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The addition theorem for locally monotileable monoid actions

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Abstract

We prove the so-called Addition Theorem for the algebraic entropy of actions of cancellative right amenable monoids S on discrete abelian groups A by endomorphisms, under the hypothesis that S is locally monotileable (that is, S admits a right Følner sequence $(F_n)_{n\in\mathbb{N}}$ such that F_n is a monotile of F_{n+1} for every $n \in \mathbb{N}$). We study in details the class of locally monotileable groups, also in relation with already existing notions of monotileability for groups, introduced by B. Weiss and developed further by other authors recently.

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1 Introduction

After a very brief and schematic introduction by Adler, Konheim and McAndrew [1], the algebraic entropy for endomorphisms of abelian groups was gradually developed by M. Weiss [46] and Peters [36, 37]. The interest in this direction increased after [16], where a rather complete description in the case of torsion abelian groups was obtained. The algebraic entropy defined by Peters [36] for automorphisms of arbitrary abelian groups was suitably extended to endomorphisms in [15] (see also [11]); this entropy is denoted by h_{alg} in the sequel. On the other hand, appropriate versions of the algebraic entropy for module endomorphisms were introduced by Salce and Zanardo [39] and studied further by Salce, Vámos and Virili [38], also in connection with length functions in the sense of Northcott and Reufel. Recently, Virili [43] extended this algebraic entropy to amenable group actions on modules and found applications to the Stable Finiteness Conjecture and the Zero Divisors Conjecture, originally stated by Kaplansky. These ideas were pushed further by Li and Liang [30].

Let S be a cancellative right amenable semigroup, A an abelian group, and $S \overset{\alpha}{\curvearrowright} A$ a left action by endomorphisms. In [9], inspired by the recent results and definitions of Ceccherini-Silbertstein, Coornaert and Krieger [5], the algebraic entropy h_{alg} was extended to such actions α as follows. For $X \in \mathcal{P}_{fin}(A)$ and $F \in \mathcal{P}_{fin}(S)$, let

$$T_F(\alpha, X) = \sum_{s\in F} \alpha(s)(X)$$

be the *α-trajectory of X with respect to F*. The *algebraic entropy of α with respect to $X \in \mathcal{P}_{fin}(A)$* is

$$H_{alg}(\alpha, X) = \lim_{i\in I} \frac{\ell(T_{F_i}(\alpha, X))}{|F_i|},$$

where $(F_i)_{i\in I}$ is a right Følner net of S and $\ell(X)$ is the natural logarithm of the cardinality of X. The limit defining $H_{alg}(\alpha, X)$ exists and does not depend on the choice of the right Følner net $(F_i)_{i\in I}$ in view of [5, Theorem 1.1]. The *algebraic entropy of α* is

$$h_{alg}(\alpha) = \sup\{H_{alg}(\alpha, X) : X \in \mathcal{P}_{fin}(A)\}.$$

This definition of algebraic entropy coincides with that for single endomorphisms (mentioned above) when those are considered as left $\mathbb{N}$-actions. Moreover, for amenable group actions on discrete abelian groups it coincides with the algebraic entropy defined in [44] on locally compact abelian groups.

A fundamental property of h_{alg}, established in [15] (and in [16] for torsion abelian groups), is the so-called Addition Theorem (or Yuzvinski's addition formula):

Theorem 1.1. *Let A be an abelian group, $\phi : A \to A$ an endomorphism and B a ϕ-invariant subgroup of G. Then $h_{alg}(\phi) = h_{alg}(\bar{\phi}) + h_{alg}(\phi\restriction_B)$, where $\bar{\phi} : A/B \to A/B$ is the endomorphism induced by ϕ.*

This result was generalized to locally finite groups that are either quasihamiltonian or FC-groups in [26], while a counterexample in the non-abelian case was given in [24].

Moreover, the same additivity property was provided in [38, 39] for the algebraic entropy of module endomorphisms under suitable conditions. This was extended in [40] to a more general setting, and to amenable group actions in [43].

In [9] the first three authors proved the Addition Theorem for left actions of cancellative right amenable monoids S on torsion abelian groups A. Here we prove it for all abelian groups A under the hypotheses that S is also countable and locally monotileable in the sense of Definition 1.5:

*The first three named authors are members of the "National Group for Algebraic and Geometric Structures, and Their Applications" (GNSAGA - INdAM)

Theorem 1.2 (Addition Theorem)**.** *Let $S \overset{\alpha}{\curvearrowright} A$ be a left action of a locally monotileable cancellative right amenable monoid S on an abelian group A. Let B be an α-invariant subgroup of A, and denote by $\alpha_{A/B}$ and α_B the induced actions of S on A/B and on B, respectively. Then*

$$h_{alg}(\alpha) = h_{alg}(\alpha_{A/B}) + h_{alg}(\alpha_B).$$

Since $\mathbb{N}$ is locally monotileable, as a corollary of Theorem 1.2 we find Theorem 1.1. While the proof in [15] was quite long and heavily used the structure of the abelian group A, the proof in the present paper is much shorter and makes no recourse to the structure of A.

The problem on whether the hypothesis "locally monotileable" can be relaxed in Theorem 1.2 remains open.

In Section 2 we prove Theorem 1.2 and give its consequences for the topological entropy. In particular, in §2.4 we offer a background on the topological entropy of (semi)group actions and its connection with the algebraic one by means of Pontryagin-van Kampen duality.

In Section 3 we study the class of countable locally monotileable groups.

Definition 1.3. For subsets T, V of a semigroup S, we say that T is a *monotile* of V if there exists a subset C of S such that $\{cT : c \in C\}$ is a partition of V.

The notion of monotile was defined in [45], in the case when $V = G$, in connection with the ε-quasi tilings from [34]. The interest in monotiles (of G) stems from the celebrated Rokhlin Lemma:

Fact 1.4 (Rokhlin Lemma)**.** *Let $T : X \to X$ be an invertible measure-preserving transformation on a probability space (X, Σ, μ). We assume T is (measurably) aperiodic, that is, the set of periodic points for T has zero measure. Then for every integer $n \in \mathbb{N}_+$ and for every $\varepsilon > 0$, there exists a measurable set E such that the sets E, TE, …, $T^{n-1}E$ are pairwise disjoint and such that $\mu(E \cup TE \cup \dots \cup T^{n-1}E) > 1 - \varepsilon$.*

An extension of Rokhlin Lemma for $\mathbb{Z}^d$-actions was proved in [7] and in [27]. A further extension of this result for amenable group actions was announced in [34] and then proved in [35]. More precisely, if a countable amenable group G acts freely on a Lebesgue measure space (X, μ), we say that Rokhlin Lemma holds for a finite subset F of G if for every $\varepsilon > 0$ there is a subset B of X such that the sets in $\{fB : f \in F\}$ are pairwise disjoint and $\mu(\bigcup_{f\in F} fB) > 1 - \varepsilon$. Rokhlin Lemma holds for F if and only if F is a monotile of G (see [35]).

For our purpose concerning the Addition Theorem, we need the following special right Følner sequences.

Definition 1.5. Let S be a monoid. A sequence $(F_n)_{n\in\mathbb{N}}$ in $\mathcal{P}_{fin}(S)$ is *locally monotileable* if $F_0 = \{1\}$ and F_n is a monotile of F_{n+1} for every $n \in \mathbb{N}$.

A countable right amenable monoid is *locally monotileable* if it admits a locally monotileable right Følner sequence.

Consider a locally monotileable sequence $(F_n)_{n\in\mathbb{N}}$ of S. By definition for every $n \in \mathbb{N}_+$ there is a finite subset K_n of F_n such that $F_n = \bigsqcup_{k\in K_n} kF_{n-1}$ (in particular, $F_n = K_n F_{n-1}$). Let $K_0 = \{1\}$. Since $F_0 = \{1\}$, we have that $K_1 = F_1$ and so by induction we conclude that, for every $n \in \mathbb{N}$,

$$F_n = K_n \dots K_1 K_0.$$

The sequence $(K_n)_{n\in\mathbb{N}}$ is the *tiling sequence* associated to the locally monotileable sequence $(F_n)_{n\in\mathbb{N}}$.

Definition 1.5 is inspired by a notion due to Weiss, that he introduced for groups in [45]:

Definition 1.6. Let S be a monoid. A countable cancellative right amenable monoid S is *monotileable amenable* (briefly, MTA) if there exists a right Følner sequence $(F_n)_{n\in\mathbb{N}}$ of S such that F_n is a monotile of S for every $n \in \mathbb{N}$.

The following special case of monotileability was introduced in [4, Definition 4] in the case of groups, we now give it using our terminology.

Definition 1.7. Let S be a monoid. A sequence $(F_n)_{n\in\mathbb{N}}$ in $\mathcal{P}_{fin}(S)$ is *congruent* if it is locally monotileable and it admits a tiling sequence $(K_n)_{n\in\mathbb{N}}$ with $1 \in K_n$ for every $n \in \mathbb{N}$. Moreover, $(F_n)_{n\in\mathbb{N}}$ is *exhaustive* if $\bigcup_{n\in\mathbb{N}} F_n = S$.

A countable right amenable monoid S is *congruent monotileable* if it admits an exhaustive congruent right Følner sequence.

If S is a cancellative monoid and $(F_n)_{n\in\mathbb{N}}$ is a locally monotileable sequence of S with associated tiling sequence $(K_n)_{n\in\mathbb{N}}$ and $1 \in K_n$ for every $n \in \mathbb{N}$, then $(F_n)_{n\in\mathbb{N}}$ is increasing, that is, $F_n \subseteq F_{n+1}$ for every $n \in \mathbb{N}$, and moreover $K_n \subseteq F_{n+1}$ for every $n \in \mathbb{N}$. Example 3.4 shows that the converse is not true in general.

For a right Følner sequence $(F_n)_{n\in\mathbb{N}}$ of a monoid S, the following (non-)implications hold.

$$\begin{tikzcd}
\text{exhaustive congruent} \arrow[r, Rightarrow] \arrow[d, Rightarrow, "\text{L.3.12}"'] & \text{congruent} \arrow[r, Rightarrow] \arrow[l, bend right, "\text{Ex.3.3}"', "/" marking] & \text{locally monotileable} \arrow[l, bend right, "\text{Ex.3.4}"', "/" marking] \arrow[dll, "/" marking] \\
\text{consisting of monotiles} \arrow[urr, "\text{Ex.3.3}", "/" marking] & &
\end{tikzcd}$$

Since an exhaustive congruent sequence of a monoid S consists necessarily of monotiles of S (see Lemma 3.12), a countable congruent monotileable monoid is MTA. In particular, the condition of monotileability in [4, Definition 4] is redundant.

For countable right amenable monoids, one has the following implications among the notions of monotileability introduced and recalled above. A counterexample witnessing that locally monotileable does not imply MTA (and so neither congruent monotileble) is given in Example 3.13.

$$\begin{tikzcd}
& \text{congruent monotileable} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow, "\text{L.3.12}"] & \\
\text{locally monotileable} \arrow[rr, "\text{Ex.3.13}", "/" marking] \arrow[dr, Rightarrow] & & \text{MTA} \arrow[dl, Rightarrow] \\
& \text{amenable} &
\end{tikzcd}$$

Restricting to the case of groups, first note that for groups the notions of local monotileability and congruent monotileability are equivalent (see Proposition 3.14).

Moreover, recall that Weiss [45] proved that every countable residually finite amenable group is MTA and that every countable solvable group is MTA. The latter result was extended by Ebli [21], showing that every countable elementary amenable group is MTA. So, the next related question is very natural.

Question 1.8. *Is every* MTA *group necessarily elementary amenable?*

The following very general question by Weiss is open.

Question 1.9 (See [45])**.** *Is every countable amenable group necessarily* MTA*?*

In this sense Downarowicz, Huczek and Zhang [20] provided a positive answer to a weaker version of this problem.

Inspired by this result from [20], Cecchi and Cortez [4] introduced the notion recalled above of congruent monotileable group. Cortez and Petite proved in [8] that residually finite amenable groups are congruent monotileable. Using this result, Cecchi and Cortez showed that every countable virtually nilpotent group is congruent monotileable (see [4, Theorem 1]). The following questions from [4], connected to the general Question 1.9, are open.

Question 1.10 (See [4])**.** *(a) Is every countable amenable group necessarily congruent monotileable?*

(b) In particular, is every countable MTA *group necessarily congruent monotileable?*

In [45], also the following notion was introduced in the case of groups.

Definition 1.11. A semigroup S is *monotileable* (briefly, MT) if every finite subset of S is contained in a finite monotile of S.

While every MTA group is necessarily MT and amenable (see Proposition 3.15), the validity of the converse implication is not known, and the following question by Weiss is open.

Question 1.12 (See [45])**.** *If a countable group G is* MT *and amenable, is G necessarily* MTA*?*

We are not aware if a negative answer of the counterpart of Question 1.12 for cancellative monoids is available.

In order to study the class $\mathfrak{M}$ of (countable) locally monotileable groups, we consider the following general problem concerning the stability of $\mathfrak{M}$ under extension.

Problem 1.13. *Consider three countable groups G, H and K, such that $0 \to H \xrightarrow{\iota} G \xrightarrow{\pi} K \to 0$ is a short exact sequence of groups.*

(a) If H and K are locally monotileable, is then G locally monotileable as well?

(b) What about splitting extensions $G = H \rtimes K$?

Moreover, we introduce the following notion of monotileability stronger than local monotileability, where for a group G we denote by $\mathrm{Aut}(G)$ its group of automorphisms, while $\mathrm{Inn}(G)$ denotes the subgroup of $\mathrm{Aut}(G)$ consisting of all inner automorphisms of G.

Definition 1.14. Let G be a group, $(F_n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{P}_{fin}(G)$ and $\mathrm{id} \in X \subseteq \mathrm{Aut}(G)$. We say that $(F_n)_{n\in\mathbb{N}}$ is an X*-monotileable* sequence of G if for all $n \in \mathbb{N}$ and $\phi \in X$, we have that $\phi(F_n)$ is a monotile of F_{n+1}. We say that G is X*-monotileable* if there exists an X-monotileable right Følner sequence $(F_n)_{n\in\mathbb{N}}$ of G.

When $X = \{\mathrm{id}, \psi, \psi^{-1}\}$, we simply write ψ*-monotileable.*

One of our main results is the following partial answer to Problem 1.13.

Theorem 1.15 (Extension Theorem)**.** *Consider three countable groups G, K and H. Suppose that*

$$0 \to H \xrightarrow{\iota} G \xrightarrow{\pi} K \to 0$$

is a short exact sequence. If K is locally monotileable and H is $\mathrm{Inn}(G)$-monotileable, then G is locally monotileable.

Using this result, we prove one of the main achievements of this paper, that is, Theorem 1.16. Recall that a group G is *hypercentral* if its upper central series terminates at the whole group, that is, there exists an ordinal α such that $Z_\alpha(G) = G$; the length of G as an hypercentral group is the minimum such α.

Theorem 1.16. *Every countable virtually hypercentral group of length $< \omega^2$ is locally monotileable (i.e., congruent monotileable).*

Clearly every nilpotent group is hypercentral. So, as an immediate corollary of Theorem 1.16 we obtain the above mentioned result from [4], that every countable virtually nilpotent group is congruent monotileable.

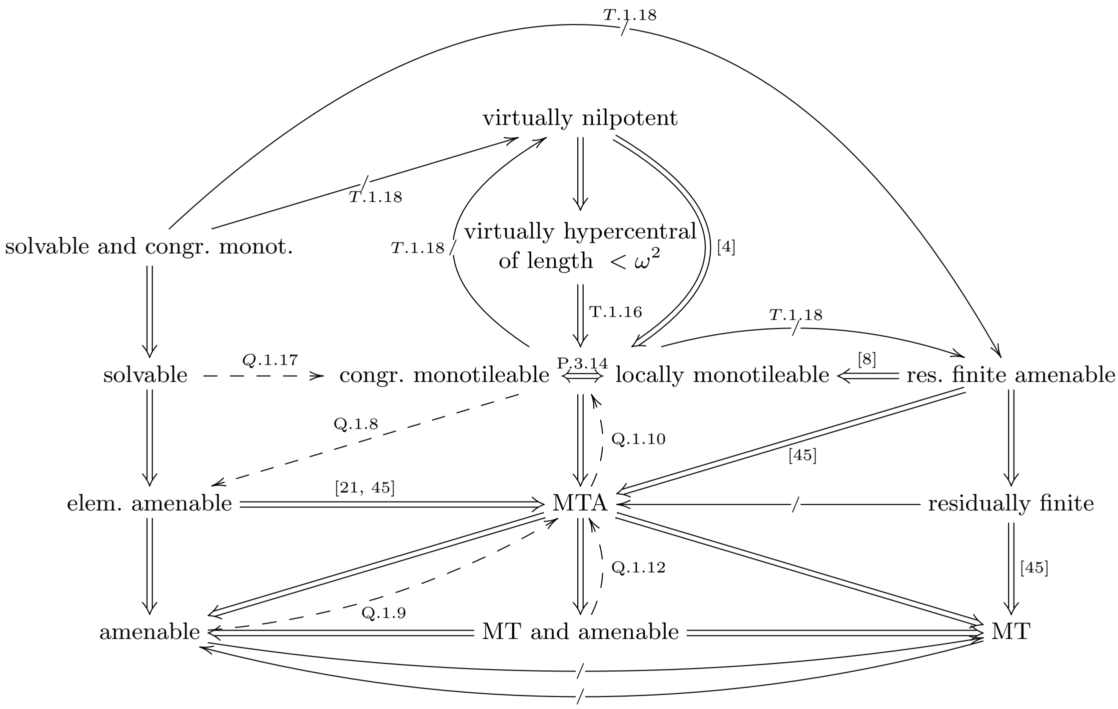
Since all countable solvable groups are known to be MTA, and in view of Theorem 1.16, it is natural to ask the following.

