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Research Paper

Liouville rigidity for higher-order elliptic operators under minimal assumptions

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ABSTRACT

In this paper we prove that any distribution $u \in \mathcal{D}'(\mathbb{R}^N)$ satisfying $u \geq 0$, $-\Delta u \geq 0$, and $(-\Delta)^m u \geq 0$, must be constant whenever $N \leq 2m$. The essential feature is that no requirement is made on the intermediate iterates $(-\Delta)^j u$ for $2 \leq j \leq m-1$. We further prove that the reverse inequality $(-\Delta)^m u \leq 0$, together with $u \geq 0$ and $-\Delta u \geq 0$, forces u to be constant, with no restriction on m or N.

Both results extend to operators of the form $P(-\Delta)$, where P is a polynomial whose roots all lie in $(-\infty, 0]$. We also establish a more general version replacing the sign conditions on u and $-\Delta u$ by a single averaged vanishing condition at infinity: any $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ satisfying such a condition together with $(-\Delta)^m u \geq 0$ or ≤ 0 (and more generally $P(-\Delta)u \geq 0$ or ≤ 0) must be constant, provided $N \leq 2m$.

1. Introduction

A cornerstone of the theory of elliptic partial differential equations is the Liouville theorem for superharmonic functions.

Proposition 1 (Liouville Theorem for superharmonic functions). *Let u be a non-negative superharmonic function on $\mathbb{R}^N$ (i.e., $u \geq 0$ and $-\Delta u \geq 0$). If $N = 1$ or $N = 2$, then u is constant.*

Clearly, this result is sharp since for $N \geq 3$ there are non-constant, non-negative superharmonic functions (e.g. $(b^2 + |x|^2)^{\alpha - N/2}$ with $b > 0$ and $1 \leq \alpha < N/2$).

The principle that solutions to certain differential inequalities are forced to be constant in "low" dimensions is a widespread phenomenon extending far beyond the Laplacian. These Liouville-type results have been established for the p-Laplace and mean curvature operators, among a general class of differential operators [1,2], and for broader classes of operators including anisotropic operators and those defined on Carnot groups [3]. For instance, for the p-Laplace operator, the analogue of Proposition 1 asserts that every non-negative p-superharmonic function u (i.e., $-\Delta_p u \geq 0$) is constant whenever $N \leq p$.

This phenomenon has also been studied extensively on manifolds. A manifold for which the analogue of Proposition 1 holds is called *parabolic*. More generally, a manifold is *p-parabolic* if the only non-negative p-superharmonic functions are constants. The study of p-parabolicity has been an active area of research, revealing deep connections to geometric and analytic properties of the manifold, including volume growth, isoperimetric inequalities, certain cohomological properties, the recurrence of Brownian motion, and the validity of Hardy-type inequalities (see, e.g., [4,5] and the references therein).

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This low-dimensional Liouville property also appears for fully nonlinear operators. For the Pucci extremal operator, Cutrì and Leoni [6] prove an analogous result, where the critical dimension depends on the ellipticity parameters. For the k-Hessian operator F_k (see [7] for definitions and related information), one can state the following.

Theorem 2. *Let $u \geq 0$ in $\mathbb{R}^N$ and assume $F_k[-u] \geq 0$ in $\mathbb{R}^N$. If $N \leq 2k$, then u is constant.*

To the best of our knowledge, Theorem 2 is not stated explicitly in the literature, but it follows from Wolff potential estimates (see Appendix A for a sketch of the argument).

A natural question is whether a similar principle holds for the higher-order linear operator $(-\Delta)^m$ (or, more generally, for $P(-\Delta)$ with P a polynomial). In most of the examples above, the critical dimension matches the homogeneity of the operator. This suggests the conjecture that if $u \geq 0$ and $(-\Delta)^m u \geq 0$ in $\mathbb{R}^N$, then u is constant in the corresponding "low" dimensions, identified as $N \leq 2m$. However, this is false: for $m = 2$, the function $u(x) = x_1^2$ is non-negative and biharmonic ($(-\Delta)^2 u = 0$) in any dimension $N \geq 1$, yet it is non-constant.

A classical Liouville-type result for the polyharmonic operator states that an entire, non-negative m-polyharmonic function ($\Delta^m u = 0$) must be a polynomial of degree at most $m-2$ (see [8]; see also [9] for earlier results on bounded m-polyharmonic functions, and [10] for a modern treatment and a historical overview). One might then conjecture that if $u \geq 0$ and $(-\Delta)^m u \geq 0$ in $\mathbb{R}^N$, then u must be a polynomial. This is also false in general. The functions $u(x) = e^{x_1}$ and $u(x) = 2 + x_1^4 + \cos(x_1)$ are positive, and for $m = 2$ they satisfy $(-\Delta)^2 u = u^{(4)}(x_1) > 0$, yet neither is a polynomial.

These examples indicate that the assumptions $u \geq 0$ and $(-\Delta)^m u \geq 0$ alone are insufficient to guarantee a Liouville property. This suggests that stronger hypotheses are needed. A natural strengthening is to require non-negativity for all intermediate iterates of the Laplacian. This condition appears in [8,9], where such functions are called *complètement surharmonique*.

Definition 3. A function u is called *completely m-superharmonic* if it satisfies

$$(-\Delta)^j u \geq 0 \quad \text{in } \mathbb{R}^N, \qquad j = 0, 1, \dots, m. \tag{1}$$

In [8], Nicolesco extends the classical Liouville theorem for harmonic functions to polyharmonic ones.

Proposition 4 ([8]). *Let u be completely m-superharmonic and assume that $(-\Delta)^m u = 0$. Then u is constant.*

One can also show that if u is completely m-superharmonic and $N \leq 2m$, then u is constant. An analogous Liouville-type result in the Heisenberg group was proved by Birindelli [11]. To the best of our knowledge, a proof in the Euclidean setting does not appear explicitly in the literature; a sketch is given in Appendix A.

