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Geproci sets on skew lines in $\mathbb{P}^3$ with two transversals

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ABSTRACT

The purpose of this work is to pursue classification of geproci sets. Specifically we classify $[m, n]$-geproci sets Z which consist of $m = 4$ points on each of n skew lines, assuming the skew lines have two transversals in common. We show in this case that $n \leq 6$. Moreover we show that all geproci sets of this type and with no points on the transversals are contained in the F_4 configuration. We conjecture that a similar result is true for an arbitrary number m of points on each skew line, replacing containment in F_4 by containment in a half grid obtained by the so-called *standard construction*.

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1. Introduction

1.1. History

The notion of a geproci set is a specific case of importing a broader, inverse scattering, perspective into algebraic geometry. Rather than asking to classify finite sets $Z \subset \mathbb{P}^n$ (over an algebraically closed field k) with a specific property (like say classifying those Z whose ideal $I(Z)$ in the homogeneous coordinate ring $k[\mathbb{P}^n]$ of $\mathbb{P}^n$ is a complete intersection, or is Gorenstein, or has a linear resolution, etc.), one can ask to classify those Z whose projection $\overline{Z} \subset H$ from a general point to a hyperplane $H \subset \mathbb{P}^n$ has the desired property. I.e., what can one say about Z given that its shadow $\overline{Z}$ generally has a certain property? (In tomography, for example, one tries to understand a 3 dimensional body from the behavior of planar data provided by x-ray shadowgraphs taken from multiple directions.)

Research exploring this idea is in its infancy; so far the only property studied from this perspective is where $\overline{Z}$ is a complete intersection (meaning $I(\overline{Z}) \subset k[\mathbb{P}^n]/I(H) = k[\mathbb{P}^{n-1}]$ is generated by $n-1$ homogeneous polynomials). In this case we say Z is *geproci*, since its GEneral PROjection is a Complete Intersection.

Trivially, if Z is already a complete intersection contained in a hyperplane, then $\overline{Z}$ is also a complete intersection, hence Z is geproci. The initial question, raised by F. Polizzi and answered by D. Panov [10,5], was whether any nondegenerate geproci sets exist even in $\mathbb{P}^3$. Panov's answer was by way of constructing examples called grids in $\mathbb{P}^3$. (We say Z is a grid exactly when Z is the intersection of two curves $A, B \subset \mathbb{P}^3$, each of which is a union of skew lines such that A and B have no common components and every component of A meets every component of B, and where $\overline{A} \cap \overline{B} = \overline{Z}$ where $\overline{A}$ and $\overline{B}$ are the projections of A and B to H.) It still remains unclear whether nondegenerate geproci sets exist in $\mathbb{P}^n$ for $n > 3$, and it remained unclear until [4, Appendix] whether any nondegenerate nongrid geproci sets exist in $\mathbb{P}^3$. (Of course, the question is not interesting for subsets of $\mathbb{P}^1$, and every finite subset of $\mathbb{P}^2$ is obviously geproci.)

The new examples given by [4] were based on work on unexpectedness [7]. They are what has come to be known as *half grids* [9], meaning (see below) that $\overline{Z}$ is the transverse intersection of two curves in H, only one of which is the image $\overline{A}$ of a curve A in $\mathbb{P}^3$ containing Z; by [3] A consists of disjoint lines. (The curve A is not always uniquely determined by Z. For example, there are 16 lines which each contain three of the points of a certain 12 point half grid coming from the D_4 root system, discussed in more detail below, and 32 choices of four of these 16 lines can play the role of A. But with respect to a given choice of A, the components of A are called half grid lines of Z. Each half grid line contains the same number of points of Z, so $|Z| = ab$ where $a = \deg A$ is the number of half grid lines and b is the number of points of Z on each half grid line.)

The half grid examples in [4] come from root systems (such as D_4 and F_4); the paper [1] gives a way (called the standard construction, modeled on the examples given by D_4 and F_4) of making many more examples of half grids and also finds three examples of nondegenerate nongrid non-half grid geproci sets (and, remarkably, these three are all connected to quantum mechanics; see [1, §8.1]). Although no examples other than these three are currently known in characteristic 0, a method of constructing many examples of nondegenerate nongrid non-half grid geproci sets in positive characteristics is given in [8], based on the combinatorial concept of minimal partial spreads of lines in $\mathbb{P}^3$ over a finite field.

Since degenerate geproci sets and grids are easy to understand, the next step is to try to understand half grids. Analyzing half grids in general remains an open problem, although, as [3] shows, skew lines, such as we have with half grid lines, have a combinatorial structure that should in the future be helpful in analyzing half grids. In the meantime it is helpful to note, as Caesar did in *de Bello Gallico* with Gaul ("Gallia est omnis divisa in partes tres"), that half grids split into three cases, defined not geographically but geometrically. Given a half grid Z with half grid lines $L_1, \ldots, L_a$, there are either 0, 1 or 2 lines (called transversals) that meet every one of the half grid lines. (Having three or more transversals would force Z

not to be a half grid but rather a grid.) The hardest situation (still to be investigated) is when there is at most one transversal. The reason it is easier to analyze the case where there are two transversals is, as we shall see, partly due to the fact that an automorphism of $\mathbb{P}^1$ fixing two points can be regarded as a scalar multiplication, thinking of the two fixed points as being 0 and ∞. (The half grid lines of the half grids Z given by the standard construction of [1] all have two transversals.)

As noted in the title and abstract, it is precisely this case of half grids with two transversals that is our focus here.

1.2. Background and conventions

Throughout this paper we work over the complex numbers, and Z will always be a reduced finite set of points in $\mathbb{P}^3$. We denote by $\overline{Z}_{P,H}$ (but often just by $\overline{Z}$) the image of Z under projection to a plane $H \cong \mathbb{P}^2$ from a general point P. When $\overline{Z}_{P,H}$ is a transverse intersection of two curves in H we say Z is *geproci*.

More specifically, following [1], we say that Z is an (a, b)-*geproci set* if $\overline{Z}$ is the transverse intersection of curves in H of degrees a and b with $a \leq b$; i.e., if $\overline{Z}$ is a complete intersection of type (a, b) with $a \leq b$, and we say that Z is $\{a, b\}$-*geproci* if we drop the condition $a \leq b$.

Definition 1.1. An $\{a, b\}$-geproci set is an $\{a, b\}$-*grid* (or (a, b)-grid if $a \leq b$) if there is a set A of a skew lines with each line containing exactly b of the points, and a set B of b skew lines with each line containing exactly a of the points (if $a = b$ we also require A and B to have no lines in common; this is automatic if $a < b$ or $b < a$). An (a, b)-*half grid* (or $\{a, b\}$-*half grid*) is an (a, b)-geproci (or $\{a, b\}$-geproci, resp.) set for which either A or B exists but not both. In addition, we say that an $\{a, b\}$-geproci set is an $[a, b]$-*half grid* if it consists of a points on each of b skew lines.

The main results of [1] establish the existence of non-grid (a, b)-geproci sets of points for all integers $4 \leq a \leq b$ and for $(a, b) = (3, 4)$. In the latter case [1] provides also the full classification: up to projective equivalence (i.e., up to choice of coordinates on $\mathbb{P}^3$), the only non-grid $(3, 4)$-geproci set in $\mathbb{P}^3$ is a $[3, 4]$-half grid denoted Z_{D_4} given by the 12 points of the D_4 root system regarded as points in $\mathbb{P}^3$. This result has a profound impact on the present note.

