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Elementary Derivation of the Nyquist Criterion for Fractional-Order Feedback Systems

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ABSTRACT It is shown that the classic Nyquist criterion can be extended in a straightforward way to feedback systems of fractional order. The proof of this extension merely requires basic notions of vector analysis and of closed-loop system transfer functions. The criterion can be used not only to ascertain the stability of a fractional-order system but also to detect the presence of closed-loop poles inside any given sector of the complex plane. The test is finally applied to three examples of didactic value.

INDEX TERMS Fractional-order systems, feedback systems, stability, root clustering, Nyquist plot.

I. INTRODUCTION

THE LAST two decades have witnessed a rapid development of the literature on fractional-order systems from both the theoretical and the practical points of view (see, e.g., [3], [4]), and dedicated journals and special issues have recently been published on this topic whose mathematical origins date back to the seventeenth century [7], [8]. Particular attention has been devoted to the stability analysis of this interesting class of systems considered either alone or in connection with other (sub)systems [9], [10]. It has long been recognised that a fractional-order system is stable if all of the roots of a polynomial strictly related to the denominator of its non-rational transfer function are outside a minor circular sector of the right half-plane with centre in the origin and radii symmetric about the real axis. To check whether this condition is satisfied, some generalisations of the criteria usually adopted to analyse the stability of integer-order systems have been suggested. In this regard, mention can be made of [11], [12], [13], where frequency-like methods are applied to the characteristic (pseudo) polynomial of a fractional-order system, of [14], where reference is made to systems in Lure's form, of [15], where non-commensurate delay systems are considered, and of [16] that present adaptations of the classical algebraic criteria of Routh and Jury. Quite understandably, these contributions naturally pertain to the broader, and by now well-established, line of research on root clustering [17], [18], [19], robust and D-stability [20], [21], [22]. Most of the aforementioned results require a fairly strong mathematical background and lead to tests that are not easily implemented.

This article focuses mainly on feedback systems of fractional order but the same arguments obviously apply to find the number of poles of an integer-order system inside a given sector. The following procedures and proofs are based on basic notions of vector analysis, complex functions and feedback systems in frequency domain. In these authors' opinion, they could well be adopted in an introductory course of control. The main result is a generalisation of the Nyquist theorem for determining, on the basis of the open-loop Nyquist diagram of a unity-feedback system, the pole distribution with respect to the boundary of a sector of its closed-loop transfer function. Its formulation exactly parallels the one traditionally used to find the pole distribution of an integer-order feedback system with respect to the imaginary axis.

The rest of this article is organised as follows. Section II recalls the essentials of fractional-order system stability. Section III briefly reviews previous results on polynomial root distribution. Section IV derives the aforementioned generalisation of the Nyquist theorem and the related stability criterion for feedback systems. This test is applied to three examples in Section V. A few concluding remarks are made in Section VI.

II. ESSENTIALS OF FRACTIONAL SYSTEM STABILITY

This section summarises only the results on frequency-domain fractional-order system stability that are strictly relevant to the following developments and explains why the consideration of circular sectors is important for stability analyses.

Consider the transfer function of a continuous-time LTI strictly-proper system in *commensurate-order form* (fractional powers with the same least common denominator):

$$\widehat{G}(w) = \frac{b_m w^{\frac{m}{q}} + b_{m-1} w^{\frac{m-1}{q}} + \cdots + b_1 w^{\frac{1}{q}} + b_0}{w^{\frac{n}{q}} + a_{n-1} w^{\frac{n-1}{q}} + \cdots + a_1 w^{\frac{1}{q}} + a_0}, \quad (1)$$

where q, m, n are positive integers, $m < n$, $q \geq 1$, and the numerator and denominator coefficients are assumed to be real.

By the change of variable

$$s = w^{\frac{1}{q}}, \quad (2)$$

function (1) is transformed into the following strictly-proper *rational* function of s:

$$G(s) = \frac{B(s)}{A(s)}, \quad (3)$$

where

$$B(s) = b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0, \quad (4)$$

$$A(s) = s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0. \quad (5)$$

The denominator of (1) is a multivalued function of w which becomes a single-valued function on a Riemann surface consisting of q sheets with branch cuts along the negative real semi-axis (see, e.g., [9]). The first, or *principal*, sheet contains the so-called physical poles of (1) [23]. The stability of the rational-order system depends on their location with respect to the imaginary axis of the w-plane. Precisely, the instability region is the right half of the principal sheet. Now, this right half-plane (RHP) maps via (2) into the minor sector of the s-plane defined by

$$\mathcal{S} \triangleq \left\{ s = \mu e^{\jmath\phi} \; : \; , \mu \in \mathbb{R}_+, \; \phi \in \left[-\frac{\pi}{2q}, \frac{\pi}{2q} \right] \right\}. \quad (6)$$

Therefore it is very important to check whether any root of (5) lies in this sector whose delimiting radii are symmetric to each other with respect to the real axis.

