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On subalgebras of the Griess algebra with alternating Miyamoto group

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ABSTRACT

We use Majorana representations to study the subalgebras of the Griess algebra that have shape $(2B, 3A, 5A)$ and whose associated Miyamoto groups are isomorphic to A_n. We prove that these subalgebras exist only if $n \in \{5, 6, 8\}$. The case $n = 5$ was already treated by Ivanov, Seress, McInroy, and Shpectorov. In case $n = 6$ we prove that these algebras are all isomorphic and provide their precise description. In case $n = 8$ we prove that these algebras do not arise from standard Majorana representations.

1. Introduction

This paper is a contribution to a project aimed at classifying, via Majorana representations, the subalgebras of the Griess algebra whose associated Miyamoto groups appear as factors over their centers of *monstralizers* (see [18]) of the subgroups H of the Monster group M that are isomorphic to the alternating group A_5 (see [13, 5.11.3]). In [18,

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Table 3], Norton listed the conjugacy classes of such subgroups H and, for each such subgroup, the isomorphism type of its monstralizer. There are 8 such classes, labeled with a triple $(2X, 3Y, 5Z)$ (with (X, Y, Z) in $\{A, B\} \times \{A, B, C\} \times \{A, B\}$), whose entries are the conjugacy classes (in Atlas notation [2]) of the 2, 3, and 5 elements of H in M. Majorana representations of monstralizers of an A_5-subgroup of type $(2A, 3A, 5A)$ (which are isomorphic to A_{12}) have been considered in [1,3,5]. In this paper we deal with the class of A_5-subgroups of type $(2B, 3A, 5A)$, whose monstralizers are isomorphic to $2.M_{22}.2$. An important inductive step towards this goal is to determine the Majorana representation induced on a submaximal subgroup $\overline{G} = Z \times G$ in the derived subgroup of such a monstralizer, where Z is a group of order 2 generated by z and G is isomorphic to A_6 (see [2, p. 39]). In this case, the set of Fischer (i.e. $2A$ in Atlas notation) involutions in $\overline{G}$ is the *diagonal* set $D := \{zs \mid s \text{ is a bitransposition in } A_6\}$ while the involutions in G are Conway (i.e. $2B$) involutions. Since the product of two elements of D does not lie in D and, by [13, p.258], central elements are contained in the kernel of a Majorana representation, by [10, Axiom (M8)], the action of $\overline{G}$ on the Griess algebra, induces a Majorana representation of $\overline{G}/\langle z \rangle$ (i.e. A_6) of shape $(2B, 3A, 5A)$ (see definition in Section 2).

More generally, in this paper we consider Majorana representations of the alternating groups A_n, $n \geq 5$, with shape $(2B, 3Y, 5A)$, with Y in $\{A, C\}$. The case $n = 5$ has been investigated by Ivanov, Seress, McInroy, and Shpectorov (see [12,22,15]). By [18, Table 3], all the Majorana representations of A_5 are based on embeddings of A_5 or $2 \times A_5$ in M. Recall from [3] that a Majorana representation of A_n is *standard* if the bitranspositions of A_n are Majorana involutions. In Section 3 we prove

Theorem 1. *The alternating group A_n has*

(a) *a standard Majorana representation of shape $(2B, 3Y, 5A)$, for every $Y \in \{A, C\}$, if and only if $n \in \{5, 6\}$.*
(b) *a nonstandard Majorana representation of shape $(2B, 3A, 5A)$, if and only if $n = 8$.*

The case $n = 6$ and shape $(2B, 3A, 5A)$ has been addressed in [15] with the use of the expansion algorithm, but the computation of the representation could not be completed within a reasonable time. In this paper we use a different approach, building up inductively a representation V of A_6 with shape $(2B, 3A, 5A)$ from its proper subalgebras whose associated Miyamoto groups are isomorphic to the maximal subgroups A_5 or S_4 of A_6. Moreover, we introduce *dormant 4-axes* (see the definition in Section 2) which are likely to be useful also in other situations, e.g. for the still open classification of the Majorana representations of A_n for $8 \leq n \leq 11$ (see [5, Conjecture 1]). Dormant 4-axes are particular idempotents of length 2 in V which turn to be 4-axes whenever V is realised as a subalgebra of the Griess algebra. This fact will be discussed in more detail in Section 2. Accordingly to what happens in the Griess algebra, we assume that dormant 4-axes are in one-to-one correspondence with the cyclic subgroups of order 4 of

A_6. We call a representation of A_6, with shape $(2B, 3Y, 5A)$ ($Y \in \{A, C\}$) satisfying this extra hypothesis a *diagonal Majorana representation.*

Theorem 2. *The alternating group A_6 has a unique diagonal Majorana representation of shape $(2B, 3A, 5A)$. This representation is 2-closed of dimension 121 and based on the embedding of $2 \times A_6$ into the Monster.*

For a partition λ of 6, denote by $\mathfrak{S}^\lambda$ the Specht module associated to λ.

Theorem 3. *Let V be the diagonal Majorana representation of A_6 with shape $(2B, 3A, 5A)$. The structure of $\mathbb{R}[A_6]$-module of V lifts naturally to a structure of $\mathbb{R}[S_6]$-module and, with the above notation, the decomposition of V into irreducible $\mathbb{R}[S_6]$-submodules is the following*

$$V \cong 3\mathfrak{S}^{(6)} \oplus 3\mathfrak{S}^{(5,1)} \oplus 5\mathfrak{S}^{(4,2)} \oplus 2\mathfrak{S}^{(3,2,1)} \oplus \mathfrak{S}^{(3^2)} \oplus 3\mathfrak{S}^{(2^3)} \oplus \mathfrak{S}^{(2,1^4)} \oplus \mathfrak{S}^{(1^6)}.$$

While this paper was in preparation, a Majorana algebra of dimension 121 and shape $(2B, 3A, 5A)$ was constructed in [7] as a subalgebra of a larger algebra. Theorem 2 answers, under the above extra hypothesis, the question posed in [7, p.703] about the uniqueness of that algebra.

In Section 2 we give the basic definitions and recall the known results that will be used in the sequel. Theorem 1 is proved in Section 3. From Section 4 onwards, we fix a diagonal Majorana representation V of A_6 with shape $(2B, 3A, 5A)$. In Section 4 we compute the inner products between axes and odd axes in V and determine the dimensions of the corresponding linear spans. In Section 5 we consider the subalgebras of V corresponding to the maximal subgroups of A_6 isomorphic to S_4 or A_5 and, by examining their intersections, we derive relations between axes, odd axes, and dormant 4-axes which are used in Section 6 to show that all dormant 4-axes are contained in the linear span U of 2-, 3-, and 4-axes. In Section 7 we prove that U has codimension 1 in the linear span V° of all axes and odd axes. Moreover, we show that the $\mathbb{R}[A_6]$-module structure of V° extends naturally to $\mathbb{R}[S_6]$ and determine its decomposition into irreducible $\mathbb{R}[S_6]$-modules. Finally, in Section 8, we show that the product in V is uniquely determined and $V^\circ = V$.

Notice that, in this paper, we use the exponential notation for automorphisms. In particular permutations act on the right.

2. Preliminary results

Let V be a real vector space endowed with a (non-associative) commutative multiplication, and a positive definite symmetric bilinear form $(\ ,\ )_V$ which associates with the algebra multiplication, i.e. for every $u, v, w \in V$

$$(u \cdot v, w)_V = (u, v \cdot w)_V.$$

Table 1
Monster fusion law.

$\star$	1	0	$\frac{1}{4}$	$\frac{1}{32}$
1	1	$\emptyset$	$\frac{1}{4}$	$\frac{1}{32}$
0	$\emptyset$	0	$\frac{1}{4}$	$\frac{1}{32}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$1, 0$	$\frac{1}{32}$
$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$1, 0, \frac{1}{4}$

For $a \in V$ let ad_a be the adjoint operator $v \mapsto av$ and, for $\lambda \in \mathbb{R}$, denote by V^a_λ the λ-eigenspace for ad_a, i.e. $V^a_\lambda := \{v \in V \mid av = \lambda v\}$. A vector $a \in V$ is called a *Majorana axis* if

Ax1. $a \cdot a = a$ and $(a, a)_V = 1$
Ax2. ad_a is semisimple with spectrum $S := \{1, 0, \frac{1}{4}, \frac{1}{32}\}$
Ax3. $V^a_1 = \{\lambda a \mid \lambda \in \mathbb{R}\}$
Ax4. for every $\lambda, \mu \in S$, the eigenspaces V^a_λ, V^a_μ satisfy the Monster fusion law in Table 1, in the sense that

$$V^a_\lambda \cdot V^a_\mu \subseteq \bigoplus_{\nu \in \lambda \star \mu} V^a_\nu.$$

Conditions Ax1 - Ax4 yield that, for every Majorana axis a, the linear map $\tau_a : V \to V$ such that

$$v^{\tau_a} = \begin{cases} v & \text{if } v \in V^a_1 \oplus V^a_0 \oplus V^a_{\frac{1}{4}} \\ -v & \text{if } v \in V^a_{\frac{1}{32}} \end{cases}$$

is an involutory algebra automorphism of V which preserves the bilinear form. The map τ_a is called *Miyamoto involution (or Majorana involution)* associated to a (see [16]).

The algebra V is called *Majorana algebra* if it is generated (as an algebra) by a set $\mathcal{A}$ of Majorana axes. Usually Norton inequality is assumed, i.e. for every $u, v \in V$

$$(u \cdot v, u \cdot v,)_V \leq (u \cdot u, v \cdot v)_V.$$

In this paper, however, we will not make use of the Norton inequality.

Let V be a Majorana algebra generated by the set of axes $\mathcal{A}$. The *automorphisms* of V are the algebra automorphisms that are also isometries of the vector space V. In particular, Miyamoto involutions are automorphisms of V. Define the *Miyamoto group* $\mathrm{Miy}(\mathcal{A})$ *of* V *with respect to* $\mathcal{A}$, as follows:

$$\mathrm{Miy}(\mathcal{A}) := \langle \tau_a \mid a \in \mathcal{A} \rangle. \tag{1}$$

Since, by [14, Lemma 3.5] $\mathrm{Miy}(\mathcal{A}) = \mathrm{Miy}(\mathcal{A}^{\mathrm{Miy}(\mathcal{A})})$, one usually assumes (as we shall do in this paper) that

$$\mathcal{A} = \mathcal{A}^{\mathrm{Miy}(\mathcal{A})}.$$

Let

$$G := \mathrm{Miy}(\mathcal{A}) \quad \text{and} \quad T := \{\tau_a \mid a \in \mathcal{A}\},$$

the quintet

$$(V, (\,,\,)_V, \cdot, \mathcal{A}, G),$$

is called a *Majorana representation of G on V with respect to T*. When the setting is clear, we shall simply say that V is a Majorana representation of G. The *dimension* of the representation is the dimension of V. For $n \in \mathbb{N}$, the *n-closure* of the representation is the subspace of V spanned by all products of length n of elements in $\mathcal{A}$. The representation is *n-closed* if it is equal to its n-closure. If H is a subgroup of G, denote by $V(H)$ the subalgebra of V generated by all the axes $a \in \mathcal{A}$ such that $\tau_a \in H$.

The Griess algebra $V^\sharp$ is a Majorana algebra and, if $2A$ denotes the set of $2A$-axes in $V^\sharp$, then $(V^\sharp, (\,,\,)_{V^\sharp}, \cdot, 2A, M)$ is a Majorana representation of the Monster group M with respect to the set of Fischer involutions (see e.g. [9]). If G is a subgroup of M generated by a set T of Fischer involutions and $\mathcal{A}$ is the set of axes corresponding to T, then $(V^\sharp, (\,,\,)_{V^\sharp}, \cdot, \mathcal{A}, G)$ is a Majorana representation of G and we say that it is *based on an embedding into the Monster*.

A fundamental result in the Majorana theory is the combination of Norton's classification of subalgebras of the Griess algebra generated by two axes [17] and a similar result of Sakuma [21] for VOA's, relying on earlier work of Miyamoto, reproved independently in [10] in the context of Majorana Theory.

Norton-Sakuma Theorem [17,21] *Let V be a Majorana algebra, let a_0 and a_1 be two axes in V, $\rho := \tau_{a_0}\tau_{a_1}$, and $N := |\rho|$. For $i \in \mathbb{N}$, let $a_{2i} = a_0^{\rho^i}$ and $a_{2i+1} = a_1^{\rho^i}$. Then $N \leq 6$ and $\langle\langle a_0, a_1 \rangle\rangle$ is one of the nine algebras in Table 2.*

Remark 2.1. The vectors a_ρ, a_{ρ^2}, a_{ρ^3} appearing in Table 2 are defined as follows:

$$a_\rho := a_0 + a_1 - 8a_0a_1, \quad a_{\rho^2} := a_0 + a_2 - 8a_0a_2, \quad a_{\rho^3} := a_0 + a_3 - 8a_0a_3.$$

An easy computation shows that the vectors a_ρ, a_{ρ^2}, a_{ρ^3} are still axes. Thus, by the Norton-Sakuma Theorem, Norton-Sakuma algebras of type $1A$, $2A$, $2B$, $3C$, and $4B$ are linearly spanned by axes, while for those of type NA, for $N \in \{3, 4, 5, 6\}$, a further vector (denoted by u_ρ, v_ρ, w_ρ, and u_{ρ^2}, respectively) is needed.

Table 2
The nine types of Norton-Sakuma algebras.

Type	Basis	Structure constants	Inner products
$1A$	a_0	$a_0 \cdot a_0 = a_0$	$(a_0, a_0) = 1$
$2A$	a_0, a_1, a_ρ	$a_0 \cdot a_1 = \frac{1}{2^3}(a_0 + a_1 - a_\rho)$, $a_0 \cdot a_\rho = \frac{1}{2^3}(a_0 + a_\rho - a_1)$, $a_\rho \cdot a_\rho = a_\rho$	$(a_0, a_1) = \frac{1}{2^3}$, $(a_0, a_\rho) = \frac{1}{2^3}$, $(a_1, a_\rho) = \frac{1}{2^3}$
$2B$	a_0, a_1	$a_0 \cdot a_1 = 0$	$(a_0, a_1) = 0$
$3A$	a_{-1}, a_0, a_1, u_ρ	$a_0 \cdot a_1 = \frac{1}{2^5}(2a_0 + 2a_1 + a_{-1}) - \frac{3^3 \cdot 5}{2^{11}} u_\rho$, $a_0 \cdot u_\rho = \frac{1}{3^2}(2a_0 - a_1 - a_{-1}) + \frac{5}{2^5} u_\rho$, $u_\rho \cdot u_\rho = u_\rho$	$(a_0, a_1) = \frac{13}{2^8}$, $(a_0, u_\rho) = \frac{1}{4}$, $(u_\rho, u_\rho) = \frac{2^3}{5}$
$3C$	a_{-1}, a_0, a_1	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-1})$	$(a_0, a_1) = \frac{1}{2^6}$
$4A$	a_{-1}, a_0, a_1, a_2, v_ρ	$a_0 \cdot a_1 = \frac{1}{2^6}(3a_0 + 3a_1 + a_{-1} + a_2 - 3v_\rho)$, $a_0 \cdot a_2 = 0$, $a_0 \cdot v_\rho = \frac{1}{2^4}(5a_0 - 2a_1 - 2a_{-1} - a_2 + 3v_\rho)$, $v_\rho \cdot v_\rho = v_\rho$	$(a_0, a_1) = \frac{1}{2^5}$, $(a_0, a_2) = 0$, $(a_0, v_\rho) = \frac{3}{2^3}$, $(v_\rho, v_\rho) = 2$
$4B$	a_{-1}, a_0, a_1, a_2, a_{ρ^2}	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-1} - a_2 + a_{\rho^2})$, $a_0 \cdot a_2 = \frac{1}{2^3}(a_0 + a_2 - a_{\rho^2})$	$(a_0, a_1) = \frac{1}{2^6}$, $(a_0, a_2) = \frac{1}{2^3}$, $(a_0, a_{\rho^2}) = \frac{1}{2^3}$
$5A$	a_{-2}, a_{-1}, a_0, a_1, a_2, w_ρ	$a_0 \cdot a_1 = \frac{1}{2^7}(3a_0 + 3a_1 - a_{-1} - a_2 - a_{-2}) + w_\rho$, $a_0 \cdot a_2 = \frac{1}{2^7}(3a_0 + 3a_2 - a_1 - a_{-1} - a_{-2}) - w_\rho$, $a_0 \cdot w_\rho = \frac{7}{2^{12}}(a_1 + a_{-1} - a_2 - a_{-2}) + \frac{7}{2^5} w_\rho$, $w_\rho \cdot w_\rho = \frac{5^2 \cdot 7}{2^{19}}(a_0 + a_1 + a_{-1} + a_2 + a_{-2})$	$(a_0, a_1) = \frac{3}{2^7}$, $(a_0, w_\rho) = 0$, $(w_\rho, w_\rho) = \frac{5^3 \cdot 7}{2^{19}}$
$6A$	a_{-2}, a_{-1}, a_0, a_1, a_2, a_3, a_{ρ^3}, u_{ρ^2}	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-1} - a_2 - a_{-2} - a_3 + a_{\rho^3}) + \frac{3^2 \cdot 5}{2^{11}} u_{\rho^2}$, $a_0 \cdot a_2 = \frac{1}{2^5}(2a_0 + 2a_2 + a_{-2}) - \frac{3^3 \cdot 5}{2^{11}} u_{\rho^2}$, $a_0 \cdot a_3 = \frac{1}{2^3}(a_0 + a_3 - a_{\rho^3})$, $a_0 \cdot u_{\rho^2} = \frac{1}{3^2}(2a_0 - a_2 - a_{-2}) + \frac{5}{2^5} u_{\rho^2}$, $a_{\rho^3} \cdot u_{\rho^2} = 0$	$(a_0, a_1) = \frac{5}{2^8}$, $(a_0, a_2) = \frac{13}{2^8}$, $(a_0, a_3) = \frac{1}{2^3}$, $(a_{\rho^3}, u_{\rho^2}) = 0$

The extra vectors u_ρ, v_ρ, w_ρ, resp. u_{ρ^2} are called *N-axes*, for $N \in \{3, 4, 5\}$, resp. 3-*axis*, for $N = 6$, or simply *odd axes*, when N needs not to be specified (note that a 4-axis is an odd axis, according to our definition) and are defined as follows (cf. Table 2):

$u_\rho := \frac{2^6}{3^3 \cdot 5}(2a_0 + 2a_1 + a_{-1}) - \frac{2^{11}}{3^3 \cdot 5} a_0 a_1;$
$v_\rho := a_0 + a_1 + \frac{1}{3}(a_{-1} + a_2) - \frac{2^6}{3} a_0 a_1;$
$w_\rho := -\frac{1}{2^7}(3a_0 + 3a_1 - a_{-1} - a_2 - a_{-2}) + a_0 a_1;$
$u_{\rho^2} := \frac{2^6}{3^3 \cdot 5}(2a_0 + 2a_2 + a_{-2}) - \frac{2^{11}}{3^3 \cdot 5} a_0 a_2.$

For $N \in \{2,3,4,5\}$, denote by $V^{(NA)}$ the subspace of V spanned by the N-axes (here the 2-axes are the axes, i.e. the elements of $\mathcal{A}$). Note that, by Table 2, the Norton-Sakuma algebras are 2-closed. More generally, the 2-closure of V, which we shall denote by V°, is the linear span of the axes and the odd axes of V.

