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Some Contributions to Generalised Descriptive Set Theory and to the Study of the Interplay between Modal Logic and Set Theory

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Introduction

This dissertation is divided into two main parts. The first part, which brings together two separate projects, contributes to the field of generalised descriptive set theory (GDST). The second part explores the relationship between modal logic and set theory. Chapter 1 is joint work with Vincenzo Dimonte and Sandra Müller. Chapter 2 is the result of joint research with Claudio Agostini and Vincenzo Dimonte. Finally, Chapter 3 presents work developed in collaboration with Juan P. Aguilera and Grigorii Stepanov.

The first part of this work focuses primarily on the topic of (generalised) regularity properties. In particular, in Chapter 1 we prove that, given a strong limit cardinal λ of countable cofinality, if every λ-coanalytic subset of the generalized Cantor space ${}^{\lambda}2$ has the λ-PSP, a straigthforward generalisation of the Perfect Set Property, then $0^{\dagger}$ exists. To prove this, we raise this lower bound, starting from the existence of an inner model with a single measurable cardinal and building up to the mentioned stronger assumption. Our analysis makes use of the inner model theory and the covering properties of the Dodd-Jensen core model and $L[U]$. Perhaps the main contribution of this work to the field of generalised descriptive set theory is not the specific mentioned result, but rather the analysis of sets of codes in ${}^{\lambda}2$ for mice in the Dodd-Jensen model and $L[U]$, along with a study of their (generalised) descriptive complexity. In Chapter 2, we investigate whether a generalised notion of measure can exist in the generalised Baire space, ultimately arriving at a negative conclusion: by assuming very mild assumptions and adopting an "umbrella definition" that encompasses all possible natural generalisations of classical measures, we conclude that no non-trivial and continuous generalised measure satisfying reasonable structural axioms can exist. In Chapter 3, we study the interplay between modal logic and set theory. Specifically, we consider two modal operators, $\Box$ and $\blacksquare$, interpreted respectively as "*it holds in every forcing extension*" and "*it holds in every ground*". We propose a modal logic to characterise this interpretation, characterise it in terms of Kripke frames and prove that it is indeed sound and complete with respect to these semantics. We also briefly discuss steps towards developing a similar framework where $\blacksquare$ is instead interpreted as "*it holds in every inner model*".

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I thank my parents, and dedicate this work to Fernando Montaner Frutos.

[1] After receiving the review reports for this thesis, I learned that Philipp Lücke was one of the reviewers. I would like to thank him and the other reviewer, Xhiangui Shi, for their valuable comments and suggestions.

I

Contributions to generalised descriptive set theory

1

The λ-PSP at λ-coanalytic sets

This chapter is based on [BDM25], a joint work with Vincenzo Dimonte and Sandra Müller.

1.1 Introduction

In recent years, the study of higher Baire and Cantor spaces has been subject of growing interest, although attempts to extend classical descriptive set theory (DST) to a wider context by varying the space under consideration and adapting the classical notions can be traced back to the 60's. For instance, in [Sto62], Stone considered countable products of discrete spaces of any size as a plausible generalised notion of the Baire space; in the 70's, Vaught ([Vau74]) and, later in the 90's, Mekler and Väänänen ([MV93]) would study definable sets in the spaces ${}^{\kappa}2$ and ${}^{\omega_1}\omega_1$ endowed with the bounded topology, respectively. More recently, Friedman, Hyttinen and Kulikov ([FHK14]), have studied Borel reducibility of Borel equivalence relations on the generalied Baire space ${}^{\kappa}\kappa$ with κ an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$; and Andretta and Motto Ros ([AMR22]) have extended classical results on Souslin quasi-orders to generalised Baire spaces. The framework has been further developed by Galeotti, Agostini, Motto Ros and Schilicht to include topological spaces of a more general form than ${}^{\kappa}\kappa$, showing that it is possible to introduce a suitable generalization of metrics in this context and that the resulting theory remains well-behaved ([Gal19, AMRS23]).

To date, most of the literature in the study of higher Baire and Cantor spaces has focused on versions of those spaces in which, roughly speaking, the role of ω is played by an uncountable regular cardinal. In this chapter, however, singular cardinals of countable cofinality take on that role. Mainly motivated by the works of Cramer, Shi and Woodin on the set-theoretic descriptive properties of $L(V_{\lambda+1})$ under the very large cardinal assumption $I0(\lambda)$, similar to those that $L(\mathbb{R})$ exhibits under $\mathsf{AD}^{L(\mathbb{R})}$, the systematic study of generalised descriptive set theory (GDST, from now on) at singular cardinals of countable cofinality has been initially carried out by Dimonte and Motto Ros in [DMRon], and has been lately enriched by works of Dimonte, Lücke, Iannella, Poveda and Thei ([DIL23],[DPT24]).

As in classical descriptive set theory, one of the mayor themes in GDST is the study of the so-called (generalised) regularity properties, to which this chapter contributes by providing a consistency strength lower bound to the satisfaction of the λ-PSP, a natural generalisation of the Perfect Set Property, at the lowest level in the λ-projective hierarchy at which it consistently fails. Recall that a subset A of a Polish space X has the PSP if it is either countable or ${}^{\omega}2$ embeds into A. Its generalised variant asserts that a subset A of a λ-Polish space X has the λ-PSP if either it is of size less than or equal to λ or ${}^{\lambda}2$ embeds into A as a closed-in-X set. It is a well known result of Souslin that all analytic sets have the PSP. In the generalised setting at singular cardinals of countable cofinality, Dimonte and Motto Ros have proved that all λ-analytic sets have the λ-PSP as well. Again, just as in the classical case where it is no longer a theorem of ZFC (assumed its consistency) that all coanalytic sets of reals have the PSP, it is neither a theorem of ZFC alone that all λ-coanalytic sets of λ-reals have the λ-PSP. Indeed, on the one hand, recent work by Dimonte, Poveda and Thei shows that, assuming a $<\theta$-supercompact cardinal λ with $\lambda<\theta$ and θ inaccessible, there exists a model of ZFC where λ is a strong limit cardinal of countable cofinality and every set in $\mathcal{P}({}^{\omega}\lambda)\cap L(V_{\lambda+1})$ has the λ-PSP ([DPT24, Main Theorem 1]); on the other hand, Dimonte and Motto Ros have proved that in L there is a λ-coanalytic subset of ${}^{\lambda}2$ without the λ-PSP. More generally, they show that if $(\lambda^+)^L=\lambda^+$, then there is a λ-coanalytic set without the λ-PSP ([DMRon, Theorem 7.2.12]). The covering properties of L thus serve to fix the lower bound of all λ-coanalytic sets of λ-reals having the λ-PSP at $0^\sharp$ ([DMRon, Corollary 7.2.13.]).

In [DMRon], the authors already noted that if there is a strong limit cardinal λ of countable cofinality such that all λ-coanalytic subsets of ${}^{\lambda}2$ have the λ-PSP, then $a^\sharp$ exists for every $a\in{}^{\lambda}2$. However, this was the furthest they could push the lower bound. By considering core models beyond $0^\sharp$, one can, however, raise the lower bound further in the large cardinal hierarchy. This is what we do in this chapter. By adapting their arguments, one can in fact prove that if the Dodd-Jensen core model K^{DJ}, the core model below a measurable, computes λ^+ correctly, then there is a λ-coanalytic subset of ${}^{\lambda}2$ in K^{DJ} without the λ-PSP. The covering properties of K^{DJ} lead to a contradiction under the assumption that, in V, all λ-coanalytic subsets of ${}^{\lambda}2$ have the λ-PSP, thus yielding the conclusion that if this assumption holds, then there must exist an inner model with a measurable cardinal. These are Theorem 1.4.9 and Corollary 1.3.24.

The natural (or easier) next step is to move to the inner model $L[U]$. The arguments involved in the proof of Theorem 1.4.9 require only minor adaptations, but one has to deal with the particularities of $L[U]$ regarding its covering properties and consider generic Prikry extensions. Once these obstacles are resolved, we prove that if there is a strong limit cardinal λ with countable cofinality such that all λ-coanalytic subsets of ${}^{\lambda}2$ have the λ-PSP, then $0^\dagger$ exists:

Theorem 1.4.9 *If there is a strong limit cardinal λ of countable cofinality such that all λ-coanalytic subsets of ${}^{\lambda}2$ have the λ-PSP, then $0^\dagger$ exists.*

The chapter is organised as follows. In Section 1.2, we recall the basics of GDST at singular cardinals of countable cofinality and review all necessary background results. This section also includes a toy example which, although not framed in a descriptive set-theoretic context, illustrates how core models can be used to obtain consistency strength lower bounds. Sections 1.3 and 1.4 deal with inner model theoretic notions and we try to provide as much detail as possible. In Section 1.3, we study codes for Dodd-Jensen premice and mice of size λ and obtain their (generalised) descriptive complexity. Section 1.4 extend our results to $L[U]$. In Section 1.5, we conclude with some final remarks and open questions.

1.2 Preliminaries

The reader of the material of this chapter may be unfamiliar to most of the literature in GDST at singular cardinals of countable cofinality. Within the first two subsections we try to remediate this. The third subsection provides the reader with a toy example on how core model theory is used for obtaining consistency strength lower bounds.

1.2.1 GDST at singular cardinals of countable cofinality

Let λ be an infinite cardinal and X be a λ-Polish space, i.e., a completely metrizable topological space with weight $wt(X)\leq\lambda$[1]:

Definition 1.2.1 ([DMRon, Definition 7.0.1]). A set $A\subseteq X$ has the λ-*Perfect Set Property* (or λ-PSP) if either $|A|\leq\lambda$ or ${}^{\lambda}2$ embeds into A as a closed-in-X set.

[1]The weight of a topological space X is κ if every dense subset Y of X is of size greater than or equal to κ.

In this chapter we focus on the case in which X is the *generalised Cantor space* ${}^\lambda 2$. If λ is a strong limit cardinal of countable cofinality, then ${}^\lambda 2$ endowed with the bounded topology is a λ-Polish space (see [AMR22, Proposition 3.12 (g,h)]).

Proviso. Throughout this chapter and unless otherwise specified, λ is assumed to be a strong limit cardinal of countable cofinality.

Given a topological space (X, τ), a set $B \subseteq X$ is λ^+-*Borel* if it belongs to the λ^+-algebra λ^+**-Bor(X)** generated by the open sets of X. When λ is singular, we can drop the $+$ from the above. If X is a λ-Polish space, a set $A \subseteq X$ is λ-*analytic* if it is the continuous image of some λ-Polish space Y, and it is λ-*coanalytic* if it is the complement of a λ-analytic set. The class of all λ-analytic (resp. λ-coanalytic) subsets of X is denoted by λ-$\boldsymbol{\Sigma}^1_1(X)$ (resp. λ-$\boldsymbol{\Pi}^1_1(X)$). Subsets of X that are both λ-analytic and λ-coanalytic are λ-*bianalytic*. The class of all λ-bianalytic subsets of X is denoted by λ-$\boldsymbol{\Delta}^1_1(X)$. These classes can be recursively extended. Indeed, for each $n \geq 1$, λ-$\boldsymbol{\Sigma}^1_{n+1}(X)$ denotes the set of those subsets A of X for which there is some λ-Polish space Y such that there exists a continuous function $f : Y \longrightarrow X$ and a λ-$\boldsymbol{\Sigma}^1_n(X)$-subset B of Y such that $f(B) = A$. Then, for each $n \geq 1$, one defines λ-$\boldsymbol{\Pi}^1_n(X) = \{A \subseteq X : X - A \in \lambda\text{-}\boldsymbol{\Sigma}^1_n\}$ and λ-$\boldsymbol{\Delta}^1_n(X) = \lambda$-$\boldsymbol{\Sigma}^1_n(X) \cap \lambda$-$\boldsymbol{\Pi}^1_n(X)$. If there is some $n < \omega$ such that $A \subseteq X$ belongs to some λ-$\boldsymbol{\Sigma}^1_n(X)$, then A is λ-*projective*. The class of λ-projective sets is the following:

$$\bigcup_{n\geq 1} \lambda\text{-}\boldsymbol{\Sigma}^1_n = \bigcup_{n\geq 1} \lambda\text{-}\boldsymbol{\Pi}^1_n = \bigcup_{n\geq 1} \lambda\text{-}\boldsymbol{\Delta}^1_n.$$

From now on, we work in the language of set theory with first and second order variables. If Q is either $\exists$ or $\forall$, Q^0 (sometimes Q alone, if there is no danger of confusion) denotes that the quantifier ranges over first order variables. By Q^1 we mean that the quantifier ranges over second order variables. Similarly, by x^0 we mean that the variable x is a first order variable, while x^1 means that x is a second order variable. This notation for variables will be ignored most of the times, though, for saying $\exists^1 x$ is immediately understood as $\exists^1 x^1$.

A formula φ is Δ^0_0 if it contains neither first nor second order quantifiers. In turn, a formula $\tilde{\varphi}$ is Π^0_1 if it is of the form $\forall x\varphi(x)$ with $\varphi(x)$ a Δ^0_0-formula. Then, recursively, we say that a formula φ is Σ^0_{n+1} with $n \in \omega$ if it is of the form $\exists x\psi(x)$, where $\psi(x)$ is Π^0_n and, analogously, a formula φ is Π^0_{n+1} if it is of the form $\forall x\psi(x)$, where $\psi(x)$ is Σ^0_n. If a formula is both Σ^0_n and Π^0_n, we say that it is a Δ^0_n formula. A formula φ is Σ^1_0 if it does not contain second order quantifiers. In turn, a formula $\tilde{\varphi}$ is Π^1_1 if it is of the form $\forall^1 x\varphi(x)$ with $\varphi(x)$ a Σ^1_0-formula. Then, recursively, we say that a formula φ is Σ^1_{n+1} with $n \in \omega$ if it is of the form $\exists^1 x\psi(x)$, where $\psi(x)$ is Π^1_n. Analogously, a formula is Π^1_{n+1} if it is of the form $\forall^1 x\psi(x)$, where $\psi(x)$ is Σ^1_n. Again, if φ is both Π^1_n and Σ^1_n for some $n \in \omega$, we say that φ is Δ^1_n.

λ-Laver-arithmetical and λ-Laver-analytical sets. Here we provide a (kind of) generalised effective (lightface) hierarchy of sets. Our approach is based on the one in [Ianed]. We use the model $\mathcal{A} := \langle V_\lambda, V_{\lambda+1}\rangle$ to interpret our formulas.

Let X be a λ-Polish space definable with a parameter a in $V_{\lambda+1}$ and let $x \in X$. For every $n \in \omega$, a set $A \subseteq X$ is λ-$\Sigma^1_n(a, x)$ in X if $A = \{y \in V_{\lambda+1} : \mathcal{A} \models (y \in X \wedge \varphi(x, y))\}$ with φ a Σ^1_n-formula. We define similarly λ-$\Pi^1_n(a, x)$ sets. We usually drop the a whenever $a = \emptyset$. Of course, the descriptive complexity of A depends on the complexity of the λ-Polish space X. For instance, the generalised Cantor space is definable in $V_{\lambda+1}$ by a $\Sigma^0_0(\emptyset)$-formula, so it does not add complexity to these definitions. Indeed, $x \in {}^\lambda 2$ if and only if $\forall y \in x \exists\alpha \in \lambda(y = \langle\alpha, 0\rangle \vee y = \langle\alpha, 1\rangle)$, where λ can be defined as the greatest ordinal in $V_{\lambda+1}$, something expressible by a Σ^0_0-formula. Sets that are λ-Σ^0_n or λ-Π^0_n-definable are λ-Laver-arithmetical, sets that are λ-Σ^1_n or λ-Π^1_n-definable are λ-Laver-analytical. The introduced hierarchy does not correspond to the *exact generalisation*[2]

[2]Of course, the problematic word here isn't *exact* but *generalisation*.

of the classical effective (lightface) hierarchy, for no notion of general recursivity is involved in our definitions. However, they relate to the λ-Borel and λ-projective hierarchies as in the classical case. For a proof of the following, check [Kan08, Proposition 12.6] and [Ianed, Proposition 6.1.28].

Proposition 1.2.2. *Let λ be a singular cardinal of countable cofinality and suppose that $A \subseteq V_\lambda$. Then:*

(1) *$A \in \lambda\text{-}\mathbf{\Sigma}^0_{\boldsymbol{\alpha}}$ if and only if $A \in \lambda\text{-}\Sigma^0_\alpha(a)$ for some $a \in V_{\lambda+1}$, and similarly for $\lambda\text{-}\mathbf{\Pi}^0_{\boldsymbol{\alpha}}$.*

(2) *$A \in \lambda\text{-}\mathbf{\Sigma}^1_{\boldsymbol{n}}$ if and only if $A \in \lambda\text{-}\Sigma^1_n(a)$ for some $a \in V_{\lambda+1}$, and similarly for $\lambda\text{-}\mathbf{\Pi}^1_{\boldsymbol{n}}$.*

1.2.2 Coding structures on ${}^\lambda 2$

Every $z \in {}^\lambda 2$ encodes a binary relation E_z on λ given by $\langle \alpha, \beta \rangle \in E_z$ if and only if $z(\langle \alpha, \beta \rangle) = 0$, where $\langle \cdot, \cdot \rangle$ is an ordinal pairing function. We define:

$$\mathsf{WO}_\lambda := \{z \in {}^\lambda 2 : z \text{ codes a well-order on } \lambda\}.$$

Its descriptive complexity is given by the following:

Proposition 1.2.3 ([DMRon, Proposition 5.1.5])**.** *WO_λ is λ-coanalytic.*

From now on, if $z \in \mathsf{WO}_\lambda$ we write $<_z$ instead of E_z. It is easy to see that for each $\alpha < \lambda^+$ there is a $z \in \mathsf{WO}_\lambda$ such that $ot(<_x) = \alpha$. We note that every set of ordinals coded by elements in a λ-analytic subset of WO_λ is bounded in λ^+:

Lemma 1.2.4 (Boundedness Lemma [DMRon, Theorem 6.2.3.])**.** *Let λ be such that $2^{<\lambda} = \lambda$. If $A \subseteq \mathsf{WO}_\lambda$ is λ-analytic, then $\sup\{ot(<_x) : x \in A\} < \lambda^+$.*

Now, let $(M, \in)$ be a transitive model of size λ and let $\pi : M \to \lambda$ be a bijection. One can define the relation E_π on λ given by $E_\pi(\alpha, \beta)$ if and only if $\pi^{-1}(\alpha) \in \pi^{-1}(\beta)$. It is then clear that $\pi : (M, \in) \to (\lambda, E_\pi)$ is an isomorphism. In fact, $(M, \in)$ is the transitive collapse of (λ, E_π). By coding E_π with some $z \in {}^\lambda 2$ one has a code for $(M, \in)$. We denote by EW_λ to set of codes for extensional and well-founded structures. It is λ-coanalytic (see Lemma 1.2.7 (2) below) and if $z \in \mathsf{EW}_\lambda$, we write $\in_z$ instead of E_π.

In the classical case, when $\lambda = \omega$, if one is to check whether $\pi_z(m) = n$ for some $m, n \in \omega$, it is enough to find a $\in$-preserving bijection between the $\in_z$-predecessors of m and n. Since the $\in_z$-predecessors of m are bounded in ω, such bijection goes from some $m' < \omega$ to n. There are countable many of such bijections, so the set $\{z \in {}^\omega 2 : \pi_z(m) = n\}$ is ω-Borel. This is not the case anymore in the generalised context: it might happen that the set of $\in_z$-predecessors of some $\alpha < \lambda$ isn't bounded in λ, so we need a full function from λ to λ to check whether $\pi_z(\alpha) < \lambda$, and this goes beyond λ-Borelness[3]. We are of course interested in maintaining as low as possible the descriptive complexity of the sets involved in our computations. In order to do so, for the set of codes for extensional and well-founded structures we impose the boundedness condition:

Definition 1.2.5 ([DMRon, Definition 7.2.7])**.** Let $z \in {}^\lambda 2$ be a code for an extensional and well-founded structure. We say that $(\lambda, \in_z)$ has a *bounded* collapse if and only if for each $b \in H_\lambda$, $\pi_z^{-1}(b)$, if it exists, has boundedly many $\in_z$-predecessors, being π_z the corresponding collapse function.

We denote by BC_λ the set of codes for such bounded structures. It is safe to consider only codes for bounded, extensional and well-founded structures:

Proposition 1.2.6 ([DMRon, Lemma 7.2.9])**.** *For any transitive structure $(M, \in)$ of size λ there exists a $z \in \mathsf{BC}_\lambda$ such that $(M, \in) \cong (\lambda, \in_z)$.*

To finish, we list some useful facts:

[3] It is indeed λ-analytic. See [DMRon, Lemma 7.2.6]).

Lemma 1.2.7.

(1) *If φ is an $\mathcal{L}_\in$-formula*[4] *and $\alpha_1, \dots, \alpha_n \in \lambda$, then $\{z \in {}^\lambda 2 : (\lambda, \in_z) \vDash \varphi(\alpha_1, \dots, \alpha_n)\}$ is λ-Borel.*

(2) *The set $\{z \in {}^\lambda 2 : (\lambda, \in_z)$ is extensional and well-founded$\}$ is λ-coanalytic.*

(3) BC_λ *is λ-coanalytic.*

(4) *The set $\{(x, y) \in \mathsf{BC}_\lambda \times {}^\lambda 2 : y \in tr(\lambda, \in_x)\}$ is λ-Borel in $\mathsf{BC}_\lambda \times {}^\lambda 2$.*

(5) *The set $\{(x, y) \in {}^\lambda 2 \times {}^\lambda 2 : (\lambda, \in_x)$ and $(\lambda, \in_y)$ are extensional and $(\lambda, \in_x) \cong (\lambda, \in_y)\}$ is λ-analytic.*

Proof. See Lemma 7.2.1, corollaries 7.2.2 and 7.2.3, Lemma 7.2.8(1), Theorem 7.2.10(3), and Theorem 7.2.10(1) in [DMRon] for (1) (2), (3), (4) and (5), respectively. □

1.2.3 A toy example

Very roughly, a core model K is a definable structure that, under suitable anti-large cardinal assumptions, is very close to V in the sense that every uncountable set in V is covered by a set in K of the same size (see Definition 1.3.8). If K is a model of AC, this implies that $(\lambda^+)^K = \lambda^+$ for every singular cardinal λ. Core models are useful to fix consistency strength lower bounds. The strategy is quite standard. If one is to fix the consistency lower bound of an arbitrary statement φ, one first shows that $\neg\varphi$ holds in the core model. As said, under the corresponding anti-large cardinal assumptions, the core model is close enough to V. This closeness might serve to show that $\neg\varphi$ also holds in V. From this, assumed φ is true in V, one derives that it must be the case that the assumed anti-large cardinal assumption fails, for otherwise one would get a contradiction. For example, Corollary 1.2.9 below, consequence of Proposition 1.2.8, says that there is a Π_2-definable set A in L that does not have the λ-PSP. By the Covering Theorem for L, if $0^\sharp$ doesn't exist, then every uncountable set in V is covered by a set in L of the same size. This in turn implies, being L a model of choice, that if $0^\sharp$ doesn't exist, then L computes correctly the successor of every singular cardinal, which is enough to prove that, under that same assumption, the set A, as seen from V, is also a Π_2-definable subset of ${}^\lambda 2$ without the λ-PSP. It thus follows that if every Π_2-definable subset of ${}^\lambda 2$ in V has the λ-PSP, then $0^\sharp$ must exist, as written in Corollary 1.2.10.

Proposition 1.2.8. *Let K be an inner model with a Q_n-definable well order $\leq_K$ with $n \geq 1$, where Q either stands for Σ or for Π. Then, there is a Π_n-definable set $A \subseteq {}^\lambda 2$ in K without the λ-PSP.*

Proof. We work in K. Let $A_K = \{x \in \mathsf{WO}_\lambda : \forall y\,((\lambda, \in_x) \cong (\lambda, \in_y) \to x \leq_K y)\}$. The statement "$(\lambda, \in_x) \cong (\lambda, \in_y)$" can be expressed as a Σ_1-formula, say $\exists w \varphi(x, y, w)$, and "$x \leq_K y$" is by assumption Q_n-definable with $n \geq 1$, so let $\tilde{R}$ denote the dual quantifier of R and say that "$x \leq_K y$" is of the form

$$Rx_n \tilde{R} x_{n-1} R x_{n-2} \dots \psi(x, y, x_0, \dots, x_n).$$

The formula $x \in \mathsf{WO}_\lambda$ is Π_1, so it can be written as $\forall z_1 \theta(x, z_1)$ with $\theta(x, z_1)$ a Δ_0-formula. It then follows that

$$\begin{aligned} x \in A_K \leftrightarrow\, & \forall z_1 \forall y \forall w\, R x_n \tilde{R} x_{n-1} R x_{n-2} \dots \\ & (\theta(x, z_1) \wedge (\neg\varphi(x, y, w) \vee \psi(x, y, x_0, \dots, x_n))). \end{aligned}$$

Note that $\forall y$ dominates the quantifiers to its right. Let Ψ denote the formula to the right of the double implication symbol above. Recall that since every Σ_n-definable well order is Δ_n-definable,

[4] As usual, $\mathcal{L}_\in$ denotes the language of set theory.

we can assume that the well order $\leq_K$ is Π_n-definable. Thus, we can simply assume that $Q = \Pi$. Therefore, we have that the quantifiers in Ψ can be written as

$$\forall z_1 \forall y \forall w \forall x_n \exists x_{n-1} \ldots,$$

which implies that Ψ is Π_n.

Now, for every $\alpha < \lambda^+$ there is some $x \in {}^\lambda 2$ coding a well-order on λ of order type α. By definition, there is a unique code in A_K for each $\alpha < \lambda^+$, so $|A_K| = \lambda^+$. Therefore, if A_K had the λ-PSP, there would exist an embedding $\pi : {}^\lambda 2 \longrightarrow A_K$. Since $\pi[{}^\lambda 2]$ is λ-analytic and $|\pi[{}^\lambda 2]| \geq \lambda^+$, the set $\{ot(<_x) : x \in \pi[{}^\lambda 2]\}$ would be unbounded in λ^+. But this contradicts the *Boundedness Lemma*, so A_K does not have the λ-PSP. □

It is well known that L has a Σ_1-definable well-ordering. By Proposition 1.2.8:

Corollary 1.2.9. *In L, there is a Π_1-definable subset of ${}^\lambda 2$ without the λ-PSP*[5].

The definability of well-orderings can vary across different core models. For example, the Jensen-Steel core model without a measurable K admits a Σ_2-definable well-order (see [JS13]). In this context, Proposition 1.2.8 shows that if there is no transitive proper class model of ZFC + "there is a Woodin cardinal", then the Jensen-Steel core model K without the measurable has a Π_3-definable subset of ${}^\lambda 2$ that fails to have the λ-PSP. Further observations concerning the Dodd–Jensen core model will follow in subsequent sections.

Now, let K in the statement of Proposition 1.2.8 be L. We check whether the complexity of A_L changes when seen from V. A_L^V is the set

$$\{x \in \mathsf{WO}_\lambda \cap L : \forall y \in {}^\lambda 2 \cap L((\lambda, \in_x) \cong (\lambda, \in_y) \to x \leq_L y))^L\}.$$

First note that the relativization to L of $(\lambda, \in_x) \cong (\lambda, \in_y)$ and $x \leq_L y$ are still Σ_1. The formula $x \in \mathsf{WO}_\lambda \cap L$ is Σ_1. Indeed, $x \in \mathsf{WO}_\lambda \cap L$ if and only if $x \in \mathsf{WO}_\lambda$, which is Δ_1, and x is constructible, which is Σ_1 (see, e.g., [Dev17, Lemma 2.13]). We then write $x \in WO_\lambda \cap L$ as $\exists z_1 \theta(x, z_1)$ and $y \in L$ is Σ_1 as $\exists z_2 \sigma(y, z_2)$. Then A_L^V is defined by the formula:

$$\begin{aligned} x \in A_L^V \leftrightarrow & \forall y \forall z_2 \forall w \exists x_0 \exists z_1 \\ & (\theta(x, z_1) \wedge (\neg\sigma(y, z_2) \vee \neg\varphi(x, y, w) \vee \psi(x, y, x_0)))), \end{aligned}$$

which is Π_2.

Now assume that $0^\sharp$ doesn't exist, so that L has the weak covering property and computes correctly the successor of every singular cardinal. Then, by an argument as in the proof above we get that A_L^V doesn't have the λ-PSP. We have proved the following:

Corollary 1.2.10. *If $0^\sharp$ doesn't exist, there is a Π_2-definable subset of ${}^\lambda 2$ without the λ-PSP.*

In other words, a sufficient condition for the existence of $0^\sharp$ is that all Π_2-definable subsets of ${}^\lambda 2$ have the λ-PSP. It is easy to see that the same holds for $a^\sharp$ for every $a \in {}^\lambda 2$.

Remark 1.2.11. Proposition 1.2.8 and Corollary 1.2.9 hold for $\lambda = \omega$. The proof of Corollary 1.2.10 doesn't work if λ is countable because the Covering Lemma for L only applies when λ is greater than ω.

[5] In his review of the first version of this thesis, Philipp Lücke notes that since L admits a Δ_1-good well-ordering, this can be further reduced to the existence of a Δ_1-definable subset of ${}^\lambda 2$ without the λ-PSP. Recall that a well-order $\leq$ on a transitive model M of ZFC is said to be Δ_1-good if for every $x \in M$ the relation $z(x) := \{y : y \leq x\}$ is Δ_1. Now, let $\leq_L$ denote a Δ_1-good well-ordering in L and note that $x \in A_L$ can be expressed as

$$x \in A_L \leftrightarrow x \in \mathsf{WO}_\lambda \cap L \wedge \exists z(z = \{y \in {}^\lambda 2 : y \leq_L x\} \wedge z \in L \wedge (\forall y \in z((\lambda, \in_x) \ncong (\lambda, \in_y))^L,$$

which is easily seen to be Σ_1 (well-foundedness is Δ_1, for it is Π_1 via the non-existence of an infinite descending chain and Σ_1 via the existence of a ranking function). It then follows that A_L is Σ_1 too, hence Δ_1.

1.3 The λ-PSP up to the existence of an inner model with a measurable cardinal

In this section we prove that if there is a strong limit cardinal λ with countable cofinality such that all λ-coanalytic subsets of ${}^\lambda 2$ have the λ-PSP, then there is an inner model with a measurable cardinal. Here we introduce the Dodd-Jensen core model[6]. To get an intuitive idea of what Dodd-Jensen premice and mice are, the reader may want to take a look to the first paragraphs of the next section.

Definition 1.3.1. Let κ be an uncountable ordinal. M is a *Dodd-Jensen premouse at* κ if it is a structure $\mathcal{J}_\alpha^U$ of the form $(J_\alpha[U], \in, U)$ such that

$$\mathcal{J}_\alpha^U \models \text{“}U \text{ is a normal measure on } \kappa\text{”}.$$

The reader, if not very familiar with Jensen's fine hierarchy, may prefer to think of J_α simply as L_α and will still be able to follow without difficulties.

Definition 1.3.2. Given M a Dodd-Jensen premouse at κ, the *lower part of* M is $lp(M) = M \cap V_\kappa$.

Let M be a Dodd-Jensen premouse at an uncountable ordinal κ and δ be an ordinal. A *linear iteration of* M *of length* δ is a sequence $\langle M_\alpha, \pi_{\alpha,\beta} : \alpha \leq \beta < \delta\rangle$ where

(1) $M_0 = M$ and $\pi_{\alpha,\alpha} = id_{M_\alpha}$ for every $\alpha < \delta$,

(2) If $\alpha + 1 < \delta$, $M_\alpha = \mathcal{J}_{\eta_\alpha}^{U_\alpha}$ is a Dodd-Jensen premouse at κ_α and $M_{\alpha+1} = \mathcal{J}_{\eta_{\alpha+1}}^{U_{\alpha+1}}$ is a Dodd-Jensen premouse at $\kappa_{\alpha+1}$, then $M_{\alpha+1}$ is the transitive collapse of the ultrapower of M_α by U_α with $\pi_{\alpha,\alpha+1} : M_\alpha \longrightarrow M_{\alpha+1}$ the corresponding Σ_1-elementary embedding, $\pi_{\alpha,\alpha+1}(\kappa_\alpha) = \kappa_{\alpha+1}$ and
$$U_{\alpha+1} = \{[f]_{U_\alpha} : f \in {}^{\kappa_\alpha}M_\alpha \cap M_\alpha \wedge \{\delta < \kappa_\alpha : f(\delta) \in U_\alpha\} \in U_\alpha\}.$$

(3) If $\gamma < \delta$ is a limit ordinal, then M_γ together with the Σ_1-elementary embeddings $\pi_{\alpha,\gamma} : M_\alpha \longrightarrow M_\gamma$ for every $\alpha < \gamma$ is the direct limit of the directed system $\langle M_\alpha, \pi_{\alpha,\beta} : \alpha \leq \beta < \gamma\rangle$.

To secure the existence of the transitive collapse of an ultrapower we need it to be well-founded. If at step α of the iteration the resulting ultrapower isn't well-founded, the iteration halts and the Dodd-Jensen premouse doesn't have an iteration of length α.

Definition 1.3.3. A Dodd-Jensen premouse is *iterable* if it has a linear iteration of length δ for every ordinal δ. M is a *Dodd-Jensen mouse* if it is an iterable Dodd-Jensen premouse.

