplicative Ideal Theory. Pure and Applied Mathematics, vol. 12. Marcel Dekker, New York (1972)
6. Heubo-Kwegna, O.A., Olberding, B., Reinhart, A.: Group-theoretic and topological invariants of completely integrally closed Prüfer domains. J. Pure Appl. Algebra **220**(12), 3927–3947 (2016)
7. Krull, W.: Beiträge zur Arithmetik kommutativer Integritätsbereiche I–II. Math. Z. **41**(1), 545–577: 665–679 (1936)
8. Nöbeling, G.: Verallgemeinerung eines Satzes von Herrn E. Specker. Invent. Math. **6**, 41–55 (1968)
9. Olberding, B.: Factorization into radical ideals. In: Chapman, S.T. (ed.) Arithmetical Properties of Commutative Rings and Monoids. Lecture Notes in Pure and Applied Mathematics, vol. 241, pp. 363–377. Chapman & Hall/CRC, Boca Raton (2005)
10. Spirito, D.: Almost Dedekind domains without radical factorization. Forum Math. **35**(2), 363–382 (2023)
11. Spirito, D.: Radical factorization in higher dimension. J. Pure Appl. Algebra **229**(11), Art. No. 108111 (2025)
12. Spirito, D.: Boundness in almost Dedekind domains. J. Algebra Appl. **25**(10), Art. No. 2650100 (2026)
13. Willard, S.: General Topology. Dover, Mineola (2004)