Remark 2.2. Let $\langle\langle a_0, a_1\rangle\rangle$ be a subalgebra of the Griess algebra of type NA, with $N \in \{3,4,5,6\}$, generated by two axes a_0 and a_1, and let $\rho := \tau_{a_0}\tau_{a_1}$. Then the odd axis u_ρ, v_ρ, w_ρ, or u_{ρ^2}, depends only on the conjugacy class of ρ in the dihedral group $\langle \tau_{a_0}, \tau_{a_1}\rangle$. In particular, if $N \in \{3,4,6\}$, we have only one conjugacy class, if $N = 5$ we have two conjugacy classes with representatives ρ and ρ^2, respectively. In the latter case $w_{\rho^2} = -w_\rho$ (see [17, Introduction and Table 1]).

This fact, together with the correspondence between axes and Miyamoto involutions is axiomatized in the following conditions, usually known as Condition 2A and Axiom M8, respectively.

(2A) For every pair of axes $a, b \in \mathcal{A}$, if $\tau_a\tau_b \in T$, then $\langle\langle a, b\rangle\rangle$ has type $2A$ and $\tau_a\tau_b = \tau_c$, where $c = a + b - 8ab$.

(M8) The vectors a_ρ, a_{ρ^2}, a_{ρ^3} in the $2A$, $4B$, and $6A$ type algebras respectively, are Majorana axes in V and, for $i \in \{1,2,3\}$, $(\tau_{a_0}\tau_{a_1})^i = \tau_{a_{\rho^i}} \in T$ for $i \in \{1,2,3\}$. The vectors u_ρ, v_ρ, and w_ρ in the $3A$, $4A$, and $5A$ type algebras depend uniquely only on the conjugacy class of ρ in the dihedral group $\langle \tau_{a_0}, \tau_{a_1}\rangle$, and $w_{\rho^2} = -w_\rho$.

Let $(V, (\,,\,)_V, \cdot, \mathcal{A}, G)$ be a Majorana representation of G and suppose that G contains a subgroup H isomorphic to S_4 such that V induces a Majorana representation $\hat{V}(H)$ on H of shape $(2B, 3A)$. By [10, Section 5], $\hat{V}(H)$ contains three idempotents $\overline{v}_{g_i}$ $(i \in \{1,2,3\})$ of length 2, uniquely depending (in $\hat{V}(H)$) on the subgroup $\langle g_i\rangle$ of order 4 in H. Note that $\overline{v}_g$ is not a 4-axis in $\hat{V}(H)$, since there is no dihedral subalgebra of type $4A$ in $\hat{V}(H)$. We call $\overline{v}_g$ a *dormant* 4-*axis*. This situation is realized when $G = M$ is the Monster group, V is the Griess algebra, and H is a subgroup of M isomorphic to S_4 such that the transpositions of H are involutions of type $2A$ in M and the bitranspositions of H are elements of type $2B$ in M. Norton showed that, in this case, the vectors $\overline{v}_g$ in $\hat{V}(H)$ are true 4-axes in V depending uniquely on $\langle g\rangle$ (see [10, p.2462]). We shall therefore assume the following

(M8D) dormant 4-axes $\overline{v}_g$ depend uniquely on the subgroup $\langle g\rangle$ of G.

We call $(V, (\ ,\ )_V, \cdot, \mathcal{A}, G)$ a *diagonal Majorana representation* if, for every $a_1, a_2 \in \mathcal{A}$ such that $|\tau_{a_1}\tau_{a_2}| = 2$, $\langle\langle a_1, a_2\rangle\rangle$ is a Norton-Sakuma algebra of type $2B$ and Condition (M8D) holds.

Given a Majorana algebra V, by the Norton-Sakuma Theorem, every subalgebra of V generated by two distinct axes is an algebra of type NL, with $NL \in \{2A, 2B, 3A, 3C, 4A, 4B, 5A, 6A\}$. Clearly, the type NL is constant on the $Aut(V)$-orbits of the 2-generated subalgebras. The function that assigns the type NL to each $Aut(V)$-orbit of the 2-generated subalgebras of V is the *shape* of V.

Lemma 2.3. *Let $(V, (,), \cdot, \mathcal{A}, A_6)$ be a diagonal Majorana representation of A_6 with respect to the set T of its bitranspositions and let $s, t \in T$. Then $|st| \in \{2, 3, 4, 5\}$, moreover*

(*i*) *if st has order 2, then $\langle\langle a_s, a_t\rangle\rangle$ has type $2B$;*
(*ii*) *if st has order 4, then $\langle\langle a_s, a_t\rangle\rangle$ has type $4A$;*
(*iii*) *if st has order 5, then $\langle\langle a_s, a_t\rangle\rangle$ has type $5A$.*

Proof. If *st* has order 2, the result follows by the definition. Assume *st* has order 4 and suppose, by contradiction, that the algebra $\langle\langle a_s, a_t\rangle\rangle$ has type $4B$. By Table 2, it follows that its subalgebra $\langle\langle a_t, a_t^s\rangle\rangle$ is of type $2A$, against (*i*). If *st* has order 5, the result follows by the Norton-Sakuma Theorem. □

We close this section by listing some elementary properties of A_6 that we assume the reader is confident with and will be used throughout this paper without further reference.

Proposition 2.4.

(*i*) *All involutions in A_6 are conjugate.*
(*ii*) *The elements of order 5 in A_6 part into two conjugacy classes which are closed under inversion.*
(*iii*) *A_6 has five conjugacy classes of maximal subgroups: two classes isomorphic to A_5, two classes isomorphic to S_4, and one class isomorphic to $3^2 : 4$ (see e.g. [2]).*
(*iv*) *Every element of order 4 in A_6 is contained in exactly two maximal subgroups isomorphic to S_4. In particular, if t is an involution in A_6, there are exactly two maximal subgroups isomorphic to S_4 whose derived subgroups contain t.*
(*v*) *If S is a subgroup of A_6 isomorphic to S_4, there are two subgroups isomorphic to A_5 containing the derived subgroup S' of S.*
(*vi*) *Let S be a subgroup of A_6 isomorphic to S_4 and let $s \in S \setminus S'$ be an element of order 2. Then there are an involution r and an element g of order 4 in S such that $C_S(s) = \{1, s, r, g^2\}$.*

Given a group G and $i \in \mathbb{N}$, let $G^{(i)}$ be a set of elements of order i in G which contains one representative from every subgroup of order i in G, and these representatives are chosen in a such a way that, if $x, y \in G^{(i)}$, $\langle x \rangle$ and $\langle y \rangle$ are conjugate in G if and only if x and y are conjugate in G. Moreover, for a subgroup H of G, $r, i \in \mathbb{N}$, set

$$H^{(i)} := H \cap G^{(i)} \text{ and } H_r^{(i)}(g) := \{h \mid h \in H^{(i)}, |gh| = r\}.$$

3. Proof of Theorem 1

We shall first prove that if $n \in \{5, 6\}$ then the alternating group A_n has a standard Majorana representation of shape $(2B, 3Y, 5A)$ for every $Y \in \{A, C\}$ and, if $n = 8$, then A_n has a nonstandard Majorana representation of shape $(2B, 3A, 5A)$. For this part we shall need the first of the next two lemmas. The second one will be used to prove the converse.

Lemma 3.1. *Let z be the permutation $(1, 7)(2, 8)(3, 9)(4, 10)(5, 11)(6, 12)$ in A_{12}. There is a subgroup H of A_{12} isomorphic to A_6 such that*

(i) $C_{A_{12}}(z) \geq \langle z \rangle \times H$
(ii) the involutions in H are permutations of type 2^4
(iii) the diagonal involutions in $\langle z \rangle \times H$ are permutations of type 2^6.

Proof. Just take H to be the set of all products gg^z where g lies in the subgroup of A_{12} fixing pointwise the set $\{7, \ldots, 12\}$. □

Lemma 3.2. *Suppose a finite group G has a Majorana representation with respect to a set of generating involutions T such that for every $r, s \in T$ with $|rs| = 2$, $\langle\langle a_r, a_s \rangle\rangle$ has type $2B$. Then, for every $r, s \in T$, rs has order at most 5.*

Proof. Let $r, s \in T$, by the Norton-Sakuma Theorem rs has order at most 6. Suppose $|rs| = 6$. Then $|rsrsrs|$=2 and so by hypothesis $\langle\langle a_r, a_{srsrs} \rangle\rangle$ has type $2B$. On the other hand, the algebra $\langle\langle a_r, a_s \rangle\rangle$ is of type $6A$. Again, by the Norton-Sakuma Theorem (see Table 2), its subalgebra $\langle\langle a_r, a_{srsrs} \rangle\rangle$ is of type $2A$, a contradiction. □

Proof of Theorem 1. By [15, Table 4], A_5 has a standard Majorana representation with shape $(2B, 3A, 5A)$. By [5, Theorem 1], there exists a (unique) Majorana representation $\mathcal{R}$ of A_{12} with respect to the set T of the permutations of cycle type 2^2 or 2^6 (which is based on an embedding of A_{12} in M). By Lemma 3.1, $\mathcal{R}$ induces a (standard) Majorana representation of A_6 with shape $(2B, 3A, 5A)$. Finally, by [5, Theorem 6], the series

$$A_8 \leq A_8 \times \langle (9, 10)(11, 12) \rangle \leq A_{12} \leq M$$

produces a nonstandard Majorana representation of A_8 with shape $(2B, 3A, 5A)$.

Conversely, let $G = A_n$ and let $\mathcal{R} := (V, (\ ,\ )_V, \cdot, \mathcal{A}, G)$ be a Majorana representation of G, with respect to a G-invariant generating set T of involutions of G, with shape $(2B, 3A, 5A)$. Since A_3 and A_4 are not generated by involutions we may assume $n \geq 5$. Assume by contradiction that the "only if" assertion in (a) and (b) is false, we shall prove that in both cases we can find a pair of involutions in T whose product has order 6, against Lemma 3.2. Assume $\mathcal{R}$ is standard, if $n \geq 7$, take

$$r := (1,2)(3,4) \text{ and } s := (1,5)(6,7),$$

then $r, s \in T$ and $|rs| = |(1,2,5)(3,4)(6,7)| = 6$. Now assume $\mathcal{R}$ is non standard, then $n \geq 8$, since, for $n \in \{5,6,7\}$ the involutions of A_n are all bitranspositions. If $n \geq 9$, take

$$r := (1,2)(3,4)(5,6)(7,8) \text{ and } s := (1,2)(3,5)(4,6)(7,9).$$

Then there exists an even permutation t fixing $\{1,\ldots,9\}$ such that $\{rt, st\} \subseteq T$. Thus $|rtst| = |rs| = |(3,6)(4,5)(7,8,9)| = 6$. □

4. The inner products between axes and odd axes

From now on let T be the set of the bitranspositions of A_6 and let

$$(V, (\ ,\ )_V, \cdot, \mathcal{A}, A_6)$$

be a Majorana representation of A_6 with respect to T, with shape $(2B, 3A, 5A)$.

In order to prove Theorem 2, our first goal is to determine the inner product $(\ ,\ )_V$ on the 2-closure V° of V. Since, by definition, V° is linearly spanned by the set of axes, 3-axes, 4-axes, and 5-axes of V, in this section we determine the inner products between the elements of that set. These inner products are displayed in Tables 3 to 11. To obtain this result we shall make precise, in the forthcoming remarks, the correspondence, mentioned in Section 2, between a certain subset of A_6 and the set of axes and odd axes. The above inner products will be used to obtain the dimensions of the subspaces $V^{(NA)}$ of V generated by the N-axes for $N \in \{2,3,4\}$.

Remark 4.1. By Axiom (M8), for $N \in \{3,4,5\}$, to each $\rho \in A_6^{(N)}$ there corresponds a unique N-axis in V. By Remark 2.2, for $N = 5$, if $\rho \in A_6^{(5)}$, then $\rho^2 \notin A_6^{(5)}$ and $w_{\rho^2} = -w_\rho$.

In this case we call w_ρ a *positive* 5-axis and w_{ρ^2} a *negative* 5-axis. For $N \in \{3,4\}$, denote by NA the set of N-axes in V and, for $n = 5$, denote by $5A$ the set of positive 5-axes. Set

$$X := \mathcal{A} \cup 3A \cup 4A \cup 5A$$

and denote by x_ρ the generic element of X corresponding to ρ.

Table 3
The inner products between 2-axes and 3-axes.

$\|th\|$	$\langle t,h\rangle$	$(a_t,u_h)_V$
2	S_3	$\frac{1}{4}$
4	S_4	$\frac{13}{180}$
3	A_4	$\frac{2}{45}$
5	A_5	$\frac{1}{30}$

Table 4
The inner products between 2-axes and 4-axes.

$\|tg\|$	$\langle t,g\rangle$	$(a_t,v_g)_V$
2	D_8	$\frac{3}{8}$
5	A_6	$\frac{5}{128}$
4	$3^2:4$	$\frac{27}{256}$
4	4	0
3	S_4	$\frac{5}{64}$

Remark 4.2. By the linearity and the fact that A_6 acts as a group of isometries on V, in order to determine the inner products between the elements of X, we can reduce ourselves to determining the inner products of unordered pairs (x_g,x_h) with $(\langle g\rangle,\langle h\rangle)$ varying in a set of representatives of their A_6-orbits (under the action induced by conjugation). Moreover, since the arguments we are using depend only on group theoretical properties that are $\mathrm{Aut}(\mathrm{A}_6)$-invariant, we may consider w.l.o.g. only those pairs in a set of representatives of the $\mathrm{Aut}(\mathrm{A}_6)$-orbits.

Remark 4.3. Let $t\in T$, $N\in\{3,4,5\}$, and $h\in A_6^{(N)}$ be such that $h^t=h^{-1}$. Then $\langle\langle a_t,x_h\rangle\rangle$ is a Norton-Sakuma algebra and the inner product $(a_t,x_h)_V$ is given in Table 2. This gives the first rows of Tables 3, 4, and 5.

Remark 4.4. A direct check in the set of subgroups of A_6 gives immediately all the columns except for the last one in Tables 3-9.

Lemma 4.5. *Let $t\in T$ and $h\in A_6^{(3)}$. The possible isomorphism classes for $\langle t,h\rangle$ and the corresponding values of the inner products $(a_t,u_h)_V$ are those listed in the second and third column of Table 3 respectively.*

Proof. If $\langle t,h\rangle\cong S_4$, the result follows by [10, Table 11]. In the remaining two cases, note that, by [15, Table 4], there is a unique standard Majorana representation $\mathcal{R}$ of A_5 with shape $(2B,3A,5A)$. By Lemma 3.1, $\mathcal{R}$ is induced by the saturated Majorana representation of A_{12}. So the values of the inner products are those given in [5, Table 6, rows 4 and 9]. □

Lemma 4.6. *Let $t \in T$ and $g \in A_6^{(4)}$. The possible isomorphism classes for $\langle t, g\rangle$ and the corresponding values of the inner products $(a_t, v_g)_V$ are those listed in the second and the third column of Table 4 respectively.*

Proof. We shall proceed as follows. For each possibility of $\langle t, g\rangle$ listed in Table 4 we choose $r, s \in T$ such that $g = rs$. Then, by Lemma 2.3, $\langle\langle a_r, a_s\rangle\rangle$ is a Norton-Sakuma algebra of type $4A$, so, by Table 2,

$$v_g = a_r + a_s + \frac{1}{3}(a_{rsr} + a_{srs}) - \frac{2^6}{3} a_r a_s.$$

Thus

$$\begin{aligned}(a_t, v_g)_V =&(a_t, a_r)_V + (a_t, a_s)_V + \tfrac{1}{3}[(a_t, a_{rsr})_V + (a_t, a_{srs})_V] - \tfrac{2^6}{3}(a_t, a_r a_s)_V \\ =&(a_t, a_r)_V + (a_t, a_s)_V + \tfrac{1}{3}[(a_t, a_{rsr})_V + (a_t, a_{srs})_V] - \tfrac{2^6}{3}(a_t a_r, a_s)_V. \qquad (2)\end{aligned}$$

Now, for each choice of r and s, all the above inner products, except for $(a_t a_r, a_s)_V$, are given by Table 2 and so computations can be reduced to find, in each case, the value of $(a_t a_r, a_s)_V$.