Question 1.17. *Are all countable solvable groups locally monotileable? What about polycyclic groups?*

A positive answer to Problem 1.13 would also be a positive answer to Question 1.17. In this context, the next theorem provides an example of a locally monotileable solvable (actually, metabelian) group that is neither virtually nilpotent nor residually finite.

Theorem 1.18. *For every automorphism ϕ of $\mathbb{Q}$, the groups $\mathbb{Q}$ and $\mathbb{Q} \rtimes_\phi \mathbb{Z}$ are locally monotileable.*

The following diagram represents all known (non-)implications among the above mentioned properties for countable groups, and the related open questions.



The following open questions, related to Problem 1.13 and Question 1.17, are motivated by Theorem 1.18.

Question 1.19. *(a) Is the group $\mathbb{Q}^n \rtimes_\phi \mathbb{Z}$ locally monotileable for every automorphism ϕ of $\mathbb{Q}^n$?*

(b) More generally, is the group $H \rtimes_\phi \mathbb{Z}$ locally monotileable for an abelian group H and $\phi \in \mathrm{Aut}(H)$?

(c) If H is a locally monotileable group and K is a finitely generated locally monotileable subgroup of $\mathrm{Aut}(H)$, is the group $H \rtimes K$ locally monotileable?

Call a group G *hereditarily locally monotileable* (briefly, *h-locally monotileable*) if every countable subgroup of G is locally monotileable. It can be easily deduced from Theorem 1.16 that every virtually nilpotent group, as well as every locally nilpotent group, is h-locally monotileable.

Problem 1.20. *Find examples of finitely generated amenable groups that are not locally monotileable, or at least h-locally monotileable.*

In §5.2 we provide further examples, showing an example of a locally monotileable group that is neither virtually solvable nor locally finite, nor residually solvable, and of a locally monotileable group that is virtually hypercentral, yet neither virtually nilpotent nor residually finite.

We show that $\mathfrak{M}$ is stable under some countable direct limits, in particular, under countable direct sums (so, under finite direct products as well). So, we conclude with the following open question about basic stability properties of $\mathfrak{M}$.

Question 1.21. *Is $\mathfrak{M}$ stable under taking subgroups or quotients?*

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Notation and terminology

For a set X, we let $\ell(X) = \log |X|$, using the convention that $\ell(X) = \infty$ if the set X is infinite. Moreover, we denote by $\mathcal{P}(X)$ the family of all subsets of X and by $\mathcal{P}_{fin}(X)$ its subfamily consisting of all non-empty finite subsets of X. For an abelian group A, let $\mathcal{P}^0_{fin}(A)$ be the family of all finite subsets of A containing 0.

In case S is a monoid with neutral element 1, a left semigroup action $S \overset{\alpha}{\curvearrowright} X$ is a *left monoid action* of S on X if $\alpha(1)(x) = x$ for all $x \in X$, i.e., $\alpha(1)$ is the identity map id. If S is a group, then this condition implies that $\alpha(s)$ is a bijection for every $s \in S$. Unless otherwise stated, all the actions of monoids considered in this paper are monoid actions.

We recall that a *right Følner net* of a semigroup S is a net $(F_i)_{i\in I}$ in $\mathcal{P}_{fin}(S)$ such that $\lim_{i\in I} |F_i s \setminus F_i|/|F_i| = 0$ for every $s \in S$. By [32, Corollary 4.3], a semigroup S is *right amenable* if and only if S admits a right Følner net. In case S is commutative we omit the adjective "right". Left Følner nets and left amenability are defined analogously.

Moreover, a group G admits admits a right Følner net if and only if it admits a left Følner net, so we simply speak of amenable groups.

2 Addition Theorem

2.1 Properties of the trajectories

Lemma 2.1. *Let S be a semigroup, A an abelian group, and $S \overset{\alpha}{\curvearrowright} A$ a left action. If $F, F' \in \mathcal{P}_{fin}(S)$ are disjoint and $X \in \mathcal{P}^0_{fin}(A)$, then $T_{F\sqcup F'}(\alpha, X) = T_F(\alpha, X) + T_{F'}(\alpha, X)$.*

Note that the inclusion $T_{FF'}(\alpha, X) \subseteq T_F(\alpha, T_{F'}(\alpha, X))$ in the next lemma holds in general.

Lemma 2.2. *Let S be a semigroup, A an abelian group, and $S \overset{\alpha}{\curvearrowright} A$ a left action. If $F, F' \in \mathcal{P}_{fin}(S)$, $X \in \mathcal{P}^0_{fin}(A)$ and the sets $\{fF' : f \in F\}$ are pairwise disjoint, then $T_{FF'}(\alpha, X) = T_F(\alpha, T_{F'}(\alpha, X))$.*

Proof. By Lemma 2.1, we have that

$$T_{FF'}(\alpha, X) = T_{\bigsqcup_{f\in F} fF'}(\alpha, X) = \sum_{f\in F} T_{fF'}(\alpha, X) = \sum_{f\in F} \alpha(f)(T_{F'}(\alpha, X)) = T_F(\alpha, T_{F'}(\alpha, X)). \qquad \square$$

Lemma 2.3. *Let S be a semigroup, A an abelian group, $S \overset{\alpha}{\curvearrowright} A$ a left action, $F \in \mathcal{P}_{fin}(S)$ and $X, Y \in \mathcal{P}^0_{fin}(A)$. Then $T_F(\alpha, X + Y) = T_F(\alpha, X) + T_F(\alpha, Y)$.*

Proof. By definition

$$\begin{aligned} T_F(\alpha, X + Y) &= \sum_{s\in F} \alpha(s)(X + Y) = \sum_{s\in F} (\alpha(s)(X) + \alpha(s)(Y)) = \\ &= \sum_{s\in F} \alpha(s)(X) + \sum_{s\in F} \alpha(s)(Y) = T_F(\alpha, X) + T_F(\alpha, Y). \qquad \square \end{aligned}$$

The proof of the next lemma is straightforward.

Lemma 2.4. *Let S be a semigroup, A an abelian group, $S \overset{\alpha}{\curvearrowright} A$ a left action, and B an α-invariant subgroup of A with $\pi : A \to A/B$ the canonical projection. If $X \in \mathcal{P}^0_{fin}(A)$ and $F \in \mathcal{P}_{fin}(S)$, then $T_F(\alpha_{A/B}, \pi(X)) = \pi(T_F(\alpha, X))$.*

2.2 The function $\ell(-,-)$

Let A be an abelian group. For $X, Y \in \mathcal{P}(A)$ let

$$\mu(X, Y) = \min \left\{ n \in \mathbb{N} : \exists a_0 = 0, a_1, \dots, a_{n-1} \in A,\ X \subseteq \bigcup_{i=0}^{n-1} (a_i + Y) \right\}; \tag{2.1}$$

if such n does not exist we put $\mu(X, Y) = \infty$ (clearly, when X is finite and Y is nonempty, $\mu(X, Y)$ is also finite); we define $\ell(X, Y) = \log \mu(X, Y)$. If X and Y are subgroups of A, then $\mu(X, Y) = \mu(X + Y, Y) = [X + Y : Y]$; in particular, if $Y \subseteq X$, then $\mu(X, Y) = [X : Y]$.

Obviously, the family

$$\mathfrak{Y} = \{a_0 + Y, a_1 + Y, \dots, a_{n-1} + Y\}$$

appearing in (2.1) is pairwise disjoint when $(C - C) \cap (Y - Y) = \{0\}$, where $C = \{a_0, a_1, \dots, a_{n-1}\}$. We say that $\mathfrak{Y}$ is *strongly pairwise disjoint* if the "fattened" family

$$\mathfrak{Y}^* = \{a_0 + Y - Y, a_1 + Y - Y, \dots, a_{n-1} + Y - Y\}$$

is still pairwise disjoint, or equivalently, when $(C - C) \cap (Y - Y + Y - Y) = \{0\}$.

In the following lemma we collect other useful properties of the function $\ell(-,-)$.

Lemma 2.5. *Let A an abelian group and $X, Y, Z, X', Y' \in \mathcal{P}^0_{fin}(A)$. Then:*

(a) $\ell(X, Y) \geq 0$, $\ell(X, X) = 0$ *and* $\ell(X) = \ell(X, \{0\})$;

(b) the function $\ell(X, Y)$ is increasing in X and decreasing in Y;

(c) $\ell(X, Y) \leq \ell(X) \leq \ell(X, Y) + \ell(Y)$;

(d) $\ell(X + X', Y + Y') \leq \ell(X, Y) + \ell(X', Y')$;

(e) $\ell(X, Y) \leq \ell(X, Z) + \ell(Z, Y)$;

(f) *if $\varphi : A \to A$ is an endomorphism, then $\ell(\varphi(X),\varphi(Y)) \leq \ell(X,Y)$;*

(g) *if $a_0 = 0, a_1,\dots,a_{n-1} \in A$ are such that $X \subseteq \bigcup_{i=0}^{n-1}(a_i+Y)$, the family $\mathfrak{Y} = \{a_0+Y, a_1+Y,\dots,a_{n-1}+Y\}$ is strongly pairwise disjoint and X meets a_i+Y for every $i \in \{0,1,\dots,n\}$, then $\mu(X,Y)=n$.*

Proof. (a), (b) and (c) are clear.

(d) Follows from the fact that if $a_0 = 0, a_1,\dots,a_{n-1} \in A$ and $a'_0 = 0, a'_1,\dots,a'_{m-1} \in A$ are such that

$$X \subseteq (a_0+Y)\cup\ldots\cup(a_{n-1}+Y) \text{ and } X' \subseteq (a'_0+Y')\cup\ldots\cup(a'_{m-1}+Y'),$$

then

$$X+X' \subseteq (a_0+Y+Y')\cup\ldots\cup(a_{n-1}+Y+Y')\cup(a'_0+Y+Y')\cup\ldots\cup(a'_{m-1}+Y+Y').$$

(e) Similarly, if $a_0 = 0, a_1,\dots,a_{n-1} \in A$ and $b_0 = 0, b_1,\dots,b_{m-1} \in A$ are such that

$$X \subseteq (a_0+Z)\cup\ldots\cup(a_{n-1}+Z) \text{ and } Z \subseteq (b_0+Y)\cup\ldots\cup(b_{m-1}+Y),$$

then

$$X \subseteq \bigcup_{i\in\{0,\dots,n-1\},j\in\{0,\dots,m-1\}} (a_i+b_j+Y).$$

(f) Let $a_0 = 0, a_1,\dots,a_{n-1} \in A$ such that $X \subseteq (a_0+Y)\cup(a_1+Y)\cup\ldots\cup(a_{n-1}+Y)$. Then

$$\begin{aligned}\varphi(X) &\subseteq \varphi((a_0+Y)\cup(a_1+Y)\cup\ldots\cup(a_{n-1}+Y))\\ &= (\varphi(a_0)+\varphi(Y))\cup(\varphi(a_1)+\varphi(Y))\cup\ldots\cup(\varphi(a_{n-1})+\varphi(Y)).\end{aligned}$$

(g) Obviously, $\mu(X,Y)\leq n$. Assume that $m := \mu(X,Y) < n$. Then there exist $b_0 = 0, b_1,\dots,b_{m-1} \in A$ such that

$$X \subseteq \bigcup_{j=0}^{m-1}(b_j+Y). \tag{2.2}$$

For every $i \in \{0,1,\dots n-1\}$ there exists $x_i \in X \cap (a_i+Y)$, by hypothesis. From (2.2) and our assumption, $m<n$ implies that there exist $0 \leq i < j < n$ and $k \in \{0,\dots,m-1\}$ such that $x_i,x_j \in b_k+Y$. Hence, $x_i - x_j \in Y-Y$. Since moreover $x_i-x_j \in a_i-a_j+Y-Y$, it follows that $a_i-a_j \in Y-Y+Y-Y$. For $C=\{a_0,a_1,\dots,a_{n-1}\}$, our hypothesis on $\mathfrak{Y}$ gives $a_i-a_j \in (C-C)\cap(Y-Y+Y-Y)=\{0\}$, and so $a_i-a_j=0$, a contradiction. ☐

Proposition 2.6. *Let A be an abelian group, B a subgroup of A and $\pi : A \to A/B$ the canonical projection. If $X \in \mathcal{P}^0_{fin}(A)$, then:*

(a) *there exists $Y \in \mathcal{P}^0_{fin}(B)$ such that $\ell(\pi(X)) = \ell(X,B) = \ell(X,Y)$;*

(b) *for $Y \in \mathcal{P}^0_{fin}(B)$ we have that $\ell(X+Y) \geq \ell(\pi(X)) + \ell(Y)$.*

Proof. (a) The equality

$$\ell(\pi(X)) = \ell(X,B) \tag{2.3}$$

is obvious.

Let $Y=(X-X)\cap B \in \mathcal{P}^0_{fin}(B)$ and let $Z=\{a_1,\dots,a_{n-1}\}\subseteq X\setminus B$ be such that $\pi(X)\subseteq\pi(Z)\cup\{0\}$ and $(Z-Z)\cap B=\{0\}$, in other words $Z\cup\{0\}$ is a set of representatives of $\pi(X)$. Then

$$|Z\cup\{0\}| = |\pi(Z\cup\{0\})| = |\pi(X)|.$$

In particular, putting $a_0=0$, one obtains

$$X \subseteq (a_0+Y)\cup(a_1+Y)\cup\ldots\cup(a_{n-1}+Y). \tag{2.4}$$

In fact, assume that $x\in X$. If $x \in B$, so $x \in X\cap B \subseteq Y$. Otherwise, if $x \notin B$, there exists $i \in \{1,\dots,n-1\}$ such that $x \in a_i+B$; since $x-a_i \in (X-X)\cap B = Y$, one concludes that $x \in a_i+Y$.

On the other hand, by the choice of Z, the family $\{a_i+Y : i\in\{0,\dots,n-1\}\}$ appearing in (2.4) is strongly pairwise disjoint. Indeed, $Y-Y+Y-Y \subseteq B$, as $Y\subseteq B$. Therefore,

$$(Z-Z)\cap(Y-Y+Y-Y)\subseteq(Z-Z)\cap B=\{0\}.$$

By Lemma 2.5(g), this shows also that $\mu(X,Y)=n=|\pi(X)|$. Therefore, $\ell(\pi(X))=\ell(X,Y)$.

(b) Let $Z\in\mathcal{P}^0_{fin}(A)$ such that $Z\subseteq X$, $\pi(Z)=\pi(X)$ and $(Z-Z)\cap B=\{0\}$; then $|Z|=|\pi(Z)|=|\pi(X)|$. The bijectivity of the map $Z\times Y\to Z+Y$, $(z,y)\mapsto z+y$ entails

$$|Z+Y| = |Z|\,|Y|. \tag{2.5}$$

The inclusion $Z+Y\subseteq X+Y$ and (2.5) yield the required inequality

$$\ell(X+Y)\geq\ell(Z+Y)=\ell(Z)+\ell(Y)=\ell(\pi(X))+\ell(Y).$$ ☐

Lemma 2.7. *Let S be a semigroup, A an abelian group, and $S \overset{\alpha}{\curvearrowright} A$ a left action. If $F\in\mathcal{P}_{fin}(S)$ and $X,Y\in\mathcal{P}^0_{fin}(A)$, then $\ell(T_F(\alpha,X),T_F(\alpha,Y))\leq|F|\ell(X,Y)$.*

Proof. By definition, Lemma 2.5(d) and Lemma 2.5(f), we have that

$$\begin{aligned}\ell(T_F(\alpha,X),T_F(\alpha,Y)) &= \ell\left(\sum_{s\in F}\alpha(s)(X),\sum_{s\in F}\alpha(s)(Y)\right)\leq\\ &\leq\sum_{s\in F}\ell(\alpha(s)(X),\alpha(s)(Y))\leq\sum_{s\in F}\ell(X,Y)=|F|\,\ell(X,Y).\end{aligned}$$ ☐

2.3 Proof of the Addition Theorem

We recall the following basic property of the algebraic entropy that is used in the sequel without referring to it each time.

Lemma 2.8 (See [9]). *Let S be a cancellative right amenable semigroup, A an abelian group, and $S \overset{\alpha}{\curvearrowright} A$ a left action. If $X, Y \in \mathcal{P}_{fin}(A)$ and $X \subseteq Y$, then $H_{alg}(\alpha, X) \leq H_{alg}(\alpha, Y)$. Consequently, if $\mathcal{F} \subseteq \mathcal{P}_{fin}(A)$ is cofinal with respect to $\subseteq$, then $h_{alg}(\alpha) = \sup\{H_{alg}(\alpha, X) : X \in \mathcal{F}\}$.*

By the above lemma, it is clear that

$$h_{alg}(\alpha) = \sup\{H_{alg}(\alpha, X) : X \in \mathcal{P}^0_{fin}(A)\}.$$

The following is the key point for the proof of the Addition Theorem.

Proposition 2.9. *Let S be a countable locally monotileable cancellative right amenable monoid, and let $(F_n)_{n\in\mathbb{N}}$ be a locally monotileable right Følner sequence of S. Let A be an abelian group and $S \overset{\alpha}{\curvearrowright} A$ a left action. Let $X, Y \in \mathcal{P}^0_{fin}(A)$. Then the following functions are decreasing:*

$$\mathbb{N} \ni n \mapsto \frac{\ell(T_{F_n}(\alpha, X))}{|F_n|} \quad \textit{and} \quad \mathbb{N} \ni n \mapsto \frac{\ell(T_{F_n}(\alpha, X), T_{F_n}(\alpha, Y))}{|F_n|}.$$

Proof. The first assertion follows from the second one by taking $Y = \{0\}$. To prove the second assertion let $n \in \mathbb{N}$. Then there exists $K = K_{n+1}$ such that $F_{n+1} = \bigsqcup_{s\in K} sF_n$; in particular, $|F_{n+1}| = |K|\,|F_n|$. Then by Lemma 2.2,

$$T_{F_{n+1}}(\alpha, X) = T_{KF_n}(\alpha, X) = T_K(\alpha, T_{F_n}(\alpha, X)).$$

The same holds for Y. Therefore, by Lemma 2.7,

$$\ell(T_{F_{n+1}}(\alpha, X), T_{F_{n+1}}(\alpha, Y)) = \ell(T_K(\alpha, T_{F_n}(\alpha, X)), T_K(\alpha, T_{F_n}(\alpha, Y))) \leq |K|\,\ell(T_{F_n}(\alpha, X), T_{F_n}(\alpha, Y)),$$

and so

$$\frac{\ell(T_{F_{n+1}}(\alpha, X), T_{F_{n+1}}(\alpha, Y))}{|F_{n+1}|} \leq \frac{|K|\,\ell(T_{F_n}(\alpha, X), T_{F_n}(\alpha, Y))}{|K|\,|F_n|} = \frac{\ell(T_{F_n}(\alpha, X), T_{F_n}(\alpha, Y))}{|F_n|};$$

this proves the second assertion. □

We are now in position to prove the Addition Theorem.

