Appl. Numer. Math. 198 (2024) 112–121

Contents lists available at ScienceDirect

Applied Numerical Mathematics

journal homepage: www.elsevier.com/locate/apnum

Research Paper

Loewner integer-order approximation of MIMO fractional-order systems

Hassan Mohamed Abdelalim Abdalla [a], Daniele Casagrande [a,*], Wiesław Krajewski [b], Umberto Viaro [a]

[a] Polytechnic Department of Engineering and Architecture, University of Udine, Via delle Scienze 206, 33100 Udine, Italy
[b] Systems Research Institute, Polish Academy of Sciences, ul. Newelska 6, 01-447 Warsaw, Poland

A R T I C L E I N F O

Keywords:
Fractional order systems
Loewner matrix
Model approximation

A B S T R A C T

A state–space integer–order approximation of commensurate–order systems is obtained using a data–driven interpolation approach based on Loewner matrices. Precisely, given the values of the original fractional–order transfer function at a number of generalised frequencies, a descriptor–form state–space model matching these frequency response values is constructed from a suitable Loewner matrix pencil, as already suggested for the reduction of high–dimensional integer–order systems. Even if the stability of the resulting integer–order system cannot be guaranteed, such an approach is particularly suitable for approximating (infinite–dimensional) fractional–order systems because: (i) the order of the approximation is bounded by half the number of interpolation points, (ii) the procedure is more robust and simple than alternative approximation methods, and (iii) the procedure is fairly flexible and often leads to satisfactory results, as shown by some examples discussed at the end of the article. Clearly, the approximation depends on the location, spacing and number of the generalised interpolation frequencies but there is no particular reason to choose the interpolation frequencies on the imaginary axis, which is a natural choice in integer–order model reduction, since this axis does not correspond to the stability boundary of the original fractional–order system.

1. Introduction

The first idea of fractional calculus is credited [1], [2] to Leibniz who in a letter to de L'Hôpital in 1695 raised the question whether the meaning of derivatives with integer order could be extended to derivatives with noninteger order. Since then, the notion of fractional–order differential operators was confined to a purely theoretical context until the last decades during which there have been an increasing attention of the engineering community to this subject (see [3–5] and references therein) mainly because models including fractional–order derivatives prove to be very effective in many applications (see, just to mention a few recent applications, [6] for the model of a multi–robot system, [7] for the model of the flexible link of a manipulator arm, and [8] for the model of a heat exchanger).

* Corresponding author.
E-mail addresses: abdalla.hma@spes.uniud.it (H.M.A. Abdalla), daniele.casagrande@uniud.it (D. Casagrande), krajewsk@ibspan.waw.pl (W. Krajewski), umberto.viaro@gmail.com (U. Viaro).

https://doi.org/10.1016/j.apnum.2023.12.011
Received 7 October 2021; Received in revised form 28 June 2023; Accepted 18 December 2023

Available online 21 December 2023

Many real world systems are characterized by a large number of states, inputs and outputs, so that their dynamic behaviour is well approximated by MIMO fractional-order systems. However, since fractional-order systems are infinite-dimensional with irrational transfer function, their analysis, simulation and control require specific methods that are often computationally hard. A reasonable way to overcome the dimensionality problem is to approximate these systems, according to well–defined criteria and over appropriate stability ranges, by means of integer–order models [9–14].

In a recent publication [15], a theoretical result concerning the integer-order approximation of SISO fractional-order systems – based on the interpolation of the fractional transfer function at prescribed points – has been shown to yield interesting results. Herein, these results are extended to the case of MIMO fractional system. More precisely, starting from a chosen set of frequency-domain input-output data pairs, an integer-order state-space model in descriptor form is obtained using a data-driven approach based on Loewner matrices. Such an approach which has been extensively used with considerable success by Antoulas and co–workers to find reduced–order models of large–scale integer–order systems (even, piecewise invariant or switched) [16–19], is particularly suited to the approximation of fractional–order systems since (i) the order of the approximating model is bounded by the number of interpolation conditions, (ii) the interpolation points can be chosen depending on the frequency range of interest, which makes the procedure fairly flexible, (iii) the numerical algorithm is fairly robust compared with other available techniques. Note that in the case of MIMO systems the state-space representation is also preferable to the transfer function description which would imply to consider a matrix of transfer functions (one for each input/output pair).

It is worth noting that in the literature, several works are available concerning the analysis and control of fractional-order MIMO systems (see, for instance, [20–22] and the survey paper [23]) and some of them are dedicated to the study and the application of approximating techniques for high dimensional systems. In [24], for instance, authors exploit iterative moment matching to find the optimal integer low-order approximation. The work [25] is dedicated to systems having an "S"-shaped step response. In [26] a genetic algorithm is used while in [27] a global heuristic method approach is followed. To the best of our knowledge, this is the first time that an approach based on Loewner matrix is followed.

The remainder of this paper is organised as follows. Section 2 briefly presents the descriptor form of noninteger–order linear time–invariant systems, suggests a broad classification of the methods for their approximation, and states the specific problem. The adopted approximation procedure is illustrated in Section 3 in the general context of MIMO systems. Section 4 applies this technique to three numerical examples, while advantages and disadvantages of the considered approach are pointed out in the final Section 5.

2. Preliminaries and problem statement

Fractional–order calculus is a generalization of integer–order differentiation and integration. Many definitions of fractional–order differentiation and integration operators have been proposed over time. Especially successful have been those of Grünwald–Letnikov, Riemann–Liouville and Caputo [1]. In particular, Caputo's definition has been often used in engineering applications despite some problems related to the initial conditions [28].

