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There is only one Farey map

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Abstract
Let A_0, A_1 be nonnegative matrices in $\mathrm{GL}_{n+1}\mathbb{Z}$ such that the subsimplexes $A_0[\Delta], A_1[\Delta]$ split the standard unit n-dimensional simplex Δ in two. We prove that, for every $n = 1, 2, \ldots$ and up to the natural action of the symmetric group by conjugation, there are precisely three choices for the pair (A_0, A_1) such that the resulting projective Iterated Function System is topologically contractive. In equivalent terms, in every dimension there exist precisely three continued fraction algorithms that assign distinct two-symbol expansions to distinct points. These expansions are induced by the Gauss-type map $G : \Delta \to \Delta$ with branches A_0^{-1}, A_1^{-1}, which is continuous in exactly one of these three cases, namely when it equals the Farey-Mönkemeyer map.

Keywords Multidimensional continued fractions · Farey map · Simplex-splitting algorithms

Mathematics Subject Classification 11J70 · 37M25

1 Introduction and preliminaries

Let $n = 1, 2, 3, \ldots$, and let $\Delta = \{x \in \mathbb{R}^{n+1}_{\geq 0} : \sum x_i = 1\}$ be the standard n-dimensional simplex, with vertices $\{e_0, \ldots, e_n\}$. We denote the monoid of all nonnegative matrices in $\mathrm{GL}_{n+1}\mathbb{Z}$ by Σ; the group of invertible elements of Σ is the symmetric group S_{n+1} of permutation matrices. Every $A \in \Sigma$ determines a continuous map of Δ into itself, still denoted by A, via $A(x) = Ax/\|Ax\|_1$; the image $A[\Delta]$ is a *unimodular simplex*. In agreement with the numbering of vertices of Δ, we number matrix rows and columns from 0 to n.

Let $\mathcal{A} = \{A_0, \ldots, A_{m-1}\}$ be a nonempty finite subset of Σ not containing invertible elements. Identifying matrices with maps as above, $\mathcal{A}$ is an *Iterated Function System* (IFS) on Δ; all IFS in this paper are of this form. We let $a, b, \ldots$ vary in the alphabet

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$\{0, \ldots, m-1\}$ and let $v, w, \ldots$ (respectively $\mathrm{a}, \mathrm{b}, \ldots$) stand for finite words (respectively infinite sequences) over that alphabet. We write $\mathrm{a} \upharpoonright t = a_0 \ldots a_{t-1}$ for the t-long prefix of a, and let $A_{\mathrm{a}\upharpoonright t}$ and $\Delta_{\mathrm{a}\upharpoonright t}$ stand for the product $A_{a_0} \cdots A_{a_{t-1}}$ and for the unimodular simplex determined by that product, respectively.

We say that $\mathcal{A}$ is *simplex-splitting* if $\Delta_0, \ldots, \Delta_{m-1}$ constitute a partition of Δ, that is, $\Delta = \bigcup_a \Delta_a$ and the interiors of Δ_a and Δ_b do not intersect for $a \neq b$. Given $x \in \Delta$, let $a = a(x)$ be the least index such that $x \in \Delta_a$ (the choice of this specific selection rule is irrelevant; we will readdress this issue in Remark 1.2(5)). Then these data define the Gauss-type map $G : \Delta \to \Delta$ by $Gx = A_{a(x)}^{-1}(x)$, and the sequence $a_0 a_1 \ldots$ given by $a_t = a(G^t x)$ is the *digit expansion* of x. As usual, we identify G with a multidimensional continued fraction algorithm; good starting points to a large literature are [2, 9, 33].

Definition 1.1 The IFS $\mathcal{A}$ is *topologically contractive* if for every infinite sequence a the intersection $\bigcap_{t \geq 0} \Delta_{\mathrm{a}\upharpoonright t}$ is a singleton.

Remark 1.2 .

(1) In the classical setting of IFS, one usually requires *uniform contractivity*, that is, the existence of a number $r \in (0, 1)$ and a metric d such that, for every $A \in \mathcal{A}$ and every pair of points x, y, we have $d\big(A(x), A(y)\big) \leq r\, d(x, y)$. Topological contractivity sits between the stronger requirement of uniform contractivity and the weaker one that the time-1 partition $\{\Delta_a : 0 \leq a < m\}$ is *generating* [14, Chapter 6] for the Gauss-type map G. It is easy to see that no simplex-splitting algorithm can be uniformly contractive; thus we never use uniform contractivity, and for simplicity we drop the adjective "topological". In the literature on multidimensional continued fractions a related concept is *topological convergence*, which means contractivity *along sequences that code a G-orbit* [33, Definition 9].