The next natural case to study is the class of half grid $(4, 4)$-geproci sets. They were fully classified in [2], where it was shown that there are only two cases.

Moreover, as noted above, all but three nondegenerate nongrid geproci sets found up to now are half grids, and most of those have transversals. Working under this “half grid with transversals” assumption, we extend the detailed classification of geproci sets to $[4, n]$-half grids for $n \geq 4$. Our main result is the following. In the statement, Z_{F_4} refers to the $[4, 6]$-half grid given by the 24 points of the F_4 root system regarded as points in $\mathbb{P}^3$ [1].

Theorem 1.2. *Let Z be a $[4, n]$-half grid (so, Z has 4 points on each of n lines) such that there are two lines, $T_1 \neq T_2$, which both meet each of the n half grid lines but do not contain any of the points of Z. Then $4 \leq n \leq 6$ and Z is projectively equivalent to a subset of Z_{F_4}.*

Our strategy for proving this is as follows. Let $L_1, \dots, L_n$ be the half grid lines for Z. Then L_1, L_2, L_3 lie on a unique quadric Q, which is smooth. It is known for any subset $L_1, L_2, L_3, L_{i_1}, \dots, L_{i_r}$ of the lines with $3 < i_1 < \cdots < i_r$, not all on Q, that $Z \cap (L_1 \cup L_2 \cup L_3 \cup L_{i_1} \cup \cdots \cup L_{i_r})$ is a $[4, r+3]$-half grid [1], in particular geproci. We use combinatorial considerations to show that if $L_{i_j} \not\subset Q$, then L_{i_j} must be one of at most 6 lines $\ell_1, \dots, \ell_6$, and that each line ℓ_i determines a unique line $\ell'_i \subset Q$ such that

$$\{L_{i_1}, \dots, L_{i_r}\} \subseteq \{\ell_1, \dots, \ell_6, \ell'_1, \dots, \ell'_6\}$$

and such that $\{L_{i_1}, \ldots, L_{i_r}\}$ contains at least one and at most two of the ℓ_i and at most one of the ℓ'_i. We then show for each Z there is a $[4,6]$-half grid Z' containing Z with half grid lines $\{L_1, L_2, L_3, \ell_{i_1}, \ell_{i_2}, \ell'_{i_1} = \ell'_{i_2}\}$ and that Z' is projectively equivalent to Z_{F_4}.

We now add a comment about the transversals, T_1, T_2. Having them constrains the combinatorics we use in our proof, and this simplifies our analysis. Also, by Proposition 2.2, if $Z \cap T_i \neq \varnothing$, then $Z'' = Z \setminus T_i$ is a $[3,n]$-half grid.

Moreover, in this case, $n \geq 4$ by Lemma 2.1 and Z'' must be projectively equivalent to Z_{D_4} by Proposition 2.2. (So the case that $Z \cap T_i = \varnothing$ for both $i = 1, 2$ is what is of interest.) Moreover, the union of Z_{D_4} with the four points of Z on T_i could not be projectively equivalent to a subset of Z_{F_4}. Indeed, Z_{F_4} consists of two disjoint copies of Z_{D_4}, and every set of 4 collinear points in Z_{F_4} intersects both copies in 2 points. Thus the statement of Theorem 1.2 would be more complicated but no more interesting if it were reformulated to cover the cases with $Z \cap T_i \neq \varnothing$.

2. Preliminaries

Here we recall some basic notions and facts we shall use in the sequel. We begin with the following useful fact.

Lemma 2.1. *Let Z be a $[4,n]$-half grid with half grid lines $L_1, \cdots, L_n$. Then $n \geq 4$, and $W = Z \cap (L_{i_1} \cup \cdots \cup L_{i_r})$ is either a $\{4,r\}$-grid or a $[4,r]$-half grid for all choices of r of the n lines. Moreover, W is a $\{4,r\}$-grid whenever $1 < r < 4$, but if W is a $[4,r]$-half grid, then there is no smooth quadric that contains all of the half grid lines L_{i_j}.*

Proof. If $r = 1$, then W consists of 4 collinear points, and thus is a degenerate case of a grid. By [1, Lemma 4.5], W is either a $\{4,r\}$-grid or a $[4,r]$-half grid, and by [1, Theorem 4.10] it is an $(r,4)$-grid whenever $r < 4$. Since Z is not a grid, we must have $n \geq 4$.

Finally, suppose W is not a grid (so $r > 3$) but that there is a smooth quadric Q that contains every line L_{i_j}. Let $X_k = W \cap (L_{i_1} \cup L_{i_2} \cup L_{i_k})$. Then X_k is a grid whose grid lines transverse to L_{i_1} and L_{i_2} are determined by L_{i_1} and L_{i_2} and thus are independent of k. Thus these transverse grid lines are the same for all k and hence W is a grid, contrary to assumption. □

As is well-known and easy to check, any set of 3 skew lines is projectively equivalent to any other set of 3 skew lines, and each given set of 3 skew lines is contained in a unique quadric surface Q, and that surface is smooth. Moreover, Q is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and the fibers of the two projections to $\mathbb{P}^1$ define two families of lines in Q called rulings. A $(3,n)$-grid is thus contained in a unique smooth quadric Q with 3 of the grid lines in one ruling and the remaining n grid lines in the other ruling.

Proposition 2.2. *Let Z be a $[4,n]$-half grid (hence $n \geq 4$) with half grid lines $L_1, \ldots, L_n$ and a transversal T (i.e., a line meeting each of the L_i). If $Z \cap T \neq \varnothing$, then $T \cap (L_1 \cup \cdots \cup L_n) \subset Z$, and $Z'' = Z \setminus T$ is a $[3,n]$-half grid, hence $n = 4$ and Z'' is projectively equivalent to Z_{D_4}.*

Proof. By Lemma 2.1, $n \geq 4$. Suppose $Z \cap T \neq \varnothing$. Since $Z \cap T \subseteq (L_1 \cup \cdots \cup L_n) \cap T$, we have $T \cap L_i \subset Z$ for some i. Pick two additional lines L_j, L_k. Then $X_{ijk} = Z \cap (L_i \cup L_j \cup L_k)$ is a $(3,4)$-grid by Lemma 2.1, and there is a unique quadric Q_{ijk} containing L_i, L_j, L_k. Since T meets these three lines, T is in the transverse ruling on Q_{ijk} and contains the point $T \cap L_i$ of X_{ijk}, hence X_{ijk} contains the points $T \cap L_j$ and $T \cap L_k$. Since $X_{ijk} \subset Z$ and this holds for all choices of j and k, we have $T \cap (L_1 \cup \cdots \cup L_n) \subset Z$. Now Z'' is $(3,n)$-geproci by [1, Lemma 4.5], and thus either a $(3,n)$-grid or a $[3,n]$-half grid. But if Z'' were a $(3,n)$-grid, then Z would be a $[4,n]$-grid, contrary to assumption. Thus Z'' is a $[3,n]$-half grid, but these have been classified [1, Theorem 4.10]. The only possibility is $n = 4$ with Z'' being projectively equivalent to Z_{D_4}. □

Next we recall two basic notions from projective geometry.