In analogy with the Laplace transform of an integer-order derivative of a function f, which can be computed as

$$\mathscr{L}\{f^{(k)}\} = s^k \mathscr{L}\{f\} - \sum_{i=0}^{k-1} s^{k-i-1} f^{(i)}(0)\,, \tag{1}$$

the Laplace transform of the (fractional) Caputo derivative of order $\alpha > 0$ of a function $x : t \mapsto x(t) \in \mathbb{R}$, denoted by $D^\alpha x$, is

$$\mathscr{L}\{D^\alpha x\} = s^\alpha \mathscr{L}\{x\} - \sum_{i=0}^{[\alpha]-1} s^{\alpha-i-1} \frac{d^i x}{dt^i}(0)\,, \tag{2}$$

where $[\alpha]$ denotes the integer part of α. Note that the index i in (2) assumes only integer values so that $\mathscr{L}\{D^\alpha x\}$ is well defined.

To simplify the exposition, consider a scalar LTI fractional–order system described by the input–output differential equation

$$y(t) + \sum_{i=1}^{n} a_i D^{\alpha_i} y(t) = \sum_{i=0}^{m} b_i D^{\beta_i} u(t)\,, \tag{3}$$

where $a_i, b_i \in \mathbb{R}$, $\alpha_i, \beta_i \in \mathbb{R}_+$.

By taking (2) into account and assuming *zero initial conditions*, the system transfer function turns out to be

$$G(s) = \frac{\mathscr{L}\{y\}(s)}{\mathscr{L}\{u\}(s)} = \frac{\sum_{i=0}^{m} b_i s^{\beta_i}}{1 + \sum_{i=1}^{n} a_i s^{\alpha_i}} \triangleq \frac{b(s)}{a(s)}\,. \tag{4}$$

If all powers in (4) are multiples of the same real number $\alpha \in (0,1)$, which qualifies (4) as a commensurate–order system, the previous transfer function can be written as

$$G(s) = \frac{\sum_{i=0}^{m} b_i (s^\alpha)^i}{1 + \sum_{i=1}^{n} a_i (s^\alpha)^i} \tag{5}$$

which is a rational function of $z := s^\alpha$ and, therefore, is sometimes called *pseudo-rational* transfer function [29].

The state–space model corresponding to (4) is (see, e.g., [1])

$$D^{(\alpha)}(\mathbf{x})(t) = F\mathbf{x}(t) + Bu(t), \tag{6}$$

$$y(t) = C\mathbf{x}(t) + d\,u(t), \tag{7}$$

where

$$D^{(\alpha)}(\mathbf{x}) = \left[D^{\alpha_1}x_1, D^{\alpha_2}x_2, \cdots, D^{\alpha_n}x_n\right]^\top,$$

$F \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$ and $d \in \mathbb{R}$.

As already said, the analysis of noninteger–order models is made difficult by the irrational nature of their transfer function and by the infinite dimensionality of their state–space representations. Many methods have been proposed to simplify such models. They can be grouped into two categories:

1. the methods leading to a simpler model that is still described by an irrational transfer function or an infinite–dimensional state–space representation,
2. the methods that approximate the noninteger–order model by means of a finite–dimensional one.

The first group of methods is particularly suited to commensurate–order systems like (5): see, for example, [30–32]. However, a serious drawback is that the approximating model may not be truly simpler than the original one in terms of number of parameters. The methods of the second group are usually based on the rational approximation of the operator s^α [33]; they often lead to high–dimensional integer–order models whose transfer function tends to be ill–conditioned since its coefficients might differ for several order of magnitude. In general, even when considering a stable fractional–order system, neither of the approaches guarantees the stability of the approximating model. Herein, an interpolation approach, leading to low–order approximating models in a numerically robust way, is adopted. The associated problem can be stated as follows.

Consider a MIMO version of system (6)-(7) i.e.

$$D^{(\alpha)}(\mathbf{x})(t) = F\mathbf{x}(t) + G\mathbf{u}(t), \tag{8}$$

$$\mathbf{y}(t) = H\mathbf{x}(t) + K\mathbf{u}(t), \tag{9}$$

where $\mathbf{y}(t) \in \mathbb{R}^{p\times 1}$, $\mathbf{u}(t) \in \mathbb{R}^{m\times 1}$, $G \in \mathbb{R}^{n\times m}$, $H \in \mathbb{R}^{p\times n}$ and $K \in \mathbb{R}^{p\times m}$, and let S denote the matrix of the transfer functions.

Problem 2.1. Given a set of N complex frequency response input–output pairs (s_i, S_i), where $S_i = S(s_i)$, for $i = 1, 2, \dots, N$, find a linear model in the descriptor form:

$$\left.\begin{array}{rcl} E\dot{\mathbf{x}}(t) &=& A\mathbf{x}(t) + B\mathbf{u}(t), \\ \mathbf{y}(t) &=& C\mathbf{x}(t), \end{array}\right\} \tag{10}$$

where $\mathbf{x}(t) \in \mathbb{R}^r$, $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{y}(t) \in \mathbb{R}^p$, in such a way that:

$$C(s_iE - A)^{-1}B = S_i, \tag{11}$$

for all $i = 1, \dots, N$.

To solve this problem, resort is made to the Loewner framework illustrated in Section 3.

3. Approximation in the Loewner framework

The Loewner framework applied to fractional–order systems is a data–driven approach that derives a linear integer–order model from a given set of data which, in the specific case, are obtained from the original frequency response (4).

Essentially, this approximation technique consists of two steps:

- collect a number of frequency–response data in an *appropriate* range of frequencies, and
- construct an integer–order model matching these data.

