J. Math. Anal. Appl. 562 (2026) 130682

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Regular Articles

Critical sinh-Gordon flow with non-negative weight functions ☆

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ARTICLE INFO

Article history:
Received 5 November 2025
Available online 7 April 2026
Submitted by J. Galvez

Keywords:
Sinh-Gordon flow
Global existence
Critical case
Non-negative weights

ABSTRACT

The aim of this article is twofold: on one side we introduce and study the properties of a critical sinh-Gordon type flow

$$\frac{\partial}{\partial t}e^u = \Delta_g u + 8\pi\left(\frac{h_1 e^u}{\int_\Sigma h_1 e^u dV_g} - 1\right) - \rho_2\left(\frac{h_2 e^{-u}}{\int_\Sigma h_2 e^{-u} dV_g} - 1\right),$$

where $\rho_2 < 8\pi$, h_1, h_2 are non-negative weight functions and Σ is a closed Riemannian surface. Secondly, under suitable geometric conditions, we prove the convergence of the flow to a solution of the critical sinh-Gordon equation, extending the result of Zhou (2008) to the case of non-negative weights. The argument is based on a careful blow-up analysis. Some remarks about a Toda flow are also given.

1. Introduction

Let (Σ, g) be a closed Riemann surface with metric g and let h_1, h_2 be smooth non-negative and non-zero functions. For simplicity, we will assume that the area $|\Sigma|_g$ of the surface equals 1 throughout the paper. We are concerned with the following sinh-Gordon equation

☆ A.J. is partially supported by INdAM-GNAMPA project "*Analisi qualitativa di problemi differenziali non lineari*" and PRIN Project 20227HX33Z "*Pattern formation in nonlinear phenomena*" and is a member of the INDAM Research Group "Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni". S.-H Moon was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2022R1C1C200982, No. 2022R1A2C4002585) and a New Faculty Research Grant of Pusan National University, 2025.

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https://doi.org/10.1016/j.jmaa.2026.130682

$$-\Delta_g u = \rho_1 \left(\frac{h_1 e^u}{\int_\Sigma h_1 e^u dV_g} - 1 \right) - \rho_2 \left(\frac{h_2 e^{-u}}{\int_\Sigma h_2 e^{-u} dV_g} - 1 \right) \quad \text{on } \Sigma, \tag{1.1}$$

where ρ_1 and ρ_2 are non-negative constants.

Derived from Onsager's vortex model [40], equation (1.1) appears in [26,42] as a model in the description of the mean field of the equilibrium turbulence with arbitrarily signed vortices from different statistical arguments, and for more physical background concerning 2D-turbulence, see [11,34,38] and the references therein. In addition to turbulent Euler flows, it also arises as a mean field equation in the description of self-dual condensates of some Chern-Simon-Higgs model, see [3,14,15,44,48]. As for conformal geometry, when $\rho_2 = 0$, (1.1) is related to the problem of prescribing the Gaussian curvature in a conformal class of metric on Σ. In particular, on the standard sphere $\mathbb{S}^2$, it is called the Nirenberg problem; see [27,7,10] and the references therein.

The sinh-Gordon equation (1.1) has a variational structure, and its solutions correspond to the critical points of the functional $I_{\rho_1,\rho_2} : H^1(\Sigma) \to \mathbb{R}$

$$I_{\rho_1,\rho_2}(u) = \frac{1}{2} \int_\Sigma |\nabla_g u|^2 dV_g - \rho_1 \log \int_\Sigma h_1 e^u dV_g - \rho_2 \log \int_\Sigma h_2 e^{-u} dV_g + (\rho_1 - \rho_2) \int_\Sigma u dV_g.$$

One fundamental tool to deal with this kind of functionals is the Moser-Trudinger type inequality

$$\log \int_\Sigma e^{u-\overline{u}} dV_g + \log \int_\Sigma e^{-u+\overline{u}} dV_g \le \frac{1}{16\pi} \int_\Sigma |\nabla_g u|^2 dV_g + C, \quad \forall u \in H^1(\Sigma), \tag{1.2}$$

where $\overline{u}$ represents the average of u and C is a constant independent of u. In the subcritical case, i.e., when $\rho_1, \rho_2 \in (0, 8\pi)$, by the Moser-Trudinger inequality (1.2), I_{ρ_1,ρ_2} is bounded from below and coercive. Thus, the global minima of I_{ρ_1,ρ_2} can be attained by the direct minimization. However, in the critical case, i.e., when $\rho_i \le 8\pi$, $i = 1, 2$ and $\max(\rho_1, \rho_2) = 8\pi$, the functional is bounded from below but not coercive. This leads to a loss of compactness and makes the existence problem quite subtle and existence of solutions typically depends on the geometry of the underlying surface. This is why the literature about this case is very limited, as we will comment later on. The goal of this paper is to introduce a new tool to address this problem and to extend some previous results.

For the supercritical case, i.e., when $\max(\rho_1, \rho_2) > 8\pi$, I_{ρ_1,ρ_2} is unbounded from below and direct minimization can not be applied to the problem. This was considered by many authors, especially for $\rho_2 = 0$, which reduces to the well-known mean-field equation

$$-\Delta_g u = \rho \left(\frac{h e^u}{\int_\Sigma h e^u dV_g} - 1 \right). \tag{1.3}$$

Indeed, many techniques have been developed like degree counting and min-max schemes. For example, we refer to the papers [9,16,17,31,37,36] and the references therein. On the other hand, there are few results about the sinh-Gordon equation (1.1) in the supercritical case. We refer to the papers [1,22,25,54] and references therein. We also remark that the problem has some analogies with the Toda system, see [1].

From now on, we will focus on the sinh-Gordon equation and the mean field equation in the critical case. For the mean field equation (1.3) with $\rho = 8\pi$ and a positive function $h \in C^\infty(\Sigma)$, Ding, Jost, Li and Wang (see [13]) proved the first existence result under a geometric condition. They considered the minimizer u_ϵ of the slightly subcritical case

$$I_{8\pi-\epsilon}(u) = \frac{1}{2} \int_M |\nabla_g u|^2 dV_g + (8\pi - \epsilon) \int_\Sigma u dV_g - (8\pi - \epsilon) \log \int_\Sigma h e^u dV_g.$$

When u_ϵ blows up and does not converge in $H^1(\Sigma)$, by blow-up analysis, they inferred the following lower bound related to the geometry of Σ

$$\inf_{H^1(M)} I_{8\pi}(u) \geq -8\pi - 8\pi\log\pi - 4\pi \max_{x_0\in M}\big(A(x_0) + 2\log h(x_0)\big). \tag{1.4}$$

On the other hand, they also constructed a test function ϕ_ϵ such that, for small $\epsilon > 0$, $I_{8\pi}(\phi_\epsilon)$ is strictly less than the right hand side of (1.4), which contradicts the blow-up property. Consequently, u_ϵ converges in $H^1(\Sigma)$ to the solution $\tilde{u}$ of (1.3) with $\rho = 8\pi$. Later, Yang and Zhu (see [50]) generalized the above result for a non-negative function h by excluding the possibility of the blow-up at zeros of $h(x)$ based on compactness-concentration lemma in [14]. Recently, in [46,55], the authors proved that such existence results still hold even when h is sign-changing.

There are also existence results for the sinh-Gordon equation in the critical case. In [53], Zhou obtains the existence result of (1.1) with $h_1, h_2 \equiv 1$, $\rho_1 = 8\pi, \rho_2 \in (0, 8\pi]$ under some geometric conditions generalizing [13]. The argument is in the spirit of Toda systems (see [24]) and exploits the compact-concentration theorem established by Ohtsuka-Suzuki [39]. However, when h_1, h_2 are non-negative functions, we can not directly follow this approach. Thus, the main goal of this paper is to provide an alternative proof of the previous results and to extend them to non-negative functions h_1, h_2.

We will base our analysis on the flow method, which was already exploited for the mean field equation (1.3). In [5,6], the author introduced the following mean field type flow

$$\frac{\partial}{\partial t}e^v = \Delta v - Q + \rho\frac{e^v}{\int_\Sigma e^v dV_g}, \qquad v(\cdot,0) = v_0(x) \in C^{2+\alpha}(\Sigma), \tag{1.5}$$

$\alpha \in (0,1)$, where $Q \in C^\infty(\Sigma)$ is a given function such that $\int_\Sigma Q dV_g = \rho$. We note that the time-independent solution satisfies a mean-field type equation, which is equivalent to (1.3). In [6], Castéras proved the global existence of the solution. Moreover, using the compactness theorem in [5], he proved the convergence of the flow $v(t)$ to a solution of the mean field equation associated to (1.5) provided that $\rho \neq 8N\pi$ for $N \in \mathbb{N}^*$. However, such compactness theorem fails in the critical case $\rho = 8\pi$. In [30], the authors used the idea of [13] to overcome this difficulty, proving a lower bound of the corresponding functional $I_{8\pi}$ when the flow is not bounded. They constructed a test function ϕ_ϵ such that $I_{8\pi}(\phi_\epsilon)$ is smaller than the lower bound under the geometric condition in [13]. This means the flow converges when we choose an appropriate initial data. Subsequently, in [47], Sun and Zhu generalized this approach for non-negative functions h. Finally, in [29], Li and Xu generalized the above result to the case of the sign-changing function h. Yang and Wang (see [49]) also considered the mean field type flow with a sign-changing function h when a finite isometric group acts on the surface and h is invariant under the group action. For other variant of mean field type flows, see [33,51,52]. For Q-curvature flows we refer to the recent result [12] and the references therein. Recently, there is also a result [41] about a parabolic system related to Toda systems.

Motivated by [29,30,47,49], we introduce the following evolution problem (1.6) to deal with the critical sinh-Gordon equation (1.1)

$$\begin{cases} \dfrac{\partial}{\partial t}e^u = \Delta_g u + 8\pi\left(\dfrac{h_1 e^u}{\int_\Sigma h_1 e^u dV_g} - 1\right) - \rho_2\left(\dfrac{h_2 e^{-u}}{\int_\Sigma h_2 e^{-u} dV_g} - 1\right), \\ u(\cdot,0) = u_0 \in C^{2+\alpha}(\Sigma), \end{cases} \tag{1.6}$$

$\alpha \in (0,1)$. We note that it is a gradient flow with respect to the functional $I_{8\pi,\rho_2}$. For simplicity, we will denote it J_{ρ_2}

$$J_{\rho_2}(u) = \frac{1}{2}\int_\Sigma |\nabla_g u|^2 dV_g - 8\pi\log\int_\Sigma h_1 e^u dV_g - \rho_2\log\int_\Sigma h_2 e^{-u} dV_g + \big(8\pi - \rho_2\big)\int_\Sigma u dV_g. \tag{1.7}$$

We investigate the properties of the latter flow and, in particular, prove the following result.

Theorem 1.1. *Fix $\alpha \in (0,1)$. For any initial data $u_0 \in C^{2+\alpha}(\Sigma)$, there exists a unique global solution $u \in C^{2+\alpha,1+\alpha/2}(\Sigma \times [0,+\infty))$ to (1.6).*

In the second part of the paper we exploit the latter result to establish existence of solutions for the critical sinh-Gordon equation with non-negative weight functions, generalizing the results of Ding-Jost-Li-Wang [13] and Zhou [53] for the mean field and sinh-Gordon equations, respectively. Before stating the theorem, we introduce some notations.

Let K denote the Gaussian curvature of Σ. For each $p \in \Sigma$, let G_p be the Green function satisfying

$$-\Delta_g G_p = 8\pi\delta_p - 8\pi \quad \text{on } \Sigma, \quad \int_\Sigma G_p \, dV_g = 0, \tag{1.8}$$

and $A(p)$ be the regular part of the Green function. More precisely, $G_p(x)$ has the following expansion in normal coordinates near p:

$$G_p(x) = -4\log \operatorname{dist}_g(x,p) + A(p) + O(r^2), \quad r = \operatorname{dist}_g(x,p). \tag{1.9}$$

For $p \in \Sigma$, let Γ_p be the set of solutions $w_p \in H^1(\Sigma)$ to the singular mean field equation

$$-\Delta_g w_p = \rho_2 \left(\frac{h_2 e^{-G_p} e^{w_p}}{\int_\Sigma h_2 e^{-G_p} e^{w_p} dV_g} - 1 \right) \quad \text{on } \Sigma, \quad \int_\Sigma w_p \, dV_g = 0. \tag{1.10}$$

We also introduce the functional

$$\tilde{J}_p(u) = \frac{1}{2}\int_\Sigma |\nabla_g u|^2 dV_g - \rho_2 \log \int_\Sigma h_2 e^{-G_p} e^u dV_g, \quad \forall u \in H^1(\Sigma) \text{ with } \int_\Sigma u = 0, \tag{1.11}$$

whose critical points solve (1.10).

Now we are prepared to state the second main theorem.

Theorem 1.2. *Let $\rho_2 \in (0,8\pi)$ and h_1, h_2 be smooth non-negative and non-zero functions. Suppose*

$$8\pi - \rho_2 - 2K(p_0) + \Delta_g \log h_1(p_0) > 0$$

for any minimizer $p_0 \in \Sigma$ of $p \mapsto \inf_{p\in\Sigma} \inf_{w\in\Gamma_p} (\tilde{J}_p(w) - 4\pi A(p) - 8\pi \log h_1(p))$. Then, there exists an initial datum $u_0 \in C^{2+\alpha}(\Sigma)$ such that the flow $u(x,t)$ converges in $C^2(\Sigma)$ to a solution u_∞ of the critical sinh-Gordon equation (1.1) with $\rho_1 = 8\pi$, $\rho_2 \in (0,8\pi)$.

In fact, when $\rho_2 = 0$, $\Gamma_p = \{0\}$ and the condition is consistent with the one for the mean field equation. However, when $\rho_2 > 0$, due to the effect of e^{-u} term in (1.1), $\tilde{J}_p(w)$ appears in the lower bound (see Proposition 3.7).

We sketch now the proofs highlighting the differences with the previous results. We first prove that the solution $u(x,t)$ of (1.6) exists globally in time. For this, we derive several a priori estimates (Proposition 2.4 – Proposition 2.6). However, this is different from previous works [6,47,49] since we have to control the e^{-u} term and $\int_\Sigma e^{-u}$ is not conserved. Even though we can perform the below explained blow up analysis for $\rho_2 = 8\pi$ and sign-changing weight functions, we use in this step $\rho_2 < 0$ and h_1, h_2 non-negative. We postpone to a future work the discussion of this point and possible extensions of this method.

Next, we prove that the time-slices $u(t_n)$ can not blow up at the zero set of h_1 (Proposition 3.3). We also show $-u(t_n)$ does not blow up (Proposition 3.5). The blow-up analysis is delicate especially when $u(t_n)$ and $-u(t_n)$ blow up at the same point. We adapted the idea of the selection process in [23,32] and used the hypothesis $\rho_2 < 8\pi$ subtly. We remark that we can not apply the result in [23] directly due to the time derivative term and the non-negativeness of h_1, h_2.

Based on this blow-up analysis, we prove the lower bound of the functional $\lim_{t\to\infty} J_{\rho_2}(u(t))$ when the time-slices $u(t_n)$ blow up (Proposition 3.7). As we mentioned after the main theorem, $\tilde{J}_p(w)$ appears in the lower bound, since $-u(t_n)$ converges to a solution of (1.10). Then we construct a test function $\tilde{\Phi}_\epsilon$ such that, under suitable geometric conditions, $J_{\rho_2}(\tilde{\Phi}_\epsilon)$ is smaller than the lower bound (Proposition 4.3). For this purpose, we have to choose a solution w of (1.10) achieving the infimum of the leading term, so we prove the compactness of the solution set of (1.10) (Proposition 4.1). Finally, we show the convergence of the flow using a priori estimates and Łojasiewicz-Simon gradient inequality [18].

We conclude the introduction with the following remark about the flow method for Toda systems.

Remark 1.3. By the same method we can address following critical $SU(3)$ Toda system:

$$\begin{cases} -\Delta_g u_1 = 2\rho_1 \left(\dfrac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right) - \rho_2 \left(\dfrac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right), \\ -\Delta_g u_2 = 2\rho_2 \left(\dfrac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right) - \rho_1 \left(\dfrac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right), \end{cases} \tag{1.12}$$

where $\rho_1 = 4\pi, \rho_2 \in (0, 4\pi)$, $h_1, h_2 \in C^\infty(\Sigma)$, $h_1, h_2 \geq 0$. Indeed, following the same strategy as before, we are able to carry out the blow up analysis and the construction of suitable test functions. However, we face a new difficulty in the global existence of the associated flow, which we describe hereafter.

We note that (1.12) is the Euler-Lagrangian equation for the following functional

$$\tilde{I}_{\rho_1,\rho_2}(u_1(x,t), u_2(x,t)) = \int_\Sigma Q(u_1, u_2) dV_g - \rho_1 \log \int_\Sigma e^{u_1 - \overline{u}_1} dV_g - \rho_2 \log \int_\Sigma e^{u_2 - \overline{u}_2} dV_g,$$

where $Q(u_1, u_2) = \frac{1}{3}\left(|\nabla_g u_1|^2 + |\nabla_g u_2|^2 + \nabla_g u_1 \nabla_g u_2\right)$.