In this paper we prove Liouville-type theorems under assumptions weaker than complete m-superharmonicity; namely, we only require superharmonicity. Moreover, we work directly in the general setting of operators $P(-\Delta)$, where P is a polynomial with all roots in $(-\infty, 0]$, namely

$$P(z) = z^m \prod_{j=1}^{\sigma} (z + \lambda_j)^{s_j}, \quad \lambda_j > 0,\ s_j \in \mathbb{N},\ \text{ for } j = 1, \dots, \sigma,\ m \geq 1, \tag{2}$$

where if $s_1 = s_2 = \cdots = s_\sigma = 0$ we mean $P(z) = z^m$. The special case $P(z) = z^m$ recovers the results for the pure polyharmonic operator $(-\Delta)^m$.

Theorem 5. *Let $m \geq 1$ be an integer and let P be as in (2). Let u be a distribution on $\mathbb{R}^N$ such that*

$$u \geq 0, \qquad -\Delta u \geq 0, \qquad P(-\Delta)u \geq 0 \quad \text{in } \mathbb{R}^N. \tag{3}$$

If $N \leq 2m$, then u is constant.

In particular, by choosing $P(t) = t^m$, we have that if u is a distribution on $\mathbb{R}^N$ such that

$$u \geq 0, \qquad -\Delta u \geq 0, \qquad (-\Delta)^m u \geq 0 \quad \text{in } \mathbb{R}^N, \tag{4}$$

and $N \leq 2m$, then u is constant.

The main novelty of Theorem 5 is a Liouville phenomenon in the *critical/low-dimensional range* $N \leq 2m$ under the *minimal sign information* (3), namely nonnegativity and superharmonicity of u together with a one-sided inequality on the top iterate $P(-\Delta)u$. No sign assumptions are imposed on the intermediate iterates $(-\Delta)^j u$, $2 \leq j \leq m-1$.

Theorem 5 is sharp in the following sense. First, the dimensional threshold $2m$ is optimal, since for $N > 2m$, non-constant completely m-superharmonic functions exist. Second, the specific sign conditions in (4) (and hence in (3)) cannot be relaxed by replacing $-\Delta u \geq 0$ with the positivity of a higher-order iterate $(-\Delta)^j u \geq 0$ for some $2 \leq j < m$. Indeed, for $m \geq 3$ the problem

$$u \geq 0, \quad (-\Delta)^2 u \geq 0, \quad (-\Delta)^m u \geq 0, \quad \text{in } \mathbb{R}^N,$$

admits a non-constant solution in dimension $N = 1$ (e.g. $u(x_1) = x_1^4$), and hence in any dimension.

The idea of replacing the full set of conditions (1) with the weaker set (4), or more generally with (3), is motivated by the work of Caristi, D'Ambrosio, and Mitidieri [12] (see [13] for analogous results). For dimensions $N > 2m$, they show that if u satisfies a suitable condition (see $(\mathcal{R})$ below) and $(-\Delta)^m u \geq 0$, then u is automatically completely m-superharmonic; moreover, this is equivalent to (4). Here we address the complementary regime $N \leq 2m$, where one cannot directly invoke those results for u, and we show that the weaker assumptions (3) already force rigidity. Nevertheless, our proof relies on the results from [12,13] to obtain information about u.

Our main effort is also to employ a minimal set of properties of solutions (for instance, we do not use the known results on polyharmonic functions, the Almansi decomposition, and so on), thereby allowing our ideas to be extended to more general settings where many standard analytical tools may not be available. Namely, the proofs are based on the integral representation of solutions and on the classical Liouville theorem for harmonic functions. These two results are strongly connected; a link is exploited in [14] even for quite general operators. We provide two distinct proofs of Theorem 5 for the special case $P(z) = z^m$. The second, given in the appendix, is more direct but relies on deep results specific to the Euclidean framework, making it less readily adaptable to more general settings.

A careful analysis of the proof of Theorem 5 reveals that the assumptions $u \geq 0$ and $-\Delta u \geq 0$ are incompatible with $P(-\Delta)u \leq 0$, unless u is constant. This leads to the following result, which holds in *any* dimension.

Theorem 6. *Let $m \geq 2$ be an integer and let P be as in (2). Let u be a distribution on $\mathbb{R}^N$ such that*

$$u \geq 0, \qquad -\Delta u \geq 0, \qquad P(-\Delta)u \leq 0 \quad \text{in } \mathbb{R}^N. \tag{5}$$

Then u is constant (with no restriction on N).

In particular, by choosing $P(t) = t^m$, we have that if u is a distribution on $\mathbb{R}^N$ such that

$$u \geq 0, \qquad -\Delta u \geq 0, \qquad (-\Delta)^m u \leq 0 \quad \text{in } \mathbb{R}^N, \tag{6}$$

then u is constant.

As an immediate consequence of Theorems 5 and 6, we have the following.

Corollary 7. *Let $N \geq 3$. If $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ is a non-constant, non-negative superharmonic function on $\mathbb{R}^N$, then for any polynomial P as in (2) with $m \geq N/2$, the distribution $P(-\Delta)u$ must necessarily change sign.*[1]

In particular, for $N = 3, 4$, and for any integer $j \geq 2$, $(-\Delta)^j u$ must change sign.

The following corollary records the equality case $P(-\Delta)u = 0$, which already forces constancy under the weaker assumptions (3). This extends Nicolesco's Proposition 4 in two directions: to general $P(-\Delta)$, and to the weaker assumptions (3) in place of complete m-superharmonicity.

Corollary 8. *Let $m \geq 1$ be an integer and let P be as in (2). Let u be a distribution on $\mathbb{R}^N$ such that*

$$u \geq 0, \qquad -\Delta u \geq 0, \qquad P(-\Delta)u = 0 \quad \text{in } \mathbb{R}^N.$$

Then u is constant.

Although Corollary 8 may be expected at least for the case $P(-\Delta) = (-\Delta)^m$, we present it as an illustration of our main theorems, to emphasise again that our proofs do not rely on known results on the polyharmonic functions or on some of their properties, nor on the Euclidean structure. They are mainly based on integral representations. We believe that similar results can be established in frameworks with fewer analytical tools, less symmetry, and richer geometric structures.

