Research Article

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Generic properties of the Rabinowitz unbounded continuum

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Abstract: In this article, we prove that, generically in the sense of domain variations, any solution to a nonlinear eigenvalue problem is either nondegenerate or the Crandall-Rabinowitz transversality condition that is satisfied. We then deduce that, generically, the unbounded Rabinowitz continuum of solutions is a simple analytic curve. The global bifurcation diagram resembles the classic model case of the Gel'fand problem in two dimensions.

Keywords: Rabinowitz continuum, bifurcation analysis, generic properties

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be an open and bounded domain of class C^4, we are concerned with generic properties of the Rabinowitz [22] unbounded continuum of $C_0^{2,r}(\overline{\Omega})$-solutions of

$$\begin{cases} -\Delta v = \mu f(v) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with $\mu \geq 0$ and f satisfying:

(**H1**) $f : (a, +\infty) \to (0, +\infty)$ of class C^2 for some $a < 0$, $f'(t) > 0, f''(t) > 0, \ \forall t \in (a, +\infty)$.

In particular, by the maximum principle, we have $v > 0$ in Ω. It follows from [22] that there exists a closed (in the $[0, +\infty) \times C_0^{2,r}(\overline{\Omega})$-topology) connected and unbounded set of solutions (μ, v_μ) of (1.1), which we denote by $\mathcal{R}_\infty$ that contains the unique solution for $\mu = 0$, which is $(\mu, v_\mu) = (0, 0)$. Of course, it is not true in general that $\mathcal{R}_\infty$ is a simple curve with no bifurcation points, see, for example, [18,19,21]. If f is real analytic, then $\mathcal{R}_\infty$ is also a path-connected set [6]. On the other side, much more is known for certain classes of nonlinearities in the radial case (see [9] and in particular [7,14,15] for a review) or, limited to $\Omega \subset \mathbb{R}^2$, for symmetric and convex geometries (see [10]). Actually, in these cases in particular, $\mathcal{R}_\infty$ is a one-dimensional connected manifold in $[0, +\infty) \times C_0^{2,r}(\overline{\Omega})$ whose boundary is $(0, 0)$. See also [1] for a more detailed description of the qualitative behavior of $\mathcal{R}_\infty$ for $f(t) = e^t$ and $N = 2$.

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Our aim here is to initiate the qualitative study (in the spirit of [1]) of $\mathcal{R}_\infty$ in a general setting. To this end, we first prove that, for "almost any domain" in a suitably defined sense, a solution (μ, u_μ) of (1.1) is either nondegenerate or the classical Crandall and Rabinowitz [4] transversality condition that is satisfied. We then deduce that, generically with respect to domain variations, under suitable regularity and growth condition on f, $\mathcal{R}_\infty$ is indeed a one-dimensional connected manifold in $[0, +\infty) \times C_0^{2,r}(\overline{\Omega})$ whose boundary is $(0, 0)$ and that asymptotically approaches the vertical axis, as shown in some particular cases in the classical result of [16]. Although this result seems to be well-known, we could not find a statement of this sort in the literature.

As far as we are concerned with generic properties with respect to domain variations, many results are by now classical, see, for example, [11,24], and references quoted therein. Among many other applications which we cannot discuss here, it follows from [11,24] that, for a fixed μ, then for "almost any domain" any solution of (1.1) is nondegenerate. This is obviously false in general if μ is not fixed ([4]). Actually, these sort of results hold for much more general semilinear elliptic partial differential equations (PDEs) and are used [11,24] to infer that the number of solutions of certain equations is, generically with respect to domain variations, either finite or at most countable.

Generic simpleness of eigenvalues and/or nondegeneracy properties with respect to variations of μ and/or coefficients of the equations are also well-known, starting with [12,17,23,24,26] as later improved in the real-analytic framework in [5] (see also [2,3]).

On the other side, results of the sort considered in this article have been obtained in [23]. First of all, as mentioned above, one can find in [23] a detailed discussion of the fact that, in a generic sense with respect to variations in coefficients, and under suitable regularity assumptions, the set of solutions of a quasilinear elliptic equation with $f(v) = d(x)v + o(\|v\|)$ as $\|v\| \to 0$ is the union of at most countably many curves of class C^{k-1} for some $k \geq 3$. Also, a short discussion of the genericity with respect to domain variations is provided in [23], claiming that coupling some arguments in [24] and those of [23] yields the same result, still for $f(v) = d(x)v + o(\|v\|)$ as $\|v\| \to 0$.

Compared to [23], our results contain some relevant differences, which we shortly describe hereafter.

First, the nonlinearity considered in [23] has essentially the form of $f(v) = d(x)v + o(\|v\|)$ and hence, in particular, the branches analyzed therein are those bifurcating from the line of trivial solutions. On the contrary, we are interested to the qualitative properties of the branch of solutions bifurcating from $(0, 0)$ with f superlinear (see (**H1**)).

Second and more importantly, we attack the problem with a different approach. Indeed, we think this could be a first promising step toward the understanding of the qualitative behavior of the solution curves (in the same spirit of [1]). Thus, we do not argue as in [23], where the fact that the set of solutions is the union of at most countably many regular curves follows at once by the argument in ([24], Section 4). Instead, as mentioned above, we first prove a result of independent interest, Theorem 1.1, showing that, in a generic sense with respect to domain variations, any solution on the continuum is either nondegenerate or the Crandall-Rabinowitz transversality condition is satisfied. Next, by the well-known bending arguments based on the Crandall-Rabinowitz transversality condition, we deduce by the analytic implicit function theorem [3] that around a singular point, the curve of solutions is real analytic and has no bifurcation points. We then apply this result to the qualitative study of $\mathcal{R}_\infty$. In particular, in view of the assumption (**H2**), we see that the branch asymptotically approaches the vertical axis. Of course, it is likely that analyticity of the solution curves considered in [23] could follow as well from the arguments in [23] under suitable assumptions.

The argument here works essentially as in [23,24], although we rely on the more recent reference [11], where one can find a detailed and self-contained exposition of the theory of domain variations (see in particular Chapter 2 in [11]).

For Ω_0, a bounded domain of class C^4 (see Section 3 for details), we denote by $\mathrm{Diff}^4(\Omega_0)$ the set of diffeomorphisms $h : \overline{\Omega}_0 \to \overline{\Omega}$ of class C^4. We recall that a subset of a metric space is said to be:

- nowhere dense, if its closure has empty interior;
- meager (or of first Baire category), if it is the union of countably many nowhere dense sets.

Once more, it is likely that this result is known to experts in the field, still we could not find a statement of this sort in the literature.