To this purpose, a procedure based on elementary vector analysis has been illustrated in [11]. In particular, a stability condition that implies the inspection of the hodograph (a Nyquist-like diagram) of a polynomial with complex coefficients has been derived. This procedure and the resulting stability criterion are briefly reviewed in the next section because they will be used in the subsequent Section IV to determine the root distribution of the characteristic equation of a feedback system with respect to the boundary of sector (6).

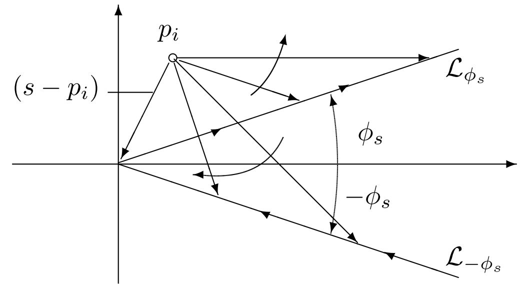

FIGURE 1. Phase variation of the vector representing factor $s - p_i$ with $p_i \notin \mathcal{S}$ as s travels along the sector boundary from right to left on radius $\mathcal{L}_{-\phi_s}$ and then from left to right on radius $\mathcal{L}_{\phi_s}$.

It has been observed in [24], [25] that the Nyquist criterion may be applied directly to the non-rational transfer function of the fractional-order system. However, the procedure described next, which involves only elementary concepts of rational function analysis, is more general since it can be used to detect the presence of closed-loop poles inside *any* circular sector of the complex plane.

III. POLYNOMIAL ROOT DISTRIBUTION

The aim of this section, largely borrowed from [11], is to provide the basis for the following developments, thus making this article self-contained as much as possible.

Consider the factored form of the nth-degree monic denominator polynomial (5):

$$A(s) = \prod_{i=1}^{n_d} (s - p_i)^{\mu_i}, \quad (7)$$

where the p_i are the distinct roots of $A(s)$, n_d is their number, and μ_i is the multiplicity of the root p_i so that $\sum_1^{n_d} \mu_i = n$. Clearly, the phase of (7) is the sum of the phases of its factors.

Let us denote the upper and lower radii of sector (6) by

$$\mathcal{L}_{\phi_s} = \left\{ s = \rho e^{\jmath\phi_s} \; : \; \rho \in \mathbb{R}_+ \right\} \quad (8)$$

and, respectively,

$$\mathcal{L}_{-\phi_s} = \left\{ s = \rho e^{-\jmath\phi_s} \; : \; \rho \in \mathbb{R}_+ \right\}, \quad (9)$$

where

$$\phi_s \triangleq \frac{\pi}{2q}, \quad (10)$$

and evaluate the phase variation of each factor $s - p_i$ of (7) as w travels along $\mathcal{L}_{-\phi_s}$ from the point at infinity to the origin (i.e., ρ going from ∞ to 0 in (9)) and then from the origin to the point at infinity along $\mathcal{L}_{\phi_s}$ (i.e., ρ going from 0 to ∞ in (8)) in the clockwise direction. As Fig. 1 shows, if the root p_i is outside the minor sector (6), i.e., $p_i \notin \mathcal{S}$, the phase of the vector representing the factor $s - p_i$ passes from $-\phi_s$ to $+\phi_s$ in the (positive) counterclockwise direction, so that its net phase variation is $+2\phi_s$.