Definition 2.3. Recall that the *cross ratio* of an ordered set of four distinct points $P_1 = [x_1 : y_1], P_2 = [x_2 : y_2], P_3 = [x_3 : y_3], P_4 = [x_4 : y_4]$ with respect to some (in fact: any) choice of coordinates on $\mathbb{P}^1$ is

$$j(P_1, P_2; P_3, P_4) = \frac{(x_1y_3 - y_1x_3)(x_2y_4 - y_2x_4)}{(x_1y_4 - y_1x_4)(x_2y_3 - y_2x_3)}.$$

Definition 2.4. We say that the points are *harmonic* if their cross ratio is -1, $1/2$ or 2 and we say that the points are *anharmonic* if their cross ratio is $(1 \pm \sqrt{3}i)/2$. (The specific value in either case depends on the ordering of the points.)

Warning 2.5. In this note (a, b, c, d) with $\{a, b, c, d\} = \{1, 2, 3, 4\}$ denotes a permutation which sends 1 to a, 2 to b, 3 to c and 4 to d. So this is not cycle notation (i.e., (a, b, c, d) does not mean $a \mapsto b \mapsto c \mapsto d \mapsto a$)!

Remark 2.6. For distinct points P_1, P_2, P_3, P_4 the cross ratio is always a complex number other than 0 or 1. If the points are not harmonic or anharmonic, the cross ratio $j(P_{\sigma(1)}, P_{\sigma(2)}; P_{\sigma(3)}, P_{\sigma(4)})$ takes on 6 distinct values depending on the permutation σ of the points. As noted above, for harmonic points, the cross ratio is either $-1, 1/2$ or 2 depending on σ and for anharmonic points, the cross ratio is either $(1\pm\sqrt{3}i)/2$ depending on σ. Indeed, it is well known and easy to check by direct calculation for non-harmonic and non-anharmonic points $P_1, \dots, P_4$ that the subgroup $V \subset S_4$ of all σ with

$$j(P_1, P_2; P_3, P_4) = j(P_{\sigma(1)}, P_{\sigma(2)}; P_{\sigma(3)}, P_{\sigma(4)})$$

has order 4. It is $V = \{id, (2, 1, 4, 3), (3, 4, 1, 2), (4, 3, 2, 1)\}$, the Klein four group. Thus, the nontrivial permutations leaving the cross-ratio invariant for every set of 4 distinct points are exactly the even involutions in S_4.

For harmonic points $P_1, \dots, P_4$ the subgroup of S_4 leaving the cross ratio invariant has order 8 (hence is one of the three 2-Sylow subgroups of S_4, which one depending on whether the cross ratio is -1, $1/2$ or 2, so isomorphic to D_8); in addition to V, it contains two 4-cycles and two 2-cycles.

For anharmonic points $P_1, \dots, P_4$, the subgroup leaving the cross ratio invariant is the alternating group $A_4 \subset S_4$ of even permutations.

We conclude this section with the following well-known and useful observation; for a proof see, e.g., [6, Paragraph 3.4.1].

Lemma 2.7. *Let $P_1, \dots, P_4$ and $R_1, \dots, R_4$ be two four-tuples of points on the projective line $\mathbb{P}^1$. Then*

$$j(P_1, P_2; P_3, P_4) = j(R_1, R_2; R_3, R_4),$$

if and only if there exists a linear projective map $F : \mathbb{P}^1 \to \mathbb{P}^1$ such that $F(P_i) = R_i$ for $i = 1, \dots, 4$.

3. Permutations on half grids

3.1. Combinatorics of $[4, n]$-half grids with transversals

Let Z be a $[4, n]$-half grid, with two distinct transversals T_1, T_2. Thus there are n skew lines $L_1, \dots, L_n$, each containing exactly 4 points from Z and all these lines intersect lines T_1 and T_2.

We assume additionally that none of the intersection points $L_i \cap T_j$ belong to Z. By Lemma 2.1, $Z \cap (L_1 \cup L_2 \cup L_3)$ is a $(3, 4)$-grid, hence contained in a unique quadric Q, which is smooth. Let's suppose that

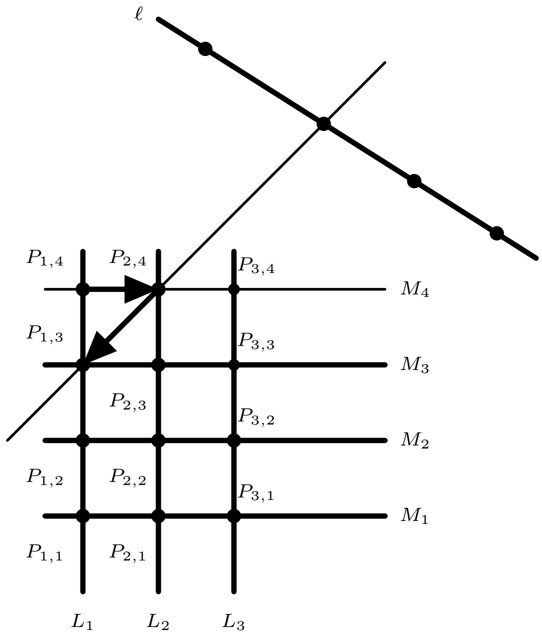

Fig. 1. The $(3,4)$-grid $Z \cap (L_1 \cup L_2 \cup L_3)$; the horizontal arrow represents $f_{L_1,L_2,L_3} : P_{1,4} \mapsto P_{2,4}$, the diagonal arrow represents $f_{L_2,L_1,\ell} : P_{2,4} \mapsto P_{1,3}$, and the composition indicates $\sigma_2^\ell : 4 \mapsto 3$.

L_1, L_2, L_3 are in the "vertical" ruling of Q; we denote by $M_1, \dots, M_4$ the grid lines in the "horizontal" ruling. The points $P_{ij} = L_i \cap M_j$, $1 \le i \le 3$, $1 \le j \le 4$, thus form the grid. The transversals T_1, T_2 are contained in Q and they are lines in the "horizontal" ruling but different from $M_1, \dots, M_4$.

Since Z is not a grid, by Lemma 2.1, $n \ge 4$ and some line L_i, $i > 3$, is not on Q. After reindexing, we may assume that Q contains $L_1, \dots, L_r$ but not $L_{r+1}, \dots, L_n$, which we denote by $\ell_1, \dots, \ell_s$, where $r + s = n$. For every such line $\ell \in \{\ell_1, \dots, \ell_s\}$ and $2 \le i \le r$ we denote by Q_i^ℓ the quadric containing L_1, L_i and ℓ. By Lemma 2.1, $Z \cap (L_1 \cup L_i \cup \ell)$ is a $(3,4)$-grid. Thus i and ℓ determine an element of the symmetric group S_4, that we denote with σ_i^ℓ, as follows. For a point P_{ij} on L_i, there is by Lemma 2.1 a line in Q_i^ℓ (in the same ruling as the transversals T_1 and T_2) passing through P_{ij} and meeting L_1 in a point of Z, say P_{1k}. We define the permutation σ_i^ℓ by putting $\sigma_i^\ell(j) = k$. By way of an example, the permutation σ_2^ℓ in Fig. 1 sends 4 to 3.