One interesting point is that a possible gradient flow of $\tilde{I}_{\rho_1,\rho_2}$ is the following one, quite different from standard (semilinear) parabolic equations,

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \dfrac{2}{3} e^{-u_1} \Delta_g u_1 + \dfrac{1}{3} e^{-u_1} \Delta_g u_2 + 4\pi \left(\dfrac{h_1}{\int_\Sigma h_1 e^{u_1} dV_g} - e^{-u_1} \right), \\ \dfrac{\partial u_2}{\partial t} = \dfrac{1}{3} e^{-u_2} \Delta_g u_1 + \dfrac{2}{3} e^{-u_2} \Delta_g u_2 + \rho_2 \left(\dfrac{h_2}{\int_\Sigma h_2 e^{u_2} dV_g} - e^{-u_2} \right). \end{cases} \tag{1.13}$$

To study the global existence we exploit the following idea. Observe that the eigenvalues of the matrix

$$\begin{pmatrix} \frac{2}{3} e^{-w_1(x,t)} & \frac{1}{3} e^{-w_2(x,t)} \\ \frac{1}{3} e^{-w_1(x,t)} & \frac{2}{3} e^{-w_2(x,t)} \end{pmatrix}, \quad w_i(x,t) \in C^{2+\alpha, 1+\alpha/2}(\Sigma \times [0,\infty)) : \text{ fixed functions}$$

are positive and distinct. Thus, one can use the eigenvalues $\lambda_i(x,t)$ and their eigenvectors to transform the linearized operator of (1.13) at (w_1, w_2) into the standard form alike

$$\frac{\partial \phi_i}{\partial t} - \lambda_i(x,t) \Delta \phi_i + L_i(x, t, \phi_1, \phi_2, \nabla \phi_1, \nabla \phi_2), \quad \lambda_i(x,t) \geq c > 0 \text{ for } x \in \Sigma, t \in [0,T], \quad i = 1, 2.$$

From this observation, we can apply the standard parabolic theory to prove the short time existence of (1.13). (We refer to [19, Chap. 9] for the linear parabolic systems and [19, Chap. 7], [21] for the quasilinear parabolic equations.)

However, there is an obstacle when proving a priori estimates for the original quasilinear system. In order to apply the well-known estimates for the standard parabolic equations (e.g. Schauder estimate), we need to transform the system again. Observe that the coefficient matrix involves e^{-u_1}, e^{-u_2}, and the coefficients of the transformed system are related to e^{-u_i} and their derivatives. Since the constant in the Schauder estimate depends on the norm of the coefficients of the differential operator, we could not apply the standard estimate. Due to this difficulty, at this point we can not prove the global existence of the flow (1.13). We postpone this to a future work.

We remark that, in [41], the authors proved the global existence of the solution of another semilinear parabolic system related to elliptic systems including Toda systems and Liouville systems.

Remark 1.4. After the submission of the present paper, two preprints on related subjects appeared online. In [45] the authors carry out blow-up analysis for sinh-Gordon equation with sign-changing weight functions, while in [35] a Toda flow is studied.

The organization of this paper is as follows. In Section 2, we prove the global existence of solutions to (1.6). In Section 3, we will carry out blow-up analysis and derive the lower bound of blow-up sequences, which is the key element in proving the existence result of (1.1). In Section 4, we construct a test function and we prove the convergence of the flow. This completes the proof of Theorem 1.2.

2. Global existence of the flow

In this section, we prove the global existence and uniqueness of solutions to the flow (1.6). The argument is divided into two main steps: we first recall the short-time existence and present several preliminary properties of the solution, and then derive a priori estimates that allow us to extend the solution globally in time.

2.1. Short-time existence and preliminaries

As the first step, by the standard parabolic theory, we can prove the short-time existence and uniqueness of the solution to (1.6) (for example, see [19]). We omit the details and refer the reader to the references.

Lemma 2.1. *Fix $\alpha \in (0,1)$. For any initial data $u_0 \in C^{2+\alpha}(\Sigma)$, there exists $\varepsilon > 0$ such that (1.6) has a unique solution $u \in C^{2+\alpha,1+\alpha/2}(\Sigma \times [0,\varepsilon])$.*

We first note that by Lemma 2.1, there exists $T > 0$ such that (1.6) has a unique solution $u \in C^{2+\alpha,1+\alpha/2}(\Sigma \times [0,T])$. In addition, we prove several basic properties of the solution, including conservation of mass and monotonicity of the energy functional. These preliminaries are crucial for the energy method employed in the next subsection and will also be used later in the paper.

Lemma 2.2. *Suppose that (1.6) admits a solution $u \in C^{2+\alpha,1+\alpha/2}(\Sigma \times [0,T])$ for some $T > 0$. Then the following properties hold:* (i) *For all $t \in [0,\ T]$, we have*

$$\int_\Sigma e^{u(t)} dV_g = \int_\Sigma e^{u_0} dV_g.$$

(ii) *The energy functional $J_{\rho_2}(u(t))$ is non-increasing in t, that is, for all $0 \le t_0 \le t_1 \le T$,*

$$J_{\rho_2}(u(t_1)) \le J_{\rho_2}(u(t_0)).$$

Proof. **(i)** By integrating both sides of (1.6) over $\Sigma \times [0,t]$, we have

$$0 = \int_\Sigma \int_0^t \frac{\partial}{\partial t}(e^{u(t)}) ds dV_g = \int_\Sigma e^{u(t)} dV_g - \int_\Sigma e^{u(0)} dV_g = \int_\Sigma e^{u(t)} dV_g - \int_\Sigma e^{u_0} dV_g.$$

(ii) By differentiating $J_{\rho_2}(u(t))$ with respect to t and integrating by parts, we get

$$\frac{\partial}{\partial t} J_{\rho_2}(u(t)) = -\int_\Sigma \left|\frac{\partial u}{\partial t}\right|^2 e^u dV_g \le 0. \tag{2.1}$$

Integrating (2.1) with respect to time from t_0 to t_1, we obtain that

$$J_{\rho_2}(u(t_1)) - J_{\rho_2}(u(t_0)) = -\int_{t_0}^{t_1} \int_\Sigma \left|\frac{\partial u}{\partial t}\right|^2 e^u dV_g dt \le 0. \quad \square$$

Lemma 2.3. *There exist $C, c > 0$, independent of T, such that for all $t \in [0,T]$,*

$$\int_\Sigma h_1 e^u dV_g,\ \frac{\int_\Sigma e^{-u} dV_g}{\int_\Sigma h_2 e^{-u} dV_g} \le C \quad \text{and} \quad \int_\Sigma h_1 e^u dV_g, \int_\Sigma h_2 e^{-u} dV_g \ge c.$$

Proof. First, we prove that $c \le \int_\Sigma h_1 e^u dV_g \le C$ for some constants $C, c > 0$, independent of $t \in [0,T]$. Indeed, by Moser-Trudinger inequality (1.2) and Jensen's inequality, it holds that

$$\frac{1}{2}\int_\Sigma |\nabla_g u|^2 dV_g \ge 8\pi \log \int_\Sigma e^{u-\overline{u}} dV_g + \rho_2 \log \int_\Sigma e^{-u+\overline{u}} dV_g - C_\Sigma. \tag{2.2}$$

Employing (2.2) and the monotonicity of $J_{\rho_2}(u(t))$, we obtain that for all $t \in [0,T]$,

$$\begin{aligned} J_{\rho_2}(u_0) \ge J_{\rho_2}(u(t)) \ge & 8\pi \log \int_\Sigma e^u dV_g + \rho_2 \log \int_\Sigma e^{-u} dV_g - 8\pi \log \int_\Sigma h_1 e^u dV_g \\ & - \rho_2 \log \int_\Sigma h_2 e^{-u} dV_g - C_\Sigma. \end{aligned} \tag{2.3}$$

This implies that $J_{\rho_2}(u(t)) \ge 8\pi \log \int_\Sigma e^u dV_g - 8\pi \log \int_\Sigma h_1 e^u dV_g - \rho_2 \log \|h_2\|_{L^\infty(\Sigma)} - C_\Sigma$. From this inequality, we derive that

$$0 < \exp\Big(-\frac{\rho_2 \log \|h_2\|_{L^\infty(\Sigma)} + J_{\rho_2}(u_0) + C_\Sigma}{8\pi}\Big) \int_\Sigma e^{u_0} dV_g \le \int_\Sigma h_1 e^u dV_g \le \|h_1\|_{L^\infty(\Sigma)} \int_\Sigma e^{u_0} dV_g. \tag{2.4}$$

Next, we prove the uniform boundedness of $\frac{\int_\Sigma e^{-u} dV_g}{\int_\Sigma h_2 e^{-u} dV_g}$. From (2.3), we also deduce that $J_{\rho_2}(u(t)) \ge \rho_2 \log \int_\Sigma e^{-u} dV_g - \rho_2 \log \int_\Sigma h_2 e^{-u} dV_g - 8\pi \log \|h_1\|_{L^\infty(\Sigma)} - C_\Sigma$. This implies that for all $t \in [0,T]$

$$0 < \frac{\int_\Sigma e^{-u} dV_g}{\int_\Sigma h_2 e^{-u} dV_g} \le \exp\Big(\frac{J_{\rho_2}(u_0) + 8\pi \log \|h_1\|_{L^\infty(\Sigma)} + C_\Sigma}{\rho_2}\Big).$$

Finally, we show that $\int_\Sigma h_2 e^{-u} dV_g \geq c$ for some $c > 0$, independent of $t \in [0, T]$. Indeed, by the Cauchy-Schwarz inequality and (2.4), we obtain

$$\int_\Sigma h_2 e^{-u} dV_g \geq \frac{\left(\int_\Sigma \sqrt{h_2 e^{-u(t)} \cdot e^{u(t)}} dV_g\right)^2}{\int_\Sigma e^{u(t)} dV_g} = \frac{\left(\int_\Sigma \sqrt{h_2} dV_g\right)^2}{\int_\Sigma e^{u_0} dV_g} > 0.$$

This completes the proof of Lemma 2.3. □

2.2. A priori estimates and global existence

We now address the global existence of solutions to the flow (1.6), i.e. Theorem 1.1.

Let us define the maximal time $T_0 > 0$ by

$$T_0 := \sup\left\{T > 0 : u \in C^{2+\alpha, 1+\alpha/2}(\Sigma \times [0, T]) \text{ is the unique solution of (1.6)}\right\}.$$

To prove global existence, it suffices to show that $T_0 = \infty$. For this purpose, we first derive a priori estimates for the solution of (1.6) in three steps: Step 1. H^1-estimate (Proposition 2.4); Step 2. H^2-estimate (Proposition 2.5); Step 3. $C^{2+\alpha,1+\alpha/2}$-estimate (Proposition 2.6).

These three steps provide progressively stronger control over the solution, which ultimately allows us to extend the solution beyond any finite maximal time.

Proposition 2.4. *Let u be the solution of (1.6) on $[0, T)$ for some $T > 0$. Then there exists a constant $C_{T,1} = C(T, \|u_0\|_{H^1(\Sigma)})$, depending on Σ, h_1 and h_2, such that*

$$\|u(t)\|_{H^1(\Sigma)} \leq C_{T,1}, \quad \forall t \in [0, T).$$

Proof. Since $J_{\rho_2}(u(t))$ is decreasing in t, by subtracting a multiple of the Moser-Trudinger inequality (1.2) from the definition of $J_{\rho_2}(u(t))$, we obtain

$$\begin{aligned} J_{\rho_2}(u_0) \geq J_{\rho_2}(u(t)) &= \frac{1}{2}\int_\Sigma |\nabla_g u|^2 dV_g - 8\pi \log \int_\Sigma h_1 e^u dV_g - \rho_2 \log \int_\Sigma h_2 e^{-u} dV_g + (8\pi - \rho_2)\overline{u} \\ &\geq \frac{8\pi - \rho_2}{16\pi} \int_\Sigma |\nabla_g u|^2 dV_g - \left(8\pi - \rho_2\right) \log \int_\Sigma e^u dV_g + \left(8\pi - \rho_2\right)\overline{u} - C \end{aligned} \tag{2.5}$$

for a constant C independent of u_0 and t. Since $\int_\Sigma e^{u(t)} dV_g = \int_\Sigma e^{u_0} dV_g$, applying Young's inequality, we obtain that for small $\epsilon > 0$

$$\begin{aligned} \|\nabla_g u(t)\|^2_{L^2(\Sigma)} &\leq \frac{16\pi}{8\pi - \rho_2} J_{\rho_2}(u_0) - 16\pi\overline{u}(t) + 16\pi \log \int_\Sigma e^{u_0} dV_g + C \\ &\leq C(\|u_0\|_{H^1(\Sigma)}) + \epsilon\|u\|^2_{L^2(\Sigma)} + C_\epsilon \end{aligned} \tag{2.6}$$

where $C(\|u_0\|_{H_1(\Sigma)})$ denotes a constant depending only on $\|u_0\|_{H^1(\Sigma)}$, and C_ϵ is a constant depending on ϵ.

Next, differentiating $\int_\Sigma e^{2u} dV_g$ with respect to t, we obtain that there exists a constant $C = C(\|u_0\|_{H^1(\Sigma)})$ such that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Sigma e^{2u(t)}dV_g &= \int_\Sigma e^u\Big[\Delta_g u(t)+8\pi\big(\frac{h_1e^u}{\int_\Sigma h_1e^udV_g}-1\big)-\rho_2\big(\frac{h_2e^{-u}}{\int_\Sigma h_2e^{-u}dV_g}-1\big)\Big]dV_g\\ &= -\int_\Sigma e^u|\nabla_g u|^2dV_g+8\pi\frac{\int_\Sigma h_1e^{2u}dV_g}{\int_\Sigma h_1e^udV_g}-\rho_2\frac{\int_\Sigma h_2dV_g}{\int_\Sigma h_2e^{-u}dV_g}-(8\pi-\rho_2)\int_\Sigma e^udV_g\\ &\le C\int_\Sigma e^{2u(t)}dV_g+C\end{aligned}$$

Integrating this differential inequality, we conclude that

$$\int_\Sigma e^{2u(t)}dV_g \le e^{Ct}\int_\Sigma e^{2u_0}dV_g+e^{Ct}\le C(T,\|u_0\|_{H^1(\Sigma)})\quad \text{for } t\in[0,T). \tag{2.7}$$

In order to estimate the average value of $u(t)$, set

$$A(t):=\Big\{x\in\Sigma : e^{u(x,t)}\ge\frac{1}{2}\int_\Sigma e^{u(t)}dV_g\Big\}.$$

By Hölder's inequality and (2.7), it follows that

$$\int_\Sigma e^{u_0}dV_g=\int_{\Sigma\setminus A(t)}e^{u(t)}dV_g+\int_{A(t)}e^{u(t)}dV_g\le\frac{1}{2}\int_\Sigma e^{u_0}dV_g+C(T,\|u_0\|_{H^1})|A(t)|_g^{1/2}.$$

Therefore, there exists $c_T>0$ such that

$$c_T\le|A(t)|\le 1,\quad\text{and}\quad |A(t)|\log\frac{\int_\Sigma e^{u_0}dV_g}{2}\le\int_{A(t)}u(t)dV_g\le\int_{A(t)}e^{u(t)}dV_g.$$

Consequently, by Hölder's inequality and (2.7), we have

$$|\overline{u}(t)|\le|\Sigma\setminus A(t)|^{1/2}\|u(t)\|_{L^2(\Sigma\setminus A(t))}+\Big|\int_{A(t)}u(t)dV_g\Big|\le\sqrt{1-c_T}\|u(t)\|_{L^2(\Sigma)}+C(T,\|u_0\|_{H^1}).$$

Combining this estimate with the Poincaré inequality, we deduce

$$\|u(t)\|_{L^2}\le c\|\nabla u(t)\|_{L^2}+\sqrt{1-c_T}\|u(t)\|_{L^2(\Sigma)}+C(T,\|u_0\|_{H^1}),$$

which implies

$$\|u(t)\|_{L^2}\le C\|\nabla u(t)\|_{L^2}+C(T,\|u_0\|_{H^1}). \tag{2.8}$$

Finally, combining (2.6) and (2.8), and choosing ϵ sufficiently small, we complete the proof. □

Proposition 2.5. *Let u be the solution of (1.6) on $[0,T)$ for some $T>0$. Then there exists a constant $C_{T,2}=C(T,\|u_0\|_{H^2(\Sigma)})$, depending on Σ, h_1 and h_2, such that*

$$\|u(t)\|_{H^2(\Sigma)}\le C_{T,2},\quad \forall t\in[0,T).$$

Proof. By Proposition 2.4, it suffices to estimate $\|\Delta_g u(t)\|_{L^2(\Sigma)}$. To this end, we introduce the auxiliary function $\nu(t) = \frac{\partial u(t)}{\partial t} e^{u(t)/2}$. Then, by a direct computation we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Sigma \Big(1+|\Delta_g u(t)|^2\Big)dV_g &= \int_\Sigma \Delta_g u(t)\Delta_g\big(\frac{\partial u(t)}{\partial t}\big)dV_g \\
&= \int_\Sigma \Big(e^{\frac{u}{2}}\nu(t) - 8\pi\big(\frac{h_1e^u}{\int_\Sigma h_1e^u dV_g}-1\big) + \rho_2\big(\frac{h_2e^{-u}}{\int_\Sigma h_2e^{-u}dV_g}-1\big)\Big)\Delta_g\big(e^{-\frac{u}{2}}\nu(t)\big)dV_g \\
&= -\int_\Sigma |\nabla_g\nu(t)|^2dV_g + \frac{1}{4}\int_\Sigma \nu(t)^2|\nabla_g u(t)|^2dV_g \\
&\quad + \frac{8\pi}{\int_\Sigma h_1e^udV_g}\int_\Sigma e^{\frac{u(t)}{2}}\big(\nabla_g h_1 + h_1\nabla_g u\big)\big(\nabla_g\nu(t) - \frac{1}{2}\nu(t)\nabla_g u\big)dV_g \\
&\quad - \frac{\rho_2}{\int_\Sigma h_2e^{-u}dV_g}\int_\Sigma e^{-\frac{3u}{2}}(\nabla_g h_2 - h_2\nabla_g u)\big(\nabla_g\nu(t) - \frac{1}{2}\nu(t)\nabla_g u\big)dV_g.
\end{aligned} \tag{2.9}$$