(2) Contractivity implies that the function $\pi : m^\omega \to \Delta$ that maps a to the only element of the singleton in Definition 1.1 is well defined; clearly, π is then continuous.
(3) In dimension $n = 1$ every $\mathcal{A}$ is contractive; see Theorem 2.1.
(4) If $\mathcal{A}$ is simplex-splitting and contractive, then every point of Δ has an expansion, and distinct points have distinct expansions.
(5) If $\mathcal{A}$ is simplex-splitting then the matrices in $\mathcal{A}$ constitute a *code* inside Σ (that is, they generate a free submonoid); this follows from the Ping-Pong Lemma [16, VII.A.2], [31, Lemma 2.5]. If, moreover, $\mathcal{A}$ is contractive, then in the definition of G we may forget about the selection rule and treat G as a multivalued map, sending x to the set $\{A_a^{-1}(x) : x \in \Delta_a\}$. This is safe because [31, Theorem 2.13] (actually, a simplified version of it for ordinary IFS, not graph-directed ones) guarantees that the number of images remains uniformly bounded: that is, there exists a bound M (depending on $\mathcal{A}$), such that for every x and every t the cardinality of set of images of x at time t is bounded by M. Moreover, if x is periodic (that is, x belongs to the set of its own images at some time $t > 0$), then G is an ordinary single-valued map along the orbit of x.
(6) Contractivity also implies that the *Hutchinson operator* $\mathcal{A}(K) = \bigcup_a A_a[K]$ on the compact space $\mathcal{K}$ of nonempty compact subsets of Δ under the Vietoris topology has a

unique fixed point, namely $\pi[m^\omega]$, and this fixed point is attractive; see the discussion in [6, §1].

Example 1.3 Let $\mathcal{A}$ be the set of the following three matrices:

$$A_0 = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 1 & & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 1 & \\ & 1 & \\ 1 & 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & & 1 \\ & 1 & 1 \\ & & 1 \end{pmatrix}.$$

Then $\mathcal{A}$ is simplex-splitting; Fig. 1 left shows the time-5 partition $\{\Delta_w : |w| = 5\}$.

The Hutchinson operator has a largest fixed point, namely Δ itself, and a smallest one *R*, which is the projection to Δ of the closure of the set of the attracting fixed points in projective 2-space of the proximal elements in the monoid generated by A_0, A_1, A_2 [7, §3.1]. It is easy to show that *R* is the image of the Rauzy gasket [4, 26] under an appropriate projective map. There are uncountably many incomparable fixed points between Δ and *R*; for example, for every *x* in the boundary of Δ the sequence $\mathcal{A}^t\{x\}$ converges to the closure of $\{A_w(x) : w \text{ is a word in } \{0,1,2\}\}$, which is a fixed point intersecting the boundary of Δ in *x* only.

The map *G* acts on $x = (x_0, x_1, x_2)$ by subtracting the smallest coordinate from the other coordinates and projecting back to Δ. Ambiguities arise when the smallest coordinate is not unique; they are irrelevant but for the case of the vertices, that in the absence of a selection rule would have uncountably many digit expansions. Contractivity is violated uncountably many times, for example at any sequence with a tail avoiding precisely one letter. Take, for example, $\mathrm{a} = (01)^\omega$; by looking at the right eigenspaces of A_{01} one readily sees that $\Delta_{\mathrm{a}\restriction t}$ converges to the line segment of vertices e_2 and $(-\tau/2 + 1, \tau/2 - 1/2, 1/2)$, with τ the golden ratio. Therefore, all points in that line segment have a as their digit expansion. This way of violating contractivity will reappear as a special case of Lemma 4.2.

In dimension greater than one simplex-splitting contractive algorithms are scarce; indeed, some experimenting shows that a "random" choice of a splitting of Δ has little probability of being contractive, even if one takes care that the sequences Δ_{a^t} determined by the generators always shrink to a point. The goal of this paper is to present an extreme case of this scarcity:

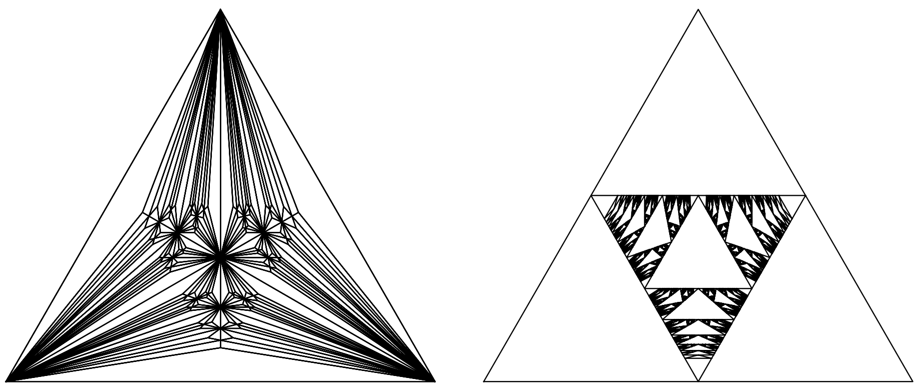


Fig. 1 The time-5 partition and the least fixed point *R*

Regardless of the dimension, and modulo the obvious symmetries of the unit simplex, there are only three contractive simplex-splitting IFS on two maps.

In order to put the above result in context, we sketch our (oversimplified, personal and biased) point of view on present-day multidimensional continued fractions. As mentioned above, we identify continued fractions with the study of Gauss-type maps, that is piecewise-defined maps on subsets of the n-dimensional real projective space whose branches are induced by matrices in $\mathrm{PGL}_{n+1}\,\mathbb{Z}$. This is a substantial oversimplification, since it excludes topics such as continued fractions over number fields (Rosen fractions and the like), over p-adic or complex numbers, the analytic theory of continued fractions, and Klein's geometric viewpoint. Nevertheless, even under these restrictions, the underlying combinatorics remain extremely intricate, leading to a proliferation of systems introduced by various authors at various times, often rediscovering minor variations of earlier approaches and repeatedly reproving basic properties from scratch.

