



Characterization of positive superharmonic functions in a half-space

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Received: 14 February 2025 / Accepted: 12 June 2026
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Abstract

We prove a representation formula for superharmonic functions on the half-space $\mathbb{R}_+^N := \mathbb{R}^{N-1} \times]0, +\infty[$. As a consequence, we derive some comparison principles and various positivity results.

Mathematics Subject Classification 35C15 · 35R06 · 35B45

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Communicated by L. Szekelyhidi.

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1 Introduction

Let $u \in L^1_{loc}(\mathbb{R}^N)$, $N \geq 3$, be a nonnegative superharmonic function, namely,

$$-\Delta u = \mu, \quad u \geq 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \quad (1)$$

here μ is a positive Radon measure.¹ From the Riesz representation theorem (see [1, Corollary 4.4.2]) it follows that u is given by

$$u(x) = l + \int_{\mathbb{R}^N} \Gamma^x(y) d\mu(y), \quad \text{for a.e. } x \in \mathbb{R}^N, \quad (2)$$

where $\Gamma^x(y)$ is the fundamental solution of $-\Delta$ (see (69)) and $l = \inf u$.

The assumption of boundedness from below of u is crucial for the validity of (2)². Therefore, finding sufficient conditions that ensure the positivity of a superharmonic function is an important and natural question. An answer to this problem is provided in [6], where the authors demonstrate:

Theorem 1.1 *Let $N \geq 3$ and let $u \in L^1_{loc}(\mathbb{R}^N)$ be a superharmonic function. Let $\mu = -\Delta u$ and $l \in \mathbb{R}$. Then (2) holds if and only if*

$$\liminf_{R \rightarrow +\infty} \frac{1}{R^N} \int_{R < |x-y| < 2R} |u(y) - l| dy = 0 \quad \text{for a.e. } x \in \mathbb{R}^N. \quad (\mathcal{R})$$

The ring condition $(\mathcal{R})$ also plays a role in the representation theorem related to distributional solutions of $L(u) \geq 0$, even when L is a higher-order operator, either homogeneous or non-homogeneous (see [6, 7]). Representation theory has also been studied for problems stated in non-Euclidean settings (see [4, 8], and [10]). Further generalizations to smooth solutions of problems associated with general second-order operators $L(u) \geq 0$, have been studied in [8], where the role of $(\mathcal{R})$ is played by special means modeled on the fundamental solution of the operator L .

One purpose of this paper is to study superharmonic functions on the half-space $\mathbb{R}_+^N := \{(x_1, \dots, x_N) \in \mathbb{R}^N, x_N > 0\}$ where $N \geq 2$, and related representation formulae. Additionally, we will prove some comparison principles (see [9] for some Liouville theorems). Here, we will focus on problems in the half-space $\mathbb{R}_+^N$; the more general case of a cone in $\mathbb{R}^N$ will be addressed in a forthcoming paper.

It is well known that for nonnegative superharmonic functions in the half-space $\mathbb{R}_+^N$, the following Riesz representation theorem holds. See Corollary 4.4.2 and Theorem 4.3.8 in [1].

Theorem 1.2 (Riesz representation) *Let u be a distributional superharmonic function in $\mathbb{R}_+^N$, that is $-\Delta u \geq 0$ in $\mathcal{D}'(\mathbb{R}_+^N)$. If $u \geq 0$, then there exist a positive Radon measure μ on $\mathbb{R}_+^N$ and a positive Radon measure ν on $\partial\mathbb{R}_+^N$ together with a constant $h \geq 0$, such that*

$$u(x) = hx_N + \int_{\partial\mathbb{R}_+^N} K^x(y') d\nu(y') + \int_{\mathbb{R}_+^N} G^x(y) d\mu(y) \quad \text{for a.e. } x \in \mathbb{R}_+^N. \quad (3)$$

Furthermore,

$$-\Delta u = \mu \quad \text{on } \mathbb{R}_+^N \quad \text{in the distributional sense.}$$

¹ As usual, by a positive Radon measure, we mean a regular Borel measure which is positive and finite on compact sets.

² Since we aim to represent u by means of (2), and the integral in the right-hand side is nonnegative, the function u must necessarily be bounded from below. Moreover, $l = \inf u$ must be finite for (2) to make sense.

Here K and G are the Poisson kernel and the Green function of $-\Delta$ on $\mathbb{R}_+^N$ respectively (see Appendix A for more details).

Even in the context of problems in a half-space, the question of the nonnegativity of superharmonic functions arises.

Throughout this paper, unless otherwise specified, we will assume that μ and ν are positive Radon measures on $\mathbb{R}_+^N$ and on $\partial\mathbb{R}_+^N = \mathbb{R}^{N-1}$ respectively. Our research will focus on the differential equation,

$$\begin{cases} -\Delta u = \mu & \text{on } \mathbb{R}_+^N, \\ u = \nu & \text{on } \partial\mathbb{R}_+^N, \end{cases} \quad (4)$$

and on the differential inequality

$$\begin{cases} -\Delta u \geq \mu & \text{on } \mathbb{R}_+^N, \\ u \geq \nu & \text{on } \partial\mathbb{R}_+^N. \end{cases} \quad (5)$$

As will become increasingly evident as we proceed, when studying problems in $\mathbb{R}_+^N$, the role of the ring condition $(\mathcal{R})$ for the problem (1) in the whole space $\mathbb{R}^N$, in $\mathbb{R}_+^N$ it is played by the following weighted ring condition:

there exists $h \in \mathbb{R}$ such that

$$\liminf_{R \rightarrow +\infty} \frac{1}{R^{N+2}} \int_{\{y_N > 0\} \cap \{R < |x-y|_* < 2R\}} y_N |u(y) - hy_N| dy = 0, \text{ for a.e. } x \in \mathbb{R}_+^N, \quad (\mathcal{R}^+)$$

where $|\cdot|_*$ is the cylindrical norm defined by,

$$|x|_* = |(x', x_N)|_* := \max\{|x'|, |x_N|\}. \quad (6)$$

To understand why we introduce the condition $(\mathcal{R}^+)$, we first note that for the problem (1), the solution u is not unique. Indeed, adding a constant to u , yields a different solution. This non-uniqueness also applies to problem (4), where a linear term like hx_N , $h \in \mathbb{R}$, can be added to a solution to produce another solution. This explains why the correction hy_N appears inside the integral in $(\mathcal{R}^+)$.

The presence of the weight y_N in $(\mathcal{R}^+)$ is instead motivated by the behavior of the Green function of $-\Delta$ on $\mathbb{R}_+^N$, indeed, $G^x(y) \approx C'_N x_N y_N |x - y|^{-N}$ as $|y| \rightarrow \infty$ (see Proposition A.1), so y_N is the natural weight for controlling the Green potential at infinity. Moreover, the necessity of the weight is confirmed a posteriori by Theorem (B), which shows that $(\mathcal{R}^+)$ is not only sufficient but also necessary for the validity of the representation formula.

Our first main result is the following.

Theorem (A) *Let $N \geq 2$, let μ and ν be positive Radon measures on $\mathbb{R}_+^N$ and $\partial\mathbb{R}_+^N$, respectively. Assume that $\mu \in \mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N)$ and let $u \in L^1_{loc}(\overline{\mathbb{R}_+^N})$ be a weak solution of (5). Suppose $x \in \mathbb{R}_+^N$ is a Lebesgue point for u . Assume that there exists $h \in \mathbb{R}$ such that $(\mathcal{R}^+)$ holds at the point x that is,*

$$\liminf_{R \rightarrow +\infty} \frac{1}{R^{N+2}} \int_{\{y_N > 0\} \cap \{R < |x-y|_* < 2R\}} y_N |u(y) - hy_N| dy = 0.$$

Then,

$$u(x) \geq hx_N + \int_{\partial\mathbb{R}_+^N} K^x(y') d\nu(y') + \int_{\mathbb{R}_+^N} G^x(y) d\mu(y). \quad (7)$$

Moreover, if u is a weak solution of (4), then the inequality in (7) becomes an equality, that is

$$u(x) = hx_N + \int_{\partial \mathbb{R}_+^N} K^x(y') dv(y') + \int_{\mathbb{R}_+^N} G^x(y) d\mu(y). \quad (8)$$

We observe that a special case arises when $(\mathcal{R}^+)$ holds with $h = 0$, i.e.,

$$\liminf_{R \rightarrow +\infty} \frac{1}{R^{N+2}} \int_{\{y_N > 0\} \cap \{|x-y|_* < 2R\}} y_N |u(y)| dy = 0 \text{ for a.e. } x \in \mathbb{R}_+^N. \quad (\mathcal{R}_0^+)$$

Notice that all bounded and measurable functions on $\mathbb{R}_+^N$ satisfy $(\mathcal{R}_0^+)$. Other examples of functions satisfying $(\mathcal{R}_0^+)$, will be provided in Appendix B below.

We point out that another interesting class of functions satisfying $(\mathcal{R}_0^+)$ is the set of solutions to the problem,

$$\begin{cases} \pm \Delta u \geq |u|^q, & \text{on } \mathbb{R}_+^N, \\ u = 0, & \text{on } \partial \mathbb{R}_+^N, \end{cases} \quad (9)$$

here $q > 1$. For a proof of this fact and related Liouville theorems see [9].

Our main result, Theorem (A), can also be applied to the representation of harmonic functions. Recently, in [17], integral representation formulae for harmonic functions on a half-space have been studied. It is worth comparing $(\mathcal{R}_0^+)$, with the condition given in [17] for representing a harmonic function by an integral formula.

We emphasize that for a superharmonic function u in a half-space $\mathbb{R}_+^N$, the condition $(\mathcal{R}^+)$ is sufficient for the validity of the representation formula (3), thereby ensuring its positivity, provided $h \geq 0$. Furthermore, the condition $(\mathcal{R}^+)$ is also necessary for the validity of the integral representation. Indeed, our second main result is the following theorem whose principal statement is the converse of the statement in Theorem (A).

Theorem (B) *Let $N \geq 2$, and let μ and ν be positive Radon measures on $\mathbb{R}_+^N$ and $\partial \mathbb{R}_+^N$, respectively. Let u be defined by (8) and assume that it is finite for a.e. $x \in \mathbb{R}_+^N$.*

Then $u \in L_{loc}^1(\overline{\mathbb{R}_+^N})$, $\mu \in \mathcal{M}_{loc}(\mathbb{R}_+^N, x_N)$ and u is a weak solution of (4). Moreover, u satisfies the condition $(\mathcal{R}^+)$, where the constant h appears in (8).

Furthermore,

(a)

$$h = \inf_{\mathbb{R}_+^N} \frac{u(x)}{x_N}$$

(b) *Let $\Omega \subset \mathbb{R}_+^N$ be a nonempty bounded open set. If $h = \inf_{\Omega} \frac{u(x)}{x_N}$, then $u(x) = hx_N$, with $\mu \equiv 0$ and $\nu \equiv 0$.*

(c) *If $u - hx_N$ is not identically zero, then there exists a constant $c_0 > 0$ such that*

$$u(x) \geq hx_N + c_0 \frac{x_N}{1 + |x|^N}, \quad \text{for all } x \in \mathbb{R}_+^N. \quad (10)$$

(d) *For almost every $x \in \mathbb{R}_+^N$,*

$$\lim_{R \rightarrow +\infty} \frac{1}{R^{N+2}} \int_{\{y_N > 0\} \cap B_R^*(x)} y_N |u(y) - hy_N| dy = 0.$$

This paper is organized as follows.

Section 2 is dedicated to defining weak solutions for problems (4) and (5), even when ν and μ are general local Radon measures, and to the concept of *lim-trace*. See Definition 2.5.

To illustrate the implications of this concept, we explore its application in Section 2.1, where we establish the relationship between distributional and weak solutions of (4) and (5), and certain specific properties (see Remark 2.13).

Section 3 formulates and proves the main result of this paper, namely the representation Theorem 3.1. The techniques we use to prove this representation result for (4) allow us to deal with measures that are general local Radon measures, without requiring positivity. See Theorem 3.8. We investigate the case of superharmonic but not necessarily nonnegative functions u in Theorem 3.11 and Corollary 3.14. Consequently, weak solutions of (5) that are bounded from below are nonnegative. See Theorem 3.15. This section also includes a simple characterization of a comparison principle (see Theorem 3.12). The remainder of the section is devoted to proving other related results.

Lastly, for completeness, in Section 4, we highlight another representation theorem for a superharmonic function on $\mathbb{R}_+^N$. The results contained in this section are based on general theorems established in [8].

As a final remark, we would like to point out that if u is a superharmonic function and there exists a superharmonic function g such that $u \geq -g$, then the results contained in Section 3, apply to $u + g$. We emphasize that in this case, by Riesz's decomposition theorem, we know that there exists a harmonic function H on $\mathbb{R}_+^N$, such that

$$u(x) = H(x) + \int_{\mathbb{R}_+^N} G^x(y) d\mu(y),$$

where $\mu := -\Delta u$. The harmonic function H can be obtained as a weighted limit of u on some rings. This construction is detailed in Section 4, see Theorem 4.1.

In Appendix A, we present basic and important information on the fundamental solution of $-\Delta$ in $\mathbb{R}_+^N$ and its Poisson kernel and the related Green function. This includes a few precise estimates on related potentials. Appendix B contains basic examples of functions fulfilling the condition $(\mathcal{R}_0^+)$.

Notation

$\mathbb{R}_+^N$	The half-space $\mathbb{R}^{N-1} \times]0, +\infty[$, $x = (x', x_N) \in \mathbb{R}_+^N$ with $x_N > 0$ and $x' \in \mathbb{R}^{N-1}$.
$\partial\mathbb{R}_+^N = \mathbb{R}^{N-1} \times \{0\}$	The boundary of $\mathbb{R}_+^N$ that we can identify with $\mathbb{R}^{N-1}$.
$ \cdot $	The Euclidean norm.
B'_R	The Euclidean ball in $\mathbb{R}^{N-1}$ centered at the origin and radius R .
$B_R^{e, N+2}(\xi_0)$	The Euclidean ball in $\mathbb{R}^{N+2}$ of radius R centered at $\xi_0 \in \mathbb{R}^{N+2}$.
$ x _*$	The cylindrical norm defined as $ x _* = (x', x_N) _* := \max\{ x' , x_N \}$.
$B_R^*(x)$	The cylindrical ball in $\mathbb{R}_+^N$ defined as $\{y \in \mathbb{R}_+^N : \max\{ y_N - x_N , y' - x' \} < R\}$.
$C_c(E), C_c^2(E)$	The spaces of continuous and C^2 functions respectively whose support is compact and contained in E .
$\mathbf{D}_0(\Omega)$	Test functions space used along the paper defined as
	$\mathbf{D}_0(\Omega) := \left\{ \varphi _\Omega : \varphi \in C_c^2(\mathbb{R}^N), \varphi = 0 \text{ on } \partial\Omega \right\}.$
$\mathcal{D}'(\mathbb{R}_+^N)$	The space of the distributions on $\mathbb{R}_+^N$.
σ_N	The measure of the unit sphere in $\mathbb{R}^N$, $\sigma_N := \frac{2\pi^{N/2}}{\Gamma(N/2)}.$

C_N, C'_N	The constants defined as $C_N^{-1} := \sigma_N \max\{N - 2, 1\}$, and $C'_N := 2/\sigma_N$.
Γ^x	The fundamental solution of $-\Delta$ at x , see (69).
G^x	The Green function of $-\Delta$ at x on $\mathbb{R}_+^N$, see (70).
K^x	The Poisson kernel, see (71).
$\mathcal{M}_{loc}(X)$	The space of local Radon measures on the σ -compact and locally compact Hausdorff space X , that is the space of linear continuous functionals on $\mathcal{C}_c(X)$. Moreover if $\mu \in \mathcal{M}_{loc}(X)$, then $\mu = \mu^+ - \mu^-$, with μ^+ and μ^- positive Radon measures and set $ \mu := \mu^+ + \mu^-$.
$\mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N)$	The space of local Radon measures $\mu \in \mathcal{M}_{loc}(\mathbb{R}_+^N)$ such that $\int_{\Omega} x_N d \mu < \infty$ for any bounded $\Omega \subset \mathbb{R}_+^N$.
$L^1_{loc}(\overline{\mathbb{R}_+^N})$	The space of locally integrable functions up to the boundary, that is the space of functions $u \in L^1_{loc}(\mathbb{R}_+^N)$ such that $u \in L^1(\Omega)$ for any open bounded set $\Omega \subset \mathbb{R}_+^N$.
$W^{1,p}_{loc}(\overline{\mathbb{R}_+^N})$	The space of functions $u \in W^{1,p}_{loc}(\mathbb{R}_+^N)$ such that $u \in W^{1,p}(\Omega)$ for any open bounded set $\Omega \subset \mathbb{R}_+^N$.
$BV_{loc}(\overline{\mathbb{R}_+^N})$	The space of functions $u \in BV_{loc}(\mathbb{R}_+^N)$ such that $u \in BV(\Omega)$ for any open bounded set $\Omega \subset \mathbb{R}_+^N$.
$Tr(u)$	The lim-trace of u , see Definition 2.5.
c	We will use the symbol c to denote a generic positive constant, the value of which may change from line to line, and is not dependent on the solutions of the problem under study and may depend on certain non-essential parameters.

2 Some qualitative properties of weak and distributional solutions

In this section, we present the formal definition of weak solutions for the problem (4) and discuss some of its properties.

The definition of weak solution of (4) can be formulated even if μ and ν are more general than positive Radon measures, extending to local Radon measures. For a σ -compact and locally compact Hausdorff space X , a local Radon measure μ is a linear continuous functional on $\mathcal{C}_c(X)$. The space of all the local Radon measures on X will be denoted by $\mathcal{M}_{loc}(X)$.