Concerning the first step, the required data can be obtained either experimentally or, when the original model is known, numerically. Concerning the second step, the approach entails the use of a suitable matrix pencil formed from two matrices called *Loewner matrices*. Details on this approach can be found, e.g., in the works by Mayo and Antoulas [17,34]. Herein, only the elements necessary to understand how the adopted approximation technique operates are provided. To simplify the exposition, assume that the input–output pairs are collected in two sets consisting of $m\nu$ and $p\nu$ pairs, respectively:

$$M = \{(\mu_j, V_j),\ j = 1, 2, \dots, m\nu\} \in \mathbb{C}^{(1+p\times m)m\nu}, \tag{12}$$

$$\Lambda = \{(\lambda_i, W_i),\ i = 1, 2, \dots, pv\} \in \mathbb{C}^{(1+p\times m)pv}, \tag{13}$$

and form the two matrices V and W defined by

$$V = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_{mv} \end{bmatrix} \in \mathbb{C}^{pmv\times m}, \quad W = \begin{bmatrix} W_1 & W_2 & \cdots & W_{pv} \end{bmatrix} \in \mathbb{C}^{p\times mpv}.$$

The aforementioned Loewner pencil combines the Loewner matrix

$$\mathbb{L} = \begin{bmatrix} \frac{V_1 - W_1}{\mu_1 - \lambda_1} & \cdots & \frac{V_1 - W_{pv}}{\mu_1 - \lambda_{pv}} \\ \vdots & \ddots & \vdots \\ \frac{V_{mv} - W_1}{\mu_{mv} - \lambda_1} & \cdots & \frac{V_{mv} - W_{pv}}{\mu_{mv} - \lambda_{pv}} \end{bmatrix} \in \mathbb{C}^{pmv\times pmv} \tag{14}$$

and the *shifted* Loewner matrix

$$\mathbb{L}_s = \begin{bmatrix} \frac{\mu_1 V_1 - \lambda_1 W_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mu_1 V_1 - \lambda_{pv} W_{pv}}{\mu_1 - \lambda_{pv}} \\ \vdots & \ddots & \vdots \\ \frac{\mu_{mv} V_{mv} - \lambda_1 W_1}{\mu_{mv} - \lambda_1} & \cdots & \frac{\mu_{mv} V_{mv} - \lambda_{pv} W_{pv}}{\mu_{mv} - \lambda_{pv}} \end{bmatrix} \in \mathbb{C}^{pmv\times pmv} \tag{15}$$

to form the matrix $s\,\mathbb{L} - \mathbb{L}_s$.

The following important theorem, whose proof can be found in [17,19], holds.

Theorem 3.1. *If* $\det(s\,\mathbb{L} - \mathbb{L}_s) \neq 0$ *for all* $s \in \{\lambda_1, \dots, \lambda_v, \mu_1, \dots, \mu_v\}$*, then the transfer function of model (10) with*

$$E = -\mathbb{L},\ A = -\mathbb{L}_s,\ B = V,\ C = W \tag{16}$$

satisfies the interpolation conditions (11). ■

However, due to redundant data, the Loewner pencil might be singular. In this case, an additional condition must be satisfied for the existence of an approximating system with the desired properties.

Precisely, by referring again to SISO systems, a necessary and sufficient condition for the existence of an approximating model interpolating the given data is the following [17]:

$$\operatorname{rank}(s\,\mathbb{L} - \mathbb{L}_s) = \operatorname{rank}\begin{bmatrix} \mathbb{L} & \mathbb{L}_s \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \mathbb{L} \\ \mathbb{L}_s \end{bmatrix} = k \tag{17}$$

for $k \le v$ and $s \in \{\lambda_1, \dots, \lambda_v, \mu_1, \dots, \mu_v\}$.

If this condition is satisfied, the determination of the approximating model requires the preliminary computation of the singular–value decompositions:

$$\begin{bmatrix} \mathbb{L} & \mathbb{L}_s \end{bmatrix} = Y \Sigma_\ell \widetilde{X}, \qquad \begin{bmatrix} \mathbb{L} \\ \mathbb{L}_s \end{bmatrix} = \widetilde{Y} \Sigma_r X \tag{18}$$

where $\operatorname{rank}(\Sigma_\ell) = \operatorname{rank}(\Sigma_r) = \operatorname{size}(\Sigma_\ell) = \operatorname{size}(\Sigma_r) = k$, $Y \in \mathbb{C}^{pmv\times k}$ and $X \in \mathbb{C}^{k\times pmv}$.

The computation of the desired approximating model in the case of redundant data is based on the following theorem.

Theorem 3.2 *([17,19]). If the condition (17) is satisfied then the approximating model (10) with*

$$\left.\begin{aligned} E &= -Y^T\,\mathbb{L}\,X^T, \\ A &= -Y^T\,\mathbb{L}_s\,X^T, \\ B &= Y^T\,V, \\ C &= W\,X^T \end{aligned}\right\} \tag{19}$$

satisfies the interpolation conditions (11). ■

Remark 3.1. Expressions (16) and (19) clearly show that the order of the approximating model is either equal to or less than mpv.

Remark 3.2. The case of redundant data occurs more often in the rational approximation of fractional–order systems because a large number of interpolation points is likely to lead to a better fit while the dimension of the resulting model will be finite anyway. However, the number of *independent* interpolation conditions cannot exceed the overall number of parameters of the original (pseudo–rational) transfer function in (irreducible) *commensurate–order* form (5), i.e., $n + m + 1$, as the classic Padé–like rational identification method due to Zobel and Levy [35,36] clearly indicates.