Here, L_μ is the linearized operator relative to (1.1) (see Section 2). Then, we have,

Theorem 1.1. *Let $f : (a, +\infty) \to (0, +\infty)$ be of class C^2 for some $a < 0$. For any $\Omega_0 \subset \mathbb{R}^N$ of class C^4, there exists a meager set $\mathcal{F} \subset \mathrm{Diff}^4(\Omega_0)$, depending on f, N, and Ω_0, such that if $h \in \mathrm{Diff}^4(\Omega_0)\backslash\mathcal{F}$ then, for any solution (μ, v_μ) of (1.1) on $\Omega := h(\Omega_0)$ with $\mu > 0$, it holds: either*

(*a*) $\mathrm{Ker}(L_\mu) = \emptyset$, *or*

(*b*) $\mathrm{Ker}(L_\mu) = \mathrm{span}\{\phi\}$ *is one-dimensional and* $\int_\Omega f(v_\mu)\phi \neq 0$.

In particular, we deduce the following result about the Rabinowitz [22] unbounded continuum of solutions of (1.1) for f real analytic, which satisfies (**H1**) and,

(**H2**) for any $\delta > 0$, there exists $C_\delta > 0$ (depending also by f, N, and Ω) such that $v_\mu \leq C_\delta$ for any solution of (1.1) with $\mu \geq \delta$.

It is well-known that, for Ω a bounded domain of class C^4, (**H2**) is satisfied under suitable growth assumptions on f, as for example those in [8] (here $F(t) = \int_0^t f(s)\mathrm{d}s$),

$$\begin{cases} \lim\limits_{t\to+\infty} \dfrac{f(t)}{t} = +\infty, \quad \lim\limits_{t\to+\infty} \dfrac{f(t)}{t^\beta} = 0, \quad \beta = \dfrac{N+2}{N-2} \ \text{ if } N \geq 3, \ \beta < +\infty \ \text{if } N = 2, \\ \limsup\limits_{t\to+\infty} \dfrac{tf(t) - \theta F(t)}{t^2 f^{\frac{2}{N}}(t)} \leq 0, \quad \text{for some } \theta \in \left[0, \dfrac{2N}{N-2}\right), \\ \text{and if } N \geq 3, \ f(t)t^{-\frac{N+2}{N-2}} \text{ is nonincreasing in } (0, +\infty) \end{cases} .$$

Clearly, the model nonlinearities $f(t) = (1+t)^p$, $1 < p < \frac{N+2}{N-2}$, $N \geq 3$, $p > 1$, $N = 2$, fit these assumptions. However, there are many other cases where (**H2**) is satisfied, as for example $f(t) = e^t$, $N = 2$ ([20]). Then, we have:

Theorem 1.2. *Let f be real analytic and satisfying* (**H1**) and (**H2**), *$\Omega_0 \subset \mathbb{R}^N$ of class C^4, $\mathcal{F} \subset \mathrm{Diff}^4(\Omega_0)$ as defined by Theorem* 1.1 *and pick $h \in \mathrm{Diff}^4(\Omega_0)\backslash\mathcal{F}$. If $\Omega = h(\Omega_0)$, then the Rabinowitz unbounded continuum $\mathcal{R}_\infty$ of solution of* (1.1) *is a one-dimensional real analytic manifold with boundary* $(\mu(0), v(0)) = (0, 0)$. *In particular,*

$$\mathcal{R}_\infty = \{[0, \infty) \ni s \mapsto (\mu(s), v(s)) \in [0, +\infty) \times C_0^{2,r}(\overline{\Omega})\},$$

is a continuous simple curve without bifurcation points where $(\mu(0), v(0)) = (0, 0)$ *and* $\mu(s) \to 0^+$ *and* $\|v(s)\|_\infty \to +\infty$ *as* $s \to \infty$.

Remark 1.3. Theorem 1.1 can be generalized to the case of uniformly elliptic operators such as $Lu := \mathrm{div}(A(\nabla u)) + \overrightarrow{b} \cdot \nabla u + cu$, where $A = (a_{ij})$, $a_{ij}(x)$, $b_j(x)$, and $c(x)$ are smooth up to the boundary. Therefore, the generic bending result (Theorem 1.2) also follows if one replaces the Laplace operator in (1.1) by uniformly elliptic operators.

2 Well-known results

Let $X = \mathbb{R} \times C_0^{2,r}(\overline{\Omega})$, we introduce the map,

$$F : X \to C^r(\overline{\Omega}), \quad F(\mu, v) := -\Delta v - \mu f(v), \tag{2.1}$$

and its differential with respect to (μ, v), that is, the linear operator,

$$D_{\mu,\nu}F(\mu,\nu) : X \to C^r(\overline{\Omega}),$$

which acts as follows:

$$D_{\mu,\nu}F(\mu,\nu)[\dot{\mu},\dot{\nu}] = D_\nu F(\mu,\nu)[\dot{\nu}] + d_\mu F(\mu,\nu)[\dot{\mu}],$$

where we have introduced the differential operators,

$$D_\nu F(\mu,\nu)[\dot{\nu}] = -\Delta\dot{\nu} - \mu f'(\nu)\dot{\nu}, \quad \dot{\nu} \in C_0^{2,r}(\overline{\Omega}),$$

$$d_\mu F(\mu,\nu)[\dot{\mu}] = -f(\nu)\dot{\mu}, \quad \dot{\mu} \in \mathbb{R}.$$

For a fixed solution (μ, ν_μ), the eigenvalues of $L_\mu := D_\nu F(\mu,\nu_\mu)$ form an increasing sequence and are denoted by σ_k, $k \in \mathbb{N}$, which satisfy:

$$L_\mu\phi = \sigma_k\phi, \ \phi \in C_0^{2,r}(\overline{\Omega}).$$

By the Fredholm alternative, the implicit function theorem applies around any solution of (1.1) as follows:

Lemma 2.1. *Let (μ_0, ν_0) be a solution of* (1.1) *with $\mu = \mu_0 \geq 0$.*

If 0 is not an eigenvalue of L_{μ_0}, then:

(*i*) *L_{μ_0} is an isomorphism;*

(*ii*) *There exists an open neighborhood $J \subset \mathbb{R}$ of μ_0 and $\mathcal{B} \subset C_0^{2,r}(\overline{\Omega})$ of ν_0, such that the set of solutions of* (1.1) *in $J \times \mathcal{B}$ is a curve of class C^2, $J \ni \mu \mapsto \nu_\mu \in \mathcal{B}$.*

Next, we state the well-known bending result of [4] for solutions of (1.1) just with an additional observation about the case where f is real analytic in $(a, +\infty)$ for some $a < 0$. The conclusions deduced in this particular case are straightforward consequences of general and well-known facts of analytic bifurcation theory [3].