It is convenient to consider the permutation σ_i^ℓ as acting on points of $Z \cap L_1$ by sending $P_{1,j}$ to $P_{1,\sigma_i^\ell(j)}$. This action preserves the cross-ratio of the four points by [2, Lemma 1], hence by Lemma 2.7 it extends to a projective linear automorphism of L_1, which, by a slight abuse of notation, we denote with the same symbol σ_i^ℓ. In the notation of [3, Section 4] it is exactly the automorphism

$f_{L_i,L_1,\ell} \circ f_{L_1,L_i,L_j}$, where $2 \le i, j \le r$, $i \ne j$. (Indeed, there is a canonical isomorphism between any two lines in the same ruling on a smooth quadric, since given a point on one line, there is a unique line in the other ruling through that point, which uniquely determines a point on the other line. Thus Q determines a unique isomorphism $L_1 \to L_i$ which is just f_{L_1,L_i,L_j}, and then Q_i^ℓ determines the unique isomorphism $f_{L_i,L_1,\ell} : L_i \to L_1$, and the composition is $\sigma_i^\ell = f_{L_i,L_1,\ell} \circ f_{L_1,L_i,L_j}$.)

Our assumptions impose strong conditions on the permutations σ_i^ℓ. As already mentioned, as automorphisms of L_1, they preserve the cross-ratio. Additional properties are given in the following lemma.

Lemma 3.1. *Each σ_i^ℓ has the following properties:*

(a) as an automorphism of L_1, σ_i^ℓ has exactly two fixed points, these being the intersection points of L_1 with the transversals T_1 and T_2;
(b) σ_i^ℓ has no fixed points on L_1 off the transversals (and so has no fixed points regarded as a permutation of the points $Z \cap L_1$);
(c) for $i \neq j$ we have $\sigma_i^\ell(p) \neq \sigma_j^\ell(p)$ for all p on L_1 not on T_1 or T_2; and
(d) at most one σ_i^ℓ, $2 \leq i \leq r$, can be an involution.

Proof. (a) We have $\sigma_i^\ell = f_{L_i,L_1,\ell} \circ f_{L_1,L_i,L_j}$. If $p \in L_1$, the $q = f_{L_1,L_i,L_j}(p)$ is the point of L_i such that the unique ruling line A on Q through p other than L_1 meets L_i at q. Likewise, $q' = f_{L_i,L_1,\ell}(q)$ is the point of L_1 such that the unique ruling line B on Q_i^ℓ through q other than L_i meets L_1 at q'. Thus $\sigma_i^\ell(p) = p$ if and only if A and B both contain the points p and q, so $A = B$. Thus a point is a fixed point of σ_i^ℓ if and only if the point lies on a line transversal to L_1, L_2, L_3, L (i.e., the point is either $L_1 \cap T_1$ or $L_1 \cap T_2$).

(b) The matrix of a linear isomorphism of $L_1 \cong \mathbb{P}^1$ with two fixed points (which we can regard as 0 and ∞) is given by a diagonal matrix. Such a map has a fixed point (other than 0 and ∞) only when the map is the identity. By (a), σ_i^ℓ has exactly two fixed points and so is not the identity. These fixed points are on the transversals hence no point of L_1 off the transversals is a fixed point (and so no point of $Z \cap L_1$ is fixed, since none are on either transversal).

(c) We have $\sigma_i^\ell(p) = f_{L_i,L_1,\ell} \circ f_{L_1,L_i,L_j}(p)$ and $\sigma_j^\ell(p) = f_{L_j,L_1,\ell} \circ f_{L_1,L_j,L_i}(p)$. By construction $q_i = f_{L_1,L_i,L_j}(p)$ and $q_j = f_{L_1,L_j,L_i}(p)$ lie on the same ruling line A of Q (in particular, A is the ruling line through p other than L_1). This line is neither T_1 nor T_2 (since p is not on either transversal). Note that the line B through $\sigma_i^\ell(p)$ and q_i meets ℓ by definition of $f_{L_j,L_1,\ell}$ and so does the line C through $\sigma_j^\ell(p)$ and q_j. If $\sigma_i^\ell(p) = \sigma_j^\ell(p)$, then B and C meet, hence define a plane that contains A and ℓ. But then ℓ meets Q in three points (since ℓ meets T_1, T_2 and A), hence we would have $\ell \subset Q$, contrary to assumption.

(d) By the proof of (a), σ_i^ℓ can be regarded as multiplication by a nonzero nonidentity complex number c. The only possible involution arises when $c = -1$, but by (b) the maps are different, so at most one can be given by $c = -1$. (Alternatively, see Lemma 6 in [2].) □

3.2. Permutations and automorphism of the projective line

Remark 3.2. By Lemma 2.7 any permutation σ of $P_1, P_2, P_3, P_4 \in \mathbb{P}^1$ which leaves the cross ratio invariant extends to a unique automorphism α_σ of $\mathbb{P}^1$ which restricts to σ on the four points. Moreover all nontrivial automorphisms arising in this way have exactly two fixed points; indeed, every automorphism of $\mathbb{P}^1$ has at least one fixed point and those with only one fixed point have all orbits infinite, with the exception of the fixed point.

Working now with specific coordinates, we will examine which permutations from the group S_4 may appear as σ_i^ℓ's. To begin with we fix projective coordinates on L_1 so that

$$P_{1,1} = [1 : 0], \qquad P_{1,2} = [0 : 1], \qquad P_{1,3} = [1 : 1], \qquad P_{1,4} = [1 : q]$$

with $q \neq 0, 1$. We first consider the four permutations V from Remark 2.6 always keeping the cross ratio invariant (which in this case is $1/q$). In Table 1 we present the associated linear maps and we determine their fixed points. Additionally we explicitly list the fixed points for $q = -1$, $q = 1/2$ and $q = 2$.

Corollary 3.3. *Under the assumptions in the first two paragraphs of Section 3 and that $P_{1,1} = [1 : 0], P_{1,2} = [0 : 1], P_{1,3} = [1 : 1], P_{1,4} = [1 : q]$, the points on each half grid line must be harmonic.*

Proof. It is enough by Lemma 2.7 to show this for the points $Z \cap L_1$, since f_{L_1,L_i,L_j} is a linear isomorphism $L_1 \to L_i$ mapping $Z \cap L_1$ to $Z \cap L_i$.

Table 1
The permutations σ preserving cross ratios, their associated automorphisms α_σ and the fixed points of α_σ.

$\sigma \in S_4$	α_σ	fixed points arbitrary q	fixed points $q=-1$	fixed points $q=1/2$	fixed points $q=2$
$(2,1,4,3)$	$\begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix}$	$[1:\pm a]$, $a^2=q$	$[1:\pm i]$	$[1:\pm\sqrt{2}/2]$	$[1:\pm\sqrt{2}]$
$(3,4,1,2)$	$\begin{pmatrix} q & -1 \\ q & -q \end{pmatrix}$	$[1:q\pm b]$, $b^2=q^2-q$	$[1:-1\pm\sqrt{2}]$	$[1:(1\pm i)/2]$	$[1:1\pm i]$
$(4,3,2,1)$	$\begin{pmatrix} 1 & -1 \\ q & -1 \end{pmatrix}$	$[1:1\pm c]$, $c^2=1-q$	$[1:1\pm\sqrt{2}]$	$[1:(2\pm\sqrt{2})/2]$	$[1:1\pm i]$