Remark 3.3. For simplicity of exposition and implementation, here all the interpolation points are chosen distinct, i.e., using the terminology in [37], only the 0–moment is matched at every interpolation point (intersection number equal to 1). For the same purposes, the interpolation points are taken real. Let us observe, in this regard, that choosing the interpolation points imaginary, as is commonly done in the model reduction of integer–order systems, is not particularly meaningful in the case of fractional–order systems whose stability boundary does not correspond to the imaginary axis [2,14].

Remark 3.4. The choice of the interpolating points clearly affects the result of the approximation method. In particular, good approximations of the transient response may be obtained by choosing the interpolating points at the low frequencies, while the steady-state behaviour is better replicated by choosing the interpolating points at the high frequencies.

The Loewner–based interpolation method is applied in Section 4 to three fractional–order systems.

4. Examples

The examples considered in this section have been suggested by [38,32,24,39] and have been used to test different approximation methods. The purpose of this section is twofold: (i) to illustrate how the proposed approach operates and (ii) to show that it may lead to satisfactory results in a fairly simple way. In order to render the effectiveness of the method more clear, as a first illustrative example a SISO system is considered. The method is applied to the descriptor form of the system. The second and the third examples, instead, are proper MIMO systems described by transfer functions. It is shown that, provided that the interpolating points are suitably chosen, the method is efficient.

4.1. Example 1

Consider the fractional order system described by the transfer function

$$G(s) = \frac{1}{0.8\, s^{2.2} + 0.5\, s^{0.9} + 1}, \tag{20}$$

and let the interpolation frequencies be $\lambda_1 = 0.1$, $\lambda_2 = 0.2$, $\lambda_3 = 0.3$, $\lambda_4 = 1$, $\lambda_5 = 10$, $\lambda_6 = 100$ and $\mu_1 = 0.01$, $\mu_2 = 0.21$, $\mu_3 = 0.41$, $\mu_4 = 0.61$, $\mu_5 = 0.81$, $\mu_6 = 0.91$. The resulting Loewner matrix is full rank and the resulting 6-th order system is characterized, in the descriptor form, by the matrices

$$E \simeq \begin{bmatrix} 0.62 & 0.61 & 0.61 & 0.56 & 0.099 & 0.0099 \\ 0.60 & 0.60 & 0.61 & 0.55 & 0.089 & 0.0087 \\ 0.61 & 0.61 & 0.62 & 0.53 & 0.077 & 0.0075 \\ 0.60 & 0.60 & 0.60 & 0.50 & 0.066 & 0.0063 \\ 0.58 & 0.58 & 0.58 & 0.46 & 0.056 & 0.0053 \\ 0.57 & 0.57 & 0.56 & 0.44 & 0.051 & 0.0048 \end{bmatrix}$$

$$A \simeq \begin{bmatrix} -0.93 & -0.87 & -0.81 & -0.43 & -0.0066 & 0 \\ -0.81 & -0.75 & -0.69 & -0.32 & 0.011 & 0.0018 \\ -0.69 & -0.63 & -0.56 & -0.22 & 0.024 & 0.003 \\ -0.57 & -0.51 & -0.45 & -0.13 & 0.033 & 0.0038 \\ -0.46 & -0.41 & -0.35 & -0.064 & 0.038 & 0.0042 \\ -0.42 & -0.36 & -0.31 & -0.038 & 0.039 & 0.0043 \end{bmatrix},$$

$$B \simeq [0.9921\ 0.8707\ 0.7481\ 0.6289\ 0.5217\ 0.4741]^\top$$

and

$$C \simeq [0.9363\ 0.8767\ 0.8158\ 0.4348\ 0.0076\ 0]\,.$$

Bode diagrams and step responses are reported in Figs. 1 and 2, respectively. It can be noted that both the Bode diagrams and the step responses of the integer order model overlap with those of the original fractional order model.

4.2. Example 2

Consider the following MIMO system (a slight modification of the MIMO system considered by [24])

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}, \tag{21}$$

where

$$G_{11}(s) = \frac{5s^{1.56} + 20}{s^{3.8} + 10s^{2.93} + 20s^{1.65} + 4}, \tag{22}$$

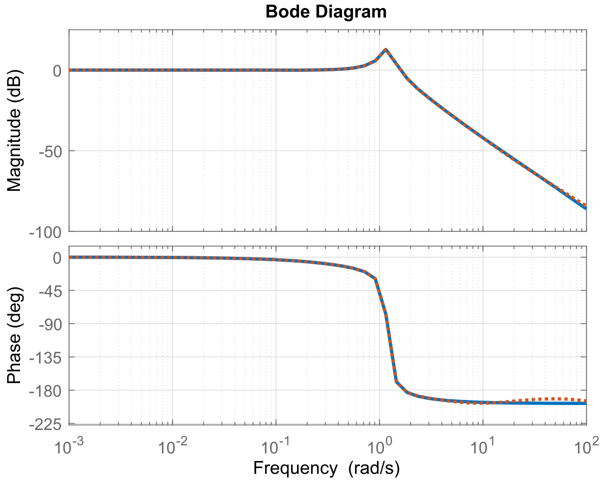

Fig. 1. Bode diagrams of the original transfer function (20) (solid blue line) and of its approximation obtained with the Loewner framework (dashed red line). (For interpretation of the colours in the figure(s), the reader is referred to the web version of this article.)