Proof of Theorem 1.2. We have to prove that if S is a countable cancellative right amenable monoid with a locally monotileable right Følner sequence $(F_n)_{n\in\mathbb{N}}$, A is an abelian group, $S \overset{\alpha}{\curvearrowright} A$ a left action, and B is an α-invariant subgroup of A, then $h_{alg}(\alpha) = h_{alg}(\alpha_{A/B}) + h_{alg}(\alpha_B)$.

First we prove the inequality

$$h_{alg}(\alpha) \geq h_{alg}(\alpha_{A/B}) + h_{alg}(\alpha_B). \tag{2.6}$$

Let $\pi : A \to A/B$ be the canonical projection, $X \in \mathcal{P}^0_{fin}(A)$ and $Y \in \mathcal{P}^0_{fin}(B)$. Pick $Z \in \mathcal{P}^0_{fin}(A)$, as in the proof of Proposition 2.6(b), i.e., with $\pi(Z) = \pi(X)$ and $(Z - Z) \cap B = \{0\}$, so that $|Z| = |\pi(Z)| = |\pi(X)|$. For every $n \in \mathbb{N}$, by Lemma 2.3, Proposition 2.6(b) and Lemma 2.4, one has the inequalities

$$\begin{aligned}\ell(T_{F_n}(\alpha, Z+Y)) &= \ell(T_{F_n}(\alpha, Z) + T_{F_n}(\alpha, Y)) \geq \\ &\geq \ell(\pi(T_{F_n}(\alpha, Z))) + \ell(T_{F_n}(\alpha, Y)) = \ell(T_{F_n}(\alpha_{A/B}, \pi(Z))) + \ell(T_{F_n}(\alpha, Y)).\end{aligned}$$

Since $\pi(X) = \pi(Z)$, after division by $|F_n|$ and letting $n \to \infty$ one obtains the inequalities

$$h_{alg}(\alpha) \geq H_{alg}(\alpha, Z+Y) \geq H_{alg}(\alpha_{A/B}, \pi(X)) + H_{alg}(\alpha_B, Y).$$

So, taking the supremum on the right-hand side of the latter inequality over all $X \in \mathcal{P}^0_{fin}(A)$ and $Y \in \mathcal{P}^0_{fin}(B)$, gives the inequality in (2.6).

It remains to prove the inequality

$$h_{alg}(\alpha) \leq h_{alg}(\alpha_{A/B}) + h_{alg}(\alpha_B). \tag{2.7}$$

Fix $\varepsilon > 0$ and $X \in \mathcal{P}^0_{fin}(A)$. By Lemma 2.9, there exists $M \in \mathbb{N}$ such that, for every $n \geq M$,

$$\frac{\ell(T_{F_n}(\alpha_{A/B}, \pi(X)))}{|F_n|} \leq H_{alg}(\alpha_{A/B}, \pi(X)) + \varepsilon \leq h_{alg}(\alpha_{A/B}) + \varepsilon. \tag{2.8}$$

By Proposition 2.6(a), there exists $Y \in \mathcal{P}^0_{fin}(B)$ such that $\ell(\pi(T_{F_M}(\alpha, X))) = \ell(T_{F_M}(\alpha, X), Y)$. Hence, we can write $\ell(T_{F_M}(\alpha, X), Y) = \ell(T_{F_M}(\alpha_{A/B}, \pi(X)))$, in view of Lemma 2.4. So, Lemma 2.5(b) and the inclusion $Y \subseteq T_{F_M}(\alpha, Y)$ allow us to conclude that

$$\ell(T_{F_M}(\alpha, X), T_{F_M}(\alpha, Y)) \leq \ell(T_{F_M}(\alpha, X), Y) = \ell(T_{F_M}(\alpha_{A/B}, \pi(X))).$$

Combining this inequality with (2.8) and Lemma 2.9, we obtain

$$\frac{\ell(T_{F_n}(\alpha, X), T_{F_n}(\alpha, Y))}{|F_n|} \leq H_{alg}(\alpha_{A/B}, \pi(X)) + \varepsilon \leq h_{alg}(\alpha_{A/B}) + \varepsilon \tag{2.9}$$

for all $n \geq M$. In view of Lemma 2.9 again, there exists $M^* \geq M$ such that, for every $n \geq M^*$,

$$\frac{\ell(T_{F_n}(\alpha, Y))}{|F_n|} \leq H_{alg}(\alpha, Y) + \varepsilon \leq h_{alg}(\alpha_B) + \varepsilon. \tag{2.10}$$

By Lemma 2.9,

$$H_{alg}(\alpha, X) = \inf_{n\in\mathbb{N}} \frac{\ell(T_{F_n}(\alpha, X))}{|F_n|} \leq \frac{\ell(T_{F_{M^*}}(\alpha, X))}{|F_{M^*}|}, \tag{2.11}$$

and by Lemma 2.5(c),

$$\ell(T_{F_{M^*}}(\alpha, X)) \leq \ell(T_{F_{M^*}}(\alpha, X), T_{F_{M^*}}(\alpha, Y)) + \ell(T_{F_{M^*}}(\alpha, Y)). \tag{2.12}$$

Therefore, by (2.11), (2.12), (2.9) and (2.10),

$$H_{alg}(\alpha, X) \leq \frac{\ell(T_{F_{M^*}}(\alpha, X))}{|F_{M^*}|} \leq \frac{\ell(T_{F_{M^*}}(\alpha, X), T_{F_{M^*}}(\alpha, Y))}{|F_{M^*}|} + \frac{\ell(T_{F_{M^*}}(\alpha, Y))}{|F_{M^*}|} \leq h_{alg}(\alpha_{A/B}) + h_{alg}(\alpha_B) + 2\varepsilon.$$

Since ε was chosen arbitrarily, this proves (2.7). □

2.4 Application to the topological entropy via the Bridge Theorem

We recall that, inspired by the work of Kolmogorov and Sinai in ergodic theory, Adler, Konheim and McAndrew [1] introduced the topological entropy for continuous selfmaps of compact topological spaces, while a different notion of topological entropy for uniformly continuous selfmaps of metric spaces was given by Bowen [3] and Dinaburg [18] independently. In the realm of topological groups, Yuzvinski [47] proved the so-called Addition Theorem (usually called Yuzvinski's addition formula) for the topological entropy of continuous endomorphisms of metrizable compact groups, that was recently extended to all compact groups in [17].

Lind, Schmidt and Ward [31] generalized for $\mathbb{Z}^d$-actions on metrizable compact groups both the definition of topological entropy by Bowen, as well as that by Adler, Konheim and McAndrew, showing that they coincide. They proved the Addition Theorem for $\mathbb{Z}^d$-actions on metrizable compact groups. Moreover, Ollagnier [33] defined the topological entropy for amenable group actions on compact spaces using open covers as in [1]. Recently, Li [29] established the Addition Theorem for actions of countable amenable groups G on metrizable compact groups K; see also Chapter 13 in the recent monograph of Kerr and Li [28]. Even if a proof seems to be not available in the literature, this result can be apparently extended to the general case, that is, without the assumption on G to be countable and on K to be metrizable.

Recently, Ceccherini-Silberstein, Coornaert and Krieger [5] extended Ornstein-Weiss Lemma from [35] to cancellative amenable semigroups, and this allowed them to define the topological entropy for amenable semigroup actions on compact spaces as follows. Let C be a compact topological space, let S be a cancellative right amenable semigroup, and consider a right action $C \overset{\gamma}{\curvearrowleft} S$ by continuous selfmaps. Let $\mathcal{U} = \{U_j\}_{j\in J}$ and $\mathcal{V} = \{V_k\}_{k\in K}$ be two open covers of C. One says that $\mathcal{V}$ refines $\mathcal{U}$, denoted by $\mathcal{U} \prec \mathcal{V}$, if for every $k \in K$ there exists $j \in J$ such that $V_k \subseteq U_j$. Moreover, let

$$\mathcal{U} \vee \mathcal{V} = \{U_j \cap V_k : (j, k) \in J \times K\}.$$

Let also $N(\mathcal{U}) = \min\{n \in \mathbb{N}_+ : \mathcal{U} \text{ admits a subcover of size } n\}$.

For a continuous selfmap $f : C \to C$ and an open cover $\mathcal{U}$ of C, let $f^{-1}(\mathcal{U}) = \{f^{-1}(U_j)\}_{j\in J}$, and for $F \in \mathcal{P}_{fin}(S)$, let

$$\mathcal{U}_{\gamma,F} = \bigvee_{s\in F} \gamma(s)^{-1}(\mathcal{U}).$$

The *topological entropy of* γ *with respect to* $\mathcal{U}$ is given by the limit

$$H_{top}(\gamma, \mathcal{U}) = \lim_{i\in I} \frac{\log N(\mathcal{U}_{\gamma,F_i})}{|F_i|},$$

where $(F_i)_{i\in I}$ is a right Følner net of S; this limit exists and does not depend on the choice of $(F_i)_{i\in I}$ by [5, Theorem 1.1]. The *topological entropy of* γ is

$$h_{top}(\gamma) = \sup\{H_{top}(\gamma, \mathcal{U}) : \mathcal{U} \text{ open cover of } C\}.$$

In order to discuss the relation of the topological entropy with the algebraic entropy, for a locally compact abelian group A, denote by $\widehat{A}$ its Pontryagin dual group, which is a locally compact group as well. For a continuous homomorphism $\phi : A \to B$, where B is another locally compact abelian group, let $\widehat{\phi} : \widehat{B} \to \widehat{A}$ be the dual of ϕ, defined by $\widehat{\phi}(\chi) = \chi \circ \phi$ for every $\chi \in \widehat{B}$. By Pontryagin-van Kampen duality, A and $\widehat{\widehat{A}}$ are canonically topologically isomorphic. Moreover, in case A is discrete (respectively, compact), $\widehat{A}$ is compact (respectively, discrete).

Now let S be a cancellative right amenable semigroup, A a locally compact abelian group and $A \overset{\gamma}{\curvearrowleft} S$ a right action by continuous endomorphisms. Then γ induces the left action $S \overset{\widehat{\gamma}}{\curvearrowright} \widehat{A}$ by continuous endomorphisms (called the *dual action* of γ), defined by

$$\widehat{\gamma}(s) = \widehat{\gamma(s)} : \widehat{A} \to \widehat{A} \quad \text{for every } s \in S.$$

Analogously, a left action $S \overset{\alpha}{\curvearrowright} A$ by continuous endomorphisms induces the right action $\widehat{A} \overset{\widehat{\alpha}}{\curvearrowleft} S$, defined by

$$\widehat{\alpha}(s) = \widehat{\alpha(s)} : \widehat{A} \to \widehat{A}, \quad \text{for every } s \in S.$$

By Pontryagin-van Kampen duality $\widehat{\widehat{\gamma}} = \gamma$ and $\widehat{\widehat{\alpha}} = \alpha$ up to conjugation (due to the above mentioned canonical isomorphisms).

Weiss [46] and Peters [36] discovered a remarkable connection, usually named Bridge Theorem, between the topological entropy and the algebraic entropy, which was proved in general in [13]. More precisely, the topological entropy of a continuous endomorphism ϕ of a compact abelian group K coincides with the algebraic entropy of its dual endomorphism $\widehat{\phi}$ of the Pontryagin dual group $\widehat{K}$ of K, which is a discrete abelian group. This connection was extended to totally disconnected locally compact abelian groups in [14].

The Bridge Theorem from [36] was recently extended by Kerr and Li [28] to actions of countable amenable groups on metrizable compact abelian groups. Then Virili [44] proved it for actions of amenable groups on locally compact abelian groups. Moreover, the one from [14] was extended in [23] to semigroup actions on totally disconnected locally compact abelian groups. In [9], generalizing the main result of [46], we proved a Bridge Theorem for left actions of cancellative left amenable monoids on totally disconnected compact abelian groups (their Pontryagin dual groups are precisely the torsion abelian groups).

The following theorem combines the Bridge Theorems from [10] and [44].

Theorem 2.10. *Let S be a cancellative right amenable semigroup, A an abelian group, and $S \overset{\alpha}{\curvearrowright} A$ a left action. Then $h_{alg}(\alpha) = h_{top}(\widehat{\alpha})$ in the following two cases:*

(a) A is torsion;

(b) S is a group.

Since the Pontryagin dual group of a compact abelian group K is torsion precisely when K is totally disconnected, we immediately obtain the following counterpart of the above theorem.

Corollary 2.11. *Let S be a cancellative right amenable semigroup, K a compact abelian group, and $K \overset{\gamma}{\curvearrowleft} S$ a right action. Then $h_{top}(\gamma) = h_{alg}(\widehat{\gamma})$ in the following two cases:*

(a) K is totally disconnected;

(b) S is a group.

From Corollary 2.11 and Theorem 1.2, we obtain the following Addition Theorem for the topological entropy.

Corollary 2.12. *Let S be a locally monotileable cancellative right amenable monoid, K a compact abelian group, and $K \overset{\gamma}{\curvearrowleft} S$ a right action. Let H be a γ-invariant closed subgroup of K, and denote by $\gamma_{K/H}$ and γ_H the induced actions of S on K/H and on H, respectively. If either S is a group or K is totally disconnected, then*

$$h_{top}(\gamma) = h_{top}(\gamma_{K/H}) + h_{top}(\gamma_H).$$

Proof. Let $A = \widehat{K}$, $\alpha = \widehat{\gamma}$ and let B be the annihilator of H in A. By Corollary 2.11 and Theorem 1.2, we have

$$h_{top}(\gamma) = h_{alg}(\alpha) = h_{alg}(\alpha_B) + h_{alg}(\alpha_{A/B}). \tag{2.13}$$

By Pontryagin-van Kampen duality $B \cong \widehat{K/H}$ and $A/B \cong \widehat{H}$, and moreover these natural isomorphisms witness that α_B is conjugated to $\widehat{\gamma_{K/H}}$ and $\alpha_{A/B}$ is conjugated to $\widehat{\gamma_H}$. The algebraic entropy is invariant under conjugation (see [10]), so, by applying also Corollary 2.11, we get $h_{alg}(\alpha_B) = h_{top}(\gamma_{K/H})$ and $h_{alg}(\alpha_{A/B}) = h_{top}(\gamma_H)$. We conclude by applying the last two equalities in (2.13). $\square$

Remark 2.13. (a) Our notion of local monotileability is inspired by that of monotileability from [45]. In both cases these are "left" conditions. Indeed, one could define that, for subsets T, V of a semigroup S, T is a *right monotile* of V if there exists a subset C of S such that $\{Tc : c \in C\}$ is a partition of V.

Then, in a monoid S; a sequence $(F_n)_{n\in\mathbb{N}}$ in $\mathcal{P}_{fin}(S)$ is *right locally monotileable* if $F_0 = \{1\}$ and F_n is a right monotile of F_{n+1} for every $n \in \mathbb{N}$. So, a left amenable monoid is *right locally monotileable* if it admits a right locally monotileable left Følner sequence.

(b) Assume that one would like to consider the topological entropy of left actions of cancellative left amenable semigroups on compact spaces as in [5], or the algebraic entropy of right actions of cancellative left amenable semigroups on abelian groups. Then one should consider left amenable semigroups that are right locally monotileable, to obtain the counterparts of the above results.

3 Locally monotileable monoids

3.1 Starting examples

We propose some basic examples.

Example 3.1. Every finite monoid S is congruent monotileable, and so locally monotileable. This is witnessed by the sequence $(F_n)_{n\in\mathbb{N}}$ with $F_0 = \{1\}$ and $F_n = S$ for all $n \in \mathbb{N}_+$, which is obviously a congruent and exhaustive right Følner sequence of S.

Remark 3.2. Let $(F_n)_{n\in\mathbb{N}}$ be a locally monotileable sequence of a cancellative monoid S and consider a tiling sequence $(K_n)_{n\in\mathbb{N}}$ associated to $(F_n)_{n\in\mathbb{N}}$. It is easy to prove by induction that $|F_n| = \prod_{i=1}^n |K_i|$ for every $n \in \mathbb{N}$ and that $|K_n \dots K_l| = \prod_{i=l}^n |K_i|$ for all positive integers $l \leq n$.

A strictly increasing sequence of natural numbers $(a_n)_{n\in\mathbb{N}}$ is an *a-sequence* if $a_0 = 1$ and $a_n \mid a_{n+1}$ for every $n \in \mathbb{N}$. It follows from Remark 3.2 that if $(F_n)_{n\in\mathbb{N}}$ is locally monotileable sequence of a monoid S, the sequence $(|F_n|)_{n\in\mathbb{N}}$ is an a-sequence.

Example 3.3. (a) The monoid $(\mathbb{N}, +)$ is congruent monotileable (so locally monotileable) and MT. Indeed, consider an a-sequence $(a_n)_{n\in\mathbb{N}}$ (for example the sequence $(n!)$) and define $F_n = [0, a_n - 1]$ for every $n \in \mathbb{N}$. Then each finite subset of $\mathbb{N}$ is contained in some F_n, and the sequence $(F_n)_{n\in\mathbb{N}}$ is a congruent and exhaustive Følner sequence of $\mathbb{N}$ consisting of monotiles.