The iterability of Dodd-Jensen mice is witnessed by any transitive structure big enough to see the well-foundedness of the ω_1-first iterated ultrapowers:

Theorem 1.3.4 ([Koe88, Theorem 2.7])**.** *Let M be a Dodd-Jensen premouse and let H be a transitive model of a sufficiently large finite part of ZFC such that $M \in H$ and $\omega_1^V \subseteq H$. Then M is iterable if and only if $H \models$ “M is iterable”.*

Dodd-Jensen mice can be compared via its corresponding iterates, that is:

Theorem 1.3.5 ([Koe88, Theorem 2.12])**.** *For every two Dodd-Jensen mice M, N there are iterates M_δ and $N_{\delta'}$ such that either M_δ is an initial segment of $N_{\delta'}$ or vice versa.*

When a Dodd-Jensen mouse M is the initial segment of a Dodd-Jensen mouse N we write $M \trianglelefteq N$. We can define an equivalence relation given by $M \sim_\trianglelefteq N$ if and only if $M \trianglelefteq N$ or $N \trianglelefteq M$. The order $\trianglelefteq$ induces a well-ordering of the $\sim_\trianglelefteq$-equivalence classes (see [Koe88, Theorem 2.12]).

[6]For other recent applications of the Dodd–Jensen core model, see [BNL25].

Definition 1.3.6. The *Dodd-Jensen core model* is the structure

$$K^{\mathsf{DJ}} := L \cup \bigcup \{lp(M) : M \text{ is a Dodd-Jensen mouse}\}.$$

$0^\sharp$ can be seen as the first mouse. Then, if $0^\sharp$ doesn't exist, the Dodd-Jensen core model is L.

Theorem 1.3.7 (Dodd-Jensen)**.**

(1) *K^{DJ} admits a Σ_1-definable well-order.*

(2) *$K^{\mathsf{DJ}} \models \mathsf{GCH}$.*

The well-ordering of the Dodd-Jensen core model is given by

$$x \leq_{\mathsf{DJ}} y \leftrightarrow \exists M(M \text{ is a Dodd-Jensen mouse} \wedge x, y \in lp(M) \wedge M \models \text{“}x \leq_M y\text{”}),$$

where $\leq_M$ denotes the constructibility well-order of the mouse M, which is uniformly $\Sigma_1(M)$ (see [Koe88, p.186]).

Definition 1.3.8. An inner model M has the *full covering property* if for every uncountable set of ordinals X in V there is a $Y \in M$ such that $X \subseteq Y$ and $|Y| = |X|$. It has the *weak covering property* if it computes correctly the successor of singular cardinals, i.e., if for every singular cardinal λ, $(\lambda^+)^M = \lambda^+$.

It is a well known fact that if an inner model satisfying the axiom of choice has the full covering property, it also has the weak covering property.

Theorem 1.3.9 (Dodd-Jensen's Covering Theorem for K^{DJ})**.** *If there is no inner model with a measurable cardinal, then K^{DJ} has the full and weak covering properties.*

As a consequence of K^{DJ}'s weak covering and its well-ordering complexity, the following corollary of Proposition 1.2.8 already provides a higher consistency strength lower bound for all Π_2-definable subsets of the generalised Cantor space ${}^\lambda 2$ having the λ-PSP than the one given in Corollary 1.2.10:

Corollary 1.3.10. *If there is no inner model with a measurable cardinal, then there is a Π_2-definable subset of ${}^\lambda 2$ without the λ-PSP.*

Proof. Let $A^V_{K^{\mathsf{DJ}}}$ be the set

$$\{x \in \mathsf{WO}_\lambda \cap K^{\mathsf{DJ}} : \forall y \in K^{\mathsf{DJ}}((\lambda, \in_x) \cong (\lambda, \in_y) \to x \leq_{\mathsf{DJ}} y)^{K^{\mathsf{DJ}}}\}.$$

The proof goes exactly as the equivalent result for L in the first section. Here, we only have to prove that $x \in K^{\mathsf{DJ}}$ is Σ_1. But this is clear: $x \in K^{\mathsf{DJ}}$ if and only if there exists a Dodd-Jensen mouse M such that $x \in lp(M)$. □

We turn our attention to the descriptive version of this problem. As shown by Dimonte and Motto Ros, in L it is possible to build a λ-coanalytic set that doesn't have the λ-PSP via the set of $<_L$-minimal codes coding a well-ordering of order type α for each $\alpha < \lambda^+$. In this section we show that this set is still λ-coanalytic in the Dodd-Jensen core model K^{DJ}. A similar argument as the one in Theorem 1.2.8 shows that it does not have the λ-PSP in K^{DJ}, either. By the weak covering property of K^{DJ}, one can show the existence of such a set also in V under the anti-large cardinal assumption of the non-existence of an inner model with a measurable cardinal. We will need the following preliminary lemmas.

Lemma 1.3.11. *For every $x, y \in {}^\lambda 2 \cap K^{\mathsf{DJ}}$ there exists a Dodd-Jensen mouse M of size λ with $x, y \in lp(M)$.*

Proof. Let $x, y \in {}^{\lambda}2 \cap K^{\mathsf{DJ}}$. Since $x, y \in K^{\mathsf{DJ}}$, there exist two Dodd-Jensen mice M_x and M_y at κ_x and κ_y respectively such that $x \in lp(M_x)$ and $y \in lp(M_y)$. Let M'_x be the collapse of $Hull^{M_x}(\lambda \cup \{x, \kappa_x\})$ and let π_x^{-1} be the collapse function. By elementarity, M'_x is a Dodd-Jensen mouse at $\pi_x^{-1}(\kappa_x)$ of size λ and $x \in lp(M'_x)$. To see the latter, note that $rank^{M'_x}(x)$ is still below $rank^{M'_x}(\pi_x^{-1}(\kappa_x))$. Analogously, we let M'_y be the collapse of $Hull^{M_y}(\lambda \cup \{y, \kappa_y\})$ with π_y^{-1} the corresponding collapse function, so that M'_y is a Dodd-Jensen mouse at $\pi_y^{-1}(\kappa_y)$ of size λ with $y \in lp(M'_y)$. By Theorem 1.3.5, by iterating both M'_x and M'_y we get two Dodd-Jensen mice $\tilde{M}_x$ and $\tilde{M}_y$ such that, with no loss of generality, $\tilde{M}_x \trianglelefteq \tilde{M}_y$. The coiteration successfully removes all disagreements[7] in $\nu < \max(|M'_x|, |M'_y|)^+ = \lambda^+$ steps (see, e.g., the proof of [Ste16, Lemma 4.8]) and it also holds that $|\tilde{M}_i| = |M'_i||\nu|$ for $i \in \{x, y\}$. It follows that $|\tilde{M}_y| = \lambda$. Moreover, $x, y \in lp(\tilde{M}_y)$ because the lower parts of M'_x and M'_y remain unmoved throughout the iteration. To finish, we let $\tilde{M}_y$ to be our desired M. $\square$

Let now $x, y \in {}^{\lambda}2 \cap K^{\mathsf{DJ}}$ such that $x \leq_{\mathsf{DJ}} y$ and recall that $x \leq_{\mathsf{DJ}} y$ if and only if there exists a Dodd-Jensen mouse M such that $x, y \in lp(M)$ and $M \vDash$ "$x \leq_M y$". Arguing as in the previous lemma, the transitive collapse M' of $Hull^M(\lambda \cup \{x, y\})$ is a Dodd-Jensen mouse of size λ with both $x, y \in lp(M')$ such that $M' \vDash$ "$x \leq_{M'} y$". Note as well that the Dodd-Jensen mouse M' contains λ. This proves the following:

Corollary 1.3.12. *For every $x, y \in {}^{\lambda}2 \cap K^{\mathsf{DJ}}$, $x \leq_{\mathsf{DJ}} y$ if and only if there exists a Dodd-Jensen mouse M of size λ with $\lambda \in M$ such that $x, y \in lp(M)$ and $M \vDash$ "$x \leq_M y$".*

Remark 1.3.13. It readily follows from the proof above that $x \leq_{DJ} y$ if and only if $M \vDash$ "$x \leq_M y$" *for all* Dodd-Jensen mice M with $x, y \in lp(M)$. That is, $\leq_{\mathsf{DJ}}$ is actually Δ_1.

Let M be a Dodd-Jensen mouse of size λ and let θ be a cardinal big enough for $M \in H_\theta$ and $H_\theta \vDash \mathsf{ZF}^-$ to hold. The collapse of $Hull^{H_\theta}(M \cup \{M\})$ is then a transitive ZF^--model of size λ containing M as an element. Therefore, every Dodd-Jensen mouse of size λ belongs to a transitive ZF^--model H of size λ. For our purposes it might be convenient to have a model satisfying only a finite fragment of ZF^-. An argument as above secures the existence of such a model. In the context of this chapter, the infinite fragment of ZF^- that we require the collapse of $Hull^{H_\theta}(M \cup \{M\})$ to satisfy is that needed to witness the iterability of its Dodd-Jensen mice elements. To sum up:

Lemma 1.3.14. *For every Dodd-Jensen mouse M of size λ there is a transitive structure H of size λ such that $M \in H$ and which models a finite fragment of ZF^- enough to witness the iterability of its Dodd-Jensen mice elements.*

If M and H are as above, then $\omega_1^V \subseteq H$ because λ is assumed to be uncountable, so, by Theorem 1.3.4, M is iterable if and only if $H \vDash$ "M is iterable".

Coding Dodd-Jensen mice. Let $x, y \in {}^{\lambda}2 \cap K^{\mathsf{DJ}}$. By Corollary 1.3.12, $x \leq_{\mathsf{DJ}} y$ if and only if there exists a Dodd-Jensen mouse M of size λ such that $x, y \in lp(M)$ and $M \vDash$ "$x \leq_M y$". Equivalently, $x \leq_{\mathsf{DJ}} y$ if and only if there is a code $z \in {}^{\lambda}2$ for a well-founded and extensional structure $(\lambda, \in_z)$ whose (bounded) transitive collapse $tr(\lambda, \in_z)$ is a Dodd-Jensen mouse such that $x, y \in lp(tr(\lambda, \in_z))$ and $tr(\lambda, \in_z) \vDash$ "$x \leq_{tr(\lambda, \in_z)} y$". Coding a Dodd-Jensen mouse requires a code for a Dodd-Jensen premouse together with a code for a transitive structure containing it which is able to witness its iterability.

By (4) in Proposition 1.2.7, the set $\{(x, y) \in \mathsf{BC}_\lambda \times {}^{\lambda}2 : y \in tr(\lambda, \in_x)\}$ is λ-Borel in $\mathsf{BC}_\lambda \times {}^{\lambda}2$, hence λ-coanalytic in ${}^{\lambda}2 \times {}^{\lambda}2$. Observe that to say of a $y \in {}^{\lambda}2$ that there exists an $x \in \mathsf{BC}_\lambda$ such that $y \in tr(\lambda, \in_x)$ is λ-$\mathbf{\Sigma}^1_2$, for it is equivalent to say that such y is in the projection of a λ-coanalytic set. However, to say $x \in \mathsf{BC}_\lambda$ and $y \in tr(\lambda, \in_x)$ is λ-coanalytic for it holds if $x \in \mathsf{BC}_\lambda \wedge \exists \alpha < \lambda \pi_x(\alpha) = y$, where π_x stands for the corresponding collapse. Note as well that a premouse is a structure of an

[7] This is inner model theory jargon. We simply mean that the iteration has finally produced two models such that one is the initial segment of the other.

augmented language with a unary predicate. We need to deal with codes for this kind of models. Let $((z)_1,(z)_2)\in{}^{\lambda}2\times{}^{\lambda}2$ be such that $(z)_1$ is the code for a bounded, extensional and well-founded structure $(\lambda,\in_{(z)_1})$ and let $M=tr(\lambda,\in_{(z)_1})$ with $\pi_{(z)_1}$ being the corresponding collapsing map $\pi_{(z)_1}:(\lambda,\in_{z_1})\longrightarrow(M,\in)$. By defining $U_{(z)_2}=\{\alpha<\lambda:(z)_2(\alpha)=1\}$, we get that $((z)_1,(z)_2)$ codes an $\mathcal{L}_\in(\dot U)$-model $(\lambda,\in_{(z)_1},U_{(z)_2})$[8]. The following extends (1) in Lemma 1.2.7 to languages augmented by a predicate.

Lemma 1.3.15. *Let $\dot U$ be a unary predicate, φ an $\mathcal{L}_\in(\dot U)$-formula, let $(\lambda,\in_{(z)_1},U_{(z)_2})$ be as above and let $\alpha_1,\dots,\alpha_n\in\lambda$. The set $\{((z)_1,(z)_2)\in\mathsf{BC}_\lambda\times{}^{\lambda}2:(\lambda,\in_{(z)_1},U_{(z)_2})\vDash\varphi(\alpha_1,\dots,\alpha_n)\}$ is λ-Borel in $\mathsf{BC}_\lambda\times{}^{\lambda}2$.*

Proof. We go by induction on the complexity of the formula. Assume that U appears in φ, for otherwise we are in the same situation as in Lemma 1.2.7(1). For the basic case, if $\varphi(x)=x\in\dot U$, then

$$\begin{aligned}&\{((z)_1,(z)_2)\in\mathsf{BC}_\lambda\times{}^{\lambda}2:(\lambda,\in_{(z)_1},U_{(z)_2})\vDash\varphi(\alpha)\}=\\&\{((z)_1,(z)_2)\in\mathsf{BC}_\lambda\times{}^{\lambda}2:(\lambda,\in_{(z)_1},U_{(z)_2})\vDash\alpha\in U_{(z)_2}\}=\\&\{((z)_1,(z)_2)\in\mathsf{BC}_\lambda\times{}^{\lambda}2:(z)_2(\alpha)=1\},\end{aligned}$$

which is λ-clopen. The rest follows. □

Remark 1.3.16. The previous can be easily generalised in the GDST context to structures with less than λ many predicates. Indeed, fixed an ordinal $\delta\in\lambda$ if $\boldsymbol{U}:=\{\dot U_\alpha:\alpha\in\delta\}$ is a collection of predicates, φ is an $\mathcal{L}_\in(\boldsymbol{U})$-formula, $(\lambda,\in_{(z)_1},(\dot U_\alpha)_{((z)_2)_\alpha})$ is a structure where $(\dot U_\alpha)_{((z)_2)_\alpha}$ represents the predicated coded by $((z)_2)_\alpha$ and $\alpha_1,\dots,\alpha_r\in\lambda$, then the set $\{((z)_1,((z)_2))\in\mathsf{BC}_\lambda\times{}^{\lambda}2:(\lambda,\in_{(z)_1},(\dot U_\alpha)_{((z)_2)_\alpha})\vDash\varphi(\alpha_1,\dots,\alpha_r)\}$ is λ-Borel in $\mathsf{BC}_\lambda\times{}^{\lambda}2$.

By $\mathsf{PM}^{\mathsf{DJ}}(\lambda)$ we denote the set of pairs in $\mathsf{BC}_\lambda\times{}^{\lambda}2$ coding Dodd-Jensen premice of size λ, that is:

$$\begin{aligned}\mathsf{PM}^{\mathsf{DJ}}(\lambda):=\{&((z)_1,(z)_2)\in\mathsf{BC}_\lambda\times{}^{\lambda}2:(\lambda,\in_{(z)_1},U_{(z)_2})\vDash\text{“}\exists\kappa(U_{(z)_2}\text{ is}\\&\text{a normal measure on }\kappa)\wedge V=L[U_{(z)_2}]\text{”}\}.\end{aligned}$$

where the last condition together with $tr(\lambda,\in_{(z)_1},U_{(z)_2})$ being transitive makes $tr(\lambda,\in_{(z)_1},U_{(z)_2})$ a structure of the form $\mathcal{J}_\alpha^{U_{(z)_2}}$, as the following tells:

Lemma 1.3.17 (See, e.g., [Sch14, Lemma 5.28])**.** *Let M be a transitive model. Then, $M\vDash V=L[\dot U]$ if and only if $M=J_\alpha[\dot U]$ for some α and $\dot U$, where $x\in\dot U$ if and only if $M\vDash x\in\dot U$ for all $x\in M$.*

Structures of the form $(\lambda,\in_{(z)_1},U_{(z)_2})$ as defined above already satisfy that $x\in U_{(z)_2}$ if and only if $(\lambda,\in_{(z)_1},U_{(z)_2})\vDash x\in U_{(z)_2}$ for all $x\in\lambda$. The formula $V=L[\dot U]$ is an $\mathcal{L}_\in(\dot U)$-sentence, so the set $\{((z)_1,(z)_2)\in{}^{\lambda}2\times{}^{\lambda}2:(\lambda,\in_{(z)_1},U_{(z)_2})\vDash\text{“}V=L[U_{(z)_2}]\text{”}\}$ is λ-Borel in $\mathsf{BC}_\lambda\times{}^{\lambda}2$ by Lemma 1.3.15, hence, provided $(z)_1\in\mathsf{BC}_\lambda$, λ-coanalytic in ${}^{\lambda}2\times{}^{\lambda}2$.

Lemma 1.3.18. $\mathsf{PM}^{\mathsf{DJ}}(\lambda)$ *is* λ-$\boldsymbol{\Pi}^1_1$.

Proof. BC_λ is λ-coanalytic by (3) in Lemma 1.2.7. It is easy routine to show that the sentence “$\exists\kappa(U_{(z)_2}$ is a normal measure on $\kappa)$” can be written as an $\mathcal{L}_\in(U_{(z)_2})$-formula without parameters. The same happens with the sentence “$V=L[\dot U]$” . By Lemma 1.3.15, both formulas define λ-Borel sets in $\mathsf{BC}_\lambda\times{}^{\lambda}2$. We conclude that $\mathsf{PM}^{\mathsf{DJ}}(\lambda)$ is λ-coanalytic. □

[8]It is possible to code a structure (M,E,U) with a single element $z\in{}^{\lambda}2$ by letting $E=\{(m,n):z(\langle m,n\rangle)=1\}$ and $U=\{n:z(n)=0\}$, but this way things become unnecessarily cumbersome.

We denote by $\mathsf{M}^{\mathsf{DJ}}(\lambda)$ the set of codes for Dodd-Jensen mice. That is:

$$\begin{aligned}\mathsf{M}^{\mathsf{DJ}}(\lambda) :=\{((z)_1, ((z)^1_2, (z)^2_2)) \in \mathsf{BC}_\lambda \times \mathsf{PM}^{\mathsf{DJ}}(\lambda) :\\ ((z)^1_2, (z)^2_2) \in tr(\lambda, \in_{(z)_1}) \wedge tr(\lambda, \in_{(z)_1}) \vDash \mathcal{T}\\ \wedge\, tr(\lambda, \in_{(z)_1}) \vDash \text{“}tr(\lambda, \in_{(z)^1_2}, U_{(z)^2_2}) \text{ is iterable”}\},\end{aligned}$$

where $\mathcal{T}$ is a finite fragment of ZF^- big enough to witness the iterability of Dodd-Jensen premice.

Lemma 1.3.19. *$\mathsf{M}^{\mathsf{DJ}}(\lambda)$ is λ-$\mathbf{\Pi}^1_1$.*

Proof. Let $((z)_1, ((z)^1_2, (z)^2_2)) \in \mathsf{M}^{\mathsf{DJ}}(\lambda)$. The existence of a structure such as the one coded by $(z)_1$ is secured by Lemma 1.3.14. First recall that both BC_λ and $\mathsf{PM}^{\mathsf{DJ}}(\lambda)$ are λ-coanalytic. The first statement is λ-coanalytic by (4) in Lemma 1.2.7, while the second is λ-Borel by Lemma 1.3.15. The last condition is λ-Borel, too, for witnessing the iterability of a Dodd-Jensen premouse is equivalent to witnessing the well-foundedness of its iterated ultrapowers, which can be expressed[9] by a $\mathcal{L}_\in(U)$-formula with parameters in λ. Then, $\mathsf{M}^{\mathsf{DJ}}(\lambda)$ is λ-$\mathbf{\Pi}^1_1$. □

A consequence of the previous two lemmas is the following:

Proposition 1.3.20. *Both ${}^\lambda 2 \cap K^{\mathsf{DJ}}$ and $\leq_{\mathsf{DJ}} \restriction ({}^\lambda 2 \times {}^\lambda 2) \cap K^{\mathsf{DJ}}$ are λ-$\mathbf{\Sigma}^1_2$.*

Proof. We have already seen that for every $x \in {}^\lambda 2 \cap K^{\mathsf{DJ}}$ there is a Dodd-Jensen mouse M of size λ such that $x \in lp(M)$. Then, $x \in {}^\lambda 2 \cap K^{\mathsf{DJ}}$ if and only if $\exists z \in \mathsf{M}^{\mathsf{DJ}}(\lambda)(x \in lp(tr(\lambda, \in_{(z)^1_2})))$. Note that $x \in lp(tr(\lambda, \in_{(z)^1_2}))$ can be written as $x \in tr(\lambda, \in_{(z)^1_2}) \wedge tr(\lambda, \in_{(z)^1_2}) \vDash$ “$x \in V_{\bigcup (z)^2_2}$”, where $(z)^2_2$ is the code of the measure as in Lemma 1.3.18. The first statement defines a λ-coanalytic set, while the second, which can be expressed using only parameters in λ, defines a λ-Borel set. From this it follows that the set $\{(x, z) \in {}^\lambda 2 \times \mathsf{M}^{\mathsf{DJ}}(\lambda) : x \in tr(\lambda, \in_{(z)^1_2}) \wedge tr(\lambda, \in_{(z)^1_2}) \vDash \text{“}x \in V_{\bigcup (z)^2_2}\text{”}\}$, whose projection is ${}^\lambda 2 \cap K^{\mathsf{DJ}}$, is λ-$\mathbf{\Pi}^1_1$. Therefore, ${}^\lambda 2 \cap K^{\mathsf{DJ}}$ is λ-$\mathbf{\Sigma}^1_2$.

For the second statement we need a slightly different variant of the argument above. Let $x, y \in {}^\lambda 2 \cap K^{\mathsf{DJ}}$, then $x \leq_D y$ if and only if $\exists z \in \mathsf{M}^{\mathsf{DJ}}(\lambda)(x, y \in lp(tr(\lambda, \in_{(z)^1_2})) \wedge tr(\lambda, \in_{(z)^1_2}) \vDash$ “$x \leq y$”), where $\leq$ stands for the constructibility order of the mouse $tr(\lambda, \in_{(z)^1_2})$. As before, $x, y \in lp(tr(\lambda, \in_{(z)^1_2}))$ is λ-coanalytic. Since the order of every mice is uniformly Σ_1, $tr(\lambda, \in_{(z)^1_2}) \vDash$ “$x \leq y$” is λ-Borel. It then follows that the set $\{(z, (x, y)) \in \mathsf{BC}_\lambda \times ({}^\lambda 2 \times {}^\lambda 2) : x, y \in tr(\lambda, \in_{(z)^1_2}) \wedge tr(\lambda, \in_{(z)^1_2}) \vDash \text{“}x, y \in V_{\bigcup (z)^2_2} \wedge x \leq y\text{”}\}$ is λ-$\mathbf{\Pi}^1_1$. Its projection, the set $\leq_{\mathsf{DJ}} \restriction {}^\lambda 2 \times {}^\lambda 2$, is λ-$\mathbf{\Sigma}^1_2$. □

Recall that the set $\{(x, y) \in {}^\lambda 2 \times {}^\lambda 2 : x \sim y\}$ is λ-$\mathbf{\Sigma}^1_1$. By the previous, the set $\leq_{\mathsf{DJ}} \restriction ({}^\lambda 2 \times {}^\lambda 2) \cap K^{\mathsf{DJ}}$ is λ-$\mathbf{\Sigma}^1_2$. Therefore, the union of its complements, the set

$$A := \{x \in \mathsf{WO}_\lambda \cap {}^\lambda 2 : \forall y \in {}^\lambda 2(y \leq_{\mathsf{DJ}} x \to y \nsim x))\},$$

is a λ-$\mathbf{\Pi}^1_2$-set. Note that A cannot have the λ-PSP in K^{DJ}: by its definition, for every $\alpha < \lambda^+$ there is a unique $x \in {}^\lambda 2$ coding a well-order on λ of order type α, so $|A| = \lambda^+$ (for GCH holds in K^{DJ}). Then, if A had the λ-PSP there would be an embedding π of ${}^\lambda 2$ into A and the λ-analytic set $\pi[{}^\lambda 2]$ would be of size λ^+. Consequently, $\{ot(<_x) : x \in \pi[{}^\lambda 2]\}$ would be an unbounded λ-analytic set in λ^+, contradicting the Boundedness Lemma. This already proves that there is a λ-$\mathbf{\Pi}^1_2$-set without the λ-PSP in K^{DJ}.

Because of how $\leq_{\mathsf{DJ}}$ is defined, if $x \in {}^\lambda 2 \cap K^{\mathsf{DJ}}$ is the $\leq_{\mathsf{DJ}}$-least element satisfying certain property P, then x is the $\leq_{\mathsf{DJ}}$-least element in some Dodd-Jensen mouse M with $x \in lp(M)$ such that $P(x)$. Conversely, if $x \in {}^\lambda 2 \cap K^{\mathsf{DJ}}$ is the $\leq_M$-least element in the lower part of some Dodd-Jensen mouse M satisfying P, then it is the $\leq_{\mathsf{DJ}}$-least. Therefore, the set A from the previous paragraph can be written instead as

$$\begin{aligned}\{x \in {}^\lambda 2 :\, &\exists z \in \mathsf{M}^{\mathsf{DJ}}(\lambda) \forall y \in lp(tr(\lambda, \in_z)) \cap {}^\lambda 2\\ &(tr(\lambda, \in_z) \vDash \text{“}y <_{tr(\lambda, \in_z)} x\text{”} \to y \nsim x)\}.\end{aligned}$$

[9] Indeed, this can be asserted through codes of countable well-orderings as in the proof of [Kan08, Theorem 20.18].

A is the projection of the intersection of the λ-$\mathbf{\Pi}^1_1$-sets

$$\{(x,z) : z \in \mathsf{M}^{\mathsf{DJ}}(\lambda)\},$$

and

$$\{(x,z) \in {}^\lambda 2 \times {}^\lambda 2 : \forall y \in lp(tr(\lambda, \in_z)) \cap {}^\lambda 2 \, (tr(\lambda, \in_z) \vDash \text{“}y <_{tr(\lambda,\in_z)} x\text{”} \to y \not\approx x)\},$$

the last one actually being the intersection

$$\bigcap_{y \in lp(tr(\lambda,\in_z)) \cap {}^\lambda 2} \{((x,z),y) : (tr(\lambda,\in_z) \vDash \text{“}y <_{tr(\lambda,\in_z)} x\text{”} \to y \not\approx x)\}.$$

Therefore, A is λ-$\mathbf{\Sigma}^1_2$.

A trick of Solovay serves to reduce the descriptive complexity of A. Let z be the $\leq_{\mathsf{DJ}}$-least code for a structure $(\lambda, \in_z)$ such that for all $y \in lp(tr(\lambda, \in_z)) \cap {}^\lambda 2$, if $tr(\lambda, \in_z) \vDash$ “$y \leq_{tr(\lambda,\in_z)} x$”, then $y \not\approx x$. To say that such z is the $\leq_{\mathsf{DJ}}$-least is not λ-coanalytic, but it is λ-coanalytic to say of some $z' \in \mathsf{PM}^{\mathsf{DJ}}(\lambda)$ that z' codes a Dodd-Jensen mouse $M_{z'}$ with $z \in lp(M_{z'})$ which witnesses that z is indeed the $\leq_{\mathsf{DJ}}$-least. Again, to specify such z' is not λ-coanalytic, so we pick a third code $z'' \in \mathsf{PM}^{\mathsf{DJ}}(\lambda)$ for a Dodd-Jensen mouse $M_{z''}$ with $z' \in lp(M_{z''})$ where z' is indeed the $\leq_{\mathsf{DJ}}$-least, and so on, and to say that of such z'' is again λ-coanalytic. This way, we inductively produce a sequence $\langle z_n : n \in \omega \rangle$. But this sequence must be unique, so instead of *there is a sequence* $\langle z_n : n \in \omega \rangle$ *such that...* we can say *for all sequences of codes* $\langle z_n : n \in \omega \rangle$ *such that...*. This makes A λ-coanalytic. More formally:

Theorem 1.3.21. *If* $(\lambda^+)^{K^{\mathsf{DJ}}} = \lambda^+$*, then there is a λ-coanalytic subset of* ${}^\lambda 2$ *without the λ-*PSP.

Proof. Let A be the set of $z \in {}^\lambda 2$ such that:

(i) $(z)_0$ is the code for an ordinal $\alpha < \lambda^+$,

(ii) $((z)_1, ((z)_2, (z)_3)) \in \mathsf{M}^{\mathsf{DJ}}(\lambda)$ and $(z)_0 \in lp(tr(\lambda, \in_{(z)_2})) \cap {}^\lambda 2$,

(iii) $(z)_0$ is the $\leq_{\mathsf{DJ}}$-least element of $lp(tr(\lambda, \in_{(z)_2})) \cap {}^\lambda 2$ coding the ordinal α in (i),

(iv) $((z)_4, ((z)_5, (z)_6)) \in \mathsf{M}^{\mathsf{DJ}}(\lambda)$ and $((z)_1, ((z)_2, (z)_3)) \in lp(tr(\lambda, \in_{(z)_5}))$,

(v) $((z)_1, ((z)_2, (z)_3))$ is the $\leq_{\mathsf{DJ}}$-least in $lp(tr(\lambda, \in_{(z)_5}))$ satisfying (iv).

And then, for every $n \in \omega$,

(vi) $((z)_{3n+4}, ((z)_{3n+5}, (z)_{3n+6})) \in \mathsf{M}^{\mathsf{DJ}}(\lambda)$,

(vii) $((z)_{3n+1}, ((z)_{3n+2}, (z)_{3n+3})) \in lp(tr(\lambda, \in_{(z)_{3n+5}}))$, and

(viii) $((z)_{3n+1}, ((z)_{3n+2}, (z)_{3n+3}))$ is the $\leq_{\mathsf{DJ}}$-least element in $lp(tr(\lambda, \in_{(z)_{3n+5}}))$ satisfying (vii).

Clause (i) is λ-coanalytic by (3) in Lemma 1.2.7. By Lemma 1.3.19 and (2) in Lemma 1.2.7, clauses (ii), (iv) and (vi) are λ-coanalytic, too. Clause (iii) holds if and only if $\forall y \in {}^\lambda 2 (tr(\lambda, \in_{(z)_1}) \vDash$ “$y \leq_{tr(\lambda,\in_{z_1})} (z)_0$” $\to tr(\lambda, \in_y) \not\cong tr(\lambda, \in_{(z)_0}))$, which is λ-coanalytic by (2) and (4) in Lemma 1.2.7. A similar argument proves that both (v) and (vii) are λ-coanalytic as well. Therefore, A is λ-$\mathbf{\Pi}^1_1$. Because of the argument right after the proof of Proposition 1.3.20, A does not have the λ-PSP in K^{DJ}. Indeed, since for every $\alpha < \lambda^+$ there is a unique $x \in {}^\lambda 2$ coding a well-order on λ of order type α and a unique $z \in A$ such that $(z)_0 = x$, A is of size λ^+ in K^{DJ}, where GCH holds. Then, if A had the λ-PSP, there would be an embedding π of ${}^\lambda 2$ into A and the λ-analytic set $\pi[{}^\lambda 2]$ would be of size λ^+. Consequently, $\{ot(<_{(x)_0}) : x \in \pi[{}^\lambda 2]\}$ would be an unbounded λ-analytic set in λ^+, contradicting the Boundedness Lemma.

Now, if $(\lambda^+)^{K^{\mathsf{DJ}}} = \lambda^+$ the set A is of size λ^+ also in V, so it cannot have the λ-PSP by an entirely analogous argument as the one above. By Lemma 1.3.20, the set A is still λ-coanalytic in V. $\square$

In particular:

Corollary 1.3.22. *In K^{DJ} there is a λ-coanalytic subset of ${}^{\lambda}2$ without the λ-*PSP*.*

Remark 1.3.23. If $0^\sharp$ doesn't exist, K^{DJ} is L. In this case, the previous is exactly what Dimonte and Motto Ros proved in [DMRon].

By the Dodd-Jensen's Covering Theorem for K^{DJ}, if there is no inner model with a measurable cardinal, K^{DJ} computes correctly the successor of every uncountable singular cardinal. That is, if there is no inner model with a measurable cardinal then for every uncountable strong limit λ of countable cofinality there is a λ-coanalytic subset of ${}^{\lambda}2$ without the λ-PSP. Equivalently, by contraposition:

Corollary 1.3.24. *If there exists an uncountable strong limit cardinal λ with $cf(\lambda) = \omega$ such that all λ-coanalytic subsets of ${}^{\lambda}2$ have the λ-*PSP*, then there is an inner model with a measurable cardinal.*

This argument doesn't work in the classical case simply because the Covering Theorem for K^{DJ} doesn't apply to the case $\lambda = \omega$. Moreover, the exact consistency strength of all coanalytic sets having the PSP is at the existence of an inaccessible cardinal. Corollary 1.3.22 still holds for ω, though:

Corollary 1.3.25. *If $\omega_1^{K^{\mathsf{DJ}}} = \omega_1$, then there is a coanalytic set of reals without the* PSP*.*

Corollary 1.3.22 extends to a general descriptive context the classical result that if $\omega_1^L = \omega_1$, then there is a coanalytic set of reals without the PSP (see [Kan08, Theorem 13.12]). Corollary 1.3.25, by going from the interpretation of ω_1 in L to its interpretation in K^{DJ}, allows this characterisation to still work under large cardinal assumptions such as $0^\sharp$ (and sharps for every real, in general). Indeed, if $0^\sharp$ doesn't exist, then $K^{\mathsf{DJ}} = L$, while if it does, then L is strictly contained in K^{DJ}.

Recall that ω_1 is said to be inaccessible from reals if $\forall a \in {}^{\omega}2(\omega_1^{L[a]} < \omega_1)$.