Since the points $P_{1,i}$ are distinct, we cannot as noted above have $q=0,1$. By Lemma 3.1, the permutations σ_2^ℓ and σ_3^ℓ are distinct, nontrivial, have the same pair of fixed points on L_1, and (by Lemma 2.7) preserve the value of the cross ratio. Thus if the points are not harmonic or anharmonic, they must be among the three permutations listed in Table 1, since no other permutations preserve the cross ratio in this case. Hence, looking at the fixed points listed in column 3 of the table, q must have a value such that $\{a,-a\}=\{q+a,q-a\}$, or $\{a,-a\}=\{1+c,1-c\}$ or $\{q+a,q-a\}=\{1+c,1-c\}$. But this would mean either that $a+(-a)=(q+b)+(q-b)$, $a+(-a)=(1+c)+(1-c)$ or $(q+b)+(q-b)=(1+c)+(1-c)$; i.e., that $0=2q$, $0=2$ or $2q=2$, all of which are impossible. Thus the points must be harmonic or anharmonic, but the permutations preserving cross ratios in the anharmonic case are the even permutations, and other than those already considered in Table 1 these are 3-cycles and so fix one of the four points $P_{1,i}$, which is excluded by Lemma 3.1. Thus we are left with the harmonic case. □

We now have:

Theorem 3.4. *Under the assumptions of Corollary 3.3, the only candidates for permutations σ_i^ℓ, depending on q, are*

$$q=-1:(2,1,4,3),\quad (3,4,2,1)\ \textit{and}\ (4,3,1,2);$$
$$q=1/2:(3,4,1,2),\quad (2,3,4,1)\ \textit{and}\ (4,1,2,3);$$
$$q=2:(4,3,2,1),\quad (2,4,1,3)\ \textit{and}\ (3,1,4,2).$$

Proof. Since the points $Z\cap L_1$ are harmonic and the cross ratio is $1/q$, we have $q\in\{-1,1/2,2\}$. For each value of q, the subgroup of permutations preserving the cross ratio is a different 2-Sylow subgroup G, each of which contains the Klein four group V. The additional four elements in each G outside V consist of the following permutations, depending on q (in each case the first two permutations listed, when converted to cycle notation, would be 2-cycles, the second two, when converted to cycle notation, would be inverse 4-cycles):

$q=-1$: $(2,1,3,4),(1,2,4,3),(3,4,2,1),(4,3,1,2)$;
$q=1/2$: $(3,2,1,4),(1,4,3,2),(2,3,4,1),(4,1,2,3)$; and
$q=2$: $(4,2,3,1),(1,3,2,4),(2,4,1,3),(3,1,4,2)$.

The first two permutations listed above for each q are excluded by Lemma 3.1 (b). The fixed points of the automorphisms generated by the third permutations listed are presented in Table 2. (The fourth permutation is the inverse of the third, so its matrix is the inverse of the matrix shown in the table and the fixed points are the same as those shown in the table.)

Table 2
The fixed points of the automorphisms α_σ for the 4 cycles σ in the harmonic case.

$\sigma \in S_4$	q	α_σ	fixed points
$(3,4,2,1)$	-1	$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$	$[1 : \pm i]$
$(2,3,4,1)$	$1/2$	$\begin{pmatrix} 0 & 2 \\ -1 & 2 \end{pmatrix}$	$[1 : (1 \pm i)/2]$
$(2,4,1,3)$	2	$\begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}$	$[1 : 1 \pm i]$

The permutations for each q listed in the statement of the theorem come from picking them from Tables 1 and 2. We get two choices for each value of q from Table 2 (namely, the one shown and its inverse), but we get only one choice for each q from Table 1. This is because for each q we must be able to choose at least two permutations from the two tables, one for σ_2^ℓ and one for σ_3^ℓ, and the two choices must have the same fixed points. But for each q, the fixed points from the first table are all different, so we get at most one choice for each q from this table, hence it must be consistent with the fixed points given in the second table. □

Theorem 3.4 has the following important consequence.

Corollary 3.5. *Let Z be the $[4,n]$-half grid under the assumptions of Theorem 3.4; its half grid lines $L_1, \dots, L_n$, and Q is the quadric containing L_1, L_2, L_3. Then Q contains at most one line L_i with $i > 3$; i.e., we must have $r \le 4$.*

Proof. Every line L_i, $2 \le i \le r$, on Q induces a permutation σ_i^ℓ which must be one of the three permutations listed in Theorem 3.4 for the given value of q, but the σ_i^ℓ are different permutations by Lemma 3.1(c). Hence $r - 1 \le 3$; i.e., $r \le 4$. □

We now address the question of how big n can be.

Proposition 3.6. *Let Z be the $[4,n]$-half grid under the assumptions of Theorem 3.4. Let the half grid lines of Z be $L_1, \dots, L_n$, with Q the quadric determined by L_1, L_2, L_3. Then at most 6 of the lines do not lie on Q, and hence $n \le 10$.*

Proof. By Theorem 3.4 and Lemma 3.1(c), there are six possibilities for the pair σ_2^ℓ and σ_3^ℓ for a line ℓ not on Q. But as we show in Remark 3.7, σ_2^ℓ and σ_3^ℓ determine ℓ, and thus there are at most 6 possibilities for ℓ. By Corollary 3.5, there are at most 4 lines on Q, and now we see there are at most 6 not on Q, so $n \le 10$. □

Remark 3.7. Here's how σ_2^ℓ and σ_3^ℓ determine ℓ, given $Z \cap (L_1 \cup L_2 \cup L_3)$. Note that $X_i = Z \cap (L_1 \cup L_i \cup \ell)$ is a $(3,4)$-grid for each line L_i, $i > 1$, on Q. The lines L_1, L_1 and ℓ are grid lines for X_i; the grid lines transverse to L_1, L_1 and ℓ are the 4 lines λ_{ij}, $1 \le j \le 4$, through the points $P_{1,\sigma_i^\ell(j)}$ and $P_{i,j}$. (For example, the line through $P_{1,3}$ and $P_{2,4}$ in Fig. 1 is a grid line for X_2.)

The 4 lines λ_{ij} for a given i are skew since each meets both L_1 and L_i, which are skew, and no two of the λ_{ij} meet L_1 or L_i at the same point. However, if $i \ne i'$, then the 4 lines λ_{ij} meet L_1 in the same four points (namely, $Z \cap L_1$) as do the 4 lines, $\lambda_{i'j}$ and likewise for ℓ. Thus the 8 lines λ_{ij} and $\lambda_{i'j}$ have altogether 8 points of intersection, these being $Z \cap L_1$ and $Z \cap \ell$. Taking $i = 2$ and $i' = 3$, we see σ_2^ℓ and σ_3^ℓ determine the 8 lines λ_{2j} and λ_{3j}, and their intersections off L_1 determine $Z \cap \ell$, and hence ℓ.

Remark 3.8. In contrast to Proposition 3.6, Theorem 1.2 implies that Z has at most 6 half grid lines. However, geproci makes sense in positive characteristics. By [8], if F is a finite field (say $|F| = q$) and K its algebraic closure, the F-rational points of $\mathbb{P}^3_K$ give a $[q+1, q^2+1]$-half grid. By thesis work of Allison Ganger (in preparation), the half grid lines can be chosen to have two transversals. When $q = 3$, we thus get a $[4, 10]$-half grid with two transversals. Thus the bound $n \le 6$ of Theorem 1.2 is specific to characteristic 0, not to the concept of half grids with transversals.

We close this section by showing how L_1, L_2, L_3 and the lines ℓ off Q determine the lines L on Q.