Clearly, $\mathcal{M}_{loc}(X)$ contains any positive Radon measure as well as any signed Radon measure on X . Furthermore, if ν and λ are two positive Radon measures on X , then $\nu - \lambda$ defines a linear functional on $\mathcal{C}_c(X)$, that is, an element of $\mathcal{M}_{loc}(X)$ (note that in general $\nu - \lambda$ is not a signed Radon measure). On the other hand, for any $\mu \in \mathcal{M}_{loc}(X)$, there exist two positive Radon measures, ν and λ , such that $\mu = \nu - \lambda$. As usual, we rewrite μ in terms of the positive measures μ^+ and μ^- , as in the classical Jordan decomposition $\mu = \mu^+ - \mu^-$. Therefore, for the functional $\mu \in \mathcal{M}_{loc}(X)$ we have the following:

$$\mu(\phi) = \int_X \phi d\mu^+ - \int_X \phi d\mu^- =: \int_X \phi d\mu \quad \forall \phi \in \mathcal{C}_c(X).$$

As usual, we set $|\mu| := \mu^+ + \mu^-$, which defines a classical positive Radon measure. The interested reader may refer to the relationships between classical measures, local Radon measures and their representation in [5] and [12].

In what follows we will need the following spaces:

$$\mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N) := \{\mu : \mu \text{ is a local Radon measure on } \mathbb{R}_+^N \text{ such that} \\ \int_{\Omega} x_N d|\mu| < \infty \text{ for any bounded } \Omega \subset \mathbb{R}_+^N\},$$

and

$$\mathbf{D}_0 = \mathbf{D}_0(\mathbb{R}_+^N) := \left\{ \phi|_{\mathbb{R}_+^N} : \phi \in C_c^2(\mathbb{R}^N), \varphi(x', 0) = 0 \right\}.$$

Next we notice that if, instead of μ and ν being measures, they are continuous functions and u is a smooth solution of (4), then by multiplying the differential equation in (4) by $\varphi \in \mathbf{D}_0$ and integrating by parts, we obtain the identity:

$$\int_{\mathbb{R}_+^N} \mu(x) \varphi(x) dx + \int_{\partial \mathbb{R}_+^N} \nu(x') \varphi_N(x', 0) dx' = \int_{\mathbb{R}_+^N} u(-\Delta \varphi). \quad (11)$$

Here, and in the following, we denote by φ_N the partial derivative of the function φ with respect to x_N , i.e. $\varphi_N(x) = \partial_N \varphi(x) = \frac{\partial}{\partial x_N} \varphi(x)$.

Similarly, if u solves (5), then by multiplying the differential inequality in (5) by a non-negative $\varphi \in \mathbf{D}_0$, we obtain

$$\int_{\mathbb{R}_+^N} \mu(x) \varphi(x) dx + \int_{\partial \mathbb{R}_+^N} \nu(x') \varphi_N(x', 0) dx' \leq \int_{\mathbb{R}_+^N} u(-\Delta \varphi). \quad (12)$$

With these preliminaries, we can now provide the formal definition of weak solutions and discuss their properties. The relations (11) and (12) justify the following definition of weak solution.

Definition 2.1 Let $\mu \in \mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N)$ and $\nu \in \mathcal{M}_{loc}(\partial \mathbb{R}_+^N)$. A function u is a *weak solution* of (4) [resp. (5)] if $u \in L^1_{loc}(\overline{\mathbb{R}_+^N})$ and for every test function $\varphi \in \mathbf{D}_0$ [resp. $\varphi \in \mathbf{D}_0$ and $\varphi \geq 0$], there holds,

$$\int_{\mathbb{R}_+^N} \varphi(x) d\mu(x) + \int_{\partial \mathbb{R}_+^N} \varphi_N(x', 0) d\nu(x') = [\leq] \int_{\mathbb{R}_+^N} u(-\Delta \varphi). \quad (13)$$

Remark 2.2 Observe that if, in the definition of weak solution, we replace μ with a general local Radon measure, $\mu \in \mathcal{M}_{loc}(\mathbb{R}_+^N)$, then it is not guaranteed that the terms in (13) are all finite. On the other hand, if $\mu \in \mathcal{M}_{loc}(\mathbb{R}_+^N)$ is a positive Radon measure, it is straightforward to recognize that requiring the term $\int_{\mathbb{R}_+^N} \varphi(x) d\mu(x)$ to be finite for any $\varphi \in \mathbf{D}_0$ is equivalent to the fact that the measure μ satisfies $\int_{\Omega} x_N d\mu(x) < \infty$ on every bounded open set $\Omega \subset \mathbb{R}_+^N$, i.e., that $\mu \in \mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N)$.

Remark 2.3 Needless to say, if u is a weak solution of (4) then

$$-\Delta u = \mu \quad \text{in } \mathcal{D}'(\mathbb{R}_+^N),$$

i.e., u is a distributional solution of $-\Delta u = \mu$. This follows by choosing $\varphi \in C_c^\infty(\mathbb{R}_+^N) \subset \mathbf{D}_0$ in (13). An analogous remark is valid for a weak solution of (5).

Remark 2.4 We point out that if u is a weak solution of

$$\begin{cases} -\Delta u = \mu_1 & \text{on } \mathbb{R}_+^N, \\ u = v_1 & \text{on } \partial \mathbb{R}_+^N, \end{cases}$$

and of

$$\begin{cases} -\Delta u = \mu_2 & \text{on } \mathbb{R}_+^N, \\ u = v_2 & \text{on } \partial\mathbb{R}_+^N, \end{cases}$$

where $\mu_1, \mu_2 \in \mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N)$ and $v_1, v_2 \in \mathcal{M}_{loc}(\partial\mathbb{R}_+^N)$, then $\mu_1 = \mu_2$ and $v_1 = v_2$. Indeed, from the definition of weak solution, we have that for any $\varphi \in \mathbf{D}_0$

$$\begin{aligned} \int_{\mathbb{R}_+^N} \varphi(x) d\mu_1(x) + \int_{\partial\mathbb{R}_+^N} \varphi_N(x', 0) dv_1(x') &= \int_{\mathbb{R}_+^N} u(-\Delta\varphi) \\ &= \int_{\mathbb{R}_+^N} \varphi(x) d\mu_2(x) + \int_{\partial\mathbb{R}_+^N} \varphi_N(x', 0) dv_2(x'). \end{aligned} \quad (14)$$

From Remark 2.3 we deduce that $\mu_1 = \mu_2$, and by (14) we get,

$$\int_{\partial\mathbb{R}_+^N} \varphi_N(x', 0) dv_1(x') = \int_{\partial\mathbb{R}_+^N} \varphi_N(x', 0) dv_2(x'), \quad \forall \varphi \in \mathbf{D}_0.$$

Next, let $\psi \in C_c^\infty(\mathbb{R}^{N-1})$. Let $\phi \in C_c^2(\mathbb{R})$ be a standard cut-off function, that is

$$0 \leq \phi \leq 1, \quad \phi(t) = 1 \text{ for } 0 \leq t \leq 1, \quad \phi(t) = 0 \text{ for } t \geq 2, \quad (15)$$

and set, $\varphi(x', x_N) := x_N \psi(x') \phi(x_N)$. Since $\partial_N \varphi(x', 0) = \psi(x') \phi(0) = \psi(x')$, we get,

$$\int_{\partial\mathbb{R}_+^N} \psi(x') dv_1(x') = \int_{\partial\mathbb{R}_+^N} \psi(x') dv_2(x').$$

Hence $v_1 = v_2$.

2.1 Weak solutions and lim-trace

In this section, we introduce the definition of lim-trace for functions belonging to $W_{loc}^{1,p}(\mathbb{R}_+^N)$ and we study its properties in connection to the weak solutions of (4) and (5).

In this respect, we observe that if u is a weak solution of (4) or (5) with μ and ν local Radon measures, then for any bounded open set $\Omega \subset \overline{\Omega} \subset \mathbb{R}_+^N$, we have $u \in W^{1,p}(\Omega)$ for any $1 \leq p < N/(N-1)$ (see e.g. [11]). This means that u has a Sobolev trace $u_{\partial\Omega}$ on $\partial\Omega$. Therefore, if u is a weak solution of (4) or (5), then $u \in W_{loc}^{1,p}(\mathbb{R}_+^N)$ and, for any $\epsilon > 0$, u has a Sobolev trace $u(\cdot, \epsilon)$ on each hyperplane with equation $x_N = \epsilon$. Mimicking the argument in [2] for harmonic functions, we define the trace of a given function on $\partial\mathbb{R}_+^N$ as the limit of the Sobolev trace $u(\cdot, \epsilon)$ as $\epsilon \rightarrow 0^+$. More precisely, we have the following.

Definition 2.5 Let $u \in W_{loc}^{1,p}(\mathbb{R}_+^N)$ for some $p > 1$ and let ν be a local Radon measure on $\partial\mathbb{R}_+^N$. We say that ν is the *lim-trace* of u on $\partial\mathbb{R}_+^N$ and we write $\nu = Tr(u)$, if for any $\psi \in C_c^2(\partial\mathbb{R}_+^N)$, we have

$$\lim_{\epsilon \rightarrow 0^+} \int_{\partial\mathbb{R}_+^N} \psi(x') u(x', \epsilon) dx' = \int_{\partial\mathbb{R}_+^N} \psi(x') d\nu(x'),$$

where $u(\cdot, \epsilon)$ is the Sobolev trace of u on the hyperplane with equation $x_N = \epsilon$.

Definition 2.5 can similarly be extended to a function $u \in BV_{loc}(\mathbb{R}_+^N)$. If $u \in C(\overline{\mathbb{R}_+^N})$ then $Tr(u) = u(x', 0)$. If $u \in W_{loc}^{1,p}(\overline{\mathbb{R}_+^N})$ then $Tr(u)$ is the Sobolev trace of u . In a more general setting, if $u \in BV_{loc}(\overline{\mathbb{R}_+^N})$, then $Tr(u)$ is the trace of u in the BV sense (see [13]).

We point out that if u and v have the lim-traces, then $u + v$ also has the lim-trace and $Tr(u + v) = Tr(u) + Tr(v)$. Similarly if $a \in \mathbb{R}$, then $Tr(au) = aTr(u)$.

The following results are similar to the ones proved in [16] in another context.

Theorem 2.6 Let $\mu \in \mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N)$ and $v \in \mathcal{M}_{loc}(\partial\mathbb{R}_+^N)$. Let u be a weak solution of

$$\begin{cases} \Delta u = \mu \text{ on } \mathbb{R}_+^N, \\ u = v \quad \text{on } \partial\mathbb{R}_+^N. \end{cases} \quad (16)$$

Then u has the lim-trace on $\partial\mathbb{R}_+^N$, and $Tr(u) = v$.

Another important result for functions that possess the lim-trace is for nonnegative superharmonic distributions considered in Corollary 3.2 below.

We need the following Lemma.

Lemma 2.7 Let $\mu \in \mathcal{M}_{loc}(\mathbb{R}_+^N)$ and let u be a distributional solution of

$$\Delta u = \mu \text{ in } \mathcal{D}'(\mathbb{R}_+^N).$$

Then, for any Lipschitz bounded open set $\Omega \subset \overline{\Omega} \subset \mathbb{R}_+^N$, it follows that $u \in W^{1,p}(\Omega)$ for $1 \leq p < N/(N-1)$. Moreover, for any $\varphi \in \mathbf{D}_0(\Omega)$ there holds,

$$\int_{\partial\Omega} u_{\partial\Omega}(\nabla\varphi \cdot n) = \int_{\Omega} u \Delta\varphi - \int_{\Omega} \varphi d\mu. \quad (17)$$

Proof Since $u \in W^{1,p}(\Omega)$, for any $\varphi \in \mathcal{C}_c^2(\mathbb{R}^N)$, we have

$$\int_{\Omega} u \Delta\varphi = - \int_{\Omega} (\nabla\varphi \cdot \nabla u) + \int_{\partial\Omega} u_{\partial\Omega}(\nabla\varphi \cdot n), \quad (18)$$

(see e.g. [15]). Next, let (m_η) be an approximation of the identity with $\text{supp}(m_\eta) \subset B_\eta(0)$, and let us choose η sufficiently small such that a compact η -neighborhood of Ω is still contained in $\mathbb{R}_+^N$, $\Omega + B_\eta \subset \overline{\Omega} + B_\eta \subset \mathbb{R}_+^N$. Since $u_\eta := m_\eta * u$ is a classical solution of $\Delta u_\eta = \mu_\eta$, where $\mu_\eta := m_\eta * \mu$, integrating by parts we obtain

$$\int_{\Omega} \varphi \mu_\eta = \int_{\Omega} \varphi \Delta u_\eta = \int_{\partial\Omega} \varphi(\nabla u_\eta \cdot n) - \int_{\Omega} (\nabla\varphi \cdot \nabla u_\eta).$$

Now, if $\varphi \in \mathbf{D}_0(\Omega)$ the first integral in the right hand side of the above identity vanishes. Letting $\eta \rightarrow 0$ in the above identity, it follows that

$$\int_{\Omega} \varphi d\mu = - \int_{\Omega} (\nabla\varphi \cdot \nabla u),$$

which, combined with (18) yields the claim. $\square$

Proof of Theorem 2.6 Fix $\psi \in \mathcal{C}_c^2(\partial\mathbb{R}_+^N)$. For $\epsilon > 0$, define $\varphi_\epsilon(x', x_N) := (x_N - \epsilon)\psi(x')\phi(x_N)$ where $\phi \in \mathcal{C}_c^2(\mathbb{R})$ is a standard cut-off function as in (15). Let $R > 0$ be sufficiently large so that $\text{supp}(\psi) \subset B'_R := \{|x'| < R\}$. For $0 < \epsilon < 1$, let Ω_ϵ be the cylinder $\Omega_\epsilon := B'_R \times (\epsilon, 2)$. Since $\varphi_\epsilon \in \mathbf{D}_0(\Omega_\epsilon)$, applying the integration by parts formula (17) gives:

$$\int_{\Omega_\epsilon} \varphi_\epsilon d\mu - \int_{\Omega_\epsilon} u \Delta\varphi_\epsilon = - \int_{\partial\Omega_\epsilon} u_{\partial\Omega_\epsilon}(\nabla\varphi_\epsilon \cdot n) = \int_{\partial\mathbb{R}_+^N} u(x', \epsilon)\psi(x')dx',$$

where we have used the fact that $\nabla \varphi_\epsilon = 0$ on $\partial \Omega_\epsilon \cap \{x_N > \epsilon\}$ and $(\nabla \varphi_\epsilon \cdot n) = -\partial_N \varphi_\epsilon = -\psi$ on $\partial \Omega_\epsilon \cap \{x_N = \epsilon\}$. Next, observe that as $\epsilon \rightarrow 0^+$, $\varphi_\epsilon \rightarrow \varphi_0$ uniformly, where $\varphi_0(x', x_N) = x_N \psi(x') \phi(x_N)$. Since $\mu \in \mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N)$, we have:

$$\int_{\Omega_\epsilon} \varphi_\epsilon d\mu \rightarrow \int_{\Omega_0} \varphi_0 d\mu \quad \text{as } \epsilon \rightarrow 0^+.$$

On the other hand, since again $\Delta \varphi_\epsilon \rightarrow \Delta \varphi_0$ uniformly, and $u \in L^1(\Omega_0)$, we obtain

$$\int_{\Omega_\epsilon} u \Delta \varphi_\epsilon \rightarrow \int_{\Omega_0} u \Delta \varphi_0 \quad \text{as } \epsilon \rightarrow 0^+.$$

Noting that $\varphi_0(x', x_N) := x_N \psi(x') \phi(x_N)$ we can conclude that there exists the limit in the definition of lim-trace, that is,

$$\begin{aligned} T(\psi) &:= \lim_{\epsilon \rightarrow 0^+} \int_{\partial \mathbb{R}_+^N} u(x', \epsilon) \psi(x') dx' = \lim_{\epsilon \rightarrow 0^+} \left(\int_{\Omega_\epsilon} \varphi_\epsilon d\mu - \int_{\Omega_\epsilon} u \Delta \varphi_\epsilon \right) \\ &= \int_{\mathbb{R}_+^N} \varphi_0 d\mu - \int_{\mathbb{R}_+^N} u \Delta \varphi_0. \end{aligned} \quad (19)$$

To complete the proof, it remains to show that T does not depend on the choice of ϕ (which implies that T is a distribution on $\mathbb{R}^{N-1}$) and can be represented by a local Radon measure. To this end, since u is a weak solution of equation (16), we utilize $\varphi_0(x', x_N) := x_N \psi(x') \phi(x_N)$ in the definition of weak solution. We deduce that

$$T(\psi) = \int_{\mathbb{R}_+^N} \varphi_0 d\mu - \int_{\mathbb{R}_+^N} u \Delta \varphi_0 = \int_{\partial \mathbb{R}_+^N} \partial_N \varphi_0 dv = \int_{\partial \mathbb{R}_+^N} \psi dv,$$

which concludes the proof. $\square$

The following can be seen as a sort of converse of Theorem 2.6.

Theorem 2.8 Let $\mu \in \mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N)$ and let $u \in L^1_{loc}(\overline{\mathbb{R}_+^N})$ be such that,

$$\Delta u = \mu \text{ in } \mathcal{D}'(\mathbb{R}_+^N).$$

If u has the lim-trace $v = \text{Tr}(u)$ with $v \in \mathcal{M}_{loc}(\partial \mathbb{R}_+^N)$, then u is a weak solution of (16).