By Lemma 2.3, Proposition 2.4 and the Moser-Trudinger inequality, there exists a constant C_T, such that for all $p\geq 1,\ t\in[0,T)$

$$\frac{1}{\int_\Sigma h_1e^{u(t)}dV_g} + \frac{1}{\int_\Sigma h_2e^{-u(t)}dV_g} + \int_\Sigma e^{pu(t)}dV_g + \int_\Sigma e^{-pu(t)}dV_g \leq C_T. \tag{2.10}$$

Now, using (2.10) together with Hölder's inequality and Young's inequality, we can estimate the third term on the right-hand side of (2.9) as follows:

$$\begin{aligned}
&\frac{8\pi}{\int_\Sigma h_1e^udV_g}\int_\Sigma e^{\frac{u(t)}{2}}\big(\nabla_g h_1 + h_1\nabla_g u(t)\big)\big(\nabla_g\nu(t) - \frac{1}{2}\nu(t)\nabla_g u(t)\big)dV_g \\
&\leq C\frac{\big(\int_\Sigma e^{2u}dV_g\big)^{\frac14}}{\int_\Sigma h_1e^udV_g}\Big(\int_\Sigma \big|\nabla_g h_1 + h_1\nabla_g u(t)\big|^4dV_g\Big)^{\frac14}\Big(\int_\Sigma |\nabla_g\nu(t) - \frac{1}{2}\nu(t)\nabla_g u(t)|^2dV_g\Big)^{\frac12} \\
&\leq C_T\big(1+\|\nabla_g u\|_{L^4(\Sigma)}\big)\left\{\|\nabla_g\nu\|_{L^2(\Sigma)} + \Big(\int_\Sigma \nu(t)^2|\nabla_g u|^2dV_g\Big)^{\frac12}\right\} \\
&\leq \epsilon\|\nabla_g\nu\|^2_{L^2(\Sigma)} + \epsilon\int_\Sigma \nu(t)^2|\nabla_g u(t)|^2dV_g + C_{\epsilon,T}\big(1+\|\nabla_g u\|^2_{L^4(\Sigma)}\big),
\end{aligned} \tag{2.11}$$

where $C_{\epsilon,T}$ depends only on $\Sigma, T, \epsilon, \|u_0\|_{H^1(\Sigma)}$. Similarly, we can estimate the fourth term on right-hand side of (2.9). More precisely,

$$\begin{aligned}
&-\frac{\rho_2}{\int_\Sigma h_2e^{-u}dV_g}\int_\Sigma e^{-\frac{3u(t)}{2}}(\nabla_g h_2 - h_2\nabla_g u(t))\big(\nabla_g\nu(t) - \frac{1}{2}\nu(t)\nabla_g u(t)\big)dV_g \\
&\leq \epsilon\|\nabla_g\nu\|^2_{L^2(\Sigma)} + \epsilon\int_\Sigma \nu(t)^2|\nabla_g u(t)|^2dV_g + C_{\epsilon,T}\big(1+\|\nabla_g u\|^2_{L^4(\Sigma)}\big).
\end{aligned} \tag{2.12}$$

Combining (2.11) and (2.12), and choosing $\epsilon > 0$ sufficiently small, we simplify (2.9) into

$$\frac{1}{2}\frac{d}{dt}\int_\Sigma \Big(1+\big|\Delta_g u(t)\big|^2\Big)dV_g \le -\frac{1}{2}\|\nabla_g \nu(t)\|^2_{L^2(\Sigma)} + \int_\Sigma \nu(t)^2|\nabla_g u|^2 dV_g + C_{\epsilon,T}\big(1+\|\nabla_g u\|^2_{L^4(\Sigma)}\big). \tag{2.13}$$

On the other hand, by Proposition 2.4, Hölder's inequality, Calderón-Zygmund inequality and Gagliardo-Nirenberg inequality, we deduce

$$\begin{aligned}\int_\Sigma \nu^2(t)|\nabla_g u|^2 dV_g &\le C\|\nu(t)\|_{L^2(\Sigma)}\|\nu(t)\|_{H^1(\Sigma)}\|u(t)\|_{H^1(\Sigma)}\|u(t)\|_{H^2(\Sigma)}\\ &\le C\big(T,\|u_0\|_{H^1(\Sigma)}\big)\|\nu(t)\|_{L^2(\Sigma)}\|\nu(t)\|_{H^1(\Sigma)}\|u(t)\|_{H^2(\Sigma)}\\ &\le \epsilon\big(\|\nabla_g\nu\|^2_{L^2(\Sigma)}+\|\nu\|^2_{L^2(\Sigma)}\big)+C_{\epsilon,T}\|\nu\|^2_{L^2(\Sigma)}\big(\|\Delta_g u(t)\|^2_{L^2(\Sigma)}+1\big),\end{aligned} \tag{2.14}$$

and similarly

$$\|\nabla_g u\|^2_{L^4(\Sigma)} \le C\|u\|_{H^1(\Sigma)}\|u\|_{H^2(\Sigma)} \le C\big(T,\|u_0\|_{H^1(\Sigma)}\big)\big(\|\Delta_g u(t)\|^2_{L^2(\Sigma)}+1\big). \tag{2.15}$$

Therefore, combining (2.13), (2.14) and (2.15) together, we obtain

$$\frac{d}{dt}\int_\Sigma \Big(1+|\Delta_g u|^2\Big)dV_g \le C\big(T,\|u_0\|_{H^1(\Sigma)}\big)\big(1+\|\nu\|^2_{L^2(\Sigma)}\big)\big(1+\|\Delta_g u\|^2_{L^2(\Sigma)}\big).$$

As a consequence, using the energy identity $J_{\rho_2}(u(T)) - J_{\rho_2}(u_0) = -\int_0^T\int_\Sigma |\frac{\partial u}{\partial t}|^2 e^u dV_g dt$ and integrating in time, we obtain

$$\begin{aligned}\log\Big(1+\|\Delta_g u\|^2_{L^2(\Sigma)}\Big) &\le C\big(T,\|u_0\|_{H^1(\Sigma)}\big)\Big(1+\int_0^T \big(1+\|\frac{\partial u}{\partial t}e^{\frac{u}{2}}\|^2_{L^2(\Sigma)}\big)dt\Big)\\ &\le C\big(T,\|u_0\|_{H^1(\Sigma)}\big)\Big(1+J_{\rho_2}(u_0)-J_{\rho_2}(u(T))\Big) \le C\big(T,\|u_0\|_{H^1(\Sigma)}\big).\end{aligned}$$

This completes the proof. □

Proposition 2.6. *Let u be the solution of (1.6) on $[0,T)$ for some $T>0$. Then there exists a constant $C_{T,3} = C(T,\|u_0\|_{C^{2+\alpha}(\Sigma)})$, depending on Σ, h_1, h_2 such that*

$$\|u(t)\|_{C^{2+\alpha,1+\alpha/2}(\Sigma\times[0,T))} \le C_{T,3}.$$

Proof. For the proof of this proposition, it suffices to show the following estimate:

$$|u(x,t_1)-u(y,t_2)| \le C\Big(|x-y|^\alpha + |t_1-t_2|^{\frac{\alpha}{2}}\Big), \quad \text{for any } x,y\in\Sigma;\ \ t_1,t_2\in[0,T), \tag{2.16}$$

where $|x-y|$ denotes $\mathrm{dist}_g(x,y)$ for simplicity. In view of (2.16), the classical parabolic Schauder estimates (see, e.g., [19, Chapter 3]) yield the desired conclusion.

Now we prove (2.16). By Proposition 2.5, we have $\|u(t)\|_{H^2(\Sigma)} \le C_{T,2}$, and by Sobolev embedding theorem, there exists a constant C_1 such that $\|u(t)\|_{C^{0,\alpha}(\Sigma)} \le C_1$ for all $t\in[0,T)$. Therefore, it is enough to prove

$$\big|u(x,t_1)-u(x,t_2)\big| \le C|t_1-t_2|^{\frac{\alpha}{2}} \quad \text{for all} \quad x\in\Sigma, t_1,t_2\in[0,T). \tag{2.17}$$

(i) If $t_2 - t_1 \geq 1$, then we have

$$\big|u(x,t_1) - u(x,t_2)\big| \leq C(T, \|u_0\|_{H^2(\Sigma)}) \leq C(T, \|u_0\|_{H^2(\Sigma)})\big|t_1 - t_2\big|^{\frac{\alpha}{2}}. \tag{2.18}$$

(ii) If $0 < t_2 - t_1 < 1$, set $s = \min\{r_0/2, \sqrt{t_2 - t_1}\}$, where r_0 is the injectivity radius of Σ. Then, we have

$$\begin{aligned}\big|u(x,t_1) - u(x,t_2)\big| &= \frac{1}{|B_s(x)|}\int_{B_s(x)} \big|u(x,t_1) - u(x,t_2)\big| dV_g(y) \\ &\leq C \int_{B_s(x)} \sum_{i=1}^{2} \frac{\big|u(x,t_i) - u(y,t_i)\big|}{t_2 - t_1} + \frac{\big|u(y,t_1) - u(y,t_2)\big|}{t_2 - t_1} dV_g(y),\end{aligned} \tag{2.19}$$

where we used $cs^2 \leq |B_s(x)| \leq Cs^2$ on (Σ, g) with constants $C, c > 0$ depending only on (Σ, g).

Let us calculate the first term of the right-hand side on (2.19). By Hölder continuity $\|u(t)\|_{C^{0,\alpha}(\Sigma)} \leq C_1$, we obtain that, for $i = 1, 2$,

$$\int_{B_s(x)} \frac{\big|u(x,t_i) - u(y,t_i)\big|}{t_2 - t_1} dV_g(y) \leq C \int_{B_s(x)} \frac{\big|x - y\big|^{\alpha}}{t_2 - t_1} dV_g(y) \leq C \frac{s^{\alpha+2}}{t_2 - t_1} \leq C\big(t_2 - t_1\big)^{\frac{\alpha}{2}}. \tag{2.20}$$

For the second term, we obtain

$$\begin{aligned}\int_{B_s(x)} \frac{\big|u(y,t_1) - u(y,t_2)\big|}{t_2 - t_1} dV_g(y) &\leq C \sup_{t_1 \leq t \leq t_2} \int_{B_s(x)} \Big|\frac{\partial u(t)}{\partial t}\Big| dV_g(y) \\ &\leq C|B_s(x)|^{1/2} \sup_{t_1 \leq t \leq t_2} \Big(\int_{B_s(x)} \Big|\frac{\partial u(t)}{\partial t}\Big|^2 dV_g(y)\Big)^{\frac{1}{2}} \\ &\leq C(T, \|u_0\|_{H^2})\big(t_2 - t_1\big)^{\frac{\alpha}{2}},\end{aligned} \tag{2.21}$$

since $\|\Delta_g u(t)\|_{L^2}, \|u(t)\|_{L^\infty} \leq C(T, \|u_0\|_{H^2})$ and $\int_\Sigma h_1 e^u dV_g, \int_\Sigma h_2 e^{-u} dV_g \geq c > 0$ by Lemma 2.3 and Proposition 2.5. Combining (2.18)–(2.21), we obtain (2.17) and this completes the proof. □

We now complete the proof of the global existence.

Proof of Theorem 1.1. Suppose, by contradiction, that $T_0 < \infty$. By the a priori estimates in Proposition 2.6, the solution $u(t)$ remains bounded in $C^{2+\alpha}(\Sigma)$ up to $t = T_0$, and hence the short-time existence lemma guarantees that u can be extended beyond T_0. This contradicts the definition of T_0 as the maximal existence time. Therefore, we conclude that $T_0 = \infty$. This completes the proof of the global existence and uniqueness of solutions to the flow (1.6). □

3. Blow-up analysis

In this section we investigate the blow-up behavior of the flow (1.6). We choose a sequence of times $t_n \to \infty$ (see (3.2)) and study the behavior of $u(t_n)$. We determine the number of blow-up points on Σ and establish a uniform upper bound for the second (normalized) component (Proposition 3.5). We also derive an energy lower bound in the blow-up regime. All normalizations and rescalings used for blow-up subsequences will be introduced where they are first needed.

To this end, we extract a sequence $t_n \to \infty$ along which the time–derivative term vanishes in a suitable sense. Since $J_{\rho_2}(u(t))$ is nonincreasing in t (Lemma 2.2(ii)) and bounded from below, we have

$$J_{\rho_2}(u(0)) - \lim_{t\to\infty} J_{\rho_2}(u(t)) = \int_0^\infty \int_\Sigma \Big|\frac{\partial u}{\partial t}(t)\Big|^2 e^{u(t)}\, dV_g\, dt \ \le C. \tag{3.1}$$

Hence there exists $t_n \to \infty$ such that

$$\int_\Sigma \Big|\frac{\partial u}{\partial t}(t_n)\Big|^2 e^{u(t_n)}\, dV_g \ \longrightarrow\ 0 \quad \text{as } n \to \infty. \tag{3.2}$$

For simplicity, set $u_n := u(t_n)$ and introduce the normalized functions

$$u_1^n := u_n - \log\int_\Sigma h_1 e^{u_n}\, dV_g, \qquad u_2^n := -u_n - \log\int_\Sigma h_2 e^{-u_n}\, dV_g, \qquad f_n := \frac{\partial u}{\partial t}(t_n)\, e^{u_n/2}. \tag{3.3}$$

With these notations, u_n solves

$$-\Delta_g u_n = 8\pi\, h_1 e^{u_1^n} - \rho_2\, h_2 e^{u_2^n} - (8\pi - \rho_2)\ -\ f_n\, e^{u_n/2} \qquad \text{on } \Sigma. \tag{3.4}$$

Moreover, the normalized functions u_i^n and f_n satisfy the identities

$$\int_\Sigma h_1 e^{u_1^n}\, dV_g = 1, \qquad \int_\Sigma h_2 e^{u_2^n}\, dV_g = 1, \qquad \|f_n\|^2_{L^2(\Sigma)} = \int_\Sigma \Big|\frac{\partial u}{\partial t}(t_n)\Big|^2 e^{u_n}\, dV_g \to 0. \tag{3.5}$$

In particular, (3.5) shows that the time–derivative term is negligible as $n \to \infty$.

Passing to a subsequence if necessary, we may assume that

$$8\pi\, h_1 e^{u_1^n} \rightharpoonup \mu_1, \qquad \rho_2\, h_2 e^{u_2^n} \rightharpoonup \mu_2 \quad \text{in the sense of measures on } \Sigma.$$

Define the singular set

$$S := \{x \in \Sigma : \mu_1(\{x\}) + \mu_2(\{x\}) \ge 4\pi\}.$$

Since $\mu_1(\Sigma) = 8\pi$ and $\mu_2(\Sigma) = \rho_2$, the singular set S is finite. In the class of stationary mean-field models (including the sinh-Gordon equation and the Toda systems), it is well-known that S coincides with the set of blow-up points

$$S_1 := \{x \in \Sigma : \exists x_n \to x \text{ with } \max(u_1^n(x_n), u_2^n(x_n)) \to +\infty\}.$$

In our setting, the equation contains an additional time–derivative term. For the sake of completeness, we include a proof that the identification $S = S_1$ still holds in this case. We recall the Brezis–Merle estimate [2], and also refer to [13] for the result on surfaces.

Lemma 3.1 *(Lemma 2.7 in [13]). Let $\Omega \subset \Sigma$ be a smooth domain. Assume that u is a solution to a Dirichlet problem*

$$-\Delta_g u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where $f \in L^1(\Omega)$. For every $0 < \delta < 4\pi$, there is a constant C depending only on δ and Ω such that

$$\int_\Omega \exp\Big(\frac{(4\pi - \delta)|u|}{\|f\|_{L^1(\Omega)}}\Big) dV_g \le C.$$

As a consequence of Lemma 3.1, we first prove that $u_n - \overline{u}_n$ is uniformly bounded on every compact subset of $\Sigma \setminus S$, and we then deduce that the singular set S coincides with the blow-up set S_1.

Lemma 3.2. *(1) For $x \notin S$, there exist a geodesic ball $B_R^g(x) \subset \Sigma \setminus S$ and a constant $C > 0$ such that*

$$\|u_n - \overline{u}_n\|_{L^\infty(B_R^g(x))} \le C \quad \textit{for all } n \in \mathbb{N}. \tag{3.6}$$

(2) $S = S_1$.