(b) Clearly, by (a) the sequence $([0, n! - 1])_{n\in\mathbb{N}}$ is a congruent and exhaustive Følner sequence of $\mathbb{N}$, while $([0, n! - 1])_{n\in\mathbb{N}}$ is a congruent Følner sequence of $\mathbb{Z}$ that is not exhaustive.

(c) The sequence $([0, n])_{n\in\mathbb{N}}$ is a Følner sequence of $\mathbb{N}$ that is not locally monotileable, while $[0, n]$ is a monotile of $\mathbb{N}$ for every $n \in \mathbb{N}$. On the other hand, consider the sequence $(F_n)_{n\in\mathbb{N}}$, where $F_0 = \{0\}$ and $F_n = \{0\}\cup[2, 3(2^{n-1})-1]\cup\{3(2^{n-1})+1\}$ for every $n \in \mathbb{N}_+$. For every $n \in \mathbb{N}_+$, we have $F_{n+1} = F_n \sqcup (3(2^{n-1}) + F_n)$. Then $(F_n)_{n\in\mathbb{N}}$ is a locally monotileable Følner sequence of $\mathbb{N}$ but F_n is not a monotile of $\mathbb{N}$ for any $n \in \mathbb{N}_+$.

The following example shows a locally monotileable Følner sequence $(F_n)_{n\in\mathbb{N}}$ of $\mathbb{Z}$ such that, for every tiling sequence $(K_n)_{n\in\mathbb{N}}$ associated to $(F_n)_{n\in\mathbb{N}}$, $0 \notin K_n$ for every $n \geq 2$; so $(F_n)_{n\in\mathbb{N}}$ cannot be congruent, even if $\mathbb{Z}$ is congruent monotileable. Note that in this case $F_n \subseteq F_{n+1}$ for every $n \in \mathbb{N}$.

Example 3.4. In $\mathbb{Z}$ let $F_0 = \{0\}$ and, for every $n \in \mathbb{N}_+$, let $F_n = [-2^{n-1} + 1, 2^{n-1}]$. Then, $K_1 = F_1 = \{0, 1\}$, but for every $n \in \mathbb{N}$ with $n \geq 2$, $K_n = \{-2^{n-2}, 2^{n-2}\}$. In particular, $0 \notin K_n$ for every $n \geq 2$.

We end this section showing that all countable locally finite groups are locally monotileable.

Proposition 3.5. *If G is a countable locally finite group, then G is congruent monotileable, and so locally monotileable.*

Proof. The group G is countable, so we enumerate its elements as $G = \{g_n : n \in \mathbb{N}\}$ with $g_0 = 1$. For every $n \in \mathbb{N}$, let $F_n = \langle g_1, \ldots, g_n\rangle$. Since G is locally finite, all subgroups F_n are finite. Clearly, $F_n \subseteq F_{n+1}$ for every $n \in \mathbb{N}$.

Fix $g_{\bar{n}} \in G$. If $n \geq \bar{n}$, then $g_{\bar{n}} \in F_n$ and so $F_n g_{\bar{n}} = F_n$, hence $|F_n g_{\bar{n}} \setminus F_n| / |F_n| = 0$. Therefore,

$$\lim_{n\to\infty} \frac{|F_n g_{\bar{n}} \setminus F_n|}{|F_n|} = \lim_{k\to\infty} \frac{|F_{\bar{n}+k} g_{\bar{n}} \setminus F_{\bar{n}+k}|}{|F_{\bar{n}+k}|} = 0. \tag{3.1}$$

The left cosets of F_n in F_{n+1} are disjoint left translates of F_n and they cover F_{n+1}. By this and by (3.1) we conclude that $(F_n)_{n\in\mathbb{N}}$ is a congruent (hence, locally monotileable) exhaustive right Følner sequence of G. □

3.2 Basic properties

Here we start with the following property of Følner sequences with respect to translates of finite sets. It applies in both Proposition 3.14 and Proposition 3.15.

Lemma 3.6. *Let S be a cancellative monoid and let $(F_n)_{n\in\mathbb{N}}$ be a right Følner sequence of S. If $X \in \mathcal{P}_{fin}(S)$, then there exists $\bar{n} \in \mathbb{N}$ such that, for every $n > \bar{n}$, F_n contains a left translate of X.*

Proof. Let $X \in \mathcal{P}_{fin}(S)$. Since $(F_n)_{n\in\mathbb{N}}$ is a right Følner sequence of S,

$$\lim_{n\to\infty} \frac{|F_n X \setminus F_n|}{|F_n|} = 0. \tag{3.2}$$

We assume by contradiction that there is an increasing sequence of natural numbers $(k_n)_{n\in\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, each F_{k_n} contains no left translates of X. Fix a $n \in \mathbb{N}$. Then, for all $f \in F_{k_n}$, the set $fX \setminus F_{k_n}$ is non-empty. Therefore, we can define a map $\phi_n : F_{k_n} \to X$, $f \mapsto \phi_n(f)$ with $f\phi_n(f) \in fX \setminus F_{k_n}$. By the pigeonhole principle there exists $x \in X$ such that $|\phi_n^{-1}(x)| \geq |F_{k_n}|/|X|$. Clearly, if f_1 and f_2 are two distinct elements of $\phi_n^{-1}(x)$, then $f_1x \neq f_2x$ because S is cancellative. This implies that

$$|F_{k_n} X \setminus F_{k_n}| \geq |\phi_n^{-1}(x)| \geq \frac{|F_{k_n}|}{|X|}.$$

Dividing both sides by $|F_{k_n}|$, we obtain $|F_{k_n} X \setminus F_{k_n}|/|F_{k_n}| \geq 1/|X|$. Since this holds for all $n \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{|F_{k_n} X \setminus F_{k_n}|}{|F_{k_n}|} \geq \frac{1}{|X|},$$

that is in contradiction with (3.2). □

We proceed with the following observation on monotiles.

Remark 3.7. Let S be a monoid and let T be a monotile of S.

(a) If S is a group, then gT is still a monotile of S for all $g \in S$.

(b) Item (a) may fail in case S is not a group. For example, if $S = \mathbb{N}$, then a monotile T of S necessarily contains 0, so $g + T$ is not a monotile of S if $g \neq 0$.

We omit the easy proof of the next two lemmas (the first one can be used for a proof of the second one, as well as further on).

Lemma 3.8. *Let S be a cancellative monoid. Consider $X \subseteq Y \subseteq Z$ subsets of S. If there exist U and V such that $Y = \bigsqcup_{u\in U} uX$ and $Z = \bigsqcup_{v\in V} vY$, then $Z = \bigsqcup_{v\in V,\ u\in U} vuX = \bigsqcup_{t\in T} tX$, where $T = VU$.*

This lemma simply asserts that if X is a monotile of Y and Y is a monotile of Z then X is a monotile of Z.

Lemma 3.9. *Let S be a cancellative monoid and $(F_n)_{n\in\mathbb{N}}$ a locally monotileable sequence of S. Then every subsequence $(F_{k_n})_{n\in\mathbb{N}}$ such that $k_0 = 0$ is still a locally monotileable sequence of S.*

Next we see that the class of locally monotileable cancellative monoids if stable under taking countable direct sums.

Proposition 3.10. *Let $\{S_i : i \in \mathbb{N}_+\}$ be a family of countable locally monotileable cancellative monoids, then $S = \bigoplus_{i\in\mathbb{N}_+} S_i$ is locally monotileable.*

Proof. Since S_i is locally monotileable for all $i \in \mathbb{N}_+$, we can fix a locally monotileable right Følner sequence $(F_{i,n})_{n\in\mathbb{N}}$ of S_i. We define a new sequence by letting, for every $n \in \mathbb{N}$, $F_n = \bigoplus_{i=1}^n F_{i,n} \oplus \bigoplus_{i\geq n+1}\{1\}$.

First we prove that $(F_n)_{n\in\mathbb{N}}$ is a right Følner sequence of S. Fix $\varepsilon > 0$ and $h = (h_n)_{n\in\mathbb{N}} \in \bigoplus_{n\in\mathbb{N}} S_n$, and let $k = \max\{n \in \mathbb{N} : h_n \neq 1\} \in \mathbb{N}$. For every $i \in \{1, \ldots, k\}$ there exists $m_i \in \mathbb{N}$ such that for all $n \geq m_i$,

$$\frac{|F_{i,n}h_i \setminus F_{i,n}|}{|F_{i,n}|} < \frac{\varepsilon}{k}. \tag{3.3}$$

If $n \geq k$, then $F_nh = \bigoplus_{i=1}^n F_{i,n}h_i \oplus \bigoplus_{i\geq n+1}\{1\}$. Moreover, since

$$F_nh \setminus F_n \subseteq \bigcup_{i=1}^n \left((F_{i,n}h_i \setminus F_{i,n}) \oplus \bigoplus_{j=1,j\neq i}^n F_{j,n}h_j \right) \oplus \bigoplus_{i\geq n+1} \{1\},$$

and since $|F_{i,n}h_i \setminus F_{i,n}| = 0$ for every $k < i \leq n$, we obtain that

$$|F_nh \setminus F_n| \leq \sum_{i=1}^n \left(|F_{i,n}h_i \setminus F_{i,n}| \prod_{j=1,j\neq i}^n |F_{j,n}h_j| \right) = \sum_{i=1}^k \left(|F_{i,n}h_i \setminus F_{i,n}| \prod_{j=1,j\neq i}^n |F_{j,n}h_j| \right). \tag{3.4}$$

By (3.3) and (3.4), observing that $|F_{j,n}h_j| = |F_{j,n}|$ for every $j \in \mathbb{N}_+$ (as the monoid S_j is cancellative) and that $|F_n| = \prod_{i=1}^n |F_{i,n}|$ for every $n \in \mathbb{N}$, we conclude that for every $n \geq m := \max\{m_i : 1 \leq i \leq k\}$,

$$\begin{aligned}\frac{|F_nh \setminus F_n|}{|F_n|} &\leq \frac{\sum_{i=1}^k \left(|F_{i,n}h_i \setminus F_{i,n}| \prod_{j=1,j\neq i}^n |F_{j,n}h_j|\right)}{\prod_{i=1}^n |F_{i,n}|} \leq \\ &\leq \frac{\sum_{i=1}^k \left(|F_{i,n}h_i \setminus F_{i,n}| \prod_{j=1,j\neq i}^n |F_{j,n}|\right)}{\prod_{i=1}^n |F_{i,n}|} \leq \sum_{i=1}^k \frac{|(F_{i,n}h_i) \setminus F_{i,n}|}{|F_{i,n}|} < \varepsilon.\end{aligned}$$

Then, when $\varepsilon \to 0$, we have $\lim_{n\to\infty} |F_nh \setminus F_n|/|F_n| = 0$, and hence, $(F_n)_{n\in\mathbb{N}}$ is a right Følner sequence of S.

By hypothesis $(F_{i,n})_{n\in\mathbb{N}}$ is locally monotileable for every $i \in \mathbb{N}_+$. Then, for all $i, n \in \mathbb{N}_+$ there exists a finite set $\widetilde{S}_{i,n} \subseteq S_i$ such that $F_{i,n} = \bigsqcup_{g_i\in\widetilde{S}_{i,n}} (g_iF_{i,n-1})$. Define $\tilde{S}_n = \{g = (g_i)_{i\in\mathbb{N}} \in S : g_i \in \widetilde{S}_{i,n} \text{ if } i \leq n,\ g_i = 1 \text{ if } i > n\}$. Then $\widetilde{S}_n$ is finite and $F_n = \bigsqcup_{g\in\widetilde{S}_n} (gF_{n-1})$. Therefore, $(F_n)_{n\in\mathbb{N}}$ is a locally monotileable right Følner sequence of S. □

By Proposition 3.10, Example 3.1 and Example 3.3, we have the following consequence.

Corollary 3.11. *The family $\mathfrak{M}$ is stable under countable direct sums. In particular, $\mathfrak{M}$ contains all finitely generated abelian groups.*

The first assertion of this corollary can be obtained also as a consequence of a more general result proved below (see Theorem 4.17).

3.3 Relations among notions of monotileability

We start this part by verifying that, for a monoid S, an exhaustive and congruent sequence in $\mathcal{P}_{fin}(S)$ consists necessarily of monotiles of S.

Lemma 3.12. *If S is a cancellative monoid and $(F_n)_{n\in\mathbb{N}}$ is an exhaustive and congruent sequence in $\mathcal{P}_{fin}(S)$, then each F_n is a monotile of S. In particular, every countable congruent monotileable monoid is* MTA.

Proof. Fix $n_0 \in \mathbb{N}$. We prove that F_{n_0} is a monotile of S.

For every $m > n_0$ let $P_m = K_m \ldots K_{n_0+1}$ and $K = \bigcup_{n>n_0} P_n$. In order to prove our assertion it suffices to prove the equality

$$S = \bigsqcup_{k\in K} kF_{n_0}. \tag{3.5}$$

To check that the above union is pairwise disjoint, pick $k, k' \in K$ with $k \neq k'$; then there exists $m > n_0$ such that $k, k' \in P_m$, and so $kF_{n_0} \cap k'F_{n_0} = \emptyset$. To prove the equality in (3.5), pick $x \in S$. Since the sequence $(F_n)_{n\in\mathbb{N}}$ is exhaustive, $S = \bigcup_{n\in\mathbb{N}} F_n$, and so there exists $k \in \mathbb{N}$ such that $x \in F_k$. If $k \leq n_0$, we have $1 \cdot x \in F_{n_0}$, where $1 \in K$ because the sequence $(F_n)_{n\in\mathbb{N}}$ is congruent and hence $1 \in \bigcap_{n\geq n_0} K_n$. If $k > n_0$, then $F_k = P_kF_{n_0} \subseteq KF_{n_0}$. Hence, we have the required equality in (3.5). □

Next we give an example of a commutative (so amenable) monoid that is locally monotileable but neither MTA nor MT, and so not congruent monotileable by Lemma 3.12.

Example 3.13. Consider the submonoid $S = (\mathbb{N} \times \mathbb{N}) \setminus (\{0\} \times \mathbb{N}_+)$ of $\mathbb{N} \times \mathbb{N}$.

(a) We prove here that if F is a finite monotile of S and $(u, v_1), (u, v_2) \in F$, then $v_1 = v_2$. As F is a monotile of S, there exists $C \subseteq S$ such that
$$S = \bigsqcup_{c \in C} (c + F). \tag{3.6}$$
Since F is finite there exists $k \in \mathbb{N}_+$ such that $(1, k + v_1), (1, k + v_2) \in S \setminus F$. By (3.6), there exist $s_1, s_2 \in C$ with $(1, k + v_1) \in s_1 + F$ and $(1, k + v_2) \in s_2 + F$. As $(1, k + v_1), (1, k + v_2) \in S \setminus F$, one has $s_1 \neq (0, 0) \neq s_2$. Then $s_1 = (1, k + v_1)$ and $s_2 = (1, k + v_2)$, as $(\{0\} \times \mathbb{N}) \cap S = \{(0, 0)\}$. Then $(1 + u, k + v_1 + v_2) \in (s_1 + F) \cap (s_2 + F)$, which implies that $s_1 = s_2$. Therefore, $v_1 = v_2$, as required.

(b) The monoid S is not MTA. Suppose by contradiction that there exists a Følner sequence $(F_n)_{n \in \mathbb{N}}$ of S such that F_n is a monotile of S for every $n \in \mathbb{N}$. Since $(F_n)_{n \in \mathbb{N}}$ is a Følner sequence of S, without loss of generality we have, for all sufficiently large $n \in \mathbb{N}$,
$$|(F_n + (1, 1)) \cap F_n| > \frac{n-1}{n} |F_n| > \frac{1}{2} |F_n| \quad \text{and} \quad |(F_n + (1, 0)) \cap F_n| > \frac{n-1}{n} |F_n| > \frac{1}{2} |F_n|.$$
This implies that $|(F_n + (1, 1)) \cap (F_n + (1, 0))| > 0$ and so that there exists $(u, v) \in F_n$ such that also $(u, v + 1) \in F_n$. This contradicts (a), since F_n is a monotile of S by assumption.

(c) The monoid S is not MT. Indeed, according to item (a), for example the finite set $\{(1, 1), (1, 2)\}$ is not contained in any finite monotile of S.

(d) The monoid S is locally monotileable. Let $(F_n)_{n \in \mathbb{N}}$ be the sequence in $\mathcal{P}_{fin}(S)$ given by $F_0 = \{(0, 0)\}$ and $F_n = [2^{n-1}, 2^n) \times [2^{n-1}, 2^n)$ for all $n \in \mathbb{N}_+$. It is easy to see that $(F_n)_{n \in \mathbb{N}}$ is a locally monotileable Følner sequence of S.

Next we see that for groups the new notion of local monotileability coincides with the existing one of congruent monotileability.

Proposition 3.14. *A countable amenable group G is locally monotileable if and only if G is congruent monotileable.*

Proof. By definition, if G is congruent monotileable, then G is locally monotileable.

Assume that G is locally monotileable and let $(F_n)_{n \in \mathbb{N}}$ be a locally monotileable right Følner sequence of G. Since G is countable, enumerate its elements as $G = \{g_n : n \in \mathbb{N}\}$ with $g_0 = 1$. For every $n \in \mathbb{N}$, let $G_n = \{g_0, \dots, g_n\}$. We build inductively a sequence $(H_n)_{n \in \mathbb{N}}$ in $\mathcal{P}_{fin}(G)$ satisfying the following conditions:

(1) $H_0 = \{1\}$;

(2) for all $n \in \mathbb{N}$, $g_n \in H_n$;

(3) for all $n \in \mathbb{N}$, there exist $g \in G$ and $m \in \mathbb{N}$ such that $H_n = gF_m$;

(4) for all $n \in \mathbb{N}$, there exists K_{n+1} with $1 \in K_{n+1}$ and $H_{n+1} = \bigsqcup_{k \in K_{n+1}} kH_n$.