Corollary 1.3.26. *If $\omega_1^{K^{\mathsf{DJ}}} = \omega_1$, there is a real $a \in {}^{\omega}\omega$ such that $\omega_1^{L[a]} = \omega_1$. In particular, if $\omega_1^{K^{\mathsf{DJ}}} = \omega_1$, then ω_1 is not inaccessible from reals.*

Proof. By the previous corollary, if $\omega_1^{K^{\mathsf{DJ}}} = \omega_1$, there exists a real a such that there is a (lightface) $\Pi^1_1(a)$-set of reals without the PSP. Since all (lightface) $\Pi^1_1(a)$ sets having the PSP is equivalent to $\omega_1^{L[a]} < \omega_1$ (Solovay, [Kan08, Theorem 14.10]), it follows that if $\omega_1^{K^{DJ}} = \omega_1$, there exists a real a such that $\omega_1^{L[a]} = \omega_1$. ☐

Solovay's full result states that for every real $a \in {}^{\omega}2$, all (lightface) $\Sigma^1_2(a)$ sets of reals have the PSP if and only if all (lightface) $\Pi^1_1(a)$ sets of reals have the PSP if and only if $\omega_1^{L[a]} < \omega_1$.

Proposition 1.3.27. *Let λ be an uncountable strong limit cardinal of countable cofinality and let $a \in {}^{\lambda}2$. Then,* (1) *implies* (2) *and* (2) *implies* (3)*.*

(1) *Every (lightface) λ-$\Sigma^1_2(a)$-subset of ${}^{\lambda}2$ has the λ-*PSP*.*

(2) *Every (lightface) λ-$\Pi^1_1(a)$-subset of ${}^{\lambda}2$ has the λ-*PSP*.*

(3) $(\lambda^+)^{L[a]} < \lambda^+$.

Proof. That (1) implies (2) is clear. We show that (2) implies (3) by contraposition. Let $(\lambda^+)^{L[a]} = \lambda^+$. As Dimonte and Motto Ros show in [DMRon], there is a λ-coanalytic A set without the λ-PSP. Every λ-$\boldsymbol{\Pi}^1_1$-set is a (lightface) λ-$\Pi^1_1(b)$-set for some parameter b. We show that A is actually a (lightface) λ-$\Pi^1_1(a)$-set. To do this, we provide the construction of the set as in the proof of Dimonte and Motto Ros' Theorem ([DMRon, Theorem 7.2.12]). We will use the already mentioned fact that sets defined with $\mathcal{L}$-formulas and variables below λ are λ-Borel. Note that if $a \in {}^{\lambda}2$ and $\bar{a}$ is its support, then $L[a] = L[\bar{a}]$, so we consider $\bar{a}$ instead of a, although we won't mention it. Let A be the set of $z \in {}^{\lambda}2$ such that:

(i) $(z)_0$ codes an ordinal $\alpha < \lambda^+$,

(ii) $(z)_1$ codes $L_{\delta_1}[a]$ for some ordinal δ_1,

(iii) $(z)_0 \in L_{\delta_1}[a]$,

(iv) δ_1 is the least such that $(z)_0 \in L_{\delta_1}[a]$,

(v) $(z)_0$ is the $<_{L[a]}$-least element of ${}^\lambda 2$ that codes α

and then, for every $n \in \omega$,

(vi) $(z)_{n+1}$ codes $L_{\delta_{n+1}}[a]$ for some δ_{n+1} limit,

(vii) $(z)_n \in L_{\delta_{n+1}}[a]$,

(viii) δ_{n+1} is the least such that $(z)_n \in L_{\delta_1}[a]$,

(ix) $(z)_{n+1}$ is the least $<_{L[a]}$-least element of ${}^\lambda 2$ coding $L_{\delta_{n+1}}[a]$.

It is easy to see that the defined set is not only λ-coanalytic but also (lightface) λ-$\Pi^1_1(a)$. □

1.4 The λ-PSP up to the existence of $0^\dagger$

We lift the previous lower bound up to the existence of $0^\dagger$.

Definition 1.4.1. If κ is an ordinal which is measurable in some inner model we say that it is an *internally measurable cardinal.*

By Corollary 1.3.24, if λ is an uncountable singular cardinal of countable cofinality such that all λ-coanalytic sets of ${}^\lambda 2$ have the λ-PSP, then there is an internally measurable cardinal. Let κ be such an ordinal and let U be a measure on κ in $L[U]$. $L[U]$ is a model of $\mathsf{ZFC} + \mathsf{GCH}$ and admits a definable well-ordering, which we denote by $\leq_{L[U]}$, where $x \leq_{L[U]} y$ if and only if there exists some limit ordinal δ such that $x, y \in L_\delta[U]$ and $L_\delta[U]$ satisfies the $\mathcal{L}_\in(\dot{U})$-sentence "$x \leq_{L[U]} y$".

The model $L[U]$ lacks of condensation. Indeed, if M is an elementary substructure of $L_\tau[U]$ for some ordinal τ, then it is, by elementarity, of the form $L_\alpha[U_\alpha]$, where

$$L_\alpha[U_\alpha] \vDash \text{“}U_\alpha \text{ is a normal ultrafilter”},$$

and $L_\alpha[U_\alpha]$ is not necessarily an initial segment of $L[U]$. However, if $L_\alpha[U_\alpha]$ satisfies enough of ZF, one can consider its iterate ultrapower $Ult^{(\kappa)}(L_\alpha[U_\alpha], U_\alpha)$ together with the corresponding elementary embedding $i_\kappa : L_\alpha[U_\alpha] \to Ult^{(\kappa)}(L_\alpha[U_\alpha], U_\alpha)$. Then one gets that $Ult^{(\kappa)}(L_\alpha[U_\alpha], U_\alpha) = L_{\alpha'}[i_\kappa(U_\alpha)]$ for some $\alpha' < \kappa^+$. But $i_\kappa(U_\alpha) \subseteq U$, because $i_\kappa(U_\alpha)$ is generated by the closed and unbounded set of closed and unbounded sets of κ[10], which belongs to U. This implies that the iterated ultrapower *is* an initial segment of $L[U]$.

Let then $x, y \in {}^\lambda 2$ and let τ be an ordinal such that $x, y \in L_\tau[U]$. Assume that τ is such that $L_\tau[U]$ satisfies enough of ZF to check the well-foundedness of its iterated ultrapowers. For this is enough that τ is a regular cardinal. Let M be the collapse of $Hull(\lambda \cup \{x, y\})$. By elementarity, $M = L_{\alpha_{x,y}}[U_{x,y}]$ for some ordinal $\alpha_{x,y} < \lambda^+$ and some $U_{x,y}$ which M sees as a normal ultrafilter on some ordinal $\kappa_{x,y}$. Moreover, $|M| = \lambda$ and $x, y \in M$. M is not an initial segment of $L[U]$. Now, if we make sure that $x, y \in lp(M)$, then $x, y \in Ult^{(\beta)}(L_{\alpha_{x,y}}[U_{x,y}], U_{x,y})$ for β big enough for the ultraproduct to be an initial segment of $L[U]$. Note now that if M satisfies those properties it actually is a Dodd-Jensen mouse such that $x, y \in lp(M)$. That is, we get that $x \leq_{L[U]} y$ if and only if there exists a Dodd-Jensen mouse M of size less than λ^+ such that $x, y \in lp(M)$ and

[10] For more details, check [Mit09, Theorem 1.11])

$x \leq_{Ult^{(\beta)}(M,U)} y$. By elementarity, $x \leq_{Ult^{(\beta)}(M,U)} y$ if and only if $x \leq_M y$. So we then have that $x \leq_{L[U]} y$ if and only if there is a Dodd-Jensen mouse M such that $x, y \in lp(M)$ and $M \vDash$ "$x \leq y$". It is then obvious that the exact same proofs for Proposition 1.3.20 and Theorem 1.3.21 serve to prove the following:

Proposition 1.4.2. *Both ${}^\lambda 2 \cap L[U]$ and $\leq_{L[U]} \restriction ({}^\lambda 2 \times {}^\lambda 2)$ are λ-$\mathbf{\Sigma}^1_2$.*

Theorem 1.4.3. *If $(\lambda^+)^{L[U]} = \lambda^+$, then there is a λ-coanalytic subset of ${}^\lambda 2$ without the λ-PSP. In particular, there is such a set in $L[U]$.*

And, as a consequence:

Corollary 1.4.4. *If λ is a strong limit cardinal of countable cofinality such that all λ-coanalytic subsets of ${}^\lambda 2$ have the λ-PSP, then $L[U]$ cannot have the weak covering property.*

Proof. Because of Theorem 1.3.21, if such a λ exists, then also $L[U]$ exists and it has a λ-coanalytic set $A \subseteq {}^\lambda 2$ without the λ-PSP. If $L[U]$ had the weak covering property, a similar argument as in the proof of Theorem 1.3.21, using this time Proposition 1.4.2, would show that A is still a λ-coanalytic set without the λ-PSP as seen from V, which contradicts the hypothesis. □

The Dodd-Jensen's Covering Theorem for $L[U]$. Let κ be an internal measurable cardinal and let U be a measure on κ in $L[U]$. Because of Prikry's Theorem (see Theorem 1.4.7 below), there always is a generic extension of $L[U]$ where κ is still a cardinal, although of countable cofinality. This means that $L[U]$ cannot have a Covering Theorem as the one of L or K^{DJ}. However, Dodd and Jensen showed that to get a Covering Theorem for $L[U]$ just the existence of such kind of forcing extensions is to be taken into account.

Definition 1.4.5. Let M be a transitive model of ZFC and let $\kappa, U \in M$ be such that $M \vDash$ "U is a normal ultrafilter on κ". Let $\mathbb{P}_U := (\mathbb{P}; \leq)$ be such that $p \in \mathbb{P}_U$ if and only if $p = (a, x)$ where $a \in [\kappa]^{<\omega}$, $x \in U$, and $\min(x) > \max(a)$ whenever $a \neq \emptyset$ and for every two $p, q \in \mathbb{P}_U$ with $p = (a, x)$ and $q = (b, y)$, let $p \leq q$ if and only if $b \supseteq a$, $y \subseteq x$, and $b \setminus a \subseteq x$. The poset $\mathbb{P}_U$ is a *Prikry forcing for the measure U*.

Definition 1.4.6. Let M be a transitive model of ZFC and let $U, \kappa \in M$ be such that $M \vDash$ "U is a normal measure on κ". A *Prikry sequence over M with respect to U* is a strictly increasing sequence $\langle \kappa_n : n \in \omega \rangle$ such that

(i) $\langle \kappa_n : n \in \omega \rangle$ is cofinal in κ, and

(ii) for all $x \in \mathcal{P}(\kappa) \cap M$, $x \in U$ if and only if $\{\kappa_n : n \in \omega\} \setminus x$ is finite.

Prikry sequences over U generate generic filters for the Prikry forcing $\mathbb{P}_U$. Let M be a transitive model of ZFC such that $M \vDash$ "U is a normal measure on κ". Mathias showed (See, e.g., [Sch14, Theorem 10.1]) that if $\langle \kappa_n : n \in \omega \rangle$ is a Prikry sequence over M with respect to U and let $G_{(\kappa_n : n \in \omega)}$ be the set of all $(\{\kappa_n : n < n_0\}, x)$ where $n_0 < \omega$, $x \in U$, $\min(x) > \kappa_{n_0 - 1}$ (if $n_0 > 0$), and $\{\kappa_n : n \geq n_0\} \subseteq x$, then $G_{\langle \kappa_n : n \in \omega \rangle}$ is $\mathbb{P}_U$-generic over M. Moreover:

Theorem 1.4.7 (Prikry. See, e.g., [Sch14, Lemma 10.6])**.** *Let M be a transitive model of* ZFC*, let $\kappa, U \in M$ be such that U witnesses that κ is measurable in M and let G be $\mathbb{P}_U$-generic over M. Then $(V_\kappa)^{M[G]} = (V_\kappa)^M$.*

The following is the Dodd-Jensen Covering Theorem for $L[U]$:

Lemma 1.4.8 (Dodd-Jensen. See, e.g., [Jec03, Theorem 35.16])**.** *Assume that there is an inner model with a measurable cardinal, let κ be the least such cardinal and let U be a measure on κ in $L[U]$. Then:*

(1) *either $0^\dagger$ exists, or*

(2) *the Covering Theorem holds for $L[U]$, or*

(3) *there exists an ω-sequence $\langle\kappa_n : n \in \omega\rangle$ Prikry generic over $L[U]$ such that the Covering Theorem holds for $L[U][\langle\kappa_n : n \in \omega\rangle]$.*

Moreover, $L[U][\langle\kappa_n : n \in \omega\rangle] = L[\langle\kappa_n : n \in \omega\rangle]$.

The fact that $L[U][\langle\kappa_n : n \in \omega\rangle] = L[\langle\kappa_n : n \in \omega\rangle]$ is, according to the comment right after Theorem 35.16 in [Jec03], a consequence of Mathias' Theorem, but no proof is given. We[11] give here a justification for this fact. Since the $\supseteq$ is evident, we only have to show that $L[U][\langle\kappa_n : n \in \omega\rangle] \subseteq L[U]$. To do this, let κ be the measurable in $L[U]$ and let

$$U' = \{A \subseteq \kappa : \{\kappa_n : n \in \omega\} \setminus A \text{ is finite}\} \cap L[\langle\kappa_n : n \in \omega\rangle].$$

Note that for every ordinal α, $L_\alpha[U] = L_\alpha[U']$. This can be proved by transfinite induction. The first step is trivial and so is every limit case, so suppose $L_\alpha[U] = L_\alpha[U']$. By a density argument, we have that

$$U = \{A \subseteq \kappa : \{\kappa_n : n \in \omega\} \setminus A \text{ is finite}\} \cap L[U].$$

Indeed, if $x \in U$ and $\mathbb{P}$ stands for the Prikry forcing, then the set $D_x := \{(t, y) \in \mathbb{P} : y \subseteq x\}$ is dense, so if $(t, y) \in G \cap D_x$ where G is Prikry generic and t is $\kappa_0{}^\frown \ldots {}^\frown\kappa_m$ for some $m \in \omega$, then $\kappa_k \in y \subseteq x$ for every $k > m$, thus x belongs to the intersection of $\{A \subseteq \kappa : \{\kappa_n : n \in \omega\} \setminus A \text{ is finite}\}$ and $L[U]$; and a similar argument shows the opposite direction.

Now, since by induction hypothesis $L_\alpha[U] \in L[\langle\kappa_n : n \in \omega\rangle]$, it follows by the definition of U' that

$$U \cap L_\alpha[U] = \{A \subseteq \kappa : \{\kappa_n : n \in \omega\} \setminus A \text{ is finite}\} \cap L_\alpha[U] = U' \cap L_\alpha[U].$$

Finally,

$$L_{\alpha+1}[U] = \mathsf{Def}(L_\alpha[U], \in, U \cap L_\alpha[U]) = \mathsf{Def}(L_\alpha[U'], \in, U' \cap L_\alpha[U]) = L_{\alpha+1}[U'].$$

In particular, $L[U] = L[U'] \subseteq L[\langle\kappa_n : n \in \omega\rangle]$, as $U' \in L[\langle\kappa_n : n \in \omega\rangle]$.

We can now prove the main theorem of this chapter:

Theorem 1.4.9. *If there is a strong limit cardinal λ of countable cofinality such that all λ-coanalytic subsets of ${}^\lambda 2$ have the λ-PSP, then $0^\dagger$ exists.*

Proof. Because of Corollary 1.4.4, $L[U]$ cannot have the weak covering property. Let $\langle\kappa_n : n \in \omega\rangle$ be an ω-sequence Prikry generic over $L[U]$ and assume that $L[U][\langle\kappa_n : n \in \omega\rangle]$ has weak covering. Let κ be the internal measurable cardinal, whose existence is secured by our hypothesis, as witnessed by $L[U]$. Assume that $L[U][\langle\kappa_n : n \in \omega\rangle]$ computes λ^+ correctly. We consider two cases. First, suppose that $\kappa \geq \lambda^+ = (\lambda^+)^{L[U][\langle\kappa_n : n\in\omega\rangle]}$. By Prikry's theorem, $L[U]$ and $L[U][\langle\kappa_n : n \in \omega\rangle]$ are the same up to κ. Now, since κ is a strong limit cardinal in $L[U][\langle\kappa_n : n \in \omega\rangle]$, it cannot be λ^+, so $\kappa > \lambda^+$, hence $\kappa > 2^{\lambda^+}$. The λ-coanalytic set A without the λ-PSP whose existence in $L[U]$ Theorem 1.4.3 secures has rank less than κ in $L[U]$, so it is a λ-coanalytic set without the λ-PSP also in $L[U][\langle\kappa_n : n \in \omega\rangle]$. Since it computes the successor of λ correctly, A is a λ-coanalytic set without the λ-PSP in V as well, contradicting the hypothesis. Suppose now that $\kappa < \lambda^+ = (\lambda^+)^{L[U][\langle\kappa_n : n\in\omega\rangle]}$. From the Covering Theorem, we have that $L[U][\langle\kappa_n : n \in \omega\rangle] = L[\langle\kappa_n : n \in \omega\rangle]$. Let $z \in {}^\lambda 2$ be a code for the sequence $\langle\kappa_n : n \in \omega\rangle$. By [DMRon, Theorem 7.2.12], in $L[z] = L[\langle\kappa_n : n \in \omega\rangle]$ there is a λ-coanalytic set A without the λ-PSP. The Covering Theorem implies that $L[z]$ has the weak covering property, and A is easily seen to be a λ-coanalytic set without the λ-PSP also in V. We conclude that, under our hypothesis, $L[U][\langle\kappa_n : n \in \omega\rangle]$ cannot compute λ^+ correctly, so it cannot have weak (nor full, of course) covering. It then follows from the Dodd-Jensen Covering Theorem for $L[U]$ that $0^\dagger$ exists. □

[11] I'd like to thank Andreas Lietz for his help here.

1.5 Final remarks and open questions

Linear iterability seems to be important for the descriptive complexity of the set of codes for premice and mice to remain at the level of λ-$\mathbf{\Pi}^1_1$-subsets of ${}^{\lambda}2$. This suggests that our method (same selector, sets of codes, etc.) should still work below the level of $0^{\P}$ (zero hand-grenade), where *almost linear iterations* occur. However, this might no longer hold unless certain restrictions are added to the anti-large cardinal assumption concerning the non-existence of $0^{\P}$, similar to that of the non-existence of 0^{long} in the context of Koepke's core model, which secures that the sequence of measures remains of the same order-type throughout iterations. If iterations generated new cardinals beyond λ, coding with elements in ${}^{\lambda}2$ would be unfeasible.

We know core models below $0^{\P}$, such as Koepke's core model, those with a measurable cardinal κ with Mitchell order κ^{++}, and core models for strong cardinals. Testing the waters with them may be a good starting point. We thus wonder:

Question 1.5.1. *Is it true that if there is a strong limit cardinal λ with countable cofinality such that all λ-coanalytic subsets of ${}^{\lambda}2$ have the λ-*PSP*, then*

(1) *there is an inner model with ω, or maybe λ,-many measurable cardinals?*

(2) *there is an inner model with a measurable cardinal κ with Mitchell order κ^{++}?, or*

(3) *there is an inner model with a strong cardinal?, or*

(4) *there is an inner model with a proper class of strong cardinals?*

Inner models beyond $0^{\P}$ give up on linear iterability in favour of iteration trees. Considerations regarding the existence of unique cofinal branches seem to imply a rise in the descriptive complexity of the codes for mice. One might consider generalising Steel's work in [Ste95] to optimise iterability complexity in the presence of finitely many Woodin cardinals. But this introduces new challenges: Steel eliminates extra quantifiers using the Gandy-Spector Theorem, but we lack of an equivalent result in the context of GDST at singular cardinals of countable cofinality, where Determinacy fails already at the very first levels of the λ-Borel hierarchy ([DMRon, Section 4.5]). One could still carry out the same work renouncing to remain at the lowest levels in the λ-projective hierarchy, but this introduces another difficulty: inner models M_n with n-many Woodin cardinals do not have nice covering properties. Other alternatives still at the level of core models with Woodin cardinals present additional problems beyond those mentioned. For instance, the Jensen-Steel core model without a measurable cardinal lacks the structural features, such as the well-behaved building blocks found in the Dodd-Jensen or in $L[U]$, that our techniques crucially rely on. As a result, the arguments we've employed in those settings may no longer apply in this context.

2

On the measure problem in generalised Baire spaces: an impossibility result

This chapter is based on joint work with Claudio Agostini and Vincenzo Dimonte.

2.1 Introduction

As already mentioned in the previous chapter, one of the major themes in DST is the study of the so-called regularity properties, and providing well-justified generalisations of these to higher spaces has been an important task for those working in GDST. While progress has been made in generalising notions such as the Baire Property or the Perfect Set Properties to the generalised Baire space, other notions have proven more difficult to extend to this context. Measure theory in particular is a notion that plays a central role in classical DST, but it has no widely accepted analogue in the generalised setting. There have been successful attempts to generalize related concepts. For example, Friedman and Laguzzi in [FL17], and Cohen and Shelah in [CS19] proposed different generalisations of the ideal of null sets to the generalised Baire space. Yet, further progress has faced significant technical and conceptual obstacles. As a result, the question of whether a satisfactory notion of measure can be developed in this context remains open. In this chapter, we address the problem of whether a generalised notion of measure can exist in the generalised Baire space from a new, different approach, and the conclusion draws a negative result. As already said in the introduction, by assuming very mild assumptions and adopting an "umbrella definition" that encompasses all possible natural generalisations of classical measures, we conclude that no non-trivial and continuous generalised measure satisfying reasonable structural axioms can exist.

To arrive at our conclusion, we begin by identifying the conditions that make the classical notion of measure possible. First, the properties of classical measures (see Fact 2.3.2) seem to be dependent on the algebraic and topological structure of $\mathbb{R}_\infty$, the extended cone of positive reals. There have been attempts to achieve a meaningful generalisation of the real numbers to uncountable settings (see, e.g., [DW96],[Gal19]), but they have failed to provide a canonical structure that can take on the role of the reals in generalisations of classical notions such as in the problem of measures. Some limitations of these attempts are studied in [Won23]. Our approach differs to those that intend to solve the problem of the lacking of a canonical codomain for measures by producing an $\mathbb{R}$-like field of higher order. Instead, noticing that the main properties that make possible the existence of measures boil down to the fact that $\mathbb{R}_\infty$ is a positively totally ordered and commutative monoid, and based on ideas of Agostini (see [AMRS23], [AMR],[Ago22]), we consider functions whose codomain belongs to the class of positively totally ordered monoids of arbitrary size, which in particular contains $\mathbb{R}_\infty$.

A second difficulty in generalising measures to higher Baire spaces is that of finding a correct notion of infinitary sum of arbitrary arity. In subsection 2.2.2, we formalise the notion of infinitary sums over positively totally ordered monoids (see Definition 2.2.10). This infinitary operation satisfies a list of minimal axioms that are sufficient to recover many features of classical measure spaces. At the same time, it is broad enough for infinitary operations to emulate the behaviour of infinite sums in $\mathbb{R}_\infty$, necessary to provide some sense of naturalness. Using this framework, in Section 2.3, we extend the notions of measure and measure space to the generalised setting, working under the assumptions that λ is an infinite cardinal of cofinality κ, that (X, τ) is a subspace of the generalised Baire space ${}^{<\kappa}\lambda$, and that $\mathbb{S} = (S, 0, +, \leq)$ is a positively totally ordered monoid. Conditioned by the fact that we, *a priori*, cannot know how these generalised notions should look like, we provide, as already mentioned, a very general definition (see Definition 2.3.4) intended to include all possible meaningful definitions. Section 2.3 explores several further aspects of λ^+-measures and weakly measure spaces. Here, for instance, we introduce the notion of continuous λ^+-measures, a straighforward generalisation of continuity for classical measures without which a generalised notion of measure would be immediately disregarded: a λ^+-measure μ on a weak λ^+-measurable space $(X, \mathfrak{M})$ is continuous if all points x in X are measurable of measure zero. Besides, we observe that sub-additivity, a property that every classical measure exhibits, does not follow from the minimal set of axioms for λ^+-measures, but introduce a reasonable assumption, λ^+-partitionability, to ensure that sub-additivity holds in weakly measure spaces (see Corollary 2.3.12). In this section, we also define a simple class of weak λ^+-measurable spaces, minimal weak λ^+-measurable spaces, to which every weak λ^+-measurable space can be restricted, in the sense that every weak λ^+-measure space $(X, \mathfrak{M}, \mu)$ contains a minimal weak λ^+-measure space measuring the same points of μ (Fact 2.3.19). This observation plays a key role in our impossibility result, for it allows us to restrict the study to such structures. We finish this section studying how weak λ^+-measure spaces can be extended and restricted to larger or smaller weak λ^+-measure spaces.

Section 2.4 addresses some existence results of the theory. Here, we try to understand the kind of weak λ^+-measures that do exist under minimal assumptions. In particular, we consider measures that in some sense are already trivial, such as Dirac measures, i.e., measures on a set X that assign a positive value to every subset of X containing a fixed element x, or countable-induced measures, i.e., measures extending classical measures defined on smaller fragments of the weak λ^+-measure space, a generalisation of Dirac measures indeed. We show that these measures do exist but show their pathological behaviour: they may fail, for instance, at measuring points.

In the final section, building upon the previous discussion, we establish our main result:

Theorem 2.1.1. *Assume $\lambda^{<\kappa} = \lambda \geq \mathfrak{c}$. Then, there is no continuous weak λ^+-measure space.*

2.2 Preliminaries

2.2.1 Totally ordered monoids

In this section we provide the reader with the necessary background on totally ordered monoids and establish some results that will play a key role after in the chapter. In particular, we provide a classification of the structures in this class which consists of three different types according to their Archimedean behaviour, their degree, and the possible existence of strongly decreasing sequences (Lemma 2.2.7). We start by recalling some of the basic notions and definitions about totally ordered monoids. Our main reference for well-known, classical results on totally ordered monoids is [Fuc63, Chapter X].

Definition 2.2.1. A *totally ordered monoid* is a structure $\mathbb{S} = \langle S, 0, +, \leq \rangle$ such that

(1) $+$ is a binary associative operation on $\mathbb{S}$ (i.e., $\langle S, + \rangle$ is a semigroup),

(2) 0 is the neutral element of the operation $+$ (i.e., $\langle S, +, 0 \rangle$ is a monoid),

(3) $\langle S, \leq \rangle$ is a total order,

(4) the order $\leq$ is (weakly) translation invariant (i.e., for every $a, b, c, d \in \mathbb{S}$, if $a \leq b$ and $c \leq d$, then $a + c \leq b + d$).

A *totally ordered group* $\mathbb{G} = \langle G, 0_{\mathbb{G}}, +_{\mathbb{G}}, \leq_{\mathbb{G}} \rangle$ is a totally ordered semigroup such that $(G, +)$ a group.

We denote by $\mathbb{S}^+$ to the *positive cone* of $\mathbb{S}$, the set $\{s \in \mathbb{S} \mid 0 < s\}$ of non-negative elements in $\mathbb{S}$.

Definition 2.2.2. The *degree of* $\mathbb{S}$, denoted by $\mathrm{Deg}(\mathbb{S})$, is the coinitiality of the positive cone $\mathbb{S}^+$ with respect to $\leq$.

That is, the *degree of* $\mathbb{S}$ is the minimal size of a set $A \subseteq \mathbb{S}^+$ such that for every $\varepsilon \in \mathbb{S}^+$ there is some $a \in A$ satisfying $a \leq \varepsilon$. By definition, $\mathrm{Deg}(\mathbb{S})$ is always a regular cardinal. We say that a totally ordered monoid $\mathbb{S}$ is *positively ordered* if $\mathbb{S}^+ \neq \emptyset$ and $\mathbb{S} = \mathbb{S}^+ \cup \{0\}$, i.e. if 0 is the minimum of the order and $\mathbb{S}$ is not the trivial monoid $\{0\}$. Notice that for every totally ordered monoid $\mathbb{S}$, we have that $\mathbb{S}^+ \cup \{0\}$ is a positively ordered totally ordered monoid. Since for the purpose of this work we only work with $\mathbb{S}^+ \cup \{0\}$, we can restrict our attention to positively ordered monoids.

Definition 2.2.3. A subset $A \subseteq \mathbb{S}$ of a positively totally ordered monoid $\mathbb{S}$ is called *Archimedean* if for every $a, b \in A \setminus \{0\}$ such that $a < b$ there is $n \in \omega$ such that $n \cdot a \geq b$, where $n \cdot a = a + ... + a$ denotes the sum of n-many elements equal to a.

We say that $\mathbb{S}$ is Archimedean if $A = \mathbb{S}$ is Archimedean. For $a, b \in \mathbb{S}$, we write $a \ll b$ when instead $n \cdot a < b$ for every $n \in \omega$. This way, a (positively totally ordered) monoid is Archimedean if and only if $a \ll b$ implies $a = 0$.

Definition 2.2.4. A positively totally ordered monoid $\mathbb{S}$ is said to be *initially Archimedean* if there is $a \in \mathbb{S}^+$ such that $(0, a)$ is Archimedean.

Definition 2.2.5. Let $\mathbb{S} = (S, 0, +, \leq)$ be a positively totally ordered monoid. Let $\bar{a} = \langle a_i \mid i < \gamma \rangle$ be a sequence of elements in $\mathbb{S}$. We say that $\bar{a}$ is:

(1) **decreasing** if $a_i \leq a_j$ for all $j < i < \gamma$,

(2) **strictly decreasing** if $a_i < a_j$ for all $j < i < \gamma$,

(3) **strongly decreasing** if $a_i + a_i < a_j$ for all $j < i < \gamma$,

(4) **nowhere Archimedean (decreasing)** if $a_i \ll a_j$ for all $j < i < \gamma$.

Remark 2.2.6. Note that, for a sequence $\bar{a}$ of elements in a positively totally ordered monoid $\mathbb{S}$, we have (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

The following lemma shows that every monoid is either pathological, initially Archimedean of countable degree, or non-Archimedean in a strong way:

Lemma 2.2.7. *Let $\mathbb{S}$ be an positively totally ordered monoid. Then, exactly one of the following holds:*

(1) *There is $c \in \mathbb{S}^+$ such that $b + b \geq c$ for every $b \in \mathbb{S}^+$.*

(2) $\mathrm{Deg}(\mathbb{S}) = \omega$ *and $\mathbb{S}^+$ contains a countable strongly decreasing Archimedean sequence $\bar{a} = \langle a_i \mid i < \omega \rangle$ coinitial in $\mathbb{S}^+$.*

(3) $\mathrm{Deg}(\mathbb{S}) = \delta \geq \omega$ *and $\mathbb{S}^+$ contains a nowhere Archimedean decreasing sequence $\bar{a} = \langle a_i \mid i < \delta \rangle$ coinitial in $\mathbb{S}^+$.*

Proof. Assume that we are not in the first case. Then, for every $c \in \mathbb{S}^+$ there is $b \in \mathbb{S}^+$ such that $b + b < c$. This implies that $\delta = \mathrm{Deg}(\mathbb{S}) \geq \omega$ is infinite (and regular). Let thus $\bar{a} = \langle a_i \mid i < \delta \rangle$ be coinitial in $\mathbb{S}^+$, with $\delta = \mathrm{Deg}(\mathbb{S})$.

We recursively define a strongly decreasing subsequence of $\bar{a}$. Let $\alpha_0 = 0$. Now consider $i < \delta$, and assume that we have already defined $\alpha_j \geq j$ for all $j < i$. Let $\gamma = \sup\{\alpha_j \mid j < i\}$. We have that $i \leq \gamma < \delta$, by the regularity of δ. Since $\gamma < \delta = \mathrm{Deg}(\mathbb{S})$, there is $c \in \mathbb{S}^+$ such that $c \leq a_j$ for every $j \leq \gamma$. Thus, by assumption, we can find some $b \in \mathbb{S}^+$ such that $b + b < c$. Now, if it exists, let b_γ be such that $b_\gamma \ll c$. Otherwise, let $b_\gamma = b$. Since $\bar{a}$ is coinitial in $\mathbb{S}^+$, there is an $\alpha_i < \delta$ such that $a_{\alpha_i} \leq b_\gamma$, and thus $a_{\alpha_i} + a_{\alpha_i} \leq b_\gamma + b_\gamma < a_j$ for every $j \leq \gamma$. This also implies $\alpha_i > \gamma$. It follows that the sequence $\bar{a}' = \langle a_{\alpha_i} \mid i < \delta \rangle$ obtained in this way is strongly decreasing and coinitial.

Now suppose first that for every $c \in \mathbb{S}^+$ there is $b \in \mathbb{S}^+$ such that $b \ll c$. By construction, this implies that $\bar{a}' = \langle a_{\beta_i} \mid i < \delta \rangle$ is nowhere Archimedean, as wanted.

Conversely, assume that there is $c \in \mathbb{S}^+$ such that for all $b \in \mathbb{S}^+$ we have $n \cdot b \geq c$ for some $n \in \omega$. This implies that the whole interval $[0, c]$ is Archimedean. Notice that if $\bar{a}' = \langle a_{\alpha_i} \mid i < \delta \rangle$ is a strongly decreasing sequence convergent to 0 for an uncountable cardinal δ, then $\bar{a}' = \langle a_{\alpha_i} \mid i < \delta, i \text{ limit} \rangle$ is a nowhere Archimedean sequence of the same length converging to 0. Thus, in order for $[0, c]$ to be Archimedean, we must have $\mathrm{Deg}(\mathbb{S}) = \delta = \omega$. Let $j < \delta$ be such that $a_{\alpha_j} < c$. Then, we have that $\bar{a}' = \langle a_{\alpha_{j+i}} \mid i < \omega \rangle$ is Archimedean, strongly decreasing, and strongly convergent to 0, as wanted. □

Notice that monoids of infinite, countable degree $\mathrm{Deg}(\mathbb{S}) = \omega$ can be of any of the three types described in Lemma 2.2.7. In this case, the only thing the makes the three cases mutually exclusive is the behaviour of the operation $+$.