Instead, if p_i is located inside the minor sector (6), the net phase variation of the vector $s - p_i$ along the same path is

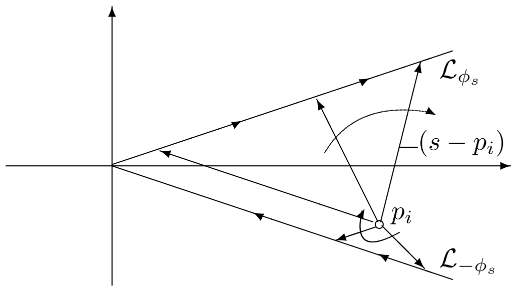

FIGURE 2. Phase variation of the vector representing $s - p_i$ with $p_i \in \mathcal{S}$ as s travels along the sector boundary from right to left on radius $\mathcal{L}_{-\phi_S}$ and then from left to right on radius $\mathcal{L}_{\phi_S}$.

$-2(\pi - \phi_s)$ because, as shown in Fig. 2, this vector rotates around p_i in the (negative) clockwise direction by such angle.

For simplicity, the following assumption is made.

Assumption 1: No root of $A(s)$ lies on the boundary of sector $\mathcal{S}$.

Therefore polynomial (7) does not vanish for $s \in \mathcal{L}_{-\phi_s} \cup \mathcal{L}_{\phi_s}$.

Since the phase variation of (7) is the sum of the phase variations of its factors, according to the previous arguments and denoting by $n_{A,\mathcal{S}}$ the number of roots (counting multiplicities) of $A(s)$ inside the minor sector (6) with central angle $2\phi_s$ the following result holds.

Theorem 1 (Phase Variation [11]): Under Assumption 1, the phase variation $\Delta\Phi_A$ of the nth-degree polynomial (7) as s travels along the sector radii (9) and (8) in the clockwise direction (from right to left on $\mathcal{L}_{-\phi_s}$ and from left to right on $\mathcal{L}_{\phi_s}$) is given by

$$\begin{aligned}\Delta\Phi_A &= \left(n - n_{A,\mathcal{S}}\right)(2\phi_s) - n_{A,\mathcal{S}}\, 2(\pi - \phi_s) \\ &= 2\left(n\,\phi_s - n_{A,\mathcal{S}}\,\pi\right),\end{aligned} \tag{11}$$

where $n_{A,\mathcal{S}}$ is the number of roots of $A(s)$ inside sector (6) with $\phi_s = \pi/(2q)$.

It is thus possible to determine the root distribution of $A(s)$ from the overall phase variation $\Delta\Phi_A$. An immediate consequence of Theorem 1 is the following stability criterion.

Corollary 1 (Stability Criterion): System (1) is stable iff $\Delta\Phi_A = 2n\phi_s$.

Remark 1: The application of the above theorem and corollary requires drawing the hodographs (Nyquist-like diagrams) of

$$\widetilde{A}_{-\phi_s}(\rho) \triangleq A\left(\rho\, e^{-\jmath\phi_s}\right), \quad \rho > 0, \tag{12}$$

and

$$\widetilde{A}_{\phi_s}(\rho) \triangleq A\left(\rho\, e^{\jmath\phi_s}\right), \quad \rho > 0, \tag{13}$$

whose coefficients are complex. Since the coefficients of $A(s)$ are assumed to be real and, thus, the complex roots of $A(s)$, if any, are in conjugate pairs, the phase variation of (12) is equal to that of (13). Note, however, that Theorem 1 and Corollary 1 also apply to polynomials with complex coefficients.

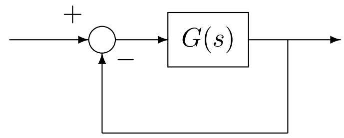

FIGURE 3. Unity-feedback system.

Remark 2: Theorem 1 can be extended to the case of asymmetric sectors whose lower radius forms with the real axis an angle $-\psi_s$ that is different from the angle ϕ_s formed by the upper radius. According to the same arguments as those used to derive Theorem 1, it turns out that the overall phase variation is in this case:

$$\Delta_a\Phi_A = n\left(\psi_s + \phi_s\right) - 2\, n_{A,\mathcal{S}}\,\pi, \tag{14}$$

where subscript a stands for asymmetric. The consideration of asymmetric sectors may turn out useful, e.g., in root-clustering problems [19] for complex polynomials [26], [27].

Remark 3: Removing Assumption 1 is not conceptually difficult but implies an increase of notational complexity at the expense of clarity and the resort to artifices similar to those adopted in the classic Mikhailov stability test to deal with the critical cases (see, e.g., [28]).

The next section uses the previous results to arrive in a natural way at a Nyquist-like stability criterion for fractional-order feedback systems or, more generally, at a criterion for finding the root distribution of the transfer function of an integer- or fractional-order feedback system with respect to a sector.