We complement the previous results with a more general formulation of our rigidity theorems in terms of the mean of the function u. We recall that if u is a superharmonic function bounded from below then necessarily u satisfies the identity

$$\lim_{R\to\infty} \fint_{|x-y|\leq R} u(y)\,dy = \inf_{\mathbb{R}^N} u, \qquad \forall x \in \mathbb{R}^N.$$

In particular, setting $l := \operatorname{ess\,inf}_{\mathbb{R}^N} u$, the function u satisfies the following condition on the behaviour of its average at infinity:

$$\liminf_{R\to\infty} \frac{1}{R^N} \int_{|x-y|\leq R} |u(y) - l|\,dy = 0, \quad \text{for a.e. } x \in \mathbb{R}^N. \tag{$\mathcal{A}_\infty$}$$

The following result generalises Theorems 5–6 by replacing the sign conditions $u \geq 0$ and $-\Delta u \geq 0$ with the weaker averaged condition $(\mathcal{A}_\infty)$.

Theorem 9. *Let $m, N \geq 1$ and let $u \in L^1_{\text{loc}}(\mathbb{R}^N)$. Assume that there exists $l \in \mathbb{R}$ such that u satisfies $(\mathcal{A}_\infty)$.*

Let P be as in (2). If $P(-\Delta)u \geq 0$ or $P(-\Delta)u \leq 0$ in $\mathcal{D}'(\mathbb{R}^N)$ and $N \leq 2m$, then $u \equiv l$ a.e.

In particular if $(-\Delta)^m u \geq 0$ or $(-\Delta)^m u \leq 0$ in $\mathcal{D}'(\mathbb{R}^N)$ and $N \leq 2m$, then $u \equiv l$ a.e.

Remark 10. Theorem 9 generalises the case $P(-\Delta)u \geq 0$ of Theorem 5, and the case $P(-\Delta)u \leq 0$ with the restriction $N \leq 2m$. However, the full strength of Theorem 6 (namely that $u \geq 0$, $-\Delta u \geq 0$, and $P(-\Delta)u \leq 0$ force constancy in *every* dimension) is not a consequence of Theorem 9. Theorem 6 is therefore an independent result.

[1] If $P(-\Delta)u$ is only a distribution, the statement "$P(-\Delta)u$ changes sign" means that $P(-\Delta)u$ has no sign: there exist non-negative test functions $\varphi_1, \varphi_2 \in \mathcal{D}(\mathbb{R}^N)$ such that $\langle P(-\Delta)u, \varphi_1\rangle > 0$ and $\langle P(-\Delta)u, \varphi_2\rangle < 0$.

Open Problem. We wish to conclude this introduction by raising a natural question in the setting of Riemannian manifolds. As recalled above, a Riemannian manifold is called *parabolic* if every non-negative superharmonic function on it is constant. This notion has been extensively studied and is known to be deeply connected to the geometry of the manifold (see, e.g., [4,5] and the references therein). In view of Theorem 5, it is natural to say that a Riemannian manifold $\mathcal{M}$ is *parabolic of order* m if every distribution u on $\mathcal{M}$ satisfying $u \geq 0$, $-\Delta_{\mathcal{M}} u \geq 0$, and $(-\Delta_{\mathcal{M}})^m u \geq 0$ must be constant. Our result shows that $\mathbb{R}^N$ is parabolic of order m whenever $N \leq 2m$. A natural open problem is to characterise parabolicity of order m for a general Riemannian manifold in terms of intrinsic geometric quantities, e.g. the volume growth. We hope this may stimulate further investigation.

2. Proofs of main results

Definition 11. Let $u \in \mathcal{D}'(\mathbb{R}^N)$ be a distribution and $k \in \mathbb{N}$. A distribution u is said to satisfy an inequality such as $(-\Delta)^k u \geq 0$ if for any non-negative test function $\phi \in \mathcal{D}(\mathbb{R}^N)$, the following holds:

$$\left\langle (-\Delta)^k u, \phi \right\rangle = \left\langle u, (-\Delta)^k \phi \right\rangle \geq 0,$$

with the convention that for $k = 0$, $(-\Delta)^0 \phi = \phi$. Hence, for a polynomial Q, a distribution u is said to satisfy the inequality $Q(-\Delta)u \geq 0$ if for any non-negative test function $\phi \in \mathcal{D}(\mathbb{R}^N)$, we have

$$\langle Q(-\Delta)u, \phi \rangle = \langle u, Q(-\Delta)\phi \rangle \geq 0.$$

In the particular case that the distribution $u \in L^1_{\text{loc}}(\mathbb{R}^N)$, this means

$$\int_{\mathbb{R}^N} u(x) Q(-\Delta)\phi(x)\, dx \geq 0.$$

Remark 12. We recall that a non-negative distribution can be identified with a positive Radon measure (see, e.g., [15]). Furthermore, if u is a superharmonic distribution, it can be shown that it has an $L^1_{\text{loc}}(\mathbb{R}^N)$ representative [16].

Our proofs rely on the integral representation for solutions to higher-order equations developed in [12] and subsequently in [13]. A central role in those papers is played by the following condition. A function $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ satisfies the *ring condition* for a constant $l \in \mathbb{R}$ if for a.e. $x \in \mathbb{R}^N$,

$$\liminf_{R\to\infty} \frac{1}{R^N} \int_{R \leq |x-y| \leq 2R} |u(y) - l|\, dy = 0. \tag{$\mathcal{R}$}$$

It is a key fact, established in [12], that a non-negative superharmonic function on $\mathbb{R}^N$ with $N \geq 3$ satisfies the ring condition ($\mathcal{R}$) with $l = \operatorname{essinf}_{\mathbb{R}^N} u$.

The key tool is the following representation result from [13], which covers all operators $P(-\Delta)$ as in (2).