As a consequence, in recent years researchers have largely lost interest in the taxonomy of this wild zoo and have adopted a more utilitarian point of view, focusing primarily on "useful" systems. The most recurrent paradigm of usability—though nothing prevents others from emerging—lies in providing renormalization schemes for other dynamical systems. In particular, just as one-dimensional continued fractions renormalize circle rotations, multidimensional continued fractions renormalize Interval Exchange Transformations via Rauzy-Veech induction. This has placed emphasys on subtractive algoritms: given non-negative vectors in $\mathbb{R}^{n+1}$, one devises a mechanism for partitioning the entries into "large" and "small", and then subtracting the small ones from the large in a consistent manner. This is the case, e.g., of the map in Example 1.3, as well as of the Brun algorithm "subtract the second-largest from the largest". We refer to [19] for an interesting treatment of simplex-splitting subtracting algoritms that formalize Rauzy-Veech induction.

Now, the three contractive IFS on two maps that we isolate in this paper correspond to three forms of an algorithm introduced by Mönkemeyer in 1954 [28]. It is not a subtractive algorithm, but results from the rule "subtract the smallest from the largest" (the Selmer algorithm from 1961 [35]) after restriction to an absorbing simplex and a further conjugation [33, Chapter 7]. Thus, it is no surprise that the Mönkemeyer algorithm has frequently appeared in recent work, possibly in disguised forms. Here, by a disguised form we mean either modulo a conjugation, or a partial acceleration (replace $\{A_0, A_1\}$ with, say, $\{A_0.A_{10}, A_{11}\}$), or a partial acceleration followed by the removal of some branches. This is, for example, the case of [13], while the orientation-preserving version is used in [20]; see also the recent [3]. The orientation-reversing version is the Cassaigne algorithm [10, 18]; the list could go on, but fortunatly can be subsumed by referring to the very recent monograph [1] that comprehensively covers this "utilitarian" viewpoint.

Our viewpoint, however, is different: we are interested in viewing Gauss-type maps as part of the theory of Iterated Function Systems. The emphasis thus shifts from the expansive behavior of G to the contractive behavior of the set $\mathcal{A}$ of its inverse branches. A priori, $\mathcal{A}$ may be an arbitrary finite subset of Σ acting projectively on Δ (or possibly several copies of Δ in the graph-directed version). The key idea is that one should be able to formulate explicit conditions on $\mathcal{A}$ ensuring that both $\mathcal{A}$ and $\mathcal{A}^\top$ (the IFS given by the transposed matrices) have tame attractors (that is, equal to the closure of their interiors), and on these attractors the maps G, $G^\top$ induced by the inverses of the matrices in $\mathcal{A}$ and $\mathcal{A}^\top$ are unambiguously defined, automatically forming a dual pair. This can indeed be accomplished in dimension one, and the explicit conditions on $\mathcal{A}$ are purely algebraic and remarkably simple; see [31].

Needless to say, in higher dimensions the situation is far more delicate, due to two key differences:

(a) While $\Sigma_2 \subset \mathrm{GL}_2\mathbb{Z}$ (we temporarily add indices for clarity) is finitely generated by the matrices N, L, F to be defined in §3, and its Caley graph reduces to two copies of the full binary tree, the monoid Σ_{n+1} is not finitely generated for $n \geq 2$ (see [15] for an algebraic proof, and [36, Remark 11.8] for a geometric one).

(b) While a strictly decreasing sequence of unimodular intervals necessarily shrinks to a point, a strictly decreasing sequence of unimodular n-dimensional simplices does not necessarily do so for $n \geq 2$ (see the ensuing Theorem 2.1).

Can the good monoid Σ_2 be embedded into the bad ones $\Sigma_3, \Sigma_4, \ldots$? Of course it can, in many useless ways (for example, diagonally). However our main result, Theorem 3.1, says that there is *precisely one* embedding (the one that sends the matrices $M_0, M_1, F \in \Sigma_2$ defined in §3 to the namesake matrices in Σ_{n+1}) that is useful, in the sense that the action of the image monoid on higher-dimensional simplexes remains contractive. This implies that every Gauss-type map induced by an $\mathcal{A}$ which is a subset of the copy of Σ_2 thus embedded into Σ_{n+1}, and satisfies the algebraic conditions of [31], is guaranteed to have many desiderable properties . As far as we know, no other continued fraction algorithm provides embeddings with similar properties. For example, the IFS given by the inverse branches of the Brun map is contractive in every dimension [33, Chapter 12], but the number of matrices involved varies with the dimension, making it unclear whether the Brun algorithm may be used to produce embeddings with comparable features.

2 Shrinking simplexes

Our first result, of independent interest, shows that the arithmetic of Σ always forces convergence to lower-dimensional simplexes.

Theorem 2.1 *Let $\Delta = K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots$ be any descending chain of simplexes; then the intersection $K = \bigcap_t K_t$ is a simplex. If, in addition, each K_t is unimodular and the chain is strictly decreasing infinitely often, then K has dimension strictly less than n.*

Proof It is easy to show directly (see the proof of [11, Lemma A.2.3] for a more general statement) that K is the limit of the sequence K_t in the Hausdorff metric. The latter induces the Vietoris topology on $\mathcal{K}$ and is defined by

$$d(J, K) = \inf\{\varepsilon > 0 : J \subseteq K_{<\varepsilon} \text{ and } K \subseteq J_{<\varepsilon}\},$$

where $K_{<\varepsilon}$ is the union of all open balls of radius ε and center a point of K. Let $\mathcal{K}'$ be the subspace of $\mathcal{K}$ whose elements are all nonempty subsets of Δ of cardinality at most $n+1$. Writing V_t for the set of vertices of K_t, the sequence $V_0, V_1, \ldots$ has an accumulation point $V = \{x_1, \ldots, x_k\}$ in $\mathcal{K}'$, because the latter is compact (it is the quotient of the product of

$n+1$ copies of Δ under the action of the symmetric group, see [12, p. 877]). Now, the map that sends an element of $\mathcal{K}$ to its convex hull is Lipschitz [17, p. 184], thus continuous. This implies that the convex hull of V, which is a simplex, equals K, thus establishing our first claim.