Remark 2.9 A further connection between superharmonic functions on $\mathbb{R}_+^N$ with a prescribed behavior on the boundary $\partial \mathbb{R}_+^N$ and weak solutions can be found in Corollary 3.14 below.

Proof of Theorem 2.8 We need to demonstrate that u is a weak solution of (16) with $v = \text{Tr}(u)$. In other words, we must prove that for any $\varphi \in \mathbf{D}_0$ the following holds:

$$\int_{\mathbb{R}_+^N} \varphi(x) d\mu(x) - \int_{\mathbb{R}_+^N} u \Delta \varphi = \int_{\partial \mathbb{R}_+^N} \varphi_N(x', 0) dv(x').$$

Let ϵ be such that $0 < \epsilon < 1$ and define $\tilde{\varphi}_\epsilon := \varphi(x', x_N - \epsilon)$. Let $\Omega_\epsilon := B'_R \times (\epsilon, R+1)$ with R large enough such that, $\text{supp}(\varphi) \subset B'_R \times (-R, R)$. Observing that $\tilde{\varphi}_\epsilon = 0$ on $\partial \Omega_\epsilon$, an application of (17) yields,

$$\begin{aligned} \int_{\Omega_\epsilon} \tilde{\varphi}_\epsilon d\mu - \int_{\Omega_\epsilon} u \Delta \tilde{\varphi}_\epsilon &= - \int_{\partial \Omega_\epsilon} u \partial_{\Omega_\epsilon} (\nabla \tilde{\varphi}_\epsilon \cdot n) = \int_{x' \in \mathbb{R}^{N-1}, x_N = \epsilon} u(x', \epsilon) \partial_N \tilde{\varphi}_\epsilon dx' \\ &= \int_{\partial \mathbb{R}_+^N} u(x', \epsilon) \partial_N \varphi(x', 0) dx'. \end{aligned}$$

Taking the limit as $\epsilon \rightarrow 0$ we obtain the desired result. $\square$

We conclude this subsection with some generalizations of Theorem 2.6 and 2.8.

Lemma 2.10 Let $\mu \in \mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N)$ and $v_1 \in \mathcal{M}_{loc}(\partial\mathbb{R}_+^N)$. Suppose u is a weak solution of

$$\begin{cases} \Delta u = \mu & \text{on } \mathbb{R}_+^N, \\ u \leq v_1 & \text{on } \partial\mathbb{R}_+^N. \end{cases} \quad (20)$$

Then, u has the lim-trace on $\partial\mathbb{R}_+^N$. Furthermore, $Tr(u) \leq v_1$, and u is a weak solution of (16) with $v = Tr(u)$.

Proof Following a similar approach to the proof of Theorem 2.6, and using the same notations, consider a function $\psi \in \mathcal{C}_c^2(\mathbb{R}^{N-1})$. The functional $T(\psi)$, defined in (19) is a distribution on $\mathbb{R}^{N-1}$. Given that u is a weak solution of (20), using φ_0 in the definition of the solution we obtain:

$$\int_{\partial\mathbb{R}_+^N} \psi dv_1 \geq \int_{\mathbb{R}_+^N} \varphi_0 d\mu - \int_{\mathbb{R}_+^N} u \Delta \varphi_0 = T(\psi).$$

This implies that the expression

$$\psi \rightarrow \int_{\partial\mathbb{R}_+^N} \psi dv_1 - T(\psi)$$

is a nonnegative distribution on $\mathbb{R}^{N-1}$. Consequently, there exists a positive Radon measure v_2 on $\mathbb{R}^{N-1}$, that represents this distribution. In other words, $v_1 - T = v_2$, indicating that T is a local Radon measure. This establishes that u has the lim-trace, and $Tr(u) \leq v_1$.

Moreover, since

$$\Delta u = \mu \text{ in } \mathcal{D}'(\mathbb{R}_+^N),$$

applying Theorem 2.8, leads to the conclusion that u is a weak solution of (16) with $v = Tr(u)$. $\square$

Theorem 2.11 Let $\mu_1 \in \mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N)$ and $v_1 \in \mathcal{M}_{loc}(\partial\mathbb{R}_+^N)$. Suppose u is a weak solution of

$$\begin{cases} \Delta u \geq \mu_1 & \text{on } \mathbb{R}_+^N, \\ u \leq v_1 & \text{on } \partial\mathbb{R}_+^N. \end{cases} \quad (21)$$

1. Assume that for any $R > 0$, there exist $M > 0$ and $\delta > 0$ such that $u \leq M$ on the cylinder $\{|x'| < R\} \times (0, \delta)$. Then u has the lim-trace on $\partial\mathbb{R}_+^N$.

2. Assume that for any $R > 0$, there exist $\tilde{M} > 0$ and $\delta > 0$ such that $\int_{|x'| < R} |u(x', \epsilon)| dx' < \tilde{M}$ for any $0 < \epsilon < \delta$. Then u has the lim-trace on $\partial\mathbb{R}_+^N$.

3. Assume that u has the lim-trace on $\partial\mathbb{R}_+^N$. Then $Tr(u) \leq v_1$, and there exist a positive Radon measure $\lambda \in \mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N)$ such that u is a solution of (16) with $\mu = \mu_1 + \lambda$ and $v = Tr(u)$.

Proof Since u satisfies

$$\Delta u \geq \mu_1 \text{ in } \mathcal{D}'(\mathbb{R}_+^N),$$

there exists a positive Radon measure $\lambda \in \mathcal{M}^+(\mathbb{R}_+^N)$ such that u solves

$$\Delta u = \mu_1 + \lambda \text{ in } \mathcal{D}'(\mathbb{R}_+^N).$$

Arguing as in the proof of Theorem 2.6, and adopting the same notations, for a nonnegative function $\psi \in \mathcal{C}_c^2(\mathbb{R}^{N-1})$, we have

$$I_1^\epsilon := \int_{\partial \mathbb{R}_+^N} u(x', \epsilon) \psi(x') dx' = \int_{\Omega_\epsilon} \varphi_\epsilon d\mu_1 - \int_{\Omega_\epsilon} u \Delta \varphi_\epsilon + \int_{\Omega_\epsilon} \varphi_\epsilon d\lambda =: I_2^\epsilon - I_3^\epsilon + I_4^\epsilon.$$

Next, as in (19) $I_2^\epsilon - I_3^\epsilon$ converges as $\epsilon \rightarrow 0$. By the Beppo Levi monotone convergence theorem, since $\varphi_\epsilon \chi_{\Omega_\epsilon} \nearrow \varphi_0$, it follows that $I_4^\epsilon \rightarrow \int_{\mathbb{R}_+^N} \varphi_0 d\lambda$, so I_1^ϵ has a finite or infinite limit. We claim that,

$$\int_{\mathbb{R}_+^N} \varphi_0 d\lambda < \infty. \quad (22)$$

This claim implies that $\lambda \in \mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N)$. Note that (22) is equivalent to I_1^ϵ having a finite limit as $\epsilon \rightarrow 0$.

Now, suppose that 1 holds. Let $R > 0$ be large enough such that $\text{supp}(\psi) \subset B'_R$. Then, by our assumption there exist $M, \delta > 0$ such that for $\epsilon < \delta$, we have $I_1^\epsilon \leq M \int_{\partial \mathbb{R}_+^N} \psi(x') dx'$. Therefore, I_4^ϵ is uniformly bounded for $\epsilon < \delta$. This implies that I_1^ϵ has a finite limit as $\epsilon \rightarrow 0$. That is, $T(\psi) := \lim_{\epsilon \rightarrow 0^+} \int_{\partial \mathbb{R}_+^N} u(x', \epsilon) \psi(x') dx'$ exists and is finite. Using a similar argument as in the proofs of Theorem 2.6 and Lemma 2.10, we conclude that T can be represented by a local Radon measure. This completes the proof.

Next, suppose that 2 holds. The claim will follow immediately by arguing as in the case 1, using the estimate $|I_1^\epsilon| \leq \|\psi\|_\infty \int_{|x'| < R} |u(x', \epsilon)| dx' < \|\psi\|_\infty M$.

Finally, suppose that 3 holds and let $v = Tr(u)$. We know that $I_1^\epsilon \rightarrow \int_{\partial \mathbb{R}_+^N} \psi(x') dv(x')$ as $\epsilon \rightarrow 0$. This implies that, the I_4^ϵ converges to a finite limit, so (22) holds. Hence, $\lambda \in \mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N)$.

The fact that u is a solution of (16) with $\mu = \mu_1 + \lambda$ and $v = Tr(u)$, is a consequence of Theorem 2.8. $\square$

By slightly modifying the above proofs, we can establish the following generalization of Theorem 2.8.

Theorem 2.12 *Let $\mu \in \mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N)$, and let $u \in L^1_{loc}(\overline{\mathbb{R}_+^N})$ satisfy*

$$\Delta u \geq \mu_1 \text{ in } \mathcal{D}'(\mathbb{R}_+^N). \quad (23)$$

If u has the lim-trace $v = Tr(u)$ with $v \in \mathcal{M}_{loc}(\partial \mathbb{R}_+^N)$, then there exist a positive Radon measure $\lambda \in \mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N)$ such that u is a solution of (16) with $\mu = \mu_1 + \lambda$ and $v = Tr(u)$.

In particular, u is a weak solution of

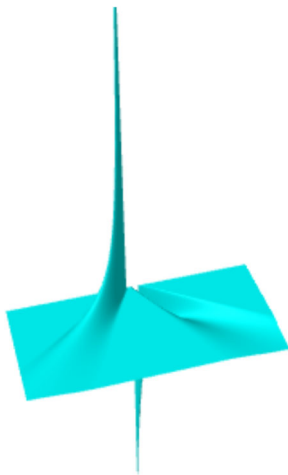
$$\begin{cases} \Delta u \geq \mu_1 & \text{on } \mathbb{R}_+^N, \\ u = v & \text{on } \partial \mathbb{R}_+^N. \end{cases}$$

Furthermore, if equality holds in (23), then $\lambda \equiv 0$.

For the lim-trace of a weak solution of (5) with positive Radon measure, refer to Theorem 3.11 and Corollary 3.14 provided below.

Remark 2.13 (on the Lim-Trace of Positive Parts of Functions) It is interesting to investigate whether a function u that has the lim-trace also ensures that its positive part u^+ possesses the same property. Generally, the answer is negative. Consider the following example: Let

Fig. 1 The plot of function u of Remark 2.13



$m_1 \in C_c^\infty(\mathbb{R})$ be an even, nonnegative standard cut off function supported in $(-1, 1)$ with $\int m_1 = 1$. For $\epsilon > 0$, define the mollifier family as $m_\epsilon(x) := \frac{1}{\epsilon} m_1(\frac{x}{\epsilon})$. Next, for $x \in \mathbb{R}$ and $\epsilon > 0$, define $u(x, \epsilon)$ as (see Figure 1)

$$u(x, \epsilon) := \frac{1}{\sqrt{\epsilon}} m_\epsilon(x - \epsilon) - \frac{1}{\sqrt{\epsilon}} m_\epsilon(x + \epsilon).$$

Clearly, $u \in C^\infty(\mathbb{R}_+^2)$, and by the Lagrange theorem, for any $\psi \in C_c^2(\mathbb{R})$, we have the estimate:

$$\left| \int_{\mathbb{R}} u(x, \epsilon) \psi(x) dx \right| \leq 2\sqrt{\epsilon} \|\psi'\|_\infty.$$

Thus, u has the lim-trace $v = Tr(u) = 0$, the trivial measure on $\partial\mathbb{R}_+^2 = \mathbb{R}$.

However, the positive part $u^+(x, \epsilon) = \frac{1}{\sqrt{\epsilon}} m_\epsilon(x - \epsilon)$ does not admit the lim-trace. For a nonnegative $\psi \in C_c^2(\mathbb{R})$ with $\psi(0) > 0$ we have:

$$\int_{\mathbb{R}} u^+(x, \epsilon) \psi(x) dx \geq \frac{1}{\sqrt{\epsilon}} \frac{\psi(0)}{4} \quad \text{for sufficiently small } \epsilon,$$

and hence,

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} u^+(x, \epsilon) \psi(x) dx \geq \lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{\epsilon}} \frac{\psi(0)}{4} = \infty.$$

On the other hand, in the general case, if both u and u^+ have the lim-traces, say $v = Tr(u)$ and $\lambda = Tr(u^+)$, it can be shown that $\lambda \geq v^+$. However, it is not necessarily the case that $\lambda = v^+$. To illustrate this, consider:

$$v(x, \epsilon) := m_\epsilon(x - \epsilon) - m_\epsilon(x + \epsilon).$$

As before, v has the lim-trace and $Tr(v) = 0$, while v^+ admits the lim-trace $Tr(v^+) = \delta_0$, the Dirac measure at the origin.

The example related to the function u highlights the challenges in relaxing the definition of lim-trace. While the lim-trace can be viewed as a weak* convergence of the traces $u(x', \epsilon)$, this convergence is achieved through smooth test functions, and the example suggests that

relaxing this to continuous test functions would require further investigation, Nonetheless, our definition of lim-trace is suitable for our purposes, as Theorems 2.6 and 2.8 establish a strong connection between the weak solutions of (16) and their lim-traces.

3 Main results

To maintain clarity and streamline the organization of this paper, we will postpone the proof of our first main result to Sections 3.1 and 3.3. The reasons for this will become apparent in due course.

For the reader's convenience we rewrite Theorems A and B of the introduction in the following form.

Theorem 3.1 *Let $N \geq 2$, and let μ and ν be positive Radon measures on $\mathbb{R}_+^N$ and $\partial\mathbb{R}_+^N$, respectively.*

- A. *Let $\mu \in \mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N)$ and let $u \in L_{loc}^1(\overline{\mathbb{R}_+^N})$ be a weak solution of (5). Suppose $x \in \mathbb{R}_+^N$ is a Lebesgue point for u . Assume that there exists $h \in \mathbb{R}$ such that $(\mathcal{R}^+)$ holds at the point x that is,*

$$\liminf_{R \rightarrow +\infty} \frac{1}{R^{N+2}} \int_{\{y_N > 0\} \cap \{R < |x-y|_* < 2R\}} y_N |u(y) - hy_N| dy = 0.$$

Then,

$$u(x) \geq hx_N + \int_{\partial\mathbb{R}_+^N} K^x(y') d\nu(y') + \int_{\mathbb{R}_+^N} G^x(y) d\mu(y). \quad (24)$$

Moreover, if u is a weak solution of (4), then the inequality in (24) becomes an equality, that is

$$u(x) = hx_N + \int_{\partial\mathbb{R}_+^N} K^x(y') d\nu(y') + \int_{\mathbb{R}_+^N} G^x(y) d\mu(y). \quad (25)$$

- B. *Let u be defined by (25) and assume that it is finite for a.e. $x \in \mathbb{R}_+^N$.*

Then $u \in L_{loc}^1(\overline{\mathbb{R}_+^N})$, $\mu \in \mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N)$ and u is a weak solution of (4). Moreover, u satisfies the condition $(\mathcal{R}^+)$, where the constant h appears in (25).

Furthermore,

(a)

$$h = \inf_{\mathbb{R}_+^N} \frac{u(x)}{x_N}.$$

(b) *Let $\Omega \subset \mathbb{R}_+^N$ be a nonempty bounded open set. If $h = \inf_{\Omega} \frac{u(x)}{x_N}$, then $u(x) = hx_N$, $\mu \equiv 0$ and $\nu \equiv 0$.*

(c) *If $u - hx_N$ is not identically zero, then there exists a constant $c_0 > 0$ such that*

$$u(x) \geq hx_N + c_0 \frac{x_N}{1 + |x|^N}, \quad \text{for all } x \in \mathbb{R}_+^N. \quad (26)$$

(d) *For almost every $x \in \mathbb{R}_+^N$,*

$$\lim_{R \rightarrow +\infty} \frac{1}{R^{N+2}} \int_{\{y_N > 0\} \cap B_R^*(x)} y_N |u(y) - hy_N| dy = 0.$$

Note that the main statement in B. is the converse of the statement in A.

As a consequence of Theorems 1.2, 3.1.B and 2.6, the following corollary holds:

Corollary 3.2 *A nonnegative superharmonic distribution on $\mathbb{R}_+^N$ possesses the lim-trace.*

Remark 3.3 We observe that Theorem 3.1 and Remark 2.4, imply the uniqueness of the measures μ and ν in the representation Theorem 1.2.

Corollary 3.4 *Let $\mu \in L_{loc}^1(\mathbb{R}_+^N, x_N)$ and $\nu \in L_{loc}^1(\mathbb{R}^{N-1})$ be nonnegative functions, and let u be a weak solution of (5) satisfying $(\mathcal{R}_0^+)$. Then for almost every $x \in \mathbb{R}_+^N$, we have*

$$u(x) \geq \int_{\mathbb{R}^{N-1}} K^x(y')\nu(y')dy' + \int_{\mathbb{R}_+^N} G^x(y)\mu(y)dy. \quad (27)$$

Corollary 3.5 *Under the assumptions of Theorem 3.1.A with $h \geq 0$, if u is a weak solution of (4), then $u(x) > hx_N$ or $u(x) \equiv hx_N$ a.e. on $\mathbb{R}_+^N$.*

Remark 3.6 In general, a superharmonic function on $\mathbb{R}_+^N$ does not belong to $L_{loc}^1(\overline{\mathbb{R}_+^N})$. For example, the function $u(x_1, x_2) = -|x|^{-2}$ does not belong to $L_{loc}^1(\overline{\mathbb{R}_+^N})$ with $N = 2$, and $-\Delta u = 4|x|^{-4} \geq 0$. See also the example defined in (38).