Proposition 2.2. [4] *Let (μ, ν_μ) be a solution of* (1.1) *with $\mu > 0$ and suppose that the kth eigenvalue of L_μ satisfies $\sigma_k = 0$ and is simple, that is, it admits only one eigenfunction, $\phi_k \in C_0^{2,r}(\overline{\Omega})$. If*

$$\int_\Omega f(\nu_\mu)\phi_k \neq 0,$$

then there exists $\varepsilon > 0$, an open neighborhood $\mathcal{U}$ of (μ, ν_μ) in X and a curve $(-\varepsilon, \varepsilon) \ni s \mapsto (\mu(s), \nu(s))$ of class C^2 such that $(\mu(0), \nu(0)) = (\mu, \nu_\mu)$ and the set of solutions of (1.1) *in $\mathcal{U}$ has the form $(\mu(s), \nu(s))$ with, $\nu(s) = \nu_\mu + s\phi_k + \xi(s)$, and*

$$\int_\Omega f(\nu(s))\xi(s)\phi_k = 0, \quad s \in (-\varepsilon, \varepsilon).$$

Moreover, it holds

$$\xi(0) \equiv 0 \equiv \xi'(0), \quad \mu'(0) = 0, \tag{2.2}$$

and there exists a continuous curve $(\sigma(s), \phi(s))$, such that $L_{\mu(s)}\phi(s) = \sigma(s)\phi(s)$, $s \in (-\varepsilon, \varepsilon)$, $\phi(0) = \phi_k$, $\sigma(0) = \sigma_k$, and

$$\sigma(s)\int_\Omega f'(\nu(s))\phi(s)\nu(s) \quad \text{and} \quad \mu'(s)\int_\Omega f(\nu(s))\phi(s)$$

have the same zeroes and, whenever $\mu'(s) \neq 0$, the same sign. In particular,

$$\frac{\sigma(s)}{\mu'(s)} = \frac{\int_\Omega f(\nu_\mu)\phi_k + o(1)}{\int_\Omega \phi_k^2 + o(1)}, \quad \text{as } s \to 0.$$

If f is real analytic in $(a, +\infty)$ for some $a < 0$, then $\mu(s)$, $\nu(s)$, $\sigma(s)$, and $\phi(s)$ are the real analytic functions of $s \in (-\varepsilon, \varepsilon)$ and in particular either $\mu(s)$ is constant in $(-\varepsilon, \varepsilon)$ or *$\mu'(s) \neq 0$, $\sigma(s) \neq 0$ in $(-\varepsilon, \varepsilon)\backslash\{0\}$ and $\sigma(s)$ is simple in $(-\varepsilon, \varepsilon)$.*

3 Generic properties of the Rabinowitz continuum

In this section, we prove Theorems 1.1 and 1.2.

Let us recall few definitions and set some notations first.

Definition 3.1. A domain Ω is of class $C^k(C^{k,r})$, $k \geq 1$, if for each $x_0 \in \partial\Omega$, there exists a ball $B = B_r(x_0)$ and a one-to-one map $\Theta : B \mapsto U \subset \mathbb{R}^N$ such that $\Theta \in C^k(B)(C^{k,r}(B))$, $\Theta^{-1} \in C^k(U)(C^{k,r}(U))$ and the following holds:

$$\Theta(\Omega \cap B) \subset \mathbb{R}^N_+ \quad \text{and} \quad \Theta(\partial\Omega \cap B) \subset \partial\mathbb{R}^N_+.$$

It is well-known (see, for example, [11]) that this is equivalent to say that there exists $r > 0$ and $M > 0$ such that, given any ball $B \subset \mathbb{R}^N$, then, after suitable rotation and translations, it holds:

$$\Omega \cap B = \{(x_1, x \dots, x_N) : x_N < f(x_1, \dots, x_{N-1})\} \cap B$$

and

$$\partial\Omega \cap B = \{(x_1, x \dots, x_N) : x_N = f(x_1, \dots, x_{N-1})\} \cap B,$$

for some $f \in C^k(\mathbb{R}^{N-1})(C^{k,r}(\mathbb{R}))$ whose norm is not larger than M.

Definition 3.2. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded domain of class C^m, $m \geq 1$. $C^m(\overline{\Omega}; \mathbb{R}^N)$ is the Banach space of continuous and m-times differentiable maps on Ω, whose derivatives of order $j = 0, 1, \dots, m$ extend continuously on $\overline{\Omega}$. $\mathrm{Diff}^m(\Omega) \subset C^m(\overline{\Omega}; \mathbb{R}^N)$ is the open subset of $C^m(\overline{\Omega}; \mathbb{R}^N)$ whose elements are C^m imbeddings on $\overline{\Omega}$, that is, of maps $h : \overline{\Omega} \mapsto \mathbb{R}^N$, which are diffeomorphisms of class C^m on their images $h(\overline{\Omega})$.

We recall that if X and Z are Banach spaces and $T : X \to Z$ is linear and continuous, then T is Fredholm (semi-Fredholm) if $R(T)$ (the range of T) is closed and both $\dim(\mathrm{Ker}(T))$ and $\mathrm{codim}(R(T))$ are finite. If T is Fredholm, then the index of T is

$$\mathrm{ind}(T) = \dim(\mathrm{Ker}(T)) - \mathrm{codim}(R(T)).$$

We refer to [13] for further details about Fredholm operators. Given a Banach space X and $x \in X$, we will denote by T_xX the tangent space at x.

Definition 3.3. Let X and Z be Banach spaces, $A \subset X$ an open set, and $F : A \to Z$ a C^1 map. Suppose that for any $x \in A$, the Fréchet derivative $D_xF(x) : T_xX \to T_\eta Z$ is a Fredholm operator. A point $x \in A$ is a regular point if $D_xF(x)$ is surjective, is a singular point otherwise. The image of a singular point $\eta = F(x) \in Z$ is a singular value. The complement of the set of singular values in Z is the set of regular values.

The following theorem is a particular case of a more general transversality result proved in [11], see also [25].