Assume now that every K_t is unimodular and that, without loss of generality, strictly contains K_{t+1}. This implies that there exists a sequence $A_1, A_2, \ldots$ of matrices in Σ, none of them a permutation matrix, such that $K_t = A_1 A_2 \cdots A_t[\Delta]$. Letting λ denote Lebesgue probability on Δ, we conclude the proof by showing that $\lambda(K) = 0$, so that K is not n-dimensional. Consider the row vector $(1 \ldots 1)A_1 \cdots A_t = \big(l_0(t) \ldots l_n(t)\big)$; it is known [29, p. 388] that $\lambda(K_t)$ equals $\big(l_0(t) \cdots l_n(t)\big)^{-1}$, thus the reciprocal of an integer. Since no matrix in the sequence is a permutation matrix, $\lambda(K_{t+1})$ is strictly less than $\lambda(K_t)$, and therefore is the reciprocal of a strictly greater integer. This holds for every t, and hence $\lambda(K) = \lim_t \lambda(K_t) = 0$. $\square$

3 Coding by two symbols

From now on we will consider only simplex-splitting algorithms on two symbols, $\mathcal{A} = \{A_0, A_1\}$. We first discuss the number of these IFS modulo the natural symmetries of Δ; this extends the discussion in [22], that treats the 2-dimensional case. We need to define a few elementary matrices in Σ:

- N is the $(n+1) \times (n+1)$ elementary matrix obtained from the identity matrix I by summing the first column to the second;
- F is obtained from I by exchanging the first two columns;
- R is obtained from I by shifting cyclically each column to the previous one, while moving the first to the last;
- $L = FNF$.

Obviously the only way of splitting Δ in two unimodular subsimplexes is by choosing an edge $\langle e_i, e_j\rangle$ and taking as first subsimplex the one with vertices $\{(e_i + e_j)/2\} \cup \{e_k : k \neq j\}$, and as second the one with vertices $\{(e_i + e_j)/2\} \cup \{e_k : k \neq i\}$. Without loss of generality, that is up to the group $S_{n+1} = \langle F, R\rangle \subset \Sigma$ of symmetries of Δ, we choose the edge $\langle e_0, e_1\rangle$, so that the first subsimplex has vertices the columns of N, and the second those of L. We agree that the matrix A_0 maps Δ to the first subsimplex, and A_1 to the second. Thus, from now on our IFS are given by *ordered* pairs (A_0, A_1) of the form (NP_0, LP_1), for arbitrary pairs $(P_0, P_1) \in S_{n+1}^2$. Two of these pairs must be identified if conjugated by an element of the subgroup S_{n-1} of S_{n+1} whose elements H fix both e_0 and e_1, and also if conjugated by F provided that, in this second case, we exchange A_0 with A_1. Note that conjugation by elements of S_{n-1} fixes both N and L, while conjugation by F exchanges them. Letting $S_2 = \{I, F\}$, this means that we must count the number of orbits of pairs $(P_0, P_1) \in S_{n+1}^2$ under the action of $S_2 \times S_{n-1}$ given by

$$
\begin{aligned}
(I, H) * (P_0, P_1) &= (HP_0H^{-1}, HP_1H^{-1}), \\
(F, I) * (P_0, P_1) &= (FP_1F^{-1}, FP_0F^{-1}).
\end{aligned}
\tag{3.1}
$$

For $n = 1, 2$ the counting can be done by hand. There are three orbits for $n = 1$, namely $\{(I, I)\}$, $\{(I, F), (F, I)\}$, $\{(F, F)\}$, and 21 orbits for $n = 2$, which have been computed in [22, §7]. For larger *n* the number of orbits grows fast; we do not know of a closed formula for this number as a function of *n*, nor expect one. However, the following snippet of Sage-Math code counts orbits for small values of *n*; for $n = 3, 4, 5$ there are 160, 1283, 11321 orbits, respectively.

```
n=5
S=SymmetricGroup([0..n])
f=S("(0,1)")
if n<3:
    H=S.subgroup([S.identity()])
else:
    H=S.subgroup([S("(2,3)"),S(str(tuple([2..n])))])
def action(x,y):
    Hxy=[(h*x*h^-1,h*y*h^-1) for h in H]
    return Set(Hxy+[(f*o[1]*f^-1,f*o[0]*f^-1) for o in Hxy])
orbits=Set([action(p0,p1) for p0 in S for p1 in S])
print(len(orbits))
```

In [28, Satz 10] Mönkemeyer proved—using, of course, a different language—that the IFS (M_0, M_1), where $M_0 = NFR$ and $M_1 = LR$, is contractive in every dimension *n*; a more modern proof is in [33, Lemma 19]. Mönkemeyer's algorithm has been rediscovered several times, notably by Selmer [35] and Baldwin [5]: see the discussion at the end of [30, §1]. In dimension 1, and identifying Δ with the real unit interval [0, 1] via $(x, 1 - x) \mapsto 1 - x$, it reduces to the classical Farey map [23–25, 27]. The following is our main result.