Proof. (1) By Lemma 2.2 (i) and (3.5), we have $\|f_n e^{u_n/2}\|_{L^1(\Sigma)} \le \|e^{u_n/2}\|_{L^2(\Sigma)}\|f_n\|_{L^2(\Sigma)} = \|f_n\|_{L^2(\Sigma)} \cdot \left(\int_\Sigma e^{u_0} dV_g\right)^{\frac{1}{2}} \to 0$ as $n \to \infty$. Fix $x \notin S$ and choose $R > 0$ so small that $\overline{B_{4R}^g(x)} \subset \Sigma \setminus S$. Then, by the definition of S, there exists some $\delta \in (0, 4\pi)$ such that for sufficiently large n

$$\int\limits_{B_{4R}^g(x)} \Big(\big|f_n e^{\frac{1}{2}u_n}\big| + 8\pi h_1 e^{u_1^n} + \rho_2 h_2 e^{u_2^n} + 8\pi - \rho_2\Big) dV_g < 4\pi - 2\delta.$$

Let ζ_n be the solution of a Dirichlet problem

$$\Delta_g \zeta_n = f_n e^{\frac{1}{2}u_n} - 8\pi h_1 e^{u_1^n} + \rho_2 h_2 e^{u_2^n} + 8\pi - \rho_2 \quad \text{in } B_{4R}^g(x), \quad \zeta_n = 0 \quad \text{on } \partial B_{4R}^g(x).$$

Applying Lemma 3.1 to ζ_n, we obtain that

$$\|\zeta_n\|_{L^p(B_{4R}^g(x))} \le \|e^{|\zeta_n|}\|_{L^p(B_{4R}^g(x))} \le C, \qquad \text{for } p = \frac{4\pi - \delta}{4\pi - 2\delta} > 1 \tag{3.7}$$

with C independent of n.

Set $\eta_n = u_n - \overline{u}_n - \zeta_n$. Then η_n is a harmonic function in $B_{4R}^g(x)$, so we have

$$\|\eta_n\|_{L^\infty(B_{2R}^g(x))} \le C\|\eta_n\|_{L^1(B_{4R}^g(x))} \le C\big(\|u_n - \overline{u}_n\|_{L^1(\Sigma)} + \|\zeta_n\|_{L^1(B_{4R}^g(x))}\big) \le C \tag{3.8}$$

From Lemma 2.2 (i), applying Jensen's inequality, we obtain $\overline{u}_n \le e^{\overline{u}_n} \le \int_\Sigma e^{u_n} dV_g = \int_\Sigma e^{u_0} dV_g \le C$. Combining this with (3.7) and (3.8) yields $\|e^{u_n}\|_{L^p(B_{2R}^g(x))} \le C\|e^{|\zeta_n|}\|_{L^p(B_{2R}^g(x))} \le C$.

Setting $s = \frac{2p}{p+1} > 1$, by Hölder's inequality, we obtain

$$\begin{aligned}\int\limits_{B_{2R}^g(x)} |f_n e^{\frac{1}{2}u_n}|^s dV_g &\le \Big(\int\limits_{B_{2R}^g(x)} |f_n|^2 dV_g\Big)^{\frac{s}{2}} \Big(\int\limits_{B_{2R}^g(x)} e^{pu_n} dV_g\Big)^{1-\frac{s}{2}} \\ &\le C\Big(\int\limits_{B_{2R}^g(x)} |f_n|^2 dV_g\Big)^{\frac{s}{2}} \to 0, \quad \text{as } n \to +\infty.\end{aligned}$$

Applying L^p-estimates (see [20, Theorem 9.11]) and the Sobolev embedding, we deduce that $\{\zeta_n\}_{n\in\mathbb{N}}$ is bounded in $W^{2,s}(B_R^g(x))$ and $L^\infty(B_R^g(x))$. Thus, $u_n - \overline{u}_n = \zeta_n + \eta_n$ is bounded in $L^\infty(B_R^g(x))$.

(2) First, we prove that $S \subset S_1$. If $x_1 \notin S_1$, then there exist $R_1 > 0$ and $C > 0$ such that $B_{R_1}^g(x_1) \subset \Sigma \backslash S_1$ and $\max\limits_{x\in B_{R_1}^g(x_1)} \{e^{u_1^n}, e^{u_2^n}\} \le C$. For any $0 < r < R_1$,

$$8\pi \int\limits_{B_r^g(x_1)} h_1 e^{u_1^n} dV_g + \rho_2 \int\limits_{B_r^g(x_1)} h_2 e^{u_2^n} dV_g \le Cr^2 \to 0, \quad \text{as } r \to 0,$$

which implies that $x_1 \notin S$.

Next, it suffices to show that $S_1 \subset S$. Suppose $x_0 \notin S$. By Jensen's inequality and Lemma 2.3, we have

$$\overline{u}_1^n \leq e^{\overline{u}_1^n} \leq \int_\Sigma e^{u_1^n} dV_g = \frac{\int_\Sigma e^{u_0} dV_g}{\int_\Sigma h_1 e^{u_n} dV_g} \leq C, \quad \overline{u}_2^n \leq e^{\overline{u}_2^n} \leq \int_\Sigma e^{u_2^n} dV_g = \frac{\int_\Sigma e^{-u_n} dV_g}{\int_\Sigma h_2 e^{-u_n} dV_g} \leq C. \tag{3.9}$$

Then, by (3.6) and (3.9), it follows that for any $x \in B_R^g(x_0) \subset \Sigma \setminus S$,

$$u_i^n(x) \leq e^{u_i^n} \leq \exp\left(\|u_i^n - \overline{u}_i^n\|_{L^\infty(B_R^g(x_0))} + \overline{u}_i^n \right) \leq C, \quad i = 1, 2.$$

Thus, $x_0 \notin S_1$. We conclude $S = S_1$. □

3.1. Asymptotic behavior of a blow-up sequence (u_1^n, u_2^n)

We now study the asymptotic behavior of a blow-up sequence (u_1^n, u_2^n) arising from (3.3). For $i = 1, 2$, let $x_i^n \in \Sigma$ be a maximum point of u_i^n, and set

$$c_i^n := \max_{x \in \Sigma} u_i^n(x) = u_i^n(x_i^n), \qquad r_i^n := e^{-c_i^n/2}. \tag{3.10}$$

Our analysis proceeds in two steps. First (Proposition 3.3), we show that blow-up is concentrated at a single point and obtain global pointwise control in terms of the distance to this point. Second (Proposition 3.5), we prove a uniform upper bound for the second component u_2^n on Σ. We begin with the first step:

Proposition 3.3. *Let (u_1^n, u_2^n) be a blow-up sequence. Then, up to a subsequence, the following hold:*

(1) $r_2^n / r_1^n \to \infty$, $c_1^n \to \infty$ *as* $n \to \infty$ *and* $h_1(x_0) > 0$ *where* $x_0 = \lim_{n\to\infty} x_1^n$;
(2) *There exists C_1, independent of n, such that*

$$u_1^n(x) + 2\log \mathrm{dist}_g(x, x_1^n) \leq C_1, \;\; u_2^n(x) + 2\log \mathrm{dist}_g(x, x_1^n) \leq C_1, \;\; \forall x \in \Sigma. \tag{3.11}$$

In particular, x_0 is the unique blow-up point, in other words, $S = \{x_0\}$.

Proof. Proof of (1) We first prove that $r_2^n / r_1^n \to \infty$. Suppose, by contradiction, that $r_2^n \leq C r_1^n$ for some $C > 0$. Taking a subsequence, we may assume that $x_2^n \to x_0 \in S$ as $n \to \infty$. Choose an isothermal coordinate system near x_0, which satisfies $g = e^{\psi(x)}|dx|^2$ and $\psi(x_0) = 0$. Since S is finite, we can fix $\tilde{r} > 0$ so small that $x_0 = 0$ is the unique blow-up point in $B_{\tilde{r}}(0)$.

Set, for $i = 1, 2$,

$$w_i^n(x) := u_i^n(x_2^n + r_2^n x) + 2\log r_2^n \qquad x \in B_{\tilde{r}/r_2^n}(0) \subset \mathbb{R}^2.$$

Then, w_2^n satisfies, in $B_{\tilde{r}/r_2^n}(0) \subset \mathbb{R}^2$,

$$\begin{aligned} -\Delta w_2^n(x) =& \rho_2 (h_2 e^\psi)(x_2^n + r_2^n x) e^{w_2^n} - 8\pi (h_1 e^\psi)(x_2^n + r_2^n x) e^{w_1^n} + \big(8\pi - \rho_2\big) e^{\psi(x_2^n + r_2^n x) - c_2^n} \\ &+ (f_n e^\psi)(x_2^n + r_2^n x) e^{w_1^n(x)/2 - c_2^n/2} \|h_1 e^{u_n}\|_{L^1(\Sigma)}^{\frac{1}{2}}, \end{aligned} \tag{3.12}$$

where Δ denotes the Laplacian in the chosen coordinate.

By the definition of w_i^n and r_2^n, we have $w_2^n(0) = 0$. Moreover, by (3.5), there exists $C' > 0$ such that $e^{w_1^n}, e^{w_2^n} \leq C'$ on each fixed ball $B_R(0)$, and

$$\|f_n(x_2^n + r_2^n x)e^{\psi(x_2^n+r_2^n x)-c_2^n/2}\|_{L^2(B_R(0))} \le \|f_n\|_{L^2(\Sigma)} \to 0 \quad \text{for any } R > 0.$$

By Harnack's type inequalities (see [20, Theorems 9.20, 9.22]) and L^p-estimates, $\{w_2^n\}_{n\in\mathbb{N}}$ is bounded in $H^2_{\text{loc}}(\mathbb{R}^2)$. Hence, up to a subsequence,

$$w_2^n \rightharpoonup w_2 \quad \text{weakly in } H^2_{\text{loc}}(\mathbb{R}^2), \quad w_2^n \to w_2 \quad \text{in } C^\alpha_{\text{loc}}(\mathbb{R}^2).$$

On the other hand, by Lemma 2.3, we have $\|h_1 e^{u_n}\|_{L^1(\Sigma)}, \|h_2 e^{u_n}\|_{L^1(\Sigma)} \ge c$ for all $n \in \mathbb{N}$, and thus

$$w_1^n(x) + w_2^n(x) \le 4\log r_2^n - \log(\|h_1 e^{u_n}\|_{L^1(\Sigma)}\|h_2 e^{-u_n}\|_{L^1(\Sigma)}) \to -\infty, \tag{3.13}$$

uniformly on compact subsets of $\mathbb{R}^2$. In particular, $w_1^n \to -\infty$ locally uniformly.

Taking the limit in (3.12) on each $B_R(0)$ (using $e^{\psi(x_2^n+r_2^n x)} \to 1$, $(h_i e^{\psi})(x_2^n + r_2^n x) \to h_i(x_0)$ uniformly, the vanishing of the terms with $e^{w_1^n}$ and $e^{-c_2^n}$, and the L^2-smallness of the f_n-term), we obtain

$$-\Delta w_2 = \rho_2\, h_2(x_0)\, e^{w_2} \quad \text{in} \quad \mathbb{R}^2. \tag{3.14}$$

We distinguish two cases according to the value of $h_2(x_0)$.

Case (i) If $h_2(x_0) = 0$, then w_2 is harmonic in $\mathbb{R}^2$. Since e^{w_2} is subharmonic and $e^{w_2}(0) = 1$, the mean-value inequality gives $\int_{B_R(0)} e^{w_2} \ge \pi R^2$ for all $R > 0$, hence $\int_{\mathbb{R}^2} e^{w_2} = \infty$. This contradicts Lemma 2.3 because

$$\int\limits_{\mathbb{R}^2} e^{w_2} = \lim_{n\to\infty} \int\limits_{B_{\tilde r/r_2^n}(0)} e^{\psi(x_2^n+r_2^n x)+w_2^n} \le \lim_{n\to\infty} \int\limits_{\Sigma} e^{u_2^n}\, dV_g = \lim_{n\to\infty} \frac{\int_\Sigma e^{-u_n}\, dV_g}{\int_\Sigma h_2 e^{-u_n} dV_g} \le C. \tag{3.15}$$

Case (ii) If $h_2(x_0) > 0$, then from (3.14) we have

$$\rho_2 h_2(x_0) \int\limits_{B_R(0)} e^{w_2} = \rho_2 \lim_{n\to\infty} \int\limits_{B_R(0)} (h_2 e^{\psi})(x_2^n + r_2^n x) e^{w_2^n(x)} dx \le \lim_{n\to\infty} \int\limits_{\Sigma} \rho_2 h_2 e^{u_2^n} dV_g = \rho_2.$$

By the classification in [8], $\rho_2 h_2(x_0) \int_{\mathbb{R}^2} e^{w_2} = 8\pi$, which contradicts the above inequality for R sufficiently large. Thus, we have $r_2^n/r_1^n \to \infty$. Finally, $c_1^n \to \infty$ since $c_1^n \ge c_2^n$ and $c_1^n + c_2^n \to \infty$.

Now we prove that $h_1(x_0) > 0$ where $x_0 = \lim_{n\to\infty} x_1^n$. Work in an isothermal coordinate system near x_0 (we will use the notation $g = e^{\psi(x)}|dx|^2$ again) and define

$$v_i^n(x) := u_i^n(x_1^n + r_1^n x) + 2\log r_1^n, \qquad x \in B_{\tilde r/r_1^n}(0) \subset \mathbb{R}^2.$$

Then $v_1^n(0) = 0$, and arguing as before, we obtain the analogue of (3.12) for v_1^n, and $v_1^n \to v_1$ weakly in $H^2_{\text{loc}}(\mathbb{R}^2)$ and strongly in $C^\alpha_{\text{loc}}(\mathbb{R}^2)$. Moreover, as in (3.13) (with r_2^n replaced by r_1^n), we have $v_2^n \to -\infty$ locally uniformly in $\mathbb{R}^2$. Taking the limit in the rescaled equation gives

$$-\Delta v_1 = 8\pi h_1(x_0) e^{v_1} \quad \text{in } \mathbb{R}^2. \tag{3.16}$$

If $h_1(x_0) = 0$, then v_1 is harmonic in $\mathbb{R}^2$, and using the same argument as in (3.15) we derive a contradiction with Lemma 2.3. Thus, $h_1(x_0) > 0$, and it follows from the classification result [8] that

$$v_1(x) = -2\log\left(1 + \pi h_1(x_0)|x|^2\right). \tag{3.17}$$

Proof of (2) Now we prove (3.11). We work in an isothermal coordinate system near $x_0 = 0 \in \mathbb{R}^2$ such that x_0 is the unique blow-up point in $B_{3\tilde r}(0)$ for some $\tilde r > 0$. By (3.17), there exists $R_n \to \infty$ such that

$$v_1^n(y) + 2\log|y| \le C, \quad \forall y \in B_{R_n}(0).$$

With a change of variables $x = x_1^n + r_1^n y$, we can find $l_1^n \to 0$ such that $l_1^n/r_1^n \to \infty$ and

$$u_1^n(x) + 2\log|x - x_1^n| \le C, \quad \forall x \in B_{l_1^n}(x_1^n). \tag{3.18}$$

Assume by contradiction that

$$\max_{i=1,2,|x|\le\tilde r} \big(u_i^n(x) + 2\log|x - x_1^n|\big) \to \infty. \tag{3.19}$$

Let $q_n \in \overline{B}_{\tilde r}(0)$ be a point where the above maximum is attained, and define, for $i = 1, 2$,

$$d_n := \tfrac{1}{2}|q_n - x_1^n|, \qquad S_i^n(x) := u_i^n(x) + 2\log\big(d_n - |x - q_n|\big) \ \text{ in } \ B_{d_n}(q_n).$$

Then $S_i^n(x) \to -\infty$ as $x \to \partial B_{d_n}(q_n)$, while, as $n \to \infty$, it follows from (3.19) that

$$\max_{i=1,2} S_i^n(q_n) = \max_{i=1,2} \big(u_i^n(q_n) + 2\log d_n\big) \ge \max_{i=1,2} \big(u_i^n(q_n) + 2\log|q_n - x_1^n|\big) - 2\log 2 \to \infty.$$

Let p_n be the point where $\max_{x\in\overline{B}_{d_n}(q_n)} \{S_1^n, S_2^n\}$ is attained. We distinguish two cases comparing $S_1^n(p_n)$ and $S_2^n(p_n)$.