Remark 3.7 i) In Theorem 3.1.A, the fact that, the integrals in (25) are well defined is an outcome of our approach. In other words, the finiteness of the integrals in (25) is a necessary condition for the existence of a solution satisfying $(\mathcal{R}^+)$.

ii) If we know a priori that the integrals in (25) are finite, then (25) describes all the solutions satisfying $(\mathcal{R}^+)$.

iii) Notice that if we require a priori that the integrals in (25) are finite without any assumption on the sign of the measures μ and ν , then (25) defines a solution of (4). More precisely, if $\int_{\partial\mathbb{R}_+^N} K^x(y')d|\nu|(y') < \infty$ and $\int_{\mathbb{R}_+^N} G^x(y)d|\mu|(y) < \infty$, then (25) defines a weak solution of (4). For a proof of this, the interested reader can follow the same steps in the proof of Theorem 3.1.B, see Section 3.3.

A more general result is given by the following.

Theorem 3.8 *Let $N \geq 2$, and let $\mu \in \mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N)$ and $\nu \in \mathcal{M}_{loc}(\partial\mathbb{R}_+^N)$.*

Writing $\mu = \mu^+ - \mu^-$ and $\nu = \nu^+ - \nu^-$, for a.e. $x \in \mathbb{R}_+^N$ we assume that

$$\int_{\mathbb{R}^{N-1}} K^x(y')d\nu^-(y') < \infty, \quad \int_{\mathbb{R}_+^N} G^x(y)d\mu^-(y) < \infty. \quad (28)$$

Let $u \in L_{loc}^1(\overline{\mathbb{R}_+^N})$ be a weak solution of (4). If u satisfies $(\mathcal{R}^+)$, then (25) holds for a.e. $x \in \mathbb{R}_+^N$.

The proof of Theorem 3.8 is a small modification of the proof of Theorem 3.1.A, which can be found in Section 3.2.

Remark 3.9 If u is given by (25), or equivalently u is a nonnegative superharmonic function, then u satisfies $(\mathcal{R}^+)$. Moreover, the $\liminf$ in the $(\mathcal{R}^+)$ condition is actually a limit, meaning u satisfies

there exists $h \in \mathbb{R}$ such that

$$\lim_{R \rightarrow +\infty} \frac{1}{R^{N+2}} \int_{\{y_N > 0\} \cap \{|x - y|_* < 2R\}} y_N |u(y) - hy_N| dy = 0, \quad \text{for a.e. } x \in \mathbb{R}_+^N. \quad (29)$$

Generally, the functional class satisfying $(\mathcal{R}^+)$ is not a linear space. However, the set of functions that fulfill (29) forms a linear space.

Theorem 3.10 *Let μ be a positive Radon measure such that $\mu \notin \mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N)$. Then the problem*

$$-\Delta u = \mu, \quad \text{in } \mathcal{D}'(\mathbb{R}_+^N), \quad u \geq 0,$$

does not admit a solution.

The proof is an immediate consequence of Theorem 3.1.B (indeed if the problem admits a solution, then necessarily $\mu \in \mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N)$).

The following result establishes a connection between the notion of weak solution and distributional solution of the problem:

$$-\Delta u = \mu, \quad \text{in } \mathcal{D}'(\mathbb{R}_+^N). \quad (30)$$

An additional result in this context is presented in Corollary 3.14 below.

Let us first recall the Riesz decomposition theorem (see [1, Theorem 4.4.1]); if u is a superharmonic function satisfying (30) and there exists a superharmonic function g such that $u \geq -g$, then there exists a harmonic function H on $\mathbb{R}_+^N$, such that

$$u(x) = H(x) + \int_{\mathbb{R}_+^N} G^x(y) d\mu(y).$$

Moreover, H is the greatest harmonic minorant of u .

We shall prove that this harmonic function H can be obtained as a weighted limit of u on certain rings, see (60) below. This construction is further discussed in Section 4, see Theorem 4.1.

The key point in order to establish a connection between a solution u of (30) and a weak solution of (4) is whether or not u possesses the lim-trace. If $u \geq -g$ with g a superharmonic function, then $u + g$ has the lim-trace as stated in Corollary 3.2. However, this does not necessarily imply that u itself has the lim-trace. A sufficient condition that ensures u has the lim-trace is if $g \geq 0$.

Theorem 3.11 *Let μ be a positive Radon measure on $\mathbb{R}_+^N$, and let $u \in L_{loc}^1(\mathbb{R}_+^N)$ be a distributional solution of (30).*

Assume there exists a nonnegative superharmonic function g on $\mathbb{R}_+^N$ such that $u \geq -g$ a.e. on $\mathbb{R}_+^N$. Then u admits the lim-trace $Tr(u)$, which is a local Radon measure (not necessarily positive) and $\mu \in \mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N)$. Moreover, setting $v := Tr(u) \in \mathcal{M}_{loc}(\partial\mathbb{R}_+^N)$, we have:

- (a) u is a weak solution of (4),
- (b) There exists $h_u \in \mathbb{R}$ such that (25) holds a.e. on $\mathbb{R}_+^N$ with $h = h_u$, i.e.

$$u(x) = h_u x_N + \int_{\partial\mathbb{R}_+^N} K^x(y') dv(y') + \int_{\mathbb{R}_+^N} G^x(y) d\mu(y). \quad (31)$$

- (c) $u + g$ satisfies $(\mathcal{R}^+)$ for a suitable $h \geq 0$.
- (d) u satisfies $(\mathcal{R}^+)$ with $h = h_u$.
- (e) If u is bounded from below, then $h_u \geq 0$.
- (f) If u is nonnegative, then the lim-trace $Tr(u)$ is a positive Radon measure.

Proof The first step relies on the same reasoning as in Corollary 3.2. Indeed, since g is non-negative and superharmonic, Theorem 1.2 applies (see also Theorem 1.37 in [1]), implying that there exists $h_g \geq 0$, a positive Radon measure ν_g on $\partial\mathbb{R}_+^N$ and a positive Radon measure μ_g on $\mathbb{R}_+^N$, such that

$$g(x) = h_g x_N + \int_{\partial\mathbb{R}_+^N} K^x(y') d\nu_g(y') + \int_{\mathbb{R}_+^N} G^x(y) d\mu_g(y), \quad \text{for a.e. } x \in \mathbb{R}_+^N.$$

Thus, by Theorem 3.1.B, it follows that $\mu_g \in \mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N)$ and g is a weak solution of

$$\begin{cases} -\Delta g = \mu_g & \text{on } \mathbb{R}_+^N, \\ g = \nu_g & \text{on } \partial\mathbb{R}_+^N. \end{cases} \quad (32)$$

Therefore, by Theorem 2.6 g has the lim-trace $Tr(g) = \nu_g$.

Next, let $v := u + g$. The function v is nonnegative and satisfies $-\Delta v = \mu + \mu_g \geq 0$ in distributional sense on $\mathbb{R}_+^N$. Using the same argument as above, there exists $h_v \geq 0$ and positive Radon measures ν_v on $\partial\mathbb{R}_+^N$, $\mu_v \in \mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N)$ such that

$$v(x) = u(x) + g(x) = h_v x_N + \int_{\partial\mathbb{R}_+^N} K^x(y') d\nu_v(y') + \int_{\mathbb{R}_+^N} G^x(y) d\mu_v(y). \quad (33)$$

Additionally, v is a weak solution of

$$\begin{cases} -\Delta v = \mu + \mu_g & \text{on } \mathbb{R}_+^N, \\ v = \nu_v & \text{on } \partial\mathbb{R}_+^N. \end{cases} \quad (34)$$

Hence, $\mu_v = \mu + \mu_g$ and v has the lim-trace $Tr(v) = \nu_v$. Consequently, $u = v - g$ has the lim-trace $Tr(u) = Tr(v) - Tr(g) = \nu_v - \nu_g \in \mathcal{M}_{loc}(\partial\mathbb{R}_+^N)$ and $\mu \in \mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N)$.

By applying linearity, from (32) and (34), we complete the proof of (a).

The proof of (b) follows by reformulating the integral expressions of v and g mentioned earlier, where $h_u = h_v - h_g$. The claim of (c) is a direct result of applying Theorem 3.1.B to (33).

Proof of (d). Utilizing Theorem 3.1.B and Remark 3.9 applied to g and v , it can be deduced that both g and v satisfy (29). Consequently, $u = v - g$ also satisfies (29) and therefore $(\mathcal{R}^+)$.

Proof of (e). If u is bounded from below, i.e., if g is a positive constant $g = a$, then it follows that $\nu_g = a$. In other words, ν_g is a positive multiple of the Lebesgue measure on $\partial\mathbb{R}_+^N$, and $h_g = 0$. Hence, $\nu_u = \nu_v - a$, and $h_u = h_v - 0 \geq 0$.

Proof of (f). This conclusion follows directly from the previous points. $\square$

The following result provides a characterization of a comparison principle.

Theorem 3.12 *Let $u, v \in L_{loc}^1(\mathbb{R}_+^N)$ such that*

$$-\Delta u \geq -\Delta v \text{ in } \mathcal{D}'(\mathbb{R}_+^N),$$

and

$$\liminf_{x \rightarrow (y', 0)} u(x) \geq \limsup_{x \rightarrow (y', 0)} v(x), \quad \text{for any } y' \in \partial\mathbb{R}_+^N. \quad (35)$$

Then $x_N |u - v| \in L_{loc}^1(\overline{\mathbb{R}_+^N})$.

Furthermore,

$$u \geq v \text{ a.e. on } \mathbb{R}_+^N, \quad (36)$$

if and only if

$$u - v \text{ satisfies } (\mathcal{R}^+) \text{ with } h \geq 0. \quad (37)$$

Moreover, if (36), or equivalently (37), holds, then $u - v \in L^1_{loc}(\overline{\mathbb{R}^N_+})$.

In particular, this comparison principle applies to functions that satisfy $(\mathcal{R}^+_0)$. For another comparison principle, see Section 4 in [9].

To avoid unnecessary complexity, we defer the proof of Theorem 3.12 to Section 3.4. The reason is that it relies on notations and results introduced in the proof of Theorem 3.1.

Remark 3.13 In the above theorem, the fact that $u - v \in L^1_{loc}(\overline{\mathbb{R}^N_+})$ is a significant consequence of (36) or (37).

Furthermore, consider $N \geq 2$ and $v \equiv 0$. Define

$$u(x) := \frac{1}{|x|^N} \left(1 - N \frac{x_N^2}{|x|^2} \right). \quad (38)$$

The function u is harmonic on $\mathbb{R}^N_+$ and satisfies $(\mathcal{R}^+_0)$. However, it is not nonnegative, which is because $u \notin L^1_{loc}(\overline{\mathbb{R}^N_+})$. In this scenario, the conditions of Theorem 3.12 are not satisfied. Indeed, if $y' \neq 0$, we have $\lim_{x \rightarrow (y', 0)} u(x) = |y'|^{-N} > 0$, while if $y' = 0$ we have $\liminf_{x \rightarrow (0, 0)} u(x) = -\infty$. This implies that condition (35) must hold for every $y' \in \partial\mathbb{R}^N_+$. The latter cannot be relaxed to hold for almost every $y' \in \partial\mathbb{R}^N_+$.

As a consequence of Theorems 3.12 and 3.1, we obtain the following result, which establishes a connection between the notion of weak solution and distributional solution.

Corollary 3.14 Let $u \in L^1_{loc}(\overline{\mathbb{R}^N_+})$ be a superharmonic function such that

$$\liminf_{x \rightarrow (y', 0)} u(x) \geq 0, \quad \text{for any } y' \in \partial\mathbb{R}^N_+. \quad (39)$$

Then $x_N |u| \in L^1_{loc}(\overline{\mathbb{R}^N_+})$.

Furthermore, $u \geq 0$ if and only if u satisfies $(\mathcal{R}^+)$ with $h \geq 0$.

Moreover, if u satisfies $(\mathcal{R}^+)$ for some $h \in \mathbb{R}$, then there exist suitable positive Radon measures μ and ν such that (25) holds and u is a weak solution of (4).

Related results on the maximum principle in bounded domains for superharmonic function with boundary condition like in (39) can be found in [3].

The following positivity result seems noteworthy on its own.

Theorem 3.15 Let μ and ν be positive Radon measures on $\mathbb{R}^N_+$, and $\partial\mathbb{R}^N_+$, respectively. Let u be a weak solution of (5) [or (4)]. If u is bounded from below, then u is nonnegative. Furthermore, u has the lim-trace, satisfies $(\mathcal{R}^+)$ with $h \geq 0$, can be represented by (24) [or (25)], and Theorem 3.1 applies.

Proof Since u is superharmonic on $\mathbb{R}^N_+$, Theorem 3.11 is applicable. From points (d) and (e), it follows that u satisfies $(\mathcal{R}^+)$ with $h \geq 0$. Therefore, applying Theorem 3.1 completes the proof. $\square$

3.1 Proof of Theorem 3.1.A

The proof employs a systematic method to demonstrate the claim by utilizing Green's functions and test functions in a weak formulation structured as follows:

Proof We adjust u by replacing it with $u(y) - hy_N$ to ensure that the function satisfies the condition $(\mathcal{R}_0^+)$ at the point x . This is a necessary step to simplify the proof.

In what follows, we use the notation of Appendix A along with the results contained therein. We select the test function $\tilde{\varphi}(y) := G_\epsilon^x(y)\varphi(y)$, where G_ϵ^x is the regularized Green's function centered at x with parameter $\epsilon > 0$ (see (75)), and $\varphi \in C_c^2(\mathbb{R}_+^N)$. Since $G_\epsilon^x \in C^\infty$ and

$$G_\epsilon^x(y', 0) = 0, \quad (40)$$

we can use $\tilde{\varphi}$ as test function in (13) obtaining,

$$\begin{aligned} \int_{\mathbb{R}_+^N} G_\epsilon^x(y)\varphi(y)d\mu(y) + \int_{\partial\mathbb{R}_+^N} K_\epsilon^x(y')\varphi(y', 0)dv(y') &\leq - \int_{\mathbb{R}_+^N} u(y)\Delta(G_\epsilon^x\varphi)(y)dy^3 \quad (41) \\ &= \int_{\mathbb{R}_+^N} u(y)\varphi(y)(-\Delta G_\epsilon^x)(y)dy + \int_{\mathbb{R}_+^N} u(y)(-\Delta\varphi)(y)G_\epsilon^x(y)dy \\ &\quad - 2 \int_{\mathbb{R}_+^N} u(y)\nabla G_\epsilon^x \cdot \nabla\varphi =: I_1 + I_2 + I_3. \end{aligned}$$

For the first integral we have,

$$\begin{aligned} I_1 &= \int_{\mathbb{R}_+^N} u(y)\varphi(y)(-\Delta G_\epsilon^x)(y)dy \\ &= \int_{\mathbb{R}_+^N} u(y)\varphi(y)(-\Delta\Gamma_\epsilon^x)(y)dy - \int_{\mathbb{R}_+^N} u(y)\varphi(y)(-\Delta\Gamma_\epsilon^{\hat{x}})(y)dy =: I_{11} + I_{12}. \end{aligned}$$

Now from the estimate,

$$|u(y)\varphi(y)(-\Delta\Gamma_\epsilon^{\hat{x}})(y)| = |u(y)\varphi(y)c\frac{\epsilon^2}{a_1^{(N+2)/2}}| \leq |u(y)\varphi(y)|c\frac{\epsilon^2}{x_N^{N+2}} \in L^1(\mathbb{R}_+^N),$$

an application of the Lebesgue dominated convergence theorem implies that $I_{12} \rightarrow 0$ as $\epsilon \rightarrow 0$.

In addition, since x is a Lebesgue point of u , $I_{11} \rightarrow u(x)\varphi(x)$ as $\epsilon \rightarrow 0$.

Finally, in order to estimate I_2 and I_3 we choose φ as

$$\varphi(y) := \phi\left(\frac{|x' - y'|}{R}\right)\phi\left(\frac{|x_N - y_N|}{R}\right),$$

where $R > 0$ and $\phi \in C_c^2(\mathbb{R})$ is a standard cut-off function as in (15).

Note that the support of φ is contained in $\{y \in \mathbb{R}^N : |y - x|_* \leq 2R\}$, and $\varphi \equiv 1$ on $B_R^*(x)$, while the support of $\nabla\varphi$ is contained in $\{y \in \mathbb{R}^N : R \leq |y - x|_* \leq 2R\}$.

³ Here, we have used the definition of K^x , (40) and the fact that

$$\frac{\partial}{\partial y_N}(G_\epsilon^x\varphi)(y', 0) = \frac{\partial G_\epsilon^x}{\partial y_N}(y', 0)\varphi(y', 0) + G_\epsilon^x(y', 0)\frac{\partial\varphi}{\partial y_N}(y', 0) = K_\epsilon^x(y')\varphi(y', 0).$$

Let L be the radial Laplacian operator in dimension $N - 1$, that is, $L(\phi)(r) := \phi''(r) + \frac{N-2}{r}\phi'(r)$, and set $t := \frac{|x_N - y_N|}{R}$, $s := \frac{|x' - y'|}{R}$. By computation we have,

$$\Delta\varphi(y) = \frac{1}{R^2} [\phi''(t)\phi(s) + \phi(t)L(\phi)(s)],$$

hence,

$$|\Delta\varphi(y)| \leq \frac{M}{R^2}.$$

Now, the supports of $\Delta\varphi$ and of $\nabla\varphi$ are such that, $\text{supp}(\Delta\varphi) \subset \text{supp}(|\nabla\varphi|) \subset \{y : R \leq |x - y|_* \leq 2R\}$. In what follows we denote by $A_R^*(x)$ the annulus $A_R^*(x) := B_{2R}^*(x) \setminus B_R^*(x) \subset \mathbb{R}_+^N$. See Figure 2 at page 29.