Theorem 3.1 *For every $n = 1, 2, 3, \ldots$, and up to the natural action of the symmetric group S_{n+1}, there are precisely three algorithms that split the unit n-dimensional simplex Δ in two, and are contractive.*

(1) *The Mönkemeyer algorithm, induced by (M_0, M_1); this is the only case in which the associated Gauss-type map G is continuous. If n is odd, G is orientation-preserving on Δ_0 and orientation-reversing on Δ_1; this behavior is reversed for even n, and also reversed by replacing (M_0, M_1) with the equivalent pair (FM_1F^{-1}, FM_0F^{-1}).*
(2) *The algorithm induced by (M_0, M_1F), whose map G is orientation-preserving for odd n and orientation-reversing for even n.*
(3) *The algorithm induced by (M_0F, M_1), whose map G is orientation-reversing for odd n and orientation-preserving for even n.*

The rest of this paper is devoted to the proof of Theorem 3.1.

4 Proof of Theorem 3.1

The orientation preserving/reversing claims amount to the fact that the matrices determining the branches of G have positive/negative determinant. They are easily checked by noting that $\det N = \det L = 1$, $\det F = -1$, and $\det R = (-1)^n$. In Case (1) the map G is continuous because, for every $i \neq 0$, we have $M_0 e_i = M_1 e_i$, and $\Delta_0 \cap \Delta_1$ is the common M_0 -image and M_1-image of the face of Δ with vertices $e_1, \dots, e_n$. In Cases (2) and (3), if G were continuous then, since its two branches are coherent with respect to orientation, the images $G[\Delta_0]$ and $G[\Delta_1]$ would lie on opposite sides of some $(n-1)$-dimensional face of Δ, which is impossible.

We have $FM_0 = M_1$ and $FM_1 = M_0$. Therefore, every product of t matrices from $(M_0, M_1 F)$, or from $(M_0 F, M_1)$, equals a product of t matrices from (M_0, M_1), possibly followed by a final F. Since $F[\Delta] = \Delta$, this implies that:

(i) The time-t partition $\{\Delta_w : |w| = t\}$ is unchanged by taking (A_0, A_1) to be either (M_0, M_1), or $(M_0, M_1 F)$, or $(M_0 F, M_1)$.
(ii) The contractivity of the original Mönkemeyer algorithm extends to its orientation-preserving and orientation-reversing versions.

Looking back at the properties of the action (3.1), the proof of Theorem 3.1 reduces then to establishing the following claim.

Claim 4.1 Let $P_0, P_1 \in S_{n+1}$ be such that $(A_0, A_1) = (NP_0, LP_1)$ is contractive. Then, under the action of $S_2 \times S_{n-1}$ given by

$$\begin{aligned} (I, H) * (A_0, A_1) &= (HA_0H^{-1}, HA_1H^{-1}), \\ (F, I) * (A_0, A_1) &= (FA_1F^{-1}, FA_0F^{-1}). \end{aligned} \tag{4.1}$$

the pair (A_0, A_1) is equivalent to precisely one of (M_0, M_1), $(M_0, M_1 F)$, $(M_0 F, M_1)$ ($(M_0 F, M_1 F)$ is equivalent to (M_0, M_1), since they are mapped to each other by (F, I)).

We assume familiarity with the basics of nonnegative matrices: reducible and irreducible matrices, their incidence graphs, the period of an irreducible matrix, the Perron-Frobenius theorem; see, for example, [21, Chapter III] or [8, Chapter 2]. The following lemma and its corollary Lemma 4.3 are our main tools.

Lemma 4.2 *Let $C, D \in \Sigma$ be such that:*

(i) *There exists a vertex e_i of Δ which is fixed by both C and D;*
(ii) *The interiors of $C[\Delta]$ and $D[\Delta]$ do not intersect.*

Then there exists a product E of C and D such that $E^t[\Delta]$ does not shrink to a point, for $t \to \infty$.

Proof It is enough to prove that the monoid Γ generated by C and D contains a matrix E with spectral radius strictly greater than 1. Indeed, if so, then the spectral radius is an eigenvalue of E, corresponding to an eigenvector $v \in \Delta$. Since clearly $v \neq e_i$, we have that $E^t[\Delta]$ does not shrink to a point.

Assume by contradiction that all matrices in Γ have spectral radius 1. By [32, §6] we may conjugate by an appropriate permutation matrix and assume without loss of generality that all matrices in Γ have block upper-triangular form

$$\begin{pmatrix} B_1 & * & * \\ & \ddots & * \\ & & B_r \end{pmatrix},$$

and, moreover, the monoids $\Gamma_1, \dots, \Gamma_r$ whose elements are the blocks appearing on the diagonal in position, respectively, $1, \dots, r$ are all irreducible by permutations (that is, none of them admits a proper nontrivial invariant coordinate subspace). Clearly, every matrix in every Γ_i still has spectral radius 1. By [32, Proposition 12], every Γ_i is a finite semigroup, thus a group of permutation matrices.