Proposition 13 ([13, Theorem. 1.1])**.** *Let $k \geq 1$ be an integer and let $N > 2k$. Let Q be a polynomial whose roots belong to $(-\infty, 0]$ and such that 0 is a root of Q of multiplicity exactly k, namely*

$$Q(z) = z^k \left(z^\sigma + a_{\sigma-1} z^{\sigma-1} + \cdots + a_1 z + a_0 \right), \qquad a_0 \neq 0,$$

and let Φ_Q denote the fundamental solution of $Q(-\Delta)$. Let μ be a positive Radon measure on $\mathbb{R}^N$ and let $v \in L^1_{\text{loc}}(\mathbb{R}^N)$ be a distributional solution of $Q(-\Delta)v = \mu$. The following are equivalent:

1. *v satisfies the ring condition ($\mathcal{R}$) for some $l \in \mathbb{R}$.*
2. *v is weakly superharmonic (i.e., $-\Delta v \geq 0$ in distributional sense) and $\operatorname{essinf}_{\mathbb{R}^N} v = l$.*
3. *v admits the representation*

$$v(x) = l + \int_{\mathbb{R}^N} \Phi_Q(x - y)\, d\mu(y) \qquad \text{for a.e. } x \in \mathbb{R}^N.$$

Furthermore, if any of these conditions holds, then v is completely k-superharmonic, meaning $(-\Delta)^j v$ is a positive Radon measure for $j = 1, \dots, k$, and

$$\operatorname{essinf}_{\mathbb{R}^N} v = l = \lim_{R\to\infty} \fint_{B_R(x)} v(y)\, dy, \quad \text{for all } x \in \mathbb{R}^N.$$

Moreover, Φ_Q is positive and satisfies the lower bound

$$\Phi_Q(x) \geq c\,|x|^{2k-N} \qquad \text{for } |x| \geq 1, \tag{7}$$

for some constant $c > 0$ depending only on Q and N.

Notice that Proposition 13 guarantees the equivalence ($\mathcal{R}$) $\Longleftrightarrow$ ($\mathcal{A}_\infty$) in the superharmonic setting.

Proof of Theorem 5. By standard mollification arguments, we can assume without loss of generality that u is a smooth function, $u \in C^\infty(\mathbb{R}^N)$, satisfying the conditions (3) in the classical sense. To this end, it suffices to recall that convolution with a standard mollifier commutes with the Laplace operator. The case $m = 1$ corresponds to Proposition 1, so we assume $m \geq 2$.

Step 1: Reduction to the case $N = 2m$. If $N < 2m$, we can employ a standard lifting argument. Define a new function $v : \mathbb{R}^{2m} \to \mathbb{R}$ by

$$v(x_1, \dots, x_N, x_{N+1}, \dots, x_{2m}) = u(x_1, \dots, x_N).$$

The function v is smooth and inherits the properties of u. Specifically, denoting by Δ_k the Laplacian in $\mathbb{R}^k$, we have $(-\Delta_{2m})^j v = (-\Delta_N)^j u$. Therefore, $v \geq 0$, $-\Delta_{2m} v \geq 0$, and $P(-\Delta_{2m})v \geq 0$ on $\mathbb{R}^{2m}$. If we can prove that v must be constant, it immediately follows that u is constant. Thus, without loss of generality, we can assume the dimension is exactly $N = 2m$. Since $m \geq 2$, we have $N \geq 4$.
Step 2: Representation of u. Let us define the functions

$$u_1 := -\Delta u, \quad u_m := P(-\Delta)u,$$

which by hypothesis are non-negative. Since u is a non-negative superharmonic function on $\mathbb{R}^N$ with $N \geq 4$, by the Riesz representation theorem there exists a constant $l_0 = \inf u \geq 0$ and a positive constant C_N (see [15] for the exact value) such that for any $x \in \mathbb{R}^N$

$$u(x) = l_0 + C_N \int_{\mathbb{R}^N} \frac{u_1(y)}{|x-y|^{N-2}} dy. \tag{8}$$

Since $u(x)$ is finite for every $x \in \mathbb{R}^N$, the integral on the right-hand side must converge. This last remark allows us to prove the following.
Step 3: u_1 satisfies $(\mathcal{R})$. We claim that u_1 satisfies

$$\lim_{R\to\infty} \frac{1}{R^N} \int_{R\leq|x-y|\leq 2R} |u_1(y)|\, dy = 0 \ \text{ for any } x \in \mathbb{R}^N, \tag{9}$$

which in turn implies that u_1 satisfies $(\mathcal{R})$ with $l = 0$. The proof is by contradiction. Assume that the condition is not met. Then there exist $\epsilon > 0$ and a sequence $R_j \to \infty$ such that for a given $x \in \mathbb{R}^N$,

$$\frac{1}{R_j^N} \int_{R_j\leq|x-y|\leq 2R_j} u_1(y)\, dy \geq \epsilon.$$

We can estimate the integral in (8) from below:

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{u_1(y)}{|x-y|^{N-2}} dy &\geq \int_{R_j\leq|x-y|\leq 2R_j} \frac{u_1(y)}{|x-y|^{N-2}} dy \geq \frac{1}{(2R_j)^{N-2}} \int_{R_j\leq|x-y|\leq 2R_j} u_1(y) dy \\ &\geq \frac{R_j^N \epsilon}{(2R_j)^{N-2}} = \frac{\epsilon}{2^{N-2}} R_j^2.\end{aligned}$$

By letting $j \to \infty$, the right-hand side tends to infinity. This implies that the integral in (8) diverges, which contradicts the fact that $u(x)$ is finite. This proves the claim.
Step 4: Representation of u_1. We have established that $u_1 \geq 0$ satisfies $(\mathcal{R})$ with $l = 0$. Setting

$$Q(z) := \frac{P(z)}{z} = z^{m-1} \prod_{j=1}^{\sigma} (z + \lambda_j)^{s_j}.$$

we have $Q(-\Delta)u_1 = Q(-\Delta)(-\Delta u) = P(-\Delta)u = u_m \geq 0$. The space dimension is $N = 2m$, so $N > 2(m-1) = 2k$ with $k = m - 1$. Hence all conditions of Proposition 13 are met, and we obtain

$$u_1(x) = \int_{\mathbb{R}^N} \Phi_Q(x-y)\, u_m(y)\, dy. \tag{10}$$

Step 5: Proof that $u_m \equiv 0$. We prove this by contradiction. Assume that u_m is not identically zero. Since u_m is a non-negative continuous function, there exists a ball B_{R_0} (centered at the origin with radius R_0) such that $\int_{B_{R_0}} u_m(y)\, dy > 0$. From (10), we obtain a lower bound for $u_1(x)$: for $|x| > R_0 + 1$

$$\begin{aligned}u_1(x) &\geq \int_{|y|<R_0} \Phi_Q(x-y)\, u_m(y)\, dy \geq c \int_{|y|<R_0} \frac{u_m(y)}{|x-y|^2} dy \\ &\geq \frac{c}{(|x|+R_0)^2} \int_{|y|<R_0} u_m(y) dy = \frac{c_0}{(|x|+R_0)^2},\end{aligned}$$

where $c_0 = c \int_{B_{R_0}} u_m(y) dy > 0$, and we have used (7) with $k = m - 1$, which for $|x - y| \geq 1$ gives $\Phi_Q(x-y) \geq c|x-y|^{2(m-1)-N} = c|x-y|^{-2}$. We now substitute this estimate for u_1 back into the representation for u from (8):

$$u(x) \geq l_0 + C_N \int_{|y|>R_0+1} \frac{c_0}{(|y|+R_0)^2} \frac{1}{|x-y|^{N-2}} dy.$$