Therefore we have,

$$|I_2| \leq \frac{M}{R^2} \int_{A_R^*(x)} |u| G_\epsilon^x(y) dy.$$

Observing that for $R \leq |x - y|_* \leq 2R$ we have, $\epsilon^2 + |x - y|^2 > R^2$, from 1. of Proposition A.1 we deduce $|G_\epsilon^x| \leq C'_n \frac{x_N y_N}{R^{N+2}}$, hence,

$$|I_2| \leq \frac{c x_N}{R^{N+2}} \int_{A_R^*(x)} y_N |u(y)| dy. \quad (42)$$

Next, we estimate $|I_3|$. We begin by computing $|\nabla\varphi \cdot \nabla G_\epsilon^x|$ in $A_R^*(x)$ with $R > x_N$. Since

$$\begin{aligned} |I_3| &\leq 2 \int_{A_R^*(x)} |u(y)| |\nabla G_\epsilon^x \cdot \nabla\varphi| \\ &\leq 2 \int_{A_R^*(x)} |u(y)| \left| \sum_{j=1}^{N-1} \partial_j \varphi(y) \partial_j G_\epsilon^x(y) \right| + 2 \int_{A_R^*(x)} |u(y)| |\partial_N \varphi(y) \partial_N G_\epsilon^x(y)| \\ &=: I_{31} + I_{32}, \end{aligned}$$

we get

$$\begin{aligned} \left| \sum_{j=1}^{N-1} \partial_j \varphi(y) \partial_j G_\epsilon^x(y) \right| &= \left| \phi\left(\frac{|x_N - y_N|}{R}\right) \phi'\left(\frac{|x' - y'|}{R}\right) \frac{1}{R} C'_N |x' - y'| \left(a_1^{-N/2} - a_2^{-N/2}\right) \right| \\ &\leq c_1 \frac{|x' - y'|}{R} \frac{a_1^{N/2} - a_2^{N/2}}{a_1^{N/2} a_2^{N/2}} \leq c_1 \frac{|x' - y'|}{R} \frac{N}{2} (a_1 - a_2) a_1^{N/2-1} \frac{1}{a_1^{N/2} a_2^{N/2}} \\ &= c_1 \frac{|x' - y'|}{R} \frac{N}{2} \frac{4x_N y_N}{a_1 a_2^{N/2}}. \end{aligned}$$

Here, we have used the convexity inequality (92) with $\alpha = N/2$, and (76). Therefore, since for $y \in A_R^*(x)$, we know that $a_1 \geq a_2 \geq R^2$, we infer

$$\left| \sum_{j=1}^{N-1} \partial_j \varphi(y) \partial_j G_\epsilon^x(y) \right| \leq c \frac{y_N}{R^{N+2}},$$

while,

$$|\partial_N \varphi(y) \partial_N G_\epsilon^x(y)| = \left| \frac{1}{R} \phi'\left(\frac{|x_N - y_N|}{R}\right) \phi\left(\frac{|x' - y'|}{R}\right) C'_N \left(\frac{y_N + x_N}{a_1^{N/2}} - \frac{y_N - x_N}{a_2^{N/2}} \right) \right|.$$

Now, we observe that the above term can be non zero only for $R \leq |x_N - y_N| \leq 2R$. Next by choosing $R \geq 2|x_N|$, it follows that $|y_N| \geq |x_N - y_N| - |x_N| \geq R/2$, and $|y_N| \leq |x_N - y_N| + |x_N| \leq 5R/2$. Hence for $a_1 \geq a_2 \geq R^2$, we obtain

$$\begin{aligned} |\partial_N \varphi(y) \partial_N G_\epsilon^x(y)| &\leq c \left[\frac{|y_N|}{R} \left(\frac{1}{a_2^{N/2}} - \frac{1}{a_1^{N/2}} \right) + \frac{|x_N|}{R} \left(\frac{1}{a_2^{N/2}} + \frac{1}{a_1^{N/2}} \right) \right] \\ &\leq c \left[\frac{5}{2} \frac{a_1^{N/2} - a_2^{N/2}}{a_1^{N/2} a_2^{N/2}} + \frac{|x_N|}{R} \left(\frac{1}{a_2^{N/2}} + \frac{1}{a_1^{N/2}} \right) \frac{2|y_N|}{R} \right] \\ &\leq c \left[\frac{5}{2} \frac{N}{2} (a_1 - a_2) a_1^{N/2-1} \frac{1}{a_1^{N/2} a_2^{N/2}} + 2x_N \frac{2}{R^N} \frac{y_N}{R^2} \right] \\ &= c \left[\frac{5N}{4} \frac{4x_N y_N}{a_1 a_2^{N/2}} + 2x_N \frac{2}{R^N} \frac{y_N}{R^2} \right] \leq cx_N [5N + 4] \frac{y_N}{R^{N+2}}. \end{aligned}$$

Therefore, we conclude that for $R > 2x_N$, we have

$$|I_3| \leq \frac{cx_N}{R^{N+2}} \int_{A_R^*(x)} y_N |u(y)| dy. \quad (43)$$

It is important to note that the constants c in (42) and (43) depends only on N and on the cut-off function ϕ .

To complete the proof of Theorem 3.1.A, we first need to take the limit as $\epsilon \rightarrow 0$ and then let $R \rightarrow \infty$ in (41).

Considering the right hand side of (41), using the estimates above and the assumption $(\mathcal{R}_0^+)$, it follows that

$$- \int_{\mathbb{R}_+^N} u(y) \Delta(G_\epsilon^x \varphi)(y) dy \xrightarrow{\epsilon \rightarrow 0} u(x) + I_2|_{\epsilon=0} + I_3|_{\epsilon=0} \xrightarrow{R \rightarrow \infty} u(x).$$

On the left hand side of (41), the limits

$$\int_{\partial \mathbb{R}_+^N} K_\epsilon^x(y') \varphi(y', 0) dv(y') \xrightarrow{\epsilon \rightarrow 0} \int_{\partial \mathbb{R}_+^N} K^x(y') \varphi(y', 0) dv(y') \xrightarrow{R \rightarrow \infty} \int_{\partial \mathbb{R}_+^N} K^x(y') dv(y'),$$

follow from the Beppo Levi monotone convergence theorem. From (41) we deduce that these limits are finite.

The same reasoning applies to the first integral in (41), yielding

$$\int_{\mathbb{R}_+^N} G_\epsilon^x(y) \varphi(y) d\mu(y) \xrightarrow{\epsilon \rightarrow 0} \int_{\mathbb{R}_+^N} G^x(y) \varphi(y) d\mu(y) \xrightarrow{R \rightarrow \infty} \int_{\mathbb{R}_+^N} G^x(y) d\mu(y).$$

This completes the proof of (24).

If u solves (4), then (41) holds with equality, and the proof follows by applying the same arguments. $\square$

3.2 Proof of Theorem 3.8

We proceed using the same approach as in the proof of Theorem 3.1.A. The key difference in this proof lies in the conclusion, specifically in the computation of the limits of the following integrals:

$$\int_{\partial \mathbb{R}_+^N} K_\epsilon^x(y') \varphi(y', 0) dv(y'), \quad \text{and} \quad \int_{\mathbb{R}_+^N} G_\epsilon^x(y) \varphi(y) d\mu(y). \quad (44)$$

Since $v = v^+ - v^-$, the first integral can be expressed as

$$\int_{\partial \mathbb{R}_+^N} K_\epsilon^x(y') \varphi(y', 0) dv(y') = \int_{\partial \mathbb{R}_+^N} K_\epsilon^x(y') \varphi(y', 0) dv^+(y') - \int_{\partial \mathbb{R}_+^N} K_\epsilon^x(y') \varphi(y', 0) dv^-(y').$$

Each of these integrals can be analyzed separately. By the Beppo Levi monotone convergence theorem, we have

$$\int_{\partial \mathbb{R}_+^N} K_\epsilon^x(y') \varphi(y', 0) dv^\pm(y') \xrightarrow{\epsilon \rightarrow 0} \int_{\partial \mathbb{R}_+^N} K^x(y') \varphi(y', 0) dv^\pm(y') \xrightarrow{R \rightarrow \infty} \int_{\partial \mathbb{R}_+^N} K^x(y') dv^\pm(y').$$

Given the assumption (28), it follows that

$$\int_{\partial \mathbb{R}_+^N} K^x dv(y') = \int_{\partial \mathbb{R}_+^N} K^x dv^+(y') - \int_{\partial \mathbb{R}_+^N} K^x dv^-(y')$$

is well-defined, and from (41), we deduce that it is finite.

The second integral in (44) can be handled similarly.

3.3 Proof of Theorem 3.1.B

Let u be defined by (25).

To prove that $u \in L^1_{loc}(\overline{\mathbb{R}_+^N})$, we start by defining

$$u_1(x) := \int_{\partial \mathbb{R}_+^N} K^x(y') dv(y'), \quad u_2(x) := \int_{\mathbb{R}_+^N} G^x(y) d\mu(y), \quad \text{for } x \in \mathbb{R}_+^N. \quad (45)$$

It suffices to show that

$$\int_{B_R^*} u_j(x) dx < \infty \quad \text{for any } R > 0, \quad (46)$$

for $j = 1, 2$.

For $y' \in \mathbb{R}^{N-1}$, set $f_1(y') := \int_{B_R^*} K^x(y') dx$. We then have

$$\begin{aligned} \int_{B_R^*} u_1(x) dx &= \int_{B_R^*} \int_{\partial \mathbb{R}_+^N} K^x(y') dv(y') dx = \int_{\partial \mathbb{R}_+^N} \left(\int_{B_R^*} K^x(y') dx \right) dv(y') \\ &= \int_{\partial \mathbb{R}_+^N} f_1(y') dv(y') = \int_{|y'| \leq S} f_1(y') dv(y') + \int_{|y'| > S} f_1(y') dv(y'), \end{aligned}$$

where $S > 0$ is chosen later. Since u_1 is finite a.e., $K^x(y') \approx |y'|^{-N}$ as $|y'| \rightarrow \infty$, and v is finite on bounded sets, we deduce that

$$\int_{\partial \mathbb{R}_+^N} \frac{1}{1 + |y'|^N} dv(y') < \infty. \quad (47)$$

Next, we claim that $f_1(y') \approx |y'|^{-N}$ as $|y'| \rightarrow \infty$. Indeed, applying the Lebesgue dominated convergence theorem and Proposition A.1(3), we obtain

$$\lim_{|y'| \rightarrow \infty} f_1(y') |y'|^N = \lim_{|y'| \rightarrow \infty} \int_{B_R^*} K^x(y') |y'|^N dx = \int_{B_R^*} C'_N x_N dx < \infty.$$

This fact, together with (47), yields, $\int_{|y'| > S} f_1(y') dv(y') < \infty$ for S large.

To prove that the remaining integral is also finite, it is sufficient to show that f_1 is locally bounded. Consider the expression

$$f_1(y') = C'_N \int_{|x'| < R} \int_0^R \frac{x_N}{(x_N^2 + |x' - y'|^2)^{N/2}} dx_N dx' = C'_N \int_{|x'| < R} \gamma(|x' - y'|) dx',$$

where we define

$$\gamma(|x' - y'|) := \int_0^R \frac{x_N}{(x_N^2 + |x' - y'|^2)^{N/2}} dx_N.$$

Thus, we have $f_1 = C'_N \chi_{B_R} * \gamma$, where the convolution is taken in $\mathbb{R}^{N-1}$. Since $\chi_{B_R} \in L^\infty(\mathbb{R}^{N-1})$ and has bounded support, if $\gamma \in L^1_{loc}(\mathbb{R}^{N-1})$, then the claim $f_1 \in L^\infty_{loc}(\mathbb{R}^{N-1})$ follows by standard arguments. Explicitly, in the case $N = 2$, we have

$$\gamma(|x' - y'|) = \int_0^R \frac{x_N}{(x_N^2 + |x' - y'|^2)} dx_N = \frac{1}{2} \ln \frac{R^2 + |x' - y'|^2}{|x' - y'|^2}.$$

For $N \geq 3$, we have

$$\begin{aligned} \gamma(|x' - y'|) &= \int_0^R \frac{x_N}{(x_N^2 + |x' - y'|^2)^{N/2}} dx_N \\ &= \frac{1}{N-2} \left(\frac{1}{|x' - y'|^{N-2}} - \frac{1}{(R^2 + |x' - y'|^2)^{(N-2)/2}} \right). \end{aligned}$$

This completes the proof of (46) for $j = 1$.

The proof of (46) for $j = 2$ follows the same steps as before. For $y \in \mathbb{R}_+^N$, let $f_2(y) := \int_{B_R^*} G^x(y) dx$. We have

$$\begin{aligned} \int_{B_R^*} u_2(x) dx &= \int_{B_R^*} \int_{\mathbb{R}_+^N} G^x(y) d\mu(y) dx = \int_{\mathbb{R}_+^N} \left(\int_{B_R^*} G^x(y) dx \right) d\mu(y) \\ &= \int_{\mathbb{R}_+^N} f_2(y) d\mu(y) = \int_{|y| \leq S} f_2(y) d\mu(y) + \int_{|y| > S} f_2(y) d\mu(y), \end{aligned}$$

where $S > 0$ will be chosen later.

Next, observe that $f_2(y) \approx y_N/|y|^N$ as $|y| \rightarrow \infty$. Indeed, since $G^x(y) \approx y_N|y|^{-N}$ as $|y| \rightarrow \infty$ (see 2. of Proposition A.1), by the dominated convergence theorem, we obtain

$$\lim_{|y| \rightarrow \infty} f_2(y)|y|^N/y_N = \lim_{|y| \rightarrow \infty} \int_{B_R^*} G^x(y)|y|^N/y_N dx = \int_{B_R^*} C'_N x_N dx < \infty.$$

Since

$$\int_{\mathbb{R}_+^N} \frac{y_N}{1 + |y|^N} d\mu(y) < \infty, \quad (48)$$

we have

$$\int_{|y| > S} f_2(y) d\mu(y) < \infty \quad \text{for } S \text{ large.}$$

Moreover,

$$f_2(y) = \int_{B_R^*} G^x(y) dx = \int_{B_R^*} \Gamma^x(y) dx - \int_{B_R^*} \Gamma^{\hat{x}}(y) dx = \chi_{B_R^*} * \Gamma^y - \chi_{B_R^*} * \Gamma^x(\hat{y}),$$

where f_2 is the sum of two functions which are convolutions of $\chi_{B_R^*}$ with locally integrable kernels. Hence, f_2 is locally bounded. This concludes the proof of (46) for $j = 2$.

To prove that u is a weak solution of (4), we choose $\varphi \in \mathbf{D}_0$ and we aim to show that (13) holds with equality sign. This requires us to compute the following integrals

$$\int_{\mathbb{R}_{+,x}^N} \left((-\Delta_x \varphi)(x) \int_{\mathbb{R}_{+,y}^N} G^x(y) d\mu(y) \right) dx + \int_{\mathbb{R}_{+,x}^N} \left((-\Delta_x \varphi)(x) \int_{\partial \mathbb{R}_{+,y'}^N} K^x(y') dv(y') \right) dx.$$

Using Fubini-Tonelli theorem, the symmetry $G^x(y) = G^y(x)$ and the fact that G is the Green function of $-\Delta$ on $\mathbb{R}_+^N$, we compute the first integral as follows:

$$\begin{aligned} \int_{\mathbb{R}_{+,x}^N} \int_{\mathbb{R}_{+,y}^N} (-\Delta_x \varphi)(x) G^x(y) d\mu(y) dx &= \int_{\mathbb{R}_{+,y}^N} \left(\int_{\mathbb{R}_{+,x}^N} (-\Delta_x \varphi)(x) G^y(x) dx \right) d\mu(y) \\ &= \int_{\mathbb{R}_{+,y}^N} \varphi(y) d\mu(y). \end{aligned}$$

Since this integral is finite for any $\varphi \in \mathbf{D}_0$, we can conclude that $\mu \in \mathcal{M}_{loc}(\overline{\mathbb{R}_+^N}, x_N)$ (see Remark 2.2).

Similarly, for the second integral, we have:

$$\begin{aligned} &\int_{\mathbb{R}_{+,x}^N} \int_{\partial \mathbb{R}_{+,y'}^N} (-\Delta_x \varphi)(x) K^x(y') dv(y') dx \\ &= \int_{\partial \mathbb{R}_{+,y'}^N} \left(\int_{\mathbb{R}_{+,x}^N} (-\Delta_x \varphi)(x) \frac{\partial}{\partial y_N} G(x, y) |_{y_N=0} dx \right) dv(y') \\ &= \int_{\partial \mathbb{R}_{+,y'}^N} \frac{\partial}{\partial y_N} \left(\int_{\mathbb{R}_{+,x}^N} (-\Delta_x \varphi)(x) G(x, y) dx \right) |_{y_N=0} dv(y') \\ &= \int_{\partial \mathbb{R}_{+,y'}^N} \frac{\partial}{\partial y_N} (\varphi(y)) |_{y_N=0} dv(y') = \int_{\partial \mathbb{R}_{+,y'}^N} \varphi_N(y', 0) dv(y'). \end{aligned}$$

Here $\varphi_N(y', 0)$ denotes the normal derivative of φ at $y_N = 0$. By summing the results of these integrals, we conclude that u satisfies the weak formulation of the equation (4), confirming that u is indeed a weak solution.