This implies that appropriate powers C^s, D^t of C, D are upper-triangular, with 1 along the diagonal; let $T = D^{-t}C^s$, which has the same form. It is easy to see that there exists a positive (that is, all entries are strictly positive) vector x such that $y = Tx$ is positive as well. Indeed, starting from any $x_n > 0$, one simply chooses recursively x_k so large that $x_k + T^k_{k+1}x_{k+1} + \cdots + T^k_n x_n > 0$ (T^k_j is the entry in line k and column j of T). Thus, the point $C^s x = D^t y$ belongs, once projected to Δ, to the intersection of the interiors of $C^s[\Delta]$ and $D^t[\Delta]$, which is a subset of the intersection of the interiors of $C[\Delta]$ and $D[\Delta]$; this contradicts assumption (ii). $\square$

Let now (A_0, A_1) be as in Claim 4.1; we begin by examining A_0. Every column of A_0 contains precisely one 1, with the exception of one column, say column h, that contains two 1 in the first two rows. We construct the *incomplete incidence graph* $\mathcal{G}_0^-$ of A_0, having nodes $0, 1, \dots, n$ and a directed edge $j \leftarrow i$ whenever $j \neq h$ and the entry in row i and column j of A_0 equals 1. We thus have $A_0(e_j) = e_i$ (this seemingly unnatural direction of arrows is the common convention in the theory of graph-directed IFS). The incomplete graph is *completed* by adding the two edges $0 \rightarrow h \leftarrow 1$, thus obtaining the ordinary incidence graph $\mathcal{G}_0$ of A_0; note that A_0 maps e_h neither to e_0 nor to e_1, but to their Farey sum $(e_0 + e_1)/2$. We let $\mathcal{G}_1^-$, $\mathcal{G}_1$ be the analogously defined incomplete and complete incidence graphs of A_1; swapping the nodes 0 and 1, all considerations above apply.

We now glue together $\mathcal{G}_0^-$ and $\mathcal{G}_1^-$, forming the *incomplete graph* $\mathcal{G}^-$ of the pair (A_0, A_1), with nodes $0, \dots, n$ and *0-edges* and *1-edges*, coming from $\mathcal{G}_0^-$ and $\mathcal{G}_1^-$, respectively. For every word $v = a_0 \dots a_{t-1} \in \{0, 1\}^t$ and every pair of nodes i, j there is at most one path in $\mathcal{G}^-$ that starts from i and follows first an a_0-edge $i_1 \leftarrow i$, then an a_1-edge $i_2 \leftarrow i_1$ and so on, reaching j after the final a_{t-1}-edge. If such a path exists we say that ${}_iv_j$ *occurs* in $\mathcal{G}^-$. If ${}_iv_i$ occurs, then it is a *loop for v at i*. Words v, w are *incomparable* if neither of them is an initial segment of the other.

Lemma 4.3 *Let $\mathcal{G}^-$ be the incomplete graph associated to the contractive pair (A_0, A_1). Then $\mathcal{G}^-$ cannot contain:*

(i) *either two loops for the same word at distinct nodes;*
(ii) *or two loops for incomparable words at the same node.*

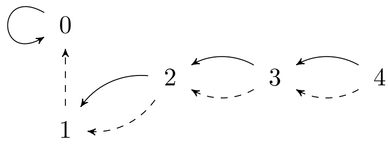


Fig. 2 The *parabolic-hyperbolic* pair (M_0, M_1)

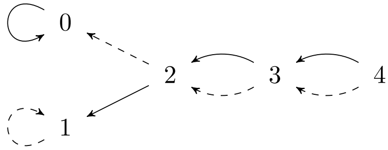


Fig. 3 The *parabolic-parabolic* pair $(M_0, M_1 F)$

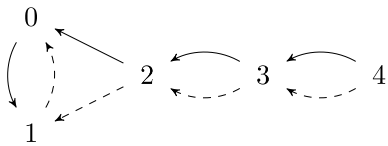


Fig. 4 The *hyperbolic-hyperbolic* pair $(M_0 F, M_1)$

Proof If there are two loops, at *i* and *j*, for the word *v*, then A_v fixes both e_i and e_j which are hence both in $\bigcap_{t \geq 0} A_v^t[\Delta]$; this contradicts contractivity.

Let *v* and *w* be incomparable. Then it is easy to see (for example by factoring out the longest common prefix) that the interiors of $A_v[\Delta]$ and $A_w[\Delta]$ do not intersect. On the other hand, if they determine loops in $\mathcal{G}^-$ at the same node *i*, then both A_v and A_w fix the vertex e_i. By Lemma 4.2 this again contradicts contractivity. $\square$

Example 4.4 As a clarifying example we draw in Figs. 2, 3 and 4 the incomplete graphs $\mathcal{G}^-$ of the matrix pairs in Theorem 3.1; 0-edges are continuous and 1-edges dashed. We use $n = 4$; larger values just require adding the obvious tail. The adjectives parabolic/hyperbolic are justified. Indeed, M_0 (and its F-conjugate $M_1 F$) has characteristic polynomial $(x-1)(x^n-1)$, and $M_0^t[\Delta]$ shrinks to the vertex e_0. On the other hand, the completed graph of M_1 is given by the full loop $n \leftarrow 0 \leftarrow 1 \leftarrow 2 \leftarrow \cdots \leftarrow n$, which is shortcut by the further edge $n \leftarrow 1$. Therefore M_1 is irreducible of period $\gcd(n+1, n) = 1$, its powers are eventually strictly positive, and $M_1^t[\Delta]$ shrinks to a point in the interior of Δ; moreover, its characteristic polynomial $x^{n+1} - x - 1$ is irreducible [34, Theorem 1].