The integral on the right-hand side diverges, implying $u(x)$ must be infinite. This contradiction completes the proof of the claim.
Step 6: Conclusion. Since $u_m \equiv 0$, from (10) we get $u_1 \equiv 0$, i.e. $-\Delta u = 0$. Consequently, u is a non-negative harmonic function on $\mathbb{R}^N$. By the classical Liouville theorem, u must be constant, concluding the proof. □

Proof of Theorem 6. The proof follows the same steps as in the proof of Theorem 5, so we shall be brief. As before, we can assume without loss of generality that u is a smooth function satisfying (5). With the same notation as in the previous proof, the hypotheses read:

$$u \geq 0, \quad u_1 := -\Delta u \geq 0, \quad u_m := P(-\Delta)u \leq 0.$$

The core of the argument is to show that $u_1 = -\Delta u \equiv 0$. The conclusion then follows immediately by the classical Liouville theorem.

First, by a lifting argument analogous to Step 1 in the proof of Theorem 5, we can assume that the dimension N satisfies $N > 2(m-1)$.

Second, as established in Steps 2 and 3 of the previous proof, u admits the representation (8), and this implies that u_1 must satisfy the ring condition $(\mathcal{R})$ with $l = 0$.

Now, set $v := -u_1$. With $Q(z) := P(z)/z$ as in Step 4 of the previous proof, the equation for v is $Q(-\Delta)v = -u_m$. Since $u_m \leq 0$, the measure $\mu := -u_m$ is a positive Radon measure, so $Q(-\Delta)v = \mu \geq 0$. The function v satisfies the ring condition $(\mathcal{R})$ with $l = 0$ (since u_1 does), and $N > 2(m-1)$, so all conditions of Proposition 13 with $k = m-1$ are met. This yields the representation

$$v(x) = \int_{\mathbb{R}^N} \Phi_Q(x-y)\, d\mu(y) = \int_{\mathbb{R}^N} \Phi_Q(x-y)\,(-u_m(y))\, dy.$$

Since $-u_m \geq 0$ and $\Phi_Q > 0$, the integral is non-negative, so $v \geq 0$. On the other hand, by definition $v = -u_1 \leq 0$. Thus $v \equiv 0$, which implies $u_1 \equiv 0$, completing the proof. □

In order to prove Theorem 9, the main idea is to regain the hypotheses of a sign on u and on $-\Delta u$ by an embedding of our problems into a higher-dimensional Euclidean space $\mathbb{R}^M$ with $M = N + K > 2m$. However, this lifting procedure is more delicate than in the previous proofs of Theorems 5 and 6.

Lemma 14. *Let $N \geq 1$ and let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Assume that there exists $l \in \mathbb{R}$ such that $(\mathcal{A}_\infty)$ holds.*

Let $K \in \mathbb{N}$ and set $M := N + K$. Define the lifted function $v : \mathbb{R}^M = \mathbb{R}^N \times \mathbb{R}^K \to \mathbb{R}$ by

$$v(x,z) := u(x).$$

Then v satisfies the condition $(\mathcal{A}_\infty)$ in $\mathbb{R}^M$ for the same constant l.

Proof of Lemma 14. In what follows we denote by $B^p_R(y)$ the Euclidean ball in $\mathbb{R}^p$ of radius R centred at y. Fix $(x_0, z_0) \in \mathbb{R}^N \times \mathbb{R}^K$ such that $(\mathcal{A}_\infty)$ holds at x_0. For $R > 0$ denote

$$Q_R := B^N_R(x_0) \times \prod_{j=1}^{K} (z_{0,j} - R,\ z_{0,j} + R) \subset \mathbb{R}^M.$$

Then $|Q_R| = |B^N_R|\,(2R)^K = 2^K \omega_N R^M$, and we have

$$\fint_{Q_R} \left|v - l\right| = \frac{1}{(2R)^K} \int_{\prod_{j=1}^K (z_{0,j}-R,\ z_{0,j}+R)} \left(\fint_{B^N_R(x_0)} |u-l| \right) dz = \fint_{B^N_R(x_0)} |u - l|.$$

Using $B^M_R(x_0, z_0) \subset Q_R$ we infer

$$\fint_{B^M_R(x_0,z_0)} |v-l| \leq \frac{|Q_R|}{|B^M_R|} \fint_{Q_R} |v-l| = \frac{2^K \omega_N}{\omega_M} \fint_{Q_R} |v-l| = C_{N,K} \fint_{B^N_R(x_0)} |u-l|.$$

Since u satisfies $(\mathcal{A}_\infty)$ in $\mathbb{R}^N$, from the last inequality we get the claim. □

Proof of Theorem 9. We first treat the case $P(-\Delta)u \geq 0$.

By replacing u with $u - l$, we may assume that $(\mathcal{A}_\infty)$ holds with $l = 0$.

Our goal is to prove that $u \geq 0$ and $-\Delta u \geq 0$ in $\mathcal{D}'(\mathbb{R}^N)$, so that the claim follows applying Theorem 5.

Fix $K \in \mathbb{N}$ that we shall choose later and set $M := N + K$. Define $v \in L^1_{loc}(\mathbb{R}^M)$ by

$$v(x,z) := u(x), \qquad (x,z) \in \mathbb{R}^N \times \mathbb{R}^K = \mathbb{R}^M.$$

By Lemma 14, v satisfies $(\mathcal{A}_\infty)$ in $\mathbb{R}^M$ with the same constant $l = 0$.

Unlike the proofs of Theorems 5 and 6, we cannot simply regularize v and argue pointwise: since v is only in L^1_{loc} and $(\mathcal{A}_\infty)$ is not known to be stable under mollification, we must perform the lifting directly in $\mathcal{D}'(\mathbb{R}^M)$, using tensor products. Therefore, preliminarily, we deduce some useful facts on the subject, see [17] for further information.