Assume first $\langle t, g\rangle \cong A_6$ and let

$$\mathcal{H} := \{(\langle t\rangle, \langle g\rangle) \mid t, g \in A_6,\ |t| = 2,\ |g| = 4,\ \text{and}\ \langle t, g\rangle = A_6\}.$$

Then $\mathcal{H}$ parts into two orbits under the action of A_6 (via conjugation) with representatives $(\langle(3,4)(5,6)\rangle, \langle(1,6)(2,5,3,4)\rangle)$ and $(\langle(3,4)(5,6)\rangle, \langle(1,5)(2,6,3,4)\rangle)$ respectively. Suppose

$$t = (3,4)(5,6) \text{ and } g = (1,6)(2,5,3,4),$$

and write $g = rs$ with $r := (3,4)(2,5)$ and $s := (2,3)(1,6)$. Since $|tr| = 3$ and V has shape $(2B, 3A, 5A)$, $\langle\langle a_t, a_r\rangle\rangle$ is a Norton-Sakuma algebra of type $3A$, thus, by Table 2

$$a_t a_r = \frac{1}{32}(2a_t + 2a_r + a_{trt}) - \frac{135}{2^{11}} u_{tr}. \qquad (3)$$

The value of $(a_t a_r, a_s)_V$ is obtained by a direct computation using Lemma 4.5 and Table 2. Swapping 5 and 6, the result for $(\langle t\rangle, \langle g\rangle) = (\langle(3,4)(5,6)\rangle, \langle(1,5)(2,6,3,4)\rangle)$ is obtained in the same way, taking $r = (3,4)(2,6)$ and $s = (2,3)(1,5)$.

Next assume $\langle t, g\rangle \cong 3^2 : 4$. As in the previous case, we may w.l.o.g. assume that

$$t = (3,4)(5,6) \text{ and } g = (1,2,3,5)(4,6).$$

Let $s := (1,3)(4,6)$ and $r := (1,5)(2,3)$. As in the previous case, by Equation (2) and Table 2, we get

Table 5
The inner products between 2-axes and 5-axes.

$\|tf\|$	$\langle t, f\rangle$	$(a_t, w_f)_V$
2	D_{10}	0
5	A_5	$-\frac{7}{2^{14}}$
3	A_5	$\frac{7}{2^{14}}$
5	A_6	$\frac{7}{2^{12}}$
4	A_6	$-\frac{7}{2^{12}}$

$$(a_t, v_g)_V = \frac{3}{128} + \frac{13}{256} + \frac{1}{3}\left(\frac{1}{32} + \frac{3}{128}\right) - \frac{2^6}{3}(a_t a_r, a_s)_V.$$

Since $|tr| = 3$, $\langle\langle a_t, a_r\rangle\rangle$ is of type $3A$, the result follows by substituting the expression for $a_t a_r$ given in Equation (3) and using Lemma 4.5.

The inner products for the last two cases are obtained in a similar way (see also [23, Lemma 4.12]). □

Lemma 4.7. *Let $t \in T$ and $f \in A_6^{(5)}$. The possible isomorphism classes for $\langle t, g\rangle$ and the corresponding values of the inner products $(a_t, w_f)_V$ are those listed in the second and the third columns of Table 5 respectively.*

Proof. We argue as in the proof of Lemma 4.6: we write f as the product of two elements r and s of T. Then, by the Norton-Sakuma Theorem, the algebra $\langle\langle a_r, a_s\rangle\rangle$ has type $5A$, so, by Table 2,

$$w_f = -\frac{1}{2^7}(3a_r + 3a_s - a_{srs} - a_{rsr} - a_{r(sr)^3}) + a_r a_s,$$

whence

$$\begin{aligned}(a_t, w_f)_V &= -\frac{1}{2^7}[3(a_t, a_r)_V + 3(a_t, a_s)_V - (a_t, a_{srs})_V - (a_t, a_{rsr})_V \\ &\quad -(a_t, a_{r(sr)^3})_V] + (a_t, a_r a_s)_V \\ &= -\frac{1}{2^7}[3(a_t, a_r)_V + 3(a_t, a_s)_V - (a_t, a_{srs})_V - (a_t, a_{rsr})_V \\ &\quad -(a_t, a_{r(sr)^3})_V] + (a_t a_s, a_r)_V.\end{aligned}$$

So, this time, computations can be reduced to find the value of $(a_t a_s, a_r)_V$.

Assume first that $|tf| = 5$ and $\langle t, f\rangle \cong A_5$, as in line 2 of Table 5. W.l.o.g., we may assume $t = (3, 4)(5, 6)$ and $f = (2, 5, 3, 6, 4)$ and choose $r := (2, 4)(5, 6)$ and $s := (4, 5)(3, 6)$, so that $f = rs$. Since $|st| = 2$, by Lemma 2.3, $\langle\langle a_t, a_s\rangle\rangle$ is a Norton-Sakuma algebra of type $2B$, whence, by Table 2, $a_t a_s = 0$. Thus, $(a_t a_s, a_r)_V = (0, a_r)_V = 0$.

Suppose that $|tf| = 3$ and $\langle t, f\rangle \cong A_5$, as in line 3 of Table 5. Then the pair (t, f^2) satisfies the conditions of line 2, thus the result follows, since, by Remark 2.2, $w_f = -w_{f^2}$.

Suppose that $|tf| = 5$ and $\langle t, f\rangle \cong A_6$, as in line 4 of Table 5. Then we may assume $t = (3, 4)(5, 6)$ and $f = (1, 2, 3, 4, 5)$ and choose $r := (2, 4)(1, 5)$ and $s := (2, 5)(3, 4)$, so that $f = rs$. Proceeding as in the previous case and using Lemma 4.5 we get the value of the inner product.

Finally, if $|tf| = 3$ and $\langle t, f\rangle \cong A_6$, as in line 5 of Table 5, the pair (t, f^2) satisfies the conditions of line 4 and we conclude as in the previous case. □

In order to treat the cases involving two odd-axes, we use three easy observations, the first of which describes a standard procedure in Majorana theory (see e.g. [12,11]).

Remark 4.8. Let $h, k \in A_6$ such that x_h and x_k are odd axes of V. Suppose there is $t \in T$ inverting by conjugation both h and k. Then the inner product $(x_h, x_k)_V$ can be computed in the following way. The algebra $\langle\langle a_t, x_h\rangle\rangle$ is contained in a Norton-Sakuma algebra of type $|h|A$ and similarly $\langle\langle a_t, x_k\rangle\rangle$ is contained in a Norton-Sakuma algebra of type $|k|A$. Let e_h be a 0-eigenvector for ad_{a_t} and e_k be a $\frac{1}{4}$-eigenvector for ad_{a_t}. Using [10, Table 4], we express e_h (resp. e_k) as a linear combination of axes and x_h (resp. x_k). Since e_h and e_k are eigenvectors relative to different eigenvalues, we get

$$(e_h, e_k)_V = 0.$$

Thus, expanding this equation by substituting e_h and e_k with their respective linear combinations, by Lemmas 4.5, 4.6, and 4.7, we can obtain the value for $(x_h, x_k)_V$.

Lemma 4.9. *Let A be a Majorana algebra and let a_i be axes for $i \in \{1, 2, 3\}$. If $e \in A$ is a 0-eigenvector for ad_{a_2}, then*

$$(a_1a_2, ea_3)_A = (a_1e, a_2a_3)_A.$$

Proof. By the associativity of the inner product and by Lemma 1.10 in [10], we have

$$(a_1a_2, ea_3)_A = ((a_1a_2)e, a_3)_A = (a_2(a_1e), a_3)_A = (a_1e, a_2a_3)_A. \quad \square$$

Note that Norton inequality and positive definiteness of the Majorana form imply that in any Majorana algebra two orthogonal idempotents multiply to zero. If one of the two idempotents is a 2-axis, this can be proved independently from Norton inequality.

Lemma 4.10. *Let A be a Majorana algebra, let a be an axis and let e be an idempotent. If $(a, e)_A = 0$, then $a \cdot e = 0$.*

Proof. Decompose e as the sum of a 1-eigenvector e_1, a 0-eigenvector e_0, a $\frac{1}{4}$-eigenvector e_α, and a $\frac{1}{32}$-eigenvector e_β for ad_a. Since $(a, e)_A = 0$ and eigenvectors relative to different eigenvalues are orthogonal, we get, by the associativity,

$$\begin{aligned}0 = (a, e)_A = (a, e^2)_A = (a \cdot e, e)_A &= (e_1 + \tfrac{1}{4}e_\alpha + \tfrac{1}{32}e_\beta, e_1 + e_0 + e_\alpha + e_\beta)_A \\ &= (e_1, e_1)_A + \tfrac{1}{4}(e_\alpha, e_\alpha)_A + \tfrac{1}{32}(e_\beta, e_\beta)_A.\end{aligned}$$

Table 6
The inner products between two 3-axes.

$\langle h, k\rangle$	$(u_h, u_k)_V$
3	$\frac{8}{5}$
3×3	0
A_4	$\frac{56}{3^3 5^2}$
A_5	$\frac{208}{3^4 5^2}$
A_6	$\frac{256}{3^4 5^2}$

Table 7
The inner products between 3-axes and 4-axes.

$\|hg\|$	$\|hg^{-1}\|$	$\langle h, g\rangle$	$(u_h, v_g)_V$
2		S_4	$\frac{1}{9}$
4		$3^2 : 4$	$\frac{3}{20}$
3	5	A_6	$\frac{2}{9}$
5	5	A_6	$\frac{1}{12}$

Table 8
The inner products between 3-axes and 5-axes.

$\|hf\|$	$\|hf^{-1}\|$	$\langle h, f\rangle$	$(u_h, w_f)_V$
2	5	A_5	$\frac{49}{2^9 3^2 5}$
5	3	A_5	$-\frac{49}{2^9 3^2 5}$
3	4	A_6	$-\frac{91}{2^9 3^2 5}$
4	5	A_6	$\frac{91}{2^9 3^2 5}$

By the positive definiteness of the form, it follows $e_1 = e_\alpha = e_\beta = 0$, $e = e_0$ and so $a \cdot e = 0$. □

Lemma 4.11. *Let $h, k \in A_6^{(3)}$, $g \in A_6^{(4)}$, $f \in A_6^{(5)}$.*

(i) The values of the inner products $(u_h, u_k)_V$ are given in the last column of Table 6.
(ii) The values of the inner products $(u_h, v_g)_V$ are given in the last column of Table 7.
(iii) The values of the inner products $(u_h, w_f)_V$ are given in the last column of Table 8.

Proof. For every $h, k \in A_6^{(3)}$, $g \in A_6^{(4)}$, and $f \in A_6^{(5)}$ there exists $t \in T$ inverting by conjugation both h and k, or h and g, or h and f. The result then follows using the argument described in Remark 4.8. □

Lemma 4.12. *Let $g_1, g_2 \in A_6^{(4)}$. The values of the inner products $(v_{g_1}, v_{g_2})_V$ are given in the last column of Table 9.*

Proof. If g_1 and g_2 are as in the first row of Table 9, then $g_1 = g_2$, whence, by Table 2, $(v_{g_1}, v_{g_1})_V = 2$.

Table 9
The inner products between two 4-axes.

$\lvert g_1g_2\rvert$	$\lvert g_1g_2^{-1}\rvert$	$\langle g_1, g_2\rangle$	$(v_{g_1}, v_{g_2})_V$
2	1	4	2
3	3	S_4	$\frac{11}{48}$
2	3	$3^2 : 4$	$\frac{53}{256}$
5	5	A_6	$\frac{5}{48}$
5	4	A_6	$\frac{89}{384}$

Suppose $\langle g_1, g_2\rangle \cong S_4$, as in the second row of Table 9. Then we may assume $g_1 = (1, 2)(3, 6, 4, 5)$ and $g_2 = (1, 2)(3, 6, 5, 4)$. Let $t = g_2^2$. Then, by Table 4 $(a_t, v_{g^2})_V = 0$ and so, by Lemma 4.10, v_{g_2} is a 0-eigenvector for ad_{a_t}. Furthermore, since t inverts g_1, tg_1 is an involution whence $\langle\langle a_t, a_{tg_1}\rangle\rangle$ is a Norton-Sakuma algebra of type $4A$ containing v_{g_1}. By [10, Table 4],

$$e := v_{g_1} - \tfrac{1}{3}a_t - \tfrac{2}{3}(a_{tg_1} + a_{tg_1^3}) - \tfrac{1}{3}a_{t^{g_1}} \tag{4}$$

is a $\frac{1}{4}$-eigenvector for ad_{a_t} in this algebra. As in Remark 4.8, it follows that $(v_{g_2}, e)_V = 0$ and, expanding this equation by substituting the expression for e given in Equation (4), we get

$$(v_{g_1}, v_{g_2})_V = \tfrac{11}{48}.$$

Now suppose $\langle g_1, g_2\rangle \cong 3^2 : 4$, as in line 3 of Table 9. W.l.o.g., we may assume $g_1 = (1, 2)(3, 5, 4, 6)$ and $g_2 = (1, 4)(2, 6, 3, 5)$. Let $t_1 = (1, 2)(3, 4)$, $r_1 = (3, 6)(4, 5)$, $t_2 = (1, 4)(2, 3)$, and $r_2 = (2, 5)(3, 6)$ so that $g_1 = t_1r_1$ and $g_2 = t_2r_2$. Then, by Table 2, for $i \in \{1, 2\}$, we have

$$v_{g_i} = a_{t_i} + a_{r_i} + \tfrac{1}{3}(a_{r_it_ir_i} + a_{t_ir_it_i}) - \tfrac{64}{3}a_{t_i}a_{r_i}$$

whence, by the linearity of the inner product and Lemmas 4.5, 4.6, 4.7, and 4.11, the computation of $(v_{g_1}, v_{g_2})_V$ reduces to that of $(a_{t_1}a_{r_1}, a_{t_2}a_{r_2})_V$. Now, $\langle\langle a_{t_1}, a_{t_2}\rangle\rangle$ is of type $2B$ and so a_{t_2} is a 0-eigenvector for $\mathrm{ad}_{a_{t_1}}$. Hence, by Lemma 4.9, we have

$$(a_{t_1}a_{r_1}, a_{t_2}a_{r_2})_V = (a_{r_1}a_{t_2}, a_{t_1}a_{r_2})_V.$$

Since $|r_1t_2| = |t_1r_2| = 3$, and V has shape $(2B, 3A, 5A)$, the two algebras $\langle\langle a_{r_1}, a_{t_2}\rangle\rangle$ and $\langle\langle a_{t_1}, a_{r_2}\rangle\rangle$ are of type $3A$. By Table 2, the vectors $a_{r_1}a_{t_2}$ and $a_{t_1}a_{r_2}$ are linear combinations of axes and 3-axes, whence, using Lemmas 4.5 and 4.11, a direct computation gives

$$(a_{r_1}a_{t_2}, a_{t_1}a_{r_2})_V = \tfrac{151}{2^{19}} \text{ and } (v_{g_1}, v_{g_2})_V = \tfrac{53}{256}.$$

Suppose $\langle g_1, g_2\rangle \cong A_6$ and $|g_1g_2| = |g_1g_2^{-1}| = 5$, as in the fourth row of Table 9. Then we may assume $g_1 = (1,2)(3,6,4,5)$ and $g_2 = (1,2,3,4)(5,6)$. Since $t := (1,2)(3,4)$ inverts both g_1 and g_2, by Remark 4.8 we get

$$(v_{g_1}, v_{g_2})_V = \tfrac{5}{48}.$$

Finally, suppose $|g_1g_2| = 5$, $|g_1g_2^{-1}| = 4$ and $\langle g_1, g_2\rangle \cong A_6$, as in the fifth row of Table 9. Then we may assume $g_1 = (1,2)(3,5,4,6)$ and $g_2 = (1,6)(2,5,3,4)$. Set

$$t_1 := (3,5)(4,6), \quad r_1 := (1,2)(3,4), \quad t_2 = (2,5)(3,4), \quad \text{and} \quad r_2 := (1,6)(2,3)$$

so that $g_1 = t_1r_1$ and $g_2 = t_2r_2$. By Table 2, we have

$$\begin{aligned}(v_{g_1}, v_{g_2})_V =& (a_{t_1}, v_{g_2})_V + (a_{r_1}, v_{g_2})_V + \tfrac{1}{3}(a_{t_1r_1t_1}, v_{g_2})_V + \tfrac{1}{3}(a_{r_1t_1r_1}, v_{g_2})_V \\ &- \tfrac{64}{3}(a_{t_1}a_{r_1}, v_{g_2})_V\end{aligned} \tag{5}$$

The first four summands of Equation (5) are inner products between axes and v_{g_2}, so by Lemma 4.6,

$$(v_{g_1}, v_{g_2})_V = \tfrac{5}{2^7} + \tfrac{5}{2^7} + \tfrac{3^2}{2^8} + \tfrac{3^2}{2^8} - \tfrac{64}{3}(a_{t_1}a_{r_1}, v_{g_2})_V = \tfrac{19}{2^7} - \tfrac{64}{3}(a_{t_1}a_{r_1}, v_{g_2})_V. \tag{6}$$