Remark 3.9. Suppose Z is a $[4, n]$-half grid with two transversals and half grid lines $L_1, \ldots, L_n$. Assume Q is the quadric determined by L_1, L_2, L_3. Let ℓ be the line, as in Remark 3.7, determined by a choice of two of the permutations listed for a given q in Theorem 3.4 σ_2^ℓ and σ_3^ℓ. Let σ be the third listed permutation. Let $L = L_i$ for some $i > 3$ with $L_i \subset Q$. Then $\sigma_i^\ell = \sigma$ and, as in Remark 3.7 the line through $P_{1,\sigma_i^\ell(j)}$ and $P_{i,j}$ is a grid line λ_{ij} for X_i. Thus $P_{i,j} \in \lambda_{ij}$, but $P_{i,j}$ is also on the ruling line R_j on Q through $P_{1,j}$ transverse to L_1. Thus $P_{i,j}$ is the intersection of λ_{ij} with R_j. But λ_{ij} is contained in the plane $\Pi_{\sigma_i^\ell(j)}$ spanned by ℓ and $P_{1,\sigma_i^\ell(j)}$, so $P_{i,j}$ is the intersection of $\Pi_{\sigma_i^\ell(j)}$ with R_j. Thus ℓ and the points of Z on L_1 determine the points of Z on L_i and hence they determine L_i.

4. Proof of Theorem 1.2

Assume $L_1, \ldots, L_n$ are the grid lines for Z. We know $n \ge 4$. We begin by choosing coordinates which standardize $Z \cap (L_1 \cup L_2 \cup L_3)$. We then find all possibilities for additional half grid lines and the points on them. Then we check which combinations of these potential half grid lines actually give half grids. And finally we check in each case the resulting half grid is contained in Z_{F_4}, up to projective equivalence.

We now start by choosing explicit coordinates. Note that any three skew lines in $\mathbb{P}^3$ with coordinates $[x : y : z : w]$ can be mapped by a projective transformation to the lines:

$$L_1 : \begin{cases} y &= 0 \\ w &= 0 \end{cases}, \qquad L_2 : \begin{cases} x &= 0 \\ z &= 0 \end{cases}, \qquad L_3 : \begin{cases} y &= x \\ w &= z \end{cases}.$$

These lines are contained in the quadric $Q : \; xw - yz = 0$. By further projective transformations the four harmonic points $P_{1,1}, \ldots, P_{1,4}$ on L_1 can be normalized to $[1 : 0 : 0 : 0]$, $[0 : 0 : 1 : 0]$, $[1 : 0 : 1 : 0]$ and $[1 : 0 : -1 : 0]$, whose cross ratio is -1. Then the rulings on Q determine the points from Z on L_2 and L_3 and we obtain our initial data as:

$$\begin{array}{lll} P_{11} = [1:0:0:0], & P_{21} = [0:1:0:0], & P_{31} = [1:1:0:0], \\ P_{12} = [0:0:1:0], & P_{22} = [0:0:0:1], & P_{32} = [0:0:1:1], \\ P_{13} = [1:0:1:0], & P_{23} = [0:1:0:1], & P_{33} = [1:1:1:1], \\ P_{14} = [1:0:-1:0], & P_{24} = [0:1:0:-1], & P_{34} = [1:1:-1:-1]. \end{array} \tag{4.1}$$

By inspection we see that: the points in column i are indeed on line L_i; the points in each row are collinear; and each point lies on Q. Thus these 12 points are indeed a $(3, 4)$-grid contained in Q.

We now identify the possible lines ℓ among the L_i off Q. By Theorem 3.4, keeping in mind that $q = -1$, there are 6 possibilities for σ_2^ℓ and σ_3^ℓ and by Remark 3.7 these determine the possibilities for ℓ. What we get by an easy direct computation is given in Table 3.

Here we give the line L on Q for each L_i possible line ℓ off Q by applying Remark 3.9. What we get by another easy direct computation is given in Table 4.

We note that the lines $T_1 : (ix - z, y + iw)$ and $T_2 : (-ix - z, y - iw)$ are transversals for L_1, L_2, L_3, and for all lines ℓ and L in Table 4. Moreover, none of the points in (4.1) or in Tables 3 and 4 lie on T_1 or T_2.

Table 3
The possible half grid lines $\ell \not\subset Q$ and the points on ℓ.

σ_2^ℓ	σ_3^ℓ	ideal of ℓ	$Z \cap \ell$
$(2,1,4,3)$	$(3,4,2,1)$	$(y+z, x-w)$	$[1:-1:1:1], [1:1:-1:1]$ $[0:1:-1:0], [1:0:0:1]$
$(2,1,4,3)$	$(4,3,1,2)$	$(y-z, x+w)$	$[0:1:1:0], [1:0:0:-1]$ $[-1:1:1:1], [1:1:1:-1]$
$(3,4,2,1)$	$(2,1,4,3)$	$(y-z+w, x-z+2w)$	$[2:1:0:-1], [0:1:2:1]$ $[1:1:1:0], [-1:0:1:1]$
$(3,4,2,1)$	$(4,3,1,2)$	$(y-2z+w, x-z+w)$	$[1:1:0:-1], [0:1:1:1]$ $[1:2:1:0], [-1:0:1:2]$
$(4,3,1,2)$	$(2,1,4,3)$	$(y+z-w, x+z-2w)$	$[2:1:0:1], [0:-1:2:1]$ $[1:0:1:1], [1:1:-1:0]$
$(4,3,1,2)$	$(3,4,2,1)$	$(y+2z-w, x+z-w)$	$[1:1:0:1], [0:-1:1:1]$ $[1:0:1:2], [1:2:-1:0]$

Table 4
The line $L \subset Q$ corresponding to each ℓ and the points on L.

σ	ideal of L	ideal of ℓ	$Z \cap L$
$(4,3,1,2)$	$(x+y, z+w)$	$(y+z, x-w)$	$[1:-1:-1:1], [-1:1:-1:1]$ $[0:0:-1:1], [1:-1:0:0]$
$(3,4,1,2)$	$(x+y, z+w)$	$(y-z, x+w)$	$[-1:1:1:-1], [1:-1:1:-1]$ $[0:0:1:-1], [1:-1:0:0]$
$(4,3,1,2)$	$(x-2y, z-2w)$	$(y-z+w, x-z+2w)$	$[2:1:-2:-1], [2:1:2:1]$ $[0:0:2:1], [2:1:0:0]$
$(2,1,4,3)$	$(2x-y, 2z-w)$	$(y-2z+w, x-z+w)$	$[1:2:-1:-2], [1:2:1:2]$ $[0:0:1:2], [1:2:0:0]$
$(3,4,1,2)$	$(x-2y, z-2w)$	$(y+z-w, x+z-2w)$	$[2:1:-2:-1], [2:1:2:1]$ $[0:0:2:1], [2:1:0:0]$
$(2,1,4,3)$	$(2x-y, 2z-w)$	$(y+2z-w, x+z-w)$	$[1:2:-1:-2], [1:2:1:2]$ $[0:0:1:2], [1:2:0:0]$

Thus T_1 or T_2 are transversals for the half grid lines for any half grid Z whose half grid lines are among the lines L_1, L_2, L_3, and the lines ℓ and L in Table 4, and none of the points of Z are on either transversal.

Let $S_{0,0}$ be the initial 12 points, given in (4.1), let $S_{1,i}$ be the 4 points given in row i of Table 3 and let $S_{2,j}$ be the 4 points given in row j of Table 4. Note that $S_{2,1} = S_{2,2}$, $S_{2,3} = S_{2,5}$, and $S_{2,4} = S_{2,6}$. We also note that the lines in Table 3 are skew; in particular, the sets $S_{1,i}$ are pairwise disjoint. In fact, all of the lines defined by either table or by the rows of (4.1) are skew.