Then (2) and (3) ensure that $(H_n)_{n \in \mathbb{N}}$ is a right Følner sequence of G, (2) that it is exhaustive, while (4) shows that $(H_n)_{n \in \mathbb{N}}$ is congruent and implies that $H_n \subseteq H_{n+1}$ for all $n \in \mathbb{N}$.

Suppose that we already defined $H_0, \dots, H_n$. Then $H_n = tF_m$ for some $t \in G$ and $m \in \mathbb{N}$. There exists $l \in \mathbb{N}$ such that $l > m$ and
$$|F_l g_{n+1} \setminus F_l| < \frac{|F_l|}{|H_n|}. \tag{3.7}$$
Note that there exists $D \in \mathcal{P}_{fin}(G)$ such that $F_l = \bigsqcup_{d \in D} dF_m$. Therefore, $F_l = \bigsqcup_{d' \in D'} d'H_n$, where $D' = Dt^{-1}$.

We show that there exists $\bar{d} \in D'$ such that $\bar{d}H_n g_{n+1} \subseteq F_l$. Otherwise, for every $d' \in D'$, there would exist a point $v \in d'H_n g_{n+1} \setminus F_l$; since the sets $d'H_n g_{n+1}$, for $d' \in D'$, are pairwise disjoint, and are contained in $F_l g_{n+1}$, we would have that
$$|F_l g_{n+1} \setminus F_l| \geq |D'| = \frac{|F_l|}{|H_n|},$$
contradicting (3.7).

Clearly $\bar{d}H_n g_{n+1} \subseteq F_l$, in conjunction with $1 \in H_n$, yields $\bar{d}g_{n+1} \in F_l$. Define $H_{n+1} = \bar{d}^{-1}F_l$ and $K_{n+1} = \bar{d}^{-1}D'$. Then $1 = \bar{d}^{-1}\bar{d} \in K_{n+1}$ and
$$H_{n+1} = \bar{d}^{-1}F_l = \bar{d}^{-1} \bigsqcup_{d' \in D'} d'H_n = \bigsqcup_{k \in K_{n+1}} kH_n.$$
□

The next result shows that for groups MTA implies MT.

Proposition 3.15. *If a countable group G is* MTA*, then G is* MT.

Proof. By hypothesis, there exists a right Følner sequence $(F_n)_{n \in \mathbb{N}}$ of G such that F_n is a monotile of G for every $n \in \mathbb{N}$. Let $K \in \mathcal{P}_{fin}(G)$. By Lemma 3.6 there exist $n \in \mathbb{N}$ and $g \in G$ such that $gK \subseteq F_n$. This immediately implies that $K \subseteq g^{-1}F_n$. By Remark 3.7 $g^{-1}F_n$ is a monotile of G and this concludes the proof. □

The implication in the above proposition cannot be inverted. Indeed, every residually finite group is MT, so free non-abelian groups are MT but fail to be amenable.

4 Extension Theorem

4.1 CIF sequences

Let G be a countable amenable group and let $\mathcal{G} = (G_n)_{n\in\mathbb{N}}$ be an increasing exhaustive sequence in $\mathcal{P}_{fin}(G)$. A sequence $(F_n)_{n\in\mathbb{N}}$ in $\mathcal{P}_{fin}(G)$ is *canonically indexed* (briefly, a CIF sequence of G) with respect to $\mathcal{G}$ if for all $n \in \mathbb{N}_+$ and for all $g \in G_n$,

$$\frac{|F_n g \setminus F_n|}{|F_n|} < \frac{1}{n}.$$

Proposition 4.1. *Let G be a countable amenable group and let $(G_n)_{n\in\mathbb{N}}$ be an increasing exhaustive sequence in $\mathcal{P}_{fin}(G)$. For every right Følner sequence $(F_n)_{n\in\mathbb{N}}$ of G there exists an increasing sequence of natural numbers $(k_n)_{n\in\mathbb{N}}$ (with $k_0 = 0$) such that $(F_{k_n})_{n\in\mathbb{N}}$ is a CIF sequence of G with respect to $(G_n)_{n\in\mathbb{N}}$.*

Proof. By recursion we build a suitable sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers. Set $k_0 = 0$. Assume that $n > 0$ and suppose that k_{n-1} is defined. Then there exists $k_n \in \mathbb{N}$ such that $k_n > k_{n-1}$ and, for all $s \geq k_n$ and for all $g \in G_n$, $|F_s g \setminus F_s| / |F_s| < 1/n$. Clearly, $(F_{k_n})_{n\in\mathbb{N}}$ is a CIF sequence of G with respect to $(G_n)_{n\in\mathbb{N}}$. □

Remark 4.2. Let G be a countable amenable group and let $(G_n)_{n\in\mathbb{N}}$ be an increasing exhaustive sequence in $\mathcal{P}_{fin}(G)$.

(a) By definition, a CIF sequence $(F_n)_{n\in\mathbb{N}}$ of G with respect to $(G_n)_{n\in\mathbb{N}}$ is also a right Følner sequence of G.

(b) Let G be a countable amenable group and $(G_n)_{n\in\mathbb{N}}$ be an increasing exhaustive sequence in $\mathcal{P}_{fin}(G)$. Then any subsequence of a CIF sequence with respect to $(G_n)_{n\in\mathbb{N}}$ is still a CIF sequence of G with respect to $(G_n)_{n\in\mathbb{N}}$.

(c) One can easily deduce from (a) and (b) that if $(F_n)_{n\in\mathbb{N}}$ is a CIF sequence of G with respect to $(G_n)_{n\in\mathbb{N}}$, then for every increasing exhaustive sequence $(H_n)_{n\in\mathbb{N}}$ in $\mathcal{P}_{fin}(G)$, an appropriate subsequence $(F_{k_n})_{n\in\mathbb{N}}$ of $(F_n)_{n\in\mathbb{N}}$ is a CIF sequence of G with respect to $(H_n)_{n\in\mathbb{N}}$, hence simultaneously a CIF sequence of G with respect to $(G_n)_{n\in\mathbb{N}}$ as well. So, in this sense, the notion of "CIF sequence of G with respect to $(G_n)_{n\in\mathbb{N}}$" essentially does not depend on $(G_n)_{n\in\mathbb{N}}$ and one can loosely speak of CIF sequence of G.

The next lemma is straightforward to prove.

Lemma 4.3. *Let G and K be countable amenable groups. Let $\pi : G \to K$ be a surjective homomorphism and $H = \ker \pi$. Let $\sigma : K \to G$ be a section for π with $\sigma(1_K) = 1_G$. Consider a right Følner sequence $(E_n)_{n\in\mathbb{N}}$ of H and a right Følner sequence $(F_n)_{n\in\mathbb{N}}$ of K with $1_K \in F_n$ for all $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$:*

(a) $H \cap (\sigma(F_n)\sigma(F_n)^{-1}) = \{1\}$;

(b) for all $i, j \in \mathbb{N}$, $E_j\sigma(F_i) = \bigsqcup_{f\in\sigma(F_i)} E_j f$;

(c) for all $i, j \in \mathbb{N}$, $|E_j\sigma(F_i)| = |E_j||\sigma(F_i)|$.

The next theorem is implicitly contained in [9, Theorem 2.27], for the sake of completeness we provide a complete and independent proof.

Theorem 4.4. *Let G and K be countable amenable groups, $\pi : G \to K$ a surjective homomorphism with $H = \ker \pi$, and $\sigma : K \to G$ a section for π with $\sigma(1_K) = 1_G$. If $(E_n)_{n\in\mathbb{N}}$ is a right Følner sequence of H and $(F_n)_{n\in\mathbb{N}}$ is a right Følner sequence of K with $1_K \in F_n$ for all $n \in \mathbb{N}$, then there exist two increasing sequences of natural numbers $(m_n)_{n\in\mathbb{N}}$ and $(k_n)_{n\in\mathbb{N}}$ such that:*

(1) the sequence given by

$$\bar{F}_n = E_{m_n}\sigma(F_{k_n}) \text{ for all } n \in \mathbb{N}, \tag{4.1}$$

is a right Følner sequence of G;

(2) for all $n \in \mathbb{N}$, $|\bar{F}_n| = |E_{m_n}||F_{k_n}|$;

(3) for every sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ there exists a sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that, for all $n \in \mathbb{N}$, $m_{h_{n+1}} > m_{h_n} + a_n$ and the sequence given by

$$F^*_n = E_{m_{h_n}}\sigma(F_{k_n}) \text{ for all } n \in \mathbb{N}, \tag{4.2}$$

is a right Følner sequence of G.

Proof. The group G is countable so we enumerate its elements as $G = \{g_n : n \in \mathbb{N}\}$ with $g_0 = 1$; let also $G_n = \{g_0, \ldots, g_n\}$ for all $n \in \mathbb{N}$. By Proposition 4.1 there exists an increasing sequence of natural numbers $(k_n)_{n\in\mathbb{N}}$ (with $k_0 = 0$) such that $(F_{k_n})_{n\in\mathbb{N}}$ is a CIF sequence of K with respect to $(\pi(G_n))_{n\in\mathbb{N}}$.

For every $n \in \mathbb{N}$ define

$$H_n = \{h \in H : h\sigma(F_{k_n}) \cap \sigma(F_{k_n})G_n \neq \emptyset\} = \sigma(F_{k_n})G_n\sigma(F_{k_n})^{-1} \cap H.$$

Since $\sigma(F_{k_n})$ and G_n are finite, also $\sigma(F_{k_n})G_n\sigma(F_{k_n})^{-1}$ is finite. This means that also H_n is finite.

Clearly, $\bigcup_{n\in\mathbb{N}} H_n \subseteq H$. On the other hand, for every $h \in H$, there is an $n_h \in \mathbb{N}$ such that $h \in G_{n_h}$. So $h \in H_{n_h}$, as $\sigma(1_K) = 1_G$ and $1_K \in F_n$ for all $n \in \mathbb{N}$. Then $H = \bigcup_{n\in\mathbb{N}} H_n$ and $H_n \subseteq H_{n+1}$ for all $n \in \mathbb{N}$.

Claim 4.5. *Let $(m_n)_{n\in\mathbb{N}}$ be an increasing sequence of natural numbers (with $m_0 = 0$) such that $(E_{m_n})_{n\in\mathbb{N}}$ is a CIF sequence of H with respect to $(H_n)_{n\in\mathbb{N}}$. Let $(\bar{F}_n)_{n\in\mathbb{N}}$ be the sequence given by (4.1). Then, $(\bar{F}_n)_{n\in\mathbb{N}}$ satisfies (1) and (2).*

Fix arbitrarily $n \in \mathbb{N}$. Lemma 4.3(c) implies that $|\bar{F}_n| = |E_{m_n}||\sigma(F_{k_n})|$. Since σ is injective, we immediately obtain item (2).

The rest of the proof is dedicated to verify that $(\bar{F}_n)_{n\in\mathbb{N}}$ is a right Følner sequence. Fix $n \in \mathbb{N}$, pick an element $g \in G_n$ and let $C_n = \bar{F}_n g \setminus \bar{F}_n$, $A_n = \pi^{-1}(F_{k_n})$, so that

$$C_n = (C_n \setminus A_n) \sqcup (C_n \cap A_n).$$

Since we need to estimate $|C_n|$, it is enough to separately estimate the cardinalities $|C_n \setminus A_n|$ and $|C_n \cap A_n|$.

Pick an $x \in C_n$, then $x = e\sigma(f)g$ with $f \in F_{k_n}$ and $e \in E_{m_n}$. If $x \in C_n \setminus A_n$, then $\pi(x) = f\pi(g) \in F_{k_n}\pi(g) \setminus F_{k_n}$. Since $(F_{k_n})_{n\in\mathbb{N}}$ is a CIF sequence with respect to $(\pi(G_n))_{n\in\mathbb{N}}$, we have $|F_{k_n}\pi(g) \setminus F_{k_n}| \leq |F_{k_n}|/n$, and so there are at most $|F_{k_n}|/n$ choices for f. Since e ranges in E_{m_n} arbitrarily, this leads to

$$|C_n \setminus A_n| \leq \frac{|E_{m_n}||F_{k_n}|}{n} \leq \frac{|\bar{F}_n|}{n}, \tag{4.3}$$

Now suppose that $x \in A_n$, i.e., $f' := \pi(x) = f\pi(g) \in F_{k_n}$. As

$$\pi(\sigma(f')) = f' = f\pi(g) = \pi(\sigma(f)g),$$

we get $\pi(\sigma(f)g\sigma(f')^{-1}) = 1$. Therefore, $h_f := \sigma(f)g\sigma(f')^{-1} \in H$. Actually, $h_f \in H_n$, as $g \in G_n$. From $\sigma(f)g = h_f\sigma(f')$ we deduce that $x = e\sigma(f)g = eh_f\sigma(f')$. As $x = (eh_f)\sigma(f') \notin \bar{F}_n = E_{m_n}\sigma(F_{k_n})$, while $\sigma(f') \in \sigma(F_{k_n})$, we conclude that $eh_f \notin E_{m_n}$. Hence, $eh_f \in E_{m_n}h_f \setminus E_{m_n}$, so $x = (eh_f)\sigma(f') \in (E_{m_n}h_f \setminus E_{m_n})\sigma(f')$. In view of $f' = f\pi(g)$ this proves that

$$C_n \cap A_n \subseteq \bigcup_{f\in F_{k_n}} (E_{m_n}h_f \setminus E_{m_n})\sigma(f\pi(g)).$$

Since $(E_{m_n})_{n\in\mathbb{N}}$ is a CIF sequence with respect to $(H_n)_{n\in\mathbb{N}}$ and $h_f \in H_n$, we get $|E_{m_n}h_f \setminus E_{m_n}| \leq |E_{m_n}|/n$. This gives

$$|C_n \cap A_n| \leq \frac{|E_{m_n}||F_{k_n}|}{n} = \frac{|\bar{F}_n|}{n}. \tag{4.4}$$

Combining (4.3) and (4.4) we obtain

$$|C_n| = |C_n \cap A_n| + |C_n \setminus A_n| \leq \frac{|\bar{F}_n|}{n} + \frac{|\bar{F}_n|}{n} \leq \frac{2|\bar{F}_n|}{n},$$

therefore

$$\frac{|\bar{F}_n g \setminus \bar{F}_n|}{|\bar{F}_n|} \leq \frac{2}{n}. \tag{4.5}$$

Since n was chosen arbitrarily and (4.5) holds for all $g \in G_n$, we have proved that $(\bar{F}_n)_{n\in\mathbb{N}}$ is a right Følner sequence of G. Therefore, we have proved Claim 4.5.

By Proposition 4.1 there exists an increasing sequence of natural numbers $(m_n)_{n\in\mathbb{N}}$ satisfying the assumptions of Claim 4.5, and therefore the sequence $(\bar{F}_n)_{n\in\mathbb{N}}$ given by (4.1) also satisfies (1) and (2).

Moreover, if $(h_n)_{n\in\mathbb{N}}$ is any strictly increasing sequence of natural numbers, with $h_0 = 0$, then Remark 4.2(b) implies that $(m_{h_n})_{n\in\mathbb{N}}$ also satisfies the assumptions of Claim 4.5, and therefore $(F_n^*)_{n\in\mathbb{N}}$ defined as in (4.2) also satisfies (1) and (2): therefore, if we choose the sequence $(h_n)_{n\in\mathbb{N}}$ in such a way that $m_{h_{n+1}} > m_{h_n} + a_n$, we get (3). □

4.2 X-monotileable sequences

According to Definition 1.14, a countable amenable group G is $\mathrm{Inn}(G)$-monotileable if it admits a right Følner sequence $(F_n)_{n\in\mathbb{N}}$ invariant under conjugation by any element $g \in G$ (i.e., $gF_n = F_n g$ for all $n \in \mathbb{N}$ and $g \in G$).

Example 4.6. Here are two examples of countable amenable groups G having a right Følner sequence $(F_n)_{n\in\mathbb{N}}$ that is $\mathrm{Aut}(G)$-monotileable.

(a) Take an infinite collection $\{S_n : n \in \mathbb{N}\}$ of simple finite groups such that for $n \neq m$ the only homomorphism $S_n \to S_m$ is the trivial one and let $H = \bigoplus_{n\in\mathbb{N}} S_n$. By our choice of the family $\{S_n : n \in \mathbb{N}\}$ one has $\mathrm{Aut}(H) = \prod_{n\in\mathbb{N}} \mathrm{Aut}(S_n)$. Consider the sequence $(\bigoplus_{i=0}^n S_i)_{n\in\mathbb{N}}$; it is easy to see that it is an $\mathrm{Aut}(H)$-monotileable right Følner sequence of H.

(b) Consider the group $K = \bigoplus_{i=1}^k \mathbb{Z}(p_i^\infty)$, where $p_1, \ldots, p_k$ are pairwise distinct primes. Then for every $n \in \mathbb{N}$ the subgroup $K[n] := \{k \in K : nk = 0\}$ of K is finite and fully invariant. Therefore the sequence $(K[n!])_{n\in\mathbb{N}}$ is $\mathrm{Aut}(K)$-monotileable.

The following lemma is the counterpart for X-monotileable groups of Lemma 3.9 and has a similar proof.

Lemma 4.7. *Let G be a group and $(F_n)_{n\in\mathbb{N}}$ an X-monotileable sequence with $\mathrm{id} \in X$. Then every subsequence $(F_{k_n})_{n\in\mathbb{N}}$ is still an X-monotileable sequence.*

We need a technical result for the proof of Theorem 5.2. Given a finitely generated group K with generating set X, as usual, let $l_X(1) = 0$ and, for $g \in G \setminus \{1\}$, $l_X(g) = \min\{n \in \mathbb{N}_+ : g = x_1^{\epsilon_1} \ldots x_n^{\epsilon_n}, x_i \in X, \epsilon_i \in \{0, 1\}\}$; if X is symmetric, that is, $X^{-1} = X$, then $l_X(g) = \min\{n \in \mathbb{N}_+ : g = x_1 \ldots x_n, x_i \in X\}$.