Theorem 3.4. [11] *Let X, $\mathcal{H}$, and Z be separable Banach spaces, $\mathcal{A} \subseteq X \times \mathcal{H}$ an open set, $\Phi : \mathcal{A} \to Z$ a map of class C^k, and $\eta \in Z$.*

Suppose that for each $(x, h) \in \Phi^{-1}(\eta)$ it holds:

(*i*) *$D_x\Phi(x, h) : T_xX \to T_\eta Z$ is a Fredholm operator with index $<k$;*

(*ii*) $D\Phi(x,h) = (D_x\Phi(x,h), D_h\Phi(x,h))\colon T_xX \times T_h\mathcal{H} \to T_\eta Z$ *is surjective.*

Let $A_h = \{x : (x,h) \in \mathcal{A}\}$ *and*

$$\mathcal{H}_{\text{crit}} = \{h : \eta \text{ is a singular value of } \Phi(\cdot,h) : A_h \to Z\}.$$

Then, $\mathcal{H}_{\text{crit}}$ *is meager in* $\mathcal{H}$.

We are ready to present the proof of Theorem 1.1.

Proof of Theorem 1.1. Let Ω_0 be as in the statement and let us define

$$X_{\Omega_0} = \mathbb{R} \times C_0^{2,r}(\overline{\Omega_0}).$$

We define the maps,

$$F_{\Omega_0} : X_{\Omega_0} \to C^r(\overline{\Omega_0}), \quad F_{\Omega_0}(\mu, v) = \Delta v + \mu f(v).$$

Next, for fixed $h \in \text{Diff}^4(\Omega_0)$ and $v \in C_0^{2,r}(\overline{h(\Omega_0)})$, we define the pull-back,

$$h^*(v)(x) = v(h(x)), \;\; x \in \overline{\Omega_0}.$$

Clearly, h^* is an isomorphism of $C_0^{2,r}(\overline{h(\Omega_0)})$ onto $C_0^{2,r}(\overline{\Omega_0})$ with inverse $h^{*-1} = (h^{-1})^*$. For any such h, it is well defined the map

$$F_{h(\Omega_0)} : X_{h(\Omega_0)} \to C^r(\overline{h(\Omega_0)})$$

and then we can set

$$h^*F_{h(\Omega_0)}h^{*-1} : X_{\Omega_0} \times \text{Diff}^4(\Omega_0) \to C^r(\overline{\Omega_0}).$$

Putting $\mathcal{H} = \text{Diff}^4(\Omega_0)$, $\eta = 0 \in Z = C^r(\overline{\Omega_0})$, we will apply Theorem 3.4 to the map $\Phi = \Phi(\mu, v, h)$ defined as follows:

$$\Phi : \mathcal{A} \to \mathbb{R} \times C^r(\overline{\Omega_0}), \quad \mathcal{A} = X_{\Omega_0} \times \mathcal{H},$$

$$\Phi(\mu, v, h) = h^*F_{h(\Omega_0)}h^{*-1}(\mu, v).$$

Step 1: Our aim is to show that assumptions (*i*) and (*ii*) of Theorem 3.4 hold.

As in [11], it is very useful for the discussion to denote by $(\dot\mu, \dot v, \dot h) \in \mathbb{R} \times C_0^{2,r}(\overline{\Omega_0}) \times C^4(\overline{\Omega_0}; \mathbb{R}^N)$ the elements of the tangent space at points $(\mu, v, h) \in X_{\Omega_0} \times \mathcal{H}$.

First of all observe that for fixed $h \in \text{Diff}^4(\Omega_0)$, the linearized operator,

$$D_{\mu,v}\Phi(\mu, v, h) : \mathbb{R} \times C_0^{2,r}(\overline{\Omega_0}) \to C^r(\overline{\Omega_0}),$$

acts as follows on $(\dot\mu, \dot v) \in \mathbb{R} \times C_0^{2,r}(\overline{\Omega_0})$,

$$D_{\mu,v}\Phi(\mu, v, h)[\dot\mu, \dot v] = h^*(\Delta \dot v^* + \mu f'(v^*)\dot v^* + f(v^*)\dot\mu),$$

where

$$v^* = (h^*)^{-1}v, \quad \dot v^* = (h^*)^{-1}\dot v.$$

Since any diffeomorphism of class C^4 maps the Laplace operator to a uniformly elliptic operator with C^2 coefficients, by standard elliptic estimates, it is not difficult to see that $D_{\mu,v}\Phi(\mu, v, h)$ is a Fredholm operator of index 1.

This fact proves (*i*) whenever we can show that $\Phi \in C^k(\mathcal{A})$ for some $k \geq 2$. The regularity of Φ with respect to h is the same as that of $F_{h(\Omega_0)}$ with respect to v, see chapter 2 in [11]. Therefore, we have $\Phi \in C^3(\mathcal{A})$, as claimed.

Next, we prove (*ii*), that is, we show that $\eta = 0$ is a regular value for the map $(\mu, v, h) \to \Phi(\mu, v, h)$. We argue by contradiction and suppose that there exists a singular point $(\overline{\mu}, \overline{v}, \overline{h})$ of Φ such that $\Phi(\overline{\mu}, \overline{v}, \overline{h}) = 0$.

First of all, let us define $\Omega = \bar{h}(\Omega_0)$, $\bar{u} = (\bar{h}^*)^{-1}\bar{v} \in C_0^{2,r}(\overline{\Omega})$, and $\widehat{\Phi}(\mu, u, h)$ on $X_\Omega \times \mathrm{Diff}^4(\Omega)$ as follows:

$$\widehat{\Phi}(\mu, u, h) = h^* F_\Omega h^{*-1}(\mu, u),$$

where

$$F_\Omega : X_\Omega \to C^r(\overline{\Omega}), \quad F_\Omega(\mu, u) = \Delta u + \mu f(u).$$

Let $i_\Omega \in \mathrm{Diff}^4(\Omega)$ be the identity map. By construction, in these new coordinates, the map $\widehat{\Phi}(\mu, u, h)$ has a singular point $(\bar{\mu}, \bar{u}, i_\Omega)$ such that $\widehat{\Phi}(\bar{\mu}, \bar{u}, i_\Omega) = 0$, that is, by assumption the derivative $D_{\mu,u,h}\widehat{\Phi}(\bar{\mu}, \bar{u}, i_\Omega)$ is not surjective. Putting

$$\bar{f} = f(\bar{u}), \quad \bar{f}' = f'(\bar{u}),$$

a subtle evaluation shows that $D_{\mu,u,h}\widehat{\Phi}(\bar{\mu}, \bar{u}, i_\Omega)$ acts on

$$(\dot{\mu}, \dot{u}, \dot{h}) \in \mathbb{R} \times C_0^{2,r}(\overline{\Omega}) \times C^4(\overline{\Omega}; \mathbb{R}^N)$$

as follows (see Theorem 2.2 in [11]):

$$\begin{aligned} D_{\mu,u,h}\widehat{\Phi}(\bar{\mu}, \bar{u}, i_\Omega)[\dot{\mu}, \dot{u}, \dot{h}] &= \Delta\dot{u} + \bar{\mu}\bar{f}'\dot{u} + \bar{f}\dot{\mu} + \dot{h}\cdot\nabla(\Delta\bar{u} + \bar{\mu}\bar{f}) - (\Delta + \bar{\mu}\bar{f}')\dot{h}\cdot\nabla\bar{u} \\ &= (\Delta + \bar{\mu}\bar{f}')\dot{u} - (\Delta + \bar{\mu}\bar{f}')\dot{h}\cdot\nabla\bar{u} + \bar{f}\dot{\mu}, \end{aligned} \tag{3.1}$$

where we used the fact that $\Delta\bar{u} + \bar{\mu}\bar{f} = \widehat{\Phi}(\bar{\mu}, \bar{u}, i_\Omega) = 0$.