In order to compute the value of $(a_{t_1}a_{r_1}, v_{g_2})_V$, note that $\langle\langle a_{t_1}, a_{t_2}\rangle\rangle$ is of type $5A$. By [10, Table 4], the two vectors

$$w_{t_2t_1} + \tfrac{3}{2^9}a_{t_2} - \tfrac{15}{2^7}(a_{t_1} - a_{t_2t_1t_2}) - \tfrac{1}{2^7}(a_{t_1t_2t_1} + a_{t_1t_2t_1t_2t_1})$$

and

$$w_{t_2t_1} - \tfrac{3}{2^9}a_{t_2} + \tfrac{1}{2^7}(a_{t_1} - a_{t_2t_1t_2}) + \tfrac{15}{2^7}(a_{t_1t_2t_1} + a_{t_1t_2t_1t_2t_1})$$

are 0-eigenvectors for $\mathrm{ad}_{a_{t_2}}$. Taking their difference multiplied by 8, we get that

$$e := -\tfrac{3}{32}a_{t_2} + a_{t_1} + a_{t_1t_2t_1} + a_{t_2t_1t_2} + a_{t_1t_2t_1t_2t_1} \tag{7}$$

is a 0-eigenvector for $\mathrm{ad}_{a_{t_2}}$ in $\langle\langle a_{t_1}, a_{t_2}\rangle\rangle$. By Equation (7) we obtain

$$\begin{aligned}(ea_{r_1}, v_{g_2})_V =& -\tfrac{3}{32}(a_{t_2}a_{r_1}, v_{g_2})_V + (a_{t_1}a_{r_1}, v_{g_2})_V + (a_{t_1t_2t_1}a_{r_1}, v_{g_2})_V \\ &+ (a_{t_2t_1t_2}a_{r_1}, v_{g_2})_V + (a_{t_1t_2t_1t_2t_1}a_{r_1}, v_{g_2})_V,\end{aligned} \tag{8}$$

whence

$$\begin{aligned}(a_{t_1}a_{r_1}, v_{g_2})_V =& (ea_{r_1}, v_{g_2})_V + \tfrac{3}{32}(a_{t_2}a_{r_1}, v_{g_2})_V - (a_{t_1t_2t_1}a_{r_1}, v_{g_2})_V \\ &- (a_{t_2t_1t_2}a_{r_1}, v_{g_2})_V - (a_{t_1t_2t_1t_2t_1}a_{r_1}, v_{g_2})_V.\end{aligned} \tag{9}$$

Now $a_{t_2}a_{r_1} \in \langle\langle a_{t_2}, a_{r_1}\rangle\rangle$ and $a_{t_1t_2t_1t_2t_1}a_{r_1} \in \langle\langle a_{t_1t_2t_1t_2t_1}, a_{r_1}\rangle\rangle$. These are Norton-Sakuma algebras of type $3A$ (since $|t_2r_1| = 3$ and $|t_1t_2t_1t_2t_1r_1| = 3$) and so, by Table 2, the products $a_{t_2}a_{r_1}$ and $a_{t_1t_2t_1t_2t_1}a_{r_1}$ are linear combinations of axes and 3-axes. Thus, we can use Lemma 4.5 and Lemma 4.11 to obtain

$$(a_{t_2}a_{r_1}, v_{g_2})_V = \tfrac{51}{2^{12}} \quad \text{and} \quad (a_{t_1t_2t_1t_2t_1}a_{r_1}, v_{g_2})_V = -\tfrac{13}{2^{12}}. \tag{10}$$

The algebra $\langle\langle a_{t_1t_2t_1}, a_{r_1}\rangle\rangle$ is of type $4A$ and $v_{(1,2,4,3)(5,6)}$ is a 4-axis in it. Hence the product $a_{t_1t_2t_1}a_{r_1}$ is a linear combination of axes and $v_{(1,2,4,3)(5,6)}$. Since the pair $(v_{(1,2,4,3)(5,6)}, v_{g_2})$ satisfies the conditions in the third row of Table 9, by the previous case we get

$$(a_{t_1t_2t_1}a_{r_1}, v_{g_2})_V = -\tfrac{3^2}{2^{13}}. \tag{11}$$

The algebra $\langle\langle a_{t_2t_1t_2}, a_{r_1}\rangle\rangle$ is of type $5A$ with 5-axis $w_{(1,2,3,6,4)}$, whence the product $a_{t_2t_1t_2}a_{r_1}$ is a linear combination of axes and $w_{(1,2,3,6,4)}$. Since $(1,6)(2,3)$ inverts both $(1,2,3,6,4)$ and g_2, by Remark 4.8, we get

$$(w_{(1,2,3,6,4)}, v_{g_2})_V = \tfrac{7\cdot 13}{2^{14}},$$

whence

$$(a_{t_2t_1t_2}a_{r_1}, v_{g_2})_V = \tfrac{23}{2^{13}}. \tag{12}$$

Thus, by Equations (9), (10), (11), and (12), we obtain

$$(a_{t_1}a_{r_1}, v_{g_2})_V = (ea_{r_1}, v_{g_2})_V + \tfrac{3\cdot 5\cdot 23}{2^{17}}. \tag{13}$$

To compute the value of $(ea_{r_1}, v_{g_2})_V$, observe that, by Table 2,

$$v_{g_2} = a_{t_2} + a_{r_2} + \tfrac{1}{3}(a_{r_2t_2r_2} + a_{t_2r_2t_2}) - \tfrac{64}{3}a_{t_2}a_{r_2}$$

and so

$$\begin{aligned}(ea_{r_1}, v_{g_2})_V =& (ea_{r_1}, a_{t_2})_V + (ea_{r_1}, a_{r_2})_V \\ &+ \tfrac{1}{3}(ea_{r_1}, a_{r_2t_2r_2})_V + \tfrac{1}{3}(ea_{r_1}, a_{t_2r_2t_2})_V - \tfrac{64}{3}(ea_{r_1}, a_{t_2}a_{r_2})_V.\end{aligned} \tag{14}$$

Now, by Equation (7), we obtain

$$ea_{r_1} = -\tfrac{3}{32}a_{t_2}a_{r_1} + a_{t_1}a_{r_1} + a_{t_1t_2t_1}a_{r_1} + a_{t_2t_1t_2}a_{r_1} + a_{t_1t_2t_1t_2t_1}a_{r_1}$$

which is a linear combination of axes and odd axes. Thus, substituting the above expression for ea_{r_1} in Equation (14), by Lemmas 4.5, 4.6, and 4.7, we get

Table 10
The inner products between 4-axes and 5-axes.

$\lvert gf\rvert$	$\lvert gf^{-1}\rvert$	$(v_g, w_f)_V$
5	3	$-\frac{91}{2^{14}}$
4	3	$\frac{91}{2^{14}}$
5	5	0
5	4	$-\frac{7}{2^{11}}$
5	2	$\frac{7}{2^{11}}$

$$(ea_{r_1}, v_{g_2})_V = -\frac{4009}{2^{18}\cdot 3} - \frac{64}{3}(ea_{r_1}, a_{t_2}a_{r_2})_V \tag{15}$$

Furthermore, by Lemma 4.9, we have

$$(ea_{r_1}, a_{t_2}a_{r_2})_V = (a_{r_1}a_{t_2}, ea_{r_2})_V.$$

Since $a_{r_1}a_{t_2}$ is a linear combination of axes and a 3-axis while ea_{r_2}, similarly as above, is a linear combination of axes and odd axes, using Lemmas 4.5, 4.6, 4.7, and 4.11 we get $(a_{r_1}a_{t_2}, ea_{r_2})_V = \frac{1133}{2^{24}}$, whence, by Equation (15),

$$(ea_{r_1}, v_{g_2})_V = -\frac{2053}{2^{17}\cdot 3}.$$

Substituting this value in Equation (13), we get

$$(a_{t_1}a_{r_1}, v_{g_2})_V = -\frac{1}{2^8}$$

and the result follows by Equation (6). □

Lemma 4.13. *Let $g \in A_6^{(4)}, f \in A_6^{(5)}$. The values of the inner products $(v_g, w_f)_V$ are given in Table 10.*

Proof. Suppose $|gf| = 5$ and $|gf^{-1}| = 3$, as in the first row of Table 10. We may assume $g = (1,6)(2,5,3,4)$ and $f = (1,3,4,2,6)$. Since these two permutations are inverted by $(1,6)(2,3)$, the result follows by Remark 4.8. Moreover, since $|gf^2| = 4$ and $|gf^{-2}| = 3$, the value in the second row of Table 10 is obtained by Remark 2.2, substituting f by f^2.

Next suppose $|gf| = 5$ and $|gf^{-1}| = 5$, as in the third row of Table 10. We may assume $g = (1,2)(3,4,5,6)$ and $f = (2,3,4,6,5)$. Set $t := (3,5)(4,6) = g^2$. Then, by the fourth row of Table 4, $(a_t, v_g)_V = 0$, whence by Lemma 4.10,

$$\mathrm{ad}_{a_t}(v_g) = 0. \tag{16}$$

Since f is inverted by t, $\langle t, f\rangle$ is a dihedral group of order 10, whence, by Lemma 2.3, $\langle\langle a_t, a_{tf}\rangle\rangle$ is a Norton-Sakuma algebra of type $5A$. Let

$$e := \frac{1}{2^7}(a_{tf} + a_{ft} - a_{tf^2} - a_{tf^3}) + w_f. \tag{17}$$

By [10, Table 4], e is a $\frac{1}{4}$-eigenvector for ad_{a_t}. Thus, by Equation (16), we get

$$(e, v_g)_V = 0. \tag{18}$$

Since, by Lemma 4.6, $(a_{tf} + a_{ft} - a_{tf^2} - a_{tf^3}, v_g)_V = 0$, substituting in Equation (18) the expression of e given in Equation (17), we obtain $(v_g, w_f)_V = 0$.

Suppose $|gf| = 5$ and $|gf^{-1}| = 4$. We may assume $g = (1,2)(3,6,5,4)$ and $f = (1,3,6,2,4)$. Set $t_1 := (3,6)(4,5)$, $r_1 := (1,2)(3,5)$, $t_2 := (1,2)(3,6)$, and $r_2 := (1,4)(2,3)$ so that $g = t_1r_1$ and $f = t_2r_2$. Then, by Lemma 2.3, the Norton-Sakuma algebra $\langle\langle a_{t_1}, a_{r_1}\rangle\rangle$ (resp. $\langle\langle a_{t_2}, a_{r_2}\rangle\rangle$) is of type $4A$ (resp. $5A$), and contains the odd axis v_g (resp. w_g). Thus, by Table 2,

$$v_g = a_{t_1} + a_{r_1} + \frac{1}{3}(a_{r_1g} + a_{gt_1}) - \frac{64}{3}a_{t_1}a_{r_1}$$

and

$$w_f = -\frac{1}{2^7}(3a_{t_2} + 3a_{r_2} - a_{t_2f^2} - a_{t_2f^3} - a_{t_2f^4}) + a_{t_2}a_{r_2}.$$

By the linearity of the inner product and the previous lemmas, the computation of $(v_g, w_f)_V$ reduces to that of $(a_{t_1}a_{r_1}, a_{t_2}a_{r_2})_V$. Since $|t_1t_2| = 2$, by Lemma 2.3, $\langle\langle a_{t_1}, a_{t_2}\rangle\rangle$ is of type $2B$, whence, by Table 2, $a_{t_1}a_{t_2} = 0$ and so, by Lemma 4.9,

$$(a_{t_1}a_{r_1}, a_{t_2}a_{r_2})_V = (a_{r_1}a_{t_2}, a_{t_1}a_{r_2})_V. \tag{19}$$

Since $|t_1r_2| = |t_2r_1| = 3$, the algebras $\langle\langle a_{t_1}, a_{r_2}\rangle\rangle$ and $\langle\langle a_{t_2}, a_{r_1}\rangle\rangle$ are of type $3A$. Using Table 2, we can express the vectors $a_{t_1}a_{r_2}$ and $a_{t_2}a_{r_1}$ as linear combinations of axes and a 3-axis. Substituting $a_{t_1}a_{r_2}$ and $a_{t_2}a_{r_1}$ with these expressions, by Lemmas 4.5, and 4.11, we obtain

$$(a_{r_1}a_{t_2}, a_{t_1}a_{r_2})_V = -\frac{1}{2^{17}}$$

and get

$$(v_g, w_f)_V = -\frac{7}{2^{11}}.$$

Finally since $|gf^2| = 5$ and $|gf^{-2}| = 2$, we obtain the value of $(v_g, w_f)_V$ in the last row as we did for the second row. □

Table 11
The inner products between two 5-axes.

$\lvert f_1f_2\rvert$	$\lvert f_1f_2^{-1}\rvert$	(w_{f_1}, w_{f_2})
5 (5)	1 (1)	$\frac{5^3 7}{2^{19}}$
3 (5)	5 (3)	$-\frac{21}{2^{19}}$
4 (4)	4 (4)	$\frac{133}{2^{19}}$
5 (2)	5 (4)	$\frac{119}{2^{20}}$

Lemma 4.14. *Let $f_1, f_2 \in A_6^{(5)}$. The values of the inner products $(w_{f_1}, w_{f_2})_V$ are given in Table 11.*[1]

Proof. If f_1 and f_2 are as in the first row of Table 11, then $f_1 = f_2$, whence, by Table 2, $(w_{f_1}, w_{f_1})_V = \frac{5^3 7}{2^{19}}$.

Suppose $|f_1f_2| = 3$ (resp. $|f_1f_2| = 4$) and $|f_1f_2^{-1}| = 5$ (resp. $|f_1f_2^{-1}| = 4$). Then we may assume $f_1 = (1, 3, 5, 2, 4)$ and $f_2 = (2, 3, 6, 4, 5)$ (resp. $f_2 = (2, 3, 4, 5, 6)$). Thus $(2, 5)(3, 4)$ inverts f_i for $i \in \{1, 2\}$ and the result for the second (resp. third) row of Table 11 follows by Remark 4.8.

Finally suppose $|f_1f_2| = |f_1f_2^{-1}| = 5$. Then we may assume $f_1 = (1, 2, 3, 4, 5)$ and $f_2 = (2, 3, 5, 4, 6)$. Set $t_1 := (2, 5)(3, 4)$, $r_1 := (1, 2)(3, 5)$, $t_2 := (2, 4)(3, 5)$, and $r_2 := (2, 6)(3, 4)$ so that $f_1 = t_1r_1$ and $f_2 = t_2r_2$. Then, by the Norton-Sakuma theorem, the linearity of the inner product and lemmas above, the computation of $(w_{f_1}, w_{f_2})_V$ reduces to that of $(a_{t_1}a_{r_1}, a_{t_2}a_{r_2})_V$. Now, $\langle\langle a_{t_1}, a_{t_2}\rangle\rangle$ is of type $2B$ and so, by Lemma 4.9 we have

$$(a_{t_1}a_{r_1}, a_{t_2}a_{r_2})_V = (a_{r_1}a_{t_2}, a_{t_1}a_{r_2})_V.$$

Since $|r_1t_2| = |t_1r_2| = 3$, the algebras $\langle\langle a_{r_1}, a_{t_2}\rangle\rangle$ and $\langle\langle a_{t_1}, a_{r_2}\rangle\rangle$ are both of type $3A$, hence the products $a_{r_1}a_{t_2}$ and $a_{t_1}a_{r_2}$ are linear combinations of axes and 3-axes. Thus, by the same argument used in Lemma 4.13 to compute the last row of Table 10, we get $(a_{r_1}a_{t_2}, a_{t_1}a_{r_2})_V = \frac{3\cdot 19}{2^{18}}$, and the result follows. □

Proposition 4.15.

(i) $\dim(V^{(2A)}) = 45$*;*
(ii) $\dim(V^{(3A)}) = 40$*;*
(iii) $\dim(V^{(4A)}) = 45$.

Proof. This follows by computing the ranks of the Gram matrices associated to the inner product with respect to the sets $\mathcal{A}$, $3A$, and $4A$, whose entries are given in Tables 2-11 (see [6, r2xA6_innerproduct.g]). □

[1] Recall that, by definition, the set $A_6^{(5)}$ depends on the choice of one of the two conjugacy classes of 5-elements in A_6. The possible values of $|f_1f_2|$ and $|f_1f_2^{-1}|$ depend on this choice. Numbers in parentheses correspond to the values associated to the other choice of the conjugacy class.

5. Maximal subalgebras

The maximal subgroups of A_6 generated by involutions are isomorphic either to A_5 or to S_4. The axes corresponding to these involutions generate Majorana subalgebras of V which have been studied in [15, Table 4] and in [10], respectively. In this section we collect some results about those subalgebras. In particular, since these subalgebras are not 2-closed, we introduce new vectors, denoted by γ_s and η_s, besides the dormant 4-axes mentioned in the Introduction. We find some technical results (Proposition 5.7 and Lemma 5.10) on the algebra product which will be used in the next sections to show that the above vectors are contained in the 2-closure V° of V.

Let S be a subgroup of A_6 isomorphic to S_4. By Proposition 2.4.(i), all nine involutions in S are Miyamoto involutions, hence $V(S)$ affords a Majorana representation of S with nine Majorana axes. Since V has shape $(2B, 3A, 5A)$, by Lemma 2.3, this representation has shape $(2B, 3A, 4A)$, hence, by [15, Table 4], it is unique and has dimension 25. Denote, as usual, by S' the derived subgroup of S and by $\hat{V}(S)$ the subalgebra of $V(S)$ generated by the set $\{a_t \,|\, t \in S^{(2)} \setminus S'\}$. Then $\hat{V}(S)$ affords the Majorana representation of S with six Majorana axes and shape $(2B, 3A)$. By [10, Section 5], $\hat{V}(S)$ has dimension 13, it is not 2-closed, and a basis for $\hat{V}(S)$ is given by the set

$$\{a_t \,|\, t \in S^{(2)} \setminus S'\} \cup \{u_h \,|\, h \in S^{(3)}\} \cup \{\overline{v}_g, \,|\, g \in S^{(4)}\}.$$

Observe that, if $g \in A_6^{(4)}$, say $g = (1,2)(3,4,5,6)$, then g is contained in exactly two subgroups of A_6 isomorphic to S_4, namely

$$S := \langle (1,2)(5,6), (3,4,5) \rangle \;\text{ and }\; S^* := \langle (1,5)(2,3), (1,3,4)(2,5,6) \rangle.$$

It follows that there are two distinct subalgebras $\hat{V}(S)$ and $\hat{V}(S^*)$ of V, each of which containing a dormant 4-axis corresponding to $\langle g \rangle$. For the remainder of the paper we assume that condition (M8D) holds.