To complete the proof of Theorem 3.1.B, we turn now to the transformation proposed by Huber [14], following an idea of Weinstein. Define a transformation of the function $v(\xi) = H[u(x)]$ in the extended space $\mathbb{R}^{N+2}$, where $\xi = (\xi_1, \dots, \xi_{N-1}, \xi_N, \xi_{N+1}, \xi_{N+2}) = (\xi', \bar{\xi}) \in \mathbb{R}^{N+2}$, with the notation $\xi' = (\xi_1, \dots, \xi_{N-1}) \in \mathbb{R}^{N-1}$, $\bar{\xi} = (\xi_N, \xi_{N+1}, \xi_{N+2}) \in \mathbb{R}^3$, and set

$$S := \{\xi \in \mathbb{R}^{N+2}, |\bar{\xi}| = 0\}.$$

Let $v(\xi) = H[u(x)]$ be defined on $\mathbb{R}^{N+2}$ by

$$v(\xi) := \begin{cases} \frac{u(\xi_1, \xi_2, \dots, \xi_{n-1}, (\xi_n^2 + \xi_{n+1}^2 + \xi_{n+2}^2)^{1/2})}{(\xi_n^2 + \xi_{n+1}^2 + \xi_{n+2}^2)^{1/2}} & \text{for } \xi \notin S, \\ \liminf_{\eta \rightarrow \xi, \eta \notin S} v(\eta) & \text{for } \xi \in S. \end{cases} \quad (49)$$

Lemma 3.16 ([14]) *If u is superharmonic on $\mathbb{R}_+^N$ and*

$$\liminf_{x \rightarrow y} u(x) \geq 0 \quad \text{for all } y \in \partial \mathbb{R}_+^N, \quad (50)$$

then v is superharmonic on $\mathbb{R}^{N+2}$.

Conversely, if v is superharmonic on $\mathbb{R}^{N+2}$ and is symmetric with respect to the subspace S^4 , then the function,

$$u(x) = x_N v(x_1, \dots, x_{N-1}, x_N, 0, 0)$$

is superharmonic on $\mathbb{R}_+^N$ and satisfies (50).

Notice that in the Lemma above, there is no assumption on the sign of u (or, equivalently, of v).

Completion of the Proof of Theorem 3.1.B

We need to show that $(\mathcal{R}^+)$ holds. By replacing $u(x)$ with $u(x) - hx_N$, i.e. assuming that $h = 0$, it follows that u is a nonnegative and superharmonic weak solution of (4).

Let $v(\xi) = H[u]$. Since u is nonnegative, we deduce that v is a nonnegative superharmonic function in $\mathbb{R}^{N+2}$. Let $c := \inf v \geq 0$. It is known that (see [6])

$$\lim_R \frac{1}{R^{N+2}} \int_{B_R^{e,n+2}(\eta_0)} (v(\xi) - c) d\xi = 0, \quad \text{for a.e. } \eta_0 \in \mathbb{R}^{N+2},$$

where $B_R^{e,n+2}(\eta_0)$ is the Euclidean ball in $\mathbb{R}^{N+2}$ of radius R centered at $\eta_0 \in \mathbb{R}^{N+2}$. In the integral above, the Euclidean ball can be replaced by the ball $B_R^{*,N+2}(\eta_0)$ associated with the norm

$$|\xi|_* = |(\xi_1, \dots, \xi_{N-1}, \xi_N, \xi_{N+1}, \xi_{N+2})|_* = |(\xi', \bar{\xi})|_* := \max\{|\xi'|, |\bar{\xi}|\}.$$

Here, we have used a notation similar to the one we have used in $\mathbb{R}_+^N$. Now, fix $x = (x', x_N) \in \mathbb{R}_+^N$ and set $\eta_0 := (\eta'_0, \bar{\eta}_0) = (x', x_N, 0, 0)$, so that $\eta'_0 = x'$, $\bar{\eta}_0 = (x_N, 0, 0)$ and $|\bar{\eta}_0| = x_N$. Therefore, we have

$$\lim_R \frac{1}{R^{N+2}} \int_{B_R^{*,N+2}(\eta_0)} (v(\xi) - c) d\xi = \lim_R \frac{1}{R^{N+2}} \int_{|\xi' - \eta'_0| < R} \int_{|\bar{\xi} - \bar{\eta}_0| < R} (v(\xi) - c) d\bar{\xi} d\xi' = 0. \quad (51)$$

Let us estimate the last integral for large R . Let $R \geq 2|\bar{\eta}_0|$ and set $\delta := |\bar{\eta}_0|/R$, $\tau := \sqrt{1 - \delta^2}$. In $\mathbb{R}^3$, with this choice, we have the inclusion

$$\{\bar{\xi} : |\bar{\xi}| < \tau R, \xi_N > 0\} \subset B_R^{e,3}(\bar{\eta}_0) = \{\bar{\xi} : |\bar{\xi} - \bar{\eta}_0| < R\}.$$

Recalling that $v(\xi) = v(\xi', |\bar{\xi}|)$, we obtain:

$$\begin{aligned} \int_{|\bar{\xi} - \bar{\eta}_0| < R} |v(\xi) - c| d\bar{\xi} &\geq \int_{\substack{\xi_N > 0 \\ |\bar{\xi}| < \tau R}} |v(\xi', |\bar{\xi}|) - c| d\bar{\xi} = \frac{\sigma_3}{2} \int_0^{\tau R} |v(\xi', r) - c| r^2 dr \\ &= \frac{\sigma_3}{2} \int_0^{\tau R} |u(\xi', r) - cr| r dr \geq \frac{\sigma_3}{2} \int_{\substack{y_N > 0 \\ |y_N - x_N| < \gamma R}} |u(\xi', y_N) - cy_N| y_N dy_N \end{aligned}$$

where $\gamma > 0$ is such that $\{y_N : |y_N - x_N| < \gamma R, y_N > 0\} \subset [0, \tau R]$. This is possible whenever $\gamma R + x_N \leq \tau R$, which holds for $0 < \gamma \leq \tau - \delta = \sqrt{1 - \delta^2} - \delta$.

⁴ This means that it depends only on $\xi_1, \xi_2, \dots, \xi_{N-1}$ and $(\xi_N^2 + \xi_{N+1}^2 + \xi_{N+2}^2)^{1/2}$.

Therefore, for any $0 < \gamma \leq \sqrt{1 - \frac{x_N^2}{R^2}} - \frac{x_N}{R}$, or equivalently for any $0 < \gamma < 1$ and $R > 2 \max\{1, (\sqrt{1 - \gamma^2} - \gamma)^{-1}\}x_N$,⁵ we have

$$\begin{aligned} \int_{B_R^{*,N+2}(\eta_0)} |v(\xi) - c| d\xi &= \int_{|\xi' - \eta'_0| < R} \int_{|\bar{\xi} - \bar{\eta}_0| < R} |v(\xi) - c| d\bar{\xi} d\xi' \\ &\geq \frac{\sigma_3}{2} \int_{|\xi' - \eta'_0| < \gamma R} d\xi' \int_{\substack{y_N > 0 \\ |y_N - x_N| < \gamma R}} |u(\xi', y_N) - cy_N| y_N dy_N \\ &= \frac{\sigma_3}{2} \int_{y \in B_{\gamma R}^*(x) \cap \mathbb{R}_+^N} |u(y) - cy_N| y_N dy. \end{aligned} \quad (52)$$

Relation (52), combined with (51) and the fact that $v \geq c$, implies by rescaling the parameter R that u satisfies $(\mathcal{R}^+)$ with a constant $c \geq 0$. We aim to prove that $c = 0$. Since u is a weak solution of (4) and satisfies $(\mathcal{R}^+)$ with a constant $c \geq 0$, u can be written as:

$$u(x) = cx_N + \int_{\partial \mathbb{R}_+^N} K^x(y') dv(y') + \int_{\mathbb{R}_+^N} G^x(y) d\mu(y).$$

However, given that u is defined by (25) with $h = 0$, it follows that $c = 0$. Finally, we observe that (52), along with (51) and the fact that $v \geq c$, confirm the statement 3.1.B.(d).

Proof of 3.1.B.(a). From the above argument, it is evident that the infimum of v is the constant h appearing in the representation of u .

Therefore, by definition of v , we have:

$$h = \inf_{\mathbb{R}^{N+2}} v = \inf_{\mathbb{R}^{N+2}} \frac{u(\xi', |\bar{\xi}|)}{|\bar{\xi}|}.$$

Proof of 3.1.B.(b). Let $\tilde{\Omega} \subset \mathbb{R}^{N+2}$ be defined as $\tilde{\Omega} := \{\xi \in \mathbb{R}^{N+2} : (\xi', |\bar{\xi}|) \in \Omega\}$. If $h = \inf_{\Omega} \frac{u(x)}{x_N}$, then $h = \inf_{\tilde{\Omega}} v$. Since v is superharmonic on the whole $\mathbb{R}^{N+2}$ and $\tilde{\Omega}$ is bounded, it follows that $v \equiv h$, thus the claim is proved.

Proof of 3.1.B.(c). To establish the estimate (26), we begin by noting that at least one of the measures μ or ν must be nontrivial. Without loss of generality we may assume that $h = 0$. We first consider the case where $\mu \not\equiv 0$ and let R_0 be such that $\mu(B_{R_0}^*) > 0$. Since $G^x(y) = G^y(x)$, from part 2 of Proposition A.1, it follows that, $G^x(y) \geq c \frac{x_N}{|x|^N}$ for large $|x|$ and for any $y \in B_{R_0}^*$. Therefore,

$$u(x) \geq \int_{\mathbb{R}_+^N} G^x(y) d\mu(y) \geq \int_{B_{R_0}^*} G^x(y) d\mu(y) \geq \int_{B_{R_0}^*} c \frac{x_N}{|x|^N} d\mu(y) \geq c \mu(B_{R_0}^*) \frac{x_N}{|x|^N}.$$

Now consider the case $\nu \not\equiv 0$ and let R_0 be such that $\nu(B'_{R_0}) > 0$. For $y' \in B'_{R_0}$, we have

$$|x' - y'| \leq |x'| + R_0 \leq \sqrt{2} \sqrt{|x'|^2 + R_0^2}.$$

Hence,

$$x_N^2 + |x' - y'|^2 \leq 2(x_N^2 + |x'|^2 + R_0^2),$$

which implies

$$K^x(y') \geq c \frac{x_N}{(x_N^2 + |x'|^2 + R_0^2)^{N/2}}, \quad \text{for } |y'| < R_0.$$

⁵ For instance, choosing $\gamma = 1/2$, (52) holds for any $R > \frac{5}{2}x_N$.

Therefore,

$$u(x) \geq \int_{\partial \mathbb{R}_+^N} K^x(y') dv(y') \geq \int_{B'_{R_0}} c \frac{x_N}{(x_N^2 + |x'|^2 + R_0^2)^{N/2}} dv(y') = cv(B'_{R_0}) \frac{x_N}{(|x|^2 + R_0^2)^{N/2}}.$$

In summary, the claim is proved for $|x|$ large, say for $|x| > R_1$.

Since 3.1.B.(a) and 3.1.B.(b) hold, the claim for any x follows by observing that, $\inf_{B_{R_1}} \frac{u(x)}{x_N} > 0 = h$.

3.4 Proof of Theorem 3.12

Proof Let $w := u - v$. Since $-\Delta w \geq 0$ in the distributional sense, there exists a positive Radon measure μ such that $-\Delta w = \mu$ and

$$\liminf_{x \rightarrow (y', 0)} w(x) \geq 0, \quad \text{for } y' \in \partial \mathbb{R}_+^N.$$

Using the same notations as on page 24, set $z := H[w]$ as in (49). From Lemma 3.16, it follows that z is superharmonic in $\mathbb{R}^{N+2}$. Fix $x = (x', x_N) \in \mathbb{R}_+^N$ and let $\eta_0 = (\eta'_0, \bar{\eta}_0) := (x', x_N, 0, 0)$. Arguing as in (52) for $R > \frac{5}{2}x_N$, we have

$$\int_{B_{2R}^{*,N+2}(\eta_0)} |z(\xi)| d\xi \geq \frac{\sigma_3}{2} \int_{B_R^*(x) \cap \mathbb{R}_+^N} |w(y)| y_N dy.$$

Now, since z is superharmonic, we know that $z \in L^1_{loc}(\mathbb{R}^{N+2})$. Hence we deduce that the above integrals are finite and $x_N |w| \in L^1_{loc}(\overline{\mathbb{R}_+^N})$. This completes the proof of the claim.

Next we prove the equivalence between (36) and (37), i.e., the equivalence between $w \geq 0$ and the fact that w satisfies $(\mathcal{R}^+)$ with $h \geq 0$.

Step 1 : Assume $w \geq 0$. Since w is a superharmonic function, by Theorem 1.2, it follows that w can be represented by (25). An application of Theorem 3.1.B completes the proof of the claim, implying $w \in L^1_{loc}(\overline{\mathbb{R}_+^N})$.

Step 2: Assume w satisfies $(\mathcal{R}^+)$ with $h \geq 0$. Without loss of generality we may suppose that w satisfies $(\mathcal{R}^+)$. Notice that if $z := H[w]$ is nonnegative, then $w \geq 0$. To prove that $z \geq 0$, we apply Theorem 1.1 to the function z with $l = 0$ in $\mathbb{R}^{N+2}$. Let $1 > \tau > 1/\sqrt{2}$ be fixed. Consider $x = (x', x_N) \in \mathbb{R}_+^N$ and let $\eta_0 = (\eta'_0, \bar{\eta}_0) := (x', x_N, 0, 0)$. Our claim is that:

$$\liminf \frac{1}{R^{N+2}} \int_{\frac{\sqrt{2}R}{\tau} < |\eta_0 - \xi| < 2R} |z(\xi)| d\xi = 0.^6$$

Noticing that $B_{2R}^{*,N+2}(\eta_0) \supset B_{2R}^{e,N+2}(\eta_0)$ and $B_R^{*,N+2}(\eta_0) \subset B_{\sqrt{2}R}^{e,N+2}(\eta_0)$, we have

$$\int_{\frac{\sqrt{2}R}{\tau} < |\eta_0 - \xi| < 2R} |z(\xi)| d\xi \leq \int_{B_{2R}^{*,N+2}(\eta_0) \setminus B_{R/\tau}^{*,N+2}(\eta_0)} |z(\xi)| d\xi.$$

Therefore, it suffices to prove that for $R > \frac{2\tau}{1-\tau}x_N$, there holds:

$$\int_{B_{2R}^{*,N+2}(\eta_0) \setminus B_{R/\tau}^{*,N+2}(\eta_0)} |z(\xi)| d\xi \leq 2\sigma_3 \int_{\substack{y_N > 0 \\ R < |x-y|_* < 2R}} |u(y)| y_N dy. \quad (53)$$

⁶ Obviously, in condition $(\mathcal{R})$ the annulus $B_{2R} \setminus B_R$ can be replaced by the annulus $B_{\gamma R} \setminus B_R$ with any $\gamma > 1$.

The claim will follow from the hypothesis $(\mathcal{R}_0^+)$.

Step 3: Proof of (53). We split the integration domain as $B_{2R}^{*,N+2}(\eta_0) \setminus B_{R/\tau}^{*,N+2}(\eta_0) = A \cup B$ where: $A := \{|\bar{\xi} - \bar{\eta}_0| < R/\tau, R/\tau \leq |\xi' - \eta'_0| < 2R\}$ and $B := \{R/\tau \leq |\bar{\xi} - \bar{\eta}_0| < 2R, |\xi' - \eta'_0| < 2R\}$. Since

$$\int_A |z(\xi)| d\xi = \int_{R/\tau \leq |\xi' - \eta'_0| < 2R} d\xi' \int_{|\bar{\xi} - \bar{\eta}_0| < R/\tau} |z(\xi', \bar{\xi})| d\bar{\xi}, \quad (54)$$

and considering that $z(\xi', \bar{\xi}) = z(\xi', |\bar{\xi}|)$, we have:

$$\begin{aligned} \int_{|\bar{\xi} - \bar{\eta}_0| < R/\tau} |z(\xi', \bar{\xi})| d\bar{\xi} &\leq \int_{|\bar{\xi}| < R/\tau + |\eta_0|} |z(\xi', \bar{\xi})| d\bar{\xi} = \sigma_3 \int_0^{R/\tau + |\eta_0|} |z(\xi', r)| r^2 dr \\ &= \sigma_3 \int_{\substack{y_N > 0 \\ |y_N - x_N| < R/\tau}} |u(\xi', y_N)| y_N dy_N \leq \sigma_3 \int_{\substack{y_N > 0 \\ |y_N - x_N| < 2R}} |u(\xi', y_N)| y_N dy_N, \end{aligned}$$

where in the last identity, we have used the fact that $R > \tau x_N$. Therefore, from (54), we can conclude that:

$$\begin{aligned} \int_A |z(\xi)| d\xi &\leq \sigma_3 \int_{R \leq |\xi' - \eta'_0| < 2R} d\xi' \int_{\substack{y_N > 0 \\ |y_N - x_N| < 2R}} |u(\xi', y_N)| y_N dy_N \\ &\leq \sigma_3 \int_{\substack{y_N > 0 \\ R < |x - y|_* < 2R}} |u(y)| y_N dy. \end{aligned}$$

Similarly, for region B , we have:

$$\int_B |z(\xi)| d\xi = \int_{|\xi' - \eta'_0| < 2R} d\xi' \int_{R/\tau \leq |\bar{\xi} - \bar{\eta}_0| < 2R} |z(\xi', \bar{\xi})| d\bar{\xi}. \quad (55)$$

Since $\{R/\tau \leq |\bar{\xi} - \bar{\eta}_0| < 2R\} \subset \{R/\tau - x_N < |\bar{\xi}| < 2R + x_N\}$, we obtain:

$$\begin{aligned} \int_{R/\tau \leq |\bar{\xi} - \bar{\eta}_0| < 2R} |z(\xi', \bar{\xi})| d\bar{\xi} &\leq \int_{R/\tau - x_N < |\bar{\xi}| < 2R + x_N} |z(\xi', \bar{\xi})| d\bar{\xi} \\ &= \sigma_3 \int_{R/\tau - x_N}^{2R + x_N} |z(\xi', r)| r^2 dr = \sigma_3 \int_{R/\tau - 2x_N < y_N - x_N < 2R} |u(\xi', y_N)| y_N dy_N \\ &\leq \sigma_3 \int_{R < y_N - x_N < 2R} |u(\xi', y_N)| y_N dy_N, \end{aligned}$$

where we used the fact that $R > 2\tau x_N$ and $R > \frac{2\tau}{1-\tau} x_N$. Therefore, from (55), we can conclude that:

$$\int_B |z(\xi)| d\xi \leq \sigma_3 \int_{|\xi' - \eta'_0| < 2R} d\xi' \int_{R < y_N - x_N < 2R} |u(\xi', y_N)| y_N dy_N \leq \sigma_3 \int_{\substack{y_N > 0 \\ R < |x - y|_* < 2R}} |u(y)| y_N dy.$$

This completes the proof. $\square$

4 Another integral representation formula

In this section we prove some representation formulae concerning regular superharmonic functions in the half-space $\mathbb{R}_+^N$. These results can be deduced from some theorems contained in [8]. In what follows μ stands for a continuous function defined on $\mathbb{R}_+^N$.