As the 1-dimensional case is trivial, from now on we assume $n \geq 2$. By construction, every node in $\mathcal{G}_0^-$ has precisely one father and one son, with the exception of *h* that has no father and of 1 that has no son. We must have $h \neq 1$, since otherwise $\mathcal{G}_0^-$ would contain a loop with at least two nodes, or two one-node loops, contrary to Lemma 4.3. Therefore $\mathcal{G}_0^-$ decomposes in a chain from *h* to 1 and at most one loop $i \leftarrow i$; if the loop is present, then necessarily is at $i = 0$, for otherwise the completed graph $\mathcal{G}_0$ would be disconnected, and $A_0^t[\Delta]$ would not shrink to a point. We relabel the nodes in $\{2, 3, \ldots, n\}$ in such a way that every $i \geq 3$ is an ancestor of $i - 1$ in $\mathcal{G}_0^-$. This amounts to replacing (A_0, A_1) with (HA_0H^{-1}, HA_1H^{-1})—which for simplicity we rename (A_0, A_1)—for an appropriate permutation matrix $H \in S_{n-1}$. Upon exchanging 0 with 1, the analysis above holds for $\mathcal{G}_1^-$, except that the nodes $2, \ldots, n$ appear along the chain of $\mathcal{G}_1^-$ in some fixed but up to now unknown order $k_2, \ldots, k_n$.

The presence or absence of an isolated one-node loop distinguishes the parabolic from the hyperbolic cases; we summarize the situation thus far.

$\mathcal{G}_0^-$ is:

(0p) either the *n*-node chain $1 \leftarrow 2 \leftarrow \cdots \leftarrow n$ plus the loop $0 \leftarrow 0$;
(0h) or an $(n+1)$-node chain $1 \leftarrow 2 \leftarrow \cdots \leftarrow 0 \leftarrow \cdots \leftarrow n$, with 0 appearing in some up to now indeterminate position (but different from the last, which is taken by 1).

$\mathcal{G}_1^-$ is:

(1p) either the *n*-node chain $0 \leftarrow k_2 \leftarrow \cdots \leftarrow k_n$ plus the loop $1 \leftarrow 1$;
(1h) or an $(n+1)$-node chain $0 \leftarrow k_2 \leftarrow \cdots \leftarrow 1 \leftarrow \cdots \leftarrow k_n$, with 1 appearing in some up to now indeterminate position (but different from the last, which is taken by 0).

Lemma 4.5 *If we are in Case (0h), then the chain $\mathcal{G}_0^-$ ends with $1 \leftarrow 0 \leftarrow 2$. Analogously, if we are in case Case (1h) the chain $\mathcal{G}_1^-$ ends with $0 \leftarrow 1 \leftarrow k_2$.*

Proof Assume we are in Case (0h); then there cannot be two nodes *i*, *j*, in the chain in $\mathcal{G}_1^-$ and such that, in $\mathcal{G}_0^-$, the node 0 is an ancestor of *i*, and *i* an ancestor of *j*. Indeed, if so, then ${}_0 0_i^r$ and ${}_i 0_j^s$ both occur in $\mathcal{G}^-$. for certain $r, s \geq 1$. Moreover, there exist *t*, *u* such that either ${}_j 1_i^t$ and ${}_i 1_0^u$ occur, or ${}_i 1_j^t$ and ${}_j 1_0^u$ occur. In the first case the incomparable words $0^s 1^t$ and $1^u 0^r$ are loops at *i*, and in the second so are the words $0^s 1^u 0^r$ and $1^t 1^u 0^r$, contradicting Lemma 4.3(ii). This establishes Lemma 4.5 whenever Case (1h) holds, or Case (1p) holds and 3 is a descendant of 0 in $\mathcal{G}_0^-$.

We have now to prove that the assumption of Cases (0h) and (1p), and of the fact that the 0-chain ends with $1 \leftarrow 2 \leftarrow 0$, leads to a contradiction.

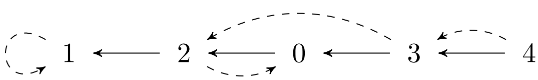
If $k_2 \neq 2$, then for certain *r*, *s* both ${}_{k_2} 0_0^r$ and ${}_2 1_{k_2}^s$ occur, creating the incomparable loops $0^r 0 1^s$ and $1 0 1^s$ at k_2, which is impossible.

Now, $k_2 = 2$ implies that $\{k_3 \ldots, k_n\}$ equals $\{3, \ldots, n\}$ as sets. The 1-chain $k_3 \leftarrow \cdots \leftarrow k_n$ must run through this set of nodes in some order, and we claim that it cannot contain any *backtrack*, that is, any edge $i \rightarrow j$ with $i < j$. Indeed:

(a) if $j = i + 1$, this would create two loops for the word 01, at the distinct nodes 0 and *j*;
(b) if $j > i + 1$, then $i + 1$ would be either a 1-ancestor of *i*, or a 1-descendant of *j*. In both cases, this would create two incomparable loops at *j*.