Following the notation from [Rei79], we call 0-continuous the non-pathological monoids, i.e. those that satisfy point (2) or point (3) of Lemma 2.2.7.

Definition 2.2.8. We call **0-continuous** a totally ordered monoid $\mathbb{S}$ such that $\mathbb{S}^+$ contains a strongly decreasing coinitial sequence.

From Lemma 2.2.7, we also get the following interesting corollary:

Corollary 2.2.9. *Every 0-continuous positively totally ordered monoid of uncountable degree is not Archimedean (and not even initially Archimedean).*

Notice that (densely ordered) groups are always 0-continuous, and if a monoid contains a dense, 0-continuous submonoid, then it is 0-continuous itself.

2.2.2 Infinite operations on monoids

To define λ^+-measures, we need a structure where sums of λ-many elements can be computed: ideally, a monoid with a sum of arity at least λ. Infinitary operations and their properties have been widely studied, mostly for the case of groups and fields, even with a specific focus on the purpose of generalising analysis and descriptive set theory (see, e.g., [Won23]). We give a minimal set of axioms that we believe are necessary to define measures, and show that structures satisfying these properties exist.

Recall that for every set A, a sequence of elements of A is a function from an ordinal γ into A. We denote by ${}^{<\mathsf{On}}A = \bigcup_{\gamma \in \mathsf{On}} {}^{\gamma}A$ the class of sequences of elements of A of any length. Given a sequence $s = (s_\alpha)_{\alpha<\gamma} \in {}^{<\mathsf{On}}A$, a sequence t is a *subsequence* of s if there is a set $I = \{\alpha_i \mid i < \delta\} \subseteq \gamma$ such that $\alpha_i < \alpha_j$ for all $i < j < \delta$ and $t = \langle s_{\alpha_i} \mid i < \delta \rangle$. Given $B \subseteq A$, denote by $s \restriction B$ the subsequence of s obtained by removing all elements outside of B from s, i.e. $s \restriction B = (s_{\alpha_i})_{i<\delta}$ where $\{\alpha_i \mid i < \delta\}$ is the increasing enumeration of the ordinals of $\{\alpha < \gamma \mid s_\alpha \in B\}$. A *reordering* or *permutation* of s is a sequence of the form $(s_{\pi(j)})_{j<\delta}$ for some bijection $\pi : \delta \to \gamma$.

We denote by Conc the extension of the binary operation of concatenation of sequences $^\frown$ to all sequences of sequences in ${}^{<\mathsf{On}}({}^{<\mathsf{On}}A)$. Formally, we define $\mathrm{Conc}(\emptyset) = \emptyset$. Then, proceeding recursively on the length, given a sequence $\bar{s} = \langle s^\alpha \mid \alpha < \beta \rangle$ of sequences $s^\alpha = (a_i^\alpha)_{i<\mathrm{lh}(s^\alpha)} \in {}^{<\mathsf{On}}A$, if β is limit, define

$$\mathrm{Conc}(\bar{s}) = \mathrm{Conc}_{\alpha<\beta}\, s^\alpha = \bigcup_{\varepsilon<\beta} \mathrm{Conc}_{\alpha<\varepsilon}\, s^\alpha = \bigcup_{\varepsilon<\beta} \mathrm{Conc}(\bar{s} \restriction \varepsilon),$$

while if $\beta = \gamma + 1$, define

$$\mathrm{Conc}(\bar{s}) = \mathrm{Conc}_{\alpha<\beta}\, s^\alpha = (\mathrm{Conc}_{\alpha<\gamma}\, s^\alpha)^\frown s^\gamma = \mathrm{Conc}(\bar{s} \restriction \gamma)^\frown s^\gamma.$$

Definition 2.2.10. Let $\mathbb{S} = (S, 0, +, \leq)$ be a positively totally ordered monoid. An *infinitary operation* or *infinitary sum* is a partial function $\mathrm{Sum} : {}^{<\mathsf{On}}\mathbb{S} \to \mathbb{S}$ satisfying:

(S1) Sum is *compatible with* $+$, i.e.,

- $\mathrm{Sum}(\emptyset) = 0$,
- $\mathrm{Sum}(s) = s_0 + \ldots + s_n$ for every non-empty finite $s = (s_i)_{i\leq n} \in \mathrm{dom}(\mathrm{Sum})$,

(S2) Sum is *continuous*, i.e., for all $s \in \mathrm{dom}(\mathrm{Sum})$ such that $\mathrm{lh}(s)$ is limit, if $s \restriction \alpha \in \mathrm{dom}(s)$ for cofinally many $\alpha < \mathrm{lh}(s)$, then

$$\mathrm{Sum}(s) = \sup\{\mathrm{Sum}(s \restriction \alpha) \mid \alpha < \mathrm{lh}(s), s \restriction \alpha \in \mathrm{dom}(\mathrm{Sum})\}.$$

Furthermore, we say that Sum is *natural* if additionally

(S3) $\mathrm{dom}(\mathrm{Sum})$ is *downward closed*, i.e., for all $s \in \mathrm{dom}(\mathrm{Sum})$ and $\alpha < \mathrm{lh}(s)$ we have

$$s \restriction \alpha \in \mathrm{dom}(s),$$

(S4) Sum is *associative*, i.e., for every $\bar{s} = (s^\alpha)_{\alpha<\gamma} \in {}^{<\mathsf{On}}\mathrm{dom}(\mathrm{Sum})$ such that $\mathrm{Conc}(\bar{s}) \in \mathrm{dom}(\mathrm{Sum})$, we have

- $\langle \mathrm{Sum}(s^\alpha) \mid \alpha < \gamma \rangle \in \mathrm{dom}(\mathrm{Sum})$,
- $\mathrm{Sum}(\mathrm{Conc}(\bar{s})) = \mathrm{Sum}(\langle \mathrm{Sum}(s^\alpha) \mid \alpha < \gamma \rangle)$,

(S5) Sum is *commutative*, i.e., for every $s \in \mathrm{dom}(\mathrm{Sum})$ and any reordering s' of s we have

- $s' \in \mathrm{dom}(\mathrm{Sum})$,
- $\mathrm{Sum}(s) = \mathrm{Sum}(s')$,

For the purpose of defining measures, we can use infinitary sums of any form. However, Proposition 2.3.7 shows that even if we define a measure using a sum that is not natural, the sum needs to be natural on the subset of its domain where the measure is defined.

Proposition 2.2.15 shows that (positively totally ordered) monoids always comes naturally equipped with an infinitary operation. We first give the explicit definition of this operation:

Definition 2.2.11. Let $\mathbb{S} = (S, 0, +, \leq)$ be a positively totally ordered monoid. The partial function $\sum : {}^{<\mathsf{On}}\mathbb{S} \to \mathbb{S}$ is defined by

$$\sum(s) = \sup\{s_{\alpha_0} + \ldots + s_{\alpha_k} \mid \alpha_0 < \ldots < \alpha_k < \gamma\}$$

for every ordinal γ and every sequence $s = \langle s_\alpha \mid \alpha < \gamma \rangle \in {}^\gamma\mathbb{S}$ such that the supremum $\sup\{s_{\alpha_0} + \ldots + s_{\alpha_k} \mid \alpha_0 < \ldots < \alpha_k < \gamma\}$ exists in $\mathbb{S}$.

Let us recall the following:

Definition 2.2.12. A linear order is *Dedekind-complete* if every bounded set has a supremum and an infimum.

Every linear order $(\mathbb{S}, \leq)$ has a (unique, up to isomorphism) Dedekind-completion $(\hat{\mathbb{S}}, \preceq)$ in which it is dense (see, e.g., [Mac37, Fuc63, FM08]).

Given $A, B \subseteq \mathbb{S}$, we write $\sup(A) \leq \sup(B)$ (even when $\sup(A)$ or $\sup(B)$ is not defined) if for every $c \in \mathbb{S}$, if $b \leq c$ for every $b \in B$, then $a \leq c$ for every $a \in A$. It is easy to check that when both $\sup(A)$ and $\sup(B)$ are defined, this coincide with the usual $\leq$ relationship of $\mathbb{S}$. We also write $\sup(A) = \sup(B)$ if both $\sup(A) \leq \sup(B)$ and $\sup(A) \geq \sup(B)$, and similarly for other expressions like $\sup(A) \leq b$ for $b \in \mathbb{S}$ (using $b = \sup(\{b\})$, and so on. With the same meaning, given $s, t \in {}^{<\mathsf{On}}\mathbb{S}$ we write $\sum(s) \leq \sum(t)$, $\sum(s) = \sum(t)$, $\sum(s) \leq b$ for some $b \in \mathbb{S}$, and so on, even when $\sum(s)$ or $\sum(t)$ is not defined.

Remark 2.2.13. This coincides with comparing sup of sets and $\sum$ of sequences in the Dedekind-completion[1].

Definition 2.2.14 ([Fuc63, Chapter XI.7])**.** A monoid $\mathbb{S}$ is *lower semi-continuous* if for every $A, B \subseteq \mathbb{S}$ such that $a = \sup A$ and $b = \sup B$ exist, then $a + b = \sup\{x + y \mid x \in A, y \in B\}$.

Proposition 2.2.15. *For every positively totally ordered monoid $\mathbb{S} = (S, 0, +, \leq)$, the operation $\sum$ is an infinitary sum which additionally satisfies the following:*

(a) 0 is the neutral element *of $\sum$, i.e., $\sum(s) = \sum(s \restriction (\mathbb{S} \setminus \{0\}))$ for every $s \in {}^{<\mathsf{On}}\mathbb{S}$.*
In particular, $s \in \mathrm{dom}(\sum)$ if and only if $s \restriction (\mathbb{S} \setminus \{0\}) \in \mathrm{dom}(\sum)$.

(b) $\sum$ is infinitary-translation invariant*, i.e.*

$$\sum(s) \leq \sum(t)$$

for every $s = (s_i)_{i<\gamma}, t = (t_i)_{i<\delta} \in {}^{<\mathsf{On}}\mathbb{S}$ such that $\gamma \leq \delta$ and $s_i \leq t_i$ for every $i < \gamma$.

(c) If the order of $\mathbb{S}$ is Dedekind-complete, then $\sum$ is total*, i.e., it is defined on every sequence of any length.*
In particular, in this case $\sum$ satisfies Axiom (S3).

(d) If the order of $\mathbb{S}$ is Dedekind-complete and the operation $+$ of $\mathbb{S}$ is lower-semicontinuous, then $\sum$ satisfies Axiom (S4),

(e) The operation $+$ of $\mathbb{S}$ is commutative if and only if $\sum$ satisfies Axiom (S5).

Proof. First, let us prove that $\sum$ is an infinitary operation. It is clear that $\sum$ satisfies Axioms (S1). To see that $\sum$ satisfies Axiom (S2), suppose that $s \in \mathrm{dom}(\sum)$ has limit length, and that there is a sequence $\langle \beta_i \mid i < \gamma \rangle$ cofinal in $\mathrm{lh}(s)$ such that $s \restriction \beta_i \in \mathrm{dom}(\sum)$ for all $i < \gamma$. For every $\alpha < \gamma$, let $i(\alpha)$ be minimal such that $\alpha < \beta_{i(\alpha)}$. Then, for each $\alpha_0 < ... < \alpha_n < \gamma$ we have $s_{\alpha_0} + ... + s_{\alpha_n} \leq \sum(s \restriction \beta_{i(\alpha_n)}) \leq \sum(s)$, by definition of $\sum$. This shows that $\sup\{\sum(s \restriction \beta_i) \mid i < \gamma\}$ exists and

$$\sum(s) = \sup\{s_{\alpha_0} + ... + s_{\alpha_n} \mid n < \omega\} \leq \sup\{\sum(s \restriction \beta_i) \mid i < \gamma\} \leq \sum(s),$$

and Axiom (S2) is satisfied, as wanted.

That $\sum$ is infinitary-translation invariant and 0 is the neutral element follow by definition. Thus statements (a) and (b) hold. It is also immediate that if the order of $\mathbb{S}$ is Dedekind-complete, then $\sum$ is total, and that the operation $+$ of $\mathbb{S}$ is commutative iff $\sum$ satisfies Axiom (S5). Thus statements (c) and (e) hold.

[1] The Dedekind-completion $(\hat{\mathbb{S}}, \leq)$ is a total order, but this is sufficient to define $\sum$ on all sequences of $\mathbb{S}$. We do not need to define $\sum$ on sequences of $\hat{\mathbb{S}}$

Finally, to prove statement (d) (and that $\sum$ satisfies Axiom (S4)), assume $+$ is lower semi-continuous and $\leq$ is Dedekind-complete. Then, by the previous point, $\sum$ is total. Consider first $s, t \in {}^{<\mathsf{On}}\mathbb{S}$. Then,

$$\begin{gathered}\sum s + \sum t = \\ = \sup\{s_{i_0} + \ldots + s_{i_k} \mid i_0 {<} \ldots {<} i_k {<} \operatorname{lh}(s)\} + \sup\{t_{j_0} + \ldots + t_{j_k} \mid j_0 {<} \ldots {<} j_k {<} \operatorname{lh}(t)\} = \\ = \sup\{s_{i_0} + \ldots + s_{i_k} + t_{j_0} + \ldots + t_{j_k} \mid i_0 < \ldots < i_k < \operatorname{lh}(s), j_0 < \ldots < j_k < \operatorname{lh}(t)\} = \\ = \sum (s \frown t)\end{gathered}$$

by definition of $\sum$ and of lower semi-continuity. By induction, it is easy to see that

$$\sum (s^0 \frown \ldots \frown s^k) = \sum s^0 + \ldots + \sum s^k$$

for all finite $\bar{s} = (s^i)_{i \leq k} \in {}^{<\omega}\operatorname{dom}(\mathrm{Sum})$. Now fix $\beta \geq \omega$. Let $\bar{s} = \langle s^\alpha : \alpha < \beta \rangle$ be such that $s^\alpha \in {}^{<\mathsf{On}}\mathbb{S}$ for every $\alpha < \beta$. Notice that

$$\sum (\operatorname{Conc}(\bar{s})) \leq \sum (\langle \sum (s^\alpha) : \alpha < \beta \rangle),$$

by definition of $\sum$. On the other hand, we have

$$\sum (\langle \sum (s^\alpha) : \alpha < \beta \rangle) = \sup\{\sum (s^{\alpha_0}) + \ldots + \sum (s^{\alpha_n}) : \alpha_0 < \ldots < \alpha_n < \beta\},$$

and by previous argument for finite sequences, we get

$$\sum (s^{\alpha_0}) + \ldots + \sum (s^{\alpha_n}) = \sum (s^{\alpha_0} \frown \ldots \frown s^{\alpha_n}) \leq \sum (\operatorname{Conc}(\bar{s})).$$

All together, this shows that $\sum(\operatorname{Conc}(\bar{s})) = \sum(\langle \sum (s^\alpha) : \alpha < \beta \rangle)$, as wanted. □

Corollary 2.2.16. *For every Dedekind-complete, lower semi-continuous, commutative positively totally ordered monoid $\mathbb{S} = (S, 0, +, \leq)$, the operation $\sum$ is a natural infinitary sum.*

In Proposition 2.2.21, we show that there are Dedekind-complete, lower semi-continuous, commutative positively totally ordered monoids of any desired degree. In particular, this shows that there are positively totally ordered monoids of any degree admitting a natural infinitary sum.

The following proposition shows that all natural infinitary sums come from restrictions of $\sum$:

Proposition 2.2.17. *Let $\mathbb{S} = (S, 0, +, \leq)$ be a positively totally ordered monoid with a natural infinitary sum $\mathrm{Sum} : {}^{<\mathsf{On}}\mathbb{S} \to \mathbb{S}$. Then, we have that $\operatorname{dom}(\mathrm{Sum}) \subseteq \operatorname{dom}(\sum)$ and $\mathrm{Sum}(s) = \sum(s)$ for every $s \in \operatorname{dom}(\mathrm{Sum})$.*

Proof. Assume $\mathrm{Sum} : {}^{<\mathsf{On}}\mathbb{S} \to \mathbb{S}$ is a natural infinitary sum. Fix $s = (s_\alpha)_{\alpha < \gamma} \in \operatorname{dom}(\mathrm{Sum})$. If γ is finite we are done, by Axiom (S1), so suppose not. Let $\alpha_0 < \ldots < \alpha_k < \gamma$, and let $t = (s_{\alpha_i})_{i<k}$. By Axiom (S5), there is a reordering $s' \in \operatorname{dom}(\mathrm{Sum})$ of s of limit length such that $t = s' \restriction k$. This shows that $t \in \operatorname{dom}(\mathrm{Sum})$, by Axiom (S3), and $\mathrm{Sum}(t) \leq \mathrm{Sum}(s') = \mathrm{Sum}(s)$ by Axioms (S2) and (S5). By Axiom (S1), we have

$$s_{\alpha_0} + \ldots + s_{\alpha_k} = \mathrm{Sum}(t) \leq \mathrm{Sum}(s).$$

In particular, this shows that $\sum(s) \leq \mathrm{Sum}(s)$, if the former is defined, and that the sum $+$ of $\mathbb{S}$ is commutative when restricted to $\{s_\alpha \mid \alpha < \operatorname{lh}(s)\}$, since Sum satisfies Axiom (S5).

By induction on γ, we now show that $s \in \operatorname{dom}(\sum)$ and

$$\mathrm{Sum}(s) = \sup\{s_{\alpha_0} + \ldots + s_{\alpha_k} \mid \alpha_0 < \ldots < \alpha_k < \gamma\} = \sum(s),$$

as wanted.

It is clear that this is true if γ is finite, by Axiom (S1). Also, if $\gamma = \beta + 1$ is infinite and the statement holds for β, then

$$\mathrm{Sum}(s) = \mathrm{Sum}(s_\beta{}^\frown(s \restriction \beta)) = \sum(s_\beta{}^\frown(s \restriction \beta)) = \sum(s),$$

by Axiom (S5), induction hypothesis, and commutativity of $\sum$ on $\{s_\alpha \mid \alpha < \mathrm{lh}(s)\}$. Suppose instead that γ is limit and the statement holds for all $\beta < \gamma$. By Axioms (S2) and (S3), we have $s \restriction \alpha \in \mathrm{dom}(\mathrm{Sum})$ for every $\alpha < \gamma$ and

$$\mathrm{Sum}(s) = \sup\{\mathrm{Sum}(s \restriction \alpha) \mid \alpha < \gamma\}.$$

By induction hypothesis, we get $s \restriction \alpha \in \mathrm{dom}(\sum)$ for every $\alpha < \gamma$ and

$$\mathrm{Sum}(s) = \sup\{\sum(s \restriction \alpha) \mid \alpha < \gamma\}.$$

This shows that

$$\sup\{\sum(s \restriction \alpha) \mid \alpha < \gamma\} = \sup\{s_{\alpha_0} + ... + s_{\alpha_k} \mid \alpha_0 < ... < \alpha_k < \gamma\}$$

is defined, and thus $s \in \mathrm{dom}(\sum)$ and $\mathrm{Sum}(s) = \sup\{\sum(s \restriction \alpha) \mid \alpha < \gamma\} = \sum(s)$ as wanted. □

In particular, by Proposition 2.2.17 we get the following:

Remark 2.2.18. Every natural infinitary sum satisfies the same properties of $\sum$ (e.g., those of Proposition 2.2.15), restricted to its domain.

2.2.3 Examples of monoids

We provide the reader with a reasonable collection of examples of positively totally ordered monoids equipped with an infinitary sum. These examples show that this class of structures is rich in its variety, and contains both well-behaved structures such as $\mathbb{R}_\infty$, but also quite pathologically behaved monoids.

Example 2.2.19. We denote by $\mathbb{R}_\infty = ([0, \infty], 0, +, \leq)$ the totally ordered monoid where $[0, \infty] \setminus \{\infty\} = [0, \infty)$ is the positive cone of the real numbers, with usual order and operation, and ∞ is the maximum of the order and an absorbing element for the operation (i.e. $a + \infty = \infty + a = \infty$ for every $a \in [0, \infty]$). Then, $\mathbb{R}_\infty$ is an initially Archimedean, Dedekind-complete, lower semi-continuous, 0-continuous, commutative, positively totally ordered monoid, thus $\sum$ is a total natural infinitary sum on it, by Proposition 2.2.15.

The operation $\sum$ can be explicitly defined for any sequence $r = \langle r_\beta \mid \beta < \gamma \rangle$ of elements of $[0, \infty]$ in the following way:

- $\sum r = 0$ if r is empty or all elements of r are zero,
- $\sum r = r_{n_1} + ... + r_{n_k}$ if r does not contain ∞ and has exactly k-many elements $\{r_{n_1}, ..., r_{n_k}\}$ different from 0,
- $\sum r = \sum_{i=0}^\infty r_{n_i}$ if r does not contain ∞ and it has countably many elements that are different from 0 and $\{r_{n_i} \mid i < \omega\}$ is an enumeration of them (by the absolute convergence theorem, the sum $\sum_{i=0}^\infty r_{n_i}$ is independent of the chosen enumeration),
- $\sum r = \infty$ otherwise.

Another class of examples is given by Dedekind-complete total orders.

Example 2.2.20. Let $(S, \leq)$ be a Dedekind-complete linear order with a minimum 0. Then, $\mathbb{S} = (S, 0, \max, \leq)$ is a Dedekind-complete, lower semi-continuous, commutative, positively totally ordered monoid, and the infinitary operation $\sum = \sup$ is a total natural infinitary sum. If $\mathrm{Deg}(\mathbb{S})$ is infinite, then $\mathbb{S}$ is also 0-continuous.

Notice that there are Dedekind-complete linear orders of any coinitiality (e.g., $\kappa+1$ with reverse order is a Dedekind-complete linear order of coinitiality κ for any cardinal κ). This shows that for any κ, there exists a Dedekind-complete, positively totally ordered commutative monoid $\mathbb{S}$ of degree $\mathrm{Deg}(\mathbb{S}) = \kappa$ with a total natural infinitary sum.

Other, less trivial examples are given by the completion of groups.

Proposition 2.2.21. *Let $\mathbb{G} = (G, 0, +, \leq)$ be a commutative densely totally ordered group. Then, the operation $+$ and the order $\leq$ can be extended to the Dedekind-completion $S_{\mathbb{G}}$ of $\mathbb{G}$ so that $\mathbb{S}_{\mathbb{G}} = (S_{\mathbb{G}}, 0, +, \leq)$ is a Dedekind-complete, lower semi-continuous, totally ordered commutative monoid having $\mathbb{G}$ as dense subgroup. In particular, $\{0\} \cup \mathbb{S}^+_{\mathbb{G}}$ is a Dedekind-complete, lower semi-continuous, 0-continuous, positively totally ordered commutative monoid such that $\mathrm{Deg}(\mathbb{S}_{\mathbb{G}}) = \mathrm{Deg}(\mathbb{G})$.*

Proof. Notice that every totally ordered group is a topological group (see, e.g., Theorem 10 and the following paragraph of [Fuc63, Chapter II.8]). In particular, this shows that the operation $+$ is lower semi-continuous on $\mathbb{G}$. By [FM08], we can extend the operation $+$ and the order $\leq$ to $\mathbb{S}_{\mathbb{G}}$ so that it is a Dedekind-complete, commutative totally ordered monoid having $\mathbb{G}$ as dense subgroup, and the operation is lower semi-continuous by definition. □

In general, the operation $\sum$ need not be natural. The following example shows that $\sum$ does not necessarily satisfy Axiom (S3) nor Axiom (S4), even among lower semi-continuous, commutative, positively totally ordered monoids.

Example 2.2.22. Let $\mathbb{Q}_\infty = (\mathbb{Q} \cup \{\infty\}) \subseteq \mathbb{R}_\infty$. Then, $\mathbb{Q}_\infty$ is a lower semi-continuous, commutative, positively totally ordered submonoid of $\mathbb{R}_\infty$. Let $(s_n)_{n<\omega}$ be a sequence of rational numbers converging to π (or your favorite irrational number). Let also $t_{\gamma+n} = s_n$ for every limit ordinal γ and natural number $n < \omega$. Let $t = (t_\alpha)_{\alpha<\omega^2}$. Then, we have $\sum(t) = \infty$, however $t \restriction \alpha \notin \mathrm{dom}(\sum)$ for every $\alpha < \mathrm{lh}(t)$.

The following example shows that $\sum$ does not necessarily satisfy Axiom (S4), even among Dedekind-complete, commutative, positively totally ordered monoids.

Example 2.2.23. Let $\mathbb{S} = ([0, \infty) \cup \{\infty_0, \infty_1\}, 0, +, \leq)$ where $[0, \infty)$ is the positive cone of the real numbers and $+$ and $\leq$ are its usual operation and order. Define also $a \leq \infty_0 < \infty_1$ for all $a \in [0, \infty)$, and $a + \infty_i = \infty_i + a = \infty_i$ for any $a \in [0, \infty)$, and $\infty_i + \infty_j = \infty_1$ for every $i, j \in \{0, 1\}$. Then, $\mathbb{S}$ is a Dedekind-complete, commutative, positively totally ordered monoid, yet $\sum$ is not associative, as

$$\sum(1^{(\omega)} {}^\frown 1^{(\omega)}) = \sum(1^{(\omega+\omega)}) = \infty_0 \neq \infty_1 = \infty_0 + \infty_0 = \sum(1^{(\omega)}) + \sum(1^{(\omega)}).$$

Example 2.2.24. Let $(\mathbb{G}, 0, +, \leq)$ be a commutative group of degree $\mathrm{Deg}(\mathbb{G}) = \kappa$, and let $\mathbb{S}_{\mathbb{G}}$ be the positive part of its Dedekind-completion. Let $\mathbb{S} = \{0\} \times \mathbb{S}_{\mathbb{G}} \cup \{1\} \times \mathbb{S}_{\mathbb{G}} \sqcup \{\infty\}$, where the order is given by $(i, a) \leq (i, b)$ for any $i \in \{0, 1\}$ and $a, b \in \mathbb{S}_{\mathbb{G}}$ satisfying $a \leq b$, and $(0, a) \leq (1, b) \leq \infty$ for any $a, b \in \mathbb{S}_{\mathbb{G}}$. Let $*$ be defined by $(i, a) * (j, b) = (i + j, a + b)$ if $i + j \leq 1$, $(1, a) * (1, b) = \infty$, and $(i, a) * \infty = \infty * (i, a) = \infty$. Then, $(\mathbb{S}, (0, 0), *, \leq)$ is a Dedekind-complete, commutative, positively totally ordered monoid, yet $\sum$ is not associative.

2.3 Measures

In this section, we extend the notions of measure and measure space to the generalised setting, working under the assumptions that λ is an infinite cardinal of cofinality κ, that (X, τ) is a subspace of the generalised Baire space ${}^{<\kappa}\lambda$, and that $\mathbb{S} = (S, 0, +, \leq)$ is a positively totally ordered monoid.

For the classical theory of measures, we refer the reader to [Kec95] and [Rud87]. Recall that a *(classical) measurable space* on a set X is a pair $(X, \mathfrak{M})$ where X is a set and $\mathfrak{M}$ is a σ-algebra on it. If X is a topological space, then we also require that $\tau \subseteq \mathfrak{M}$, i.e., $\mathfrak{M}$ extends the σ-algebra of Borel sets of X. On classical measurable spaces, one usually defines measures in the following way.

Definition 2.3.1. A **(classical) measure** μ on a measurable space $(X, \mathfrak{M})$ is a partial function

$$\mu : \mathscr{P}(X) \to \mathbb{R}_\infty$$

satisfying that $\mathfrak{M} \subseteq \mathrm{dom}(\mu)$ and

$$\mu(\bigcup_{i\in\omega} A_i) = \sum_{i\in\omega} \mu(A_i)$$

for any countable family $(A_i)_{i<\omega} \subseteq M$ of disjoint sets.

We call **(classical) measure space** a triple $(X, \mathfrak{M}, \mu)$ where $(X, \mathfrak{M})$ is a measurable space and $\mu : \mathfrak{M} \to \mathbb{R}_\infty$ is a measure on X.

Let us recall the following result:

Fact 2.3.2 ([Rud87, Theorem 1.19])**.** Let $(X, \mathfrak{M}, \mu)$ be a measure space. Then:

(1) $\mu(\emptyset) = 0$,

(2) $\mu(\bigcup_{i\in\alpha} A_i) = \sum_{i\in\alpha} \mu(A_i)$ for any family $(A_i)_{i<\alpha} \subseteq M$ of disjoint sets of any size $\alpha \leq \omega$.

(3) $A \subseteq B$ implies $\mu(A) \leq \mu(B)$ for all $A, B \in \mathfrak{M}$,

(4) $\mu(A) = \sup_{i\in\omega} \mu(A_i)$ for every chain $(A_i)_{i<\omega} \subseteq M$ such that $A_i \subseteq A_{i+1}$ for every $i \in \omega$.

(5) If $\mu(A_0) < \infty$, then $\mu(A) = \inf_{i\in\omega} \mu(A_i)$ for every chain $(A_i)_{i<\omega} \subseteq M$ such that $A_i \supseteq A_{i+1}$ for every $i \in \omega$.

We intend to extend the definition of classical measure to the uncountable case. The properties of classical measures showed in Fact 2.3.2 heavily depend on the structure of $\mathbb{R}$ and the fact that $\mathfrak{M}$ is a σ-algebra. When trying to move from classical measures to some kind of more general notion and work in a setting where the properties of $\mathbb{R}$ may no longer be present, we need to impose these conditions as part of the extended definition. For example, Axiom (M5), that we assume as part of our definition of weak λ^+-measure, is not a generalisation of the axioms of classical measure, but rather a generalisation of (a weakening of) Fact 2.3.2(5). On the other hand, since we aim to the most general possible notion of measure, we relax the hypotheses as much as we can. As the following definitions show, we do not need to require in general that $\mathsf{Bor}(X) \subseteq \mathfrak{M}$, or even that $\mathfrak{M}$ be closed under complements or intersections.

Definition 2.3.3. A *weakly λ^+-measurable space* is a pair $(X, \mathfrak{M})$ where X is a topological space and $\mathfrak{M}$ is a family of subsets of X closed under unions of size $\leq \lambda$ and containing all open subsets of X.

We will sometimes call *weakly measurable space* to pairs $(X, \mathfrak{M})$ that are weakly λ^+-measurable for some λ^+, if we don't want to focus our attention on that particular λ.

Definition 2.3.4. Given a weakly λ^+-measurable space $(X, \mathfrak{M})$, we say that μ is a *λ^+-measure* on $(X, \mathfrak{M})$, and that $(X, \mathfrak{M}, \mu)$ is a *weak λ^+-measure space*, if μ is a partial function $\mu : \mathscr{P}(X) \to \mathbb{S}$ into some infinitary monoid $(\mathbb{S}, \mathrm{Sum})$ such that $\mathfrak{M} \subseteq \mathrm{dom}(\mu)$ and:

(M1) $\mu(\emptyset) = 0$,

(M2) $\mu(X) > 0$,

(M3) $A \subseteq B$ implies $\mu(A) \leq \mu(B)$ for all $A, B \in \mathfrak{M}$,

(M4) for every family of disjoint sets $(A_i)_{i<\gamma} \subseteq \mathfrak{M}$ of order type $\gamma < \lambda^+$, we have

- $\langle \mu(A_i) \mid i < \gamma \rangle \in \mathrm{dom}(\mathrm{Sum})$,
- $\mu(\bigcup_{i\in\gamma} A_i) = \mathrm{Sum}_{i\in\gamma}\, \mu(A_i)$,

(M5) μ is *point-regular*, i.e., for every $x \in X$ such that $x \in \mathfrak{M}$ and for every (open) local basis $\{A_i \mid i < \gamma\}$ at x of order type $\gamma < \lambda^+$,

$$\mu(x) = \inf_{i<\gamma} \mu(A_i)^{2}.$$

As before, we adopt the convention of calling *weak measure space* to triples $(X, \mathfrak{M}, \mu)$ that are weak λ^+-measure space for some λ^+ if we are not interested in a particular λ.

Remark 2.3.5. The notion of ω^+-measure properly extends the notion of classical measure by allowing one to use monoids other than $\mathbb{R}_\infty$ and measurable structures that are not σ-algebras.

Adopting the notation of [Kec95, Exercise 17.4], we also define the following.

Definition 2.3.6. Given a weak λ^+-measure space $(X, \mathfrak{M}, \mu)$, we say that μ is *continuous* if all points are measurable of measure $\mu(x) = 0$.

Even though the definition of a weak λ^+-measure can use any kind of infinitary sum Sum, the next proposition shows that this Sum must be natural at least on the range of μ. Since the values of Sum outside the range of μ do not matter in this context, Proposition 2.2.17 then implies that $\sum$ is essentially the only natural infinitary sum that can be used to define measures.

Proposition 2.3.7. *Let $\mathbb{S}$ be a positively totally ordered monoid with an infinitary operation* Sum, *and let $(X, \mathfrak{M}, \mu)$ be a weakly λ^+-measurable space such that μ takes values in $(\mathbb{S}, \mathrm{Sum})$. Let*

$$\mathcal{D} = \{\langle \mu(A_i) \mid i < \gamma \rangle \mid (A_i)_{i<\gamma} \text{ is a family of disjoint sets of } \mathfrak{M} \text{ of length } \gamma < \lambda^+\}$$

be the set of sequences of length $< \Lambda^+$ in the range of μ. Then, we have that $\mathrm{Sum} \restriction \mathcal{D}$ is natural, and thus

$$\mu(\bigcup_{i\in\gamma} A_i) = \sum_{i\in\gamma} \mu(A_i)$$

for every family of disjoint sets $(A_i)_{i<\gamma} \subseteq \mathfrak{M}$ of order type $\gamma < \lambda^+$.