Note that $|S^{(3)}| = 4$ and $|S^{(4)}| = 3$. Let $S^{(3)} = \{h_1, h_2, h_3, h_4\}$ and for an involution $s \in S'$ define

$$\gamma_s := a_s \cdot (u_{h_1} + u_{h_2} + u_{h_3} + u_{h_4}). \tag{20}$$

Similarly, let $S^{(4)} = \{g_1, g_2, g_3\}$ and for $s \in S^{(2)} \setminus S'$ define

$$\eta_s := a_s \cdot (v_{g_1} + v_{g_2} + v_{g_3}). \tag{21}$$

Since, every $x \in C_{A_6}(s)$ acts trivially on $v_{g_1} + v_{g_2} + v_{g_3}$ and on $u_{h_1} + u_{h_2} + u_{h_3} + u_{h_4}$, we have

Lemma 5.1. *η_s and γ_s depend only on S and s.*

Proposition 5.2. *With the above notation, the following assertions hold:*

(i) *$V(S)$ has dimension 25 and it is 3-closed;*
(ii) *a basis of $V(S)$ is given by the union of the following sets*
  a) $\{a_s \mid s \in S^{(2)}\}$ *(2-axes),*
  b) $\{u_h \mid h \in S^{(3)}\}$ *(3-axes),*
  c) $\{v_{g_1}, v_{g_2}, v_{g_3}\}$ *(4-axes),*
  d) $\{\overline{v}_{g_1}, \overline{v}_{g_2}, \overline{v}_{g_3}\}$ *(dormant 4-axes),*
  e) $\{\eta_s \mid s \in S^{(2)} \setminus S'\}$;
(iii) *for every $g \in S^{(4)}$, we have $a_{g^2} \cdot \overline{v}_g = 0$ and $v_g \cdot \overline{v}_g = 0$;*
(iv) *for every $g \in S^{(4)}$, we have*

$$\gamma_{g^2} = \tfrac{4}{15}a_{g^2} + \tfrac{26}{135}\sum_{s\in S'\setminus\langle g^2\rangle} a_s - \tfrac{44}{135}\sum_{s\in C_S(g^2)^{(2)}\setminus S'} a_s + \tfrac{1}{16}\sum_{h\in S^{(3)}} u_h \\ -\tfrac{2}{9}v_g - \tfrac{2}{15}\sum_{l\in S^{(4)}\setminus\{g\}} v_l + \tfrac{32}{45}\sum_{s\in C_S(g^2)^{(2)}\setminus S'} \eta_s. \tag{22}$$

Proof. Claims (i), (ii), and (iii) have been proved in [23] and can be checked using [6, algebra2xS4.g]. The formula for γ_{g^2} in (iv) has been computed using the construction of the algebra $V(S)$ in [23] (see [6, algebra2xS4.g]). □

Corollary 5.3. *For every $g \in A_6^{(4)}$ we have $a_{g^2} \cdot \overline{v}_g = 0$ and $v_g \cdot \overline{v}_g = 0$.*

Proof. If $g \in A_6^{(4)}$, then g is contained in maximal subgroup of A_6 isomorphic to S_4 and the result follows by Proposition 5.2.(iii). □

Lemma 5.4. *With the notation above, let $g \in S^{(4)}$ and let g^2, t, r be the involutions of S'. Then*

$$a_t \cdot \overline{v}_g = \tfrac{5}{36}a_{g^2} + \tfrac{1}{16}a_t + \tfrac{1}{48}a_r + \tfrac{1}{9}\sum_{s\in N_S(g)^{(2)}\setminus S'} a_s - \tfrac{11}{72}\sum_{s\in S^{(2)}\setminus C_S(g^2)} a_s \\ +\tfrac{15}{512}\sum_{h\in S^{(3)}} u_h - \tfrac{5}{48}\sum_{l\in S^{(4)}\setminus\{g\}} v_l + \tfrac{1}{24}v_g \\ -\tfrac{1}{3}\left(\sum_{r\in N_S(g)^{(2)}\setminus S'} \eta_r - \sum_{r\in S^{(2)}\setminus C_S(g^2)} \eta_r\right)$$

Proof. This formula has been computed in [6, algebra2xS4.g]. □

Let

$$U := V^{(2A)} + V^{(3A)} + V^{(4A)} \quad \text{and} \quad U^\bullet := U + \langle \overline{v}_l \mid l \in G^{(4)}\rangle. \tag{23}$$

Since $\mathrm{Miy}(\mathcal{A}) = A_6$, U and $U^\bullet$ are invariant under the action of $\mathrm{Miy}(\mathcal{A})$.

Lemma 5.5. *With the above notation, let $s \in S^{(2)} \setminus S'$ and let r, t, and rt be the involutions in S'. Then there are elements h and h_1 in $S^{(3)}$, $s \in S^{(2)} \setminus S'$ and a unique element $g_1 \in S^{(4)}$, such that*

(i) $rs = sr$,
(ii) $h^s = h^{-1}$ *and* $h_1^s = h_1^{-1}$,
(iii) $rg_1 = g_1 r$,

moreover

$$a_r \cdot u_h = \tfrac{1}{15} a_r + \tfrac{13}{270}(a_t + a_{rt}) - \tfrac{49}{270} a_s + \tfrac{1}{54} a_{rs} + \tfrac{3}{64} u_h + \tfrac{1}{64} u_{h_1} - \tfrac{1}{30} \sum_{g \in S^{(4)} \setminus \{g_1\}} v_g - \tfrac{1}{18} v_{g_1} + \tfrac{16}{45} \eta_s. \tag{24}$$

In particular, $\eta_s \in a_r \cdot u_h + U$.

Proof. The first part follows by a direct inspection in the group $S \cong S_4$, the last formula has been computed in [6, algebra2xS4.g]. □

Corollary 5.6. *Let $s \in A_6^{(2)}$. Then there exist $r \in A_6^{(2)}$ and $h \in A_6^{(3)}$ such that $\eta_s \in a_r \cdot u_h + U$.*

Proof. This follows immediately by Equation (24). □

Proposition 5.7. *Let $g \in S^{(4)}$. Then $\gamma_{g^2} \in U^\bullet$.*

Proof. By Proposition 5.2.(*iv*) it is enough to prove that

$$\sum_{r \in C_S(g^2)^{(2)} \setminus S'} \eta_r \in U^\bullet. \tag{25}$$

We may assume w.l.o.g. that $S = \langle (1,2)(5,6), (3,4,5) \rangle$ and $g = (1,2)(3,4,5,6)$, so that

$$C_S(g^2)^{(2)} \setminus S' = \{(1,2)(3,5), (1,2)(4,6)\}.$$

Let $S^* := \langle (1,5)(2,3), (1,3,4)(2,5,6) \rangle$. Then, as mentioned at the beginning of this section, S and S^* are the two subgroups of A_6 isomorphic to S_4 that contain g. Since $S \cap S^* = N_{A_6}(\langle g \rangle) \cong D_8$, the Klein subgroups of S and S^* are both contained in $S \cap S^*$ and they intersect exactly in $\langle g^2 \rangle$. Let $t := (3,6)(4,5)$. Then t belongs to the Klein subgroup K of S, but t does not belong to the Klein subgroup of S^*. Let $\hat{V}(S^*)$ be the subalgebra of $V(S^*)$ generated by the axes a_t associated to the involutions t of S^* that are not contained in its derived subgroup. By the discussion at the beginning of this section, $\hat{V}(S^*)$ affords a Majorana representation of S_4 of type $(2B, 3A)$. Hence, by the formula in [10, p.2462], the product $a_t \cdot \overline{v}_g$ belongs to $U^\bullet$. Thus, by Lemma 5.4, we get

$$\sum_{r \in N_S(g)^{(2)} \setminus S'} \eta_r - \sum_{r \in S^{(2)} \setminus C_S(g^2)} \eta_r \in U^\bullet$$

that is

$$\eta_{(1,2)(3,5)} + \eta_{(1,2)(4,6)} - \eta_{(1,2)(3,4)} - \eta_{(1,2)(3,6)} - \eta_{(1,2)(4,5)} - \eta_{(1,2)(5,6)} \in U^\bullet. \quad (26)$$

Applying $(1,2)(3,4)$ to Equation (26) we get

$$-\eta_{(1,2)(3,5)} - \eta_{(1,2)(4,6)} - \eta_{(1,2)(3,4)} + \eta_{(1,2)(3,6)} + \eta_{(1,2)(4,5)} - \eta_{(1,2)(5,6)} \in U^\bullet \quad (27)$$

whence, taking the sum and dividing by -2, we get

$$\eta_{(1,2)(3,4)} + \eta_{(1,2)(5,6)} \in U^\bullet$$

which is the condition in Equation (25). $\square$

We consider now the subalgebras of V corresponding to the maximal subgroups of A_6 isomorphic to A_5. So let L be a subgroup of A_6 with $L \cong A_5$ and let $V(L)$ be the subalgebra of V generated by the axes a_t with $t \in T \cap L$. By Lemma 2.3, it follows that $V(L)$ affords a Majorana representation of A_5 with shape $(2B, 3A, 5A)$.

Lemma 5.8. *The alternating group A_5 has a unique Majorana representation W of shape $(2B, 3A, 5A)$. The space W has dimension 46 and a basis for W is given by the set consisting of all 2-axes, 3-axes, 5-axes and the vectors $\gamma_{(i,j)(k,l)}$ as defined in Equation (20), for every $(i,j)(k,l) \in A_5$.*

Proof. By [15, Table 4] there is a unique Majorana representation of A_5 with shape $(2B, 3A, 5A)$ and this representation has dimension 46. Using the GAP package "MajoranaAlgebras" [19] one sees that the set consisting of all 2-axes, 3-axes, 5-axes, and the vectors $\gamma_{(i,j)(k,l)}$ has size 46 and its elements are linearly independent (see [6, repA5(2B).g]). $\square$

Lemma 5.9. *Let L be a subgroup of A_6 isomorphic to A_5. Let $t \in L^{(2)}$ and $h \in L^{(3)}$ be such that $\langle t, h \rangle \cong A_4$. Let $r \in C_L(t) \setminus \{1, t\}$ and let*

(i) $L_2^{(3)}(h) \cap L_2^{(3)}(t) = \{h_1\}$,
(ii) $L_5^{(3)}(h) \cap L_2^{(3)}(t) = \{h_2\}$,
(iii) $L_2^{(5)}(h) \cap L_2^{(5)}(t) = \{f\}$.

In the algebra $V(L)$, we have

$$
\begin{aligned}
a_t \cdot u_h = \tfrac{1}{32} u_h + \tfrac{1}{45} & \sum_{s \in L_3^{(2)}(t) \cap L_5^{(2)}(h)} (a_s - a_{s^r}) \\
& + \tfrac{1}{36} \sum_{s \in L_5^{(2)}(t) \cap L_3^{(2)}(h)} (a_s + a_{s^t} - a_{s^r} - a_{s^{tr}}) \\
& + \tfrac{1}{64} \left(u_{h_1} - u_{h_2} - \sum_{k \in L_3^{(3)}(t) \setminus \{h, h^t, (h^{-1})^t\}} u_k \right) \\
& - \tfrac{32}{45} (w_f - w_{f^r}) \\
& + \tfrac{1}{4} \gamma_t - \tfrac{1}{8} \sum_{s \in L_3^{(2)}(t) \cap L_5^{(2)}(h)} (\gamma_s - \gamma_{s^r}) \\
& - \tfrac{1}{8} \sum_{s \in L_5^{(2)}(t) \cap L_3^{(2)}(h)} (\gamma_s + \gamma_{s^t} - \gamma_{s^r} - \gamma_{s^{tr}}) .
\end{aligned}
$$

In particular, $a_t \cdot u_h \in (w_f - w_{f^r}) + U^{\bullet}$.

Proof. The formula has been computed in GAP using the package "MajoranaAlgebras" (see [6, repA5(2B).g]). □

Lemma 5.10. *For every $s \in A_6^{(2)}$, there are $f_1, f_2 \in A_6^{(5)}$ such that*

$$\eta_s \in (w_{f_1} - w_{f_2}) + U^{\bullet}.$$

Proof. By Corollary 5.6, there exist $t \in A_6^{(2)}$ and $h \in A_6^{(3)}$ such that $\langle t, h\rangle \cong A_4$ and $\eta_s \in a_t \cdot u_h + U^{\bullet}$. Let L be a maximal subgroup of A_6 isomorphic to A_5 and containing $\langle t, h\rangle$. By Lemma 5.9, there exist $f_1, f_2 \in L^{(5)}$ such that $a_t \cdot u_h \in (w_{f_1} - w_{f_2}) + U^{\bullet}$ and the result follows. □

6. More inner products

Let U and $U^{\bullet}$ be defined as in Equation (23). In this section we show that

$$U^{\bullet} = U,$$

in particular this implies that every dormant 4-axis is contained in the 2-closure V° of V. This fact will be used in Section 8 to prove that $V^{\circ} = V$. Since by definition $U \leq U^{\bullet}$, it is enough to show that the two subspaces have the same dimension. To do that, we first compute the inner products between N-axes, $N \in \{2, 3, 4, 5\}$, and dormant 4-axes. In order to simplify the notation, for the remainder of this paper we let

$$G := A_6.$$

Lemma 6.1. *Let $t \in G^{(2)}$, $h \in G^{(3)}$, $g, l \in G^{(4)}$. The following holds*

(i) *if $t = l^2$, then $(a_t, \overline{v}_l)_V = 0$;*
(ii) *if $\langle t, l\rangle \cong D_8$, then $(a_t, \overline{v}_l)_V = \frac{1}{24}$;*
(iii) *if $\langle t, l\rangle \cong S_4$, then $(a_t, \overline{v}_l)_V = \frac{31}{2^6 \cdot 3}$;*
(iv) *if $\langle h, l\rangle \cong S_4$, then $(u_h, \overline{v}_l)_V = \frac{11}{27}$;*
(v) *$(v_g, \overline{v}_g)_V = 0$;*
(vi) *if $\langle g, l\rangle \cong S_4$, then $(v_g, \overline{v}_l)_V = \frac{11}{48}$ and $(\overline{v}_g, \overline{v}_l)_V = \frac{9}{16}$;*
(vii) *$(\overline{v}_l, \overline{v}_l)_V = 2$;*
(viii) *$(\eta_t, \eta_t)_V = \frac{4565}{3 \cdot 2^{12}}$.*

Proof. These inner products can be obtained inside a Majorana subalgebra $V(S)$, with $S \cong S_4$, of dimension 25 and shape $(2B, 3A, 4A)$. Hence the result follows by [23, Section 4.1], where all these inner products have been computed. □

Lemma 6.2. *Let $t \in G^{(2)}$, $h \in G^{(3)}$, $g, l \in G^{(4)}$, and $f \in G^{(5)}$. The following holds*

(i) *if $\langle t, l\rangle \cong 3^2 : 4$, then $(a_t, \overline{v}_l)_V = \frac{27}{2^8}$;*
(ii) *if $\langle t, l\rangle \cong A_6$, then $(a_t, \overline{v}_l)_V = \frac{31}{384}$;*
(iii) *if $h \in G_3^{(3)}(l) \cup G_5^{(3)}(l^{-1})$, then $(u_h, \overline{v}_l)_V = \frac{2}{27}$;*
(iv) *if $\langle g, l\rangle \cong 3^2 : 4$, then $(v_g, \overline{v}_l)_V = \frac{117}{256}$;*
(v) *if $g \in G_5^{(4)}(l) \cap G_5^{(4)}(l^{-1})$, then $(v_g, \overline{v}_l)_V = \frac{5}{48}$;*
(vi) *if $\langle g, l\rangle \cong 3^2 : 4$, then $(\overline{v}_g, \overline{v}_l)_V = \frac{53}{256}$;*
(vii) *if $\langle h, l\rangle \cong 3^2 : 4$, then $(u_h, \overline{v}_l)_V = \frac{3}{20}$;*
(viii) *if $h \in G_5^{(3)}(l) \cap G_5^{(3)}(l^{-1})$, then $(u_h, \overline{v}_l)_V = \frac{1}{12}$;*
(ix) *if $g \in G_5^{(4)}(l) \cap G_5^{(4)}(l^{-1})$, then $(\overline{v}_g, \overline{v}_l)_V = \frac{31}{144}$;*
(x) *if $g \in G_4^{(4)}(l) \cup G_5^{(4)}(l^{-1})$, then $(\overline{v}_g, \overline{v}_l)_V = \frac{107}{1152}$.*

Proof. For $i \in \{1, \dots, 10\}$, denote by x_i the inner product in claim *(i)*. We shall first express x_7, x_8, x_9, and x_{10} in terms of $x_1, \dots, x_6$. Then, from the fusion law, we shall derive linear relations involving $x_1, \dots, x_6$ and determine the values of $x_1, \dots, x_6$. We refer to [6, algebraA6(2B).g] for the explicit computations.