At this point, observe that, by the Fredholm property of the operator $\Delta + \bar{\mu}\bar{f}$ on $C_0^{2,r}(\overline{\Omega})$, we have that the subspace $\{D_{\mu,u,h}\widehat{\Phi}(\bar{\mu}, \bar{u}, i_\Omega)[(0, \dot{u}, 0)], \dot{u} \in C_0^{2,r}(\overline{\Omega})\}$, is closed and has finite codimension. Next, since $\bar{u} \in C_0^{2,r}(\overline{\Omega})$ and $\partial\Omega$ is of class C^4, then by standard elliptic regularity theory, we find that $\bar{u} \in C_0^{3,r}(\overline{\Omega})$ and then $\dot{h}\cdot\nabla\bar{u} \in C^{2,r}(\overline{\Omega})$. As a consequence, we can prove that the subspace $\{D_{\mu,u,h}\widehat{\Phi}(\bar{\mu}, \bar{u}, i_\Omega)[(0, 0, \dot{h})], \dot{h} \in C^4(\overline{\Omega}; \mathbb{R}^N)\}$ is closed with finite codimension as well. Indeed, let us define $K : C^{2,r}(\overline{\Omega}) \mapsto C^{2,r}(\overline{\Omega})$ as the linear operator, which, to any $\phi \in C^{2,r}(\overline{\Omega})$, associates the unique solution $\phi_b = K[\phi]$ of $\Delta\phi_b = 0$, $\phi_b = \phi$ on $\partial\Omega$. Clearly, this is always well posed since Ω is of class C^4 and $\phi \in C^{2,r}(\overline{\Omega})$. Then,

$$\Delta\phi + \bar{\mu}\bar{f}'\phi = g \in C^r(\overline{\Omega}),$$

if and only if

$$\phi \in C^{2,r}(\overline{\Omega}) \quad \text{and} \quad \phi + T[\phi] = G[g] \in C^{2,r}(\overline{\Omega}),$$

where $G[g] = \int_\Omega G(x, y)g(y)$ and $T : C^{2,r}(\overline{\Omega}) \mapsto C^{2,r}(\overline{\Omega})$, $T(\phi) = G[\bar{\mu}\bar{f}'\phi] - K[\phi]$. Since Ω is of class C^4, then by standard elliptic estimates, T maps $C^{2,r}(\overline{\Omega})$ into $C^{3,r}(\overline{\Omega})$. Therefore, T is compact, and then, we conclude by the Fredholm alternative that the range of $(\Delta + \bar{\mu}\bar{f}')(\dot{h}\cdot\nabla\bar{u})$, $\dot{h}\cdot\nabla\bar{u} \in C^{2,r}(\overline{\Omega})$, is closed in $C^r(\overline{\Omega})$ and has finite codimension.

At this point, we deduce from these two facts that there exists a nontrivial $\phi_\perp \in C^r(\overline{\Omega})$, which is orthogonal to the image of $D_{\mu,u,h}\widehat{\Phi}(\bar{\mu}, \bar{u}, i_\Omega)$, that is,

$$\int_\Omega \phi_\perp((\Delta + \bar{\mu}\bar{f}')\dot{u} - (\Delta + \bar{\mu}\bar{f}')\dot{h}\cdot\nabla\bar{u} + \bar{f}\dot{\mu}) = 0, \quad \forall(\dot{\mu}, \dot{u}, \dot{h}). \tag{3.2}$$

Putting $(\dot{\mu}, \dot{u}, \dot{h}) = (\dot{\mu}, 0, 0)$ in (3.2), we find $\int_\Omega \bar{f}\phi_\perp = 0$, and then if we choose $\dot{h} = 0$, we find that

$$\int_\Omega \phi_\perp(\Delta + \bar{\mu}\bar{f}')\dot{u} = 0, \ \ \forall\dot{u} \in C_0^{2,r}(\overline{\Omega}),$$

which shows that $\phi_\perp$ is a $C^r(\overline{\Omega})$ distributional solution of $\Delta\phi_\perp + \bar{\mu}\bar{f}'\phi_\perp = 0$. Therefore, by standard elliptic estimates (where we recall that $\partial\Omega$ is of class C^4), $\phi_\perp$ is a $C_0^2(\overline{\Omega})$ solution of $\Delta\phi_\perp + \bar{\mu}\bar{f}'\phi_\perp = 0$. As a consequence we observe that (3.2) is reduced to

$$\int_\Omega \phi_\perp(\Delta + \bar{\mu}\bar{f}')\dot{h}\cdot\nabla\bar{u} = 0, \quad \forall\dot{h} \in C^4(\overline{\Omega}; \mathbb{R}^N),$$

which allows us to deduce that

$$\begin{aligned}
0 &= \int_{\Omega} \phi_{\perp}(\Delta + \overline{\mu}\overline{f}')\dot{h}\cdot\nabla\overline{u} \\
&= \int_{\Omega} \phi_{\perp}(\Delta + \overline{\mu}\overline{f}')\dot{h}\cdot\nabla\overline{u} - \int_{\Omega} (\Delta\phi_{\perp} + \overline{\mu}\overline{f}'\phi_{\perp})\dot{h}\cdot\nabla\overline{u} \\
&= \int_{\Omega} \phi_{\perp}\Delta(\dot{h}\cdot\nabla\overline{u}) - \int_{\Omega} (\Delta\phi_{\perp})\dot{h}\cdot\nabla\overline{u} \\
&= \int_{\partial\Omega} (\phi_{\perp}\partial_{\nu}(\dot{h}\cdot\nabla\overline{u}) - \dot{h}\cdot\nabla\overline{u}(\partial_{\nu}\phi_{\perp})) \\
&= -\int_{\partial\Omega} (\partial_{\nu}\phi_{\perp})\dot{h}\cdot\nabla\overline{u} \\
&= -\int_{\partial\Omega} (\partial_{\nu}\phi_{\perp})(\partial_{\nu}\overline{u})\dot{h}\cdot\nu, \quad \forall \dot{h} \in C^4(\overline{\Omega}, \mathbb{R}^N).
\end{aligned}$$