We now see that any $[4, n]$-half grid Z with two transversals but no points on the transversals is projectively equivalent to a union of some selection of the sets $S_{i,j}$, where this union includes $S_{0,0}$ and at least one of the sets $S_{1,j}$ and at most one of the sets $S_{1,j}$.

In particular, note that $Z_1 = S_{0,0} \cup S_{1,1} \cup S_{1,2} \cup S_{2,1}$ is exactly Z_{F_4}; its points are the roots of the F_4 root system, regarded as points in $\mathbb{P}^3$. In particular, Z is indeed geproci [1]. If any additional sets $S_{i,j}$ are added to Z_1, then what we get will no longer be geproci, because at most 4 of the half grid lines of Z can lie on Q (thus no other $S_{2,j}$ can be added) and (according to Table 4) no other set $S_{1,i}$ is compatible with $S_{2,1}$. Thus Z_1 is a maximal union which is a half grid. By Lemma 2.1, $S_{0,0} \cup S_{1,1} \cup S_{1,2}$, $S_{0,0} \cup S_{1,1} \cup S_{2,1}$, $S_{0,0} \cup S_{1,2} \cup S_{2,1}$, $S_{0,0} \cup S_{2,1}$ and $S_{0,0} \cup S_{1,2}$ are also half grids, with the last two being minimal half grid unions.

We now show that $Z_2 = S_{0,0} \cup S_{1,3} \cup S_{1,5} \cup S_{2,3}$ and $Z_3 = S_{0,0} \cup S_{1,4} \cup S_{1,6} \cup S_{2,4}$ are half grids, projectively equivalent to Z_1 and hence to Z_{F_4}.

For this purpose it is useful to note that $S_{0,0} \cup S_{2,j}$ is, for each j, a $(4,4)$-grid. In fact the points of $S_{2,j}$ can be added as a fourth column to (4.1) so that every row and column gives a set of 4 collinear points. If one does this the cross ratio of each column is -1 and the cross ratio of the rows is q, with $q = -1$ if $j = 1$ but $q = 1/2$ if $j = 3$ and $q = 2$ if $j = 4$.

Thus the $(4,4)$-grid is "biharmonic" with "bi-cross ratios" of either $(-1,-1)$, $(-1,1/2)$ or $(-1,2)$. Regarding the grid as a subset of $\mathbb{P}^1 \times \mathbb{P}^1$, there is an automorphism of $V = \mathbb{P}^1 \times \mathbb{P}^1$ (obtained from an automorphism acting on the second $\mathbb{P}^1$) which takes the grid for $j = 1$ to the grids with $j = 3$ and $j = 4$. This induces an automorphism on the global sections of $\mathcal{O}_V(1,1)$ which in turn induces an automorphism of $\mathbb{P}^3$ preserving Q, showing that the three $(4,4)$-grids are projectively equivalent. It also takes Z_1 to Z_2 and Z_3 (due to Z_1 being the unique $[4,6]$-half grid containing $S_{0,0} \cup S_{1,1} \cup S_{1,2}$).

We now also see that any $[4,6]$-half grid with two transversals and no points in the transversal, where four of the half grid lines give a $(4,4)$-grid, projectively equivalent to Z_{F_4}.

There remains the question of which unions of $S_{0,0}$ with a selection of 3 or more of the sets $S_{1,i}$ are half grids. There are only two additional maximal unions, namely: $Z_4 = S_{0,0} \cup S_{1,1} \cup S_{1,5} \cup S_{1,6}$ and $Z_5 = S_{0,0} \cup S_{1,2} \cup S_{1,3} \cup S_{1,4}$.

Let $S_{0,i}$ be the points given in column i of (4.1). Then for Z_4 it turns out (among others) that $S_{0,1} \cup S_{0,2} \cup S_{1,5} \cup S_{1,6}$ lie on a quadric, and for Z_5 it turns out (among others) that $S_{0,1} \cup S_{0,2} \cup S_{1,3} \cup S_{1,4}$ lie on a quadric. Thus Z_4 and Z_5 are also projectively equivalent to Z_{F_4}.

5. Remarks and questions

In this short final section we discuss where future research is needed to expand our understanding of half grids, and provide some specific questions. There are no new results in this section, but we include it as a resource for researchers interested in advancing the theory.

Theorem 1.2 shows that every $[4,s]$-half grid with two transversals and no points on the transversals is contained in the $[4,6]$-half grid Z_{F_4}. Note by the proof of the theorem we see that Z_{F_4} is obtained from a $(4,4)$-grid by adding two sets of 4 collinear points off the quadric containing the grid. In [1], a procedure, called the *standard construction*, is given for producing half grids from grids by generalizing this fact. The standard construction always produces examples of half grids whose half grid lines have transversals. Here we show there is no $[4,s]$-half grid with $s > 6$ even if we drop the condition on there being two transversals, and we raise the general question of maximality of the half grids given by the standard construction.

Assume $m \geq 3$. We now recall the standard construction in more detail. It constructs examples of $[m,n]$-half grids (where $n = m+1$ if m is odd and $n = m+2$ if m is even) of m points on each of n lines. It starts with a $(2,2)$-grid; let S_1, S_2, T_1, T_2 be the grid lines, so S_1 and S_2 are skew, T_1 and T_2 are skew, and S_i and T_j meet in a single point for each i and j.

There is a linear action of $\mathbb{C}^*$ on $\mathbb{P}^3$ associated to T_1 and T_2 given as follows. The action is the identity on $T_1 \cup T_2$. For each point $p \notin T_1 \cup T_2$, there is a unique line L_p through p meeting both T_1 and T_2. We can choose a coordinate system on $L_p \cong \mathbb{P}^1$ such that $T_1 \cap L_p$ is $[0:1]$, $T_2 \cap L_p$ is $[1:0]$, and $p = [1:1]$. Then for each $u \in \mathbb{C}^*$ we set $up = [1:u]$. If we choose a coordinate system on $\mathbb{P}^3$ such that $T_1 : x, y = 0$ and $T_2 : z, w = 0$, then the action just defined has matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{pmatrix}$.

There is similarly a linear action of $\mathbb{C}^*$ on $\mathbb{P}^3$ associated to S_1 and S_2 given analogously. In particular, the action is the identity on $S_1 \cup S_2$. For each point $p \notin S_1 \cup S_2$, there is a unique line L_p through p meeting both S_1 and S_2. We can choose a coordinate system on $L_p \cong \mathbb{P}^1$ such that $S_1 \cap L_p$ is $[0:1]$, $S_2 \cap L_p$ is $[1:0]$, and $p = [1:1]$. Then for each $u \in \mathbb{C}^*$ we set $up = [1:u]$. If we choose a coordinate system on $\mathbb{P}^3$ such that S_1 is $x = z = 0$ and S_2 is $y = w = 0$, then the action just defined has matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & u \end{pmatrix}$.