Let $h \in G^{(3)}$ be such that $\langle h, l\rangle \cong 3^2 : 4$, as in (vii). W.l.o.g. we may assume $l = (1, 2)(3, 6, 4, 5)$ and $h = (2, 3, 4)$. Then $h = l^2 s$ with $s = (2, 3)(5, 6)$, hence $\langle\langle a_{l^2}, a_s\rangle\rangle$ is a Norton-Sakuma algebra of type $3A$. Set

$$\alpha := u_h - \frac{8}{45} a_{l^2} - \frac{2^5}{45}(a_s + a_{l^2 s l^2}). \tag{28}$$

Then, by [10, Table 4],

$$a_{l^2} \cdot \alpha = \frac{1}{4}\alpha,$$

and, by Corollary 5.3,

$$a_{l^2} \cdot \overline{v}_l = 0.$$

Whence (see Remark 4.8)

$$(\alpha, \overline{v}_l)_V = 0. \tag{29}$$

Substituting in Equation (29) the expression of α given in Equation (28) and using Lemma 2.3 and Table 2 to obtain the values of the inner products between axes, we get

$$x_7 = (u_h, \overline{v}_l)_V = \frac{64}{45}(a_t, \overline{v}_l)_V = \frac{64}{45}x_1. \tag{30}$$

Let $h \in G_5^{(3)}(l) \cap G_5^{(3)}(l^{-1})$ as in $(viii)$. We may assume $l = (1,2)(3,6,4,5)$ and $h = (1,2,5)$. Then $h = rs$ with $r = (2,5)(4,6)$ and $s = (1,2)(4,6)$. By [10, Table 4]

$$\beta := u_h - \frac{10}{27}a_s + \frac{2^5}{27}(a_r + a_{srs})$$

and

$$\hat{\beta} := u_h - \frac{8}{45}a_s - \frac{2^5}{45}(a_r + a_{srs})$$

are, respectively, a 0-eigenvector and a $\frac{1}{4}$-eigenvector for ad_{a_s}. Since $\overline{v}_l + \overline{v}_l^s$ is invariant for s, by [20, Equations (3.5) and (3.6)], the vectors

$$\hat{\delta} := a_s(\overline{v}_l + \overline{v}_l^s) - (\overline{v}_l + \overline{v}_l^s, a_s)_V a_s$$

and

$$\delta := \overline{v}_l + \overline{v}_l^s - (\overline{v}_l + \overline{v}_l^s, a_s)_V a_s - 4\hat{\delta}$$

are, respectively, a $\frac{1}{4}$-eigenvector and a 0-eigenvector for ad_{a_s}. Hence $(\beta, \hat{\delta})_V = 0$ and $(\hat{\beta}, \delta)_V = 0$. Since $\langle s, l\rangle \cong S_4$, using [10, Section 5], we obtain expressions of the vectors δ and $\hat{\delta}$ as linear combinations of N-axes and dormant 4-axes in a basis of the subalgebra $V(\langle s, l\rangle)$. Thus, as in the previous case, a straightforward computation gives, by the identities $(\hat{\beta}, \delta)_V = 0$ and $(\beta, \hat{\delta})_V = 0$, respectively the relations

$$x_8 = (u_h, \overline{v}_l)_V = \frac{64}{81}x_1 - \frac{256}{81}x_2 + \frac{62}{243} \tag{31}$$

and

$$x_2 = \frac{1}{10}x_1 + \frac{539}{7680}. \tag{32}$$

Let $g \in G_5^{(4)}(l) \cap G_5^{(4)}(l^{-1})$ as in (ix). We may assume $g = (1,2)(3,5,6,4)$ and $l = (1,4,2,6)(3,5)$. Then, $l^2 = (1,2)(4,6) = s$, and so, by Corollary 5.3, $\overline{v}_l$ is a 0-eigenvector for a_s, whence $(\overline{v}_l, \hat{\delta})_V = 0$. From this equality, we get, as in the previous cases,

$$x_9 = (\overline{v}_g, \overline{v}_l)_V = \tfrac{4}{9}x_1 + \tfrac{97}{576}. \tag{33}$$

To compute the product $(\overline{v}_g, \overline{v}_l)_V$ when $g \in G_4^{(4)}(l) \cup G_5^{(4)}(l^{-1})$, as in claim (x), we may assume $l = (1,2)(3,6,4,5)$ and $g = (1,4,5,6)(2,3)$. Let $s := (1,2)(3,5)$. Then

$$\langle s, g\rangle \cong S_4 \cong \langle s, l\rangle.$$

Proceeding as above, taking, in place of δ and $\hat{\delta}$, two orthogonal eigenvectors for ad_{a_s}, one in the subalgebra $V(\langle s, g\rangle)$ and the other one in the subalgebra $V(\langle s, l\rangle)$, we get, using the structure of these subalgebras given in [10, Section 5],

$$x_{10} = (\overline{v}_g, \overline{v}_l)_V = \tfrac{43}{420}x_1 - \tfrac{9}{44}x_3 - \tfrac{1}{10}x_6 + \tfrac{961}{9216}. \tag{34}$$

We are now ready to compute $x_1, \dots, x_6$. Let $s := (1,2)(4,6)$ and

$$\begin{aligned}
\hat{\beta}_1 &:= v_{(1,2)(3,4,5,6)} - \tfrac{1}{3}a_s - \tfrac{2}{3}(a_{z(3,4)(5,6)} + a_{z(3,6)(4,5)}) - \tfrac{1}{3}a_{z(1,2)(3,5)},\\
\hat{\beta}_2 &:= v_{(1,2,4,6)(3,5)} - \tfrac{1}{3}a_s - \tfrac{2}{3}(a_{z(1,4)(3,5)} + a_{z(2,6)(3,5)}) - \tfrac{1}{3}a_{z(1,6)(2,4)},\\
\hat{\delta}_1 &:= a_s(u_{(2,3,5)} + u_{(1,5,3)}) - (u_{(2,3,5)} + u_{(1,5,3)}, a_s)_V a_s,\\
\beta_1 &:= v_{(1,2)(3,4,5,6)} - \tfrac{1}{2}a_s + 2(a_{z(3,4)(5,6)} + a_{z(3,6)(4,5)}),\\
\beta_2 &:= v_{(1,2,4,6)(3,5)} - \tfrac{1}{2}a_s + 2(a_{z(1,4)(3,5)} + a_{z(2,6)(3,5)}),\\
\delta_1 &:= u_{(2,3,5)} + u_{(1,5,3)} - (u_{(2,3,5)} + u_{(1,5,3)}, a_s)_V a_s - 4\hat{\delta}_1.
\end{aligned}$$

By [10, Table 4] and, since $u_{(2,3,5)} + u_{(1,5,3)}$ is invariant for s, by [20, Equations (3.5) and (3.6)], $\hat{\beta}_1$, $\hat{\beta}_2$, and $\hat{\delta}_1$ are $\frac{1}{4}$-eigenvectors for ad_{a_s}, while β_1, β_2, and δ_1 are 0-eigenvectors for ad_{a_s}. Hence, by Remark 4.8, we have

$$(\delta, \hat{\delta}_1)_V = (\hat{\delta}, \delta_1)_V = (\delta_1, \hat{\beta}_1)_V = (\hat{\delta}_1, \beta_1)_V = (\delta, \hat{\beta}_2)_V = 0.$$

The identities $(\delta, \hat{\delta}_1)_V = 0$ and $(\hat{\delta}, \delta_1)_V = 0$ imply respectively

$$\tfrac{3392}{6075}x_1 - \tfrac{2}{9}x_3 - \tfrac{16}{27}x_6 + \tfrac{7801}{97200} = 0$$

and

$$-\tfrac{2272}{6075}x_1 + \tfrac{7}{9}x_3 + \tfrac{112}{135}x_6 - \tfrac{18461}{97200} = 0,$$

whence $x_6 = \frac{80}{63}x_1 + \frac{131}{1792}$ and $x_3 = -\frac{1376}{1575}x_1 + \frac{6283}{37800}$. The identities $(\delta_1, \hat{\beta}_1)_V = 0$ and $(\hat{\delta}_1, \beta_1)_V = 0$ imply $x_1 = \frac{27}{256}$ and $x_5 = \frac{5}{48}$, whence $x_2 = \frac{31}{384}$, $x_3 = \frac{2}{27}$ and $x_6 = \frac{53}{256}$. Finally, the identity $(\delta, \hat{\beta}_2)_V = 0$ implies $x_4 = \frac{117}{256}$. □

Lemma 6.3. *Let $g, l \in G^{(4)}$ and $f \in G^{(5)}$. The following hold*

(i) if $f \in G_5^{(5)}(l) \cap G_5^{(5)}(l^{-1})$, then $(w_f, \overline{v}_l)_V = 0$;
(ii) if $f \in G_3^{(5)}(l) \cap G_5^{(5)}(l^{-1})$, then $(w_f, \overline{v}_l)_V = \frac{175}{49152}$;
(iii) if $f \in G_4^{(5)}(l) \cap G_3^{(5)}(l^{-1})$, then $(w_f, \overline{v}_l)_V = -\frac{175}{49152}$;
(iv) if $f \in G_4^{(5)}(l) \cap G_5^{(5)}(l^{-1})$, then $(w_f, \overline{v}_l)_V = \frac{49}{6144}$;
(v) if $f \in G_2^{(5)}(l) \cap G_5^{(5)}(l^{-1})$, then $(w_f, \overline{v}_l)_V = -\frac{49}{6144}$;
(vi) if $g \in G_4^{(4)}(l) \cup G_5^{(4)}(l^{-1})$, then $(v_g, \overline{v}_l)_V = \frac{41}{384}$.

Proof. Let l and f be as in (i). We may assume $l = (1,2)(3,4,5,6)$ and $f = (2,4,5,3,6)$. By Corollary 5.3, $\overline{v}_l$ is a 0-eigenvector for $\mathrm{ad}_{a_{l^2}}$. Let

$$\hat{\beta} := w_f + \frac{1}{2^7}(a_{l^2 f} + a_{l^2 f^4} - a_{l^2 f^2} - a_{l^2 f^3}).$$

Since $\langle l^2, f\rangle \cong D_{10}$, $\langle\langle a_{l^2}, a_{l^2 f}\rangle\rangle$ is a Norton-Sakuma algebra of type $5A$ and by [10, Table 4], $\hat{\beta}$ is a $\frac{1}{4}$-eigenvector for $\mathrm{ad}_{a_{l^2}}$. Thus $(\overline{v}_l, \hat{\beta})_V = 0$. Since, by Lemmas 6.1 and 6.2, $(\overline{v}_l, a_{l^2 f} + a_{l^2 f^4} - a_{l^2 f^2} - a_{l^2 f^3})_V = 0$, we get

$$(\overline{v}_l, w_f)_V = (\overline{v}_l, w_f + \frac{1}{2^7}(a_{l^2 f} + a_{l^2 f^4} - a_{l^2 f^2} - a_{l^2 f^3}))_V = (\overline{v}_l, \hat{\beta})_V = 0.$$

Let l and f be as in (ii). We may assume $l = (1,2)(3,6,4,5)$ and $f = (1,4,3,5,6)$. Set $s := (3,5)(4,6)$,

$$\beta_1 := w_f + \frac{3}{2^9}a_s - \frac{15}{2^7}(a_{sf} + a_{sf^4}) - \frac{1}{2^7}(a_{sf^2} + a_{sf^3}) \tag{35}$$

and

$$\hat{\beta}_1 := w_f + \frac{1}{2^7}(a_{sf} + a_{sf^4} - a_{sf^2} - a_{sf^3}).$$

Then $f^s = f^{-1}$ and so, by [10, Table 4], β_1 is a 0-eigenvector and $\hat{\beta}_1$ is a $\frac{1}{4}$-eigenvector for ad_{a_s}, respectively. Moreover, since $l^s = l^{-1}$, $\overline{v}_l$ is s-invariant. By [20, Equations (3.5) and (3.6)], it follows that the vectors

$$\hat{\delta}_1 := a_s\overline{v}_l - (\overline{v}_l, a_s)_V a_s \quad \text{and} \quad \delta_1 := \overline{v}_l - (\overline{v}_l, a_s)_V a_s - 4\hat{\delta}_1$$

are, respectively, a $\frac{1}{4}$-eigenvector and a 0-eigenvector for ad_{a_s}, whence

$$(\beta_1, \hat{\delta}_1)_V = (\delta_1, \hat{\beta}_1)_V = 0.$$

Now, substituting the expressions for $\beta_1, \hat{\beta}_1, \delta_1$, and $\hat{\delta}_1$ in the identity $(\beta_1, \hat{\delta}_1)_V + (\delta_1, \hat{\beta}_1)_V = 0$, by the results in Section 4 and Lemmas 6.1 and 6.2, we obtain $(\overline{v}_l, w_f)_V = \frac{175}{49152}$.

If l and f are as in (iii), then l and f^2 satisfy the hypotheses of claim (ii) and the result follows by Remark 2.2.

Let l and f be as in (iv). We may assume $l = (1,2)(3,6,4,5)$ and $f = (1,6,2,5,4)$. Set $s := (1,2)(4,5)$ and define β_1 as in Equation (35). By [10, Table 4], β_1 is a 0-eigenvector for ad_{a_s}. Moreover, by [20, Equations (3.5) and (3.6)],

$$\hat{\delta}_2 := a_s(\overline{v}_l + \overline{v}_{l^s}) - (\overline{v}_l + \overline{v}_{l^s}, a_s)_V a_s$$

is a $\frac{1}{4}$-eigenvector for ad_{a_s} and it can be computed explicitly as a linear combination of axes and dormant 4-axes in the subalgebra $V(\langle s, l\rangle)$. From the identity $(\beta_1, \hat{\delta}_2)_V = 0$ we get $(\overline{v}_l, w_f)_V = \frac{49}{6144}$.

If l and f are as in (v), then l and f^2 satisfy the hypotheses of claim (iv) and the result follows by Remark 2.2.

Finally, assume g and l are as in (vi). Fix $s \in G^{(2)}$, then, by Lemma 6.1.(ix),

$$(\eta_s, \eta_s)_V = \tfrac{4565}{12288}. \tag{36}$$

By Lemma 5.10, η_s has an expression as a linear combination of axes, odd axes and dormant 4-axes, which can be explicitly obtained using Proposition 5.2, Lemma 5.5, Proposition 5.7, and Lemma 5.9 (see [6, algebraA6(2B).g]). Substituting this expression in Equation (36), by the linearity of the inner product and the values of the inner products obtained so far, we get

$$\tfrac{4565}{12288} = (\eta_s, \eta_s)_V = \tfrac{36643}{98304} - \tfrac{3}{256}(v_g, \overline{v}_l)_V,$$

whence

$$(v_g, \overline{v}_l)_V = \tfrac{41}{384}. \quad \square$$

Let $g \in G^{(4)}$. Let K_1 and K_2 denote the two subgroups of G isomorphic to S_4 containing g, such that, for $i \in \{1,2\}$, the elements of order 3 in K_i have cycle structure 3^i. Further, let Y be the set of all 2-axes, 3-axes and 4-axes of V (i.e. $Y = X \setminus 5A$, in the notation used at the beginning of Section 4). Set

$$\begin{aligned}
[\overline{v}_g]_2 := & -\tfrac{200}{81} a_{g^2} - \tfrac{176}{81} \sum_{t \in K_1 \cap G_2^{(2)}(g)} a_t - \tfrac{68}{81} \sum_{t \in K_2 \cap G_2^{(2)}(g)} a_t + \tfrac{40}{81} \sum_{t \in G_2^{(5)}(g)} a_t \\
& - \tfrac{98}{81} \sum_{t \in K_1 \cap G_2^{(3)}(g)} a_t + \tfrac{10}{81} \sum_{t \in K_2 \cap G_2^{(3)}(g)} a_t \\
& - \tfrac{2}{81} \sum_{t \in G_2^{(4)}(g), [g,t] \in K_2 \setminus \{1_G\}} a_t + \tfrac{52}{81} \sum_{t \in G_2^{(4)}(g), [g,t] \in K_1 \setminus \{1_G\}} a_t;
\end{aligned}$$

$$[\overline{v}_g]_3 = -\tfrac{15}{16} \sum_{h\in(1,2,3)^G, |[g,h]|=5} u_h + \tfrac{25}{24} \sum_{h\in K_1^{(3)}} u_h + \tfrac{5}{6} \sum_{h\in(1,2,3)^G, |[g,h]|=2} u_h$$
$$+\tfrac{5}{48} \sum_{h\in K_2^{(3)}} u_h - \tfrac{5}{48} \sum_{h\in(1,2,3)(4,5,6)^G, |[g,h]|=2} u_h;$$
$$[\overline{v}_g]_4 = -\tfrac{14}{27} \sum_{h\in G^{(4)}, |[g,h]|=5} v_h + \tfrac{40}{27} \sum_{h\in G_2^{(4)}(g^2), |[g,h]|=3} v_h$$
$$+\tfrac{10}{27} \sum_{h\in G^{(4)}, |[g,h]|=2} v_h - \tfrac{2}{27} \sum_{h\in G_4^{(4)}(g^2), |[g,h]|=3} v_h.$$

Note that, for $N \in \{2, 3, 4\}$, $[\overline{v}_g]_N \in V^{(NA)} \leq \langle Y\rangle$.

Lemma 6.4. *For every $g \in G^{(4)}$, $\overline{v}_g$ admits the following expression as a linear combination of elements of Y:*

$$\overline{v}_g = [\overline{v}_g]_2 + [\overline{v}_g]_3 + [\overline{v}_g]_4. \tag{37}$$

Proof. The inner products computed in this section and in Section 4 imply

$$(\overline{v}_g - [\overline{v}_g]_2 - [\overline{v}_g]_3 - [\overline{v}_g]_4 \, , \, \overline{v}_g - [\overline{v}_g]_2 - [\overline{v}_g]_3 - [\overline{v}_g]_4)_V = 0,$$

whence $\overline{v}_g = [\overline{v}_g]_2 + [\overline{v}_g]_3 + [\overline{v}_g]_4$ by the positive definiteness of the inner product. (for the explicit computation see [6, algebraA6(2B).g]). □

Corollary 6.5. *$U^\bullet = U$.*

Proof. By Lemma 6.4 $\overline{v}_g \in U$. Now U is invariant under the action of G, so the claim follows because, by Hypothesis (M8D), $\overline{v}_g$ depends uniquely on $\langle g\rangle$ and G is transitive on its subgroups of order 4. □

7. 5-axes

Let U be defined as in Equation (23), let H and S be maximal subgroups of A_6 isomorphic to A_5 and S_4, respectively, and let $V(H)$ and $V(S)$ be their associated subalgebras of V, as defined in Section 2. In this section we prove that the subspace U is a hyperplane of V° and that $V(H)$ and $V(S)$ are contained in V°. Further, we show that V° has dimension 121 and give its decomposition into irreducible $\mathbb{R}[S_6]$-modules. These results will be used in Section 8 to prove that $V = V^\circ$, from which Theorems 2 and 3 will follow.