Case (i) Assume $S_1^n(p_n) \ge S_2^n(p_n)$. Then

$$u_1^n(p_n) + 2\log\big(d_n - |p_n - q_n|\big) = S_1^n(p_n) \ge \max\{S_1^n(q_n),\ S_2^n(q_n)\} \to +\infty. \tag{3.20}$$

Let $l_n = \frac{1}{2}\big(d_n - |p_n - q_n|\big)$. For any $y \in B_{l_n}(p_n)$ and $i = 1, 2$,

$$\begin{aligned} &u_i^n(y) + 2\log\big(d_n - |y - q_n|\big) \le u_1^n(p_n) + 2\log(2l_n), \\ &d_n - |y - q_n| \ge d_n - |p_n - q_n| - |y - p_n| \ge l_n, \end{aligned} \tag{3.21}$$

hence

$$u_i^n(y) \le u_1^n(p_n) + 2\log 2, \quad \text{for all } y \in B_{l_n}(p_n), \quad i = 1, 2.$$

Define the rescaled functions $\tilde u_i^n(z) := u_i^n(p_n + r_n z) + 2\log r_n$, $i = 1, 2$, where $r_n := e^{-u_1^n(p_n)/2}$. Then, by (3.20) and (3.21),

$$\tilde u_i^n(z) \le 2\log 2, \quad \text{for all } |z| \le l_n/r_n,\ i = 1, 2, \quad \text{and} \quad r_n \to 0,\ \ l_n/r_n \to \infty.$$

Moreover, $\tilde u_1^n$ satisfies on $B_{l_n/r_n}(0)$ the rescaled equation analogous to (3.12). By L^p-estimates and Sobolev embedding, we obtain that $\tilde u_1^n \rightharpoonup \tilde u_1$ in $H^2_{loc}(\mathbb{R}^2)$ and $\tilde u_1^n \to \tilde u_1$ in $C^\alpha_{loc}(\mathbb{R}^2)$. Arguing as in (3.13), we have $\tilde u_1^n + \tilde u_2^n \to -\infty$ locally in $\mathbb{R}^2$, hence $\tilde u_2^n \to -\infty$ locally uniformly in $\mathbb{R}^2$. Taking the limit, we obtain

$$-\Delta \tilde u_1 = 8\pi h_1(x_0) e^{\tilde u_1} \quad \text{in } \ \mathbb{R}^2, \tag{3.22}$$

since $u_1^n(p_n) \to \infty$ and $x_0 = 0$ is the unique blow-up point in $B_{\tilde r}(0)$. Hence, by (3.22) and the classification result [8],

$$1 = \beta_1 \int_{\mathbb{R}^2} e^{\tilde u_1} dy = \lim_{n\to\infty} \int_{B_{l_n/2r_n}(0)} (h_1 e^{\psi})(p_n + r_n y) e^{\tilde u_1^n(y)} dy = \lim_{n\to\infty} \int_{B_{l_n/2}(p_n)} h_1 e^{u_1^n + \psi} dy.$$

On the other hand, from (3.17) (the blow-up at x_1^n) we have

$$1 = h_1(x_0)\int_{\mathbb{R}^2} e^{v_1}dy = \lim_{n\to\infty}\int_{B_{l_1^n/2r_1^n}(0)} (h_1e^{\psi})(x_1^n + r_1^n y)e^{v_1^n(y)}dy = \lim_{n\to\infty}\int_{B_{l_1^n/2}(x_1^n)} h_1 e^{u_1^n+\psi}dy,$$

where we used $l_1^n/2r_1^n \to \infty$.

Since $u_1^n(p_n) + 2\log|p_n - x_1^n| \geq u_1^n(p_n) + 2\log(2l_n) - C \to \infty$, the inequality (3.18) implies $p_n \notin B_{l_1^n(x_1^n)}$. Moreover, by the definition of d_n, l_n, we also have $x_1^n \notin B_{l_n}(p_n)$. Therefore, for large n, $B_{l_1^n/2}(x_1^n) \cap B_{l_n/2}(p_n) = \emptyset$, so combining the above integration identities derives a contradiction to the fact that $\int_\Sigma h_1 e^{u_1^n} dV_g = 1$. Hence, Case (i) cannot occur.

Case (ii) Suppose that $S_2^n(p_n) \geq S_1^n(p_n)$. Then we define $\tilde{u}_i^n(y) := u_i^n(p_n + r_n y) + 2\log r_n$, $i = 1, 2$, where $r_n := e^{-u_2^n(p_n)/2}$. As in Case (i), by L^p-estimates, we have that $\tilde{u}_2^n \rightharpoonup \tilde{u}_2$ in $H^2_{loc}(\mathbb{R}^2)$, $\tilde{u}_2^n \to \tilde{u}_2$ in $C^\alpha_{loc}(\mathbb{R}^2)$, and $\tilde{u}_2$ satisfies

$$-\Delta\tilde{u}_2 = \rho_2\beta_2 e^{\tilde{u}_2} \quad \text{in } \mathbb{R}^2, \quad \text{where } \beta_2 = \lim_{n\to\infty} h_2(p_n).$$

By the same argument used for (3.14), we derive a contradiction. Hence, Case (ii) cannot occur.

Combining the above, we obtain (3.11) on $B_{\tilde{r}}(x_0)$. If there was another blow-up point $\tilde{x}_0 \neq x_0$, then the same blow-up analysis at $\tilde{x}_0$ yields, for every $\delta > 0$,

$$\int_{B^g_\delta(\tilde{x}_0)} h_1 e^{u_1^n}\, dV_g \to 1 \quad \text{or} \quad \int_{B^g_\delta(\tilde{x}_0)} \rho_2 h_2 e^{u_2^n}\, dV_g \to 8\pi.$$

However, we already have $\int_{B^g_\delta(x_0)} h_1 e^{u_1^n}\, dV_g \to 1$, while $\int_\Sigma h_1 e^{u_1^n}\, dV_g = 1$ and $\int_\Sigma \rho_2 h_2 e^{u_2^n}\, dV_g = \rho_2 < 8\pi$, a contradiction. Hence x_0 is the unique blow-up point.

Consequently, u_1^n and u_2^n are uniformly bounded above on $\Sigma \setminus B_{\tilde{r}}(x_0)$; This proves (3.11) and completes the proof of Proposition 3.3. □

Remark 3.4. From the proof of Proposition 3.3, we also obtain a useful consequence: In contrast to the blow-up behavior $u_1^n(x_1^n) \to \infty$, it holds that

$$\overline{u}_1^n \to -\infty, \quad \text{and} \quad \mu_1 = 8\pi\delta_{x_0},$$

where δ_{x_0} is the Dirac measure concentrated at x_0. Consequently, Lemma 3.2 implies that, for any compact subset $K \Subset \Sigma \setminus \{x_0\}$, $u_1^n \to -\infty$ uniformly on K.

In fact, using (3.16) and the classification result [8] (see (3.17)) in the proof of Proposition 3.3, we further deduce that, for any $\delta > 0$,

$$8\pi = 8\pi\int_{\mathbb{R}^2} h_1(x_0)e^{v_1}dx = 8\pi\lim_{n\to\infty}\int_{B^g_\delta(x_0)} h_1 e^{u_1^n} dV_g = \mu_1(B^g_\delta(x_0)) \leq \mu_1(\Sigma) = 8\pi.$$

Hence $\mu_1(B^g_\delta(x_0)) = 8\pi$ for all sufficiently small δ, which implies that $\mu_1 = 8\pi\delta_{x_0}$.

To see $\overline{u}_1^n \to -\infty$, fix $\delta > 0$ so small that $h_1(x) \geq \epsilon > 0$ on the annulus $A_\delta := B^g_\delta(x_0) \setminus B^g_{\delta/2}(x_0)$. By Lemma 3.2, there exists $C_\delta > 0$ such that

$$e^{\overline{u}_1^n}\int_{A_\delta} h_1 dV_g \leq C_\delta \int_{A_\delta} h_1 e^{u_1^n} dV_g \to 0 \quad \text{as } n \to \infty.$$

Since $\int_{A_\delta} h_1 dV_g > 0$, we conclude that $\overline{u}_1^n \to -\infty$. □

Next, we focus on the second component u_2^n. Recall that x_i^n is the maximum point of u_i^n, with $c_i^n = u_i^n(x_i^n)$, $r_i^n = e^{-c_i^n/2}$. We work in an isothermal coordinate around the unique blow-up point $x_0 = 0$ (so $g = e^{\psi(x)}|dx|^2$ with $\psi(0) = 0$), and set $s_n = |x_1^n - x_2^n|$. Rescaling at the scale s_n around x_2^n, define

$$w_2^n(x) := u_2^n(x_2^n + s_n x) + 2\log r_2^n, \quad w_1^n(x) := u_1^n(x_2^n + s_n x) + 2\log s_n.$$

Set also the unit vector $\tilde{x}_n := (x_1^n - x_2^n)/s_n \in \partial B_1(0)$ (so $w_2^n(0) = 0 = \sup w_2^n$). After passing to a subsequence if necessary, assume $\tilde{x}_n \to \tilde{x}_0 \in \partial B_1(0)$.

Proposition 3.5. *The sequence u_2^n is uniformly bounded above on Σ, i.e. there exists $C > 0$ such that $u_2^n(x) \le C$ for all $x \in \Sigma$ and all $n \in \mathbb{N}$.*

Proof. Suppose that u_2^n also blows up, i.e. $r_2^n \to 0$ as $n \to \infty$. Substituting $x = x_2^n$ into the inequality in (3.11) for u_2^n, we obtain $|x_1^n - x_2^n|/r_2^n \le C$ for some C independent of n. Taking a subsequence if necessary, we may assume $s_n/r_2^n \to A \in [0, \infty)$ as $n \to \infty$.

Before analyzing the functions w_i^n, $i = 1, 2$, we claim that $s_n/r_1^n \to \infty$ as $n \to \infty$. To see this, suppose not. Then, up to a subsequence, $(x_2^n - x_1^n)/r_1^n \to z_0 \in \mathbb{R}^2$. As in the proof of Proposition 3.3 (see (3.17)), we have

$$v_1^n(x) = u_1^n(x_1^n + r_1^n x) + 2\log r_1^n \to v_1(x) = -2\log\left(1 + \pi h_1(x_0)|x|^2\right) \quad \text{in } C^\alpha_{\text{loc}}(\mathbb{R}^2).$$

Since $v_1(x)$ is radially symmetric and strictly decreasing, $u_2^n(x_1^n + r_1^n x) = -v_1^n(x) + \text{const}$, the functions $u_2^n(x_1^n + r_1^n x)$ cannot have a maximum at $x = (x_2^n - x_1^n)/r_1^n$ for large n. This contradicts the definition of x_2^n. Hence, the claim holds.

Step 1. Asymptotic behavior of w_1^n. We claim that $w_1^n \to -\infty$ locally uniformly in $\mathbb{R}^2 \setminus \{\tilde{x}_0\}$.

Suppose not. Then there exist $D, \delta > 0$ such that $\max\limits_{x \in B_D(\tilde{x}_0)\setminus B_\delta(\tilde{x}_0)} w_1^n \ge c$ for all $n \in \mathbb{N}$. On the other hand, by the definition of $\tilde{x}_n = (x_1^n - x_2^n)/s_n$, we have $w_1^n(x) \le C - 2\log|x - \tilde{x}_n|$. Since $\tilde{x}_n \to \tilde{x}_0$, for large n, we have $\max\limits_{x \in B_{2D}(\tilde{x}_0)\setminus B_{\delta/2}(\tilde{x}_0)} w_1^n \le C$. Hence, by Harnack type inequalities ([20, Theorems 9.20, 9.22]) applied on a compact subset $K \Subset B_D(\tilde{x}_0)\setminus B_\delta(\tilde{x}_0)$, there exists C independent of n such that $\min\limits_{x \in B_D(\tilde{x}_0)\setminus B_\delta(\tilde{x}_0)} w_1^n \ge -C$. In particular, by the fact $h_1(x_0) > 0$, there exists $\epsilon_0 > 0$ independent of n such that

$$\int_K (h_1 e^\psi)(x_2^n + s_n x) e^{w_1^n(x)} dx \ge 2\epsilon_0 > 0. \tag{3.23}$$

Next, by the classification result (see (3.16)–(3.17)), there exists $R > 0$ such that $1 - \epsilon_0 = h_1(x_0)\int_{B_R(0)} e^{v_1} dx$. By a change of variables, it follows that

$$1 - \epsilon_0 = \lim_{n\to\infty} \int_{B_R(0)} (h_1 e^\psi)(x_1^n + r_1^n x) e^{v_1^n(x)} dx = \lim_{n\to\infty} \int_{B_{Rr_1^n/s_n}(\tilde{x}_n)} (h_1 e^\psi)(x_2^n + s_n x) e^{w_1^n(x)} dx. \tag{3.24}$$

Since $s_n/r_1^n \to \infty$ as $n \to \infty$, the sets $B_{Rr_1^n/s_n}(\tilde{x}_n)$ and K are disjoint for large n. Combining (3.23) and (3.24), we then obtain that, for large n,

$$1 = \int_\Sigma h_1 e^{u_1^n} dV_g \ge \int_{B_{Rr_1^n/s_n}(\tilde{x}_n)\cup K} (h_1 e^\psi)(x_2^n + s_n x) e^{w_1^n(x)} dx \ge 1 + \epsilon_0 > 1. \tag{3.25}$$

It is a contradiction. This completes the proof of the claim.

Step 2. Analysis of the PDE for w_2^n. First, by the definition, $w_2^n(x) \le 0$ and $w_2^n(0) = 0$. We also note that, in the isothermal coordinate around $x_0 = 0$, the rescaled function w_2^n satisfies, in $B_{\tilde{r}/s_n}(0) \subset \mathbb{R}^2$

$$\begin{aligned} -\Delta w_2^n(x) =& \rho_2 (h_2 e^{\psi})(x_2^n + s_n x) e^{w_2^n(x)} \Big(\frac{s_n}{r_2^n}\Big)^2 - 8\pi (h_1 e^{\psi})(x_2^n + s_n x) e^{w_1^n(x)} \\ &+ (8\pi - \rho_2) e^{\psi(x_2^n + s_n x)} s_n^2 + (f_n e^{\psi})(x_2^n + s_n x) e^{w_1^n(x)/2} s_n \|h_1 e^{u_n}\|_{L^1(\Sigma)}^{\frac{1}{2}}. \end{aligned} \tag{3.26}$$

Fix an annulus $A_{R,\delta} := B_R(\tilde{x}_0) \setminus B_\delta(\tilde{x}_0)$ with $0 < \delta < R < \infty$. By Step 1, $w_1^n \to -\infty$ locally uniformly on $\mathbb{R}^2 \setminus \{\tilde{x}_0\}$; hence $(h_1 e^{\psi})(x_2^n + s_n x)\, e^{w_1^n(x)} \to 0$ uniformly on $A_{R,\delta}$. Since $\|f_n\|_{L^2(\Sigma)} \to 0$ and $\int_\Sigma e^{u_n} = \int_\Sigma e^{u_0}$, we also have

$$\int\limits_{B_R(\tilde{x}_0)\setminus B_\delta(\tilde{x}_0)} \Big| (f_n e^{\psi})(x_2^n + s_n x) e^{\frac{1}{2} w_1^n(x)} s_n \|h_1 e^{u_n}\|_{L^1(\Sigma)}^{\frac{1}{2}} \Big|^2 \to 0.$$

Moreover, by $s_n / r_2^n \to A \in [0, \infty)$, the remaining terms of the right-hand side of (3.26) are uniformly bounded in $L^2(A_{R,\delta})$.

By the Harnack type inequality and L^p-estimate, the sequence $\{w_2^n\}$ is bounded in $H^2_{loc}(\mathbb{R}^2 \setminus \{\tilde{x}_0\})$. After passing to a subsequence, we see that $w_2^n \rightharpoonup w_2$ weakly in $H^2_{loc}(\mathbb{R}^2 \setminus \{\tilde{x}_0\})$ and strongly in $C^\alpha_{loc}(\mathbb{R}^2 \setminus \{\tilde{x}_0\})$ for some $\alpha \in (0,1)$, where w_2 satisfies

$$\Delta w_2 + A^2 \rho_2 h_2(x_0) e^{w_2} = 0 \quad \text{in } \mathbb{R}^2 \setminus \{\tilde{x}_0\}.$$

Applying the L^p-estimates again, we obtain that $w_2^n \to w_2$ strongly in $H^2_{loc}(\mathbb{R}^2 \setminus \{\tilde{x}_0\})$.

It now remains to show that w_2 satisfies the equation in $B_1(\tilde{x}_0) \subset \mathbb{R}^2$. To see this, we decompose $w_2^n := w_r^n + w_s^n$, where the singular part w_s^n solves the Dirichlet problem

$$\begin{cases} \Delta w_s^n = 8\pi (h_1 e^{\psi})(x_2^n + s_n x) e^{w_1^n} - (f_n e^{\psi})(x_2^n + s_n x) e^{w_1^n/2} s_n \|h_1 e^{u_n}\|_{L^1(\Sigma)}^{\frac{1}{2}} & \text{in } B_1(\tilde{x}_0), \\ w_s^n = 0 & \text{on } \partial B_1(\tilde{x}_0). \end{cases}$$

By Cauchy-Schwarz inequality and (3.5), we have that

$$\int\limits_{B_1(\tilde{x}_0)} f_n(x_2^n + s_n x) s_n e^{w_1^n(x)/2} dx \le C \|f_n\|_{L^2(\Sigma)} (\int\limits_\Sigma e^{u_0} dV_g)^{\frac{1}{2}} \to 0.$$

Moreover, by Step 1, we have

$$h_1(x_2^n + s_n x) e^{w_1^n(x)} \rightharpoonup \delta_{\tilde{x}_0},$$

in the sense of measures.

By potential estimates (e.g. [20, Lemma 7.12]), the sequence $\{w_s^n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(B_1(\tilde{x}_0))$ for any $1 < p < 2$. Hence, taking to a subsequence, we may assume that $w_s^n \to w_s$ weakly in $W_0^{1,p}(B_1(\tilde{x}_0))$, and

$$\Delta w_s = 8\pi \delta_{\tilde{x}_0} \quad \text{in } \mathbb{R}^2$$

in the sense of distributions. Consequently, $w_s(x) = 4 \log |x - \tilde{x}_0|$.