Let $1_{\mathbb{R}^K} \in \mathcal{D}'(\mathbb{R}^K)$ be the distribution induced by the constant function 1, i.e.

$$\langle 1_{\mathbb{R}^K}, \psi \rangle := \int_{\mathbb{R}^K} \psi(z)\, dz, \qquad \psi \in \mathcal{D}(\mathbb{R}^K).$$

For $T \in \mathcal{D}'(\mathbb{R}^N)$ we denote by $T \otimes 1_{\mathbb{R}^K} \in \mathcal{D}'(\mathbb{R}^{N+K})$ the tensor product defined by

$$\langle T \otimes 1_{\mathbb{R}^K}, \Phi \rangle := \left\langle T,\ x \mapsto \int_{\mathbb{R}^K} \Phi(x,z)\, dz \right\rangle, \qquad \Phi \in \mathcal{D}(\mathbb{R}^{N+K}). \tag{11}$$

For $T \in \mathcal{D}'(\mathbb{R}^N)$ we deduce the equivalence

$$T \geq 0 \ \text{ in } \ \mathcal{D}'(\mathbb{R}^N) \Longleftrightarrow T \otimes 1_{\mathbb{R}^K} \geq 0 \ \text{ in } \ \mathcal{D}'(\mathbb{R}^{N+K}). \tag{12}$$

Indeed, the implication $\Longrightarrow$ is immediate. For $\Longleftarrow$: if $\langle T, \varphi\rangle < 0$ for some nonnegative $\varphi \in \mathcal{D}(\mathbb{R}^N)$, then taking $\Phi(x,z) := \varphi(x)\psi(z) \geq 0$ with $\psi \in \mathcal{D}(\mathbb{R}^K)$ nonnegative and $\psi \not\equiv 0$, from (11) we have $\langle T \otimes 1_{\mathbb{R}^K}, \Phi\rangle = \langle T, \varphi\rangle \int_{\mathbb{R}^K} \psi(z)dz < 0$, a contradiction.

With the definition (11), one checks that $v = u \otimes 1_{\mathbb{R}^K}$ in $\mathcal{D}'(\mathbb{R}^M)$, since for every $\Phi \in \mathcal{D}(\mathbb{R}^M)$,

$$\langle v, \Phi\rangle = \int_{\mathbb{R}^M} u(x)\Phi(x,z)\,dx\,dz = \Big\langle u,\ x \mapsto \int_{\mathbb{R}^K} \Phi(x,z)\,dz\Big\rangle = \langle u \otimes 1_{\mathbb{R}^K}, \Phi\rangle.$$

Denote by $\Delta_{x,z}$ [resp. Δ_x] the Laplace operator in $\mathbb{R}^M$ [resp. $\mathbb{R}^N$]. We now prove the key identity

$$P(-\Delta_{x,z})\, v = \big(P(-\Delta_x)u\big) \otimes 1_{\mathbb{R}^K} \qquad \text{in } \mathcal{D}'(\mathbb{R}^M). \tag{13}$$

Let $\Phi \in \mathcal{D}(\mathbb{R}^M)$. By definition of distributional derivatives,

$$\langle P(-\Delta_{x,z})v, \Phi\rangle = \langle v, P(-\Delta_{x,z})\Phi\rangle = \Big\langle u,\ x \mapsto \int_{\mathbb{R}^K} P(-\Delta_{x,z})\Phi(x,z)\,dz\Big\rangle.$$

Since Φ is compactly supported and smooth, integration by parts in z shows that

$$\begin{aligned}\int_{\mathbb{R}^K} (-\Delta_{x,z})\Phi(x,z)\,dz &= \int_{\mathbb{R}^K} (-\Delta_x)\Phi(x,z)\,dz + \int_{\mathbb{R}^K} (-\Delta_z)\Phi(x,z)\,dz \\ &= \int_{\mathbb{R}^K} (-\Delta_x)\Phi(x,z)\,dz = (-\Delta_x)\Big(\int_{\mathbb{R}^K} \Phi(x,z)\,dz\Big),\end{aligned}$$

and iterating we have

$$\int_{\mathbb{R}^K} (-\Delta_{x,z})^j \Phi(x,z)\,dz = (-\Delta_x)^j\Big(\int_{\mathbb{R}^K} \Phi(x,z)\,dz\Big)$$

for every integer $j \geq 0$, hence by linearity the same identity holds with $P(-\Delta_{x,z})$:

$$\int_{\mathbb{R}^K} P(-\Delta_{x,z})\Phi(x,z)\,dz = P(-\Delta_x)\Big(\int_{\mathbb{R}^K} \Phi(x,z)\,dz\Big).$$

Therefore,

$$\begin{aligned}\langle P(-\Delta_{x,z})v, \Phi\rangle &= \Big\langle u,\ P(-\Delta_x)\Big(\int_{\mathbb{R}^K} \Phi(\cdot,z)\,dz\Big)\Big\rangle \\ &= \Big\langle P(-\Delta_x)u,\ \int_{\mathbb{R}^K} \Phi(\cdot,z)\,dz\Big\rangle \\ &= \langle (P(-\Delta_x)u) \otimes 1_{\mathbb{R}^K}, \Phi\rangle,\end{aligned}$$

which proves (13).

Now, choose K so large that $M = N + K > 2m$. Since $P(-\Delta_x)u \geq 0$ in $\mathbb{R}^N$ in the distributional sense, by relations (13) and (12) we have that

$$P(-\Delta_{x,z})v \geq 0 \ \text{ in } \ \mathcal{D}'(\mathbb{R}^M).$$

By Lemma 14 the function v satisfies $(\mathcal{A}_\infty)$ in $\mathbb{R}^M$ with constant $l = 0$. Since $(\mathcal{A}_\infty)$ immediately implies the ring condition $(\mathcal{R})$, the lifted function v satisfies $(\mathcal{R})$ in $\mathbb{R}^M$ with $l = 0$. Since $M > 2m$, we can apply Proposition 13 to v obtaining that $v \geq 0$ and $-\Delta_{x,z}v \geq 0$ in $\mathcal{D}'(\mathbb{R}^M)$. Using again relations (13) and (12) we deduce that $u \geq 0$ and $-\Delta u \geq 0$ in $\mathcal{D}'(\mathbb{R}^N)$. Thus we are in the position to apply Theorem 5 (in the low-dimensional regime $N \leq 2m$) to u and conclude

$$u \equiv l \qquad \text{in } \mathbb{R}^N.$$

The case $P(-\Delta)u \leq 0$. Assume $P(-\Delta)u \leq 0$ in $\mathcal{D}'(\mathbb{R}^N)$. Set $\tilde{u} := -u$. Then $\tilde{u}$ satisfies $(\mathcal{A}_\infty)$ with constant $-l$ and

$$P(-\Delta_x)\tilde{u} = -\,P(-\Delta_x)u \geq 0.$$

By the first part of this proof, $\tilde{u} \equiv -l$, hence $u \equiv l$. ☐

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Appendix A. Supplementary proofs