Let

$$w := \sum_{e\in G^{(5)}} w_e \tag{38}$$

be the sum of all the 36 positive 5-axes of V. Moreover, for $f \in G^{(5)}$ set

$$[w_f]_2 := \tfrac{5}{768}\left(\sum_{t\in G_3^{(2)}(f)} a_t - \sum_{t\in G_5^{(2)}(f),\langle f,t\rangle\cong A_5} a_t\right) + \tfrac{5}{192}\left(\sum_{t\in G_5^{(2)}(f),\langle f,t\rangle=G} a_t - \sum_{t\in G_4^{(2)}(f)} a_t\right); \tag{39}$$

$$[w_f]_3 := \tfrac{165}{16384}\left(\sum_{h\in G^{(3)},[f,h]\in f^G} u_h - \sum_{h\in G^{(3)},|[f,h]|=3} u_h\right) + \tfrac{105}{16384}\left(\sum_{h\in G^{(3)},|[f,h]|=5,[f,h]\notin f^G} u_h - \sum_{h\in G^{(3)},|[f,h]|=2} u_h\right); \tag{40}$$

$$[w_f]_4 := \tfrac{1}{64}\left(\sum_{h\in G^{(4)},|[f,h]|=2} v_h + \sum_{h\in G^{(4)},[f,h]\in f^G} v_h\right) - \tfrac{1}{64}\left(\sum_{h\in G^{(4)},|[f,h]|=4,(w_f,v_h)_V\neq 0} v_h + \sum_{h\in G^{(4)},|[f,h]|=5,[f,h]\notin f^G} v_h\right). \tag{41}$$

Lemma 7.1. *For every $f \in G^{(5)}$, w_f admits the following expression as a linear combination of w and elements of Y:*

$$w_f = \tfrac{1}{36}w + [w_f]_2 + [w_f]_3 + [w_f]_4 \tag{42}$$

Proof. Using the values of the inner products between axes obtained in Sections 4 and 6, we get

$$(w_f - \tfrac{1}{36}w - [w_f]_2 - [w_f]_3 - [w_f]_4\,,\, w_f - \tfrac{1}{36}w - [w_f]_2 - [w_f]_3 - [w_f]_4)_V = 0$$

(see [6, algebraA6(2B).g]), whence $w_f = \frac{1}{36}w + [w_f]_2 + [w_f]_3 + [w_f]_4$, by the positive definiteness of the inner product. □

Now let K be a subgroup of G isomorphic to S_4 and let s be an element of order 2 in $K \setminus K'$. By Proposition 2.4, there are $r \in K'$ of order two and $g \in G^{(4)}$ such that $C_K(s) = \{1_G, s, r, g^2\}$. Let $\bar{t} \in K^{(2)}$ be such that the Klein subgroup of K is $\{1_G, r, \bar{t}, \bar{t}^s\}$. Then, set

$$[\eta_s]_2 := \tfrac{15}{32}a_s - \tfrac{1}{24}a_r - \tfrac{1}{36}(a_{\bar{t}} + a_{\bar{t}^s}) - \tfrac{23}{288}a_{g^2} + \tfrac{11}{144}\sum_{t\in C_G(s)^{(2)}\setminus C_G(r)} a_t$$

$$+\tfrac{1}{8}\sum_{t\in K^{(2)}\setminus\{s,r,\bar{t},\bar{t}^s,g^2\}} a_t - \tfrac{17}{144}\sum_{t\in C_G(g^2)^{(2)}\setminus C_G(s)} a_t$$

$$+\tfrac{11}{144}\sum_{t\in C_G(s)^{(2)}\setminus K} a_t + \tfrac{1}{16}\sum_{\substack{t\in G^{(2)},\\ |ts|=4,\,|tr|=3}} a_t \tag{43}$$

$$+\tfrac{1}{72}\sum_{t\in G^{(2)},|tg^2|=5} a_t + \tfrac{1}{36}\sum_{\substack{t\in G^{(2)},\\ |ts|=5,\,|tr|=3}} a_t;$$

$$[\eta_s]_3 := -\tfrac{15}{128}\sum_{\substack{h\in G^{(3)},\\ |\langle h,s\rangle|\le 24,\,\langle h,g^2\rangle\cong S_3}} u_h + \tfrac{5}{64}\sum_{\substack{h\in G^{(3)},\\ \langle h,s\rangle\cong A_4,\,\langle h,g^2\rangle\cong S_4}} u_h$$

$$+\tfrac{5}{128}\sum_{\substack{h\in G^{(3)},\\ \langle h,s\rangle\cong S_4,\,\langle h,g^2\rangle\cong A_4}} u_h + \tfrac{5}{128}\sum_{\substack{h\in G^{(3)},\\ \langle h,s\rangle\cong S_3,\,\langle h,g^2\rangle\cong A_4}} u_h$$

$$+\tfrac{5}{512}\sum_{\substack{h\in G^{(3)},\\ \langle h,s\rangle\cong\langle h,g^2\rangle\cong S_4}} u_h - \tfrac{15}{512}\sum_{\substack{h\in G^{(3)}\cap(1,2,3)^G,\\ \langle h,s\rangle\cong\langle h,g^2\rangle\cong A_5}} u_h \tag{44}$$

$$-\tfrac{5}{512}\sum_{\substack{h\in G^{(3)},\\ \langle h,s\rangle\cong S_3,\,\langle h,g^2\rangle\cong S_4}} u_h - \tfrac{5}{512}\sum_{\substack{h\in G^{(3)},\\ \langle h,s\rangle\cong\langle h,g^2\rangle\cong A_4}} u_h;$$

$$[\eta_s]_4 := \tfrac{1}{24}v_g - \tfrac{1}{48}\sum_{\substack{l\in G^{(4)},\\ l^s=l^{-1},\,\langle l,g^2\rangle\cong S_4}} v_l - \tfrac{1}{48}\sum_{\substack{l\in K^{(4)},\\ \langle l,s\rangle=K}} v_l$$

$$-\tfrac{1}{16}\sum_{\substack{l\in G^{(4)},\\ |\langle l,g^2\rangle|=36,\,|\langle l,s\rangle|\in\{24,36\}}} v_l - \tfrac{1}{48}\sum_{\substack{l\in G^{(4)},\\ \langle l,g^2\rangle=G}} v_l$$

$$+\tfrac{1}{48}\sum_{\substack{l\in G^{(4)},\\ \langle l,s\rangle=G,\,|\langle l,g^2\rangle|=36}} v_l + \tfrac{5}{48}\sum_{\substack{l\in G^{(4)},\\ |\langle l,s\rangle|=36,\,\langle l,g^2\rangle\cong S_4}} v_l \tag{45}$$

$$+\tfrac{7}{48}\sum_{\substack{l\in G^{(4)},\\ l^{g^2}=l^{-1},\,\langle l,s\rangle\cong S_4}} v_l + \tfrac{5}{32}\sum_{\substack{l\in G^{(4)},\\ l^s=l^{-1},\,l^{g^2}=l^{-1}}} v_l.$$

Corollary 7.2. *With the above notation, the following assertions hold:*

(*i*) $V^\circ = \langle w\rangle + U$ *and* $\dim(V^\circ) = \dim(U) + 1 = 121$;
(*ii*) *if* $f_1, f_2 \in G^{(5)}$, *then* $w_{f_1} - w_{f_2} \in U$;
(*iii*) *if* S *is a subgroup of* G *isomorphic to* S_4 *and* $s \in S^{(2)} \setminus S'$, *then* $\eta_s \in U$ *and admits the following expression as a linear combination of elements of* Y:

$$\eta_s = [\eta_s]_2 + [\eta_s]_3 + [\eta_s]_4, \tag{46}$$

in particular, $V(S) \leq U$*;*

(iv) if H *is a subgroup of* G *isomorphic to* A_5*, then* $V(H) \leq V^\circ$*.*

Proof. Since $V^\circ = U + V^{(5A)}$, by Lemma 7.1 we have $V^\circ = \langle w \rangle + U$. A direct computation in GAP of the Gram matrices Γ and Γ^* associated to Y and $Y \cup \{w\}$, respectively, gives (i) (see [6, algebraA6(2B).g]). The second assertion follows immediately by Equation (42).

Now let S be a subgroup of G isomorphic to S_4 and let $s \in S^{(2)} \setminus S'$ as in (iii). By Corollary 5.10, $\eta_s \in w_{f_1} - w_{f_2} + U^\bullet$ for some $f_1, f_2 \in G^{(5)}$. Then, by (ii) and Proposition 6.5, $\eta_s \in U$, whence, by Proposition 5.2.(ii) and Corollary 6.5, $V(S) \leq U$. The explicit formula for η_s has been computed with GAP in [6, algebraA6(2B).g].

Finally, let H be a subgroup of G isomorphic to A_5 as in (iv) and let $t \in H^{(2)}$. By Proposition 5.7 and Corollary 6.5, $\gamma_t \in U^\bullet = U$, so the result follows by Lemma 5.8. □

Let now $t \in G^{(2)}$ and let K_1 and K_2 be the two subgroups of G isomorphic to S_4 such that t is contained in the derived subgroup of K_i, $i \in \{1, 2\}$. Denote by $\gamma_{t,i}$ the element γ_t defined as in Equation (20) in the subalgebra $V(K_i)$.

Corollary 7.3. *With the above notation, let* $t \in G^{(2)}$*. Then, for* $N \in \{2, 3, 4\}$ *and* $i \in \{1, 2\}$*, there exist* $[\gamma_{t,i}]_N \in V^{(NA)}$ *such that* $\gamma_{t,i} = [\gamma_{t,i}]_2 + [\gamma_{t,i}]_3 + [\gamma_{t,i}]_4$*.*

Proof. This follows from Equations (22) and (46). □

We conclude this section by determining the decomposition into irreducible submodules of V°.

For a partition λ of 6, denote by $V^{(\lambda)}$ the subspace of V spanned by the N-axes in V associated to a permutation of cycle type λ.

Proposition 7.4. *The structure of* $\mathbb{R}[A_6]$*-module of* V° *lifts naturally to a structure of* $\mathbb{R}[S_6]$*-module. With the above notation, the decomposition of* V° *into irreducible* $\mathbb{R}[S_6]$*-submodules is the following*

$$V^\circ \cong 3\mathfrak{S}^{(6)} \oplus 3\mathfrak{S}^{(5,1)} \oplus 5\mathfrak{S}^{(4,2)} \oplus 2\mathfrak{S}^{(3,2,1)} \oplus \mathfrak{S}^{(3^2)} \oplus 3\mathfrak{S}^{(2^3)} \oplus \mathfrak{S}^{(2,1^4)} \oplus \mathfrak{S}^{(1^6)}.$$

Proof. By Equation (23) and Corollary 7.2,

$$V^\circ = (V^{(2A)} + V^{(3A)} + V^{(4A)}) \oplus \langle w \rangle. \tag{47}$$

Let

$$\zeta_w \colon S_6 \to GL(\langle w \rangle)$$

be the alternating representation and denote by $\langle w \rangle_\zeta$ the corresponding $\mathbb{R}[S_6]$-module, so that

$$\langle w\rangle_\zeta \cong \mathfrak{S}^{(1^6)}. \tag{48}$$

Further, for $N \in \{2,3,4\}$, let $\mathfrak{P}_N$ be the $\mathbb{R}[S_6]$-permutation module induced by the action by conjugation of S_6 on the set $\mathcal{C}_N$ of the cyclic subgroups of order N in A_6, and let $\mathfrak{P}$ be the direct sum of $\mathbb{R}[S_6]$-modules $\mathfrak{P}_2 \oplus \mathfrak{P}_3 \oplus \mathfrak{P}_4$. Since U is a homomorphic image of $\mathfrak{P}$ (as $\mathbb{R}[A_6]$-modules), the inner product on V induces in an obvious way an inner product

$$\kappa\colon \mathfrak{P} \times \mathfrak{P} \to \mathbb{R}$$

on $\mathfrak{P}$ (see e.g. [5, Section 2]). By Remark 4.2, $\mathrm{rad}(\kappa)$ is a $\mathbb{R}[S_6]$-submodule of $\mathfrak{P}$ and $U \cong \mathfrak{P}/\mathrm{rad}(\kappa)$, thus the permutation representation

$$\zeta_{\mathfrak{P}}\colon S_6 \to GL(\mathfrak{P})$$

defines a representation

$$\zeta_U\colon S_6 \to GL(U).$$

By construction, $\zeta_U \oplus \zeta_w$ is a representation of S_6 on $U \oplus \langle w\rangle \cong V^\circ$, whose restriction to A_6 is induced by the identity map on A_6 as the Miyamoto group of V, proving the first assertion.

Let $N \in \{2,4\}$, by Proposition 4.15,

$$\dim(V^{(NA)}) = 45 = |\mathcal{C}_N| = \dim(\mathfrak{P}_N),$$

whence, by [3, Lemma 6],

$$V^{(NA)} \cong \mathfrak{P}_N \cong \mathfrak{S}^{(6)} \oplus \mathfrak{S}^{(5,1)} \oplus 2\mathfrak{S}^{(4,2)} \oplus \mathfrak{S}^{(3,2,1)} \oplus \mathfrak{S}^{(2^3)}. \tag{49}$$

Let now $N = 3$. By Proposition 4.15,

$$\dim(V^{(3A)}) = 40 = |\mathcal{C}_3| = \dim(\mathfrak{P}_3),$$

and, since the exceptional automorphism of A_6 swaps 3-cycles with cycles of type 3^2, we have

$$V^{(3A)} = V^{(3)} \oplus V^{(3^2)} \text{ and } \dim(V^{(3)}) = \dim(V^{(3^2)}) = 20.$$

Let

$$V_+^{(3^2)} := \langle u_{ab} + u_{ab^{-1}} \mid a, b \text{ disjoint 3-cycles in } G\rangle$$

and

$$V_-^{(3^2)} := \langle u_{ab} - u_{ab^{-1}} \mid a, b \text{ disjoint 3-cycles in } G \rangle.$$

Clearly $V^{(3^2)}$ is the sum of $V_+^{(3^2)}$ and $V_-^{(3^2)}$. Since

$$\dim(V_+^{(3^2)}) = \dim(V_-^{(3^2)}) = 10 = \tfrac{1}{2}\dim(V^{(3^2)}),$$

we have

$$V^{(3^2)} = V_+^{(3^2)} \oplus V_-^{(3^2)}. \tag{50}$$

Since $V_+^{(3^2)}$ is isomorphic to the permutation module associated to the action of S_6 on the 10 conjugates of $\langle (1,2,3),(4,5,6)\rangle$, an easy computation shows that

$$V_+^{(3^2)} \cong \mathfrak{S}^{(6)} \oplus \mathfrak{S}^{(4,2)}. \tag{51}$$

By [4, Theorem 1.1], $V^{(3^2)} \cong \mathfrak{S}^{(6)} \oplus \mathfrak{S}^{(4,2)} \oplus \mathfrak{S}^{(2^3)} \oplus \mathfrak{S}^{(2,1^4)}$, whence by Equation (50),

$$V_-^{(3^2)} \cong V^{(3^2)}/V_+^{(3^2)} \cong \mathfrak{S}^{(2^3)} \oplus \mathfrak{S}^{(2,1^4)}. \tag{52}$$

By the Pasechnik's relations [11, Lemma 3.4], $V_+^{(3^2)} \subseteq V^{(2A)} \oplus V^{(3)}$, whence

$$\begin{aligned}\dim(U) &\leq \dim(V^{(2A)}) + \dim(V^{(3)}) + \dim(V_-^{(3^2)}) + \dim(V^{(4A)}) \\ &\leq 45 + 20 + 10 + 45 = 120.\end{aligned}$$

By Corollary 7.2, $\dim(U) = 120$. It follows that

$$U = V^{(2A)} \oplus V^{(3)} \oplus V_-^{(3^2)} \oplus V^{(4A)}.$$

Finally, since $V^{(3)}$ is isomorphic to the permutation module of S_6 on its subsets of order 3, we have

$$V^{(3)} \cong \mathfrak{S}^{(6)} \oplus \mathfrak{S}^{(5,1)} \oplus \mathfrak{S}^{(4,2)} \oplus \mathfrak{S}^{(3^2)}. \tag{53}$$

The result follows by Equations (49), (52), and (53). □

8. The algebra closure of V°

In this section we prove that there is a unique possibility for the algebra product in V, that $V^\circ = V$, and prove Theorems 2 and 3. We first show that $a_t \cdot V^\circ \subseteq V^\circ$, for every $t \in G^{(2)}$.

Lemma 8.1. *For every $t \in G^{(2)}$, $a_t \cdot (V^{(2A)} + V^{(3A)}) \subseteq V^\circ$.*

Proof. By the definition of V°, we have $a_t \cdot V^{(2A)} \subseteq V^\circ$. Since for every $h \in G^{(3)}$, $\langle t, h\rangle$ is contained in a maximal subgroup of G isomorphic to S_4 or to A_5 (see Table 3), by Corollary 7.2, we get $a_t \cdot V^{(3A)} \subseteq V^\circ$. □

To deal now with $V^{(4A)}$ and w, we show we can find a set Λ of vectors generating $V^{(4A)}$ and such that, for every $v \in \Lambda$, the product $a_t \cdot v$ lies in V°.