IV. EXTENDED NYQUIST THEOREM AND CRITERIA

The celebrated Nyquist criterion [29] is a powerful tool for the analysis of the stability and robustness of feedback systems. It simply requires plotting the Nyquist diagram of the open-loop system. Since this system may incorporate the controller, the Nyquist criterion provides very useful information also for control system design. The test is easy to implement and, in general, does not require the solution of algebraic or differential equations. As is customary, in this section reference is made to the negative unity-feedback system of Fig. 3, where $G(s)$ is an *irreducible* rational function like (3) that may be obtained from the original irrational function (1) via (2), in which case (6) represents the instability region for both the forward-path fractional-order transfer function and the overall feedback fractional-order transfer function.

As is well known, the transfer function of the overall system obtained by the unity-feedback connection in Fig. 3 is

$$W(s) = \frac{G(s)}{1 + G(s)}. \tag{15}$$

The location of the poles of (15) characterises the system dynamics very well. The problem considered here is that of determining how many of these poles are inside sector (6).

The poles of $W(s)$ coincide with the roots of the equation

$$1 + G(s) = 0 \tag{16}$$

which, using the notation in (4)–(5), is equivalent to

$$\frac{A(s) + B(s)}{A(s)} = 0. \tag{17}$$

Consider the phase variation $\Delta\Phi_{1+G}$ of the left-hand side of (16) or, equivalently, (17) as the variable s travels along the radii of sector (6) in the clockwise direction. This phase variation is the difference between the phase variations $\Delta\Phi_{A+B}$ of the numerator and the phase variation $\Delta\Phi_A$ of the denominator of (17). Therefore, (i) denoting by $n_{A+B,\mathcal{S}}$ and $n_{A,\mathcal{S}}$ the roots of $A(s) + B(s)$ and $A(s)$ inside sector (6), (ii) taking into account that the degree of both the numerator and the denominator of (17) is n, and (iii) invoking Theorem 1, which assumes that there are no polynomial roots on the path, we have:

$$\begin{aligned}\Delta\Phi_{1+G} &= \Delta\Phi_{A+B} - \Delta\Phi_A \\ &= 2\left(n\,\phi_s - n_{A+B,\mathcal{S}}\,\pi\right) - 2\left(n\,\phi_s - n_{A,\mathcal{S}}\,\pi\right) \\ &= -2\,\pi\left(n_{A+B,\mathcal{S}} - n_{A,\mathcal{S}}\right). \end{aligned} \tag{18}$$

By taking the round angle as the unit of measure for angles (i.e., expressing angles in terms of complete circular rotations), and assuming as positive the clockwise direction, which is customary in the application of the Nyquist criterion, we can restate the above result in terms of the *encirclements* of the origin. More precisely, the hodograph of $1 + G(s)$ encircles the origin a number of times N_{1+G} given by

$$N_{1+G} \triangleq \frac{\Delta\Phi_{1+G}}{-2\pi} = n_{A+B,\mathcal{S}} - n_{A,\mathcal{S}}. \tag{19}$$

Since the rotations of $1 + G(s)$ around the origin correspond to the rotations of $G(s)$ around the "critical point" $-1 + \jmath 0$, relation (19) leads immediately to the following generalisation of the classical Nyquist theorem.

Theorem 2 (Extended Nyquist Theorem): The diagram of the forward-path transfer function $G(s)$ as the variable s varies along the s-plane contour $\mathcal{L}_{-\phi_s} \cup \mathcal{L}_{\phi_s}$, formed by the radii of sector $\mathcal{S}$ and travelled in the clockwise direction, encircles the critical point $(-1, 0)$ a number of times N_G (positive in the clockwise direction) equal to the difference between the number of poles $n_{W,\mathcal{S}}$ of the closed-loop transfer function (15) inside $\mathcal{S}$ and the number of poles $n_{G,\mathcal{S}}$ of $G(s)$ inside $\mathcal{S}$, i.e.,

$$N_G = n_{W,\mathcal{S}} - n_{G,\mathcal{S}}, \tag{20}$$

where $n_{W,\mathcal{S}} = n_{A+B,\mathcal{S}}$ and $n_{G,\mathcal{S}} = n_{A,\mathcal{S}}$.