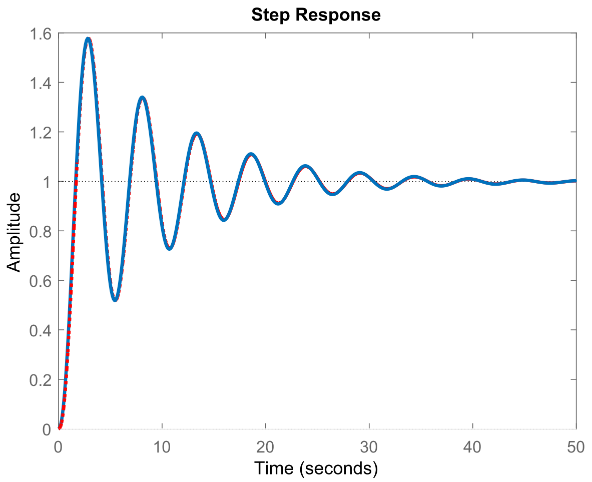

Fig. 2. Step responses of the original transfer function (20) (solid blue line) and of its approximation obtained with the Loewner framework (dashed red line).

$$G_{12}(s) = \frac{10s^{2.93} + 5s^{1.65} + 5}{s^{3.8} + 10s^{2.93} + 20s^{1.65} + 4}, \tag{23}$$

$$G_{21}(s) = \frac{0.8s^{2.15} + 3s^{1.65} + 8s^{1.28} + 27.2}{s^{3.8} + 10s^{2.93} + 20s^{1.65} + 4}, \tag{24}$$

$$G_{22}(s) = \frac{N_{22}(s)}{s^{3.8} + 10s^{2.93} + 20s^{1.65} + 4}, \tag{25}$$

with

$$N_{22}(s) = 0.5s^{3.8} + 11s^{2.93} + 0.2s^{2.15} + 13s^{1.65} + 0.4s^{1.28} + 8.2,$$

and let the interpolating frequencies be $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = 10$, $\lambda_4 = 100$, $\lambda_5 = 1000$, $\lambda_6 = 10000$ and $\mu_1 = 0.2$, $\mu_2 = 0.4$, $\mu_3 = 0.6$, $\mu_4 = 0.8$, $\mu_5 = 0.9$, $\mu_6 = 0.999$. The Loewner pencil is full rank and formulas (16) can be applied leading to a $6th$–order model.

The Bode diagrams and the step responses of the original fractional order system and of the reduced order model have been reported in Figs. 3 and 4, respectively. It can be noted that the reduced order model provides a good approximation of the original system. Fig. 5 report the relative error of the step responses. It shows, as well as the Bode diagrams do, that the selected interpolating points do not perform very well at high-frequencies. A different choice of the interpolating sets could possibly lead to better results.

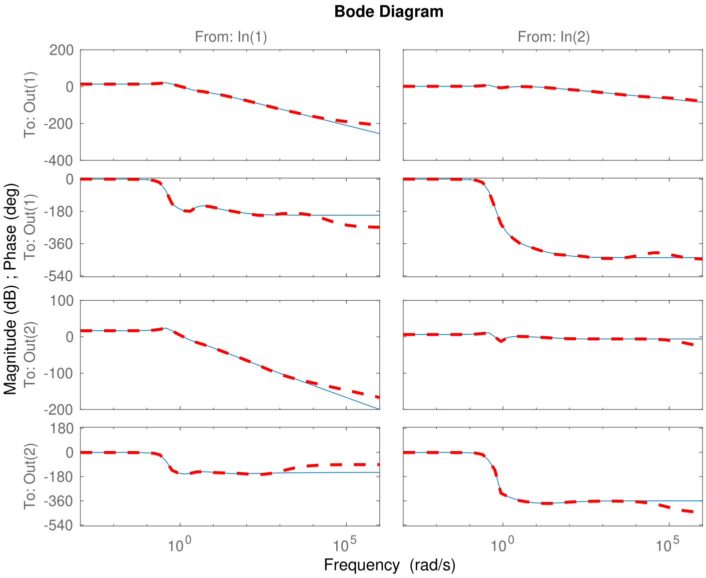

Fig. 3. Bode diagrams of the original system (21) (solid blue line) and of the reduced-order system (dashed red line).