Lemma 4.8. *Let H be a countable group and K a finitely generated group with a symmetric generating set $X = \{f_1, \ldots, f_m\}$. Consider a group homomorphism $\phi : K \to \mathrm{Aut}(H)$ and let $\tilde{X} = \{\mathrm{id}, \phi(f_1), \ldots, \phi(f_m)\}$. If $(E_n)_{n\in\mathbb{N}}$ is an $\tilde{X}$-monotileable sequence of H, then for all $f \in K$ the set $\phi(f)(E_n)$ is a monotile of E_{n+s}, where $s = l_X(f)$.*

Proof. Fix $n \in \mathbb{N}$. We proceed by induction on $s \in \mathbb{N}$. If either $l_X(f) = 0$ or $l_X(f) = 1$, the statement follows by hypothesis. Suppose we have already proved the thesis for all the elements of K of length $s-1$ and let f be such that $l_X(f) = s$. This means that exist an index $i \in \{1, \dots, m\}$ and an element $\bar{f}$ such that $f = f_i \bar{f}$ and $l_X(\bar{f}) = s-1$. So there exists a subset $\tilde{E}$ of H such that

$$E_{n+s-1} = \bigsqcup_{\tilde{e} \in \tilde{E}} \tilde{e}\phi(\bar{f})(E_n). \tag{4.6}$$

By hypothesis, $\phi(f_i)(E_{n+s-1})$ is a monotile of E_{n+s}, therefore there exists a subset E' of H such that

$$E_{n+s} = \bigsqcup_{e' \in E'} e'\phi(f_i)(E_{n+s-1}). \tag{4.7}$$

Combining (4.6) and (4.7), we obtain

$$\begin{aligned} E_{n+s} &= \bigsqcup_{e' \in E'} e'\phi(f_i)\left(\bigsqcup_{\tilde{e} \in \tilde{E}} \tilde{e}\phi(\bar{f})(E_n)\right) \\ &= \bigsqcup_{e' \in E'} e'\left(\bigsqcup_{\tilde{e} \in \tilde{E}} \phi(f_i)(\tilde{e})\phi(f_i)(\phi(\bar{f})(E_n))\right) \\ &= \bigsqcup_{e' \in E'} e'\left(\bigsqcup_{\phi(f_i)(\tilde{e}) \in \phi(f_i)(\tilde{E})} \phi(f_i)(\tilde{e})\phi(f)(E_n)\right). \end{aligned}$$

Let $E'' = E'\phi(f_i)(\tilde{E})$. By Lemma 3.8, $E_{n+s} = \bigsqcup_{e'' \in E''} e''\phi(f)(E_n)$. $\square$

4.3 Proof of the Extension Theorem

Consider two countable groups G and K and a surjective homomorphism $\pi : G \to K$. Fix a section $\sigma : K \to G$ for π, such that $\sigma(1_K) = 1_G$. Let $(F_n)_{n\in\mathbb{N}}$ be a locally monotileable sequence of K and let $(K_n)_{n\in\mathbb{N}}$ be a tiling sequence associated to $(F_n)_{n\in\mathbb{N}}$. We recall that any element $f \in F_n$ can be written in a unique way as $f = \prod_{j=1}^n k_j$, where $k_j \in K_{n+1-j}$ for $1 \le j \le n$. For all $n \in \mathbb{N}$ we define

$$\sigma_n : F_n \to G \quad \text{as} \quad \sigma_n(f) = \prod_{j=1}^n \sigma(k_j) \quad \text{where } f = \prod_{j=1}^n k_j.$$

Since $F_n \subseteq F_{n+1}$ for all $n \in \mathbb{N}$, we obtain that $\sigma_{n+1}\restriction_{F_n} = \sigma_n$ for all $n \in \mathbb{N}$. Then the map $\bar{\sigma} : \bigcup_{n\in\mathbb{N}} F_n \to G$, given by $\bar{\sigma}(f) = \sigma_n(f)$ for $n \in \mathbb{N}$ such that $f \in F_n$, is well defined.

It is straightforward to verify that $\bar{\sigma}(1_K) = \sigma_0(1_K) = \sigma(1_K) = 1_G$ and that, for every $f \in \bigcup_{n\in\mathbb{N}} F_n$, $\pi(\bar{\sigma}(f)) = f$, that is, $\pi \circ \bar{\sigma} = \mathrm{id}_K\restriction_{\bigcup_{n\in\mathbb{N}} F_n}$; in particular $\bar{\sigma}$ is injective. Hence, we can extend the map $\bar{\sigma}$ to a section $\tilde{\sigma} : K \to G$ for π and we call it the *section associated* to $(F_n)_{n\in\mathbb{N}}$ and σ. Clearly, this $\tilde{\sigma}$ need not be unique. In case $\bigcup_{n\in\mathbb{N}} F_n = K$ (so, $\tilde{\sigma} = \bar{\sigma}$) and $\tilde{\sigma} = \sigma$, we simply say that σ is associated to $(F_n)_{n\in\mathbb{N}}$.

Lemma 4.9. *Let G and K be countable groups, $\pi : G \to K$ a surjective homomorphism and $\sigma : K \to G$ a section for π with $\sigma(1_K) = 1_G$. For a locally monotileable sequence $(F_n)_{n\in\mathbb{N}}$ of K with associated tiling sequence $(K_n)_{n\in\mathbb{N}}$ consider also the section $\tilde{\sigma}$ associated to $(F_n)_{n\in\mathbb{N}}$ and σ. Then $(\tilde{\sigma}(F_n))_{n\in\mathbb{N}}$ is a locally monotileable sequence of G with associate tiling sequence $(\sigma(K_n))_{n\in\mathbb{N}}$.*

Proof. To verify that $(\tilde{\sigma}(F_n))_{n\in\mathbb{N}}$ is a locally monotileable sequence we have to prove that

$$\sigma(K_{n+1})^{-1}\sigma(K_{n+1}) \cap \tilde{\sigma}(F_n)\tilde{\sigma}(F_n)^{-1} = \{1\}, \tag{4.8}$$

for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ and $g \in \sigma(K_{n+1})^{-1}\sigma(K_{n+1}) \cap \tilde{\sigma}(F_n)\tilde{\sigma}(F_n)^{-1}$. So there are $k_1, k_2 \in K_{n+1}$ and $f_1, f_2 \in \tilde{\sigma}(F_n)$ such that $g = f_1 f_2^{-1}$ and $\sigma(k_1)f_1 = \sigma(k_2)f_2$. Thus, applying π,

$$k_1\pi(f_1) = \pi(\sigma(k_1)f_1) = \pi(\sigma(k_2)f_2) = k_2\pi(f_2).$$

Since $(F_n)_{n\in\mathbb{N}}$ is a locally monotileable sequence of K, from $k_1\pi(f_1) = k_2\pi(f_2)$ we conclude that $k_1 = k_2$ and $\pi(f_1) = \pi(f_2)$. This implies $\sigma(k_1) = \sigma(k_2)$ and $f_1 = f_2$. Therefore, $g = 1$ and then (4.8) holds. $\square$

We are now in position to prove Theorem 1.15, that is the Extension Theorem stated in the introduction. Actually, here we prove the more technical claim below, from which the theorem follows immediately. Recall that, for a group G and a subset X of G, $c_G(X) = \{g \in G : gx = xg \text{ for every } x \in G\}$ is the centralizer of X in G.

Claim 4.10. *Suppose that $0 \to H \xrightarrow{\iota} G \xrightarrow{\pi} K \to 0$ is a short exact sequence of countable groups, where K, H are locally monotileable. Fix a section $\sigma : K \to G$ for π such that $\sigma(1_K) = 1_G$, a locally monotileable Følner sequence $(F_n)_{n\in\mathbb{N}}$ of K and a locally monotileable Følner sequence $(E_n)_{n\in\mathbb{N}}$ of H. Let $(K_n)_{n\in\mathbb{N}}$ be a tiling sequence associated to $(F_n)_{n\in\mathbb{N}}$ and $\tilde{\sigma}$ a section associated to $(F_n)_{n\in\mathbb{N}}$ and σ. For convenience, for every $n \in \mathbb{N}$, we set $F_n' = \tilde{\sigma}(F_n)$ and $K_n' = \sigma(K_n)$. If one of the following two conditions holds:*

(a) $(E_n)_{n\in\mathbb{N}}$ *is* $\mathrm{Inn}(G)$*-monotileable,*

(b) $K'_n = \sigma(K_n) \subseteq c_G(H)$ *for all* $n \in \mathbb{N}$,

then there exist a locally monotileable sequence $(F^\#_n)_{n\in\mathbb{N}}$ *of* G *and a strictly increasing sequence* $(m_n)_{n\in\mathbb{N}}$ *in* $\mathbb{N}$ *such that:*

(1) $(\pi(F^\#_n))_{n\in\mathbb{N}}$ *is a locally monotileable right Følner sequence of* K*;*

(2) $(\bar{F}_n)_{n\in\mathbb{N}} = (E_{m_n}F^\#_n)_{n\in\mathbb{N}}$ *is a locally monotileable right Følner sequence of* G*;*

(3) for all $n \in \mathbb{N}$, $\bar{F}_n = \bigsqcup_{f\in F^\#_n} E_{m_n} f$.

Proof. By Theorem 4.4 applied to the section $\tilde{\sigma}$, there exist two strictly increasing sequences of natural numbers $(m_n)_{n\in\mathbb{N}}$ and $(k_n)_{n\in\mathbb{N}}$ such that the sequence $(\bar{F}_n)_{n\in\mathbb{N}}$ given by $\bar{F}_n = E_{m_n}F'_{k_n}$ for all $n \in \mathbb{N}$ is a right Følner sequence of G. So, for all $n \in \mathbb{N}$ let $F^\#_n = F'_{k_n}$. Moreover, Lemma 4.3 (with $j = m_n$ and $i = n$) gives $\bar{F}_n = \bigsqcup_{f^\#\in F^\#_n} E_{m_n} f^\#$ for all $n \in \mathbb{N}$, and so also (3) holds. Since, for every $n \in \mathbb{N}$, $\pi(F^\#_n) = \pi(F'_{k_n}) = \pi(\tilde{\sigma}(F_{k_n})) = F_{k_n}$, the condition in (1) is satisfied by Lemma 3.9.

It remains to prove only that $(\bar{F}_n)_{n\in\mathbb{N}}$ is locally monotileable, so that also (2) is satisfied. The sequence $(F'_n)_{n\in\mathbb{N}}$ is locally monotileable by Lemma 4.9 and so also $(F'_{k_n})_{n\in\mathbb{N}}$ is locally monotileable by Lemma 3.9, that is, for every $n \in \mathbb{N}$ there exists a finite subset $\bar{K}_n$ of G such that

$$F^\#_n = F'_{k_n} = \bigsqcup_{f\in\bar{K}_n} fF'_{k_{n-1}}. \tag{4.9}$$

If (a) holds, then $(E_{m_n})_{n\in\mathbb{N}}$ is $\mathrm{Inn}(G)$-monotileable by Lemma 4.7 and so, for every $n \in \mathbb{N}$ and $f \in \bar{K}_n$, the subset $fE_{m_{n-1}}f^{-1}$ is a monotile of E_{m_n}. If (b) holds, we immediately conclude that $\bar{K}_n \subseteq c_G(H)$, and so, for every $n \in \mathbb{N}$ and $f \in \bar{K}_n$, $E_{m_{n-1}} = fE_{m_{n-1}}f^{-1}$ is a monotile of E_{m_n}. Hence, in both cases, for every $n \in \mathbb{N}$ and each $f \in \bar{K}_n$ there exists a finite subset $\bar{E}_{f,n}$ of H such that

$$E_{m_n} = \bigsqcup_{e\in\bar{E}_{f,n}} efE_{m_{n-1}}f^{-1}. \tag{4.10}$$

Fixed $n \in \mathbb{N}_+$, (4.9) and Lemma 4.3(b) yield

$$\bar{F}_n = E_{m_n}F'_{k_n} = E_{m_n} \bigsqcup_{f\in\bar{K}_n} fF'_{k_{n-1}} = \bigsqcup_{f\in\bar{K}_n} E_{m_n} fF'_{k_{n-1}}, \tag{4.11}$$

and by (4.10), (4.11) and Lemma 4.3(b), we have

$$\begin{aligned}\bar{F}_n &= \bigsqcup_{f\in\bar{K}_n} E_{m_n} fF'_{k_{n-1}} = \bigsqcup_{f\in\bar{K}_n} \left(\left(\bigsqcup_{e\in\bar{E}_{f,n}} efE_{m_{n-1}}f^{-1}\right) fF'_{k_{n-1}}\right) = \\ &= \bigsqcup_{f\in\bar{K}_n}\bigsqcup_{e\in\bar{E}_{f,n}} efE_{m_{n-1}}F'_{k_{n-1}} = \bigsqcup_{f\in\bar{K}_n}\bigsqcup_{e\in\bar{E}_{f,n}} ef\bar{F}_{n-1} = \bigsqcup_{g\in\bar{G}_n} g\bar{F}_{n-1},\end{aligned}$$

where $\bar{G}_n = \bigsqcup_{f\in\bar{K}_n} \bar{E}_{f,n}f$ (this union is disjoint since from the last equality we obtain that $|\bar{G}_n| = \sum_{f\in\bar{K}_n} |\bar{E}_{f,n}|$). □

The following corollary about the stability of $\mathfrak{M}$ under central extensions coincides with [4, Lemma 6].

Corollary 4.11. *Consider two countable groups* G *and* K*. Let* G *be a central extension of* K*, and let* H *be a subgroup of* G *with* $H \subseteq Z(G)$ *and such that* $0 \to H \to G \to K \to 0$ *is a short exact sequence. If* H *and* K *are locally monotileable, then also* G *is locally monotileable.*

Proof. Since H is abelian, every locally monotileable sequence of H is trivially $\mathrm{Inn}(G)$-monotileable. Therefore Claim 4.10 applies. □

Example 4.12. The Heisenberg group $H_3(\mathbb{Z})$ is the group of 3×3 upper unitriangular matrices in $M_3(\mathbb{Z})$. Since $H_3(\mathbb{Z})$ is a central extension of $\mathbb{Z}$ by $\mathbb{Z}^2$, so $H_3(\mathbb{Z})$ is locally monotileable by Corollary 4.11.

If in Claim 4.10 H is finite, we can consider the locally monotileable Følner sequence $(E_n)_{n\in\mathbb{N}}$ given by $E_0 = \{1_G\}$ and $E_n = H$ for all $n > 0$. Since $H = \ker\pi$, we have that H is normal in G. Therefore the sequence $(E_n)_{n\in\mathbb{N}}$ is invariant under conjugation by any element $g \in G$ and so it is $\mathrm{Inn}(G)$-monotileable. Thus we can apply Claim 4.10 and we obtain the following result.

Corollary 4.13. *Consider two countable groups* G *and* K*. Let* $\pi : G \to K$ *be a surjective homomorphism with* $\ker\pi$ *finite. If* K *is locally monotileable then also* G *is locally monotileable.*

4.4 Countable virtually nilpotent groups are locally monotileable

In this section we use the Extension Theorem (actually Claim 4.10) to prove first that a countable group with a locally monotileable normal subgroup of finite index is necessarily locally monotileable, and then that all countable abelian groups are locally monotileable. These results together give that all countable virtually nilpotent groups are locally monotileable. Note that the virtually nilpotent finitely generated groups are precisely those of polynomial growth by the celebrated Gromov Theorem.

Proposition 4.14. *If* G *is a countable group having a normal subgroup of finite index* H *which is locally monotileable, then so is* G*.*

Proof. Let $(E_n)_{n\in\mathbb{N}}$ be a locally monotileable right Følner sequence of H. Let $K = G/H$, let $\pi : G \to K$ be the canonical projection and $\iota : H \to G$ the inclusion of H in G. Fix a section $\sigma : K \to G$ for π such that $\sigma(1_K) = 1_G$, and let $\sigma(K) = R$. Since K is finite, consider the locally monotileable Følner sequence $(F_n)_{n\in\mathbb{N}}$ given by $F_0 = \{1_K\}$ and $F_n = K$ for all $n \in \mathbb{N}_+$. By Theorem 4.4 there is an increasing sequence of natural numbers $(m_n)_{n\in\mathbb{N}}$ such that the sequence $(\bar{F}_n)_{n\in\mathbb{N}}$, given by $\bar{F}_0 = \{1\}$ and $\bar{F}_n = E_{m_n}R$ for all $n \in \mathbb{N}_+$, is a right Følner sequence of G (note that R is finite and so also $\bar{F}_n$ is finite for every $n \in \mathbb{N}$).

It remains to prove that $(\bar{F}_n)_{n\in\mathbb{N}}$ is locally monotileable. Clearly $\bar{F}_0$ is a monotile of $\bar{F}_1$, so we can suppose $n > 1$. The sequence $(E_n)_{n\in\mathbb{N}}$ is locally monotileable by hypothesis so by Lemma 3.9 also $(E_{m_n})_{n\in\mathbb{N}}$ is locally monotileable. Therefore, for every $n > 1$ there is a finite subset $\bar{E}_n$ of H such that

$$E_{m_n} = \bigsqcup_{\bar{e}\in\bar{E}_n} \bar{e}E_{m_{n-1}}. \tag{4.12}$$

For $\bar{e}_1, \bar{e}_2 \in \bar{E}_n$ with $\bar{e}_1 \neq \bar{e}_2$, we have $\bar{e}_1E_{m_{n-1}} \cap \bar{e}_2E_{m_{n-1}} = \emptyset$. Therefore, since R is a set of right coset representatives of H in G and $E_{m_n} \subseteq H$,

$$\bar{e}_1E_{m_{n-1}}R \cap \bar{e}_2E_{m_{n-1}}R = \emptyset. \tag{4.13}$$

By (4.12) and (4.13) we conclude that, for all $n \in \mathbb{N}$, $\bar{F}_n = E_{m_n}R = \bigsqcup_{\bar{e}\in\bar{E}_n} \bar{e}E_{m_{n-1}}R = \bigsqcup_{\bar{e}\in\bar{E}_n} \bar{e}\bar{F}_{n-1}$. □

Example 4.15. Consider the group $G = \mathbb{Z} \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on $\mathbb{Z}$ as the automorphism $t(x) = -x$. The group G is the semidirect product between $\mathbb{Z}$ and $\mathbb{Z}_2$ therefore $0 \to \mathbb{Z} \to G \to \mathbb{Z}_2 \to 0$ is a short exact sequence. We know that $\mathbb{Z}$ and $\mathbb{Z}_2$ are both locally monotileable groups, so by Proposition 4.14 we have that also G is locally monotileable.