With reference to a unity-feedback system of fractional order with forward-path transfer function (1), for which sector (6) represents the instability region in the s-plane, Theorem 2 results directly in the following stability criterion ($n_{W,\mathcal{S}} = 0$).

Corollary 2 (Extended Nyquist Stability Criterion): The unity-feedback fractional-order system with forward-path transfer function (1) is stable iff

$$N_G = -n_{G,\mathcal{S}}, \tag{21}$$

where N_G is the number of times (positive in the clockwise direction) the diagram of the rational function $G(s)$ in (3) encircles point $-1 + \jmath 0$ as the variable s travels along the contour of sector (6) in the clockwise direction and $n_{G,\mathcal{S}}$ is the number of poles of $G(s)$ inside the same sector.

Often, $n_{G,\mathcal{S}} = 0$, i.e., the forward path is stable, so that the following extended *reduced* Nyquist criterion applies.

Corollary 3 (Extended Reduced Nyquist Criterion): The unity-feedback fractional-order system whose forward-path is stable is in turn stable if and only if

$$N_G = 0, \tag{22}$$

where N_G is defined as in Corollary 2.

Remark 4: To apply the criterion in Corollary 2, $n_{G,\mathcal{S}}$ need be known. Usually, this precondition is not difficult to satisfy because the forward-path transfer function $G(s)$ is typically the product of simpler transfer functions expressed in factored form. If this is not the case, $n_{G,\mathcal{S}}$ can be found by means of the procedure in Section III.

Remark 5: The previous theorem and criteria require plotting the hodographs of the rational functions

$$\widetilde{G}_{-\phi_s}(\rho) \triangleq G\big(\rho\, e^{-\jmath\phi_s}\big), \quad \rho \geq 0, \tag{23}$$

and

$$\widetilde{G}_{\phi_s}(\rho) \triangleq G\big(\rho\, e^{\jmath\phi_s}\big), \quad \rho \geq 0, \tag{24}$$

which have complex coefficients and may be regarded as functions of the generalised frequency ρ. However, this task is not computationally demanding since it does not require the implementation of an algorithm but entails only the operations that are necessary to evaluate the given functions at different values of ρ, as is the case in frequency-sweeping techniques. In particular, no root-finding algorithm need be employed.

Remark 6: A critical case in the application of the Nyquist test occurs when, for some value $\hat{\rho}$ of ρ, $\widetilde{G}_{-\phi_s}(\hat{\rho}) = -1$ (which also implies $\widetilde{G}_{\phi_s}(\hat{\rho}) = -1$ because (23) and (24) are complex conjugate). This critical situation reveals the presence of poles of $W(s)$ on the boundary of the sector, since its denominator $1 + G(s)$ vanishes for $s = \hat{\rho} e^{\pm\jmath\phi_s}$. In this case, to find the number of poles of $W(s)$ in the interior of $\mathcal{S}$, resort can be made to the same procedures as those adopted in the standard Nyquist test for integer-order systems. Another critical case occurs when the aforementioned functions go to infinity (poles of G(s) on the sector boundary). Again, the usual artifice adopted in the standard Nyquist test can be employed.

The next section shows by means of three simple examples of didactic value how the previous rules can be applied to find the number of poles of $W(s)$, if any, inside a given sector of the s-plane.

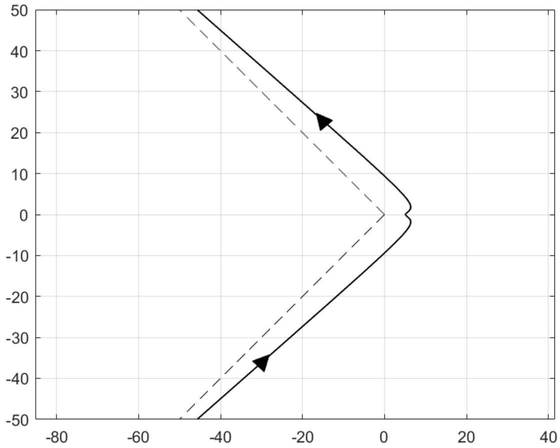

FIGURE 4. Hodograph (bold lines) of $A(s) = s^3 - s^2 + 3s + 5$ for evaluating $\Delta\Phi_A$. Dashed lines represent the bisectors of the second and third quadrants.