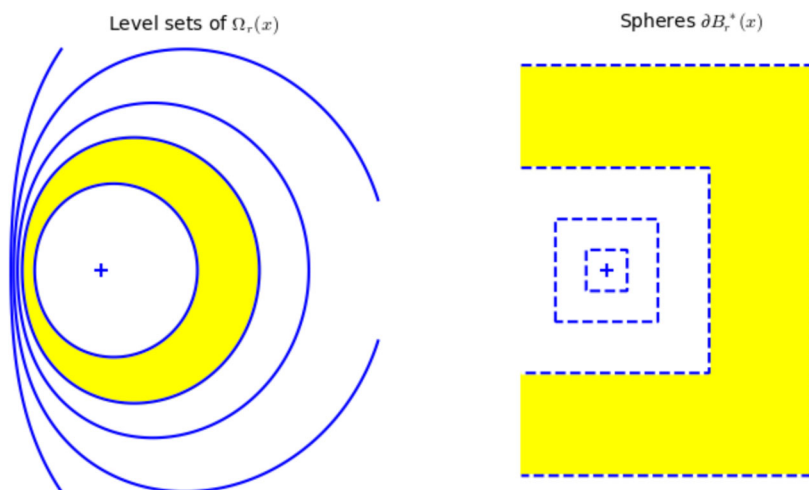


Fig. 2 On left: In blue the level sets $\partial\Omega_r(x)$ with $x = (0, 5)$ and $r = 1, 2, 3, 4, 5$. The yellow filled region is the “annulus” $\Omega_2(x) \setminus \Omega_1(x)$. On right: In dotted line the surface of the ball $B_r^*(0, 5)$ (in the $|\cdot|_*$ norm) for $r = 1, 3, 6, 12$. The yellow filled region is the “annulus” $A_6^*(x) = B_{12}^*(x) \setminus B_6^*(x)$

The possibility to represent a superharmonic function u in the whole space $\mathbb{R}^N$ or on the half-space $\mathbb{R}_+^N$ with an integral formula, is linked to the asymptotic behavior of some weighted integrals of the function u on some rings, see Theorem 1.1 and 3.1 respectively.

Let $x \in \mathbb{R}_+^N$ and $r > 0$. We set

$$\Omega_r(x) := \left\{ y \in \mathbb{R}_+^N \mid G^x(y) > \frac{1}{r} \right\} \cup \{x\}.$$

Throughout this section we shall call the set $\Omega_{2R}(x) \setminus \Omega_R(x)$ the standard annulus. See Fig. 2.

Notice also that the rings of condition $(\mathcal{R})$ are modeled in a similar way modulo a rescaling. Indeed the integration domain appearing in $(\mathcal{R})$ is given by $B_{2R}(x) \setminus B_R(x)$. Notice that

$$B_r(x) = \left\{ y \in \mathbb{R}_+^N \mid \Gamma^x(y) > \frac{C_N}{r^{N-2}} \right\} \cup \{x\}.$$

Theorem 4.1 *Let $u \in \mathcal{C}^2(\mathbb{R}_+^N)$ be such that $-\Delta u =: \mu \geq 0$.*

A. Let $x \in \mathbb{R}_+^N$ and assume that

$$l_x := \frac{1}{\ln 2} \liminf_{R \rightarrow +\infty} \int_{\Omega_{2R}(x) \setminus \Omega_R(x)} \frac{|\nabla G^x(y)|^2}{G^x(y)} u(y) dy \in \mathbb{R}, \quad (56)$$

then

$$u(x) = l_x + \int_{\mathbb{R}_+^N} G^x(y) \mu(y) dy. \quad (57)$$

Therefore assuming that (56) holds for any $x \in \mathbb{R}_+^N$, and setting $l(x) := l_x$, we deduce that,

(a) $\inf u = \inf l$ (finite or infinite), and the following alternative holds,

$$\text{either } u(x) > l(x), \forall x \in \mathbb{R}_+^N, \quad \text{or} \quad u \equiv l \text{ and } \mu \equiv 0. \quad (58)$$

(b) l is harmonic in $\mathbb{R}_+^N$.

(c) If l_x does not depend on x , i.e. $l_x = l \in \mathbb{R}$, then for any $x \in \mathbb{R}_+^N$

$$u(x) = l + \int_{\mathbb{R}_+^N} G^x(y) \mu(y) dy. \quad (59)$$

B. If u is bounded from below, then (56) is fulfilled for any $x \in \mathbb{R}_+^N$. Hence the claims in A. hold true.

C. If there exists a harmonic function $H \in C^2(\mathbb{R}_+^N)$ such that

$$u(x) = H(x) + \int_{\mathbb{R}_+^N} G^x(y) \mu(y) dy,$$

then for any $x \in \mathbb{R}_+^N$

$$H(x) = \frac{1}{\ln 2} \liminf_{R \rightarrow +\infty} \int_{\Omega_{2R}(x) \setminus \Omega_R(x)} \frac{|\nabla G^x(y)|^2}{G^x(y)} u(y) dy. \quad (60)$$

D Let $x \in \mathbb{R}_+^N$ and $l_x \in \mathbb{R}$. Then

$$\liminf_{R \rightarrow +\infty} \int_{\Omega_{2R}(x) \setminus \Omega_R(x)} \frac{|\nabla G^x(y)|^2}{G^x(y)} |u(y) - l_x| dy = 0, \quad (61)$$

if and only if

$$\liminf_{R \rightarrow +\infty} R \int_{\Omega_{2R}(x) \setminus \Omega_R(x)} \frac{1}{|x - y|^{2N}} |u(y) - l_x| dy = 0. \quad (62)$$

Moreover, if one of these assumptions is satisfied, then (56) holds.

Remark 4.2 The existence of a harmonic function H in the hypothesis of point C. above, is guaranteed by the Riesz decomposition theorem provided there exists a subharmonic minorant of u . See also [1, Theorem 4.4.1] and the comment preceding Theorem 3.11.

The following result is a consequence of Theorem 14 and Remark 17 of [8]. Set

$$X := \{u \in C^2(\overline{\mathbb{R}_+^N}) : u(x', 0) = 0, \text{ and } \liminf_{x_N \rightarrow \infty} \frac{u(x)}{x_N} = 0\}.$$

Theorem 4.3 The following statements hold.

A. Let $u \in X$. Then $-\Delta u =: \mu \geq 0$ and $\inf u = 0$ if and only if

$$u(x) = \int_{\mathbb{R}_+^N} G^x(y) \mu(y) dy \quad \forall x \in \mathbb{R}_+^N.$$

B. Let $u \in X$ be such that $-\Delta u \geq 0$. Then, $\inf u = 0$ if and only if

$$\liminf_{R \rightarrow +\infty} \int_{\Omega_{2R}(x) \setminus \Omega_R(x)} \frac{|\nabla G^x|^2}{G^x} u dy = 0 \quad \forall x \in \mathbb{R}_+^N, \quad (63)$$

if and only if

$$\lim_{R \rightarrow +\infty} \int_{\Omega_{2R}(x) \setminus \Omega_R(x)} \frac{|\nabla G^x|^2}{G^x} |u| dy = 0 \quad \forall x \in \mathbb{R}_+^N, \quad (64)$$

if and only if

$$\liminf_{R \rightarrow +\infty} \int_{\partial\Omega_R(x)} |\nabla G^x| u dH_{N-1} = 0 \quad \forall x \in \mathbb{R}_+^N, \quad (65)$$

if and only if

$$\lim_{R \rightarrow +\infty} \int_{\partial\Omega_R(x)} |\nabla G^x| |u - l| dH_{N-1} = 0 \quad \forall x \in \mathbb{R}_+^N, \quad (66)$$

if and only if

$$l = \lim_{R \rightarrow +\infty} \frac{1 + \alpha}{R^{1+\alpha}} \int_{\Omega_R(x)} \frac{|\nabla G^x|^2}{(G^x)^{2+\alpha}} u dy \quad \alpha > -1 \quad \forall x \in \mathbb{R}_+^N. \quad (67)$$

C. Let $u \in X$. Suppose that u is bounded from below and $-\Delta u \geq 0$. Set

$$c_x := \liminf_{R \rightarrow +\infty} \frac{1}{\ln 2} \int_{\Omega_{2R}(x) \setminus \Omega_R(x)} \frac{|\nabla G^x|^2}{G^x} u dy \quad \forall x \in \mathbb{R}_+^N, \quad (68)$$

then c_x does not depend on x and $c_x = 0$.

Proof of Theorem 4.1 The theorem is a consequence of Remark 17 and Theorem 6 of [8]. Indeed, the Green function on the half-space $\mathbb{R}_+^N$ satisfies the assumptions listed in [8] H1.–H7.

The statements A, Aa, Ac, B and C are a direct consequence of Theorem 6 in [8].

The claim in Ab follows again from Theorem 6 in [8] provided

$$w(x) := \int_{\mathbb{R}_+^N} G^x(y) \mu(y) dy \in L_{loc}^1(\mathbb{R}_+^N).$$

This is easily seen to be true since w is a superharmonic function.

Proof of D. First we observe that the equivalence between (61) and (62) is a consequence of the following argument. For $x, y \in \mathbb{R}_+^N$, $x \neq y$, from the estimates (88) and (89) we have

$$\left(\frac{C'_N}{2}\right)^2 x_N^2 \leq |\nabla G^x(y)|^2 |x - y|^{2N} \leq C_N'^2 x_N^2 [(1 + N)^2 + N^2].$$

Let $x \in \mathbb{R}_+^N$ be fixed, let $R > 0$ and consider $\Omega_{2R}(x) \setminus \Omega_R(x)$. If $y \in \Omega_{2R}(x) \setminus \Omega_R(x)$ we have $2R > \frac{1}{G^x(y)} > R$. Hence, for $y \in \Omega_{2R}(x) \setminus \Omega_R(x)$ we deduce

$$\frac{|\nabla G^x(y)|^2}{G^x(y)} \approx \frac{R}{|x - y|^{2N}}.$$

This completes the proof of the equivalence between (61) and (62).

Finally, the implication (61) $\Rightarrow$ (56) follows from Lemma 26 in [8].

Proof of Theorem 4.1 From Theorem 14 and Remark 17 of [8] it follows that the statements A., B. and C. are equivalent to the following classical Liouville property:

Let $u \in X$. If u is harmonic and bounded from below, then u is constant.

Indeed let $c > 0$ be such that $u + c > 0$. By Theorem 1.2 it follows that

$$u(x) + c = hx_N + \int_{\partial\mathbb{R}_+^N} K^x(y') dv(y'),$$

where ν is a suitable positive Radon measure and $h \geq 0$. Since $u \in X$ we have $\liminf_{x_N \rightarrow \infty} u(x)/x_N = 0$. Hence $h = 0$. In addition, by the fact that $u(x', x_N) \rightarrow 0$ as $x_N \rightarrow 0$ it follows that

$$\int_{\partial \mathbb{R}_+^N} K^x(y') d\nu(y') \rightarrow c \quad \text{as } x_N \rightarrow 0, \quad \forall x' \in \mathbb{R}^{N-1}.$$

In other words the harmonic function $w(x) := \int_{\partial \mathbb{R}_+^N} K^x(y') d\nu(y')$ has boundary value identically equal to the constants c . Therefore $w \equiv c$ on $\mathbb{R}_+^N$, that is, $u \equiv 0$.

Appendix A Estimates related to Poisson kernel and Green function

Here we collect a few results related to the fundamental solution of $-\Delta$ on $\mathbb{R}_+^N$, to the Poisson kernel, and its Green function.

Let $R > 0$ and set,

$$B_R^*(x) := \left\{ y \in \mathbb{R}_+^N : \max\{|y_N - x_N|, |y' - x'|\} < R \right\},$$

the “cylindrical” ball in $\mathbb{R}_+^N$ centered at $x \in \mathbb{R}_+^N$ and radius R generated by the norm $|\cdot|_*$ defined in (6).

In this paper G and K are defined as follows. Let $\Gamma^x(y)$ be the fundamental solution of $-\Delta$ at x , that is, $-\Delta_y \Gamma^x(y) = \delta_x$ where δ_x is the Dirac measure at x . It follows that,

$$\Gamma^x(y) = \begin{cases} \frac{C_N}{|x-y|^{N-2}}, & \text{for } N \geq 3, \\ -C_2 \ln|x-y|, & \text{for } N = 2, \end{cases} \quad \text{for } x \neq y. \quad (69)$$

Let $x = (x', x_N) \in \mathbb{R}_+^N$. We set $\hat{x} = (x', -x_N)$. For $y \in \mathbb{R}_+^N \setminus \{x\}$ we put,

$$G(x, y) := G^x(y) := \Gamma^x(y) - \Gamma^{\hat{x}}(y), \quad (70)$$

and for $x \in \mathbb{R}_+^N$ and $y' \in \partial \mathbb{R}_+^N$, we define,

$$K^x(y') := \frac{C'_N x_N}{(x_N^2 + |x' - y'|^2)^{N/2}} = \frac{\partial G^x}{\partial y_N}(y', 0). \quad (71)$$

By computation, we notice the following properties of the Green's function G :

$$G^x(y) = G^y(x), \quad \forall x \neq y, \quad (72)$$

$$G^x(y', 0) = 0, \quad \forall (x', x_N) \neq (y', 0), \quad (73)$$

$$\frac{\partial G}{\partial y_N}(x', 0, y) = 0. \quad (74)$$

For further applications, we need a regularized version of the kernels G and K . For $0 \leq \epsilon < 1$, $x, y \in \mathbb{R}_+^N$, we define the regularized fundamental solution $\Gamma_\epsilon^x(y)$ and the regularized Green's function $G_\epsilon^x(y)$, as follows

$$\Gamma_\epsilon^x(y) := \begin{cases} \frac{C_N}{(\epsilon^2 + |x-y|^2)^{(N-2)/2}}, & N \geq 3, \\ -\frac{C_2}{2} \ln(\epsilon^2 + |x-y|^2), & N = 2, \end{cases}$$

and

$$G_\epsilon^x(y) := \Gamma_\epsilon^x(y) - \Gamma_\epsilon^{\hat{x}}(y). \quad (75)$$

That is,

$$G_\epsilon^x(y) := \frac{C_N}{(\epsilon^2 + |x - y|^2)^{(N-2)/2}} - \frac{C_N}{(\epsilon^2 + |\hat{x} - y|^2)^{(N-2)/2}}, \quad \text{for } N \geq 3,$$

$$G_\epsilon^x(y) := -\frac{C_2}{2} \ln(\epsilon^2 + |x - y|^2) + \frac{C_2}{2} \ln(\epsilon^2 + |\hat{x} - y|^2), \quad \text{for } N = 2.$$

For $x \in \overline{\mathbb{R}_+^N}$ and $y' \in \partial \mathbb{R}_+^N$ we define the regularized Poisson kernel as

$$K_\epsilon^x(y') := \frac{\partial G_\epsilon^x}{\partial y_N}(y', 0) = \frac{C'_N x_N}{(\epsilon^2 + x_N^2 + |x' - y'|^2)^{N/2}}.$$

These regularized kernels $G_\epsilon^x(y)$ and $K_\epsilon^x(y)$ help in dealing with potential singularities and are useful in various analytical settings. Observe that,

$$G_\epsilon^x(y) \nearrow G^x(y), \quad \text{and} \quad K_\epsilon^x(y') \nearrow K^x(y') \quad \text{as } \epsilon \rightarrow 0.$$

Let us introduce the auxiliary functions,

$$a_1 := \epsilon^2 + |\hat{x} - y|^2, \quad a_2 := \epsilon^2 + |x - y|^2.$$

It follows that G_ϵ can be written as,

$$G_\epsilon^x(y) = \frac{C_N}{a_2^{(N-2)/2}} - \frac{C_N}{a_1^{(N-2)/2}}, \quad \text{for } N \geq 3,$$

$$G_\epsilon^x(y) = \frac{C_2}{2} \ln \frac{a_1}{a_2}, \quad \text{for } N = 2.$$