We apply twice the following easy observation in the proof of Theorem 4.17.

Lemma 4.16. *Let G be a group and $A, B \in \mathcal{P}_{fin}(G)$. If $g \in G$, then*

$$|(AgB) \setminus (AB)| \leq |(Ag) \setminus A|\,|B| \quad \textit{and} \quad |(ABg) \setminus (AB)| \leq |(Bg) \setminus B|\,|A|.$$

Proof. Fix $g \in G$ and let $a \in A$ and $b \in B$ be such that $agb \notin AB$. This clearly implies that $ag \notin A$. So, $(AgB) \setminus (AB) \subseteq (Ag \setminus A)B$, and it is straightforward to deduce the first inequality. The second inequality can be proved analogously. □

Theorem 4.17. *Assume that the group G is the increasing union of its subgroups $G_0 \leq G_1 \leq G_2 \leq \dots$, and that:*

(1) every G_n is locally monotileable,

(2) every quotient G_{n+1}/G_n is locally monotileable,

(3) every quotient G_{n+1}/G_n admits a section $\sigma_n : G_{n+1}/G_n \to G_{n+1}$ such that $\sigma_n(1) = 1$ and $\sigma_n(G_{n+1}/G_n) \subseteq c_{G_{n+1}}(G_n)$.

Then G is locally monotileable.

Proof. We build recursively a locally monotileable right Følner sequence $(E_{n,j})_{j\in\mathbb{N}}$ of G_n for every $n \in \mathbb{N}$. Let $(E_{0,j})_{j\in\mathbb{N}}$ be a locally monotileable right Følner sequence of G_0. Fix $n \in \mathbb{N}$ and suppose we have already defined a locally monotileable right Følner sequence $(E_{n,j})_{j\in\mathbb{N}}$ of G_n. Consider the exact sequence

$$0 \to G_n \to G_{n+1} \to G_{n+1}/G_n \to 0. \tag{4.14}$$

Since $\sigma_n(G_{n+1}/G_n) \subseteq c_{G_{n+1}}(G_n)$, by Claim 4.10 there exist a locally monotileable sequence $(F^{\#}_{n+1,j})_{j\in\mathbb{N}}$ of G_{n+1} and a strictly increasing sequence of natural numbers $(m_{n+1,j})_{j\in\mathbb{N}}$ with $m_{n+1,j} \geq j$ for all $j \in \mathbb{N}$, such that the sequence $(E_{n+1,j})_{j\in\mathbb{N}}$, defined letting, for every $j \in \mathbb{N}$,

$$E_{n+1,j} = E_{n,m_{n+1,j}}F^{\#}_{n+1,j},$$

is a locally monotileable right Følner sequence of G_{n+1}. Since $F^{\#}_{n+1,j} \subseteq \sigma_n(G_{n+1}/G_n) \subseteq c_{G_{n+1}}(G_n)$, then

$$E_{n+1,j} = \bigsqcup_{f\in F^{\#}_{n+1,j}} fE_{n,m_{n+1,j}}. \tag{4.15}$$

It remains to prove that the diagonal sequence $(E_{n,n})_{n\in\mathbb{N}}$ is a locally monotileable right Følner sequence of G.

Claim 4.18. *$(E_{n,n})_{n\in\mathbb{N}}$ is a locally monotileable sequence of G.*

For every $n \in \mathbb{N}$, (4.15) yields

$$E_{n+1,n+1} = \bigsqcup_{f\in F^{\#}_{n+1,n+1}} fE_{n,m_{n+1,n+1}}. \tag{4.16}$$

For all $n \in \mathbb{N}$, let $(K_{n,j})_{j\in\mathbb{N}}$ be a tiling sequence associated to $(E_{n,j})_{j\in\mathbb{N}}$. For every $n \in \mathbb{N}$ and $j \in \mathbb{N}$, for $m_{n+1,j} = j$ let $M_{n,j} = \{1\}$, while for $m_{n+1,j} > j$ the product $M_{n,j} = K_{n,m_{n+1,j}} \dots K_{n,j+1}$ makes sense, and so for every $n \in \mathbb{N}$ and $j \in \mathbb{N}$,

$$E_{n,m_{n+1,j}} = M_{n,j}E_{n,j}. \tag{4.17}$$

Fix $n \in \mathbb{N}$ and $j = n + 1$. Then $M_{n,n+1} = K_{n,m_{n+1,n+1}} \dots K_{n,n+2}$, and letting $M = M_{n,n+1}K_{n,n+1}$,

$$E_{n,m_{n+1,n+1}} = M_{n,n+1}K_{n,n+1}E_{n,n} = ME_{n,n} = \bigsqcup_{m\in M} mE_{n,n}. \tag{4.18}$$

Combining (4.16) and (4.18), we get

$$E_{n+1,n+1} = \bigsqcup_{f\in F^{\#}_{n+1,n+1}} f\left(\bigsqcup_{m\in M} mE_{n,n}\right).$$

For $A_{n+1} = F^{\#}_{n+1,n+1}M$, by Lemma 3.8 we obtain that $E_{n+1,n+1} = \bigsqcup_{a\in A_{n+1}} aE_{n,n}$. This proves Claim 4.18.

Claim 4.19. *$(E_{n,n})_{n\in\mathbb{N}}$ is a right Følner sequence of G.*

Take $g \in G$. Since $G = \bigcup_{n\in\mathbb{N}} G_n$, there exists $t \in \mathbb{N}$ such that $g \in G_t$. Pick $n \in \mathbb{N}$ with $n \geq t$. Then $g \in G_n$, so $gF^{\#}_{n+1,j} = F^{\#}_{n+1,j}g$ for every $j \in \mathbb{N}$. Moreover, for every $j \in \mathbb{N}$,

$$\begin{aligned}\frac{|E_{n+1,j}g \setminus E_{n+1,j}|}{|E_{n+1,j}|} &= \frac{|E_{n,m_{n+1,j}}F^{\#}_{n+1,j}g \setminus (E_{n,m_{n+1,j}}F^{\#}_{n+1,j})|}{|E_{n,m_{n+1,j}}F^{\#}_{n+1,j}|} = \frac{|E_{n,m_{n+1,j}}gF^{\#}_{n+1,j} \setminus (E_{n,m_{n+1,j}}F^{\#}_{n+1,j})|}{|E_{n,m_{n+1,j}}F^{\#}_{n+1,j}|} \leq \\ &\leq \frac{|E_{n,m_{n+1,j}}g \setminus E_{n,m_{n+1,j}}||F^{\#}_{n+1,j}|}{|E_{n,m_{n+1,j}}||F^{\#}_{n+1,j}|} = \frac{|E_{n,m_{n+1,j}}g \setminus E_{n,m_{n+1,j}}|}{|E_{n,m_{n+1,j}}|},\end{aligned} \tag{4.19}$$

where the inequality holds by Lemma 4.16. By Remark 3.2 and by (4.17) we obtain that, for $m_{n+1,j} > j$,

$$|E_{n,m_{n+1,j}}| = |M_{n,j}E_{n,j}| = |K_{n,m_{n+1,j}} \dots K_{n,j+1}E_{n,j}| = |K_{n,m_{n+1,j}}| \dots |K_{n,j+1}||E_{n,j}| = |M_{n,j}||E_{n,j}|; \tag{4.20}$$

the equality $|E_{n,m_{n+1,j}}| = |M_{n,j}||E_{n,j}|$ holds trivially for $m_{n+1,j} = j$. Now (4.20) and Lemma 4.16 give

$$\frac{|E_{n,m_{n+1,j}}g \setminus E_{n,m_{n+1,j}}|}{|E_{n,m_{n+1,j}}|} = \frac{|M_{n,j}E_{n,j}g \setminus (M_{n,j}E_{n,j})|}{|M_{n,j}E_{n,j}|} \leq \frac{|E_{n,j}g \setminus E_{n,j}||M_{n,j}|}{|E_{n,j}||M_{n,j}|} = \frac{|E_{n,j}g \setminus E_{n,j}|}{|E_{n,j}|}, \tag{4.21}$$

and so (4.19) and (4.21) yield

$$\frac{|E_{n+1,j}g \setminus E_{n+1,j}|}{|E_{n+1,j}|} \leq \frac{|E_{n,j}g \setminus E_{n,j}|}{|E_{n,j}|}.$$

Using this inequality for $n = t$, by induction we get that, for every $n \geq t$,

$$\frac{|E_{n,j}g \setminus E_{n,j}|}{|E_{n,j}|} \leq \frac{|E_{t,j}g \setminus E_{t,j}|}{|E_{t,j}|}. \tag{4.22}$$

By the choice of the sequence $(E_{t,j})_{j\in\mathbb{N}}$,

$$\lim_{j\to\infty} \frac{|E_{t,j}g \setminus E_{t,j}|}{|E_{t,j}|} = 0. \tag{4.23}$$

By (4.22) and (4.23) we conclude that

$$\lim_{n\to\infty} \frac{|E_{n,n}g \setminus E_{n,n}|}{|E_{n,n}|} = \lim_{j\to\infty} \frac{|E_{t+j,t+j}g \setminus E_{t+j,t+j}|}{|E_{t+j,t+j}|} \leq \lim_{j\to\infty} \frac{|E_{t,t+j}g \setminus E_{t,t+j}|}{|E_{t,t+j}|} = \lim_{j\to\infty} \frac{|E_{t,j}g \setminus E_{t,j}|}{|E_{t,j}|} = 0.$$

This proves Claim 4.19, and so concludes the proof of the theorem. □

Now Corollary 3.11 can be obtained as a consequence of Theorem 4.17. Moreover, by Theorem 4.17 we have that $\mathbb{Q} \in \mathfrak{M}$ (for an alternative proof see Theorem 1.18) and the following more general results.

Corollary 4.20. *If G is a countable abelian group, then $G \in \mathfrak{M}$.*

Proof. By hypothesis G is the increasing union of an increasing chain $\{G_n : n \in \mathbb{N}\}$ of finitely generated subgroups; moreover, G_{n+1}/G_n is finitely generated for every $n \in \mathbb{N}$. By Corollary 3.11, for every $n \in \mathbb{N}$, G_n and G_{n+1}/G_n are locally monotileable. Hence G is locally monotileable by Theorem 4.17. □

Corollary 4.21. *If G is a countable hypercentral group of length $< \omega^2$, then $G \in \mathfrak{M}$.*

Proof. By hypothesis $G = Z_\alpha(G)$ for some countable ordinal $\alpha < \omega^2$. We prove by induction that $Z_\kappa(G)$ is locally monotileable for every ordinal $\kappa < \omega^2$. Indeed, $Z_0(G) = \{1\}$ is locally monotileable. Moreover, if $\kappa = \beta + 1$ for some ordinal β, since $Z_\beta(G)$ is locally monotileable by the inductive hypothesis and $Z_\kappa(G)/Z_\beta(G)$ is locally monotileable by Corollary 4.20, $Z_\kappa(G)$ is locally monotileable by Corollary 4.11. If κ is a limit ordinal, then $\kappa = m\omega$ for some $m \in \mathbb{N}_+$, and so $Z_\kappa(G)$ is increasing union of its subgroups $\{Z_{(m-1)\omega+n}(G) : n \in \mathbb{N}\}$. By the inductive hypothesis and by Corollary 4.20, those subgroups satisfy the hypotheses of Theorem 4.17, so $Z_\kappa(G)$ is locally monotileable. □

Corollary 4.21 and Proposition 4.14 give Theorem 1.16.

5 Applications of the Extension Theorem

5.1 Extensions of $\mathbb{Q}$ by $\mathbb{Z}$

The following technical lemma is needed in the proof of Theorem 5.2.

Lemma 5.1. *Let H, K be countable groups, $\phi : K \to \mathrm{Aut}(H)$ a group homomorphism and $(E_n)_{n\in\mathbb{N}}$ a locally monotileable sequence of H. Consider also a locally monotileable sequence $(F_n)_{n\in\mathbb{N}}$ of K and let $(K_n)_{n\in\mathbb{N}}$ be an associated tiling sequence. Let $(\bar{F}_n)_{n\in\mathbb{N}}$ be the sequence in $\mathcal{P}_{fin}(H \times K)$ given by $\bar{F}_n = E_n \times F_n$. If for all $n \in \mathbb{N}$ and $k \in K_{n+1}$, $\phi(k)(E_n)$ is a monotile of E_{n+1}, then $(\bar{F}_n)_{n\in\mathbb{N}}$ is a locally monotileable sequence of $H \rtimes_\phi K$.*

Proof. Fix $n \in \mathbb{N}$. By hypothesis $\phi(k)(E_n)$ is a monotile of E_{n+1} for all $k \in K_{n+1}$. Therefore for all $k \in K_{n+1}$, there is $\tilde{E}_{n+1,k}$ such that $E_{n+1} = \bigsqcup_{\tilde{e}\in\tilde{E}_{n+1,k}} \tilde{e}\phi(k)(E_n)$. Consider $\bar{K}_{n+1} = \bigsqcup_{k\in K_{n+1}} \tilde{E}_{n+1,k} \times \{k\}$. Then

$$\bar{F}_{n+1} = E_{n+1} \times F_{n+1} = \bigsqcup_{k\in K_{n+1}} E_{n+1} \times (kF_n) = \bigsqcup_{\bar{k}\in\bar{K}_{n+1}} \bar{k}(E_n \times F_n) = \bigsqcup_{\bar{k}\in\bar{K}_{n+1}} \bar{k}\bar{F}_n. \qquad \square$$

Theorem 5.2. *Let H be a countable group and K a locally monotileable finitely generated group with a symmetric generating set $X = \{f_1, \dots, f_m\}$. Consider a group homomorphism $\phi : K \to \mathrm{Aut}(H)$ and let $\tilde{X} = \{\mathrm{id}, \phi(f_1), \dots, \phi(f_m)\}$. If H is $\tilde{X}$-monotileable then the group $G = H \rtimes_\phi K$ is locally monotileable.*

Proof. Let $(E_n)_{n\in\mathbb{N}}$ be an $\tilde{X}$-monotileable right Følner sequence of H and $(F_n)_{n\in\mathbb{N}}$ a locally monotileable right Følner sequence of K. By Theorem 4.4 there exist two sequences $(m_n)_{n\in\mathbb{N}}$ and $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that the sequence $(\bar{F}_n)_{n\in\mathbb{N}}$, given by $\bar{F}_n = E_{m_n} \times F_{k_n}$ for every $n \in \mathbb{N}$, is a right Følner sequence of $H \rtimes_\phi K$.

Consider a tiling sequence $(K_n)_{n\in\mathbb{N}}$ associated to $(F_{k_n})_{n\in\mathbb{N}}$. For all $n \in \mathbb{N}$ let $a_n = \max\{l_S(k) : k \in K_{n+1}\}$. By Theorem 4.4(3) there exists a strictly increasing sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that:

(1) $m_{h_{n+1}} > m_{h_n} + a_n$ for every $n \in \mathbb{N}$;

(2) the sequence $(F_n^*)_{n\in\mathbb{N}}$, given by $F_n^* = E_{m_{h_n}} \times F_{k_n}$ for every $n \in \mathbb{N}$, is a right Følner sequence of $H \rtimes_\phi K$.

By Lemma 4.7 the subsequence $(E_{m_{h_n}})_{n\in\mathbb{N}}$ is still an $\tilde{X}$-monotileable sequence of H. By Lemma 4.8 we know that $\phi(k)(E_{m_{h_n}})$ is a monotile of $E_{m_{h_n}+l_S(k)}$ for all $k \in K_{n+1}$. On the other hand, since $m_{h_n} + l_S(k) \le m_{h_{n+1}}$, by Lemma 3.9 $E_{m_{h_n}+l_S(k)}$ is a monotile of $E_{m_{h_{n+1}}}$. Finally by Lemma 3.8 we have that $\phi(k)(E_{m_{h_n}})$ is a monotile of $E_{m_{h_{n+1}}}$ for all $k \in K_{n+1}$.

The sequences $(E_{m_{h_n}})_{n\in\mathbb{N}}$, $(F_{k_n})_{n\in\mathbb{N}}$ and $(F_n^*)_{n\in\mathbb{N}}$ satisfy the hypothesis of Lemma 5.1, so we apply it to conclude. $\square$

The following is an immediate consequence of Theorem 5.2 (take $X = \{1, -1\}$ as a finite generating set of $\mathbb{Z}$).

Corollary 5.3. *Let H be a countable locally monotileable group and $\phi : H \to H$ an automorphism. If H is ϕ-monotileable, then $H \rtimes_\phi \mathbb{Z}$ is locally monotileable.*

Now we prove Theorem 1.18 stating that for every automorphism ϕ of $\mathbb{Q}$, the group $\mathbb{Q} \rtimes_\phi \mathbb{Z}$ is locally monotileable. First we need the following folklore lemma; recall that an a-sequence $(a_n)_{n\in\mathbb{N}}$ is *geometric* if for every $k \in \mathbb{N}$ there exists $\bar{n} \in \mathbb{N}$ with $k \mid a_{\bar{n}}$.

Lemma 5.4. *Let $(a_n)_{n\in\mathbb{N}}$ be a geometric a-sequence.*

(a) For every $q \in \mathbb{Q}$ there is a minimum $n_q \in \mathbb{N}$ such that $q \in \langle \frac{1}{a_{n_q}} \rangle$.