For the reader's convenience, this appendix collects the proofs of several results that supplement the main text. We begin by sketching the proof of Theorem 2 (the k-Hessian case), referenced in the introduction.

Sketch of proof of Theorem 2. For details on the definition of solutions and other concepts, see [7].

Before entering into the proof, we begin with general considerations about the Wolff potential. We recall that the Wolff potential of a Radon measure μ with parameters $\alpha \in \mathbb{R}, \gamma > 0$ is defined as

$$W^{\alpha,\gamma}[\mu](x, R) = \int_0^R \left(\frac{\mu(B(x,r))}{r^\alpha} \right)^{1/\gamma} \frac{dr}{r}, \quad \text{for } x \in \mathbb{R}^N, R > 0.$$

Assume that a function $u : \mathbb{R}^N \to \mathbb{R}$ is finite for some $x \in \mathbb{R}^N$ and satisfies, at that x, the lower bound

$$u(x) \geq c_1 W^{\alpha,\gamma}[\mu](x, R) \text{ for all large } R, \tag{A.1}$$

for a suitable Radon measure μ and a constant $c_1 > 0$. If $\alpha \leq 0$, then μ must be identically zero. Indeed, if this is not the case, there exists $R_0 > 0$ such that $\mu(B(x, R_0)) = c_2 > 0$. Hence, for any R large

$$u(x) \geq c_1 W^{\alpha,\gamma}[\mu](x, R) \geq c_1 \int_{R_0}^R \left(\frac{\mu(B(x,r))}{r^\alpha} \right)^{1/\gamma} \frac{dr}{r} \geq c_1 c_2^{1/\gamma} \int_{R_0}^R r^{-\frac{\alpha}{\gamma}} \frac{dr}{r} \geq c_1 c_2^{1/\gamma} R_0^{-\frac{\alpha}{\gamma}} \ln \frac{R}{R_0}.$$

By letting $R \to +\infty$, the last chain of inequalities contradicts the finiteness of $u(x)$.

Now, in the case of the k-Hessian operator, if u is non-negative and $F_k[-u] \geq 0$, then results from [7] show that u satisfies a two-sided estimate involving the Wolff potential. Namely, after normalizing so that $\inf u = 0$, u satisfies both a lower bound of the form (A.1) and an upper bound

$$u(x) \leq c_2 W^{\alpha,\gamma}[\mu](x, 2R) + c_2 \inf_{B(x,R)} u \tag{A.2}$$

for large R, with $\gamma = k$, $\alpha = N - 2k$, and a suitable Radon measure μ for a.e. $x \in \mathbb{R}^N$. The dimensional condition $N \leq 2k$ implies $\alpha \leq 0$, which in turn forces the associated measure μ to be zero and consequently, from (A.2), leads to the constancy of u. □

In the introduction, we mentioned the following result.

Theorem 15 (Liouville theorem for completely m-superharmonic functions). *Let $m \geq 1$ be an integer and let $u \in \mathcal{D}'(\mathbb{R}^N)$ be a distribution such that*

$$u \geq 0, \qquad -\Delta u \geq 0, \qquad (-\Delta)^2 u \geq 0, \ \ldots, \ (-\Delta)^m u \geq 0 \qquad \text{in } \mathbb{R}^N.$$

If $N \leq 2m$, then u is constant.

Although this result is a corollary of Theorem 5, we provide here two alternative proofs. The first is based not on Proposition 13, but on the simpler classical Riesz representation for non-negative superharmonic functions. This approach can be used to obtain similar results for higher-order differential operators obtained by iterating fairly general second-order operators (for which holds the analogue of the Riesz representation formula). For instance, this class includes the polyharmonic operators on Carnot groups (we leave the straightforward generalization of Theorem 15 to the sub-Laplacian on Carnot groups to the interested reader).

The second proof relies solely on fundamental properties of superharmonic functions, thereby avoiding the representation theory. This approach may be useful for operators where such integral formulas are not available, provided there is a suitable notion of radialization. This second approach is quite similar to the one used in [11] in the Heisenberg group framework.

Sketch of Proof of Theorem 15 by Riesz representation. If $N = 1, 2$, then the claim follows from Theorem 1.

Assume $N > 2$. The claim is equivalent to showing that if u is not constant, then $N > 2m$. For $j = 0, 1, \ldots, m$ set

$$u_j := (-\Delta)^j u. \tag{A.3}$$

In particular, $u_1 \geq 0$ and $-\Delta u_1 = u_2 \geq 0$, hence by the Riesz representation theorem there exists $\ell_1 \geq 0$ such that

$$u_1(x) = \ell_1 + C(N) \int_{\mathbb{R}^N} \frac{u_2(y)}{|x - y|^{N-2}} \, dy \qquad \forall x \in \mathbb{R}^N. \tag{A.4}$$

On the other hand, since $u \geq 0$ and $-\Delta u = u_1 \geq 0$ (and $N > 2$), there exists $\ell_0 \geq 0$ such that

$$u(x) = \ell_0 + C(N) \int_{\mathbb{R}^N} \frac{u_1(y)}{|x - y|^{N-2}} \, dy \qquad \forall x \in \mathbb{R}^N. \tag{A.5}$$

Plugging (A.4) into (A.5) gives

$$u(x) = \ell_0 + C(N) \int_{\mathbb{R}^N} \frac{\ell_1}{|x - y|^{N-2}} \, dy + C(N)^2 \int_{\mathbb{R}^N} \frac{1}{|x - y|^{N-2}} \left(\int_{\mathbb{R}^N} \frac{u_2(z)}{|y - z|^{N-2}} \, dz \right) dy. \tag{A.6}$$

Clearly $\ell_1 = 0$. Indeed if $\ell_1 \neq 0$ then the integral $\int_{\mathbb{R}^N} \frac{\ell_1}{|x-y|^{N-2}} dy$ is divergent for any x and from (A.6) we have the contradiction $u \equiv \infty$.