Thus, the absence of backtracks forces 0-edges and 1-edges to run in parallel along the nodes $\{3, \ldots, n\}$, so that $\mathcal{G}^-$ appears as in Fig. 5 where, as in Figs. 2, 3 and 4, continuous and dashed edges are 0- and 1-edges, respectively. We set $n = 4$ for concreteness: add

Fig. 5 No backtracks along $\{3, \ldots, n\}$



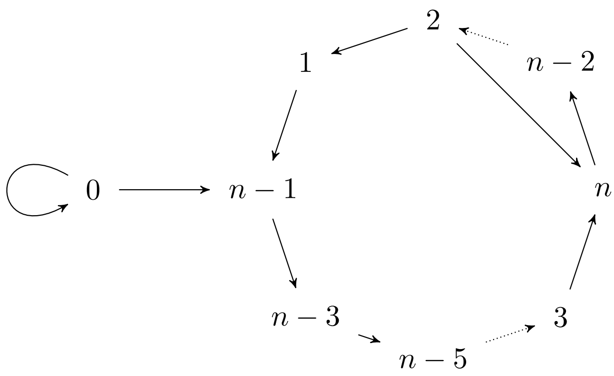


Fig. 6 The incidence graph of A_{01}

the obvious tail for larger *n*, and remove nodes in excess and all edges related to them for $n = 2, 3$.

The choice $n = 4$ in Fig. 5 is however appropriate, for *n* has to be even. Indeed, by completing $\mathcal{G}_0^-$, that is by adding the edges $0 \to n \leftarrow 1$, we obtain a loop with $n + 1$ edges, which is shortcut by a loop of $n - 1$ edges. Thus the period $\gcd(n + 1, n - 1)$ of the irreducible matrix A_0 is 1 if and only if *n* is even; if so, $A_0^t[\Delta]$ shrinks to a point in the interior of Δ. Otherwise the period is 2, and A_0^2 has nontrivial block-diagonal form, preventing contractivity.

We complete our quest for a contradiction by showing that $A_{01}^t[\Delta]$ does not shrink to a point. If $n = 2$ this is established by computing the right eigenspaces of

$$A_{01} = \begin{pmatrix} 1 & & 1 \\ & & 1 \\ & 1 & 1 \end{pmatrix}.$$

If $n = 4, 6, 8, \ldots$ we see by direct inspection that the incidence graph of A_{01} has the form in Fig. 6, with a loop of length *n* shortcut to one of length *n*/2. The greatest common divisor of *n* and *n*/2 is *n*/2. Thus $A_{01}^{n/2}$ has upper block-triangular form, with irreducible blocks of period 1 along the diagonal. The first block, corresponding to the node 0, has order 1, and the remaining *n*/2 blocks order 2. In particular, A_{01} has a dominant eigenvalue strictly greater than 1; since the vertex e_0 is fixed, the chain $A_{01}^t[\Delta]$ cannot shrink to a point. We have thus reached a final contradiction and settled Case (0h).

Of course Cases (0p) and (1p), as well as Cases (0h) and (1h), are exchanged by exchanging 0 with 1, and every $i \in \{2, \ldots, n\}$ with the corresponding k_i. Therefore the argument above applies to Case (1h) and establishes that 1 must appear as second-to-last node in $\mathcal{G}_1^-$.
□

The proof of Claim 4.1, and thus of Theorem 3.1, is almost complete. We have started from a contracting IFS (A_0, A_1) such that $\Delta_0 = N[\Delta]$ and $\Delta_1 = L[\Delta]$, and have established that an appropriate conjugation reduces the pair of incomplete graphs $(\mathcal{G}_0^-, \mathcal{G}_1^-)$ (which determine the completed incidence graphs, and thus A_0, A_1 themselves) to one of the three possibilities (0p,1h), (0p,1p), (0h,1h) (the possibility (0h,1p) is equivalent to (0p,1h)). Moreover, we have proved that in the hyperbolic cases (0h) and (1h) the node chains are $1 \leftarrow 0 \leftarrow 2 \leftarrow 3 \leftarrow \cdots \leftarrow n$ and $0 \leftarrow 1 \leftarrow k_2 \leftarrow k_3 \leftarrow \cdots \leftarrow k_n$, respectively. Cases (0p) and (0h) correspond then to the Mönkemeyer matrices M_0 and M_0F; if we establish that

$k_i = i$ for every $i \in \{2, \dots, n\}$, Cases (1h) and (1p) will correspond to M_1 and $M_1 F$, thus settling Claim 4.1.

As in the proof of Lemma 4.5, no matter if we are in the parabolic or hyperbolic cases, the 1-chain $k_2 \leftarrow \dots \leftarrow k_n$ must run along the nodes $\{2, \dots, n\}$ in some order. We claim that it does not contain backtracks. Indeed, if $i \to j$ were a long backtrack, that is with $i + 1 < j$, then the argument in (b) in the proof of Lemma 4.5 would yield the existence of two incomparable loops at *j*, which is impossible. There cannot be two short backtracks $i \to i + 1$ and $j \to j + 1$, because this would create two loops for the word 10 at *i* and *j*, contrary to Lemma 4.3(i). Finally, not even a single short backtrack $i \to i + 1$ is possible. Indeed, direct inspection shows that, no matter if we are in the parabolic or hyperbolic cases, the incidence graph of A_{10} would then be disconnected with an isolated loop at *i*, so that $A^t_{10}[\Delta]$ would not shrink to a point. Therefore there are no backtracks, and $k_i = i$ for every *i* in $\{2, \dots, n\}$.

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