The subgroup $U_m \subset \mathbb{P}GL_4(\mathbb{C})$ generated by the two matrices above, where u is a primitive mth root of 1, is isomorphic to $C_m \times C_m$, where C_m is the multiplicative cyclic group of order m. The orbit of a point contained in the plane spanned by the lines S_i, T_j is contained in that plane, but the orbit of a point p_{00} not contained in any of those four planes is an (m,m)-grid G. Indeed, by appropriately scaling the variables x, y, z, w, the point p_{00} has coordinates $[1:1:1:1]$ and the U_m-orbit of p_{00} consists of the points

$p_{ij} = [1 : u^j : u^i : u^{i+j}]$ for $0 \le i, j < m$. Note that this set of points is an (m, m)-grid. To this end note that given i, the points $[1 : u^j : u^i : u^{i+j}]$ for $0 \le j < m$ are collinear; denote the line containing them by M_i (it is defined by $w - u^i y = u^i x - z = 0$). Similarly, given j, the points $[1 : u^j : u^i : u^{i+j}]$ for $0 \le i < m$ are also collinear; denote the line containing them by L_j (it is defined by $w - u^j z = u^j x - y = 0$). The lines M_i are pair-wise skew, as are the lines L_j, but $M_i \cap L_j = \{p_{ij}\}$.

The question now is: what collinear sets of m points can be added to G to obtain a half grid of m points on $m + 1$ lines. In terms of the coordinates used above, the standard construction gives two subsets: Y_1, consisting of the points $[-1 : 0 : 0 : u^j]$ for $0 \le j < m$, and Y_2, consisting of the points $[0 : -1 : u^j : 0]$ for $0 \le j < m$. For any $m \ge 3$, $G \cup Y_i$ is an $[m, m + 1]$-half grid for either $i = 1$ or $i = 2$. When m is even, then $G \cup Y_1 \cup Y_2$ is an $[m, m + 2]$-half grid.

There remains the question of whether Y_1 and Y_2 are the only two subsets. To explore this question, note that a necessary condition for a set Z to be an $[m, r]$-half grid on r lines $A_1, \dots, A_r$, is for $Z \cap (A_i \cup A_j \cup A_k)$ to be a $(3, m)$-grid. So suppose L is a line containing a set of m collinear points $q_1, \dots, q_m$ such that $Z = G \cup \{q_1, \dots, q_m\}$ is an $[m, m + 1]$-half grid with half grid lines $L_0, \dots, L_{m-1}$ and L. We will not assume that $L \cap M_i = \varnothing$ for all i (although this is the case for Y_1, Y_2 in the standard construction). Since L is not contained in the quadric containing G (because Z is a half grid), and since $m \ge 3$, there must be a line M_i disjoint from L.

The lines M_i and L_j (for any j) span the plane Π_{ij} defined by $w - u^j z - u^i y + u^{i+j} x = 0$. Pick any point $p_{ik} \in M_i$ (but not p_{ij} so $k \ne j$). Then $p_{ik} \in L_k$, so the points of Z on L, L_j and L_k give a $(3, m)$-grid, and this grid has a transverse grid line $T \subset \Pi_{ij}$ through p_{ik} which meets L in a point q_r for some r (since the points q_r are the points of the $(3, m)$-grid on L). But L is skew to L_j, so L meets Π_{ij} in a single point, which thus must be the same point q_r where T meets L. This is true for each point p_{ik}, $k \ne j$, so the point $L \cap \Pi_{ij}$ is a point of concurrence of $m - 1$ lines where each line goes through the point $L \cap \Pi_{ij}$ and through a point of both M_i and L_j (but not through p_{ij}).

Question 5.1. *Given grid lines M_i and L_j, how many points of concurrence in the plane Π_{ij} are there (meaning a point $q \in \Pi_{ij}$ not on $M_i \cup L_j$ such that for each point $p_{ik} \in M_i$, $k \ne j$, the line through q and p_{ik} also contains a point $p_{lj} \in L_j$)?*

For a given m, this is a purely computational question. We know there are at least two, namely $Y_1 \cap \Pi_{ij} = \{[-1 : 0 : 0 : u^{i+j}]\}$ and $Y_2 \cap \Pi_{ij} = \{[0 : -1 : u^{i-j} : 0]\}$, based on the fact the standard construction gives an $[m, m+1]$-half grid. If these are the only two for some choice of i and j, then there are only two for each i and j (since U_m is a group of linear automorphisms of $\mathbb{P}^3$ which acts transitively on the points p_{ij}). And if there are only two, then the standard configuration with m points per line is contained in no larger half grid with m points per line, even if we do not require transversals for the half grid lines.

We checked by brute force computation for $3 \le m \le 11$ and indeed there are only two points of concurrency in these cases. Thus the $[m, r]$-half grid given by the standard construction (with $r = m + 1$ if m is odd and $r = m + 2$ if m is even) is contained in no $[m, s]$-half grid with $s > r$ when $3 \le m \le 11$.

Question 5.2. *Is the previous sentence true for all m?*

We conclude with a conjecture, which if verified, would finish the classification of half grids with two transversals:

Conjecture 5.3. *Let Z be an $[m, n]$-half grid with two transversals where the points on each half grid line are a single C_m orbit. Then Z is contained in an $[m, r]$-half grid given by the standard construction. (So, in particular, with $r = m + 1$ if m is odd and $r = m + 2$ if m is even.)*

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Declaration of competing interest

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References

[1] L. Chiantini, Ł. Farnik, G. Favacchio, B. Harbourne, J. Migliore, T. Szemberg, J. Szpond, Configurations of points in projective space and their projections, arXiv:2209.04820.

[2] L. Chiantini, Ł. Farnik, G. Favacchio, B. Harbourne, J. Migliore, T. Szemberg, J. Szpond, On the classification of certain geproci sets, in: U. Nagel, K. Adiprasito, R. Di Gennaro, S. Faridi, S. Murai (Eds.), Lefschetz Properties. SLP-WLP 2022, in: Springer INdAM Series, vol. 59, Springer, Singapore, 2024.

[3] L. Chiantini, Ł. Farnik, G. Favacchio, B. Harbourne, J. Migliore, T. Szemberg, J. Szpond, Geproci sets and the combinatorics of skew lines in $\mathbb{P}^3$, arXiv:2308.00761.

[4] L. Chiantini, J. Migliore, Sets of points which project to complete intersections, and unexpected cones, Trans. Am. Math. Soc. 374 (4) (2021) 2581–2607, With an appendix by A. Bernardi, L. Chiantini, G. Denham, G. Favacchio, B. Harbourne, J. Migliore, T. Szemberg and J. Szpond.

[5] C. Ciliberto, T. Dedieu, F. Flamini, R. Pardini, C. Galati, S. Rollenske, Open problems, Boll. Unione Mat. Ital. 11 (1) (2018) 5–11.

[6] D. Eisenbud, J. Harris, 3264 and All That–a Second Course in Algebraic Geometry, Cambridge University Press, 2016.

[7] B. Harbourne, J. Migliore, U. Nagel, Z. Teitler, Unexpected hypersurfaces and where to find them, Mich. Math. J. 70 (2) (2021) 301–339.

[8] J. Kettinger, The geproci property in positive characteristic, Proc. Am. Math. Soc. 152 (2024) 3229–3242.

[9] P. Pokora, T. Szemberg, J. Szpond, Unexpected properties of the Klein configuration of 60 points in $\mathbb{P}^3$, Mich. Math. J. 74 (3) (2024) 599–615.

[10] F. Polizzi, D. Panov, When is a general projection of d^2 points in $\mathbb{P}^3$ a complete intersection?, MathOverflow Question 67265, 2011.