Therefore, since $\dot{h}$ is arbitrary, we conclude that,

$$(\partial_{\nu}\phi_{\perp})(\partial_{\nu}\overline{u}) \equiv 0 \quad \text{on } \partial\Omega.$$

At this point, we observe that since $\overline{f} > 0$ on $\overline{\Omega}$ and $\overline{u} = 0$ on $\partial\Omega$, then, by the strong maximum principle, we have $\overline{u} > 0$ in Ω. Since $\partial\Omega$ is of class C^4, we can apply the Hopf boundary lemma and conclude that $\partial_{\nu}\overline{u} < 0$ on $\partial\Omega$. Therefore, we conclude that necessarily $\partial_{\nu}\phi_{\perp} \equiv 0$ on $\partial\Omega$, which is in contradiction with the Hopf boundary lemma. This contradiction shows that *(ii)* holds, and then, we can apply Theorem 3.4 and conclude that there exists a meager set $\mathcal{F} \subset \mathrm{Diff}^4(\Omega_0)$ such that if $h(\Omega_0) \notin \mathcal{F}$, then $\eta = 0$ is a regular value of $\Phi(\mu, \nu, h)$.

Step 2: We have from step 1 that there exists a meager set $\mathcal{F} \subset \mathrm{Diff}^4(\Omega_0)$ such that if $h \notin \mathcal{F}$ and $\Omega := h(\Omega_0)$, then $\eta = 0$ is a regular value of the map $\Phi(\cdot,\cdot,h)$. As a consequence, for any $(\overline{\mu}, \overline{v})$, which solves

$$\Phi(\mu, v) = F_{\Omega}(\mu, v) = 0$$

and setting $\overline{f} = f(\overline{v})$, $\overline{f}'(\overline{v})$, then the differential

$$\overline{L}[\dot{\mu}, \dot{v}] := D_{\mu,v}\Phi(\overline{\mu}, \overline{v})[\dot{\mu}, \dot{v}] = \Delta\dot{v} + \overline{\mu}\overline{f}'(\overline{v})\dot{v} + \overline{f}\dot{\mu}.$$

is surjective. On the other side, since $\overline{v}$ solves (1.1), then the operator,

$$\Delta\dot{v} + \overline{\mu}\overline{f}'(\overline{v})\dot{v}$$

is just $L_{\overline{\mu}}$ for which the Fredholm alternative holds. Let us define $R = R(L_{\overline{\mu}}) \subseteq C^r(\overline{\Omega})$ to be the range of $L_{\overline{\mu}}$. Now if $L_{\overline{\mu}}$ is surjective, then, by the Fredholm alternative, we have $\mathrm{Ker}(L_{\overline{\mu}}) = \emptyset$, which is *(a)* in the statement of Theorem 1.1. Therefore, we can assume without loss of generality that $L_{\overline{\mu}}$ is not surjective, let $\overline{d} = \mathrm{codim}(R)$ be the codimension of R. Since $\overline{L}$ is surjective, by the Fredholm alternative, it is not difficult to see that $\overline{d} \leq 1$, and since $L_{\overline{\mu}}$ is not surjective, then necessarily $\overline{d} = 1$. We will conclude the proof by showing that *(b)* holds in this case. Indeed, obviously, the kernel must be one-dimensional, $\mathrm{Ker}(\overline{L}) = \mathrm{span}\{\overline{\phi}\}$, for some $\overline{\phi} \in C_0^{2,r}(\overline{\Omega})$, which satisfies $L_{\overline{\mu}}[\overline{\phi}] = 0$. Since $\overline{L}$ is surjective, then $\overline{f}'\overline{\phi}$ must be an element of its range and then there exists $\phi \in C_0^{2,r}(\overline{\Omega})$, which satisfies

$$L_{\overline{\mu}}[\phi] + \dot{\mu}\overline{f} = \overline{f}'\overline{\phi}.$$

Multiplying this equation by $\overline{\phi}$ and integrating by parts, we find that

$$\dot{\mu}\int_{\Omega}\overline{f}\,\overline{\phi} = \int_{\Omega}\overline{f}'\overline{\phi}^2,$$

and since $f'(t) > 0, \forall\, t$, then we deduce that necessarily $\int_{\Omega} \overline{f}\, \overline{\phi} \neq 0$. In other words, (b) of Theorem 1.1 holds and the proof is concluded. □

We are ready to present the proof of Theorem 1.2.

Proof of Theorem 1.2. It is well-known [4] that, due to (**H1**), there exists $\mu_* < +\infty$ such that $\mu \leq \mu_*$ for any solution of (1.1) and in particular that there exists a continuous simple curve of solutions of (1.1) (the branch of minimal solutions) for any $\mu < \mu_*$ which emanates from $(\mu, v_\mu) = (0, 0)$, which we denote by $\mathcal{G}(\Omega)$. In particular, with the notations of Section 2, $\mathcal{G}(\Omega)$ is characterized by the fact that the first eigenvalue of the linearized equation, which we denote by $\sigma_1(\mu, v_\mu)$, satisfies $\sigma_1(\mu, v_\mu) > 0$ for any $(\mu, v_\mu) \in \mathcal{G}(\Omega)$. In view of (**H2**) and standard elliptic theory, we have that $v_* = v_\mu|_{\mu=\mu_*}$ is a classical solution and $\sigma_1(\mu_*, v_*) = 0$. By Theorem 1.1, we have that (b) holds for (μ_*, v_*), and then by Proposition 2.2, we can continue $\mathcal{G}(\Omega)$ to a continuous and simple curve without bifurcation points, $[0, s_1 + \delta_1) \ni s \mapsto (\mu(s), v(s))$, which locally around any point $s_0 > 0$ admits a real analytic reparametrization, that is, an injective and continuous map $\gamma_0 : (-1, 1) \to (s_0 - \varepsilon, s_0 + \varepsilon)$, $s = \gamma_0(t)$, such that $\gamma_0(0) = s_0$ and $(\mu(\gamma_0(t)), v(\gamma_0(t)))$ is real analytic. Therefore, locally, this branch has also the structure of a one-dimensional real analytic manifold and we denote it by,

$$\mathcal{G}^{(s_1+\delta_1)} = \{[0, s_1 + \delta_1) \ni s \mapsto (\mu(s), v(s))\},$$

which satisfies

$$\overline{\mathcal{G}^{(s_1+\delta_1)}} = \{[0, s_1 + \delta_1] \ni s \mapsto (\mu(s), v(s))\},$$

where, for some $s_1 > 0$ and $\delta_1 > 0$, we have:

$(A1)_0$ $(\mu(s), v(s))$ is continuous and locally (up to reparametrization) real analytic for $s \in [0, s_1 + \delta_1]$;
$(A1)_1$ $v(s)$ is a solution of (1.1) with $\mu = \mu(s)$ for any $s \in [0, s_1 + \delta_1]$;
$(A1)_2$ $\mu(s) = s$ for $s \leq s_1$, $\mu(s_1) = \mu_*$;
$(A1)_3$ the inclusion $\{(\mu, v_\mu),\ \mu \in [0, \mu_*]\} \equiv \overline{\mathcal{G}(\Omega)} \subset \mathcal{G}^{(s_1+\delta_1)}$, holds;
$(A1)_4$ $\inf_{[s_1, s_1+\delta_1)} \mu(s) > 0$ and $0 < \mu(s) \leq \mu_*$, $\forall s \in (0, s_1 + \delta_1)$;
$(A1)_5$ $0 \notin \Sigma(L_{\mu(s)})$, $\forall s \in (0, s_1 + \delta_1) \setminus \{s_1\}$,
$(A1)_6$ $\mathrm{Ker}(L_{\mu(s_1)}) = \mathrm{span}\{\phi_1\}$ and $\int_{\Omega} f(v(s_1))\phi_1 \neq 0$,

where $\Sigma(L_{\mu(s)})$ denotes the spectrum of $L_{\mu(s)}$. Clearly, $(A1)_6$ follows from (b) of Theorem 1.1. Concerning $(A1)_5$, we recall that, by Proposition 2.2, either $\sigma_1(s)$ vanishes identically around s_1 or its zero must be isolated. In particular, since $\sigma_1(s)$ is (locally up to a reparametrization) real analytic, its level sets cannot have accumulation points unless $\sigma_1(s)$ is locally constant and consequently unless it is constant on $[0, s_1 + \delta_1)$. However, we can rule out this case since, in view of $(A1)_2$, for $s < s_1$, we have $\sigma_1(\mu(s), v(s)) > 0$ and then no $\sigma_k(s)$ can vanish identically, which shows that $(A1)_5$ holds as well. Therefore, it is well defined

$$s_2 := \sup\{t > s_1 : \inf_{s \in [s_1, t)} \mu(s) > 0, \quad 0 \notin \Sigma(L_{\mu(s)}),\ \forall (\mu(s), v(s)) \in \mathcal{G}^{(t)},\ \forall s_1 < s < t\}.$$

At this point, either $\inf_{s \in [s_1, s_2)} \mu(s) = 0$ or $\inf_{s \in [s_1, s_2)} \mu(s) > 0$.

If $\inf_{s \in [s_1, s_2)} \mu(s) = 0$, we set $s_\infty = s_2$,

$$\mathcal{G}^{(s_\infty)} = \{[0, s_\infty) \ni s \mapsto (\mu(s), v(s))\}, \tag{3.3}$$

and claim that in this case, necessarily $\mu(s) \to 0^+$ and $\|v(s)\|_\infty \to +\infty$ as $s \to s_\infty$.

We first prove that $\mu(s) \to 0^+$ and argue by contradiction, assuming that there exists a sequence $\{s_n\} \subset (0, s_\infty)$ such that $s_n \to s_\infty$, as $n \to +\infty$ and $\mu(s_n) \geq \delta > 0$ for some $\delta > 0$. In view of $(A1)_4$, passing to a subsequence if necessary, we can assume that $\mu(s_n) \to \overline{\mu} \in [\delta, \mu_*]$. By (**H2**) and passing to a further subsequence, we would deduce that $v(s_n) \to \overline{v}$, where $(\overline{\mu}, \overline{v})$ is a solution of (1.1). By Theorem 1.1, we see that either (a) or (b) holds and then, possibly with the aid of Proposition 2.2, we would deduce that locally around $(\overline{\mu}, \overline{v})$, the set of solutions of (1.1) is a real analytic parametrization of the form $(\overline{\mu}(t), \overline{v}(t))$, $t \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ with $(\overline{v}(0), \overline{v}(0)) = (\overline{\mu}, \overline{v})$. In particular, for any fixed $\overline{n}$ large enough, we can assume without loss of generality that $(\overline{\mu}(t), \overline{v}(t))$, $t \in (-\varepsilon, 0)$ coincides with $(\mu(s), v(s))$, $s \in (s_{\overline{n}}, s_\infty)$. Now by construction, $\mu(s) > 0$ in $[s_1, s_{\overline{n}}]$, and since $\mu(s)$ is continuous, we have $\inf_{s\in[s_1,s_{\overline{n}}]}\mu(s) \geq \overline{\delta} > 0$ for some $\overline{\delta} > 0$. On the other side, possibly taking a larger $s_{\overline{n}}$, we have $\inf_{s\in[s_{\overline{n}},s_\infty)}\mu(s) \geq \frac{\delta}{2}$. In other words, we have a contradiction to $\inf_{s\in[s_1,s_\infty)}\mu(s) = 0$ and the claim is proved.

Next, we show that $\|v(s)\|_\infty \to +\infty$ and argue by contradiction. If this was not the case, we could find a sequence $\{s_n\} \subset (0, s_\infty)$ such that $s_n \to s_\infty$, as $n \to +\infty$ and $\|v(s_n)\|_\infty \leq C$ for some $C > 0$. Since we have shown that $\mu(s) \to 0^+$ as $s \to s_\infty$, then passing to a subsequence, we would deduce that $v(s_{n_k}) \to \overline{v}$, where $\overline{v}$ solves (1.1) with $\mu = 0$. However by $(A1)_4$, this fact implies that $(\mu, v_\mu) = (0, 0)$ would be a bifurcation point, which is clearly impossible, which proves the claim. At this point, since by definition, $\mathcal{R}_\infty$ is a closed and connected set, it is not difficult to see that $\mathcal{R}_\infty \equiv \mathcal{G}^{(s_\infty)}$.