Now we characterize the regular part w_r^n, which satisfies the following Dirichlet problem:

$$\begin{cases} \Delta w_r^n = -\rho_2(h_2e^{\psi})(x_2^n + s_n x)e^{w_2^n}(\frac{s_n}{r_2^n})^2 - (8\pi - \rho_2)e^{\psi(x_2^n+s_n x)}s_n^2 & \text{in } B_1(\tilde{x}_0), \\ w_r^n = w_2^n & \text{on } \partial B_1(\tilde{x}_0). \end{cases}$$

Since $w_2^n \leq 0$ and $w_2^n \to w_2$ in $H^2_{loc}(\mathbb{R}^2 \setminus \{\tilde{x}_0\})$ near $\partial B_1(\tilde{x}_0)$, applying standard elliptic estimates, we obtain that $w_r^n \to w_r$ in $H^2(B_1(\tilde{x}_0))$ and w_r satisfies

$$\Delta w_r + A^2\rho_2 h_2(x_0)e^{w_2} = 0 \ \text{ in } B_1(\tilde{x}_0), \quad w_r = w_2 \quad \text{on } \partial B_1(\tilde{x}_0).$$

Thus, $w_2 = w_r + 4\log|\cdot - \tilde{x}_0|$ satisfies the following equation in distribution sense

$$\Delta w_2 + A^2\rho_2 h_2(x_0)e^{w_2} = 8\pi\delta_{\tilde{x}_0} \ \text{ in } \ \mathbb{R}^2. \tag{3.27}$$

We distinguish two cases according to the limiting coefficient:

Case (i) When $A^2h_2(x_0) = 0$. Then (3.27) reduces to $\Delta w_2 = 8\pi\,\delta_{\tilde{x}_0}$, hence w_2 is harmonic on $\mathbb{R}^2 \setminus \{\tilde{x}_0\}$. By construction $w_2 \leq 0$ and $w_2(0) = \sup_{\mathbb{R}^2\setminus\{\tilde{x}_0\}} w_2 = 0$; hence by the strong maximum principle $w_2 \equiv 0$ on $\mathbb{R}^2 \setminus \{\tilde{x}_0\}$, which contradicts $\Delta w_2 = 8\pi\,\delta_{\tilde{x}_0}$.

Case (ii) When $A^2h_2(x_0) > 0$. We rewrite (3.27) into

$$\begin{cases} -\Delta w_r(x) = A^2\rho_2 h_2(x_0)|x - \tilde{x}_0|^4 e^{w_r(x)} & \text{in } \mathbb{R}^2, \\ \int\limits_{\mathbb{R}^2} |x - \tilde{x}_0|^4 e^{w_r} = \int\limits_{\mathbb{R}^2} e^{w_2} \leq \lim\limits_{n\to\infty} (\frac{r_2^n}{s_n})^2 \int\limits_{\Sigma} e^{u_2^n} dV_g = \frac{1}{A^2} \lim\limits_{n\to\infty} \frac{\int_\Sigma e^{-u_n} dV_g}{\int_\Sigma h_2 e^{-u_n} dV_g} < \infty. \end{cases}$$

By the classification result in [43], it holds that $A^2\rho_2 h_2(x_0)\int_{\mathbb{R}^2} |x - \tilde{x}_0|^4 e^{w_r} = 24\pi$. Therefore, we can choose $R \gg 1$ and $0 < \delta \ll 1$ such that $A^2\rho_2 h_2(x_0)\int_{B_R(0)\setminus B_\delta(\tilde{x}_0)} |x - \tilde{x}_0|^4 e^{w_r} > 8\pi$. However, this leads to a contradiction:

$$8\pi < \rho_2 \lim_{n\to\infty} \frac{s_n^2}{(r_2^n)^2} \int\limits_{B_R(0)\setminus B_\delta(\tilde{x}_0)} (h_2e^{\psi})(x_2^n + s_n x)e^{w_2^n(x)} \leq \rho_2 \int\limits_{\Sigma} h_2 e^{u_2^n} dV_g = \rho_2 < 8\pi.$$

Both cases lead to contradictions, hence u_2^n is uniformly bounded from above on Σ. This completes the proof of Proposition 3.5. □

Now we can describe the global weak limit of the sequence $u_n - \overline{u}_n$. The next proposition identifies the limiting profile as a Green function plus a smooth correction, which will be the key input for the lower bound.

Proposition 3.6. *For any $1 < p < 2$, $u_n - \overline{u}_n + w_n$ converges to G_{x_0} weakly in $W^{1,p}(\Sigma)$ and strongly in $W^{2,2}_{loc}(\Sigma\setminus\{x_0\})$, where G_{x_0} is the Green function in (1.8) with $p = x_0$, and w_n is the solution of the following equation*

$$-\Delta_g w_n = \rho_2\Big(\frac{h_2e^{-u_n}}{\int_\Sigma h_2 e^{-u_n} dV_g} - 1\Big) \quad on \quad \Sigma, \quad \int\limits_{\Sigma} w_n dV_g = 0. \tag{3.28}$$

In addition, up to a subsequence, $w_n \to w_{x_0}$ in $C^{1,\alpha}(\Sigma)$ for some constant $0 < \alpha < 1$, where w_{x_0} satisfies the singular mean field equation (1.10) with $p = x_0$.

Proof. Observe that $u_n - \overline{u}_n + w_n$ solves

$$-\Delta_g\big(u_n - \overline{u}_n + w_n\big) = 8\pi\big(\frac{h_1 e^{u_n}}{\int_\Sigma h_1 e^{u_n} dV_g} - 1\big) - f_n e^{\frac{1}{2}u_n} \quad \text{on} \quad \Sigma. \tag{3.29}$$

By Remark 3.4 and (3.5), we have $h_1 e^{u_n} / \int_\Sigma h_1 e^{u_n} dV_g \rightharpoonup \delta_{x_0}$ in the sense of measure, and $\|f_n e^{\frac{1}{2}u_n}\|_{L^1(\Sigma)} \le \|f_n\|_{L^2(\Sigma)}(\int_\Sigma e^{u_0} dV_g)^{\frac{1}{2}} \to 0$, as $n \to \infty$. Consequently, by the potential estimates, $u_n - \overline{u}_n + w_n \rightharpoonup G_{x_0}$ weakly in $W^{1,p}(\Sigma)$ for any $p \in (1,2)$.

On the other hand, by Remark 3.4, $u_n \to -\infty$ uniformly on any compact set $K \Subset \Sigma \setminus \{x_0\}$. Therefore,

$$\int_K \Big|8\pi \frac{h_1 e^{u_n}}{\int_\Sigma h_1 e^{u_n} dV_g} - f_n e^{\frac{1}{2}u_n}\Big|^2 dV_g \le C\|f_n\|^2_{L^2(\Sigma)} \to 0 \quad \text{as} \quad n \to \infty.$$

This estimate, combined with standard elliptic regularity theory, yields the strong convergence in $W^{2,2}_{loc}(\Sigma \setminus \{x_0\})$.

Finally, Proposition 3.5 implies the boundedness of $e^{-u_n} / \int_\Sigma h_2 e^{-u_n} dV_g$. Applying elliptic regularity theory to (3.28), we conclude that $\{w_n\}_{n\in\mathbb{N}}$ is bounded in $W^{2,q}(\Sigma)$ for every $q \in (1,\infty)$. By the Sobolev embedding theorem, it follows that $w_n \to w_{x_0}$ in $C^{1,\alpha}(\Sigma)$ for some $\alpha \in (0,1)$. This completes the proof. □

3.2. Energy lower bound in the blow-up regime

We are ready to establish a lower bound for the energy functional $J_{\rho_2}(u_n)$ along the blow-up sequence u_n. This will be important in Section 4, where we select initial conditions that prevent blow-up and show that the flow converges to a solution of the stationary problem (1.1).

Recall that, for $p \in \Sigma$, Γ_p denotes the set of solutions to the singular mean field equation (1.10).

Proposition 3.7. *Let $u_n \in H^2(\Sigma)$ be a blow-up sequence. Then*

$$\lim_{n\to\infty} J_{\rho_2}(u_n) \ge \inf_{p\in\Sigma} \inf_{w_p \in \Gamma_p} \big\{\tilde{J}_p(w_p) - 4\pi A(p) - 8\pi \log h_1(p)\big\} - 8\pi \log \pi - 8\pi, \tag{3.30}$$

where G_p is the Green function in (1.8), $A(p)$ is the regular part of G_p defined in (1.9), and $\tilde{J}_p$ is the functional (1.11) for the singular mean field equation (1.10).

Proof. Recalling the definition of the energy functional J_{ρ_2} and the definition of u_1^n in (3.3), we have

$$J_{\rho_2}(u_n) = \frac{1}{2}\int_\Sigma |\nabla_g u_1^n|^2 dV_g - \rho_2 \log \int_\Sigma h_2 e^{-u_n + \overline{u}_n} dV_g + 8\pi \overline{u}_1^n. \tag{3.31}$$

In order to compute the second term in (3.31), we claim that a sequence $(-u_n + \overline{u}_n)$ is uniformly bounded above. If not, then by (3.6) and the fact that $S = \{x_0\}$, there exists a sequence of points $y_n \to x_0$, such that $-u_n(y_n) + \overline{u}_n \to \infty$. On the other hand, by Lemma 2.3 and Remark 3.4, we know that $\overline{u}_1^n \to -\infty$, while $\int_\Sigma h_1 e^{u_n} dV_g$ remains bounded. Hence, $\overline{u}_n = \overline{u}_1^n + \log \int_\Sigma h_1 e^{u_n} dV_g \to -\infty$. This implies that $u_n(y_n) \to -\infty$, as $n \to \infty$. However, by Proposition 3.3, x_0 is the unique blow-up point of u_1^n. Since $\int_\Sigma h_1 e^{u_n} dV_g$ is bounded, we conclude that $u_n(y_n) \to \infty$ as $n \to \infty$. The two conclusions contradict each other, and therefore the claim is proved.

With this claim, we can compute the second term. By Proposition 3.6, we know that $u_n - \overline{u}_n + w_n \to G_{x_0}$ in $C^\alpha_{loc}(\Sigma \setminus \{x_0\})$, and $w_n \to w_{x_0}$ in $C^\alpha(\Sigma)$. Hence, for sufficiently small $\delta > 0$, we obtain

$$\lim_{n\to\infty}\int_\Sigma h_2 e^{-u_n+\overline{u}_n}dV_g=\int_{\Sigma\setminus B^g_\delta(x_0)} h_2 e^{-G_{x_0}+w_{x_0}}dV_g+o_\delta(1)=\int_\Sigma h_2 e^{-G_{x_0}+w_{x_0}}dV_g+o_\delta(1). \tag{3.32}$$

We next compute the first term in (3.31). To this end, fix an isothermal chart $\Psi : B_{\tilde r}(0)\to\Sigma$ with $\Psi(0)=x_0$ and $g=e^{\psi(x)}|dx|^2$. Set $x_n := x^n_1$ and $r_n := r^n_1$ as in (3.10), and choose $0<\delta<\tilde r<Rr_n^{-1}\tilde r$. All balls are taken in the chart and then pushed forward by Ψ; for brevity, in the remainder of the proof, we write $B_r(x_n)$ both for the Euclidean ball $B_r(\Psi^{-1}(x_n))\subset\mathbb{R}^2$ and for its image $\Psi(B_r(\Psi^{-1}(x_n)))\subset\Sigma$. With this convention, we decompose $\Sigma=B_{Rr_n}(x_n)\cup\big(B_\delta(x_n)\setminus B_{Rr_n}(x_n)\big)\cup\big(\Sigma\setminus B_\delta(x_n)\big)$.

Near the blow-up point x_0, considering the rescaled solution v^n_1 in Proposition 3.3, we obtain that

$$\begin{aligned}\lim_{n\to\infty}\frac12\int_{B_{Rr_n}(x_n)}|\nabla_g u^n_1|^2dV_g&=\frac12\lim_{n\to\infty}\int_{B_R(0)}|\nabla v^n_1|^2dx=\pi\int_0^R\Big|\frac{4\pi h_1(x_0)r}{\pi h_1(x_0)r^2+1}\Big|^2rdr\\&=8\pi\log\pi-8\pi+16\pi\log R+8\pi\log h_1(x_0)+o_R(1),\end{aligned}\tag{3.33}$$

where $o_R(1)\to0$ as $R\to\infty$.

For the domain $\Sigma\backslash B^g_\delta(x_n)$, i.e. away from the blow-up point x_0, Proposition 3.6 together with the identity $\int_\Sigma\nabla_g G_{x_0}\nabla_g w_{x_0}dV_g=8\pi w_{x_0}(x_0)$ yields

$$\begin{aligned}&\lim_{n\to\infty}\frac12\int_{\Sigma\backslash B_\delta(x_n)}|\nabla_g u^n_1|^2dV_g=\lim_{n\to\infty}\frac12\int_{\Sigma\backslash B_\delta(x_0)}|\nabla_g\big(u_n-\overline{u}_n+w_n\big)-\nabla_g w_n|^2dV_g\\&=\frac12\int_\Sigma|\nabla_g w_{x_0}|^2dV_g+\frac12\int_{\Sigma\backslash B_\delta(x_0)}|\nabla_g G_{x_0}|^2dV_g-8\pi w_{x_0}(x_0)+o_\delta(1).\end{aligned}\tag{3.34}$$

Integrating by parts the second term of (3.34) and using the local expansion (1.9) (with $p=x_0$), we obtain

$$\begin{aligned}\frac12\int_{\Sigma\backslash B_\delta(x_0)}|\nabla_g G_{x_0}|^2dV_g&=-\frac12\int_{\Sigma\backslash B_\delta(x_0)}(\Delta_g G_{x_0})G_{x_0}dV_g-\frac12\int_{\partial B_\delta(x_0)}\frac{\partial G_{x_0}}{\partial\nu}G_{x_0}dS_g\\&=-16\pi\log\delta+4\pi A(x_0)+o_\delta(1),\end{aligned}\tag{3.35}$$

where $o_\delta(1)\to0$ as $\delta\to0$.

For the neck domain, we compare u_n with harmonic functions (see [30,47]). Define the spherical mean u^*_n of u_n by

$$u^*_n(r):=\frac{1}{2\pi}\int_0^{2\pi}u^n_1(x_n+re^{i\theta})d\theta.$$

Then u^n_1 and u^*_n satisfy the following inequality (e.g. see [28, inequality (3.4)])

$$\int_{B_s\backslash B_r}|\nabla u^*_n|^2dx\le\int_{B_s\backslash B_r}\Big|\frac{\partial u^n_1}{\partial r}\Big|^2dx.$$

Let $\tilde u^*_n$ be the harmonic function on the neck domain with boundary conditions $\tilde u^*_n(r)=u^*_n(r)$ for $r=\delta$ and $r=Rr_n$. Then it satisfies the following inequality (e.g. see [30, equation (31)])

$$\frac12\int_{B^g_\delta(x_n)\backslash B^g_{Rr_n}(x_n)}|\nabla_g u^n_1|^2dV_g\ge\frac12\int_{B^g_\delta(x_n)\backslash B^g_{Rr_n}(x_n)}|\nabla_g\tilde u^*_n|^2dV_g=\frac{\pi\big(u^*_n(\delta)-u^*_n(Rr_n)\big)^2}{\log\delta-\log(Rr_n)}.$$

Define $\tau_n := u_n^*(\delta)-u_n^*(Rr_n)-\overline{u}_1^n-2\log r_n$ and $u_1^n-\overline{u}_1^n+w_n \to G_{x_0}$ in $C^\alpha_{loc}(\Sigma\setminus\{x_0\})$ from Proposition 3.6, we obtain that

$$\begin{aligned}\lim_{n\to\infty}\left(u_n^*(Rr_n)+2\log r_n\right) &= -2\log\left(\pi h_1(x_0)R^2\right)+o_R(1),\\ \lim_{n\to\infty}\left(u_n^*(\delta)-\overline{u}_1^n\right) &= -4\log\delta+A(x_0)-w_{x_0}(x_0)+o_\delta(1),\end{aligned} \tag{3.36}$$

hence

$$\tau_n \to 4\log\frac{R}{\delta}+A(x_0)-w_{x_0}(x_0)+2\log\pi+2\log h_1(x_0)+o_\delta(1)+o_R(1) \quad \text{as } n\to\infty. \tag{3.37}$$

Since $\overline{u}_1^n\to-\infty$ and $r_n\to 0$ as $n\to\infty$, by straightforward calculation (e.g. see [30,47]), we obtain that for large n,

$$\begin{aligned}\frac{\pi\left(u_n^*(\delta)-u_n^*(Rr_n)\right)^2}{\log\delta-\log(Rr_n)} &= \frac{\pi\left(\tau_n+\overline{u}_1^n+2\log r_n\right)^2}{-\log r_n}\left(1-\frac{\log R-\log\delta}{-\log r_n}\right)^{-1}\\ &\geq \frac{\pi\left(\tau_n+\overline{u}_1^n+2\log r_n\right)^2}{-\log r_n}\left(1+\frac{\log R-\log\delta}{-\log r_n}+\frac{A'}{(\log r_n)^2}\right)\\ &\geq -\pi\log r_n\left(2-\frac{\overline{u}_1^n}{\log r_n}\right)^2-8\pi\overline{u}_1^n-2\pi\tau_n\left(2+\frac{\overline{u}_1^n}{\log r_n}\right)+\pi\left(2+\frac{\overline{u}_1^n}{\log r_n}\right)^2\log\frac{R}{\delta}\\ &\quad+\frac{2\pi\overline{u}_1^n\tau_n}{(\log r_n)^2}\log\frac{R}{\delta}+\frac{\pi A'}{-\log r_n}\left(2+\frac{\overline{u}_1^n}{\log r_n}\right)^2+\frac{2\pi A'\tau_n}{-(\log r_n)^2}\left(2+\frac{\overline{u}_1^n}{\log r_n}\right)\\ &\quad+o_R(1)+o_\delta(1)+o_n(1).\end{aligned} \tag{3.38}$$

Since $J_{\rho_2}(u(t))$ is decreasing with respect to t, letting $n\to\infty$, we must have $\overline{u}_1^n/\log r_n\to 2$. Indeed, if this were not the case, then as $r_n\to 0$ and $\overline{u}_1^n\to-\infty$, the first term $-\pi\log r_n\left(2-\frac{\overline{u}_1^n}{\log r_n}\right)^2$ and the second term $-8\pi\overline{u}_1^n$ in (3.38) would dominate all the remaining terms. Moreover, combining (3.31)–(3.38), we obtain

$$J_{\rho_2}(u_n)\geq -\pi\log r_n\left(2-\frac{\overline{u}_1^n}{\log r_n}\right)^2(1+o_n(1)),$$

since the contribution of $8\pi\overline{u}_1^n$ in $J_{\rho_2}(u_n)$ is canceled by the second term on the right-hand side of (3.38). This contradicts the boundedness of the energy $J_{\rho_2}(u_n)$. Therefore, we conclude that $\overline{u}_1^n/\log r_n\to 2$.