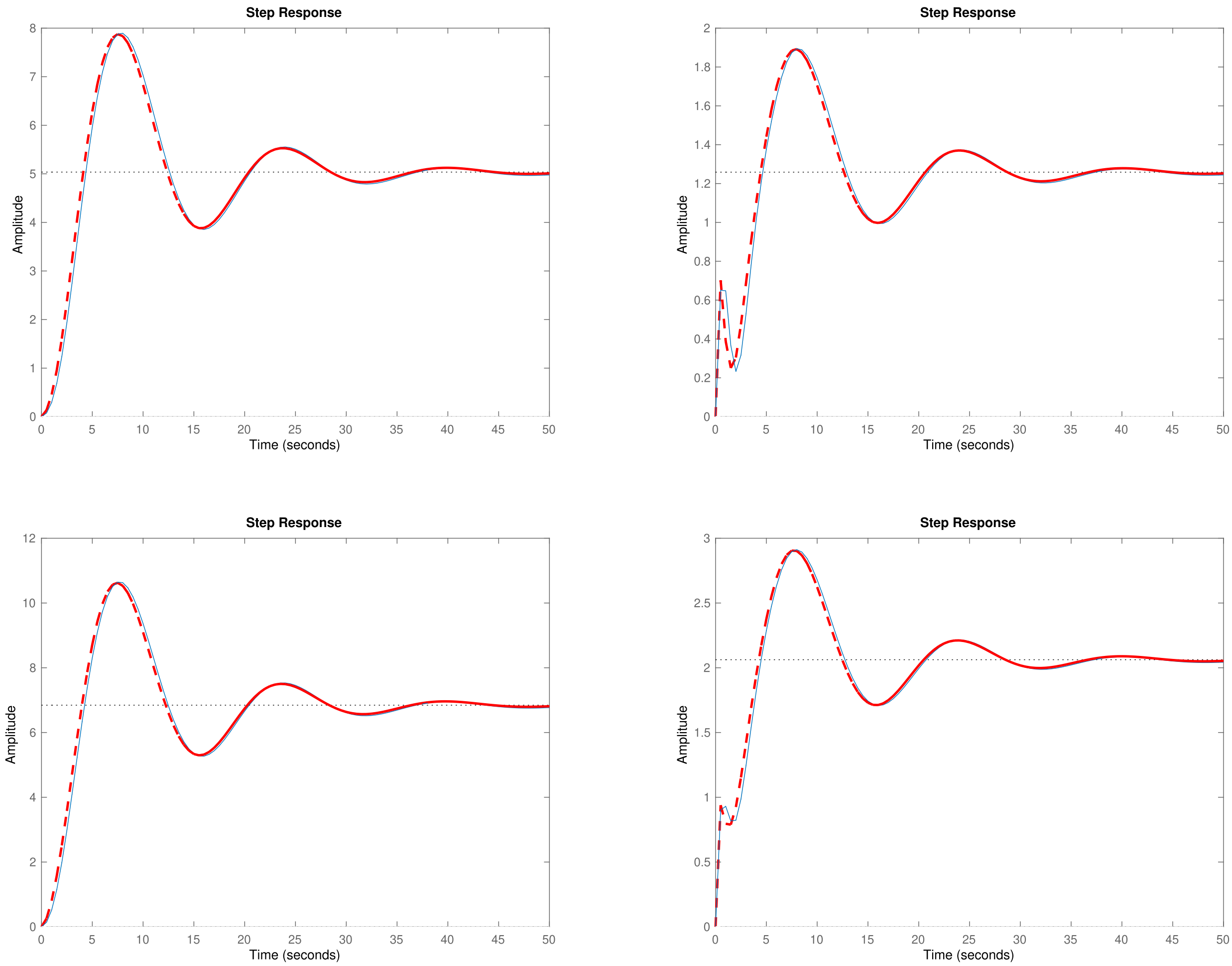

Fig. 4. Step responses of the original system (21) (solid blue line) and of the reduced-order system (dashed red line).

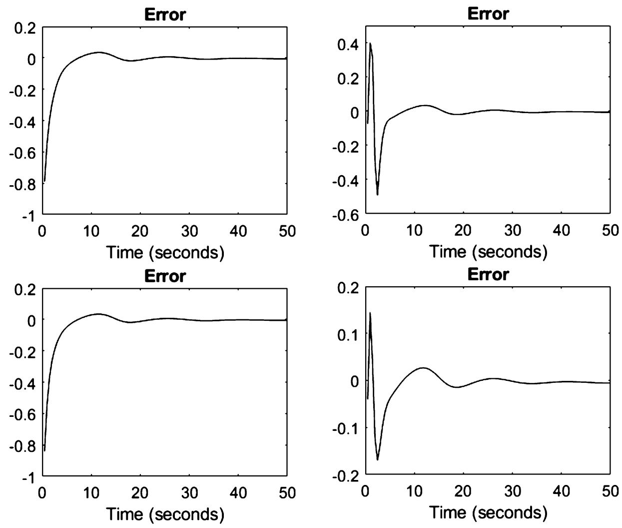

Fig. 5. Errors (normalized to 1) between the step response of the original system and that of the approximated one.

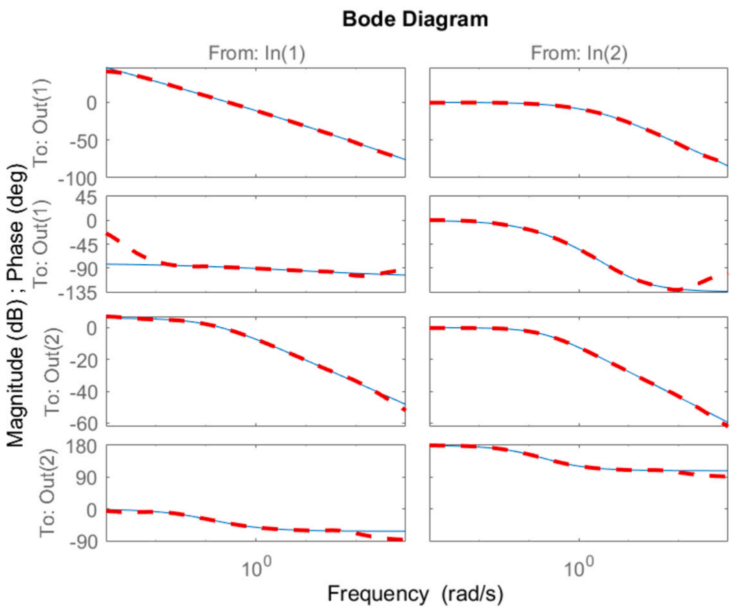

Fig. 6. Bode diagrams of the original system (26) (solid blue line) and of the reduced-order system (dashed red line).

4.3. Example 3

Consider the following MIMO system [40]

$$G(s) = \begin{bmatrix} \dfrac{1}{1.35s^{1.2} + 2.3s^{0.9} + 1} & \dfrac{2}{4.13s^{0.7} + 1} \\ \dfrac{1}{0.52s^{1.5} + 2.03s^{0.7} + 1} & -\dfrac{1}{3.8s^{0.8} + 1} \end{bmatrix}. \tag{26}$$

In this case the interpolating frequencies have been chosen as $\lambda_1 = 0.1$, $\lambda_2 = 0.2$, $\lambda_3 = 0.3$, $\lambda_4 = 0.4$, $\lambda_5 = 0.5$, $\lambda_6 = 6$ and $\mu_1 = 1.1$, $\mu_2 = 2.2$, $\mu_3 = 3.3$, $\mu_4 = 4.4$, $\mu_5 = 5.5$ and $\mu_6 = 6.6$. The Loewner pencil is full rank and formulas (16) can be applied leading to a $6th$–order model.