After a suitable reparametrization, we can assume without loss of generality that $s_2 = +\infty$ and we conclude that statement of Theorem 1.2 is true as far as $\inf_{s\in(0,s_2)}\mu(s) = 0$. Therefore, we can assume without loss of generality that $\inf_{s\in(0,s_2)}\mu(s) > 0$. In this case, in view of $(A1)_4$, (**H2**), Theorem 1.1, and Proposition 2.2, it is not difficult to see that $(\mu(s), v(s))$ converges to a solution (μ_2, v_2) as $s \to s_2$ and that $0 \in \Sigma(L_{\mu_2})$, and in particular that we can continue the branch $\mathcal{G}^{(s_2)}$ in a right neighborhood of s_2 to a continuous curve, which admits local real analytic reparametrizations. In particular, by arguing as above, we see that $0 \notin \Sigma(L_{\mu(s)})$ for $s \notin \{s_1, s_2\}$ and we can argue by induction defining, for $k \geq 3$,

$$s_k := \sup\left\{t > s_{k-1} : \inf_{s\in[s_1,t)} \mu(s) > 0, \quad 0 \notin \Sigma(L_{\mu(s)}),\ \forall(\mu(s), v(s)) \in \mathcal{G}^{(t)},\ \forall s_{k-1} < s < t\right\}.$$

If there exists some $k \geq 3$ such that $\inf_{s\in(0,s_k)}\mu(s) = 0$, then as for (3.3) we are done. Otherwise by using $(A1)_4$, (**H2**), Theorem 1.1, and Proposition 2.2, we can find sequences s_k and $\delta_k > 0$ such that, for any $k \in \mathbb{N}$ we have, $s_{k+1} > s_k > \cdots > s_2 > s_1$, $s_k + \delta_k < s_{k+1}$ and

$(Ak)_0$ $(\mu(s), v(s))$ is continuous and simple curve without bifurcation points (which admits local real analytic reparametrizations) defined for $s \in [0, s_k + \delta_k]$;

$(Ak)_1$ $v(s)$ is a solution of (1.1) with $\mu = \mu(s)$ for any $s \in [0, s_k + \delta_k]$;

$(Ak)_2$ $\mu(s) = s$ for $s \leq s_1$, $\mu(s_1) = \mu_*$;

$(Ak)_3$ the inclusion $\{(\mu(s), v(s)),\ s \in [0, s_k]\} \equiv \overline{\mathcal{G}^{(s_k)}(\Omega)} \subset \mathcal{G}^{(s_k+\delta_k)}$, holds;

$(Ak)_4$ $\inf_{[s_1,s_k+\delta_k)}\mu(s) > 0$ and $0 < \mu(s) \leq \mu_*$, $\forall s \in (0, s_k + \delta_k)$;

$(Ak)_5$ $0 \notin \Sigma(L_{\mu(s)})$, $\forall s \in (0, s_k + \delta_k)\setminus\{s_1, s_2, \ldots, s_k\}$;

$(Ak)_6$ $\mathrm{Ker}(L_{\lambda(s_k)}) = \mathrm{span}\{\phi_k\}$ and $\int_\Omega f(v(s_k))\phi_k \neq 0$.

Let $s_\infty = \lim_{k\to+\infty}s_k$, we claim that:

Claim: $\mu(s) \to 0^+$ as $s \to s_\infty$.

We argue by contradiction and assume that along an increasing sequence $\{\hat{s}_j\}$ such that $\hat{s}_j \to s_\infty$, it holds $\mu(\hat{s}_j) \geq \delta > 0$ for some $\delta > 0$. Clearly, we can extract a subsequence $\{s_{k_j}\} \subset \{s_k\}$ such that $s_{k_j} < \hat{s}_j \leq s_{k_{j+1}}$. By $(Ak)_4$ and (**H2**), we can extract an increasing subsequence (which we will not relabel) such that $(\mu(\hat{s}_j), v(\hat{s}_j))$ converges to a solution $(\hat{\mu}, \hat{v})$ of (1.1) as $j \to +\infty$, where $\delta \leq \hat{\mu} \leq \mu_*$.

By Theorem 1.1, we can apply either Lemma 2.1 or Proposition 2.2 and conclude that locally around $(\hat{\mu}, \hat{v})$ the set of solutions of (1.1) is a real analytic parametrization of the form $(\hat{\mu}(t), \hat{v}(t))$, $t \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ with $(\hat{\mu}(0), \hat{v}(0)) = (\hat{\mu}, \hat{v})$. In particular for j large enough, we can assume without loss of generality that $(\hat{\mu}(t), \hat{v}(t))$, $t \in (-\varepsilon, 0)$ coincides with $(\mu(s), v(s))$, $s \in (\hat{s}_j, s_\infty)$. Let $\{\hat{\sigma}_n\}_{n\in\mathbb{N}}$ be the eigenvalues corresponding to $(\hat{\mu}, \hat{v})$ and $\{\hat{\sigma}_n(t)\}_{n\in\mathbb{N}}$ be those corresponding to $(\hat{\mu}(t), \hat{v}(t))$. On the one side, since by construction $0 \in \sigma(L_{\lambda(s_{k_j})})$ and $s_{k_j} < \hat{s}_j \leq s_{k_{j+1}}$ for any j, then we have that $0 \in \sigma(L_{\hat{\lambda}})$. Indeed, if this was not the case, then, by Lemma 2.1 and since the eigenvalues are isolated, we would have that there exists a fixed full neighborhood of 0 with empty intersection with $\sigma(L_{\lambda(s_{k_j})})$ for any j large enough, which is a contradiction since the number of negative eigenvalues is, locally around each positive solution, uniformly bounded. As a consequence, there exists $n \in \mathbb{N}$ such that $\widehat{\sigma}_n = 0$. On the other side, since $\widehat{\sigma}_n(t)$ is, in particular, a continuous function of t, by using once more the fact that the eigenvalues are isolated, possibly passing to a further subsequence if necessary, we must obviously have $\widehat{\sigma}_n(\widehat{t}_j) = 0$ for some $\widehat{t}_j \to 0^-$ as $j \to +\infty$. Whence $\widehat{\sigma}_n$ must vanish identically in $(-\varepsilon, 0]$. In particular, the nth eigenvalue relative to $(\mu(s), v(s))$ must vanish identically for $s \in (\widehat{s}_j, s_\infty)$ and therefore in $[0, s_\infty)$. This is again a contradiction to $(Ak)_2$ since for $s < s_1$ we have $\sigma_1(\mu(s), v(s)) > 0$ and then no eigenvalue can vanish identically. Therefore, a contradiction arises, which shows that $\mu(s) \to 0^+$ as $s \to s_\infty$.

At this point, arguing as above, it is not difficult to see that $\|v(s)\|_\infty \to +\infty$ as $s \to s_\infty$ and, defining

$$\mathcal{G}^{(s_\infty)} = \{[0, s_\infty) \ni s \mapsto (\mu(s), v(s))\},$$

that $\mathcal{R}_\infty \equiv \mathcal{G}^{(s_\infty)}$. After a suitable reparametrization, we can assume without loss of generality that $s_2 = +\infty$, which concludes the proof. □

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