Lemma 8.2. *Let $t \in G^{(2)}$ and $g \in G^{(4)}$ and suppose that $\langle t, g\rangle \cong S_4$. Then, with the notation of Equation (37), $a_t \cdot [\overline{v}_g]_4 \in V^\circ$.*

Proof. Since $\langle t, g\rangle \cong S_4$, by Corollary 7.2, V° contains $a_t \cdot \overline{v}_g$. Hence, by Lemma 6.4 and Corollary 8.1,

$$a_t \cdot [\overline{v}_g]_4 = a_t \cdot \overline{v}_g - a_t \cdot ([\overline{v}_g]_2 + [\overline{v}_g]_3) \in V^\circ. \quad \square$$

Lemma 8.3. *Let $t \in G^{(2)}$ and $f \in G^{(5)}$ and suppose that $\langle t, f\rangle$ is properly contained in G. Then, with the notation of Equation (41), $a_t \cdot [w_f]_4 \in V^\circ$.*

Proof. Set

$$E_t := \{(f_1, f_2) \in G^{(5)} \times G^{(5)} \mid \langle t, f_i\rangle < G,\ \forall i \in \{1, 2\}\},$$

and

$$F_t := \{(f_1, f_2) \in G^{(5)} \times G^{(5)} \mid \langle f_1^t\rangle = \langle f_2\rangle\}.$$

We first prove that

$$\text{for every } (f_1, f_2) \in E_t \cup F_t,\ \ a_t \cdot (w_{f_1} - w_{f_2}) \in V^\circ. \tag{54}$$

Assume $(f_1, f_2) \in E_t$ and let H_i be a maximal subgroup of G containing $\langle t, f_i\rangle$. Then $H_i \cong A_5$ and a_t and w_{f_i} are contained in the subalgebra $V(H_i)$. Hence, by Lemma 5.8, there is a unique possibility for the product $a_t \cdot w_{f_i}$, and, by Corollary 7.2.(*iv*), this product belongs to V°. Next assume $(f_1, f_2) \in F_t$. Since $\langle f_1^t\rangle = \langle f_2\rangle$, by the choice of the set $G^{(5)}$, f_1 and f_2 are conjugate in G. Hence, either $f_1^t = f_2$, or $f_1^t = f_2^{-1}$. In both cases $(w_{f_1})^t = w_{f_1^t} = w_{f_2}$. Thus $w_{f_1} - w_{f_2}$ is a $\frac{1}{32}$-eigenvector for ad_{a_t} and so $a_t \cdot (w_{f_1} - w_{f_2}) = \frac{1}{32}(w_{f_1} - w_{f_2}) \in V^\circ$, proving (54).

Set

$$W_t := \langle [w_{f_1}]_4 - [w_{f_2}]_4 \ \mid\ (f_1, f_2) \in E_t \cup F_t\rangle,$$

$$W_t' := \langle v_g \ \mid\ \langle t, g\rangle \cong D_8 \text{ or } \langle t, g\rangle \cong S_4\rangle,$$

$$W''_t := \langle v_g - v_{g^t} \mid g \in G^{(4)} \rangle, \text{ and}$$

$$\overline{W}_t := W_t + W'_t + W''_t.$$

A direct calculation using GAP (see [6, algebraA6(2B).g]) shows that, for $f \in G^{(5)}$ such that $\langle t, f \rangle < G$, we have $\dim(\langle [w_f]_4, \overline{W}_t \rangle) = \dim(\overline{W}_t) = 36$, whence

$$[w_f]_4 \in \overline{W}_t. \tag{55}$$

By Equation (42), for every $(f_1, f_2) \in G^{(5)} \times G^{(5)}$, we have

$$a_t \cdot ([w_{f_1}]_4 - [w_{f_2}]_4) = a_t \cdot (w_{f_1} - w_{f_2}) - a_t \cdot ([w_{f_1}]_2 + [w_{f_1}]_3 - [w_{f_2}]_2 - [w_{f_2}]_3),$$

whence, by Equation (54) and Corollary 8.1, we get

$$a_t \cdot W_t \subseteq V^\circ. \tag{56}$$

By claim (iii) of Corollary 7.2, we have also

$$a_t \cdot W'_t \subseteq V^\circ, \tag{57}$$

and finally

$$a_t \cdot W''_t \subseteq V^\circ, \tag{58}$$

since $v_g - v_{g^t}$ is a $\frac{1}{32}$-eigenvector for ad_{a_t}. Thus, by Equations (55), (56), (57), and (58), $a_t \cdot [w_f]_4 \in a_t \cdot \overline{W}_t \subseteq V^\circ$. □

Lemma 8.4. *Let w be as in Equation (38). For every $t \in G^{(2)}$, $a_t \cdot w \in V^\circ$.*

Proof. Let $f \in G^{(5)}$ be such that $f^t = f^{-1}$. Then, w_f is contained in the subalgebra $\langle\langle a_t, a_{tf} \rangle\rangle$, which is of type $5A$. By Table 2, $a_t \cdot w_f \in V^\circ$. Hence, by Equation (42), Corollary 8.1 and Lemma 8.3, we have

$$a_t \cdot w = 36 a_t \cdot (w_f - [w_f]_2 - [w_f]_3 - [w_f]_4) \in V^\circ. \quad \square$$

Lemma 8.5. *For every $t \in G^{(2)}$, $a_t \cdot V^{(4A)} \subseteq V^\circ$.*

Proof. Let $t \in G^{(2)}$ and let K_1 and K_2 be the two subgroups of G isomorphic to S_4 such that t is contained in the derived subgroup of K_i, $i \in \{1, 2\}$. Let $\gamma_{t,i}$ and $[\gamma_{t,i}]_N$, for $N \in \{2, 3, 4\}$, be as in Corollary 7.3. Moreover, for $i \in \{1, 2\}$, let L_i and L_i^* be the two subgroups of G isomorphic to A_5 containing the derived subgroup of K_i. Let

$$Q_t := \langle [\gamma_{t,i}]_4^x \mid x \in L_i \cup L_i^*, i \in \{1,2\} \rangle. \tag{59}$$

For each $i \in \{1,2\}$, the vectors a_t and $\gamma_{t,i}$ belong to $V(L_i) \cap V(L_i^*)$. Hence, for each $x \in L_i \cup L_i^*$,

$$(\gamma_{t,i})^x \in V(L_i) \cup V(L_i^*)$$

and so, by Corollary 7.2,

$$a_t \cdot (\gamma_{t,i})^x \in V(L_i) \cup V(L_i^*) \subseteq V^\circ.$$

Thus, by Corollaries 7.3 and 8.1,

$$a_t \cdot Q_t \subseteq V^\circ. \tag{60}$$

Let W_t' and W_t'' be the subspaces defined in the proof of Lemma 8.3 and set

$$R_t := \langle [w_f]_4 \mid f \in G^{(5)}, \langle t, f \rangle \leq G \rangle \text{ and } \overline{R}_t := R_t + W_t' + W_t''.$$

As in the proof of Lemma 8.3, by Equation (60), Lemma 8.3 and Corollary 7.2,

$$a_t \cdot (Q_t + \overline{R}_t) \subseteq V^\circ.$$

A direct computation using GAP (see [6, algebraA6(2B).g]) shows that

$$\dim(Q_t + \overline{R}_t) = 45.$$

Since $Q_t + \overline{R}_t \leq V^{(4A)}$ and, by Proposition 4.15, $\dim(V^{(4A)}) = 45$, it follows that $Q_t + \overline{R}_t = V^{(4A)}$ and the result follows. □

Proposition 8.6. *For every $t \in G^{(2)}$, $a_t \cdot V^\circ \subseteq V^\circ$.*

Proof. The result follows immediately by Corollary 7.2.(i), Corollary 8.1, Lemma 8.4, and Lemma 8.5. □

We can now use the resurrection principle in its simpler version given in [10, Lemma 1.8] to get the remaining products between two odd axes.

Lemma 8.7. *Let $h, k \in G$ such that x_h and x_k are odd axes of V. Suppose there is $t \in G^{(2)}$ inverting by conjugation both h and k. Then $x_h \cdot x_k \in V^\circ$.*

Proof. $\langle\langle a_t, x_h \rangle\rangle$ is contained in a Norton-Sakuma algebra of type $|h|A$ and similarly $\langle\langle a_t, x_k \rangle\rangle$ is contained in a Norton-Sakuma algebra of type $|k|A$. Let e_h be a 0-eigenvector for ad_{a_t} and let e_k and $\tilde{e}_k$ be a 0-eigenvector and a $\frac{1}{4}$-eigenvector for ad_{a_t}, respectively. Using [10, Table 4], we can express e_h (resp. e_k and $\tilde{e}_k$) as a linear combination of axes and x_h (resp. x_k and $\tilde{x}_k$). It follows that there exist y and z in V°, such that

$$e_h \cdot e_k = x_h \cdot x_k + y \text{ and } e_h \cdot \tilde{e}_k = x_h \cdot x_k + z.$$

By the fusion law, $e_h \cdot e_k$ is a 0-eigenvector for ad_{a_t}, while $e_h \cdot e_k$ is a $\frac{1}{4}$-eigenvector for ad_{a_t}. By [10, Lemma 1.8] and Proposition 8.6,

$$x_h \cdot x_k = -[4a_t \cdot (y - z) + z] \in V^\circ. \quad \square$$

Corollary 8.8. $V^{(3)} \cdot V^\circ \subseteq V^\circ$

Proof. The result follows by Lemma 8.7 since, for every $h, k \in G^{(3)}$, $g \in G^{(4)}$, and $f \in G^{(5)}$ there exists $t \in G^{(2)}$ inverting by conjugation both h and k, or h and g, or h and f. $\square$

Proposition 8.9. *For every $g, l \in G^{(4)}$ and $f_1, f_2 \in G^{(5)}$ the following products belong to V°:*

(i) $v_g \cdot v_g$ *and* $w_{f_1} \cdot w_{f_1}$*;*
(ii) $v_g \cdot v_l$*, if* $\langle g, l\rangle \le S_4$*;*
(iii) $v_g \cdot v_l$*, if* $\langle g, l\rangle \cong A_6$ *and* $l \in G_5^{(4)}(g) \cap G_5^{(4)}(g^{-1})$*;*
(iv) $v_g \cdot w_{f_1}$*, if* $\{f_1, f_1^2\} \cap G_5^{(5)}(g) \cap G_3^{(5)}(g^{-1}) \neq \emptyset$*;*
(v) $w_{f_1} \cdot w_{f_2}$*, if* $f_2 \notin G_5^{(5)}(f_1) \cap G_5^{(5)}(f_1^{-1})$*;*
(vi) $v_g \cdot w_{f_1}$*, if* $f_1 \in G_5^{(5)}(g) \cap G_5^{(5)}(g^{-1})$*.*

Proof. Claim (i) follows by the Norton-Sakuma Theorem and Table 2. Claim (ii) follows by Corollary 7.2. Claims (iii), (iv), and (v) follow by Lemma 8.7 since, for every $g, l \in G^{(4)}$, and $f_1, f_2 \in G^{(5)}$ as in the statement, there exists $t \in G^{(2)}$ inverting by conjugation both g and l, or g and f_1, or f_1 and f_2. Finally, suppose $f \in G_5^{(5)}(g) \cap G_5^{(5)}(g^{-1})$. Then, as in the proof of Lemma 4.13, we may assume $g = (1, 2)(3, 4, 5, 6)$ and $f = (2, 3, 4, 6, 5)$. Set $t := g^2 = (3, 5)(4, 6)$. By Table 4, it follows that $(a_t, v_g) = 0$, whence, by Lemma 4.10, v_g is a 0-eigenvector for ad_{a_t}. Moreover, since t inverts f, the algebra $\langle\langle a_t, a_{tf}\rangle\rangle$ is a Norton-Sakuma algebra of type $5A$, whence, by Table 2,

$$w_f = a_t \cdot a_{tf} - \frac{1}{2^7}(3a_t + 3a_{tf} - a_{tf^2} - a_{tf^3} - a_{tf^4}).$$

By [10, Lemma 1.10] and Proposition 8.6, it follows that

$$\begin{aligned} v_g \cdot w_f =& v_g \cdot (a_t \cdot a_{tf} - \tfrac{1}{2^7}(3a_t + 3a_{tf} - a_{tf^2} - a_{tf^3} - a_{tf^4})) \\ =& v_g \cdot (a_t \cdot a_{tf}) - \tfrac{1}{2^7} v_g \cdot (3a_t + 3a_{tf} - a_{tf^2} - a_{tf^3} - a_{tf^4}) \\ =& (v_g \cdot a_{tf}) \cdot a_t - \tfrac{1}{2^7} v_g \cdot (3a_t + 3a_{tf} - a_{tf^2} - a_{tf^3} - a_{tf^4}) \in V^\circ. \quad \square \end{aligned}$$

Proposition 8.10. $V^{(4A)} \cdot V^{(4A)} \subseteq V^\circ$.

Proof. Let $g \in G^{(4)}$. Define

$$F_g := \{f \in G^{(5)} \mid \text{ either } \{f, f^2\} \cap G_5^{(5)}(g) \cap G_3^{(5)}(g^{-1}) \neq \emptyset \\ \text{ or } f \in G_5^{(5)}(g) \cap G_5^{(5)}(g^{-1})\}.$$

By Proposition 8.9, for every $f \in F_g$, we have $v_g \cdot w_f \in V^\circ$, whence, by Equation (42), Proposition 8.6 and Proposition 8.9, $v_g \cdot ([w_{f_1}]_4 - [w_{f_2}]_4) \in V^\circ$ for every $f_1, f_2 \in F_g$. Moreover, for every $x \in N_G(\langle g \rangle)$, $(v_g)^x = v_g$, whence

$$v_g \cdot ([w_{f_1}]_4 - [w_{f_2}]_4)^x \in V^\circ \text{ for every } f_1, f_2 \in F_g \text{ and } x \in N_G(\langle g \rangle). \tag{61}$$

Let K_1 and K_2 be the two subgroups of G isomorphic to S_4 containing g. For every involution $s \in K_i \setminus K_i'$, denote by $\eta_{s,i}$ the element in the subalgebra $V(K_i)$ defined as in Equation (21). By Corollary 7.2 (Equation (46)), for $N \in \{2, 3, 4\}$, there exist $[\eta_{s,i}]_N \in V^{(NA)}$ such that

$$\eta_{s,i} = [\eta_{s,i}]_2 + [\eta_{s,i}]_3 + [\eta_{s,i}]_4.$$

By Corollary 7.2, $v_g \cdot \eta_{s,i} \in V(K_i) \subseteq V^\circ$, whence, by Corollary 8.1 and Proposition 8.9,

$$v_g \cdot [\eta_{s,i}]_4 \in V^\circ \text{ for every } s \in K_i \setminus K_i',\ i \in \{1, 2\}. \tag{62}$$

Now define

$$\begin{aligned} Q_g^1 &:= \langle v_l \mid \langle g, l \rangle \leq S_4 \rangle \\ Q_g^2 &:= \langle v_l \mid l \in G_5^{(4)}(g) \cap G_5^{(4)}(g^{-1}) \rangle \\ Q_g^3 &:= \langle ([w_{f_1}]_4 - [w_{f_2}]_4)^x \mid f_1, f_2 \in F_g, x \in N_G(\langle g \rangle) \rangle \\ Q_g^4 &:= \langle [\eta_{s,i}]_4 \mid s \in K_i \setminus K_i', i \in \{1, 2\} \rangle \\ Q_g &:= Q_g^1 + Q_g^2 + Q_g^3 + Q_g^4. \end{aligned}$$

A direct check (see [6, algebraA6(2B).g]) and Proposition 4.15 show that $\dim(Q_g) = 45 = \dim(V^{(4A)})$, whence

$$Q_g = V^{(4A)}.$$

By Proposition 8.9, Equation (61), and Equation (62), $v_g \cdot Q_g^i \subseteq V^\circ$ for every $i \in \{1, 2, 3, 4\}$, whence $v_g \cdot V^{(4A)} = v_g \cdot Q_g \subseteq V^\circ$. □

Lemma 8.11. *The following assertions hold:*

(i) $V^{(4A)} \cdot V^{(5A)} \subseteq V^\circ$*;*

(ii) $w \cdot w \in V^\circ$.

Proof. Let $g \in G^{(4)}$ and $\bar{f} \in G_5^{(5)}(g) \cap G_3^{(5)}(g^{-1})$. By Lemma 7.1, Proposition 8.6, Corollary 8.8, Proposition 8.9, and Proposition 8.10,

$$v_g \cdot w = v_g \cdot 36(w_{\bar{f}} - [w_{\bar{f}}]_2 - [w_{\bar{f}}]_3 - [w_{\bar{f}}]_4) \in V^\circ. \tag{63}$$

It follows that, for every $f \in G^{(5)}$,

$$v_g \cdot w_f = v_g \cdot (\tfrac{1}{36}w + [w_f]_2 + [w_f]_3 + [w_f]_4) \in V^\circ,$$

proving claim (i). Claim (ii) follows again by Lemma 7.1, Proposition 8.6, Corollary 8.8, Proposition 8.10 and claim (i) since

$$w \cdot w = 36^2 w_f \cdot w_f - 36^2([w_f]_2 + [w_f]_3 + [w_f]_4) \cdot (w_f - [w_f]_2 - [w_f]_3 - [w_f]_4)$$

and $w_f \cdot w_f \in V^\circ$ by the Norton-Sakuma Theorem. □

Proposition 8.12. *$V = V^\circ$ and the algebra product in V is unique.*

Proof. By Corollary 7.2.(i), Proposition 8.6, Corollary 8.8, Proposition 8.10, and Lemma 8.11, V° is closed under the algebra product. Since it contains the generating axes a_t, $t \in T$, we have $V^\circ = V$. In particular V has a basis $\mathcal{B}$ contained in the set X defined in Section 4 after Remark 4.1. The arguments used in this section now show that the structure constants of the algebra V with respect to the basis $\mathcal{B}$ are uniquely determined. □

Proofs of Theorems 2 and 3. Theorem 2 follows by Proposition 8.12 and Corollary 7.2. Theorem 3 follows by Propositions 7.4 and 8.12. □

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Data availability

Computations were performed in GAP [8] run on a MacBook Pro with 2.2 GHz quad-core Intel Core i7 processor and 16GB 1600 MHz DDR3 memory. The additional code, developed by the authors, is available at [6].

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