The Bode diagrams and the step responses of the original fractional order system and of the reduced order model have been reported in Figs. 6 and 7, respectively. Also in this case the reduced order model provides a good approximation of the original system.

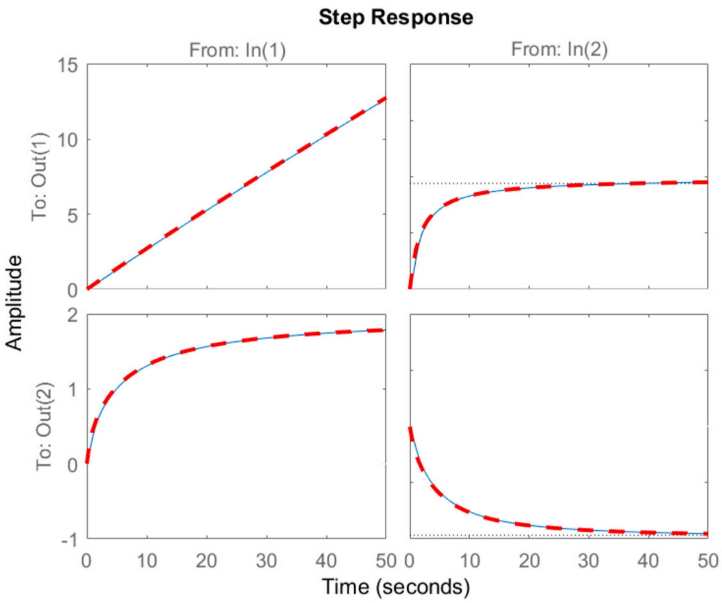

Fig. 7. Step responses of the original system (26) (solid blue line) and of the reduced-order system (dashed red line).

5. Concluding remarks

It has been shown that the approximation of (infinite–dimensional) fractional–order systems in the Loewner framework leads to low–order models whose time and frequency responses match the original responses quite accurately. Another remarkable advantage of this approach over alternative rational approximations is represented by its numerical efficiency. Note, in this regard, that alternative methods based on the Oustaloup approximation of fractional–order operators, as is done in the first step of the procedure suggested by [41], usually lead to high–order transfer functions whose computation is ill–conditioned because the magnitude of their coefficients may be extremely different: for example, the ratio between the largest and the smallest coefficient of the denominator of the intermediate transfer function derived by [41] is larger than 10^{80}. The Loewner procedure described herein could possibly be successfully applied to the description of variable-order and distributed order MIMO fractional operators [42] since a low order model, in general, is associated with a lower computational load. Future directions of investigation may concern the extension of the results to these kind of systems.

References

[1] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, CA, USA, 1999.
[2] Y. Chen, I. Petras, D. Xue, Fractional order control - a tutorial, in: Proc. American Control Conf., St. Louis, MO, USA, 2009, pp. 1397–1411.
[3] T. Kaczorek, Fractional positive continuous–time linear systems and their reachibility, Int. J. Appl. Math. Comput. Sci. 18 (2) (2008) 223–228.
[4] J. Tenreiro Machado, V. Kiryakova, The chronicles of fractional calculus, Fract. Calc. Appl. Anal. 20 (2) (2017) 307–336.
[5] A. Azar, A. Radwan, S. Vaidyanathan (Eds.), Mathematical Techniques of Fractional Order Systems, Elsevier, Amsterdam, the Netherlands, 2018.
[6] B. Goodwine, Modeling a multi–robot system with fractional-order differential equations, in: Proc. 2014 IEEE Int. Conf. Robotics & Automation (ICRA), Hong Kong, China, 2014, pp. 1763–1768.
[7] D. Casagrande, W. Krajewski, S. Miani, U. Viaro, Performance evaluation of an LQG controller of a robotic link with fractional dampers based on their integer–order approximation, in: A. Gasparetto, M. Ceccarelli (Eds.), Mechanism Design for Robotics, Springer, Chan, Switzerland, 2019, pp. 243–250.
[8] D. Casagrande, G. Cortella, W. Krajewski, S. Miani, U. Viaro, Investigation on the fractional nature of a refrigeration evaporator, Appl. Therm. Eng. 179 (2020) 115626.
[9] A. Oustaloup, F. Levron, B. Mathieu, F. Nanot, Frequency–band complex noninteger differentiator: characterization and synthesis, IEEE Trans. Circuits Syst. I, Fundam. Theory Appl. 47 (1) (2000) 25–39.
[10] T. Poinot, J. Trigeassou, A method for modelling and simulation of fractional systems, Signal Process. 83 (11) (2003) 2319–2333.
[11] R. Mansouri, M. Bettayeb, S. Djennoune, Optimal reduced–order approximation of fractional dynamical systems, in: Hichem Arioui, Rochdi Merzouki, Hadj Ahmed Abbassi (Eds.), Intelligent Systems and Automation, 1st Mediterranean Conference, American Institute of Physics, 2008, pp. 127–132.
[12] B. Krishna, Studies on fractional order differentiators and integrators: a survey, Signal Process. 91 (3) (2011) 386–426.
[13] M. Rydel, R. Stanisławski, G. Bialic, K.J. Latawiec, Modeling of discrete-time fractional-order state space systems using the balanced truncation method, in: S. Domek, P. Dworak (Eds.), Theoretical Developments and Applications of Non-Integer Order Systems, Springer, 2016.
[14] D. Casagrande, W. Krajewski, U. Viaro, Fractional–order system forced–response decomposition and its application, in: A. Azar, A. Radwan, S. Vaidyanathan (Eds.), Mathematical Techniques of Fractional Order Systems, Elsevier, Amsterdam, the Netherlands, 2018, pp. 75–102.
[15] D. Casagrande, W. Krajewski, U. Viaro, An interpolation approach to the integer-order approximation of fractional-order systems, in: Proc. of the 24th International Conference on Methods and Models in Automation and Robotics, MMAR, 2019.
[16] A. Antoulas, Approximation of Large-Scale Dynamical Systems, Advances in Design and Control, vol. 6, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, USA, 2005.
[17] A. Mayo, A. Antoulas, A framework for the generalized realization problem, Linear Algebra Appl. 426 (2007) 634–662.
[18] A. Antoulas, The Loewner framework and transfer functions of singular/rectangular systems, Appl. Math. Lett. 54 (2016) 36–47.