Substituting $\lim_{n\to\infty}\overline{u}_1^n/\log r_n=2$ and the expression of τ_n from (3.37) into (3.38), we derive the inequality over the neck region:

$$\begin{aligned}\frac{1}{2}\int_{B_\delta(x_n)\setminus B_{Rr_n}(x_n)}|\nabla_g u_1^n|^2 dV_g \geq &-16\pi\log r_n-8\pi A(x_0)+8\pi w_{x_0}(x_0)-16\pi\log\pi\\ &-16\pi\log h_1(x_0)-16\pi\log\frac{R}{\delta}+O\left(\frac{1}{\log r_n}\right).\end{aligned} \tag{3.39}$$

Finally, combining (3.31)–(3.35) and (3.39) together, and letting $n\to\infty$ first and $R\to\infty$, $\delta\to 0$ next, we obtain the desired lower bound (3.30). This completes the proof. □

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Guided by the limit profile in Proposition 3.6 and the energy lower bound (3.30), we construct test profiles whose energy lies strictly below the barrier level. This provides initial data for which blow-up is precluded and the flow converges to a stationary solution.

4.1. Construction of a sub-barrier test function

Define the energy barrier level

$$L_* := \inf_{p\in\Sigma}\inf_{w\in\Gamma_p}\left\{\tilde{J}_p(w) - 4\pi A(p) - 8\pi\log h_1(p)\right\} - 8\pi\log\pi - 8\pi,$$

where Γ_p is the solution set of the singular mean field equation (1.10). Thus L_* is the barrier below which blow-up cannot occur by (3.30).

We first show that the infimum is attained by a minimizing pair with $p_0 \in \Sigma$ and $w_{p_0} \in \Gamma_{p_0}$.

Proposition 4.1. *Let Γ_p be the solution set of the singular mean field equation (1.10), then there exist $p_0 \in \Sigma$ and $w_{p_0} \in \Gamma_{p_0}$ such that*

$$\tilde{J}_{p_0}(p_0) - 4\pi A(p_0) - 8\pi\log h_1(p_0) = \inf_{p\in\Sigma}\inf_{w_p\in\Gamma_p}\left\{\tilde{J}_p(w_p) - 4\pi A(p) - 8\pi\log h_1(p)\right\}. \tag{4.1}$$

Proof. Let (p_n, w_{p_n}) be a minimizing sequence of L_*, and up to a subsequence, we may assume that $p_n \to p_0 \in \Sigma$.

Define $\xi_n := w_{p_n} - G_{p_n} - \log\int_\Sigma h_2\exp(-G_{p_n} + w_{p_n})dV_g$, and substitute ξ_n into (1.10). Then ξ_n satisfies a singular Liouville type equation

$$-\Delta_g\xi_n = \rho_2 h_2 e^{\xi_n} - \rho_2 - 8\pi(\delta_{p_n} - 1) \quad \text{on } \Sigma, \quad \int_\Sigma e^{\xi_n}dV_g < C. \tag{4.2}$$

We claim that the sequence $\{\xi_n\}_{n\in\mathbb{N}}$ is uniformly bounded above. If the claim is true, consider the following equation that w_{p_n} satisfies

$$-\Delta_g w_{p_n} = \rho_2 h_2 e^{\xi_n} - \rho_2 \text{ on } \Sigma, \quad \int_\Sigma w_{p_n}dV_g = 0.$$

By standard elliptic estimates, we have that $\{w_{p_n}\}_{n\in\mathbb{N}}$ is uniformly bounded in $H^1(\Sigma)$. Moreover, by L^p-estimates, $\{w_{p_n}\}_{n\in\mathbb{N}}$ is uniformly bounded in $W^{2,p}(\Sigma)$ and, up to a subsequence, $w_{p_n} \to w_{p_0}$ in $C^1(\Sigma)$. Therefore, the pair (p_0, w_{p_0}) attains the infimum.

Now it only remains to prove the claim. We first note that, for each $n \in \mathbb{N}$, by standard elliptic estimates, $w_{p_n} \in W^{2,p}(\Sigma)$ and $\max_{x\in\Sigma} w_{p_n}(x) < \infty$, and it follows that $\max_{x\in\Sigma}\xi_n(x) < \infty$. The remaining point is to show that this upper bound can be chosen uniformly in n. To the contrary, suppose that $\max_{x\in\Sigma}\xi_n(x) := \xi_n(y_n) \to \infty$. Taking a subsequence, we may assume $y_n \to y_0$. Pick an isothermal coordinate system centered at y_0, such that $g = e^\varphi\left(dx_1^2 + dx_2^2\right)$ and $\varphi(0) = 1$. Set $r_n := \exp(-\xi_n(y_n)/2) \to 0$.

We distinguish two cases according to the relative position of p_n and y_n:

Case 1. $|p_n - y_n|/r_n \to \infty$. Define the rescaled function $\psi_n(x) := \xi_n(r_n x + y_n) + 2\log r_n$. By standard blow-up analysis, we may assume that $\psi_n \to \psi$ weakly in $H^2_{loc}(\mathbb{R}^2)$ and strongly in $C^\alpha_{loc}(\mathbb{R}^2)$, and ψ satisfies

$$-\Delta_{\mathbb{R}^2}\psi(y) = \rho_2 h_2(y_0)e^{\psi(y)} \quad \text{in} \quad \mathbb{R}^2, \quad \int_{\mathbb{R}^2} h_2(y_0)e^{\psi(y)}dy \le 1, \quad \int_{\mathbb{R}^2} e^{\psi(y)}dy < \infty.$$

If $h_2(y_0) > 0$, this contradicts the classification result [8] since $\rho_2 < 8\pi$. If $h_2(y_0) = 0$, then ψ is harmonic, and it contradicts the fact that $\int_{\mathbb{R}^2} e^{\psi(y)}dy < \infty$.

Case 2. $|p_n - y_n|/r_n \to A \in [0,\infty)$. We define the rescaled function $\psi_n(x) := \xi_n\big(|p_n - y_n|x + y_n\big) + 2\log r_n$. By the arguments in Proposition 3.5, we have that $\psi_n \to \psi$ weakly in $H^2_{loc}(\mathbb{R}^2 \setminus \{\overline{z}_0\})$ and strongly in

$C^{\alpha}_{loc}(\mathbb{R}^2 \setminus \{\overline{z}_0\})$ where $\overline{z}_0 := \lim_{n\to\infty}(p_n - y_n)/|p_n - y_n| \in \mathbb{R}^2 \setminus \{0\}$. Therefore, $\psi(x) \le 0$ and $\psi(0) = 0$. Moreover, by the elliptic regularity theory, ψ satisfies

$$-\Delta_{\mathbb{R}^2}\psi(y) = A^2\rho_2 h_2(y_0)e^{\psi(y)} - 8\pi\delta_{\tilde{x}_0} \quad \text{in} \quad \mathbb{R}^2, \quad \int\limits_{\mathbb{R}^2} h_2(y_0)e^{\psi(y)}dy \le 1, \quad \int\limits_{\mathbb{R}^2} e^{\psi(y)}dy < \infty.$$

If $A^2h_2(y_0) > 0$, then this contradicts the classification result in [43]. If $A^2h_2(y_0) = 0$, then it contradicts the maximum principle.

All two cases lead to a contradiction. Thus, the sequence ξ_n must be uniformly bounded above, and this completes the proof of the claim. □

Remark 4.2. For each $p \in \Sigma$, the functional

$$\tilde{J}_p(u) = \tfrac{1}{2}\int\limits_{\Sigma} |\nabla_g u|^2\, dV_g - \rho_2 \log\int\limits_{\Sigma} h_2\, e^{-G_p} e^u\, dV_g$$

admits a uniform lower bound, independent of p, i.e. there exists $C > 0$ such that $\tilde{J}_p(u) \ge -C$ for all $p \in \Sigma$ and all $u \in H^1(\Sigma)$ with $\int_\Sigma u = 0$. Indeed, in an isothermal coordinate centered at p, one has $e^{-G_p} \sim r^4\, \tilde{h}_p(x)$ with $\tilde{h}_p$ smooth and strictly positive; by compactness of Σ and smooth dependence on p, the weighted singular Moser–Trudinger inequality (see [4]) holds with constants uniform in p. In particular, any minimizer $w_p \in \Gamma_p$ satisfies $\tilde{J}_p(w_p) > -\infty$ uniformly in p.

By Proposition 4.1 there exists a minimizing pair (p_0, w_{p_0}) such that

$$\tilde{J}_{p_0}(w_{p_0}) - 4\pi A(p_0) - 8\pi \log h_1(p_0) = \inf_{p\in\Sigma}\ \inf_{w\in\Gamma_p} \left\{\tilde{J}_p(w) - 4\pi A(p) - 8\pi \log h_1(p)\right\} < \infty.$$

Hence $-8\pi \log h_1(p_0) < \infty$ and therefore $h_1(p_0) > 0$.

With the minimizing pair (p_0, w_{p_0}) from Proposition 4.1 in hand, we construct a sub-barrier test function $\tilde{\Phi}_\epsilon$ with $J_{\rho_2}(\tilde{\Phi}_\epsilon) < L_*$. Unlike [13], Proposition 3.6 shows that $u_n - \overline{u}_n \rightharpoonup G_{x_0} - w_{x_0}$; motivated by this decomposition, we construction a function Φ_ϵ centered at p_0 and subtract w_{p_0}.

We work in normal coordinates (r, θ) centered at p_0 and we will repeatedly use the standard expansions, uniform in θ:

$$G_{p_0}(x) = -4\log r + A(p_0) + b_1 r\cos\theta + b_2 r\sin\theta + O(r^2), \tag{4.3}$$

$$dV_g = \Big(1 - \frac{K(p_0)}{6}r^2 + O(r^3)\Big) r\,dr\,d\theta. \tag{4.4}$$

For $0 < \epsilon \ll 1$, we define $\tilde{\Phi}_\epsilon := \Phi_\epsilon - w_{p_0}$, where Φ_ϵ constructed in [13] is given as follows:

$$\Phi_\epsilon := \begin{cases} -2\log\left(r^2+\epsilon\right) + b_1 r\cos\theta + b_2 r\sin\theta + \log\epsilon, & x \in B_{\alpha\sqrt{\epsilon}}(p_0), \\ \left(G_{p_0} - \eta\beta(r,\theta)\right) - 2\log\left(\frac{\alpha^2+1}{\alpha^2}\right) - A(p_0) + \log\epsilon, & x \in B_{2\alpha\sqrt{\epsilon}}(p_0)\backslash B_{\alpha\sqrt{\epsilon}}(p_0), \\ G_{p_0} - 2\log\left(\frac{\alpha^2+1}{\alpha^2}\right) - A(p_0) + \log\epsilon, & x \in \Sigma\backslash B_{2\alpha\sqrt{\epsilon}}(p_0). \end{cases} \tag{4.5}$$

Here $\eta \in C_0^\infty(B_{2\alpha\sqrt{\epsilon}}(p_0))$ satisfies $\eta \equiv 1$ in $B_{\alpha\sqrt{\epsilon}}(p_0)$ and $|\nabla_g \eta| \le C/(\alpha\sqrt{\epsilon})$, and $\alpha = \alpha(\epsilon) \gg 1$ is chosen so that $\alpha^4\epsilon = 1/\log(-\log\epsilon) \to 0$ as $\epsilon \to 0$.

Proposition 4.3. *Suppose that* $8\pi - 2K(x) + \Delta_g \log h_1(x) - \rho_2 > 0$ *for all* $x \notin h_1^{-1}(\{0\})$*. Then, for* ϵ *small enough, the following inequality holds:*

$$J_{\rho_2}(\tilde{\Phi}_\epsilon) < \inf_{p\in\Sigma} \inf_{w_p\in\Gamma_p} \Big(\tilde{J}_p(w_p) - 4\pi A(p) - 8\pi \log h_1(p)\Big) - 8\pi\log\pi - 8\pi \tag{4.6}$$

where Γ_p *is the solution set of the singular mean field equation* (1.10)*.*

Proof. By substituting the test function $\tilde{\Phi}_\epsilon$ into the energy functional J_{ρ_2} in (1.7), we obtain that

$$\begin{aligned} J_{\rho_2}(\widetilde{\Phi}_\epsilon) =& \frac{1}{2}\int_\Sigma |\nabla_g \Phi_\epsilon|^2 dV_g + \frac{1}{2}\int_\Sigma |\nabla_g w_{p_0}|^2 dV_g - \int_\Sigma \nabla_g \Phi_\epsilon \nabla_g w_{p_0} dV_g - 8\pi \log \int_\Sigma h_1 e^{(\Phi_\epsilon - w_{p_0})} dV_g \\ & - \rho_2 \log \int_\Sigma h_2 e^{-(\Phi_\epsilon - w_{p_0})} dV_g + (8\pi - \rho_2)\int_\Sigma \Phi_\epsilon dV_g. \end{aligned} \tag{4.7}$$

Using the expansion of $G_{p_0}(r,\theta)$ in (4.3) together with the identities in [13], the Φ_ϵ–only part can be computed as

$$\begin{aligned} & \frac{1}{2}\int_\Sigma |\nabla_g \Phi_\epsilon|^2 dV_g + 8\pi \int_\Sigma \Phi_\epsilon dV_g - 8\pi \log \int_\Sigma h_1 e^{\Phi_\epsilon - w_{p_0}} dV_g \\ =& -8\pi - 8\pi\log\pi - 4\pi A(p_0) - 8\pi \log(h_1 e^{-w_{p_0}})(p_0) + 16\pi^2 \Big(1 - \frac{K(p_0)}{4\pi} + \frac{b_1^2 + b_2^2}{8\pi} \\ & + \frac{\Delta_g (h_1 e^{-w_{p_0}})(p_0)}{8\pi h_1 e^{-w_{p_0}}(p_0)} + \frac{(k_1 b_1 + k_2 b_2)}{4\pi h_1 e^{-w_{p_0}}(p_0)}\Big) \cdot \epsilon(-\log\epsilon) + o(\epsilon(-\log\epsilon)), \end{aligned} \tag{4.8}$$

where $(k_1, k_2) := \nabla_g (h_1 e^{-w_{p_0}})(p_0)$, and

$$\begin{aligned} \int_\Sigma \Phi_\epsilon dV_g =& \log\epsilon - 2\pi\alpha^2\epsilon \log\big(\frac{\alpha^2+1}{\alpha^2}\big) - 2\pi\epsilon\log\big(\alpha^2+1\big) - A(p_0) - 2\log\big(\frac{\alpha^2+1}{\alpha^2}\big)\big(1 - |B_{\alpha\sqrt{\epsilon}}|\big) \\ & + O\big(\alpha^4\epsilon^2 \log(\alpha^2\epsilon)\big). \end{aligned} \tag{4.9}$$

Moreover, we have

$$\int_\Sigma \nabla_g \Phi_\epsilon \nabla_g w_{p_0} dV_g = \int_\Sigma \nabla_g G_{p_0} \nabla_g w_{p_0} dV_g + 2\rho_2 \pi\epsilon \log(\alpha^2+1) + O(\epsilon), \tag{4.10}$$

$$\log \int_\Sigma h_2 e^{-(\Phi_\epsilon - w_{p_0})} dV_g = -\log\epsilon + 2\log(\frac{\alpha^2+1}{\alpha^2}) + A(p_0) + \log\int_\Sigma h_2 e^{-G_{p_0} + w_{p_0}} dV_g + O(\alpha^4\epsilon^3). \tag{4.11}$$

Substituting (4.8)–(4.11) into (4.7) and using $\int_\Sigma \nabla_g G_{p_0} \nabla_g w_{p_0} dV_g = 8\pi w_{p_0}(p_0)$, we obtain

$$\begin{aligned} J_{\rho_2}(\tilde{\Phi}_\epsilon) =& \Big(\tilde{J}_{p_0}(w_{p_0}) - 4\pi A(p_0) - 8\pi\log h_1(p_0)\Big) - 8\pi\log\pi - 8\pi - 16\pi^2\Big(1 - \frac{1}{4\pi}K(p_0) \\ & + \frac{b_1^2+b_2^2}{8\pi} + \frac{\Delta_g(h_1 e^{-w_{p_0}})(p_0)}{8\pi h_1 e^{-w_{p_0}}(p_0)} + \frac{k_1 b_1 + k_2 b_2}{4\pi h_1 e^{-w_{p_0}}(p_0)}\Big) \cdot \epsilon\big(-\log\epsilon\big) + o\big(\epsilon(-\log\epsilon)\big). \end{aligned} \tag{4.12}$$

We now analyze the coefficient of $\epsilon(-\log\epsilon)$ in (4.12). Using the fact $\Delta_g w_{p_0}(p_0) = \rho_2$ (from (1.10)), we deduce that

$$
\begin{aligned}
&-16\pi^2\Big(1-\frac{1}{4\pi}K(p_0)+\frac{b_1^2+b_2^2}{8\pi}+\frac{\Delta_g(h_1e^{-w_{p_0}})(p_0)}{8\pi h_1e^{-w_{p_0}}(p_0)}+\frac{k_1b_1+k_2b_2}{4\pi h_1e^{-w_{p_0}}(p_0)}\Big)\\
&=-2\pi\Big(8\pi-2K(x)-\rho_2+\sum_{i=1}^{2}\big(b_1+\frac{k_1}{h_1e^{-w_{p_0}}(x)}\big)^2+\frac{\Delta_g h_1}{h_1}-\frac{|\nabla_g h_1|^2}{h_1^2}\Big)|_{x=p_0} \qquad (4.13)\\
&=-2\pi\Big(8\pi-2K(p_0)+\Delta_g\log h_1(p_0)-\rho_2+\sum_{i=1}^{2}\big(b_i+\frac{k_i}{h_1e^{-w_{p_0}}(p_0)}\big)^2\Big)<0,
\end{aligned}
$$

where the strict negativity follows from the assumption $8\pi-2K(x)+\Delta_g\log h_1(x)-\rho_2>0$ for all $x\notin h_1^{-1}(\{0\})$ since $h_1(p_0)>0$ by Remark 4.2.