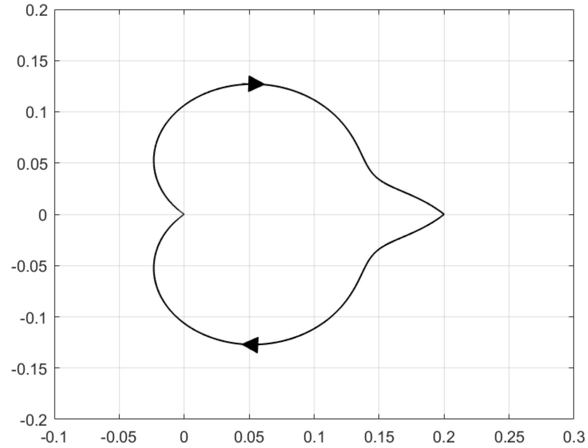

FIGURE 5. Hodograph of (23) and (24) with $G(s)$ as in (27).

V. EXAMPLES

A. EXAMPLE 1

Consider the fractional-order unity-feedback system whose forward-path transfer function is

$$\hat{G}(w) = \frac{1}{w^{3/2} - w^{2/2} + 3w^{1/2} + 5}. \tag{25}$$

According to (6), the instability region in the s–plane is the sector

$$\mathcal{S} \triangleq \left\{ s = \rho e^{\jmath\phi} : \rho \in \mathbb{R}_+, \ \phi \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \right\}. \tag{26}$$

Now, by the change of variable $s = w^{1/2}$, (25) is transformed into

$$G(s) = \frac{1}{s^3 - s^2 + 3s + 5} = \frac{1}{(s+1)(s-1+\jmath 2)(s-1-\jmath 2)} \tag{27}$$

which has no pole in (26). As observed in Remark 4, if this information is not readily available, it can be obtained from the hodograph of the denominator $A(s)$ of $G(s)$ as s travels along both radii of the instability sector in accordance with Theorem 1. Note that, when s travels along $\mathcal{L}_{\pi/4}$ and $\rho \to \infty$, the dominant term in $A(s)$ is

$$s^3 = \rho^3\left[\cos(3\pi/4) + \jmath \sin(3\pi/4)\right] = \rho^3 \frac{\sqrt{2}}{2}[-1 + \jmath].$$

Hence the hodograph tends asymptotically to the bisector of the second quadrant. Analogously, when s travels along $\mathcal{L}_{-\pi/4}$ and $\rho \to \infty$, the hodograph tends asymptotically to the bisector of the third quadrant. Therefore, the overall phase variation $\Delta\Phi_A$ of the vector connecting the origin with the current point on this hodograph, depicted in Fig. 4, is $\Delta\Phi_A = \frac{3}{2}\pi$.

As a consequence, from (11), the number $n_{A,\mathcal{S}}$ of roots of $A(s)$ in $\mathcal{S}$ (and, thus, the number $n_{G,\mathcal{S}}$ of the poles of $G(s)$ in $\mathcal{S}$) is $n_{A,\mathcal{S}} = n_{G,\mathcal{S}} = 0$.

Fig. 5 shows the hodograph of $G(s)$:

$$\left\{G\left(\rho e^{-\jmath\phi_s}\right) : \rho \geq 0\right\} \cup \left\{G\left(\rho e^{\jmath\phi_s}\right) : \rho \geq 0\right\}$$

which is the union of two curves that are symmetric to each other about the real axis because (23) and (24) have the same real parts and opposite imaginary parts. The orientation of this Nyquist-like diagram corresponds to the direction in which the sector radii are travelled in the s plane according to Theorem 2, precisely, from right to left along the lower radius (ρ decreasing from $+\infty$ to 0 in $G(\rho e^{-\jmath\phi_s})$) and from left to right along the upper radius (ρ increasing from 0 to $+\infty$ in $G(\rho e^{\jmath\phi_s})$).

Since the critical point $-1 + \jmath 0$ is not encircled by the diagram, by the extended reduced Nyquist criterion (see Corollary 1) the fractional-order feedback system is stable ($n_{W,\mathcal{S}} = 0$). Indeed, the (transformed) closed-loop transfer function is

$$W(s) = \frac{G(s)}{1 + G(s)} = \frac{1}{s^3 - s^2 + 3s^1 + 6} \tag{28}$$

whose poles are -1.1179 and $1.0589 \pm \jmath 2.0606$. Fig. 6 shows how these poles are distributed with respect to the instability sector.