Proof. We want to show that $\mathrm{Sum} \restriction \mathcal{D}$ is natural, and the rest will follow from Proposition 2.2.17. First, it is clear that $\mathrm{Sum} \restriction \mathcal{D}$ satisfies Axioms (S1) and (S2) since Sum does, and it satisfies Axiom (S3) by definition of $\mathcal{D}$. It is also immediate to check that it satisfies Axiom (S5), by Axiom (M4) and since $\bigcup$ is commutative.

Finally, for Axiom (S4) notice that if $\bar{s} = (s^\alpha)_{\alpha<\gamma} \in {}^{<\mathsf{On}}\mathcal{D}$ and $\mathrm{Conc}(\bar{s}) \in \mathcal{D}$, and $\{A_i^\alpha \mid \alpha < \gamma, i < \mathrm{lh}(s^\alpha)\}$ are disjoint sets in $\mathfrak{M}$ witnessing that $\mathrm{Conc}(\bar{s}) \in \mathcal{D}$, then $\langle \bigcup_{i<\mathrm{lh}(s^\alpha)} A_i^\alpha \mid \alpha < \gamma \rangle$ is still a sequence of disjoint sets in $\mathfrak{M}$ of length $\gamma < \lambda^+$. Thus,

$$\langle \mathrm{Sum}(s^\alpha) \mid \alpha < \gamma \rangle = \langle \mu(\bigcup_{i<\mathrm{lh}(s^\alpha)} A_i^\alpha) \mid \alpha < \gamma \rangle \in \mathcal{D}$$

and

$$\begin{aligned} \mathrm{Sum}(\mathrm{Conc}(\bar{s})) = \mu(\bigcup_{\alpha<\gamma} \bigcup_{i<\mathrm{lh}(s^\alpha)} A_i^\alpha) = \\ = \mathrm{Sum}(\langle \mu(\bigcup_{i<\mathrm{lh}(s^\alpha)} A_i^\alpha) \mid \alpha < \gamma \rangle) = \\ = \mathrm{Sum}(\langle \mathrm{Sum}(s^\alpha) \mid \alpha < \gamma \rangle) \end{aligned}$$

as wanted. □

Thus, without loss of generality, for the rest of the chapter we work under the assumption that all measures are defined using the infinitary sum $\sum$ from Definition 2.2.11. By restricting the domain of $\sum$ to $\mathcal{D}$, in practice we can always assume that $\sum$ is natural (on the relevant part of its range).

[2] As customary in the literature, we write $x \in \mathfrak{M}$ and $\mu(x)$ in place of $\{x\} \in \mathfrak{M}$ and $\mu(\{x\})$.

2.3.1 Properties of measures

Every classical measure is also countable subadditive, i.e., $\mu(\bigcup \mathcal{V}) \leq \sum_{V \in \mathcal{V}} \mu(V)$ for every countable family of measurable sets $\mathcal{V} \subseteq \mathfrak{M}$. We can define an analogue property in this context.

Definition 2.3.8. A weak λ^+-measure space $(X, \mathfrak{M}, \mu)$ is λ^+*-subadditive* if for every family $\mathcal{V} \subseteq \mathfrak{M}$ of size $|\mathcal{V}| \leq \lambda$, we have $\mu(\bigcup \mathcal{V}) \leq \sum_{V \in \mathcal{V}} \mu(V)$.

In general, from the axioms of Definition 2.3.4 alone we can not conclude that every weak λ^+-measure space $(X, \mathfrak{M}, \mu)$ is λ^+-subadditive, as the following shows:

Example 2.3.9. Let $\mathbb{Q} + \pi = \{\pi + q \mid q \in \mathbb{Q}\}$, and let

$$\mathfrak{M} = \{O \cup D \mid O \in \tau, D \in \{\emptyset, \mathbb{R} \setminus \mathbb{Q}, \mathbb{R} \setminus (\mathbb{Q} + \pi)\}\}.$$

Then, it is immediate to check that $\mathfrak{M}$ is a weak ω^+-measurable structure on $\mathbb{R}$. Let $\mu_L : \mathfrak{M}_L \to \mathbb{R}_\infty$ be the usual Lebesgue measure on $\mathbb{R}$. For every $M \in \mathfrak{M}$, define

$$\mu(M) = \begin{cases} \mu_L(M) & \text{if } M \in \tau, \\ \mu_L(\text{int}(M)) + 1 & \text{otherwise.} \end{cases}$$

It is easy to check that μ satisfies Axiom (M1)-(M2), since μ_L does. Also notice that if $M \in \mathfrak{M} \setminus \tau$, then for every $M' \in \mathfrak{M}$ such that $M \subseteq M'$ we have either $M' \notin \tau$, or $\mu_L(M') = \infty$, thus Axiom (M3) holds as well. Since no point is measurable in $\mathfrak{M}$, then μ satisfies Axiom (M5). Finally, if $\mathcal{A}$ is a family of disjoint measurable sets of size ≥ 2, then $\mathcal{A} \subseteq \tau$, since $\mathbb{R} \setminus \mathbb{Q}$ and $\mathbb{R} \setminus (\mathbb{Q} + \pi)$ are dense in $\mathbb{R}$ and have non-empty intersection. Thus μ satisfies Axiom (M4). However,

$$\mu((\mathbb{R} \setminus \mathbb{Q}) \cup (\mathbb{R} \setminus (\mathbb{Q} + \pi))) = \mu(\mathbb{R}) = \infty \not\leq 2 = \mu(\mathbb{R} \setminus \mathbb{Q}) + \mu(\mathbb{R} \setminus (\mathbb{Q} + \pi)),$$

thus μ is not even finitely subadditive.

However, in every weak measure space $(X, \mathfrak{M}, \mu)$ where $\mathfrak{M}$ is sufficiently well-behaved, the weak measure μ is indeed subadditive.

Definition 2.3.10. Given a weak λ^+-measure space $(X, \mathfrak{M}, \mu)$, we say that a family $\mathcal{V} \subseteq \mathfrak{M}$ is λ^+*-partitionable* if there exists a partition $\mathbb{P}_{\vec{\mathcal{U}}} \subseteq \mathfrak{M}$ of size $< \lambda^+$ of $\bigcup \mathcal{V}$ refining $\mathcal{V}$. We say that $(X, \mathfrak{M}, \mu)$ is λ^+*-partitionable* if every family $\mathcal{V} \subseteq \mathfrak{M}$ of size $|\mathcal{V}| < \lambda^+$ is λ^+-partitionable.

Every weak λ^+-measure space $(X, \mathfrak{M}, \mu)$ where $\mathfrak{M}$ is closed under complements (and thus, a λ^+-algebra) is also λ^+-partitionable. However, these are not the only cases of λ^+-partitionable, weak λ^+-measure spaces (see Proposition 2.3.24).

Proposition 2.3.11. *Let $(X, \mathfrak{M}, \mu)$ be a weak λ^+-measure space. Then, for every λ^+-partitionable family $\mathcal{V} \subseteq \mathfrak{M}$ we have $\mu(\bigcup \mathcal{V}) \leq \sum_{V \in \mathcal{V}} \mu(V)$.*

Recall that we follow the convention introduced in Remark 2.2.13 and preceding paragraph: the fact that we write $\mu(\bigcup \mathcal{V}) \leq \sum_{V \in \mathcal{V}} \mu(V)$ does not imply that $\sum_{V \in \mathcal{V}} \mu(V)$ exists in $\mathbb{S}$.

Proof. Let $\mathbb{P}_{\vec{\mathcal{U}}} \subseteq \mathfrak{M}$ be a family of disjoint sets of size $< \lambda^+$ refining $\mathcal{V}$ such that $\bigcup \mathbb{P}_{\vec{\mathcal{U}}} = \bigcup \mathcal{V}$. We can find a partition $\mathbb{P}_{\vec{\mathcal{U}}} = \bigcup_{V \in \mathcal{V}} \mathbb{P}_{\vec{\mathcal{U}}V}$ of $\mathbb{P}_{\vec{\mathcal{U}}}$ such that $P \subseteq V$ for every $P \in \mathbb{P}_{\vec{\mathcal{U}}V}$. Thus, $(\bigcup \mathbb{P}_{\vec{\mathcal{U}}V})_{V \in \mathcal{V}}$ is still a partition of $\bigcup \mathcal{V}$ refining $\mathcal{V}$ with $\leq \lambda$ elements different from $\emptyset$, and each $\bigcup \mathbb{P}_{\vec{\mathcal{U}}V}$ is measurable since $\mathfrak{M}$ is closed under unions of size $\leq \lambda$. By Axiom (M3), we get $\mu(\bigcup \mathbb{P}_{\vec{\mathcal{U}}V}) \leq \mu(V)$ for every $V \in \mathcal{V}$, and so

$$\mu(\bigcup \mathcal{V}) = \mu(\bigcup_{\substack{V \in \mathcal{V}, \\ \mathbb{P}_{\vec{\mathcal{U}}V} \neq \emptyset}} \bigcup \mathbb{P}_{\vec{\mathcal{U}}V}) = \sum_{\substack{V \in \mathcal{V}, \\ \mathbb{P}_{\vec{\mathcal{U}}V} \neq \emptyset}} \mu(\bigcup \mathbb{P}_{\vec{\mathcal{U}}V}) = \sum_{V \in \mathcal{V}} \mu(\bigcup \mathbb{P}_{\vec{\mathcal{U}}V}) \leq \sum_{V \in \mathcal{V}} \mu(V)$$

by Axiom (M4) and by Proposition 2.2.15 (using that $\sum$ is infinitary-translation invariant and $0 = \mu(\emptyset)$ is the neutral element of $\sum$). $\square$

Corollary 2.3.12. *Let $(X, \mathfrak{M}, \mu)$ be a λ^+-partitionable, weak λ^+-measure space. Then, μ is λ^+-subadditive.*

Another important property of classical measure spaces is that adding a set of measure zero should not increase the measure of a set. Here, we get something similar under the additional assumption that the weak λ^+-measure space is λ^+-subadditive.

Definition 2.3.13. Let $(X, \mathfrak{M}, \mu)$ be a weak λ^+-measure space. A subset $N \subseteq X$ is said to be *essentially null* if for every $M, M' \in \mathfrak{M}$, we have that

$$M \setminus N = M' \setminus N \qquad \text{implies} \qquad \mu(M) = \mu(M').$$

Remark 2.3.14. Let $(X, \mathfrak{M}, \mu)$ be a weak λ^+-measure space. If $N \subseteq X$ is essentially null and $N' \subseteq N$, then N' is essentially null as well, i.e., essentially null sets are closed under subsets.

Lemma 2.3.15. *Let $(X, \mathfrak{M}, \mu)$ be a λ^+-subadditive, weak λ^+-measure space. Then, every set of measure zero is also essentially null.*

Proof. Let $M, M', N \in \mathfrak{M}$ be such that $\mu(N) = 0$ and $M \setminus N = M' \setminus N$. Then $M \subseteq M' \cup N$ and $M' \subseteq M \cup N$. Thus, if μ is λ^+-subadditive, we get

$$\mu(M) \leq \mu(M' \cup N) \leq \mu(M') + \mu(N) = \mu(M') + 0 = \mu(M')$$

and viceversa, therefore $\mu(M) = \mu(M')$, as wanted. □

Since these properties seem quite important, it might make sense to assume then. However, in practice this is not needed, for they are automatically satisfied in a substructure of any weak λ^+-measure space (see Proposition 2.3.24).

2.3.2 Minimal λ^+-measurable spaces

In order to prove our impossibility result, we introduce a simple class of weak λ^+-measurable spaces, to which we can always restrict when needed.

Definition 2.3.16. A *minimal λ^+-measurable space* is a weakly λ^+-measurable space $(X, \mathfrak{M})$ such that for every $M \in \mathfrak{M}$ there is a partition $M = D \cup O$ satisfying that $|D| \leq \lambda$, $x \in \mathfrak{M}$ for every $x \in D$, and $O \in \tau$ is open. A *minimal λ^+-measure space* is a weak λ^+-measure space $(X, \mathfrak{M}, \mu)$ where $(X, \mathfrak{M})$ is called a minimal λ^+-measurable space.

Notice that given a weakly λ^+-measurable space $(X, \mathfrak{M})$, if D and O are subsets of X satisfying that $|D| \leq \lambda$, $x \in \mathfrak{M}$ for every $x \in D$, and $O \in \tau$, then we get $D \in \mathfrak{M}$ and $O \in \mathfrak{M}$, by definition of weak λ^+-measure. Thus, in a minimal λ^+-measurable space $(X, \mathfrak{M})$ we have $M \in \mathfrak{M}$ if and only if there is a partition $M = D \cup O$ satisfying that $|D| \leq \lambda$, $x \in \mathfrak{M}$ for every $x \in D$, and $O \in \tau$ is open.

Definition 2.3.17. Given $A \subseteq X$, let $\mathfrak{M}_p^\lambda(A)$ be the minimal family of subsets of X containing all open sets and all points of A and closed under unions of size $\leq \lambda$. Equivalently,

$$\mathfrak{M}_p^\lambda(A) := \bigcap \{\mathfrak{M} : (X, \mathfrak{M}) \text{ is a weakly } \lambda^+\text{-measurable space} \wedge x \in \mathfrak{M} \text{ for all } x \in A\}.$$

It is easy to check then that $(X, \mathfrak{M}_p^\lambda(A))$ is a minimal λ^+-measure space. Conversely, given weakly λ^+-measurable space $(X, \mathfrak{M})$, if $A = \{x \in X \mid x \in \mathfrak{M}\}$, then we get $\mathfrak{M}_p^\lambda(A) \subseteq \mathfrak{M}$. Thus, for every minimal λ^+-measure space $(X, \mathfrak{M}, \mu)$, we must have $\mathfrak{M} = \mathfrak{M}_p^\lambda(A)$ for $A = \{x \in X \mid x \in \mathfrak{M}\}$. In particular, $(X, \mathfrak{M}_p^\lambda(X))$ is the unique minimal λ^+-measurable space measuring all points of X, and any continuous minimal λ^+-measure space must be of the form $(X, \mathfrak{M}_p^\lambda(X), \mu)$ for some μ. This also shows that every weakly λ^+-measurable space contains a minimal one.

Remark 2.3.18. Every weakly λ^+-measurable space $(X, \mathfrak{M})$ contains a minimal λ^+-measurable space $(X, \mathfrak{M}_p^\lambda(A))$ such that $\mathfrak{M}$ and $\mathfrak{M}_p^\lambda(A)$ measures the same family of points A.

Notice that if $(X, \mathfrak{M})$ is a weakly λ^+-measurable subspace of $(X, \mathfrak{M}')$ (i.e., $\mathfrak{M}' \subseteq \mathfrak{M}$) and μ is a weak λ^+-measure on $\mathfrak{M}'$, then $\mu \restriction \mathfrak{M}$ is still a weak λ^+-measure. Thus, we get the following:

Fact 2.3.19. If there exists a weak λ^+-measure space $(X, \mathfrak{M}, \mu)$ on X, then there exists a minimal λ^+-measure space on X measuring the same points of μ.

As a consequence, without loss of generality, we can restrict our study to minimal λ^+-measurable spaces: if we show that finding such a minimal structure is impossible, then we get automatically that finding anything else is impossible.

Minimal λ^+-measurable spaces have a nicer behaviour than most weakly λ^+-measurable spaces, which makes them ideal candidates to work with:

Proposition 2.3.20. *For every minimal λ^+-measurable space $(X, \mathfrak{M})$ we have that $\mathfrak{M}$ is closed under intersections of size $< \kappa$.*

Proof. Let $(X, \mathfrak{M})$ be a λ^+-measurable space, and let $(M_i)_{i<\gamma}$ be a sequence of elements of $\mathfrak{M}$ of length $\gamma < \kappa$. For every $i < \gamma$, let $D_i, O_i \in \mathfrak{M}$ be such that $M_i = D_i \cup O_i$, $|D_i| \leq \lambda$, $x \in \mathfrak{M}$ for every $x \in D_i$, and $O \in \tau$. Let $M = \bigcap_{i<\gamma} M_i$. Then, if $O = \bigcap_{i<\gamma} O_i$, we have that O is open, by κ-additivity of ${}^\kappa\lambda$. Also,

$$D = M \setminus O \subseteq \bigcup_{i<\gamma} D_i$$

has size $\leq \lambda$, since $\gamma < \kappa \leq \lambda$, and $x \in \mathfrak{M}$ for every $x \in D$. Thus $M = D \cup O \in \mathfrak{M}$ as well, as wanted. □

For a subspace $X \subseteq {}^\kappa\lambda$, we denote by $\mathcal{B}(X)$ the canonical basis of X made of clopen cones $\mathbf{N}_s(X)$.

The following well-known fact (see, e.g., [Nyi99, Theorem 2.7] or [AMR]) shows that such spaces are zero-dimensional in a strong sense: every open cover can be refined by a clopen partition made of basic clopen cones.

Lemma 2.3.21. *For every family $\mathcal{V}$ of open sets of X there is a family $\mathbb{P}_{\vec{\mathcal{U}}} \subseteq \mathcal{B}(X)$ of disjoint cones which refines $\mathcal{V}$ and satisfies that $\bigcup \mathbb{P}_{\vec{\mathcal{U}}} = \bigcup \mathcal{V}$.*

As a corollary, we get the following:

Corollary 2.3.22. *Let $(X, \mathfrak{M}, \mu)$ be a weak λ^+-measure space. Then every family of open sets is partitionable.*

As an immediate consequence, we get the following:

Corollary 2.3.23. *Let $(X, \mathfrak{M}, \mu)$ be a weak λ^+-measure space. Then, the family $\mathcal{N} = \{V \in \tau \mid \mu(V) = 0\}$ is closed under unions of size $\leq \lambda$.*

In minimal λ^+-measure spaces, all families of measurable sets are essentially made of open sets, modulo a set of small size. By Lemma 2.3.21, we get then that they are λ^+-partitionable:

Proposition 2.3.24. *Assume $\lambda^{<\kappa} = \lambda$. Under this assumption, every minimal λ^+-measure space is λ^+-partitionable.*

Proof. Let $(X, \mathfrak{M}, \mu)$ be a minimal λ^+-measure space. Consider $\mathcal{V} \subseteq \mathfrak{M}$ of size $\leq \lambda$. Then, for each $M \in \mathcal{V}$ we can find two disjoint sets $O(M) \in \tau$ and $D(M) \in \mathfrak{M}$ such that $M = O(M) \cup D(M)$, $|D(M)| \leq \lambda$, and $x \in \mathfrak{M}$ for every $x \in D(M)$. By Lemma 2.3.21, we can find a clopen partition $\mathbb{P}_{\vec{\mathcal{U}}1}$ of $O = \bigcup_{M \in \mathcal{V}} O(M)$ refining $\{O(M) \mid M \in \mathcal{V}\}$ (and thus $\mathcal{V}$). Notice that $\mathbb{P}_{\vec{\mathcal{U}}1} \subseteq \mathfrak{M}$, since $\mathfrak{M}$ contains all open subsets of X. Also, $|\mathbb{P}_{\vec{\mathcal{U}}1}| \leq \lambda$, since $\lambda^{<\kappa} = \lambda$ implies that ${}^\kappa\lambda$ (and thus X) has weight $\leq \lambda$. Let also $D = \bigcup \mathcal{V} \setminus O$, and $D' = \bigcup_{M \in \mathcal{V}} D(M)$. Notice that $D \subseteq D'$: this implies both that $|D| \leq \lambda$, and that $x \in \mathfrak{M}$ for every $x \in D$. Therefore, $\mathbb{P}_{\vec{\mathcal{U}}2} = \{\{x\} \mid x \in D\}$ is a partition of D refining $\mathcal{V}$. Thus, $\mathbb{P}_{\vec{\mathcal{U}}1} \cup \mathbb{P}_{\vec{\mathcal{U}}2}$ is a partition of $\bigcup \mathcal{V}$ refining $\mathcal{V}$, as wanted. □

By Corollary 2.3.12, we get the following:

Corollary 2.3.25. *Every minimal λ^+-measure space is λ^+-subadditive.*

2.3.3 Extensions and restrictions of measures

Given a weak λ^+-measure space $(Y, \mathfrak{M}, \mu)$, we define a way to induce a structure on a superspace $X \supseteq Y$.

Definition 2.3.26. Given two spaces $Y \subseteq X$ and a weak λ^+-measure space $(Y, \mathfrak{M}, \mu)$, define the set

$$\mathfrak{M} \uparrow X = \{A \subseteq X \mid A \cap Y \in \mathfrak{M}\},$$

and let

$$\mu \uparrow X : \mathfrak{M} \uparrow X \to \mathbb{S}$$

be the function given by

$$(\mu \uparrow X)(B) = \mu(B \cap Y) \text{ for all } B \in \mathfrak{M} \uparrow X.$$

Conversely, given a weak λ^+-measure space $(X, \mathfrak{M}, \mu)$, we define the a way to induce a structure on a subspace $Y \subseteq X$.

Definition 2.3.27. Given two spaces $Y \subseteq X$ and a weak λ^+-measure space $(X, \mathfrak{M}, \mu)$, define

$$\mathfrak{M} \downarrow Y = \{M \cap Y \mid M \in \mathfrak{M}\}$$

and let $\mu \downarrow Y$ be the partial function given by

$$(\mu \downarrow Y)(A) = \inf\{\mu(M) \mid M \in \mathfrak{M}, M \cap Y = A\}$$

for all $A \in \mathfrak{M} \downarrow Y$ such that the above inf exists.

The following lemmata give sufficient conditions for $(X, \mathfrak{M} \uparrow X, \mu \uparrow X)$ or $(Y, \mathfrak{M} \downarrow Y, \mu \downarrow Y)$ to be weak λ^+-measure spaces.

Lemma 2.3.28. *For every topological space X and every weak λ^+-measure space $(Y, \mathfrak{M}, \mu)$ such that $Y \subseteq X$, we have that $(X, \mathfrak{M} \uparrow X)$ is a weakly λ^+-measurable space, and $\mu \uparrow X$ satisfies Axioms (M1)-(M4). Furthermore, if Y is closed in X, then $(X, \mathfrak{M} \uparrow X, \mu \uparrow X)$ is a weak λ^+-measure space.*

Proof. Since $\mathfrak{M}$ is closed under unions of size $\leq \lambda$, so is $\mathfrak{M} \uparrow X$. Besides, $\mathfrak{M} \uparrow X$ contains all open sets of X, since the restriction of an open set of X to Y is still open in Y and $\mathfrak{M}$ contains all open sets of Y. It is also immediate to check that $\mu \uparrow X$ satisfies Axioms (M1)-(M4), since μ satisfies them. Now if Y is closed in X, then for every $x \in X$ we have that either $x \in Y$ and we are done because μ satisfies Axiom (M5); or $x \notin Y$ and thus

$$(\mu \uparrow X)(x) = 0 = (\mu \uparrow X)(X \setminus Y),$$

which shows once again that $\mu \uparrow X$ satisfies Axiom (M5) for x. □

Lemma 2.3.29. *Let $(X, \mathfrak{M}, \mu)$ be a weak λ^+-measure space such that $\mathfrak{M}$ is closed under finite intersections. Let $Y \in \mathfrak{M}$ be a set of positive measure $\mu(Y) > 0$. Then, $(Y, \mathfrak{M} \downarrow Y, \mu \downarrow Y)$ is a weak λ^+-measure space.*

Proof. Since $Y \in \mathfrak{M}$ and $\mathfrak{M}$ is closed under intersections, we get $M \cap Y \in \mathfrak{M}$ for every $M \in \mathfrak{M}$, and so $\mathfrak{M} \downarrow Y = \{A \in \mathfrak{M} \mid A \subseteq Y\} \subseteq \mathfrak{M}$. Also, $\mu(M \cap Y) \leq \mu(M)$ for every $M \in \mathfrak{M}$ by Axiom (M3), thus $\inf\{\mu(M) \mid M \in \mathfrak{M}, M \cap Y = A\} = \mu(A)$ exists for every $A \in \mathfrak{M} \downarrow Y$, and so $\mu \downarrow Y = \mu \upharpoonright (\mathfrak{M} \downarrow Y)$. Therefore, it is immediate to check that $(Y, \mathfrak{M} \downarrow Y)$ is a weakly λ^+-measurable space, since $(X, \mathfrak{M})$ is, and that $\mu \downarrow Y$ is a weak λ^+-measure on it, since μ is. □

Lemma 2.3.30. *Let $(X, \mathfrak{M}, \mu)$ be a λ^+-partitionable, weak λ^+-measure space, and let $Y \subseteq X$ be such that $X \setminus Y$ is essentially null. Then, $(Y, \mathfrak{M} \downarrow Y, \mu \downarrow Y)$ is a weak λ^+-measure space.*

Proof. By definition of essentially null set, we get that $\mu(M) = \mu(M')$ for all $M, M' \in \mathfrak{M}$ such that $M \cap Y = M' \cap Y$, thus $\mu \downarrow Y$ is defined on the whole $\mathfrak{M} \downarrow Y$. It is then immediate to check that $(Y, \mathfrak{M} \downarrow Y)$ is a weakly λ^+-measurable space, since $(X, \mathfrak{M})$ is. It is also easy to check that $\mu \downarrow Y$ satisfies Axioms (M1), (M2), and (M5), since μ does.

Let $A, B \in \mathfrak{M} \downarrow Y$ be such that $A \subseteq B$, and let $M_A, M_B \in \mathfrak{M}$ be such that $A = M_A \cap Y$ and $B = M_B \cap Y$. Notice that $Y \cap (M_A \cup M_B) = B$, and $M_A \cup M_B \in \mathfrak{M}$ since $\mathfrak{M}$ is closed under unions. Thus, we may assume $M_A \subseteq M_B$. Then, by Axioms (M3) applied to μ we have

$$(\mu \downarrow Y)(A) = \mu(M_A) \leq \mu(M_B) = (\mu \downarrow Y)(B)$$

and Axioms (M3) holds for $\mu \downarrow Y$ as well.

Now let $\mathcal{A} \subseteq \mathfrak{M} \downarrow Y$ be a family of disjoint measurable sets of size $|\mathcal{A}| < \lambda^+$. Let $\mathcal{A}' = \{M_A \mid A \in \mathcal{A}\} \subseteq \mathfrak{M}$ be a family of measurable sets such that $M_A \cap Y = A$ for every $A \in \mathcal{A}$. Since $(X, \mathfrak{M}, \mu)$ is λ^+-partitionable, there is a family of disjoint sets $\mathbb{P}_{\vec{\mathcal{U}}} \subseteq \mathfrak{M}$ refining $\mathcal{A}'$ and such that $\bigcup \mathbb{P}_{\vec{\mathcal{U}}} = \bigcup \mathcal{A}'$. Since $\mathbb{P}_{\vec{\mathcal{U}}}$ refines $\mathcal{A}'$, using the axiom of choice we can partition $\mathbb{P}_{\vec{\mathcal{U}}}$ into families $\{\mathbb{P}_{\vec{\mathcal{U}}A} \mid A \in \mathcal{A}\}$ such that $P \in \mathbb{P}_{\vec{\mathcal{U}}A}$ implies $P \subseteq M_A$. Notice that since $\mathcal{A}$ is made of disjoint sets, we have that $P \cap A \neq \emptyset$ implies $P \in \mathbb{P}_{\vec{\mathcal{U}}A}$ for every $P \in \mathbb{P}_{\vec{\mathcal{U}}}$. Since $Y \cap \bigcup \mathbb{P}_{\vec{\mathcal{U}}} = \bigcup \mathcal{A}$, then we get $Y \cap \bigcup \mathbb{P}_{\vec{\mathcal{U}}A} = A$. Therefore,

$$(\mu \downarrow Y)(\bigcup \mathcal{A}) = \mu(\bigcup \mathcal{A}') = \mu(\bigcup_{A \in \mathcal{A}} \bigcup \mathbb{P}_{\vec{\mathcal{U}}A}) = \sum_{A \in \mathcal{A}} \mu(\bigcup \mathbb{P}_{\vec{\mathcal{U}}A}) =$$
$$= \sum_{A \in \mathcal{A}} (\mu \downarrow Y)(\bigcup \mathbb{P}_{\vec{\mathcal{U}}A} \cap Y) = \sum_{A \in \mathcal{A}} (\mu \downarrow Y)(A)$$

and Axiom (M4) is satisfied. □

Then, by Corollary 2.3.12 and Lemma 2.3.15:

Corollary 2.3.31. *Let $(X, \mathfrak{M}, \mu)$ be a λ^+-partitionable, weak λ^+-measure space, and let $Y \subseteq X$ be such that $X \setminus Y \in \mathfrak{M}$ and $\mu(X \setminus Y) = 0$. Then, $(Y, \mathfrak{M} \downarrow Y, \mu \downarrow Y)$ is a weak λ^+-measure space.*

By Proposition 2.3.20, minimal λ^+-measure spaces are closed under finite intersections. By Proposition 2.3.24, they are λ^+-partitionable. Then, by Lemma 2.3.29 and Corollary 2.3.31, the following holds:

Corollary 2.3.32. *Let $(X, \mathfrak{M}, \mu)$ be a minimal λ^+-measure space, and let $Y \subseteq X$ be such that one of the following holds:*

- *$Y \in \mathfrak{M}$ and $\mu(Y) > 0$,*
- *$X \setminus Y \in \mathfrak{M}$ and $\mu(X \setminus Y) = 0$.*

Then, $(Y, \mathfrak{M} \downarrow Y, \mu \downarrow Y)$ is a minimal λ^+-measure space.

When we extend a measure from Y to X using $\uparrow$, the set $X \setminus Y$ is always essentially null. For these sets, the operations $\uparrow$ and $\downarrow$ are then the inverses of each other:

Remark 2.3.33. For every weak λ^+-measure space $(Y, \mathfrak{M}, \mu)$ and $X \supseteq Y$, we have $(Y, \mathfrak{M}, \mu) = (Y, (\mathfrak{M} \uparrow X) \downarrow Y, (\mu \uparrow X) \downarrow Y)$. Conversely, for every λ^+-partitionable, weak λ^+-measure space $(X, \mathfrak{M}, \mu)$ and every $Y \supseteq X$ such that $X \setminus Y$ is essentially null, we have $\mathfrak{M} \subseteq (\mathfrak{M} \downarrow Y) \uparrow X$ and $\mu \subseteq (\mu \downarrow Y) \uparrow X$.

2.4 On the existence of (trivial) λ^+-measures

In this section, we provide explicit examples of λ^+-measures on (subspaces of) ${}^\kappa\lambda$ that are, in one way or another, not really satisfactory for the purposes of generalised descriptive set theory.

In the classical case, one usually regards as trivial those measures that are focused on a single point. Let us introduce them:

Definition 2.4.1. A **Dirac measure** is a measure $\delta : \mathscr{P}(X) \to \mathbb{S}$ on $(X, \mathscr{P}(X))$ such that there is $x \in X$ satisfying $\delta(M) > 0$ if and only if $x \in M$ for every $M \subseteq X$.

Notice that every Dirac measure δ on X is constant on the family of subsets of X of positive measure. Indeed, if x is the only point of positive measure of δ, then $\delta(X) = \delta(X \setminus \{x\}) + \delta(x) = \delta(x)$, and thus $\delta(x) \leq \delta(M) \leq \delta(X) = \delta(x)$ for every $M \subseteq X$ with $x \in M$. In particular, $(X, \mathscr{P}(X), \delta)$ is a weak λ^+-measure space for every cardinal λ.

Dirac measures are essentially measures defined on a single point and then extended to a larger space (this can even be formalized using the operation $\uparrow$ from Definition 2.3.26). For this reason, among others, they are not a particularly interesting example of a classical measure.

In a similar way, in the study of ${}^\kappa\lambda$, those measures that are essentially defined on a small part of ${}^\kappa\lambda$ (according to some notion of "small") and then extended to the whole space are not particularly interesting. Such measures do exist, and are not limited to Dirac measures. For example, if $\kappa = \omega$, then ${}^\omega\omega$ is a subspace of ${}^\kappa\lambda$, so any measure on ${}^\omega\omega$ can be used to define a measure on all of ${}^\kappa\lambda$:

Example 2.4.2. Let $\mu : {}^\omega\omega \to [0, \infty]$ be a classical measure on $({}^\omega\omega, \mathsf{Bor}({}^\omega\omega))$. Notice that the restriction of a Borel set of ${}^\omega\lambda$ to ${}^\omega\omega$ is Borel in ${}^\omega\omega$. Thus, for every Borel subset $B \subseteq {}^\omega\lambda$, we can define

$$\mu_\omega(B) = \mu(B \cap {}^\omega\omega).$$

Then, $\mu_\omega : {}^\omega\lambda \to [0, \infty]$ is a (classical) measure on $({}^\omega\lambda, \mathsf{Bor}({}^\omega\lambda))$.

The same process can be done starting from any separable subspace $Y \subseteq {}^\kappa\lambda$ and any weak measurable structure on it. We isolate the class of measures obtained this way in the following definition:

Definition 2.4.3. We say that a weak λ^+-measurable space $(X, \mathfrak{M}, \mu)$ is *countable-induced* if there is a separable subspace $Y \subseteq X$ such that $X \setminus Y$ is measurable of measure zero.

Notice that countable-induced measures extend the notion of Dirac measures, for a measure δ on $\mathscr{P}(X)$ is a Dirac measure if and only if there is a subspace $\{x\} \subseteq X$ of density 1 (instead of separable) such that $\mu(X \setminus \{x\}) = 0$:

Remark 2.4.4. Every Dirac measure is countable-induced.