(b) For q and n_q as in item (a), there exist unique $k_1, \dots, k_{n_q} \in \mathbb{N}$ and $k_0 \in \mathbb{Z}$ such that $q = \sum_{i=0}^{n_q} \frac{k_i}{a_i}$ and $0 \le k_i < q_i = a_i/a_{i-1}$ for all $i \in \{1, \dots, n_q\}$.

Proof. (a) Fix $q = \frac{s}{t} \in \mathbb{Q}$, where $(s,t) = 1$ and $0 < t = p_1^{k_1} \dots p_m^{k_m}$. Since $(a_n)_{n\in\mathbb{N}}$ is a geometric a-sequence, for every $i = 1, 2, \dots, m$ there exists a minimal $n_i \in \mathbb{N}$ such that $p_i^{k_i} \mid a_{n_i}$. Let $n_q = \max\{n_i : 1 \le i \le m\}$, then $q \in \langle 1/a_{n_q} \rangle$ and n_q is minimal with this property.

(b) We proceed by induction on n_q. If $n_q = 0$ then $q \in \mathbb{Z}$ and the statement is known to be true. Fix $n \in \mathbb{N}$ and suppose that we already proved the statement for all the $q \in \mathbb{Q}$ such that $n_q \le n$. Fix $q = \frac{s}{a_{n+1}} \in \langle \frac{1}{a_{n+1}} \rangle$ such that $n_q = n+1$, then there are unique $0 \le k_{n+1} < q_{n+1}$ and $s' \in \mathbb{Z}$ such that $s = k_{n+1} + s'q_{n+1}$ and so, by inductive hypothesis,

$$q = \frac{k_{n+1}}{a_{n+1}} + \frac{s'}{a_n} = \frac{k_{n+1}}{a_{n+1}} + \sum_{i=0}^{n} \frac{k_i}{a_i} = \sum_{i=0}^{n+1} \frac{k_i}{a_i}. \qquad \square$$

In order to prove Theorem 1.18, we start with the following fact.

Claim 5.5. *Let $(a_n)_{n\in\mathbb{N}}$ be a geometric a-sequence and $(c_n)_{n\in\mathbb{N}}$ an a-sequence, then the sequence $(F_n)_{n\in\mathbb{N}}$, with $F_0 = \{0\}$ and $F_n = \langle \frac{1}{a_n} \rangle \cap [0, c_n)$ for every $n \in \mathbb{N}_+$, is a Følner sequence of $\mathbb{Q}$.*

Proof. Fix $0 \le q = \frac{s}{t} \in \mathbb{Q}$. There exists n' such that $q < c_{n'}$ and by Lemma 5.4 there is n_q such that $\frac{1}{t} \in F_{n_q}$. Let $\bar{n} = \max\{n_q, n'\}$, then $q \in F_{\bar{n}}$. Note that $F_{\bar{n}+k}$ can be covered by $c_{\bar{n}+k}/c_{\bar{n}}$ disjoint sets of the form

$$F_{\bar{n}+k} \cap [lc_{\bar{n}}, (l+1)c_{\bar{n}}) = \left\langle \frac{1}{a_{\bar{n}+k}} \right\rangle \cap [lc_{\bar{n}}, (l+1)c_{\bar{n}}),$$

where $l \in [0, c_{\bar{n}+k}/c_{\bar{n}} - 1]$. In each of these sets there are exactly $|F_{\bar{n}+k}|\, c_{\bar{n}}/c_{\bar{n}+k}$ elements. If we translate $F_{\bar{n}+k}$ by $c_{\bar{n}}$, all these sets, except the last one, are shifted exactly in the next one. So,

$$(c_{\bar{n}} + F_{\bar{n}+k}) \setminus F_{\bar{n}+k} = \left(\left\langle \frac{1}{a_{\bar{n}+k}} \right\rangle \cap [c_{k+\bar{n}} - c_{\bar{n}}, c_{k+\bar{n}})\right) + c_{\bar{n}},$$

and then

$$|(c_{\bar{n}} + F_{\bar{n}+k}) \setminus F_{\bar{n}+k}| = \frac{|F_{\bar{n}+k}|\, c_{\bar{n}}}{c_{\bar{n}+k}}. \tag{5.1}$$

Moreover $q < c_{\bar{n}}$, so $q \in \langle \frac{1}{a_{\bar{n}+k}} \rangle$ for all $k \in \mathbb{N}$. Thus, $(q + F_{\bar{n}+k}) \cup F_{\bar{n}+k} \subseteq (c_{\bar{n}} + F_{\bar{n}+k}) \cup F_{\bar{n}+k}$, and this implies

$$(q + F_{\bar{n}+k}) \setminus F_{\bar{n}+k} \subseteq (c_{\bar{n}} + F_{\bar{n}+k}) \setminus F_{\bar{n}+k}. \tag{5.2}$$

By (5.1) and (5.2) we obtain that

$$|(q + F_{\bar{n}+k}) \setminus F_{\bar{n}+k}| \leq |(c_{\bar{n}} + F_{\bar{n}+k}) \setminus F_{\bar{n}+k}| = \frac{|F_{\bar{n}+k}|\, c_{\bar{n}}}{c_{\bar{n}+k}},$$

and so

$$\lim_{n\to\infty} \frac{|(q + F_n) \setminus F_n|}{|F_n|} = \lim_{k\to\infty} \frac{|(q + F_{\bar{n}+k}) \setminus F_{\bar{n}+k}|}{|F_{\bar{n}+k}|} \leq \lim_{k\to\infty} \frac{c_{\bar{n}}}{c_{\bar{n}+k}} = 0.$$

If we consider instead a negative $q \in \mathbb{Q}$, we can proceed in a similar way. We find $\bar{n} \in \mathbb{N}$ as before but using $-q$ and then we have

$$(F_{\bar{n}+k} - c_{\bar{n}}) \setminus F_{\bar{n}+k} = \left(\left\langle \frac{1}{a_{\bar{n}+k}} \right\rangle \cap [0, c_{\bar{n}})\right) - c_{\bar{n}}.$$

As before we conclude that

$$|(q + F_{\bar{n}+k}) \setminus F_{\bar{n}+k}| \leq |(F_{\bar{n}+k} - c_{\bar{n}}) \setminus F_{\bar{n}+k}| = \frac{|F_{\bar{n}+k}|\, c_{\bar{n}}}{c_{\bar{n}+k}},$$

and then also in this case

$$\lim_{n\to\infty} \frac{|(q + F_n) \setminus F_n|}{|F_n|} \leq \lim_{k\to\infty} \frac{c_{\bar{n}}}{c_{\bar{n}+k}} = 0.$$

Therefore $(F_n)_{n\in\mathbb{N}}$ is a Følner sequence of $\mathbb{Q}$. □

One can verify that the above Følner sequence of $\mathbb{Q}$ is also locally monotileable (see [41]). This is written in details in the next proof in a particular case (but in a more general setting), where one should take $q = 1$.

Proof of Theorem 1.18. Each automorphism ϕ of the group $(\mathbb{Q}, +)$ is of the form $\phi_q : x \mapsto qx$ for some $q \in \mathbb{Q} \setminus \{0\}$. Fix $q = a/b \in \mathbb{Q} \setminus \{0\}$, with $(a, b) = 1$, $a \in \mathbb{Z} \setminus \{0\}$ and $b \in \mathbb{N} \setminus \{0\}$. For every $n \in \mathbb{N}$, we define E_n by

$$E_n = \left\langle \frac{1}{|a|^n b^n (n!)} \right\rangle \cap [0, 2^n |a|^n b^n).$$

First $(E_n)_{n\in\mathbb{N}}$ is a Følner sequence of $\mathbb{Q}$, since $(|a|^n b^n (n!))_{n\in\mathbb{N}}$ is a geometric a-sequence and $(2^n |a|^n b^n)_{n\in\mathbb{N}}$ is an a-sequence, and so Claim 5.5 applies.

Now we verify that $(E_n)_{n\in\mathbb{N}}$ is ϕ_q-monotileable, To this end, fix $n \in \mathbb{N}$. If $a = |a| > 0$, then

$$\phi_q(E_n) = qE_n = \left\langle \frac{1}{b^{n+1} |a|^{n-1} (n!)} \right\rangle \cap [0, 2^n |a|^{n+1} b^{n-1}).$$

We note that

$$E_{n+1} = \bigsqcup_{j=0}^{2b^2} 2^n |a|^{n+1} b^{n-1} j + \left(E_{n+1} \cap [0, 2^n |a|^{n+1} b^{n-1})\right). \tag{5.3}$$

Moreover,

$$E_{n+1} \cap [0, 2^n |a|^{n+1} b^{n-1}) = \bigsqcup_{i=0}^{|a|^2(n+1)-1} \frac{i}{|a|^{n+1} b^{n+1} (n+1)!} + \phi_q(E_n) \tag{5.4}$$

Combining (5.3) and (5.4), we obtain

$$E_{n+1} = \bigsqcup_{j=0}^{2b^2-1} \left(2^n |a|^{n+1} b^{n-1} j + \left(\bigsqcup_{i=0}^{|a|^2(n+1)-1} \frac{i}{|a|^{n+1} b^{n+1} (n+1)!} + \phi_q(E_n)\right)\right). \tag{5.5}$$

Define

$$\bar{E}^+_{n+1} = \left\{2^n |a|^{n+1} b^{n-1} j + \frac{i}{|a|^{n+1} b^{n+1} (n+1)!} : 0 \leq i < |a|^2 (n+1) - 1,\ 0 \leq j < 2b^2 - 1\right\}.$$

By Lemma 3.8 and (5.5),

$$E_{n+1} = \bigsqcup_{\bar{e} \in \bar{E}^+_{n+1}} \bar{e} + \phi_q(E_n), \tag{5.6}$$

i.e., $\phi_q(E_n)$ is a monotile of E_{n+1}. Exactly in the same way we obtain that $\phi_q^{-1}(E_n)$ is a monotile of E_{n+1}.

If $a < 0$, then

$$\phi_q(E_n) = qE_n = \left\langle \frac{1}{|a|^{n-1}b^{n+1}(n!)} \right\rangle \cap (-2^n|a|^{n+1}b^{n-1}, 0] \quad \text{and} \quad \phi_{-q}(E_n) = 2^n|a|^{n+1}b^{n-1} - \frac{1}{|a|^{n-1}b^{n+1}(n!)} + \phi_q(E_n).$$

Since $a < 0$, clearly $-q > 0$, and so (5.6) yields

$$E_{n+1} = \bigsqcup_{\bar{e}\in \bar{E}^+_{n+1}} \bar{e} + \phi_{-q}(E_n) = \bigsqcup_{\bar{e}\in \bar{E}^+_{n+1}} \left(\bar{e} + 2^n|a|^{n+1}b^{n-1} - \frac{1}{|a|^{n-1}b^{n+1}(n!)} + \phi_q(E_n)\right). \tag{5.7}$$

Define

$$\bar{E}^-_{n+1} = 2^n|a|^{n+1}b^{n-1} - \frac{1}{|a|^{n-1}b^{n+1}(n!)} + \bar{E}^+_{n+1}.$$

By (5.7) we have $E_{n+1} = \bigsqcup_{\bar{e}\in \bar{E}^-_{n+1}} \bar{e} + \phi_q(E_n)$, and so $\phi_q(E_n)$ is a monotile of E_{n+1}. In the same way we find that $\phi_q^{-1}(E_n)$ is a monotile of E_{n+1}. Hence, $(E_n)_{n\in\mathbb{N}}$ is ϕ_q-monotileable.

To conclude, apply Corollary 5.3. □

Given a finitely generated subgroup K of $\mathrm{Aut}(\mathbb{Q})$, one can prove that $\mathbb{Q} \rtimes K$ is locally monotileable. Suppose that K is generated by $\{\frac{a_1}{b_1}, \dots, \frac{a_m}{b_m}\}$. Consider the sequence $(E_n)_{n\in\mathbb{N}}$ given by

$$E_n = \left\langle \frac{1}{(|a_1 \dots a_m|\, b_1 \dots b_m)^n(n!)} \right\rangle \cap [0, (2^n|a_1 \dots a_m|^n(b_1 \dots b_m)^n).$$

Proceeding as in the proof of Theorem 1.18 one could prove that $(E_n)_{n\in\mathbb{N}}$ is K-monotileable and then apply Theorem 5.2.

5.2 Examples of locally monotileable groups that are not virtually nilpotent

Here we show that the general Extension Theorem can be proved for extensions

$$0 \to H \xrightarrow{\iota} G \xrightarrow{\pi} K \to 0 \tag{5.8}$$

when H has a property a bit stronger than local finiteness.

Corollary 5.6. *Suppose that in the short exact sequence* (5.8) *the group H has the property that every finite subset of H is contained in a finite characteristic subgroup of H. If K is locally monotileable, then G is locally monotileable as well.*

Proof. Clearly, H is locally finite, hence locally monotileable by Proposition 3.5. Moreover, there exists an exhausting increasing sequence $(E_n)_{n\in\mathbb{N}}$ of finite characteristic subgroups of H. In particular, E_n is invariant under conjugations in G. So $(E_n)_{n\in\mathbb{N}}$ is an $\mathrm{Aut}(G)$-monotileable right Følner sequence of G. Therefore Claim 4.10 gives that G is locally monotileable. □

This corollary provides some new examples of locally monotileable groups.

Example 5.7. Take an infinite collection $\{S_n : n \in \mathbb{N}\}$ of simple finite groups such that for $n \neq m$ the only homomorphism $S_n \to S_m$ is the trivial one and $\exp(S_n)$ is not bounded. Let $H = \bigoplus_{n\in\mathbb{N}} S_n$ be as in Example 4.6(a). Then H satisfies the hypotheses of Corollary 5.6. In particular, H is locally finite, hence H is locally monotileable by Proposition 3.5.

Let us see that for every residually finite countable abelian group K one can define an appropriate faithful action θ of K on H, such that $H \rtimes_\theta K$ is locally monotileable. Indeed, K can be identified with a subgroup of $P = \prod_{m\geq 2} \mathbb{Z}(m)$. Since $\mathrm{Aut}(H) = \prod_{n\in\mathbb{N}} \mathrm{Aut}(S_n)$, each $\mathrm{Aut}(S_n)$ contains a copy of S_n, hence the orders of these groups are not bounded. Therefore, the product $\mathrm{Aut}(H)$ contains a subgroup isomorphic to P. In particular, P (hence, $\mathrm{Aut}(H)$ as well) contains an isomorphic copy of the group K, which gives rise to a faithful action θ of K on H. Since the group K is locally monotileable by Corollary 4.20, we deduce that $G = H \rtimes_\theta K$ is locally monotileableby Corollary 5.6.

For example, we can just pick a non-torsion element $\phi \in P$ and put $K = \langle\phi\rangle \cong \mathbb{Z}$; then $G = H \rtimes K \in \mathfrak{M}$. Note that G is neither locally finite, nor virtually solvable, nor residually solvable.

Example 5.8. We build here examples of locally monotileable amenable groups that are neither virtually nilpotent nor residually finite.

(a) Suppose that in the short exact sequence (5.8) the group H is abelian and satisfies the descending chain condition on subgroups. If K is locally monotileable, then G is locally monotileable as well. Indeed, it is well known that the hypothesis on H implies that $H \cong F \oplus \bigoplus_{i=1}^k \mathbb{Z}(p_i^\infty)$, where $p_1, \dots, p_k$ are not necessarily distinct primes (see [22]). Then for every $n \in \mathbb{N}$ the subgroup $H[n] = \{h \in H : nh = 0\}$ is finite and fully invariant. Since every finite subset of H is contained in some of the subgroups $H[n]$, the group H satisfies the hypotheses of Corollary 5.6. Hence, G is locally monotileable.

(b) Take as in Example 4.6(b) $H = \bigoplus_{i=1}^k \mathbb{Z}(p_i^\infty)$, where $p_1, \dots, p_k$ are distinct primes. Then $K^* = \mathrm{Aut}(H) \cong \prod_{i=1}^k U(\mathbb{J}_{p_i})$, where $U(\mathbb{J}_{p_i})$ is the group of units of the ring $\mathbb{J}_{p_i}$, and in particular $K^* = \mathrm{Aut}(H)$ is abelian. So any countable subgroup K of K^* is locally monotileable. Now consider the semidirect product $G = H \rtimes K$, where the action of K is that induced by the natural one of $\mathrm{Aut}(H)$ on H. By item (a) G is locally monotileable.

(c) Now take for simplicity $k = 1$ and $p = p_1 > 2$ in item (b), so $H = \mathbb{Z}(p^\infty)$ and $K^* = U(\mathbb{J}_p)$. Since $p > 2$, we can choose $K \not\subseteq 1 + p\mathbb{J}_p$, and so the natural action of K on H is fixed-point free. Therefore, $G = H \rtimes K$ is center-free, so G is locally monotileable and G is not nilpotent.

To see that G is not virtually nilpotent consider a finite index subgroup N of G contained in $K_1 = (1 + p^k\mathbb{J}_p) \cap K$. It is enough to show that N is not nilpotent. Since H is divisible, we deduce that $H \leq N$. Hence, $N = H \rtimes (N \cap K)$ and $m = [K : (N \cap K)] < \infty$, so $N \subseteq mK$. In particular, if $m = p^l m_1$, with $l \in \mathbb{N}$ and $(m_1, p) = 1$, then there exists $\xi \in N \cap (1 + p^l\mathbb{J}_p)$. This implies that $Z(N) = Z_1(N) = \mathbb{Z}(p^l) \times \{1\}$. Arguing by induction one can see that $Z_s(N) = Z_1(N) = \mathbb{Z}(p^{ls}) \times \{1\}$. Hence, $N \neq Z_s(N)$ for every $s \in \mathbb{N}$. Thus N is not nilpotent. Since $H = \bigcup_{s\in\mathbb{N}} Z_s(N)$, N is hypercentral. Therefore, G is virtually hypercentral, yet not virtually nilpotent.

Finally, it is easy to see that G is not residually finite, as H is a non-trivial divisible subgroup of G.

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