B. EXAMPLE 2

Consider the fractional-order unity-feedback system whose forward-path transfer function is

$$\hat{G}(w) = \frac{30\left(w^{1/2} + 1\right)}{w^{4/2} + 2w^{3/2} - 7w^{2/2} - 8w^{1/2} + 12}. \tag{29}$$

Choosing, again, $s = w^{1/2}$, the instability sector is again (26) while the corresponding transformed rational function is

$$G(s) = \frac{30(s+1)}{s^4 + 2s^3 - 7s^2 - 8s + 12} = \frac{30(s+1)}{(s+3)(s+2)(s-1)(s-2)}. \tag{30}$$

The number of poles of (30) in $\mathcal{S}$ can be determined by plotting the hodograph of $A(s) = s^4 + 2s^3 - 7s^2 - 8s + 12$, which is shown in Fig. 7, and evaluating its phase variation $\Delta\Phi_A$ as s travels along the sector boundary according to Theorem 1.

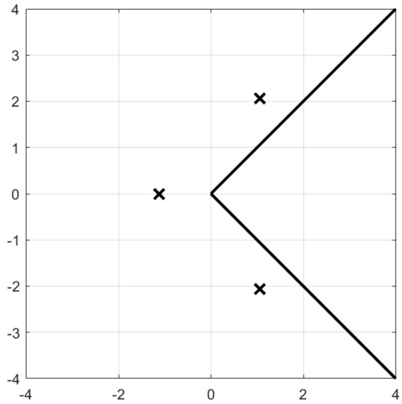

FIGURE 6. Pole distribution of (28) with respect to the sector (26).

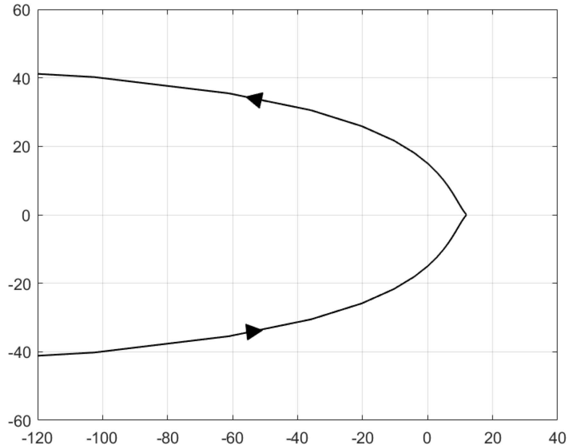

FIGURE 7. Hodograph of the denominator of (30).

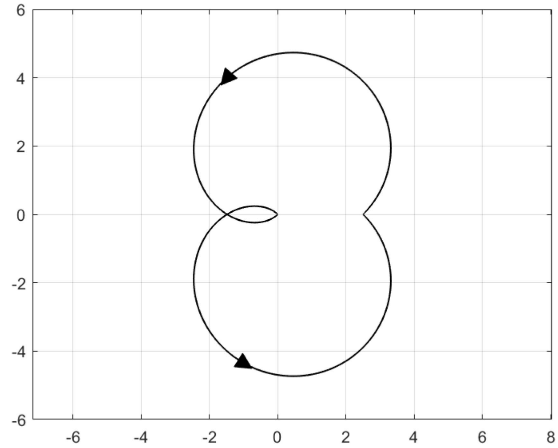

FIGURE 8. Hodograph of (23) and (24) with $G(s)$ as in (30).

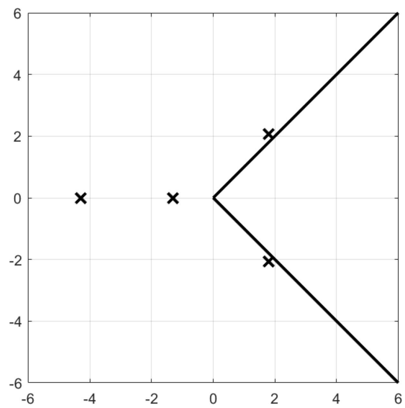

FIGURE 9. Pole distribution of (31) with respect to the sector (26).