We claim that $u_2 \not\equiv 0$. Indeed, if $u_2 \equiv 0$, then (A.4) gives that u_1 is constant; since $\ell_1 = 0$, we have $u_1 \equiv 0$, and then (A.5) yields that u is constant, a contradiction.

Therefore the identity (A.6), by the Fubini–Tonelli theorem, reads as

$$u(x) = \ell_0 + C(N)^2 \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2}} \frac{1}{|y-z|^{N-2}}\, dy \right) u_2(z) dz.$$

Since $u_2 \not\equiv 0$, the inner integral

$$\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2}} \frac{1}{|y-z|^{N-2}}\, dy$$

must be finite on a set where u_2 does not vanish, and this is possible if and only if $N > 4$.

Iterating this argument we get that $N > 2m$. □

Sketch of Proof of Theorem 15 by radialization. Without loss of generality, we can assume that $N > 2$, u is smooth and radial by standard mollification and spherical averaging arguments, and that $\inf_{\mathbb{R}^N} u = 0$.

We recall a key property of non-negative radial superharmonic functions v in $\mathbb{R}^N$ with $N > 2$. Setting $\mu := -\Delta v \geq 0$ we have

$$v(r) \geq \frac{1}{(N-2)r^{N-2}} \int_0^r \tau^{N-1} \mu(\tau) d\tau. \tag{A.7}$$

Indeed, adapting the proof of [18, Proposition 2.1], integrating the identity $-\left(r^{N-1} v'(r)\right)' = r^{N-1}\mu(r)$ on $(0,t)$ gives

$$-t^{N-1} v'(t) = \int_0^t \tau^{N-1} \mu(\tau)\, d\tau =: F(t).$$

Since $v \geq 0$ and F is nondecreasing, integrating once more on (r,ρ) yields

$$v(r) \geq v(r) - v(\rho) = \int_r^\rho t^{1-N} F(t) dt \geq \int_r^\rho t^{1-N}\, dt\, F(r) = \int_r^\rho t^{1-N} dt \int_0^r \tau^{N-1} \mu(\tau)\, d\tau,$$

and letting $\rho \to \infty$ yields (A.7).

Let u_j be as in (A.3). For $j = 0, \dots, m-1$, each u_j is a non-negative superharmonic function.

We claim that $u_m \equiv 0$. Assume, for the sake of contradiction, that $u_m \not\equiv 0$. Then there exists $R > 0$ such that $\int_0^R \tau^{N-1} u_m(\tau) d\tau > 0$. Applying (A.7) with $v = u_{m-1}$ and $\mu = u_m$, we find that for $r > R$, $u_{m-1}(r) \geq c_{m-1} r^{2-N}$. Applying (A.7) again with $v = u_{m-2}$ and $\mu = u_{m-1}$, we find that for large r, $u_{m-2}(r) \geq c_{m-2} r^{4-N}$. Iterating this procedure m times, we obtain that for large r, $u(r) \geq c_0 r^{2m-N}$. Since $2m \geq N$, we have that $u(r)$ is bounded below by a positive constant for large r. Since u is superharmonic we have that the infimum of u on any bounded set is positive, and we conclude that $\inf_{\mathbb{R}^N} u > 0$, contradicting the normalization $\inf_{\mathbb{R}^N} u = 0$. Therefore, we must have $u_m = (-\Delta)^m u \equiv 0$.

Since $-\Delta u_{m-1} = u_m \equiv 0$, we get that u_{m-1} must be a constant by the classical Liouville theorem. Plugging this information into (A.7), with $\mu = u_{m-1} = c = \text{const}$ and $v = u_{m-2}$, we obtain that $u_{m-2}(r) \geq \frac{c}{N(N-2)} r^2$. Since u_{m-2} is superharmonic and cannot have a minimum unless it is constant, we conclude that the constant c must be 0, that is, $u_{m-1} \equiv 0$. Iterating this argument, we conclude the proof. □

The following is an alternative proof of Theorem 5 for the case $P(z) = z^m$ that, unlike the proof in the main text, does not depend on the integral representation of solutions of super-polyharmonic functions, but instead relies on results that are specific to the Euclidean setting. However, we notice that this alternative proof cannot be adapted to prove Theorem 6.

Sketch of an alternative proof of Theorem 5 for $P(z) = z^m$. As before, we assume that u is smooth, and without loss of generality, we normalize $\inf_{\mathbb{R}^N} u = 0$. It is a known fact that since u is non-negative and superharmonic, we have that

$$\lim_{R\to\infty} \frac{1}{R^N} \int_{R<|y|<2R} u(y)\, dy = \inf_{\mathbb{R}^N} u = 0. \tag{A.8}$$

See [12], and also [14] for analogous properties related to quite general second-order operators.

Next, let $\phi \in C_0^\infty(\mathbb{R})$ be a standard cut-off function such that $\phi(t) = 1$ for $|t| \leq 1$, $\phi(t) = 0$ for $|t| \geq 2$, and $0 \leq \phi \leq 1$. By using $\varphi_R := \phi(|x|/R)$ as a test function in the equation $(-\Delta)^m u = \mu \geq 0$, we have

$$\int_{|y|<R} \mu(y) dy \leq \int_{\mathbb{R}^N} \mu(y)\varphi_R(y) dy = \int_{\mathbb{R}^N} u(-\Delta)^m \varphi_R \leq C R^{N-2m} \frac{1}{R^N} \int_{R<|y|<2R} |u(y)| dy.$$

The right hand side in the last chain of inequalities vanishes as $R \to \infty$ (since $N \leq 2m$ and (A.8)); that is, $\mu = 0$.

Hence, u is a non-negative m-polyharmonic function. By the result in [8], u is a non-negative polynomial, and since it is superharmonic, it must be constant. (For completeness, one may justify the last claim by reduction to the radial case.) □

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