In Definition 2.4.3, we used "separable" as a notion of smallness. However, we could have used, equivalently, "second countable". Indeed, when $\kappa > \omega$ all separable and all second countable subspaces of ${}^\kappa\lambda$ are countable and discrete, since ${}^\kappa\lambda$ is κ-additive. When $\kappa = \omega$ instead, we get that ${}^\kappa\lambda$ is metrizable, and thus a subspace $Y \subseteq {}^\kappa\lambda$ is separable if and only if it is second countable (see, e.g., [Eng89, Theorem 4.1.15]). Either way, we get the following:

Remark 2.4.5. For $Y \subseteq {}^\kappa\lambda$, the following are equivalent:

- Y is separable,
- Y is second countable,
- Y is homeomorphic to a subspace of ${}^\omega\omega$.

In particular, a weak λ^+-measure space $(X, \mathfrak{M}, \mu)$ is countable-induced if and only if there an embedding $f : Y \subseteq {}^\omega\omega \to X$ such that $X \setminus Y \in \mathfrak{M}$ and $\mu(X \setminus Y) = 0$.

Notice that if $Y \subseteq X$ is separable, then $\mathrm{cl}(Y)$ is separable as well. Since $X \setminus \mathrm{cl}(Y)$ is open, it is in particular measurable in every weakly measurable space $(X, \mathfrak{M})$, and by Axiom (M3) we get $\mu(X \setminus \mathrm{cl}(Y)) \leq \mu(X \setminus Y)$. Therefore, the following holds:

Remark 2.4.6. A weak λ^+-measure space $(X, \mathfrak{M}, \mu)$ is countable-induced if and only if there a closed, separable subspace $Y \subseteq X$ such that $\mu(X \setminus Y) = 0$.

Recall that every countable, discrete space is Polish. Thus, when $\kappa > \omega$, any separable subspace of $X \subseteq {}^\kappa\lambda$ is Polish. If X is G_δ in ${}^\omega\lambda$, then every closed subspace of X is G_δ in ${}^\omega\lambda$ as well, thus Polish (see, e.g., [Eng89, Theorem 4.3.23]). Therefore, by Remark 2.4.6, we get the following:

Remark 2.4.7. Let $(X, \mathfrak{M}, \mu)$ be a weak λ^+-measure space such that X is G_δ in ${}^\kappa\lambda$. Then, $(X, \mathfrak{M}, \mu)$ is countable-induced if and only if there is a Polish closed subspace $Y \subseteq X$ such that $\mu(X \setminus Y) = 0$.

As anticipated, all measures defined through the $\uparrow$ operation described in Definition 2.3.26 starting from a separable subspace of X are countable-induced. For λ^+-partitionable spaces, the converse is true as well:

Proposition 2.4.8. *Let $(X, \mathfrak{M}, \mu)$ be a weak λ^+-measurable space. Then, the first statement implies the second:*

(a) *There is a separable subspace $Y \subseteq X$ such that $(Y, \mathfrak{M} \downarrow Y, \mu \downarrow Y)$ is a weak λ^+-measurable space, $\mathfrak{M} \subseteq (\mathfrak{M} \downarrow Y) \uparrow X$, and $\mu \subseteq (\mu \downarrow Y) \uparrow X$.*

(b) *$(X, \mathfrak{M}, \mu)$ is countable-induced.*

Furthermore, if λ^+-partitionable, then the converse (b)⇒(a) holds as well.

Proof. First, let $Y \subseteq X$ be a separable subspace such that $(Y, \mathfrak{M} \downarrow Y, \mu \downarrow Y)$ is a weak λ^+-measurable space, $\mathfrak{M} \subseteq (\mathfrak{M} \downarrow Y) \uparrow X$, and $\mu \subseteq (\mu \downarrow Y) \uparrow X$. Then, $Y' = \mathrm{cl}(Y)$ is separable as well and $\mu(X \setminus Y') = 0$ since $\mu \subseteq (\mu \downarrow Y) \uparrow X$, as wanted. Conversely, assume $(X, \mathfrak{M}, \mu)$ is countable-induced. By Remark 2.4.6, there is a closed, separable $Y \subseteq X$ such that $\mu(X \setminus Y) = 0$. By Corollary 2.3.31, we get that $(Y, \mathfrak{M} \downarrow Y, \mu \downarrow Y)$ is a weak λ^+-measurable space, and the inclusions follows by Remark 2.3.33. □

Since $\mathfrak{M} \downarrow Y$ is a λ^+-algebra whenever $\mathfrak{M}$ is, we get the following:

Remark 2.4.9. In Proposition 2.4.8, if $\mathfrak{M}$ is a σ-algebra, then $(Y, \mathfrak{M} \downarrow Y)$ is a classical measurable space. Furthermore, if μ takes values in $\mathbb{S} = \mathbb{R}_\infty$, then $\mu \downarrow Y$ is a classical measure.

Notice that if (Y, τ) is a topological space and $\mathfrak{M}_\tau = \tau$, then $(Y, \mathfrak{M}_\tau)$ is a weak λ^+-measurable space for any λ. In a second countable space Y, every family of disjoint open sets is at most countable, and a point $x \in X$ is measurable if and only if it is isolated, in which case Axiom (M5) is trivially satisfied. Thus, if ν is a weak ω^+-measure on $(Y, \mathfrak{M}_\tau)$, then it is also a weak λ^+-measure on it.

Proposition 2.4.10. *If (Y, τ) is a second countable space, then any minimal ω^+-space $(Y, \mathfrak{M}_\tau, \nu)$ is also a minimal λ^+-space (for any λ).*

Corollary 2.4.11. *If $(X, \mathfrak{M}_\tau, \mu)$ is a countable-induced weak ω^+-space, then it is also a weak λ^+-measure space (for any λ).*

This shows that weak λ^+-measures in general do exists, although they may fail to measure points. However, if $\lambda < \mathfrak{c}$, then countable-induced measures can also measure points.

Proposition 2.4.12. *Assume $\kappa = \omega$ and $\lambda < \mathfrak{c}$. Then, there is a (countable-induced) continuous weak λ^+-measure space $({}^\kappa\lambda, \mathfrak{M}, \mu)$ on ${}^\kappa\lambda$.*

Proof. Let μ' be a classical measure on ${}^\omega\omega$. For every $A \subseteq {}^\kappa\lambda$, denote by $\mathrm{int}(A)$ the interior of A, i.e. the union of all open subsets U contained in A. Let $\mathfrak{M} = \mathfrak{M}_p^\lambda({}^\kappa\lambda)$, and define

$$\mu : \mathfrak{M} \to \mathbb{R}_\infty, \qquad \mu(A) = \mu'(\mathrm{int}(A) \cap {}^\omega\omega) \text{ for every } A \in \mathfrak{M}_p^\lambda({}^\kappa\lambda).$$

Then, μ is a λ^+-measure: indeed, $\mu(\emptyset) = 0$, $\mu({}^\kappa\lambda) = \mu'({}^\omega\omega) > 0$, and μ is decreasing since μ' is. Now let $(A_i)_{i<\alpha} \subseteq \mathfrak{M}_p^\lambda({}^\kappa\lambda)$ be a family of disjoint sets of order type $\alpha < \lambda^+$. Notice that either $\mathrm{int}(A_i) \cap {}^\omega\omega = \emptyset$ or $|\mathrm{int}(A_i) \cap {}^\omega\omega| \geq \mathfrak{c} > \lambda$, since $\lambda < \mathfrak{c}$, and $\mathrm{int}(A_i) \cap {}^\omega\omega \neq \emptyset$ happens for at most countably many indexes $I \subseteq \alpha$, since ${}^\omega\omega$ is second countable.

We claim $\mu'(\mathrm{int}(\bigcup_{i<\alpha} A_i) \cap {}^\omega\omega) = \mu'(\bigcup_{i<\alpha}(\mathrm{int}(A_i) \cap {}^\omega\omega))$. Notice that for every $i < \alpha$ we have $|A_i \setminus \mathrm{int}(A_i)| \leq \lambda$, by definition of $\mathfrak{M}_p^\lambda({}^\kappa\lambda)$. Since

$$\bigcup_{i<\alpha}(\mathrm{int}(A_i) \cap {}^\omega\omega) \subseteq \mathrm{int}(\bigcup_{i<\alpha} A_i) \cap {}^\omega\omega \subseteq \bigcup_{i<\alpha} A_i \cap {}^\omega\omega,$$

we get

$$\begin{aligned}
&|(\mathrm{int}(\bigcup_{i<\alpha} A_i) \cap {}^\omega\omega) \setminus \bigcup_{i<\alpha}(\mathrm{int}(A_i) \cap {}^\omega\omega)| = \\
&|{}^\omega\omega \cap (\mathrm{int}(\bigcup_{i<\alpha} A_i) \setminus \bigcup_{i<\alpha} \mathrm{int}(A_i))| \leq \\
&|{}^\omega\omega \cap \bigcup_{i<\alpha}(A_i \setminus \mathrm{int}(A_i))| \leq \\
&\sum_{i<\alpha} |A_i \setminus \mathrm{int}(A_i)| \leq \\
&\lambda \cdot \lambda = \lambda.
\end{aligned}$$

Notice that $\mathrm{int}(\bigcup_{i<\alpha} A_i) \cap {}^\omega\omega$ and $\bigcup_{i<\alpha}(\mathrm{int}(A_i) \cap {}^\omega\omega)$ are both open in ${}^\omega\omega$. Since $\lambda < \mathfrak{c}$, and Borel subsets of ${}^\omega\omega$ have the perfect set property, we get that

$$|\mathrm{int}(\bigcup_{i<\alpha} A_i) \cap {}^\omega\omega \setminus \bigcup_{i<\alpha}(\mathrm{int}(A_i) \cap {}^\omega\omega)| \leq \omega,$$

thus $\mu'(\mathrm{int}(\bigcup_{i<\alpha} A_i) \cap {}^\omega\omega \setminus \bigcup_{i<\alpha}(\mathrm{int}(A_i) \cap {}^\omega\omega)) = 0$ as wanted. Therefore,

$$\mu(\bigcup_{i<\alpha} A_i) = \mu'(\mathrm{int}(\bigcup_{i<\alpha} A_i) \cap {}^\omega\omega) = \sum_{i\in I} \mu'(\mathrm{int}(A_i) \cap {}^\omega\omega) = \sum_{i<\alpha} \mu(A_i),$$

as wanted. □

2.5 On the non-existence of continuous λ^+-measures

In this section we prove our impossibility result (Theorem 2.1.1), the main result of our chapter. We will repeatedly make use of the following straightforward consequence of Corollary 2.3.23:

Corollary 2.5.1. *Assume $\lambda^{<\kappa} = \lambda$. Let $(X, \mathfrak{M}, \mu)$ be a weak λ^+-measure space and let $U \subseteq X$ be an open set in X such that $\mu(U) > 0$. Then, for every family of open sets $\mathcal{V}$ of X of measure zero there is $x \in U \setminus \bigcup \mathcal{V}$.*

Proof. The assumption $\lambda^{<\kappa} = \lambda$ implies that ${}^\kappa\lambda$, and thus X, has weight $\leq \lambda$. By Corollary 2.3.23, we have that $\mu(\bigcup \mathcal{V}) = 0$, thus $U \nsubseteq \bigcup \mathcal{V}$ by Axiom (M3). □

We split the proof in multiple cases. These cases depend on the algebraic structure of $\mathbb{S}$ and on the value of κ.

2.5.1 On measures in monoids of wrong degree

Lemma 2.5.2. *If* $\mathrm{Deg}(\mathbb{S}) \neq \kappa$*, then for every weak* λ^+*-measure space* $(X, \mathfrak{M}, \mu)$ *and for every open set* U *of positive measure* $\mu(U) > 0$ *there is* $x \in U$ *such that either* x *is not measurable or* $\mu(x) > 0$*.*

Proof. Let $\mathcal{V} = \{V \in \tau \mid \mu(V) = 0\}$ be the set of open sets of X of measure 0. By Corollary 2.5.1, there is $x \in U \setminus \bigcup \mathcal{V}$. By definition of $\mathcal{V}$, we get $\mu(A) > 0$ for every $A \in \tau$ such that $x \in A$. In particular, for every $\alpha < \beta < \kappa$ we have $0 < \mu(\mathbf{N}_{x\restriction\beta}) \leq \mu(\mathbf{N}_{x\restriction\alpha})$ by Axiom (M3), thus the sequence $\langle \mu(\mathbf{N}_{x\restriction\alpha}) \mid \alpha < \kappa \rangle$ is decreasing in $\mathbb{S}^+$. Now, if x is measurable of measure 0, this also implies that $\langle \mu(\mathbf{N}_{x\restriction\alpha}) \mid \alpha < \kappa \rangle$ is coinitial in $\mathbb{S}^+$ by Axiom (M5), hence $\mathrm{Deg}(\mathbb{S}) = \kappa$. □

2.5.2 On measures in non-0-continuous monoids

Recall that a positively totally ordered monoid $\mathbb{S}$ is not 0-continuous if and only if it contains an element c such that $b + b \geq c$ for all $b \in \mathbb{S}^+$.

Lemma 2.5.3. *Assume* $\lambda^{<\kappa} = \lambda$*. Let* $(X, \mathfrak{M}, \mu)$ *be a weak* λ^+*-measure. Let* $U \subseteq X$ *be an open subset of* X *of positive measure* $\mu(U) = c > 0$ *whose value satisfies* $b + b \geq c$ *for every* $b \in \mathbb{S}^+$*. Then, one of the following holds:*

- *U contains a point $x \in U$ that is not measurable,*
- *U contains a measurable point $x \in U$ of measure $\mu(x) = \mu(U)$,*
- *U contains two measurable points $x, y \in U$ such that $\mu(x) + \mu(y) = \mu(U)$.*

Proof. If U contains a non-measurable point we are done, so assume instead that all points of U are measurable. Also, if there is $x \in U$ such that $\mu(U) = \mu(O)$ for every open neighbourhood O of x contained in U, then we get $\mu(x) = \mu(U)$, by Axiom (M5), and we are done. So suppose instead that every $x \in U$ has an open neighbourhood O of measure $\mu(O) < \mu(U) = c$. Let $\mathcal{V} = \{V \in \tau \mid \mu(V) = 0\}$ be the set of open sets of X of measure 0. Then, $U \setminus \bigcup \mathcal{V}$ is non-empty, by Corollary 2.5.1. Let $x_1 \in U \setminus \bigcup \mathcal{V}$. Then, $\mu(x_1) < c$, by Axiom (M5) and since $\mu(O) < \mu(U) = c$ for some neighbourhood O of x_1. This also implies that $\mu(U \setminus \{x_1\}) > 0$, since otherwise

$$\mu(U) = \mu(x_1) + \mu(U \setminus \{x_1\}) = \mu(x_1) + 0 = \mu(x_1) < c = \mu(U),$$

contradiction. Thus, there is $x_2 \in (U \setminus \{x_1\}) \setminus \bigcup \mathcal{V}$, by Corollary 2.5.1. Let $N_1, N_2 \subseteq U$ be disjoint cones such that $x_i \in N_i$ and $0 < \mu(N_i) < c$ for each $i \in \{1, 2\}$. Fix $i \in \{1, 2\}$. Notice that $\mu(V) = 0$ for every clopen $V \subseteq N_i \setminus \{x_i\}$, for otherwise both $\mu(V) \in \mathbb{S}^+$ and $\mu(N_i \setminus V) \in \mathbb{S}^+$, and thus $\mu(N_i) = \mu(V) + \mu(N_i \setminus V) \geq c > \mu(N_i)$ by assumption on c, contradiction. Then, $\mu(N_i) = \mu(O) + \mu(N_i \setminus O) = \mu(O)$ for every clopen neighbourhood O of x_i contained in U, and thus $\mu(x_i) = \mu(N_i)$, as before. Then, by Axiom (M3) and by the assumption on c, we get

$$\mu(U) = c \leq \mu(N_1) + \mu(N_2) \leq \mu(N_1 \cup N_2) \leq \mu(U).$$

and thus $\mu(U) = \mu(N_1) + \mu(N_2) = \mu(x_1) + \mu(x_2)$ as wanted. □

Corollary 2.5.4. *Assume* $\lambda^{<\kappa} = \lambda$*. Let* $(X, \mathfrak{M}, \mu : \mathfrak{M} \to \mathbb{S})$ *be a weak* λ^+*-measure space where the measure takes values in a non-0-continuous monoid* $\mathbb{S}$*. Then, for every open set* U *of positive measure* $\mu(U) > 0$ *there is* $x \in U$ *such that either* x *is not measurable or* $\mu(x) > 0$*.*

Proof. Let c be such that $b + b \geq c$ for every $b \in \mathbb{S}^+$, that is, a witness of the non-0-continuity of $\mathbb{S}$. Assume by contradiction that every point of U is measurable of measure 0. Then, $\mathcal{V} = \{V \in \tau \mid V \subseteq U, \mu(V) \leq c\}$ is a cover of U, by Axiom (M5). Let also $\mathcal{V}' = \{V \in \tau \mid \mu(V) = 0\}$ be the set of open sets of X of measure 0, then we can find $U' \in \mathcal{V} \setminus \mathcal{V}'$, by Corollary 2.5.1. The result then follows from Lemma 2.5.3 applied to U'. □

2.5.3 On measures in non-Archimedean monoids

Lemma 2.5.5. *Assume $\lambda^{<\kappa} = \lambda$. Let $(X, \mathfrak{M}, \mu)$ be a weak λ^+-measure space. Assume $U \in \tau$ and $a, b \in \mathbb{S}$ are such that $a \ll b < \mu(U)$. Then, there is $x \in U$ such that either x is not measurable or it has measure $\mu(x) \geq a$.*

Proof. Assume by contradiction that $x \in \mathfrak{M}$ and $\mu(x) < a$ for every $x \in U$. Let $\mathcal{V} = \{V \in \tau \mid \mu(V) \leq a\}$. Proceeding as in Corollary 2.5.1, if $\mathcal{V}$ covers U, then by Lemma 2.3.21 we can find a clopen partition $\mathbb{P}_{\vec{\mathcal{U}}}$ of U refining $\mathcal{V}$, from which it follows that

$$\mu(U) = \sum_{P \in \mathbb{P}_{\vec{\mathcal{U}}}} \mu(P) \leq \sum_{i<\lambda} a \leq b < c,$$

contradiction. Thus $\mathcal{V}$ cannot cover U. Let $x \in U \setminus \bigcup \mathcal{V}$. This implies $\mu(A) > a$ for every $A \in \tau$ such that $x \in A$. Therefore, we have $\mu(x) = \inf\{A \in \tau \mid x \in A\} \geq a$ by Axiom (M5), contradiction. ☐

Notice that if $\bar{a} = \langle a_i \mid i < \delta \rangle$ is a nowhere Archimedean decreasing sequence that is coinitial in $\mathbb{S}^+$, then we have that for every $c > 0$ there are $i, j \in \mathbb{S}^+$ such that $0 < a_i \ll a_j < c$. Thus, we get the following:

Corollary 2.5.6. *Assume $\lambda^{<\kappa} = \lambda$. Let $(X, \mathfrak{M}, \mu : \mathfrak{M} \to \mathbb{S})$ be a weak λ^+-measure space, and assume $\mathbb{S}^+$ contains a nowhere Archimedean decreasing coinitial sequence. Then, for every open set U of positive measure $\mu(U) > 0$ there is $x \in U$ such that either x is not measurable or $\mu(x) > 0$.*

2.5.4 On countable-induced measures

Lemma 2.5.7. *Assume $\lambda \geq \mathfrak{c}$. Let $(X, \mathfrak{M}, \mu)$ be a countable-induced, weak λ^+-measure space. Then, for every open set U of positive measure $\mu(U) > 0$ there is $x \in U$ such that either x is not measurable or $\mu(x) > 0$.*

Proof. Since $(X, \mathfrak{M}, \mu)$ is countable-induced, by Remark 2.4.6 there is a separable closed subspace $Y \subseteq X$ such that $\mu(X \setminus Y) = 0$. By Remark 2.4.5, it is easy to see that $|Y| \leq \mathfrak{c} \leq \lambda$. If all points of U are measurable of measure 0, then we have that $\mathcal{A} = \{U \cap (X \setminus Y)\} \cup \{\{x\} \mid x \in U \cap Y\}$ is a cover of U made of disjoint measurable sets (recall that $U \cap (X \setminus Y)$ is open) of size $\leq \lambda$. By Axiom (M3) and assumption on Y, we have $\mu(U \cap (X \setminus Y)) \leq \mu(X \setminus Y) = 0$. Thus, by Axiom (M4) we get

$$\mu(U) = \mu(U \cap (X \setminus Y)) + \sum_{x \in U \cap Y} \mu(x) = 0 + 0 = 0$$

as wanted. ☐

2.5.5 Proof of the impossibility theorem

The main theorem is proved through a more general, technical result (Proposition 2.5.8) which wipes out the possible existence of non-trivial λ^+-measures under the assumption that $\lambda < \mathfrak{c}$.

Proposition 2.5.8. *Assume $\lambda^{<\kappa} = \lambda$. Let $(X, \mathfrak{M}, \mu)$ be a continuous, λ^+-partitionable, weak λ^+-measure space such that $\mathfrak{M}$ is closed under finite intersections. Then, there is a clopen partition $\mathbb{P}_{\vec{\mathcal{U}}}$ of X such that $(P, \mathfrak{M} \downarrow P, \mu \downarrow P)$ is a countable-induced, continuous, weak λ^+-measure space for every $P \in \mathbb{P}_{\vec{\mathcal{U}}}$ and μ is the sum of $(\mu \downarrow P)_{P \in \mathbb{P}_{\vec{\mathcal{U}}}}$, i.e., for every $M \in \mathfrak{M}$ we have*

$$\mu(M) = \sum_{P \in \mathbb{P}_{\vec{\mathcal{U}}}} (\mu \downarrow P)(M \cap P).$$

Proof. By Corollary 2.5.4, we get that $\mathbb{S}$ is 0-continuous and of infinite degree. By Corollary 2.5.6, $\mathbb{S}$ contains no coinitial nowhere Archimedean sequence. Thus, by Lemma 2.2.7, we have that $\mathrm{Deg}(\mathbb{S}) = \omega$ and $\mathbb{S}$ is initially Archimedean. By Lemma 2.5.2, we have that $\kappa = \mathrm{Deg}(\mathbb{S}) = \omega$.

Say that μ is *essentially countable on* V if for every family $\mathcal{A}$ of disjoint open subsets of V there are at most countably many elements of $\mathcal{A}$ of positive measure.

Claim 2.5.8.1. Assume $\kappa = \omega$. Let $(X, \mathfrak{M}, \mu)$ be a weak λ^+-measure space. Suppose that μ is essentially countable on X. Then, $(X, \mathfrak{M}, \mu)$ is countable-induced.

Proof. Let $T = \{s \in {}^{<\omega}\lambda \mid \mu(\mathbf{N}_s(X)) > 0\}$. By assumption, each level $\mathrm{Lev}_n(T) = \{s \in {}^{n}\lambda \mid \mu(\mathbf{N}_s(X)) > 0\}$ must be countable, since $\{\mathbf{N}_s(X) \mid s \in {}^{n}\lambda\}$ is a family of disjoint open sets of X. This implies that T is countable as well, and thus $Y = [T] \cap X$ is a closed, second countable subspace of X. This implies $X \setminus Y$ is open, and thus $X \setminus Y \in \mathfrak{M}$. Notice that $\mathcal{V} = \{\mathbf{N}_s(X) \mid s \in {}^{<\omega}\lambda \setminus T\}$ is a cover of $X \setminus Y$ of cones of measure zero, thus (either by Lemma 2.3.21, or refining directly $\mathcal{V}$) we can find a clopen partition $\mathbb{P}_{\vec{\mathcal{U}}}$ of $X \setminus Y$ made of disjoint cones of measure zero. Since $\lambda^{<\omega} = \lambda$, we have that $|\mathbb{P}_{\vec{\mathcal{U}}}| \leq \lambda$, and thus by Axiom (M4), we get

$$\mu(X \setminus Y) \leq \sum_{P \in \mathbb{P}_{\vec{\mathcal{U}}}} \mu(P) = 0,$$

as wanted. □

Let $\mathcal{V} = \{V \in \tau \mid \mu \text{ is essentially countable on } V\}$. Assume first that $\mathcal{V}$ does not cover X. Let $a, b \in \mathbb{S}^+$ be such that the interval $[0, b]$ is Archimedean and $0 < a < b$. Let $\mathcal{A} = \{A \in \tau \mid \mu(A) < a\}$, then we have that $\mathcal{A}$ is a cover of X, by Axiom (M5) and since $\mu(x) = 0$ for every $x \in X$. Therefore, there is $A \in \mathcal{A} \setminus \mathcal{V}$. Since μ is not essentially countable on A, there is an uncountable family $\mathbb{P}_{\vec{\mathcal{U}}}$ of disjoint clopen subsets of A such that $0 < \mu(P)$ for every $P \in \mathbb{P}_{\vec{\mathcal{U}}}$. By Axiom (M3), we also have $\mu(P) \leq \mu(A) < a$ for every $P \in \mathbb{P}_{\vec{\mathcal{U}}}$. By the pigeonhole principle, since $\mathrm{Deg}(\mathbb{S}) = \omega$, there is $c \in (0, a)$ such that $\mu(P) > c$ for uncountably many $P \in \mathbb{P}_{\vec{\mathcal{U}}}$. Since $[0, b]$ is Archimedean and $c \in [0, b]$, we can find $n \in \omega$ such that $n \cdot c \geq b$. But then,

$$\mu(A) \geq \sum_{P \in \mathbb{P}_{\vec{\mathcal{U}}}} \mu(P) \geq n \cdot c \geq b > a,$$

contradicting that $A \in \mathcal{A}$.

It follows that $\mathcal{V}$ does cover X. By Lemma 2.3.21, we can find a clopen partition $\mathbb{P}'_{\vec{\mathcal{U}}}$ of X refining $\mathcal{V}$. Notice that since $\mathbb{P}'_{\vec{\mathcal{U}}}$ covers X and is made of disjoint sets, there is $A \in \mathbb{P}'_{\vec{\mathcal{U}}}$ such that $\mu(A) > 0$, by Axioms (M2) and (M4). Let

$$\mathbb{P}_{\vec{\mathcal{U}}} = \{P \in \mathbb{P}_{\vec{\mathcal{U}}} \setminus \{A\} \mid \mu(P) > 0\} \cup \{A \cup \bigcup \{P \in \mathbb{P}_{\vec{\mathcal{U}}} \setminus \{A\} \mid \mu(P) = 0\}\}.$$

This way, $\mu(P) > 0$ for every $P \in P$. It is also easy to check that μ is still essentially countable on every $P \in \mathbb{P}_{\vec{\mathcal{U}}}$, by Axiom (M4) and since $\mathfrak{M}$ is closed under finite intersections. By Lemma 2.3.29, we have that $(P, \mathfrak{M} \downarrow P, \mu \downarrow P)$ is a (continuous) weak λ^+-measure space for every $P \in \mathbb{P}_{\vec{\mathcal{U}}}$. By Claim 2.5.8.1, for every $P \in \mathbb{P}_{\vec{\mathcal{U}}}$ the space $(P, \mathfrak{M} \downarrow P, \mu \downarrow P)$ is countable-induced. Finally, from Axiom (M4) and the fact that $\mathfrak{M}$ is closed under finite intersections, we get

$$\mu(M) = \sum_{P \in \mathbb{P}_{\vec{\mathcal{U}}}} \mu \downarrow P(M \cap P),$$

for every $M \in \mathfrak{M}$, i.e. μ is the sum of $(\mu \downarrow P)_{P \in \mathbb{P}_{\vec{\mathcal{U}}}}$, as wanted. □

Thanks to Propositions 2.3.20 and 2.3.24, we get the following.

Corollary 2.5.9. *Assume $\lambda^{<\kappa} = \lambda$. Let $(X, \mathfrak{M}, \mu)$ be a continuous, minimal λ^+-measure space. Then, there is a clopen partition $\mathbb{P}_{\vec{\mathcal{U}}}$ of X such that $(P, \mathfrak{M} \downarrow P, \mu \downarrow P)$ is a countable-induced, continuous weak λ^+-measure space for every $P \in \mathbb{P}_{\vec{\mathcal{U}}}$, and μ is the sum of $(\mu \downarrow P)_{P \in \mathbb{P}_{\vec{\mathcal{U}}}}$.*

This, together with Fact 2.3.19 and Lemma 2.5.7 leads immediately to Theorem 2.1.1.

From Theorem 2.1.1, we can also obtain the following:

Corollary 2.5.10. *Assume $\lambda^{<\kappa} = \lambda \geq \mathfrak{c}$. Let $(X, \mathfrak{M}, \mu)$ be a weak λ^+-measure space. Then, there is a partition $X = C \cup O$ such that O is open, $\mu(O) = 0$, and $x \in \mathfrak{M}$ for every $x \in O$, while C is closed and contains a dense subset $D \subseteq C$ such that for every $x \in D$, either x is not measurable or $\mu(x) > 0$.*

Proof. Let

$$\mathcal{V} = \{V \in \tau \mid \text{ For all } x \in V, x \notin \mathfrak{M} \vee \mu(x) > 0\},$$
$$\mathcal{U} = \{U \in \tau \setminus \mathcal{V} \mid \mu(U) = 0\}.$$

Suppose by contradiction there is $Y \in \tau \setminus (\mathcal{U} \cup \mathcal{V})$: then, $\mu(Y) > 0$ and for every $x \in Y$, we have $x \in \mathfrak{M}$ and $\mu(x) = 0$. Without loss of generality, we may assume that $(X, \mathfrak{M}, \mu)$ is minimal - if not, consider $(X, \mathfrak{M}_p^\lambda(A), \mu \upharpoonright (\mathfrak{M}_p^\lambda(A)))$ for $A = \{x \in X \mid x \in \mathfrak{M}\}$ instead of $(X, \mathfrak{M}, \mu)$. By Corollary 2.3.32, we have that $(Y, \mathfrak{M} \downarrow Y, \mu \downarrow Y)$ is a weak λ^+-measure space, and $\mu \downarrow Y$ is continuous since $Y \notin \mathcal{V}$, contradicting Theorem 2.1.1.

Therefore, we must have that $\mathcal{U} \cup \mathcal{V}$ is a cover of X. Let $O = \bigcup \mathcal{U}$, then $\mu(O) = 0$ by Corollary 2.3.23 and since $|\mathcal{U}| \leq \lambda^{<\kappa} = \lambda$, and furthermore $x \in \mathfrak{M}$ and $\mu(x) = 0$ for every $x \in O$, since $U \notin \mathcal{V}$ for every $U \in \mathcal{U}$, by definition. Let $C = X \setminus O$. Now for every $V \in \tau$ such that $V \cap C \neq \emptyset$, we have that $V \in \mathcal{V}$, since $\mathcal{U} \cup \mathcal{V}$ covers of X, and thus we may chose $x_V \in V$ such that $x \notin \mathfrak{M}$ or $\mu(x) > 0$. Then, $D = \{x_V \mid V \in \tau, V \cap C \neq \emptyset\}$ is as wanted. □

II

On the interplay between modal logic and set theory

3

The modal logics of forcing and grounds, and forcing and inner models

This chapter is based on joint work with Juan P. Aguilera and Grigorii Stepanov.

3.1 Introduction

By iteratively generating forcing extensions and identifying their respective ground models, one generates the *generic multiverse*[1]. The fact that the generic multiverse can be seen as a Kripke structure of models whose accessibility relations are that of being a forcing extension and being a ground suggests modal logic as a natural tool for investigating it. This idea motivates the work of Hamkins and Löwe, who in [HL08] first approached the generic multiverse using modal logics focusing initially on the forcing relation. In that paper, they showed that the modal logic of forcing, that is, the collection of modal formulas valid under the interpretation of $\Box\varphi$ as *"φ holds in all forcing extensions"* is exactly S4.2, assuming the consistency of ZFC. More precisely, let $\varphi(q_0, \ldots, q_n)$ be a propositional modal formula in variables q_i, and interpret each variable as a sentence ψ_i in the language of set theory. Then, $\varphi(q_0, \ldots, q_n)$ is said to be a ZFC-provable principle of forcing if ZFC proves that $\varphi(\psi_0, \ldots, \psi_n)$ holds under the forcing interpretation for all substitutions ψ_i. Hamkins and Löwe proved that the collection of such provable principles coincides exactly with the modal logic S4.2

This result has been extended in several directions. In [HLL15], Hamkins, Leibman, and Löwe examined the modal logic of particular classes of forcing, such as Cohen forcing, collapse forcing, and c.c.c. forcing. Parallel to this, Hamkins and Löwe investigated the modal logic of grounds in [HL13], and showed that the ZFC-provable principles of the ground model relation are again exactly those in S4.2 [2]. Also in [HL13], the authors explored the modal logic of forcing and grounds "together", studying the two respective mono-modal fragments and their possible combinations, although without considering axioms capturing the interplay between the two. We address this in this chapter. We consider two modal operators, $\Box$ and $\blacksquare$, interpreted respectively as "*it holds in every forcing extension*" and "*it holds in every ground*", and introduce a modal logic that captures this interpretation with axioms capturing their interactions (see Definition 3.3.1), characterise it

[1] What the generic multiverse is may change depending on who defines it. When Steel defines it, he imposes the condition of *generic amalgamation*: if M_1 and M_2 are two models, then there exists a $\mathbb{P}$-generic filter over M_1 and a $\mathbb{Q}$-generic filter over M_2, with $\mathbb{P}$ and $\mathbb{Q}$ forcing notions in M_1 and M_2 such that $M_1[G] = M_2[G]$. That is, Steel requires *a* witness for each forcing notion. In Woodin's generic multiverse, there is no amalgamation. In his definition, every generic extension of every model M is added to the collection. We use Woodin's definition. For more, see [Mea21].

[2] Hamkins and Löwe could only give S4.2 as a lower bound, but pointed out to the fact that if grounds were downward set-directed, this would be the exact modal logic of grounds. This turned out to be true, as Usuba would prove in [Usu17] shortly after.

in terms of Kripke frames (Theorem 3.3.16), and prove that it is both sound (Proposition 3.3.20) and complete with respect to the given interpretations (Theorem 3.3.28).