Notice that

$$a_1 - a_2 = 4x_N y_N. \quad (76)$$

Therefore, in the case $N = 2$, we have

$$G_\epsilon^x(y) = \frac{C_2}{2} \ln \frac{a_1}{a_2} = \frac{C_2}{2} \ln \left(1 + \frac{a_1 - a_2}{a_2} \right) = \frac{C_2}{2} \ln \left(1 + \frac{4x_N y_N}{a_2} \right), \quad (77)$$

while for $N \geq 3$, we have

$$G_\epsilon^x(y) = \frac{C_N}{a_2^{(N-2)/2}} - \frac{C_N}{a_1^{(N-2)/2}} = C_N \frac{a_1^{(N-2)/2} - a_2^{(N-2)/2}}{a_1^{(N-2)/2} a_2^{(N-2)/2}} \quad (78)$$

$$= C_N \frac{a_1^{N-2} - a_2^{N-2}}{a_1^{(N-2)/2} a_2^{(N-2)/2} (a_1^{(N-2)/2} + a_2^{(N-2)/2})} \quad (79)$$

$$= C_N \frac{4x_N y_N (a_1^{N-3} + a_1^{N-4} a_2 + \cdots + a_2^{N-3})}{a_1^{(N-2)/2} a_2^{(N-2)/2} (a_1^{(N-2)/2} + a_2^{(N-2)/2})}. \quad (80)$$

For future reference, let us compute the derivatives of G_ϵ^x . We have,

$$\frac{\partial}{\partial y_N} G_\epsilon^x(y) = \frac{C'_N}{2} \left(\frac{y_N + x_N}{a_1^{N/2}} - \frac{y_N - x_N}{a_2^{N/2}} \right), \quad (81)$$

and for $j = 1, \dots, N-1$,

$$\frac{\partial}{\partial y_j} G_\epsilon^x(y) = \frac{C'_N}{2} \left(\frac{1}{a_1^{N/2}} - \frac{1}{a_2^{N/2}} \right) (y_j - x_j).$$

So that,

$$|\nabla G_\epsilon^x|^2 = \left(\frac{C'_N}{2} \right)^2 \left| \frac{y - \hat{x}}{a_1^{N/2}} - \frac{y - x}{a_2^{N/2}} \right|^2 \quad (82)$$

$$= \left(\frac{C'_N}{2} \right)^2 \left[\left| \frac{1}{a_1^{N/2}} - \frac{1}{a_2^{N/2}} \right|^2 |x' - y'|^2 + \left(\frac{y_N + x_N}{a_1^{N/2}} - \frac{y_N - x_N}{a_2^{N/2}} \right)^2 \right]. \quad (83)$$

This detailed computation of the gradients and their magnitudes provides a clear understanding of the behavior of the regularized Green's function and its derivatives. This will be useful in further analytical work involving these functions.

Proposition A.1 *Let $0 \leq \epsilon < 1$.*

1. *For $y, x \in \mathbb{R}_+^N$, $x \neq y$, we have*

$$0 \leq G_\epsilon^x(y) \leq C'_N \frac{x_N y_N}{(\epsilon^2 + |x - y|^2)^{N/2}}. \quad (84)$$

2. *We have,*

$$\lim_{|y| \rightarrow \infty} G_\epsilon^x(y) \frac{|y|^N}{y_N} = C'_N x_N, \quad (85)$$

uniformly with respect to ϵ and x in bounded sets.

3. *We have,*

$$\lim_{|y'| \rightarrow \infty} K_\epsilon^x(y') |y'|^N = C'_N x_N, \quad (86)$$

uniformly with respect to ϵ and x in bounded sets.

4. *For $y, x \in \mathbb{R}_+^N$, $x \neq y$, we have*

$$|\nabla G_\epsilon^x|^2(y) \leq \left(\frac{C'_N}{2} \right)^2 \frac{x_N^2}{a_2^N} \left[\left(1 + \left(\frac{a_2}{a_1} \right)^{N/2} + 2N \frac{y_N^2}{a_1} \right)^2 + 4N^2 \frac{y_N^2 |x' - y'|^2}{a_1^2} \right] \quad (87)$$

$$\leq C_N'^2 \frac{x_N^2}{(\epsilon^2 + |x - y|^2)^N} [(1 + N)^2 + N^2], \quad (88)$$

$$|\nabla G_\epsilon^x|^2(y) \geq \left(\frac{C'_N}{2} \right)^2 \frac{x_N^2}{(\epsilon^2 + |x - y|^2)^N}. \quad (89)$$

5. *For $y, x \in \overline{\mathbb{R}_+^N}$, $x \neq y$, we have*

$$|\partial_N G_\epsilon^x(y)| \leq \frac{C'_N}{2} \frac{x_N}{a_2^{N/2}} \left[1 + \left(\frac{a_2}{a_1} \right)^{N/2} + 2N \frac{y_N^2}{a_1} \right] \leq (N+1) C'_N \frac{x_N}{(\epsilon^2 + |x - y|^2)^{N/2}}. \quad (90)$$

Proof Proof of claim 1. We begin by observing that since $a_1 \geq a_2$, we have $G_\epsilon^x \geq 0$.

In the case $N = 2$, by the concavity of the logarithmic function, we obtain:

$$G_{\epsilon}^x = \frac{C_2}{2} \ln \frac{a_1}{a_2} = \frac{C_2}{2} \ln(1 + \frac{a_1 - a_2}{a_2}) \leq \frac{C_2}{2} \frac{a_1 - a_2}{a_2} = \frac{C_2}{2} \frac{4x_N y_N}{a_2},$$

which proves (84).

Let $N \geq 3$. By the convexity of the function, $t \mapsto t^{-\alpha}$ for $t > 0$ and $\alpha > 0$, we have:

$$\frac{1}{s^{\alpha}} - \frac{1}{t^{\alpha}} \leq \alpha \frac{1}{s^{\alpha+1}}(t - s), \quad \text{for } t, s > 0. \quad (91)$$

Applying (91) with $\alpha = (N - 2)/2$, $s = a_2$ and $t = a_1$, we deduce:

$$G_{\epsilon}^x = C_N \left(\frac{1}{a_2^{(N-2)/2}} - \frac{1}{a_1^{(N-2)/2}} \right) \leq C_N \frac{N-2}{2} \frac{1}{a_2^{N/2}} (a_1 - a_2) = C_N (N-2) 2 \frac{x_N y_N}{a_2^{N/2}}.$$

This completes the proof of (84) for $N \geq 3$.

Proof of claim 2. Let $C \subset \mathbb{R}_+^N$ be bounded and let $x \in C$. For $|y| > 2 \sup\{|x| : x \in C\}$, we have,

$$\sqrt{1 + |y - \hat{x}|^2} \geq \sqrt{a_1} \geq \sqrt{a_2} \geq |y - x| \geq |y| - |x| \geq |y|/2.$$

Next, let $N = 2$. Since $\frac{4x_N y_N}{a_2} \rightarrow 0$ uniformly with respect to $x \in C$ and ϵ , whenever $|y| \rightarrow \infty$, and taking into account that $\ln(1 + t) \cong t$ as $t \rightarrow 0$,⁷ from (77) it follows that

$$G_{\epsilon}^x(y) \cong \frac{C_2}{2} \frac{4x_N y_N}{a_2} \cong 2C_2 x_N \frac{y_N}{|y|^2}, \quad \text{as } |y| \rightarrow \infty,$$

completing the proof of the claim.

Next we consider the case $N \geq 3$. From (80) we have,

$$\begin{aligned} G_{\epsilon}^x(y) \frac{|y|^N}{y_N} &= 4C_N x_N \frac{(a_1^{N-3} + a_1^{N-4} a_2 + \dots + a_2^{N-3})}{|y|^{2N-6}} \frac{|y|^{2N-4}}{(a_1 a_2)^{(N-2)/2}} \frac{|y|^{N-2}}{a_1^{(N-2)/2} + a_2^{(N-2)/2}} \\ &\cong 4C_N x_N (N-2) \frac{1}{2}, \quad \text{as } |y| \rightarrow \infty, \end{aligned}$$

this completes the proof of claim 2.

Proof of claim 3. Proceeding as in the proof of claim 2., we have

$$K_{\epsilon}^x(y') |y'|^N = C_N' x_N \frac{|y'|^N}{(\epsilon^2 + x_N^2 + |x' - y'|^2)^{N/2}} \cong C_N' x_N, \quad \text{as } |y'| \rightarrow \infty.$$

Proof of claim 4. and claim 5. From the expression of the derivative of G_{ϵ}^x , it follows that we must estimate the quantity, $a_2^{-N/2} - a_1^{-N/2}$. We observe that since the function t^{α} is convex for $\alpha \geq 1$, we have

$$t^{\alpha} - s^{\alpha} \leq \alpha(t - s)t^{\alpha-1}, \quad \text{for } t, s > 0. \quad (92)$$

Taking into account (76), the latter inequality, with $t = a_1$, $s = a_2$ and $\alpha = N/2$, produces

$$\frac{1}{a_2^{N/2}} - \frac{1}{a_1^{N/2}} = \frac{a_1^{N/2} - a_2^{N/2}}{a_1^{N/2} a_2^{N/2}} \leq \frac{N}{2} (a_1 - a_2) \frac{a_1^{N/2-1}}{a_1^{N/2} a_2^{N/2}} = 2N \frac{x_N y_N}{a_1 a_2^{N/2}}.$$

⁷ $a(t) \cong b(t)$ as $t \rightarrow t_0$ means $\lim_{t \rightarrow t_0} a(t)/b(t) = 1$.

Next from (81) we have,

$$\begin{aligned} |\partial_N G_\epsilon^x(y)| &= \frac{C'_N}{2} \left(y_N \left(\frac{1}{a_2^{N/2}} - \frac{1}{a_1^{N/2}} \right) + x_N \left(\frac{1}{a_2^{N/2}} + \frac{1}{a_1^{N/2}} \right) \right) \\ &\leq \frac{C'_N}{2} \left(y_N 2N \frac{x_N y_N}{a_1 a_2^{N/2}} + x_N \left(\frac{1}{a_2^{N/2}} + \frac{1}{a_1^{N/2}} \right) \right). \end{aligned}$$

This proves the first inequality in (90), while the second one is due to the fact that, $a_1 \geq a_2$ and $a_1 \geq y_N^2$. This concludes the proof of claim 5.

Similarly from (83) we have,

$$\begin{aligned} \frac{|\nabla G_\epsilon^x|^2}{(C'_N/2)^2} &= \left| \frac{1}{a_1^{N/2}} - \frac{1}{a_2^{N/2}} \right|^2 |x' - y'|^2 + \left(y_N \left(\frac{1}{a_2^{N/2}} - \frac{1}{a_1^{N/2}} \right) + x_N \left(\frac{1}{a_2^{N/2}} + \frac{1}{a_1^{N/2}} \right) \right)^2 \\ &\leq 4N^2 \frac{x_N^2 y_N^2}{a_1^2 a_2^N} |x' - y'|^2 + \frac{x_N^2}{a_2^N} \left(1 + \left(\frac{a_2}{a_1} \right)^{N/2} + 2N \frac{y_N^2}{a_1} \right)^2, \end{aligned}$$

which is (87). The remaining inequality (88) is a consequence of the fact that, $a_1 \geq a_2$, $a_1 \geq y_N^2$ and $a_1 \geq |x' - y'|^2$.

Second, we prove the estimate (89). Indeed we have,

$$\begin{aligned} \frac{|\nabla G_\epsilon^x|^2}{(C'_N/2)^2} &= \left| \frac{1}{a_1^{N/2}} - \frac{1}{a_2^{N/2}} \right|^2 |x' - y'|^2 + \left(y_N \left(\frac{1}{a_2^{N/2}} - \frac{1}{a_1^{N/2}} \right) + x_N \left(\frac{1}{a_2^{N/2}} + \frac{1}{a_1^{N/2}} \right) \right)^2 \\ &\geq \left(x_N \left(\frac{1}{a_2^{N/2}} + \frac{1}{a_1^{N/2}} \right) \right)^2 \geq x_N^2 \frac{1}{a_2^N}. \end{aligned}$$

□

Appendix B Examples of functions satisfying $(\mathcal{R}_0^+)$

Here we summarize examples of functions satisfying $(\mathcal{R}_0^+)$.

Remark B.1 We notice that a function $u \in L^1_{loc}(\overline{\mathbb{R}_+^N})$ can be extended to the whole space $\mathbb{R}^N$ by setting $u(x) = 0$ for $x \notin \overline{\mathbb{R}_+^N}$. We will still denote this extension with u . If u satisfies the ring condition

$$\liminf \frac{1}{R^N} \int_{R < |x-y|_* < 2R} |u(y)| dy = 0, \quad \text{for a.e. } x \in \mathbb{R}_+^N, \quad (93)$$

then u satisfies also $(\mathcal{R}_0^+)$. It is easy to see that (93) is satisfied by any function belonging to the following global spaces: $L^p(\mathbb{R}_+^N)$, the Marcinkiewicz spaces $L^p_w(\mathbb{R}_+^N)$, and to the Campanato-Morrey spaces $M^p_q(\mathbb{R}_+^N)$ (see [6]). This implies that such a function fulfills $(\mathcal{R}_0^+)$. In effect, essentially bounded functions fulfill $(\mathcal{R}_0^+)$ but not (93).

Theorem B.2 1. Let $1 \leq p \leq \infty$. If $u \in L^p(\mathbb{R}_+^N)$, then u satisfies $(\mathcal{R}_0^+)$.

2. Let $1 \leq p < \infty$. If $x_N |u|^p \in L^1(\mathbb{R}_+^N)$, then u satisfies $(\mathcal{R}_0^+)$.

3. If $x_N |u| \in L^\infty(\mathbb{R}_+^N)$, then u satisfies $(\mathcal{R}_0^+)$.

4. If $u \in L^1_{loc}(\overline{\mathbb{R}^N_+})$ and $|u(x)| \leq c|x|^p$ for a suitable $c > 0$, $0 < p < 1$ and $|x|$ large, then u satisfies $(\mathcal{R}_0^+)$.

5. If $u \in M^p_q(\mathbb{R}^N_+)$ with $1 \leq q \leq p$, then u satisfies $(\mathcal{R}_0^+)$.

Proof We begin by proving statements 2 and 4, since they imply the others. Let $x \in \mathbb{R}^N_+$. For $R > 0$, define $A^*_R(x) := B^*_{2R}(x) \setminus B^*_R(x)$. See Figure 2 at page 29.

Proof of claim 2. From our assumption, we know that

$$\lim_{R \rightarrow \infty} \int_{A^*_R(x)} y_N |u(y)|^p dy = 0.$$

This directly implies the claim for $p = 1$. Now consider the case $p > 1$. By Hölder inequality with exponent p , we have

$$\begin{aligned} \frac{1}{R^{N+2}} \int_{A^*_R(x)} y_N |u(y)| dy &= \frac{1}{R^{N+1}} \int_{A^*_R(x)} |u(y)| \frac{y_N}{R} dy \\ &\leq \frac{1}{R^{N+1}} \left(\int_{A^*_R(x)} |u(y)|^p \frac{y_N}{R} dy \right)^{1/p} \left(\int_{A^*_R(x)} \frac{y_N}{R} dy \right)^{1/p'} \\ &\leq \frac{R^{N/p'}}{R^{N+1}} \left(\int_{A^*_R(x)} |u(y)|^p \frac{y_N}{R} dy \right)^{1/p} \rightarrow 0, \end{aligned}$$

and the claim follows.

Proof of claim 4. Let R_0 be such that $|u(x)| \leq c|x|^p \leq c(|x'|^p + |x_N|^p)$ for $|x| > R_0$. We have

$$\begin{aligned} \int_{B^*_R \setminus B^*_{R_0}} y_N |u(y)| dy &\leq c \int_{B^*_R \setminus B^*_{R_0}} y_N (|y'|^p + |y_N|^p) dy \leq c \int_{B^*_R} y_N (|y'|^p + |y_N|^p) dy \\ &\leq \int_{|y'| < R} dy' \int_0^R y_N (|y'|^p + |y_N|^p) dy_N \leq c \int_{|y'| < R} (|y'| R^2 + R^{p+2}) dy' \leq c R^{N+1+p}. \end{aligned}$$

Therefore, we can complete the proof by noticing that

$$\frac{1}{R^{N+2}} \int_{B^*_R \setminus B^*_{R_0}} y_N |u(y)| dy \leq c R^{p-1} \rightarrow 0, \text{ as } R \rightarrow \infty.$$

Proof of claims 1. and 5. The case $1 \leq p < \infty$ is contained in Remark B.1. While the case $p = \infty$ follows from point 4.

Proof of claim 3. It follows by using the same argument as above. $\square$

Another example of functions which naturally meet the condition $(\mathcal{R}_0^+)$ is the following.

Theorem B.3 Let $1 < q < \infty$. Let $u \in L^q_{loc}(\mathbb{R}^N_+)$ be a weak solution of

$$\begin{cases} -\Delta u \geq |u|^q, & \text{on } \mathbb{R}^N_+, \\ u \geq 0, & \text{on } \partial \mathbb{R}^N_+. \end{cases} \quad (94)$$

Then u satisfies $(\mathcal{R}_0^+)$.

For a proof of this theorem see [9].

Funding Open access funding provided by Università degli Studi di Udine within the CRUI-CARE Agreement.

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