Therefore, substituting (4.13) into (4.12) and using Proposition 4.1, we conclude that

$$
\begin{aligned}
J_{\rho_2}(\tilde{\Phi}_\epsilon)&<\Big(\tilde{J}_{p_0}(w_{p_0})-4\pi A(p_0)-8\pi\log h_1(p_0)\Big)-8\pi\log\pi-8\pi\\
&=\inf_{p\in\Sigma}\inf_{w_p\in\Gamma_p}\big\{\tilde{J}_p(w_p)-4\pi A(p)-8\pi\log h_1(p)\big\}-8\pi\log\pi-8\pi.
\end{aligned}
$$

This completes the proof of Proposition 4.3. □

4.2. Convergence to a stationary solution

In this subsection, we are now in position to complete the proof of our main theorem. In particular, we shall prove the existence of a solution to (1.1), provided the initial data u_0 is chosen suitably.

Proof of Theorem 1.2. Let $\tilde{\Phi}_\epsilon$, as defined in Proposition 4.3, be the initial datum of the flow. Using the monotonicity of the flow together with Proposition 3.7 and Proposition 4.3, the sequence $\{u_n\}$ does not blow up. By Lemma 3.2 together with standard elliptic estimates, we conclude that $\{\|u_n\|_{L^\infty(\Sigma)}\}$ is uniformly bounded. Hence, up to a subsequence, $u_n\to u_\infty$ weakly in $H^2(\Sigma)$ and strongly in $C^\alpha(\Sigma)$ as $n\to\infty$. In particular, the limit u_∞ satisfies the mean-field type equation (1.1).

Now it remains to prove the convergence of the flow. Using the estimates in Section 2, we first prove the boundedness of $\{\|u(t)\|_{C^{2,\alpha}(\Sigma)}\}$ Then, by a standard argument in parabolic theory, we show that $u(t)\to u_\infty$ in $L^2(\Sigma)$ sense. Finally, applying the Arzela-Ascoli theorem yields that $u(t)\to u_\infty$ in $C^2(\Sigma)$. Thus, the proof is completed once we establish the boundedness of $\{\|u(t)\|_{C^{2,\alpha}(\Sigma)}\}$ (Step 1) and the convergence $L^2(\Sigma)$ (Step 2).

Step 1. To the contrary, we suppose that there exists a sequence $t_n\to\infty$ such that $\|u(t_n)\|_{C^{2,\alpha}(\Sigma)}\to\infty$. For a fixed $T>0$, by (3.1), we can choose a sequence $s_n\to\infty$ such that

$$
t_n-T<s_n<t_n, \qquad \text{and} \qquad \lim_{n\to\infty}\int_\Sigma\left|\frac{\partial u(s_n)}{\partial t}\right|^2 e^{u(s_n)}dV_g=0.
$$

Since $J_{\rho_2}(u_0)$ is less than the lower bound in Proposition 3.7, the sequence $u_n:=u(s_n)$ does not blow up. Applying the results in Section 3 to $u(s_n)$ (see Lemma 3.2 (2)), we obtain

$$
u_1^n=u(s_n)-\log\int_\Sigma h_1e^{u(s_n)}dV_g,\quad u_2^n=-u_n-\log\int_\Sigma h_2e^{-u(s_n)}dV_g\le C \quad \text{on} \quad \Sigma,
$$

for all $n\in\mathbb{N}$. Moreover, since $\int_\Sigma h_1e^{u(t)}dV_g\le C$ for all $t\ge0$ (see Lemma 2.3), applying the L^p-estimate for elliptic equations to (3.4), we obtain the boundedness of $\{\|u(s_n)\|_{H^2(\Sigma)}\}$.

By Proposition 2.5, we deduce that $\|u(t)\|_{H^2(\Sigma)} \leq C$ for all $t \in [s_n, s_n + 2T)$ and $n \in \mathbb{N}$. In addition, by the arguments in the proof of Proposition 2.6, we have

$$\|u(x,t)\|_{C^{\alpha,\alpha/2}(\Sigma\times[s_n,s_n+2T))} \leq C,$$

uniformly in $n \in \mathbb{N}$. Then, by the Schauder estimates for parabolic equations, we have

$$\|u(x,t)\|_{C^{2+\alpha,1+\alpha/2}(\Sigma\times[s_n,s_n+2T))} \leq C,$$

and it contradicts the assumption for t_n. This completes the proof of Step 1.

Step 2. First, we observe that the energy functional $J_{\rho_2} : H^1(\Sigma) \to \mathbb{R}$ is analytic and $J'_{\rho_2}(H^2(\Sigma)) \subset L^2(\Sigma)$. Moreover, for any critical point $u_\infty \in C^\infty(\Sigma)$ of J_{ρ_2}, the second derivative $J''_{\rho_2}(u_\infty) : H^1(\Sigma) \to H^{-1}(\Sigma)$ is a Fredholm operator with index 0. By Łojasiewicz-Simon gradient inequality (see [18, Theorem 2]), there exist constants $Z \in (0,\infty)$, $\sigma \in (0,1]$ and an exponent $\theta \in [\frac{1}{2}, 1)$, all of which are fixed by the structure of J_{ρ_2} and the critical point u_∞, such that for all $u \in H^2(\Sigma)$ with $\|u - u_\infty\|_{H^2(\Sigma)} < \sigma$, we have

$$Z\big|J_{\rho_2}(u) - J_{\rho_2}(u_\infty)\big|^\theta \leq \|J'_{\rho_2}(u)\|_{L^2(\Sigma)}.$$

Since $u_n = u(t_n) \to u_\infty$, we can apply this inequality to the flow $u(t)$ for $t \in [t_n, T]$, where $\|u_n - u_\infty\|_{L^2(\Sigma)} \ll \sigma$ and $T := \inf\big\{t > t_n : \ \|u(t) - u_\infty\|_{L^2(\Sigma)} \geq \sigma\big\}$. Then it follows that

$$\begin{aligned} -\frac{d}{dt}\big(J_{\rho_2}(u(t)) - J_{\rho_2}(u_\infty)\big)^{1-\theta} &= -(1-\theta)\big(J_{\rho_2}(u(t)) - J_{\rho_2}(u_\infty)\big)^{-\theta}\left\|e^{u(t)/2}\frac{\partial u(t)}{\partial t}\right\|^2_{L^2(\Sigma)} \\ &\geq c(1-\theta)\left\|\frac{\partial u(t)}{\partial t}\right\|_{L^2(\Sigma)}, \end{aligned}$$

for some $c > 0$. Consequently, for $s \in (t_n, T)$, we obtain

$$\|u(s) - u(t_n)\|_{L^2(\Sigma)} \leq \int_{t_n}^{s}\left\|\frac{\partial u(t)}{\partial t}\right\|_{L^2(\Sigma)} dt \leq \frac{1}{c(1-\theta)}\big(J_{\rho_2}(u(s)) - J_{\rho_2}(u_\infty)\big)^{1-\theta}.$$

Choose n sufficiently large so that $T = \infty$ and $\int_{t_n}^{\infty}\left\|\frac{\partial u(t)}{\partial t}\right\|_{L^2(\Sigma)} dt < \infty$. This completes the proof of the convergence in $L^2(\Sigma)$. □

References

[1] L. Battaglia, A. Jevnikar, A. Malchiodi, D. Ruiz, A general existence result for the Toda system on compact surfaces, Adv. Math. 285 (2015) 937–979.
[2] H. Brezis, F. Merle, Uniform estimates and blow-up behaviour for solutions of $-\Delta u = V(x)e^u$ in two dimensions, Commun. Partial Differ. Equ. 16 (1991) 1223–1253.
[3] L. Caffarelli, Y. Yang, Vortex condensation in the Chern-Simons Higgs model: an existence theorem, Commun. Math. Phys. 168 (1995) 321–336.
[4] A. Carlotto, A. Malchiodi, Weighted barycentric sets and singular Liouville equations on compact surfaces, J. Funct. Anal. 262 (2012) 409–450.
[5] J.B. Castéras, A mean field type flow I: compactness of solutions to a perturbed mean field type equation, Calc. Var. Partial Differ. Equ. 53 (2015) 221–246.
[6] J.B. Castéras, A mean field type flow II: existence and convergence, Pac. J. Math. 276 (2015) 321–345.
[7] S. Chang, P. Yang, Prescribing Gaussian curvature on S^2, Acta Math. 159 (1987) 215–259.
[8] W.X. Chen, C.M. Li, Classification of solutions of some nonlinear elliptic equations, Duke Math. J. 63 (1991) 615–622.
[9] C. Chen, C.S. Lin, Topological degree for a mean field equation on Riemann surfaces, Commun. Pure Appl. Math. 56 (2003) 1667–1727.

[10] K. Cheng, C. Lin, Conformal metrics with prescribed nonpositive Gaussian curvature on $\mathbb{R}^2$, Calc. Var. Partial Differ. Equ. 11 (2000) 203–231.
[11] A.J. Chorin, Vorticity and Turbulence, Springer, New York, 1994.
[12] L. Cui, C. Gui, H. Yan, W. Yang, Critical prescribed Q-curvature flow on closed even-dimensional manifolds with sign-changing functions, J. Funct. Anal. 289 (2025) 111133.
[13] W. Ding, J. Jost, J. Li, G. Wang, The differential equation $\Delta u = 8\pi - 8\pi h e^u$ on a compact Riemann surface, Asian J. Math. 1 (1997) 230–248.
[14] W. Ding, J. Jost, J. Li, G. Wang, An analysis of the two-vortex case in the Chern-Simons Higgs model, Calc. Var. Partial Differ. Equ. 7 (1998) 87–97.
[15] W. Ding, J. Jost, J. Li, G. Wang, Multiplicity results for the two-vortex Chern-Simons Higgs model on the two-sphere, Comment. Math. Helv. 74 (1999) 118–142.
[16] W. Ding, J. Jost, J. Li, G. Wang, Existence results for mean field equations, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 16 (1999) 653–666.
[17] Z. Djadli, Existence result for the mean field problem on Riemann surfaces of all genuses, Commun. Contemp. Math. 10 (2008) 205–220.
[18] P. Feehan, M. Maridakis, Lojasiewicz-Simon gradient inequalities for analytic and Morse-Bott functions on Banach spaces, J. Reine Angew. Math. 765 (2020) 35–67.
[19] A. Friedman, Partial Differential Equations of Parabolic Type, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1964.
[20] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, Berlin, 2001.
[21] G. Huisken, A. Polden, Geometric evolution equations for hypersurfaces, in: Calculus of Variations and Geometric Evolution Problems, Cetraro, 1996, in: Lecture Notes in Math., vol. 1713, Springer, Berlin, 1999, pp. 45–84.
[22] A. Jevnikar, An existence result for the mean-field equation on compact surfaces in a doubly supercritical regime, Proc. R. Soc. Edinb., Sect. A 143 (2013) 1021–1045.
[23] A. Jevnikar, J.C. Wei, W. Yang, Classification of blow-up limits for the sinh-Gordon equation, Differ. Integral Equ. 31 (2018) 657–684.
[24] J. Jost, C.S. Lin, G. Wang, Analytic aspects of the Toda system: II. Bubbling behavior and existence of solutions, Commun. Pure Appl. Math. 59 (2006) 526–558.
[25] J. Jost, G. Wang, D. Ye, C. Zhou, The blow up analysis of solutions of the elliptic sinh-Gordon equations, Calc. Var. Partial Differ. Equ. 31 (2008) 263–276.
[26] G. Joyce, D. Montgomery, Negative temperature states for the two-dimensional guiding-centre plasma, J. Plasma Phys. 10 (1973) 107–121.
[27] J. Kazdan, F. Warner, Curvature functions for compact 2-manifolds, Ann. Math. 99 (1974) 14–47.
[28] Y.X. Li, Y. Wang, A weak energy identity and the length of necks for a sequence of Sacks-Uhlenbeck α-harmonic maps, Adv. Math. 225 (2010) 1134–1184.
[29] M. Li, X. Xu, A flow approach to mean field equation, Calc. Var. Partial Differ. Equ. 61 (2022) 143.
[30] J. Li, C. Zhu, The convergence of the mean field type flow at a critical case, Calc. Var. Partial Differ. Equ. 58 (2019) 60.
[31] C.S. Lin, Topological degree for mean field equations on S^2, Duke Math. J. 104 (2000) 501–536.
[32] C.S. Lin, J.C. Wei, L. Zhang, Classification of blow up limits for SU(3) singular Toda systems, Anal. PDE 8 (2015) 807–837.
[33] Y. Lin, Y. Yang, A heat flow for the mean field equation on a finite graph, Calc. Var. Partial Differ. Equ. 60 (6) (2021) 206.
[34] P.L. Lions, On Euler Equations and Statistical Physics, Scuola Normale Superiore, Pisa, 1997.
[35] Y. Luo, L. Sun, G. Wang, A flow approach to the Toda system, arXiv:2602.20707v1.
[36] A. Malchiodi, Morse theory and a scalar field equation on compact surfaces, Adv. Differ. Equ. 13 (2008) 1109–1129.
[37] A. Malchiodi, D. Ruiz, A variational analysis of the Toda system on compact surfaces, Commun. Pure Appl. Math. 66 (2013) 332–371.
[38] P.K. Newton, The N-Vortex Problem: Analytical Techniques, Springer, New York, 2001.
[39] H. Ohtsuka, T. Suzuki, Mean field equation for the equilibrium turbulence and a related functional inequality, Adv. Differ. Equ. 11 (2006) 281–304.
[40] L. Onsager, Statistical hydrodynamics, Nuovo Cimento 6 (1949) 279–287.
[41] W. Park, L. Zhang, Long time existence of a flow of elliptic systems, arXiv:2508.00551.
[42] Y. Pointin, S. Lundgren, Statistical mechanics of tow-dimensional vortices in a bounded container, Phys. Fluids 19 (1976) 459–1470.
[43] J. Prajapat, G. Tarantello, On a class of elliptic problems in R2: symmetry and uniqueness results, Proc. R. Soc. Edinb., Sect. A 131 (2001) 967–985.
[44] M. Struwe, G. Tarantello, On multivortex solutions in Chern-Simons gauge theory, Boll. Unione Mat. Ital. 1 (1998) 109–121.
[45] L. Sun, X. Zhu, Critical mean field equations for equilibrium turbulence with sign-changing prescribed functions, arXiv: 2505.16414v1.
[46] L. Sun, J. Zhu, Existence of Kazdan-Warner equation with sign-changing prescribed function, arXiv:2012.12840.
[47] L. Sun, J. Zhu, Global existence and convergence of a flow to Kazdan-Warner equation with non-negative prescribed function, Calc. Var. Partial Differ. Equ. 60 (1) (2021) 42.
[48] G. Tarantello, Multiple condensate solutions for the Chern-Simons-Higgs theory, J. Math. Phys. 37 (1996) 3769–3796.
[49] Y. Wang, Y. Yang, A mean field type flow with sign-changing prescribed function on a symmetric Riemann surface, J. Funct. Anal. 282 (2022) 109449.
[50] Y. Yang, X. Zhu, A remark on a result of Ding-Jost-Li-Wang, Proc. Am. Math. Soc. 145 (2017) 3953–3959.
[51] P. Yu, Y. Yang, Existence of the heat flow with sign-changing prescribed function, J. Math. Anal. Appl. 535 (1) (2024) 128118.

[52] M. Zhang, Y. Yang, Existence results for the mean field equation on a closed symmetric Riemann surface, J. Math. Anal. Appl. 514 (1) (2022) 126263.
[53] C. Zhou, Existence of solution for mean-field equation for the equilibrium turbulence, Nonlinear Anal. 69 (2008) 2541–2552.
[54] C. Zhou, Existence result for mean field equation of the equilibrium turbulence in the super critical case, Commun. Contemp. Math. 13 (2011) 659–673.
[55] X.B. Zhu, Another remark on a result of Ding-Jost-Li-Wang, Proc. Am. Math. Soc. 152 (2024) 639–651.