[19] I. Gosea, M. Petreczky, A. Antoulas, Data–driven model order reduction of linear switched systems in the Loewner framework, SIAM J. Sci. Comput. 40 (2) (2018) B572–B610.
[20] K. Hcheichi, F. Bouani, Robust MPC for fractional MIMO systems, Energy Proc. 162 (2019) 211–220.
[21] S. Lakshmanaprabu, U. Sabura Banu, P. Hemavathy, Fractional order IMC based PID controller design using Novel Bat optimization algorithm for TITO process, in: Proc. of the 1st International Conference on Power Engineering, Computing and Control, 2017.
[22] R. Mohsenipour, M. Fathi Jegarkandi, Fractional order MIMO controllers for robust performance of airplane longitudinal motion, Aerosp. Sci. Technol. 91 (2019) 617–626.
[23] A. Marques de Almeida, M. Kaminski Lenzi, E. Kaminski Lenzi, A survey of fractional order calculus applications of multiple-input, multiple-output (MIMO) process control, Fractal Fract. 4 (2) (2020) 22.
[24] M. Khanra, J. Pal, K. Biswas, Reduced order approximation of MIMO fractional order systems, IEEE J. Emerg. Sel. Top. Circuits Syst. 3 (3) (2013) 451–458.
[25] M. Tavakoli-Kakhki, M. Haeri, M. Tavazoei, Simple fractional order model structures and their applications in control system design, Eur. J. Control 6 (2010) 680–694.
[26] H. Nasiri Soloklo, N. Bigdeli, Direct approximation of fractional order systems as a reduced integer/fractional-order model by genetic algorithm, Sādhanā 45 (2020) 277.
[27] S. Ganguli, G. Kaur, P. Sarkar, Global heuristic methods for reduced-order modelling of fractional-order systems in the delta domain: a unified approach, Ric. Mat. (2021), https://doi.org/10.1007/s11587-021-00644-7.
[28] J. Sabatier, C. Farges, Misconceptions in using Riemann-Liouville's and Caputo's definitions for the description and initialization of fractional partial differential equations, in: Proc. 20th IFAC, World Congress, Toulouse, France, 2017, pp. 8574–8579.
[29] C. Monje, Y. Chen, B. Vinagre, D. Xue, V. Feliu, Fractional–Order Systems and Control: Fundamentals and Applications, Springer, London, UK, 2010.
[30] M. Tavakoli-Kakhki, M. Haeri, Model reduction in commensurate fractional–order linear systems, Proc. Inst. Mech. Eng., Part I, J. Syst. Control Eng. 223 (2009) 493–505.
[31] M. Tavazoei, M. Haeri, Model reduction in commensurate rational–order linear systems, Proc. Inst. Mech. Eng., Part I, J. Syst. Control Eng. 223 (4) (2009) 493–505.
[32] Y. Jiang, Z. Xiao, Arnoldi–based model reduction for fractional order linear systems, Int. J. Syst. Sci. 46 (2015) 1411–1420.
[33] M. Tavazoei, M. Haeri, Rational approximations in the simulation and implementation of fractional–order dynamics: a descriptor system approach, Automatica 46 (2010) 94–100.
[34] A. Antoulas, C. Beattie, S. Gugercin, Interpolatory model reduction of large-scale systems, in: K. Grigoriadis, E.J. Mohammadpour (Eds.), Efficient Modeling and Control of Large-Scale Systems, Springer, New York, USA, 2010, pp. 3–58.
[35] A. Ferrante, A. Lepschy, U. Viaro, Introduzione ai Controlli Automatici (Introduction to Automatic Control), UTET, Torino, Italy, 2000.
[36] R. Isermann, M. Münchhof, Identification of Dynamic Systems, an Introduction with Applications, Springer, Berlin, Germany, 2011.
[37] A. Astolfi, Model reduction by moment matching for linear and nonlinear systems, IEEE Trans. Autom. Control 55 (2010) 2321–2336.
[38] M. Rachid, B. Maamar, D. Said, Comparison between two approximation methods of state space fractional systems, Signal Process. 91 (3) (2011) 461–469.
[39] W. Krajewski, U. Viaro, A method for the integer–order approximation of fractional–order systems, J. Franklin Inst. 351 (1) (2014) 555–564.
[40] D. Xue, T. Li, An approach to design controllers for MIMO fractional-order plants based on parameter optimization algorithm, ISA Trans. 82 (2018) 145–152.
[41] W. Krajewski, U. Viaro, A new method for the integer order approximation of fractional order models, in: S. Domek, P. Dworak (Eds.), Theoretical Developments and Applications of NonInteger Order Systes, in: Lecture Notes in Electrical Engineering, vol. 357, Springer, 2016, pp. 81–92.
[42] C. Lorenzo, T. Hartley, Variable order and distributed order fractional operators, Nonlinear Dyn. 29 (2002) 57–98.