In this case, when s travels along $\mathcal{L}_{\pi/4}$ and $\rho \to \infty$, the dominant term in $A(s)$ is

$$s^4 = \rho^4\big[\cos(\pi) + \jmath\,\sin(\pi)\big] = -\rho^4$$

and the overall phase variation is $\Delta\Phi_A = -2\pi$. From (11) with $n = 4$ and $\phi_s = \pi/4$ one obtains $n_{A,\mathcal{S}} = n_{G,\mathcal{S}} = 2$. The hodograph of the transformed forward-path function $G(s)$ as s travels along the sector radii is depicted in Fig. 8. The critical point $-1 + \jmath 0$ is encircled two times in the counterclockwise direction, which is the negative direction in the application of the Nyquist criterion, so that $N_G = -2 = -n_{G,\mathcal{S}}$ and the stability condition (21) is satisfied. Indeed, the closed-loop transfer function is

$$W(s) = \frac{30\,(s+1)}{s^4 + 2s^3 - 7s^2 + 22s + 42} \tag{31}$$

whose poles are -4.293, -1.301, and $1.797 \pm \jmath 2.072$. Their distribution with respect to $\mathcal{S}$ is shown in Fig. 9.

C. EXAMPLE 3

Consider finally the fractional-order unity-feedback system whose forward-path transfer function is

$$\hat{G}(w) = \frac{10\left(w^{1/2}+1\right)}{w^{4/2} + 2w^{3/2} - 7w^{2/2} - 8w^{1/2} + 12} \tag{32}$$

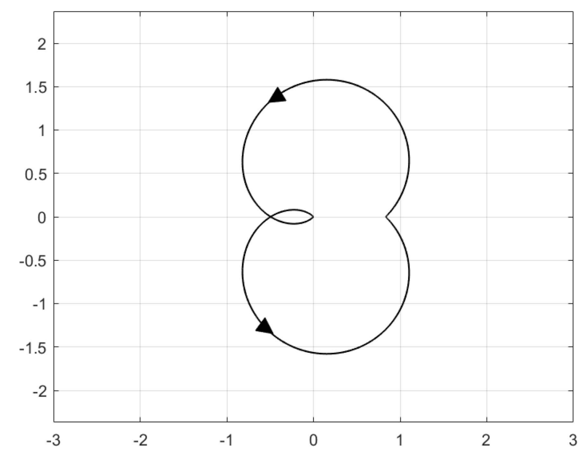

FIGURE 10. Hodograph of (23) and (24) with $G(s)$ as in (32).

which differs from (29) only for the gain. Therefore, the instability sector in the s-plane is again (26) and $n_{G,\mathcal{S}} = 2$.

The hodograph of (23) and (24) for the transformed function $G(s)$ as s travels along the radii of the instability sector is depicted in Fig. 10. This time the critical point is not encircled ($N_G = 0$) and the stability criterion is not satisfied. To find the number $n_{W,\mathcal{S}}$ of poles of $W(s)$ inside $\mathcal{S}$,

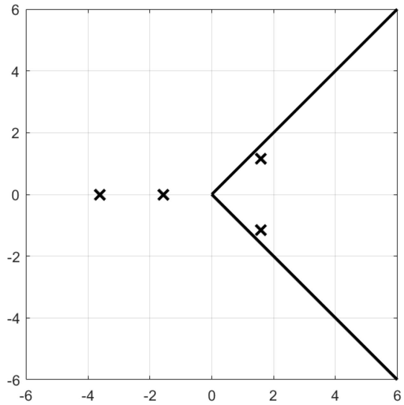

FIGURE 11. Pole distribution of (34) with respect to the sector (26).

resort can be made to Theorem 2. From (20) we get

$$n_{W,\mathcal{S}} = N_G + n_{G,\mathcal{S}} = 2. \tag{33}$$

Indeed, the closed-loop transfer function is

$$W(s) = \frac{10\,(s+1)}{s^4 + 2s^3 - 7s^2 + 2s + 22} \tag{34}$$

whose poles are -3.622, -1.568, and $1.595 \pm \jmath 1.154 \in \mathcal{S}$. Their distribution with respect to $\mathcal{S}$ is shown in Fig. 11.

VI. CONCLUSION

A Nyquist-like criterion for checking the stability of a fractional-order feedback system has been derived using only elementary vector analysis and basic notions of feedback theory. Its practical implementation is computationally easy since it only requires plotting the hodograph of rational functions without roots computation. Attention has been focused on fractional-order systems, but the same procedures can be applied to find the pole distribution with respect to a sector of an integer-order rational function and in polynomial root clustering methods especially when only roots in a given region are sought.

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