In [IL16], Inamdar and Löwe studied the modal logic of inner models under the interpretation of $\Box\varphi$ as "φ holds in all inner models" and demonstrated that the corresponding ZFC-provable modal principles are those of S4.2 augmented with an additional axiom, denoted Top, designed to capture a specific structural aspect of the inclusion relation among inner models ([IL16, Theorem 5.6]). We also briefly explore how a bi-modal logic as the one introduced for forcing and grounds might be developed when $\blacksquare$ is instead interpreted as "*it holds in every inner model*".

3.2 Preliminaries

The intended reader of this dissertation is not assumed to have a background in modal logic. We therefore provide a brief overview of fundamental concepts and results in the field. Notions and results that play a significant role in this chapter will be introduced as they are needed, not in this section. Not everything in this section will be used in the rest of the chapter, but they may provide a better understanding of the topic. That is why we have added some proofs, as well. Most of the material in this section is taken from [BDRV01].

The *basic modal language* is defined using a set of propositional variables Φ and a unary operator $\Diamond$. The elements in Φ are usually denoted by letters in the Latin alphabet. The set Φ is normally considered to be of countable size. The *well-formed formulas* ϕ of the basic modal language are given by the rule

$$\phi ::= p \,|\, \bot \,|\, \neg\phi \,|\, \psi \vee \phi \,|\, \Diamond\phi,$$

where p ranges over Φ. We also have the dual modal operator for $\Diamond$, which is denoted by $\Box$:

$$\Box\phi := \neg\Diamond\neg\phi.$$

The rest of Boolean connectives ($\to, \vee$, etc.) are easily obtained from the ones above by the typical arguments.

Definition 3.2.1. A *modal similarity type* is a pair $\tau = (O, \rho)$ where O is a non-empty set of modal operators and ρ is the *arity function* $\rho : O \to \mathbb{N}$ which assigns a natural number to each element in O. The *arity* of a modal operator indicates the number of arguments that it can be applied to. A *modal language* $ML(\tau, \Phi)$ is the basic modal language whose formulas are obtained from variables in Φ and modal operators in the modal similarity type τ according to the generating rule mentioned above.

In the following sections of this chapter, we will consider logics in which the set O contains at most two unary modal operators, together with their duals. In this section, we present results for mono-modal logics, that is, logics where O consists of a single unary modal operator and its dual. These results can be easily extended to logics with a richer modal similarity type.

Definition 3.2.2. Suppose we are working in a modal language with modal similarity type τ and collection of propositional variables Φ. A *substitution* is a function $\sigma : \Phi \longrightarrow Form(\tau, \Phi)$.

A substitution induces a function

$$(\cdot)^\sigma : Form(\tau, \Phi) \longrightarrow Form(\tau, \Phi)$$

recursively defined:

1. $(p)^\sigma = \sigma(p)$ for every $p \in \Phi$,
2. $(\bot)^\sigma = \bot$,

3. $(\neg\psi)^\sigma = \neg(\psi)^\sigma$,
4. $(\phi \vee \psi)^\sigma = (\phi)^\sigma \vee (\psi)^\sigma$, and
5. $(\triangle(\phi_0, \ldots, \phi_n))^\sigma = \triangle((\phi_0)^\sigma, \ldots, (\phi_n)^\sigma)$, where $\triangle$ denotes a modality operator of arity $n+1$.

We still use $(\cdot)^\sigma$ to denote the extended function.

Definition 3.2.3. A formula θ is a *substitution instance* of ϕ if there is a substitution σ such that $\theta = (\phi)^\sigma$.

Of course, the substitution instance of a formula is completely determined by the substitution function σ, meaning that if θ is a substitution instance of ϕ by σ, θ is what one obtains from ϕ by substituting in ϕ every occurrence of p by $\sigma(p)$ for every $p \in \Phi$.

Definition 3.2.4. A *frame* for the basic modal language is a pair $\mathcal{F} = (W, R)$ where

- W is a non-empty set, and
- R is a binary relation on W.

Definition 3.2.5. A *model* for the basic modal language is a pair $\mathcal{M} = (\mathcal{F}, V)$ where $\mathcal{F}$ is a frame and V is a function which assigns to each propositional variable in Φ a subset $V(p)$ of W. The function V is called *valuation*.

A model $\mathcal{M}$ of the form $(\mathcal{F}, V)$ is said to be based on the frame $\mathcal{F}$, or that $\mathcal{F}$ is the frame underlying $\mathcal{M}$.

Definition 3.2.6. Let w be a world in a model $\mathcal{M} = (W, R, V)$. We define inductively the notion of a formula ϕ being *satisfied* or *true* in the model $\mathcal{M}$ at w, denoted $M, w \Vdash \phi$, as follows:

(i) $\mathcal{M}, w \Vdash p$ if and only if $w \in V(p)$, with $p \in \Phi$,

(ii) never happens that $M, w \Vdash \bot$,

(iii) $\mathcal{M}, w \Vdash \neg\phi$ if and only if not $\mathcal{M}, w \Vdash \phi$, o

(iv) $\mathcal{M}, w \Vdash \phi \wedge \psi$ if and only if $\mathcal{M}, w \Vdash \phi$ and $\mathcal{M}, w \Vdash \psi$,

(v) $\mathcal{M}, w \Vdash \diamond\phi$ if and only if for some $v \in W$ with Rwv we have $\mathcal{M}, v \Vdash \phi$.

A set of formulas Σ is satisfied or true in the model $\mathcal{M}$ at w if $M, w \Vdash \phi$ for every $\phi \in \Sigma$.

When the context is clear, we write $w \Vdash \phi$ instead of $\mathcal{M}, w \Vdash \phi$. We also write $\mathcal{M}, w \nVdash \phi$ instead of not $\mathcal{M}, w \Vdash \phi$. When $\mathcal{M}, w \nVdash \phi$ we say that ϕ is *false* or *refuted* at w. Since $\Box\phi$ is $\neg\Diamond\neg\phi$, it follows from (iii) and (v) above that $\mathcal{M}, w \Vdash \Box\phi$ if and only if $\mathcal{M}, w \nVdash \Diamond\neg\phi$ if and only if there is no world v with Rwv such that $\mathcal{M}, v \Vdash \neg\phi$ if and only if for every world v R-accessible from w we have $\mathcal{M}, v \Vdash \phi$.

The notion of satisfaction at a world in a model is contingent both upon the valuation and the world, which leave us possible different ways to look for a more general notion of validity. If we still want to consider validity in a model, that is, a structure with a particular valuation, the corresponding notion is universal satisfiability:

Definition 3.2.7. A formula ϕ is *globally* or *universally true* in a model $\mathcal{M} = (W, R, V)$ if it is satisfied at every world in $\mathcal{M}$, that is, if $\mathcal{M}, w \Vdash \phi$ for every $w \in W$. A set of formulas Σ is globally or universally true if all of its formulas are universally true.

By $\mathcal{M} \Vdash \phi$ we mean ϕ is universally true in $\mathcal{M}$, that is, $\forall w \in W(\mathcal{M}, w \Vdash \phi)$.

We may want to consider a notion of validity that is not contingent upon a particular valuation. This necessarily forces us to move to a frame-based set up.

Definition 3.2.8. Let $\mathcal{F}$ be a frame. A formula ϕ is *valid at a world w in $\mathcal{F}$* (in symbols, $\mathcal{F}, w \Vdash \phi$) if it is true at w in every model $(\mathcal{F}, V)$, that is, if $(\mathcal{F}, V), w \Vdash \phi$ for every valuation V.

We can abstract ourselves further first by not considering anymore validity in a particular world, then by considering validity not in a frame but in a class of frames and, finally, by considering validity in the class of all frames:

Definition 3.2.9. Let $\mathcal{F}$ be a frame. The formula ϕ is *valid in $\mathcal{F}$* (in symbols, $\mathcal{F} \Vdash \phi$) if ϕ is true at every world $w \in \mathcal{F}$; it is *valid on a class of frames* F (in symbols, $\mathsf{F} \Vdash \phi$) if it is valid in every frame $\mathcal{F} \in \mathsf{F}$; it is *valid* (in symbols, $\Vdash \phi$) if it is valid on the class of all frames.

Example 3.2.10. Let $\mathcal{F}$ be a frame $(W, \leq)$, where $W = \{w_n : n \in \omega\}$ and $w_i \leq w_j$ if and only if $i \leq j$. Note that $\leq$ is a transitive relation. It is easy to see that the formula $\Box\varphi \to \Box\Box\varphi$ is valid in $\mathcal{F}$. The fact is that, no matter what frame we consider, as long as its accessibility relation, the formula $\Box\varphi \to \Box\Box\varphi$ is going to be valid in it. That is, $\Box\varphi \to \Box\Box\varphi$ is valid on the class of transitive frames. However, that formula cannot be valid on the class of all frames, for it of course fails in every non-transitive frame.

The class of formulas that are valid in a class F of frames is called the *logic of* F. In this section, we denote it by Λ_{F}. Note that Λ_{F} depends as well on the chosen collection Φ of variables, that is, on the basic modal language considered beforehand.

Remark 3.2.11 (Validity and satisfaction are different concepts). Let $\phi \vee \psi$ be a formula that is true at a world w and valid on a frame $\mathcal{F}$. Because it is true at w, either ϕ or ψ is true at w. Validity on the frame $\mathcal{F}$ implies that $\phi \vee \psi$ is true at every world w in $\mathcal{F}$ for every valuation V, but this doesn't imply that either ϕ or ψ is valid in $\mathcal{F}$. As an example, consider the formula $p \vee \neg p$, clearly valid in $\mathcal{F}$, and note that neither p nor $\neg p$ are valid in $\mathcal{F}$.

Validity is preserved by uniform substitution:

Proposition 3.2.12. *Let F be a class of frames. A formula ϕ is valid in F if and only if so is every substitution instance of ϕ.*

Proof. The right to left direction is immediate because every formula is a trivial substitution instance of itself. For the left to right direction, let $\{p_i : i \in I\}$ be the collection of variables present in ϕ and let $\sigma(p_i) = \theta_i$ be the formula assigned to p_i for each $i \in I$. Let then χ be the corresponding σ-substitution instance of ϕ. Consider the model $(\mathcal{F}, V')$ where $V'(p_i) = \{w : (\mathcal{F}, V) \Vdash \theta_i\}$, then for every $w \in \mathcal{F}$ it happens that $(\mathcal{F}, V), w \Vdash \phi$ if and only if $(\mathcal{F}, V'), w \Vdash \chi$. This implies that it cannot happen that ϕ is valid while there exists a substitution instance of it that it isn't. Indeed, assume that is the case and let $w \in \mathcal{F}$ and V a valuation such that $(\mathcal{F}, V), w \nVdash \chi$ and let V' be as before. Then $(\mathcal{F}, V'), w \nVdash \phi$, which contradicts the validity of ϕ. □

From this, one can easily infer that propositional variables cannot be valid principles in general, which in turn implies that models do not provide the necessary level of abstraction to isolate logically strong invariant statements. Suppose as an example a model M such that $M \Vdash p$ but $M \nVdash q$. While q is obtainable from p by uniform substitution, we have that $q \notin \Lambda_{\mathsf{M}}$. The proposition above tells that the notion of validity wipes out all these issues.

Definition 3.2.13. A *modal logic* Λ is a set of modal formulas that contains all propositional tautologies and is closed under modus ponens and uniform substitution.

A set Λ of formulas is closed under uniform substitution if whenever ϕ belongs to Λ then so do all its substitution instances. For instance, since the propositional tautology $p \vee \neg p$ belongs to every modal logic by the definition above, so does $\Diamond p \vee \neg\Diamond p$.

From now on and unless otherwise specified Λ denotes a modal logic.

Definition 3.2.14. If a formula ϕ belongs to Λ then it is a *theorem of* Λ. We write $\vdash_\Lambda \phi$. If $\phi \notin \Lambda$, we write $\nvdash_\Lambda \phi$.

We have put no restriction on the sets of modal formulas one can consider to make a modal logic. The set of all formulas is then a modal logic in spite of its inconsistency. We call this modal logic the *inconsistent logic*.

Lemma 3.2.15. *Let* $\{\Lambda_i : i \in I\}$ *be a collection of modal logics indexed by a set* I*. Then* $\bigcap_{i\in I} \Lambda_i$ *is a modal logic.*

Proof. Every Λ_i contains all propositional tautologies, so they all belong to $\bigcap_{i\in I} \Lambda_i$ as well. Suppose $\phi, \phi \to \psi \in \bigcap_{i\in I} \Lambda_i$. Then $\phi, \phi \to \psi \in \Lambda_i$ for every $i \in I$. Since each Λ_i is a modal logic, it is closed under modus ponens, hence $\psi \in \Lambda_i$ for every $i \in I$ and $\psi \in \bigcap_{i\in I} \Lambda_i$. This proves that $\bigcap_{i\in I} \Lambda_i$ is closed under modus ponens. A similar argument shows that it is closed under uniform substitution. □

Let Γ be an arbitrary set of formulas. There are always modal logics containing Γ, being the inconsistent logic an example. Consider then the collection $\mathsf{ML}(\Gamma) := \{\Lambda : \Lambda \text{ is a modal logic} \wedge \Gamma \subseteq \Lambda\}$. By the previous lemma, the intersection $\bigcap \mathsf{ML}(\Gamma)$ is a modal logic, the smallest logic containing Γ. We call $\bigcap \mathsf{ML}(\Gamma)$ the *modal logic generated by* Γ.

Proposition 3.2.16. *If* F *is a class of frames, then* $\Lambda_{\mathsf{F}} := \{\phi : \forall \mathcal{F} \in \mathsf{F}(\mathcal{F} \Vdash \phi)\}$ *is a logic. If* M *is a class of models,* Λ_{M} *need not be a logic.*

Proof. Propositional tautologies are trivially valid, so they belong to Λ_{F}. Let $\phi, \phi \to \psi \in \Lambda_{\mathsf{F}}$ so that both ϕ and $\phi \to \psi$ are valid in every frame $\mathcal{F}$. This means that for every such frame, for every world $w \in \mathcal{F}$ and every valuation V we have $(\mathcal{F}, V), w \Vdash \phi, \phi \to \psi$. Since $(\mathcal{F}, V), w \Vdash \phi \wedge (\phi \to \psi)$ implies that $(\mathcal{F}, V), w \Vdash \psi$, we have that ψ is valid in $\mathcal{F}$ for every $\mathcal{F} \in \mathsf{F}$, so Λ_{F} is closed under modus ponens. Because of Proposition 3.2.12, Λ_{F} is closed under uniform substitution. That classes of models are not enough to define logics has been mentioned already after Proposition 3.2.12. □

The Proposition above puts together the semantical and syntatic perspective on modal logics. The set of formulas satisfied by a given class of frames is indeed a logic. This will be important when proving soundness and completeness results. In order to introduce these notions, we first need the following:

Definition 3.2.17. Let $\psi_1, \ldots, \psi_n$ be modal formulas. A formula ϕ is *deducible in propositional logic from assumptions* $\psi_1, \ldots, \psi_n$ if $(\psi_1 \wedge \ldots \wedge \psi_n) \to \phi$ is a tautology[3]. If $\Gamma \cup \{\phi\}$ is a set of formulas, ϕ is *deducible in* Λ *from* Γ or Λ*-deducible from* Γ (in symbols, $\Gamma \vdash_\Lambda \phi$) if either $\vdash_\Lambda \phi$ or there are $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\vdash_\Lambda (\psi_1 \wedge \ldots \wedge \psi_n) \to \phi$.

As always, if ϕ is not Λ-decidable from Γ, we write $\Gamma \nvdash_\Lambda \phi$.

Definition 3.2.18. A set Γ is Λ*-consistent* if $\Gamma \nvdash_\Lambda \bot$ and Λ*-inconsistent* otherwise. A formula ϕ is Λ-consistent or Λ-inconsistent if $\{\phi\}$ is.

In this chapter, we work with normal modal logics:

Definition 3.2.19. A modal logic Λ is a *normal modal logic* if it contains the formulas

K: $\Box(p \to q) \to (\Box p \to \Box q)$, and

Dual: $\Diamond p \leftrightarrow \neg\Box\neg p$.

We need the Dual axiom because we've introduced modal logics based on a single modal quantifier. A normal modal logic can be enriched with more modal axioms. This will be seen in the next sections.

The following is obvious:

[3] Recall that a tautology is either a propositional tautology or a substitution instance of a propositional tautology.

Lemma 3.2.20. *Let $\{\Lambda_i : i \in I\}$ be a collection of normal modal logics indexed by a set I. Then $\bigcap_{i\in I} \Lambda_i$ is a normal modal logic.*

It makes sense again to consider the normal modal logic generated by a given set of formulas Γ. The minimal normal modal logic, that one generated by the empty set, is called K.

Definition 3.2.21 (Soundness)**.** Let S be a class of frames (or models). A normal modal logic Λ is *sound* with respect to S if $\Lambda \subseteq \Lambda_{\mathsf{S}}$.

So Λ is sound with respect to S if for all formulas ϕ, $\vdash_\Lambda \phi$ implies $\Vdash_{\mathsf{S}} \phi$ (that is, for all $S \in \mathsf{S}$, $S \Vdash \phi$). In other words, a normal modal logic is sound with respect to a class of frames if every theorem of that logic is valid on that class of frames.

Let S be a class of frames. Since modus ponens and uniform substitution preserve validity on any such class S, proving the soundness of a normal modal logic Λ boils down to proving the validity of its axioms in S.

Definition 3.2.22 (Strong and weak completeness)**.** Let S be a class of frames (or models). A normal modal logic Λ is *strongly complete* with respect to S if for every set of formulas $\Gamma \cup \{\phi\}$, $\Gamma \Vdash_{\mathsf{S}} \phi$ implies $\Gamma \vdash_\Lambda \phi$. It is *weakly complete* with respect to S if for every formula ϕ, $\Vdash_{\mathsf{S}} \phi$ (or $\mathsf{S} \Vdash \phi$) implies $\vdash_\Lambda \phi$.

Weak completeness is strong completeness restricted to Γ being the empty set. One can reformulate weak completeness by saying that Λ is weakly complete with respect to S if $\Lambda_{\mathsf{S}} \subseteq \Lambda$. Therefore, a normal modal logic Λ is sound and weakly complete with respect to S if $\Lambda = \Lambda_{\mathsf{S}}$. In order words, Λ is sound and weakly complete with respect to S if the syntactical methods of Λ provide the statements that are semantically valid in S, and vice versa. The following will be useful when proving completeness results:

Proposition 3.2.23. *A logic Λ is strongly complete with respect to a class of structures S if and only if every Λ-consistent set of formulas is satisfiable in some $S \in \mathsf{S}$. A logic Λ is weakly complete with respect to S if and only if every Λ-consistent formula is satisfiable in some $S \in \mathsf{S}$.*

Proof. We prove the first assertion. The second follows easily after. Let Λ be strongly complete with respect to S and let Γ be an arbitrary Λ-consistent set of formulas. By strong completeness, since $\Gamma \nvdash_\Lambda \bot$, it then happens that $\Gamma \nVdash_{\mathsf{S}} \bot$, so there exists a structure $S \in \mathsf{S}$ such that $S \Vdash \Gamma$ and $S \nVdash \bot$. In particular, Γ is satisfiable in S (note, moreover, that every structure S satisfies that $S \nVdash \bot$). For the right to left direction we argue by contraposition. Assume that Γ isn't strongly complete with respect to S and let $\Gamma \cup \{\phi\}$ be an arbitrary set of formulas such that $\Gamma \Vdash_{\mathsf{S}} \phi$ but $\Gamma \nvdash_\Lambda \phi$.

Lemma 3.2.24. *If Γ is Λ-consistent and $\Gamma \nvdash_\Lambda \phi$, then $\Gamma \cup \{\neg\phi\}$ is Λ-consistent.*

Proof of the lemma. Towards a contradiction, suppose $\Gamma \cup \{\neg\phi\}$ is not Λ-consistent. Then, there is a finite subset U of $\Gamma \cup \{\neg\phi\}$ such that $\vdash_\Lambda \bigwedge U \to \bot$. Γ is Λ-consistent, so it must be the case that $\neg\phi \in U$. Let then U' be such that $U = U' \cup \{\neg\phi\}$. We then have that $\vdash_\Lambda \bigwedge U' \wedge \neg\phi \to \bot$. By propositional logic we then have that $\vdash_\Lambda \neg(\bigwedge U' \wedge \neg\phi)$, that is, $\vdash_\Lambda (\neg \bigwedge U' \vee \phi)$ or, equivalently, $\vdash_\Lambda \bigwedge U' \to \phi$, which implies $\Gamma \vdash_\Lambda \phi$, a contradiction. □

It follows that $\Gamma \cup \{\neg\phi\}$ is Λ-consistent. But since $\Gamma \Vdash_{\mathsf{S}} \phi$, there is no structure in S where $\Gamma \cup \{\neg\phi\}$ is satisfied. This finishes the proof. □

Lemma 3.2.25. *For every Λ-consistent set of formulas Γ and every formula ϕ, either $\Gamma \cup \{\phi\}$ or $\Gamma \cup \{\neg\phi\}$ is Λ-consistent.*

Proof. First note that it cannot happen that both $\Gamma \cup \{\phi\}$ and $\Gamma \cup \{\neg\phi\}$ are Λ-inconsistent, for then $\Gamma \vdash_\Lambda \phi \wedge \neg\phi$ by Lemma 3.2.24, which contradicts the Λ-consistency of Γ. Similarly, it cannot happen that both $\Gamma \cup \{\phi\}$ and $\Gamma \cup \{\neg\phi\}$ are simultaneously Λ-consistent. Indeed, if $\Gamma \cup \{\phi\}$ is

Λ-consistent, there is no $U' \subseteq \Gamma$ such that $\vdash_\Lambda \bigwedge U' \wedge \phi \to \bot$; and we can analogously prove that there is no $V' \subseteq \Gamma$ such that $\vdash_\Lambda \bigwedge V' \wedge \neg\phi \to \bot$. So, in general, there is no $Y \subseteq \Gamma$ such that $\vdash_\Lambda \bigwedge Y \wedge \phi \wedge \neg\phi \to \bot$, which is not true because $\bigwedge \psi \wedge \phi \wedge \neg\phi \to \bot$ is a tautology no matter what $\bigwedge \psi$ is. □

Proposition 3.2.23 provides a strategy to show the strong (equiv. weak) consistency of a logic Λ. If one is to check the completeness of the logic Λ with respect to a class of structures S, it is enough to prove that for every Λ-consistent set of formulas Γ (equiv. every Λ-consistent formula ϕ) there is a structure $S \in \mathsf{S}$ where Γ (equiv. ϕ) holds. The question then moves to the existence and construction of such a suitable satisfying model for each Λ-consistent set of formulas (equiv. formula). For this purpose, canonical models are one of the most useful tools at our disposal. Canonical models are defined in the next section, but rely on the following ideas:

Definition 3.2.26. A set of formulas Γ is *maximal* Λ*-consistent* if it is Λ-consistent and any proper extension $\Gamma' \supsetneq \Gamma$ is Λ-inconsistent.

The following are easy but important features of maximal Λ-consistent sets of formulas:

Proposition 3.2.27. *Let Λ be a logic and Γ be a maximal Λ-consistent sets of formulas. Then:*

(1) *Γ is closed under modus ponens,*

(2) *$\Lambda \subseteq \Gamma$,*

(3) *for every formula ϕ, either $\phi \in \Gamma$ or $\neg\phi \in \Gamma$,*

(4) *for every two formulas ϕ, ψ, if $\phi \vee \psi \in \Gamma$ then either $\phi \in \Gamma$ or $\psi \in \Gamma$.*

The following tells that every consistent set formulas can be extended to a maximal consistent set of formulas.

Lemma 3.2.28 (Lindenbaum's Lemma)**.** *If Σ is a Λ-consistent set of formulas, then there is a Σ^+ maximal Λ-consistent sets of formulas such that $\Sigma \subseteq \Sigma^+$.*

3.3 The modal logic of forcing and grounds

We begin by examining the modal logic interpretation of the generic multiverse, which consists of the collection of models of set theory derived from each model by closing under forcing extensions and grounds. The relationships between these models are thus those of *being a forcing extension* and *being a ground*. In [HL08] and [HL13], Hamkins and Löwe investigated the modal logics of forcing and grounds as well as possible combinations of both, though they did not incorporate modal axioms capturing the interplay between the two relations. In this section, we introduce such a connection by adding a single axiom that establishes one relation as the inverse of the other, and evaluate its suitability.

Definition 3.3.1. LFG is the minimal set of $\mathcal{L}_{\Box,\blacksquare}$-formulas closed under modus ponens and substitutions containing propositional tautologies and

1. $\mathsf{S4.2}(\Box)$,
2. $\mathsf{S4.2}(\blacksquare)$,
3. $\mathsf{Inv_{FG}}$: $\varphi \to \Box\blacklozenge\varphi \wedge \blacksquare\Diamond\varphi$,

where $\mathsf{S4.2}(\Box)$ is the normal modal logic obtained from the axioms:

K: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$,

Dual: $\neg\Diamond\varphi \leftrightarrow \Box\neg\varphi$, S: $\Box\varphi \to \varphi$,

4: $\Box\varphi \to \Box\Box\varphi$, and

.2: $\Diamond\Box\varphi \to \Box\Diamond\varphi$.

Recall that a *lattice* is a partial order in which every two nodes have a greatest lower bound (or meet), and a least upper bound (or join). If a, b are two elements in a finite lattice, $a \wedge b$ denotes its meet and $a \vee b$ denotes its join. A *pre-lattice* is a partial pre-order (i.e., reflexive and transitive) relation $\leq$ on a set F such that the quotient of F by the equivalent relation $a \equiv b$ if and only if $a \leq b \leq a$ is a lattice under the induced quotient relation, that is, the relation $\leq_\equiv$ given by $[a]_\equiv \leq_\equiv [b]_\equiv$ if and only if $\exists a \in [a] \exists b \in [b](a \leq b)$. Sometimes we refer to the $[\cdot]_\equiv$ classes of equivalence as *clusters*. In [HL08], the authors provided several frame characterisations of the logic S4.2 alone. Namely, they showed that this system is complete with respect to the class of finite directed partial pre-orders, finite pre-lattices, finite baled pre-trees and finite pre-Boolean algebras. This motivates the following:

Definition 3.3.2. Let $\mathcal{C}_{\mathsf{FG}}$ be the class of frames $\mathcal{F} = (W, \leq, \blacktriangleleft)$ such that

1. $(W, \leq)$ is a finite pre-lattice,
2. $(W, \blacktriangleleft)$ is a finite pre-lattice,
3. $\leq = \trianglelefteq^{-1}$.

Remark 3.3.3. A frame F of the form (W, R, S), where S is the converse of R, is called *bidirectional.* All frames in the class $\mathcal{C}_{\mathsf{FG}}$ are of this type. Although useful, bidirectionality makes it redundant to write (2) in the definition above: it already follows from (1) and (3) (equivalently, assumed (2) and (3), (1) follows). Assume (1) and (3). By the former, $\leq$ is reflexive and transitive. By reflexivity on $\leq$ and (3), $\trianglelefteq$ is reflexive, too. Let $u, v, w \in W$ such that $u \trianglelefteq v \trianglelefteq w$. By (3), $w \leq v \leq u$. Then, by transitivity on $\leq$, $w \leq u$. By (3) again, $u \trianglelefteq w$. This shows that $\trianglelefteq$ is a pre-order. Now, let $\equiv_\blacktriangleleft$ be the equivalence relation given by $u \equiv_\blacktriangleleft v$ if and only if $u \trianglelefteq v \trianglelefteq u$. Let $[u]_{\equiv_\blacktriangleleft}$ denote the $\equiv_\blacktriangleleft$-equivalence class of u. Analogously, define $\equiv_\leq$ and $[\cdot]_\equiv$ for $\leq$. It is easy to see that $[u]_{\equiv_\blacktriangleleft} = [u]_{\equiv_\leq}$ for every $u \in W$. Indeed, if $u' \in [u]_{\equiv_\blacktriangleleft}$, then $u' \trianglelefteq u \trianglelefteq u'$, and (3) implies $u' \leq u \leq u'$, so $u' \in [u]_{\equiv_\leq}$; the same argument shows that $[u]_{\equiv_\leq} \subseteq [u]_{\equiv_\blacktriangleleft}$. We can then drop the $\leq$ and $\trianglelefteq$ from the $\equiv_\leq$ and $\equiv_\blacktriangleleft$. It is as well easy to see that $\leq_\equiv = \trianglelefteq_\equiv^{-1}$. Then, since the $\leq_\equiv$-meet of every two clusters $[u]_\equiv, [v]_\equiv$ is their $\trianglelefteq_\equiv$-join and vice versa and these always exist because $(W, \leq)$ is a pre-lattice, $(W, \trianglelefteq)$ is a pre-lattice, too.

Remark 3.3.4. Bidirectional transitive frames characterise *temporal logics*. A temporal logic is a bi-modal logic with two modalities interpreted as *it will happen that* and *it happened that*. It is perhaps worth noticing that our interpretation of the modalities as *it holds in every forcing extension* and *it holds in every ground* can be seen as a temporal logic: forcing extensions of a given model are its possible future states, while grounds are in the past.

Theorem 3.3.16 shows that the logic LFG is Kripke complete with respect to the class of frames $\mathcal{C}_{\mathsf{FG}}$. A few preliminary lemmas are needed. Recall that if φ is a modal formula and P is a property, then φ is *canonical for* P if φ is valid precisely on those frames that exhibit the property P (see, e.g., [BDRV01, Definition 4.31]).

Lemma 3.3.5. *Axiom* $\mathsf{Inv}_{\mathsf{FG}}$ *is canonical for* $\leq = \trianglelefteq^{-1}$.

Proof. Let $F = (W, \leq, \trianglelefteq)$ be a frame. Assume first that $\leq = \trianglelefteq^{-1}$ and let $u \in W$ such that $u \Vdash \varphi$. Let $v \in W$ be such that $u \trianglelefteq v$. By assumption, $v \leq u$, hence $v \Vdash \Diamond\varphi$, so $u \Vdash \blacksquare\Diamond\varphi$. If $v \in W$ is such that $u \leq v$, then $v \trianglelefteq u$, for $\leq^{-1} = (\trianglelefteq^{-1})^{-1} = \trianglelefteq$. Then, $v \Vdash \blacklozenge\varphi$, hence $u \Vdash \Box\blacklozenge\varphi$. Altogether, this means that $u \Vdash \varphi$ implies $u \Vdash \Box\blacklozenge\varphi \wedge \blacksquare\Diamond\varphi$. We prove the converse by contraposition. Assume that there exist $u, v \in W$ such that $u \trianglelefteq v$ but $v \nleq u$ and let $u \Vdash \varphi$. Let V be a valuation in F such that $V(\varphi) = \{u\}$. Then $u \nVdash \blacksquare\Diamond\varphi$, hence $u \nVdash \mathsf{Inv}_{\mathsf{FG}}$. $\Box$

It is well known that axiom .2 is canonical for the weak directeness property. Recall that a relation R is weakly directed if it holds that $\forall u \forall v \forall w(uRv \wedge uRw \rightarrow \exists t(vRt \wedge wRt))$. Note that, in mono-modal logics[4], if R is transitive and reflexive, as it happens in S4-frames, this property can be weakened to strong directeness ($\forall v \forall w \exists t(vRt \wedge wRt)$) because generated submodels preserve validity and the generated submodel of a transitive, reflexive and weak directed frame is always transitive, reflexive and strong directed. In connected bidirectional frames, weak and strong directedness are equivalent, too:

Lemma 3.3.6. *Let $(W, \leq, \blacktriangleleft)$ be a connected bidirectional frame. Then, $\leq$ and $\blacktriangleleft$ are weak-directed partial orders if and only if they are strong-directed.*

Proof. The right to left direction holds in general. For the other, note that for every two distinct worlds u and v there exists a path $\{u_0, \ldots, u_n\}$ such that $u_0 = u$, $u_n = v$, and either $u_i \leq u_{i+1}$ or $u_i \blacktriangleleft u_{i+1}$ for every $i < n$. If the path is of length 2, either $u \leq v$ or $u \blacktriangleleft v$. If the former, we have $v \blacktriangleleft u$. Otherwise, we have $v \leq u$. In any case, u and v have a common $\leq$-successor and a common $\blacktriangleleft$-predecessor. Assume the length of the path is 3. If the path is a $\leq$ or $\blacktriangleleft$-path, we are done. If the path is of the form $u \leq t \blacktriangleleft v$, since $\leq = \blacktriangleleft^{-1}$ we have that $u, v \leq t$ and $t \blacktriangleleft u, v$. In the first case, both u and v have a common $\leq$-successor; in the second case, the existence of a common $\blacktriangleleft$-successor follows from weak-directeness on $\blacktriangleleft$. An analogous argument shows that the same happens if the path is of the form $u \blacktriangleleft t \leq v$. Now, assumed this happens for any path of length $\leq n$, suppose there is a path from u to v. Consider the path after removing v, a path of length n. By induction hypothesis, u and u_{n-1} have a common $\leq$-successor v' and a $\blacktriangleleft$-successor v''. Then, one gets the paths $u \leq v' \blacktriangleleft v_{n-1} R v$ and $u \blacktriangleleft v'' \leq v_{n-1} Q v$, where R and Q can either be $\leq$ or $\blacktriangleleft$. By induction hypothesis, since this is a path of length 4, u and v have a common $\leq$-successor and a common $\blacktriangleleft$-successor and we can conclude that both relations are strong directed. □

Remark 3.3.7. We adopt the convention that the length of a path is the number of nodes in that path, cointing the first and the last one. It follows from the proof above that in any bidirectional frame $F = (W, \leq, \blacktriangleleft)$ with $\leq$ and $\blacktriangleleft$ weak-directed partial orders, every two elements in the same connected component are connected by a $(\leq \cup \blacktriangleleft)$-path of length at most 4. It is clear that if transitivity for both $\leq$ and $\blacktriangleleft$ is assumed, the maximal length of the shortest $\leq \cup \blacktriangleleft$-path connecting two worlds can be reduced to 3.

Remark 3.3.8. Conditions (1) and (2) in Definition 3.3.2 imply that $(W, \leq)$ and $(W, \blacktriangleleft)$ are connected.

Remark 3.3.9. It is easy to see that bidirectional strong-directed and connected frames $(W, \leq, \blacktriangleleft)$, have a $\leq$-least and a $\leq$-greatest element (we sometimes refer to them as $\leq$-top and $\leq$-bottom, respectively). By strong-directedness, there must be top and botom elements, and finiteness forces them to be unique. By the previous discussion, the $\leq$-least and the $\leq$-greatest element of $(W, \leq)$ in such frames are the $\blacktriangleleft$-greatest and the $\blacktriangleleft$-least element of $(W, \blacktriangleleft)$, respectively.

Recall that the *canonical model* of LFG is the structure $(W^{\mathsf{can}}, \leq^{\mathsf{can}}, \blacktriangleleft^{\mathsf{can}}, V^{\mathsf{can}})$ where

(i) W^{can} is the set of all maximal LFG-consistent sets of formulas,

(ii) $\leq^{\mathsf{can}}$ (equiv. $\blacktriangleleft^{\mathsf{can}}$) is the binary relation on W^{can} defined by $u \leq^{\mathsf{can}} v$ (equiv. $u \blacktriangleleft^{\mathsf{can}} v$) if for all formulas ψ, $\Diamond\psi \in u$ whenever $\psi \in v$, (equiv. $\blacklozenge\psi \in u$ whenever $\psi \in v$)[5],

(iii) V^{can} is the valuation given by $V^{\mathsf{can}}(p) = \{w \in W^{\mathsf{can}} : p \in w\}$.

The *canonical frame* of LFG is $(W^{\mathsf{can}}, \leq^{\mathsf{can}}, \blacktriangleleft^{\mathsf{can}})$.

Let Φ be a set of formulas. By $[w]_\Phi$ we denote the equivalence class

$$\{u \in W : \forall \phi \in \Phi(M, u \Vdash \phi \text{ if and only if } M, w \Vdash \phi)\}.$$

[4]That is, a modal logic with a single modal quantifier $\Box$ (modulo its dual $\Diamond$).

[5]Instead, we could have written that $u \leq^{\mathsf{can}} v$ (equiv. $u \blacktriangleleft^{\mathsf{can}} v$) if for all formulas ψ, $\psi \in v$ whenever $\Box\psi \in u$ (equiv. $\psi \in v$ whenever $\blacksquare\psi \in u$) (see [BDRV01, Lemma 4.19]).

The *filtration* of a model $\mathcal{M} = (W, R, S, V)$ through a subformula-closed set of formulas Φ is a model $\mathcal{M}_\Phi = (W', R', S', V')$ where:

1. $W' = \{[w]_\Phi : w \in W\}$;
2. R' is a binary relation in W' satisfying that
   (a) if wRv then $[w]_\Phi R' [v]_\Phi$, and
   (b) if $[w]_\Phi R'[v]_\Phi$, then for all $\Diamond\phi \in \Phi$, if $M, v \Vdash \phi$ then $M, w \Vdash \Diamond\phi$; and

   We define S' similarly.
3. $V'(p) = \{[w]_\Phi : M, w \Vdash p\}$.

Filtrations preserve satisfaction in the following sense:

Theorem 3.3.10 (Filtration Theorem)**.** *Let M_Φ be a filtration of a model M through a sub-formula closed set Φ. Then, for all formulas $\phi \in \Phi$ and all nodes $w \in M$, we have*

$$M, w \Vdash \phi \textit{ if and only if } M_\Phi, [w]_\Phi \Vdash \phi.$$

Proof. See, e.g., [BDRV01, Theorem 2.39]. □

That is, the filtration technique constructs finite models by removing superfluous structure from large, maybe infinite models. It does so by identifying as many states as possible according to the equivalence relation induced by the filtration set. In sum, filtrating through a finite set of formulas allows one to represent an infinite amount of information in a finite, manageable form.

There may be different filtrations of a model through a subformula-closed set of formulas, depending on how R' is defined. This may determine which of the properties present in the initial model remain in its filtration.

Example 3.3.11. The relation R', when defined by

$$[u]_\Phi R_m [v]_\Phi \text{ if and only if } \exists u \in [u]_\Phi \exists v \in [v]_\Phi (uRv),$$

is called the *minimal relation*. It is well-known that the minimal relation preserves reflexivity. However, it does not preserve in general other properties such as, for instance, transitivity. Let R_t be the relation in W_Φ given by

$$[u]_\Phi R_t [v]_\Phi \text{ if and only if } (\Diamond\varphi \in \Phi \wedge M, v \Vdash \varphi \vee \Diamond\varphi) \to M, u \Vdash \Diamond\varphi.$$

Note that if R is transitive, then R_t is a filtration for R. It is clear that (2a) holds. For (2a), let $u, v \in W$ such that uRv and let φ be a formula such that $\Diamond\varphi \in \Phi$ and $v \Vdash \varphi \vee \Diamond\varphi$. If $v \Vdash \varphi$ it must be the case that $u \Vdash \Diamond\varphi$, for otherwise there would exist no world $w \in W$ with uRw and $w \Vdash \varphi$, which is not possible for v is such a world. It $v \Vdash \Diamond\varphi$, then there exists some $w \in W$ such that vRw and $w \Vdash \varphi$. Again, if $u \nVdash \Diamond\varphi$, there would be no world satisfying φ to which u has access, but uRw by transitivity. Furthermore, the relation R_t is always transitive[6], for if $[u]_\Phi R_f [v]_\Phi R_t [w]_\Phi$ and φ is a formula such that $\Diamond\varphi \in \Phi$ and $w \Vdash \varphi \vee \Diamond\varphi$, then $v \Vdash \Diamond\varphi$, hence $v \Vdash \varphi \vee \Diamond\varphi$, so $u \Vdash \Diamond\varphi$ and $[u]_\Phi R_t [w]_\Phi$ (see, e.g., [BDRV01, Lemma 2.42]). This justifies naming R_t the *filtration relation.*

In this setting, we need a filtration that also preserves bidirectionality and strong-directedness. Let $(W, \leq, \blacktriangleleft)$ be a bidirectional frame and let Φ be a subformula closed set of formulas. Let W_Φ be the quotient induced by Φ and let $\leq_t$ and $\blacktriangleleft_t$ denote be the transitive relation for $\leq$ and $\blacktriangleleft$, respectively. Now, note that if $u, v \in W$ are such that $u \blacktriangleleft v$, then $v \leq u$ by bidirectionality. By

[6]Indeed, observe that the transitivity of R is not required for R_t to be transitive but for it to be a filtration relation.

(2a), $[v]_\Phi \leq_t [u]_\Phi$, so for every formula φ with $\Diamond\varphi \in \Phi$, if $u \Vdash \varphi \vee \Diamond\varphi$, we have $v \Vdash \Diamond\varphi$. Let then $\underline{\blacktriangleleft}_t^*$ be given by

$$[u]_\Phi \underline{\blacktriangleleft}_t^* [v]_\Phi \text{ if and only if } (\Diamond\varphi \in \Phi \wedge u \Vdash \varphi \vee \Diamond\varphi) \rightarrow v \Vdash \Diamond\varphi.$$

It is clear that $\leq_t = (\underline{\blacktriangleleft}_t^*)^{-1}$. Then, the transitivity of $\leq_t$ implies the transitivity of $\underline{\blacktriangleleft}_t$. This means that the transitive filtration of a bidirectional transitive frame is a bidirectional transitive frame (see, e.g., [BDRV01, Exercise 2.3.7]). However, the relation $\underline{\blacktriangleleft}_t^*$ may not be a filtration relation for $\underline{\blacktriangleleft}$, for although it clearly satisfies (2a), it doesn't talk about $\blacklozenge$ modalities. We thus proceed differently:

Define the relation $[u]_\Phi \leq_t^* [v]_\Phi$ if and only if

$$[(\Diamond\varphi \in \Phi \wedge v \Vdash \varphi \vee \Diamond\varphi) \rightarrow u \Vdash \Diamond\varphi] \wedge [(\blacklozenge\varphi \in \Phi \wedge u \Vdash \varphi \vee \blacklozenge\varphi) \rightarrow v \Vdash \blacklozenge\varphi].$$

Note that $\leq_t^* = \leq_t \cap \underline{\blacktriangleleft}_t^{-1}$. We define similarly the relation $\underline{\blacktriangleleft}_t^*$. That is, $[u]_\Phi \underline{\blacktriangleleft}_t^* [v]_\Phi$ if and only if

$$[(\blacklozenge\varphi \in \Phi \wedge v \Vdash \varphi \vee \blacklozenge\varphi) \rightarrow u \Vdash \blacklozenge\varphi] \wedge [(\Diamond\varphi \in \Phi \wedge u \Vdash \varphi \vee \Diamond\varphi) \rightarrow v \Vdash \Diamond\varphi].$$

This time, $\underline{\blacktriangleleft}_t^* = \leq_t^{-1} \cap \underline{\blacktriangleleft}_t$. By construction, both $\leq_t^*$ and $\underline{\blacktriangleleft}_t^*$ are transitive and $\leq_t^*$ is the converse of $\underline{\blacktriangleleft}_t^*$. Moreover, $\leq_t^*$ and $\underline{\blacktriangleleft}_t^*$ are filtration relations for $\leq$ and $\underline{\blacktriangleleft}$, respectively. It is enough to prove it for only one of them. That (2b) holds is immediate. So assume $u \leq v$ and let φ be an arbitrary formula. If $\Diamond\varphi \in \Phi$ and $v \Vdash \varphi \vee \Diamond\varphi$, then the transitivity of $\leq$ implies that $u \Vdash \Diamond\varphi$. Similarly, if $\blacklozenge\varphi \in \Phi$ and $u \Vdash \varphi \vee \blacklozenge\varphi$, we get $v \Vdash \blacklozenge\varphi$. Therefore, (2a) holds, too. We state this as a lemma:

Lemma 3.3.12. *The relations $\leq_t^*$ and $\underline{\blacktriangleleft}_t^*$ are transitive filtration relations for $\leq$ and $\underline{\blacktriangleleft}$, respectively, provided $\leq$ and $\underline{\blacktriangleleft}$ are transitive. Furthermore, $\leq_t^* = (\underline{\blacktriangleleft}_t^*)^{-1}$.*

To sum up, the filtration frame $(W, \leq_t^*, \underline{\blacktriangleleft}_t^*)$ of a bidirectional transitive model $(W, \leq, \underline{\blacktriangleleft})$ is bidirectional and transitive. By Lemma 3.3.6, if the bidirectional frame $(W, \leq, \underline{\blacktriangleleft})$ is such that both $\leq$ and $\underline{\blacktriangleleft}$ are weak-directed, then both relations are actually strong-directed. The bidirectional filtration also preserves strong-directedness:

Lemma 3.3.13. *Both $\leq_t^*$ and $\underline{\blacktriangleleft}_t^*$ preserve strong-directedness.*

Proof. This is a consequence of the fact that every filtration relation extends the minimal filtration (see, e.g., [BDRV01, Lemma 2.40]). Let $u, v \in W$. By strong-directedness, there exists $w \in W$ such that $u \leq w$ and $v \leq w$. Then, by (2a) on the minimal filtration, $[u]_\Phi \leq_m [w]_\Phi$ and $[v]_\Phi \leq_m [w]_\Phi$, so $[u]_\Phi \leq_t^* [w]_\Phi$ and $[v]_\Phi \leq_t^* [w]_\Phi$. The same goes for $\underline{\blacktriangleleft}_t^*$. □

Connectedness is also preserved, and so it is reflexivity. As a consequence:

Corollary 3.3.14. *The bidirectional transitive filtration of a $\mathcal{C}_{\mathsf{FG}}$-frame is a bidirectional, connected, reflexive, transitive and strong-directed frame.*

If $(W, \leq, \underline{\blacktriangleleft})$ is a bidirectional transitive frame, we call $(W, \leq_t^*, \underline{\blacktriangleleft}_t^*)$ the *bidirectional transitive filtration* of $(W, \leq, \underline{\blacktriangleleft})$.

The theorem below, announced at the beginning of this section, proves the frame completeness of LFG with respect to the class of frames $\mathcal{C}_{\mathsf{FG}}$. Before we get into the proof, recall that two Kripke models $M_1 = (W_1, R_1^1, \ldots, R_1^n, V_1)$ and $M_2 = (W_2, R_2^1, \ldots, R_2^n, V_2)$ of the same similarity type are said to be *bisimilar* if there exists a non-empty binary relation $Z \subseteq W_1 \times W_2$ such that:

1. if $Z(w_1, w_2)$, then w_1 and w_2 satisfy the same proposition letters;
2. if $Z(w_1, w_2)$ and $R_1^i(w_1, v_1, \ldots, v_r)$ where $i \in \{1, \ldots, n\}$, $r + 1$ is the arity of R_1^i and $v_1, \ldots, v_r \in W_1$, then there are $v_1', \ldots, v_n' \in W_2$ such that $Z(v_i, v_i')$ for all $i = 1, \ldots, n$ $R_2^i(w_2, v_1', \ldots, v_n')$;

3. if $Z(w_1, w_2)$ and $R_2^i(w_2, v_1', \ldots, v_r')$ where $i \in \{1, \ldots, n\}$, $r + 1$ is the arity of R_2^i and $v_1', \ldots, v_r' \in W_2$, then there are $v_1, \ldots, v_n \in W_1$ such that $Z(v_i, v_i')$ for all $i = 1, \ldots, n$ and $R_1^i(w_1, v_1, \ldots, v_n)$.

Modal formulas are invariant under bisimulations in the following sense:

Proposition 3.3.15. *Let Z be a bisimulation between two Kripke models M_1 and M_2 as those above. Then, for every $w_1 \in W_1$ and every $w_2 \in W_2$, $Z(w_1, w_2)$ implies that, for every formaula φ, $M_1, w_1 \Vdash \varphi$ if and only if $M_2, w_2 \Vdash \varphi$.*

Proof. See [BDRV01, Theorem 2.20]. □

Theorem 3.3.16. LFG *is Krikpe complete with respect to the class $\mathcal{C}_{\mathsf{FG}}$.*

Proof. Let $\mathcal{M} := (W^{\mathsf{can}}, \leq^{\mathsf{can}}, \trianglelefteq^{\mathsf{can}})$ be the canonical frame of LFG. Well-known canonicity arguments imply that $\mathcal{M}$ is such that both $(W^{\mathsf{can}}, \leq^{\mathsf{can}})$ and $(W^{\mathsf{can}}, \trianglelefteq^{\mathsf{can}})$ are transitive, reflexive and weak-directed. By Lemma 3.3.5, $\leq^{\mathsf{can}} = (\trianglelefteq^{\mathsf{can}})^{-1}$.

Let φ be a formula consistent with LFG, let $w \in W$ be such that $\varphi \in w$ and let $\mathcal{M}_\Phi = (W_\Phi, \leq_\Phi, \trianglelefteq_\Phi)$ be the bidirectional transitive filtration of the submodel $\mathcal{M}[w]$ generated by w with respect to the set

$$\Phi = \{\psi : \psi \text{ is a subformula of } \varphi\}.$$

Note that the generated submodel $\mathcal{M}[w]$ is connected and inherits reflexivity, transitivity, weak-directedness and invertibility for both the restrictions of $\leq^{\mathsf{can}}$ and $\trianglelefteq^{\mathsf{can}}$ to $\mathcal{M}[w]$. Then, by Lemma 3.3.6, $\leq^{\mathsf{can}}_{\restriction \mathcal{M}[w]}$ and $\trianglelefteq^{\mathsf{can}}_{\restriction \mathcal{M}[w]}$, the restrictions of $\leq^{\mathsf{can}}$ and $\trianglelefteq^{\mathsf{can}}$ to $\mathcal{M}[w]$, are strong-directed. Then, $(\mathcal{M}[w], \leq^{\mathsf{can}}_{\restriction \mathcal{M}[w]}, \trianglelefteq^{\mathsf{can}}_{\restriction \mathcal{M}[w]})$ is a bidirectional, reflexive, transitive and strong-directed frame. Recall that in the bidirectional transitive filtration, $\leq_\Phi$ and $\trianglelefteq_\Phi$ are $\leq_t^*$ and $\trianglelefteq_t^*$, respectively. Since Φ is finite, the filtrated model is finite. By Corollary 3.3.14, the filtration $\mathcal{M}_\Phi$ is finite, bidirectional, connected, reflexive, transitive and strong-directed.

Let $\equiv$ be the equivalence relation in $\mathcal{M}_\Phi$ given by $u \equiv v$ if and only if $u \leq_\Phi v \leq_\Phi u$ and let $W_\equiv$ denote the quotient $W/\equiv$ together with the induced quotient relation $R_\leq$. Since $(W_\Phi, \leq_\Phi)$ is finite and strong-directed, so is $(W_\equiv, R_\leq)$. Let $[a]_\equiv, [b]_\equiv \in W_\equiv$. Since M_Φ is strong-directed for $\leq_\Phi$, there exists some $c \in M_\Phi$ such that $a, b \leq_\Phi c$, hence $[a]_\equiv, [b]_\equiv R_\leq [c]_\equiv$. That is, $[a]_\equiv$ and $[b]_\equiv$ have common upper bounds. However, it may be the case that there is no least common upper bound for $[a]_\equiv$ and $[b]_\equiv$. We now follow a similar approach to that in the proof of [HL08, Lemma 6.5], and apply a slight variation of the unravelling method described there. With it, we obtain a bisimilar frame $(\tilde{W}_\Phi, \tilde{\leq}_\Phi, \tilde{\trianglelefteq}_\Phi)$ whose two mono-relational fragments are finite pre-lattices. Assume that $[a]_\equiv$ and $[b]_\equiv$ have two different upper bounds $[c]_\equiv$ and $[d]_\equiv$ that are incomparable. Then, we proceed as follows (see Figure 3.1):

(a) Substitute each node $[a], [b], [c], [d]$ for a pair of nodes $[a]_1, [a]_2, [b]_1, [b]_2, [c]_1, [c]_2$, and $[d]_1, [d]_2$. Each node is an equivalence class. Note that we are actually substituting each element in the cluster for a pair of two identical elements.

(b) Define the relation $R'_\leq$ given by

- $[a]_1 R'_\leq [c]_1$, $[a]_1 R'_\leq [d]_1$,
- $[b]_1 R'_\leq [c]_1$, $[b]_1 R'_\leq [d]_2$,
- $[a]_2 R'_\leq [c]_2$, $[a]_2 R'_\leq [d]_1$,
- $[b]_2 R'_\leq [c]_2$, $[b]_2 R'_\leq [d]_2$,

(c) its inverses (i.e., $[c]_1 R'_\leq [a]_1$, $[d]_1 R'_\leq [a]_1$, $[c]_1 R'_\leq [b]_1$, and so on);

(d) and $R'_\leq = R_\leq$ for the other nodes.

The resulting model is bisimilar to the original one, the bisimulation Z being

- $Z(x, x_1)$, $Z(x, x_2)$ when $x = [a], [b], [c], [d]$,

- and the identity, otherwise.

It is easy to verify that the process preserves bidirectionality, reflexivity, transitivity and strong-directedness. Moreover, it secures that every two nodes have a single upper bound. By repeatedly doing this, after a finite number of times one eventually obtains a model with the mentioned properties in which every two elements have a join. Analogously, one gets that every two elements have a meet. Thus, after finitely many partial unravellings, one obtains a bisimilar frame $(\tilde{M}, \tilde{\lesssim}, \tilde{\blacktriangleleft})$ such that $(\tilde{M}, \tilde{\lesssim})$ is a finite pre-lattice, where $\tilde{\lesssim}$ is the relation induced by $R'_{\leq}$[7]. By substituting each node for the cluster of elements in the equivalence relation, we get the frame $(\tilde{W}_\Phi, \tilde{\lesssim}_\Phi, \tilde{\blacktriangleleft}_\Phi)$ where $(\tilde{W}_\Phi, \tilde{\lesssim}_\Phi)$ is a finite pre-lattice. By Remark 3.3.3, the bisimulation is such that $(W_\Phi, \blacktriangleleft_\Phi)$ is a pre-lattice. This finishes the proof. □

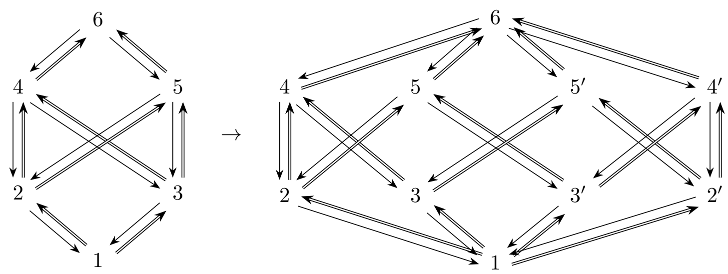

Figure 3.1: A partial (and total) unravelling.

3.3.1 Set-theoretic completeness.

As usual, $\mathcal{L}_\in$ denotes the language of set theory. We now assign to each modal formula a sentence of $\mathcal{L}_\in$ by interpreting $\Box$ as *in all forcing extensions* and $\blacksquare$ as *in all grounds*. Because of Laver's theorem on the first order expressibility of grounds [Lav07], *being a ground* can be written as an $\mathcal{L}_\in$-formula.

Definition 3.3.17. Let $v : \mathsf{var} \to \mathcal{L}_\in$ assign a $\mathcal{L}_\in$-formula to each propositional variable. Recursively, we extend v to arbitrary bimodal formulas as follows:

- $v(\varphi \wedge \psi) = v(\varphi) \wedge v(\psi)$;
- $v(\varphi \vee \psi) = v(\varphi) \vee v(\psi)$;
- $v(\neg\varphi) = \neg v(\varphi)$;
- $v(\blacklozenge\varphi) = \exists M(M \text{ is a ground } \wedge M \models v(\varphi))$;
- $v(\Diamond\varphi) = \exists \mathbb{P} \exists p \in \mathbb{P}(\mathbb{P} \text{ is a poset } \wedge p \Vdash_{\mathbb{P}} v(\varphi))$;

We will sometimes abuse notation by identifying formulas with their valuations.

Remark 3.3.18. Because of the Forcing Theorem, to say of a countable transitive model M that $\exists \mathbb{P} \exists p \in \mathbb{P}(\mathbb{P} \text{ is a poset } \wedge p \Vdash_{\mathbb{P}} v(\varphi))$ holds in M is equivalent to saying that φ holds in some forcing extension of M.

As in [HL08], we give the following definition:

Definition 3.3.19. A modal assertion $\varphi(p_0, \dots, p_n)$ is a *valid principle of* FG if for all sentences of set theory $\varphi(\psi_0, \dots, \psi_n)$ holds under the interpretation of forcing and grounds. We say that $\varphi(p_0, \dots, p_n)$ is a ZFC-*provable principle of* FG if ZFC proves $\varphi(v(p_0), \dots, v(p_n))$ for all assignments v. For any model W of ZFC, a modal assertion $\varphi(p_0, \dots, p_n)$ is a *valid principle of* FG *in* W if $\varphi(v(p_0), \dots, v(p_n))$ holds in W for all assignments v.

[7]Note that could have defined the relation $R'_{\leq}$ by directly drawing the new relations between each element in each duplicated cluster.

Proposition 3.3.20 (Soundness). *The theorems of* LFG *are valid principles of* FG.

Proof. Every theorem of S4.2($\Box$) and S4.2($\blacksquare$) is a valid principle of FG (see [HL08], [HL13] and the remark below). Under our interpretation of the modalities $\Box$ and $\blacksquare$ (and its duals $\Diamond$ and $\blacklozenge$), the statement $\mathsf{Inv}_{\mathsf{FG}}$ is also valid. Indeed, on the one hand, if M is a model of ZFC and it satisfies φ, then every forcing extension $M[G]$ of M has a ground, which is M itself, satisfying φ and, on the other hand, every ground of M has a forcing extension, which is M itself again, where φ holds. $\Box$

Remark 3.3.21. For completeness, we provide a rather informal an intuitive idea of why S4.2 is sound with respect to the modal logic of forcing. To reason about the modal logic of grounds one can proceed in an analogous way. Normality clearly holds, and the fact that Dual, the scheme $\neg\Diamond\varphi \leftrightarrow \Box\neg\varphi$, is valid follows easily form the fact that a statement φ of set theory is not forceable if and only if $\neg\varphi$ holds in every forcing extension. Axiom T, the scheme $\Box\varphi \to \varphi$, expresses that if φ is true in every forcing extension then it holds in the current model, but this is valid simply because every model is a trivial forcing extension of itself. Axiom 4, $\Box\varphi \to \Box\Box\varphi$, is valid as well, for every forcing extension of a forcing extension of M is a forcing extension of M. Finally, Axiom .2, the schema $\Diamond\Box\varphi \to \Box\Diamond\varphi$, is valid because if φ is necessary in a forcing extension M' of M and M'' is an arbitrary forcing extension, then φ follows in the corresponding product extension M''' of M' and M'', so it is forceable over M''.

The rest of this section is devoted to proving the converse of the previous proposition. Our proof is a generalisation of that of Hamkins and Löwe in [HL08].

Definition 3.3.22 (Hamkins-Löwe). A statement φ in $\mathcal{L}_\in$ is a *button* if it is $\Box$-necessarily possibly necessary. It is a *switch* if both φ and $\neg\varphi$ are $\Box$-necessarily possible.

That is, φ is a button if whenever it holds, it does hold in every subsequent forcing extension. We say that the button is *pushed* when it holds, otherwise we say that it is *unpushed*. A collection of buttons and switches is *independent* in a model if all the buttons are unpushed in the model and, necessarily, any of the buttons can be pushed and any of the switches can be switched without affecting the value of any of the other buttons and switches.

The corresponding notions for the $\blacksquare$ relation are that of *downward button* and *downward switch*. Note that, provided M has non-trivial grounds, a statement φ is a button on a model M if and only if $\neg\varphi$ is a downward button on M. If M does not have non-trivial grounds, no statement can be an unpushed downward button in M. However, a statement φ is an unpushed button on an arbitrary model of set theory M if and only if $\neg\varphi$ is a pushed downward button on M. Regarding switches, it is clear that it only makes sense to talk about downward switches for models with infinitely many non-trivial grounds. In the current context it is enough to consider solely upwards buttons and switches, to which we will refer simply as buttons and switches. It is a trivial but important fact that every statement of set theory is either a switch, a button or the negation of a button.

As proved by Hamkins and Löwe (see [HL08, Lemma 6.2]), if $F = (W, \leq)$ is a finite lattice and W is a model of set theory with a sufficiently large independent family of buttons b_i, then to each node $w \in F$ we can assign an assertion p_w such that W satisfies that in every forcing extension of W exactly one of the p_w is true and $W \vDash p_{w_0}$, where w_0 denotes the minimal node of F, and in any model satisfying p_w, the statement p_v is forceable if and only if $w \leq v$ in F. Since in bidirectional frames one can just focus on one of its mono-modal fragments, we can use the same result.

Lemma 3.3.23. *Let $F = (F, \leq, \trianglelefteq)$ be a finite bidirectional frame where $(F, \leq)$ (hence $(F, \trianglelefteq)$, too) is a finite lattice and let W be a model of set theory with a sufficiently large independent family of buttons b_n. Then, to each element $w \in F$ we can assign a statement p_w such that $W \vDash p_{w_0}$, where w_0 is the $\leq$-least of $(F, \leq)$ and:*

1. *in any model, exactly one of the p_w is true;*

2. *in any model satisfying* p_w*, the statement* p_v *is forceable if and only if* $w \leq v$ *(eq.* $v \blacktriangleleft w$*) in* F *and holds in a ground if and only if* $w \blacktriangleleft v$ *in* F *(eq.* $v \leq w$*).*

Proof. The proof is identical to that of [HL08, Lemma 6.2], and is included here for completeness. The statements p_w are those asserting that the pattern of buttons that have been pushed corresponds to a set in the lattice $(F, \leq)$ whose upper bound is w. More in detail, one associates a button b_u to each element $u \in F$, something we can do because there are sufficiently many of them. Now, if $A \subseteq F$, let $b_A = (\bigwedge_{u \in A} \Box b_u) \wedge (\bigwedge_{u \notin A} \neg \Box b_u)$. The statement b_A asserts that the buttons pushed are those corresponding exactly to the set A. Then, define $p_w = \bigvee \{b_A : w = \bigwedge A\}$ to be the statement asserting that the pushed buttons are determined by a set A whose least upper bound is w. Since each model has only one combination of pushed buttons, each model satisfies exactly one p_w, which proves (1); moreover, since all buttons are unpushed in W (the family of buttons is assumed to be independent in W, so all of them are unpushed), we get $W \vDash p_{w_0}$. If p_w is true in a forcing extension $M[G]$, let A be the set $\{u \in F : M[G] \vDash b_u\}$ of buttons pushed in $M[G]$. Since $M[G] \vDash p_w$, then w is the join of A in F. Let $w \leq v$ in F and push the button b_v. Let $M[G][H]$ be the forcing extension whose pushed buttons are those determined by the set $A' = A \cup \{v\}$. The join of A' is v, so p_v is true in $M[G][H]$. This means that p_v is forceable in $M[G]$. For the converse, if p_v was forceable in $M[G]$, there would be a forcing extension $M[G][H]$ such that $M[G][H] \vDash p_v$. Let B be the subset of F determining the buttons pushed in $M[G][H]$. The join of B is v, and since $M[G][H]$ is a forcing extension of $M[G]$, the buttons pushed in $M[G]$ remain pushed in $M[G][H]$, so $A \subseteq B$. This implies that v must be at least as large as the join of A, which is w, hence $w \leq v$ in F. This proves the first assertion in (2). By reversing the arrows, one gets the contrapositive, which is the second assertion in (2). □

Remark 3.3.24. Hamkins and Löwe worked within a mono-modal logic framework, where frames are equipped with a single accessibility relation. This is not the case in our setting. Their reference to a least element w_0 in the frame relies on the fact that any generated submodel of a mono-modal frame is rooted at the generating node. With multiple accessibility relations, this property no longer holds. However, due to the bidirectionality of the relations in our context, we can restrict attention to the fragment corresponding to just one of the two relations. Within that fragment, it remains meaningful to refer to the $\leq$-least element of the frame.

Lemma 3.3.23 can be extended to independent families of buttons and switches, as in [HL08, Lemma 6.3]. Roughly, if F is a finite pre-lattice and W is a model of set theory with a sufficiently large independent family $\{b_n, s_m\}$ of buttons and swtiches, then to each $w \in F$ we may assign an assertion p_w, this time a Boolean combination of buttons and switches, such that W satisfies that in any forcing extension exactly one p_w is true and $W \vDash p_{w_0}$, where w_0 is a node in the minimal cluster of F, and that in any model satisfying p_w the statement p_v is forceable if and only if $w \leq v$ in F. We adapt this to our case. The proof is that in [HL08, Lemma 6.3]:

Lemma 3.3.25. *Let* F *be as in Lemma 3.3.23 and let* $\{b_n, s_r\}$ *be a sufficiently large independent family of buttons and switches in a model of set theory* W*. Let* w_0 *be any node in the* $\equiv_{\leq}$*-cluster of* F *generated by* w_0 *from Lemma 3.3.23. Then, to each* $w \in F$ *we may assign an assertion* p_w *such that:* $W \vDash p_{w_0}$ *and*

1. *In any forcing extension, exactly one of the* p_w *is true.*
2. *In any model satisfying* p_w*, the statement* p_v *is forceable if and only if* $w \leq v$ *in* F *and holds in a ground if and only if* $v \leq w$ *in* F*.*

Proof. The proof can be found in [HL08, Lemma 6.3], but add it for completeness. The construction proceeds by combining buttons and switches in such a way that buttons determine which cluster (equivalence class in the quotient lattice $F/\equiv$) the current world is in, while switches distinguish between individual nodes within each cluster. Let $[u]$ denote the equivalence class of u in $F/\equiv$, and let $p_{[u]}$ be the label assigned to $[u]$ as in Lemma 3.3.23. Suppose the largest cluster in F has k elements and choose n such that $k \leq 2^n$. For each subset $A \subseteq 0, \ldots, n-1$, define s_A to be the assertion that exactly the switches indexed by A are on. This is expressed as the conjunction of all s_i for $i \in A$ and the negations of s_i for $i \notin A$. Since switches are independent, any such s_A is necessarily possible, and in any forcing extension exactly one s_A holds. For each cluster $[u]$, we partition the collection of such subsets A among the elements $w \in [u]$ by assigning to each w a nonempty collection A_w of such subsets in a way that the sets A_w partition all of $\mathcal{P}(n)$. Then for each w, we define s_w as the disjunction of all s_A for $A \in \mathcal{A}w$, and finally define p_w as $p[w] \wedge s_w$. By construction, in any forcing extension exactly one $p[u]$ is true and exactly one s_w is true, so exactly one p_w is true. We choose the actual switch pattern A that holds in W so that it belongs to A_{w_0}, ensuring that $W \models p_{w_0}$. Now, suppose $W[G]$ is a forcing extension satisfying p_w. This means $p_{[w]}$ and s_w both hold in $W[G]$. If $w \leq v$ in F, then $p_{[v]}$ is forceable from $p_{[w]}$ by the button construction of Lemma 3.3.23, and s_v is forceable by changing the switches independently, so $p_v = p_{[v]} \wedge s_v$ is forceable over $W[G]$. Conversely, if p_v is forceable from $W[G]$, then in some extension $W[G][H]$ it holds, which means $p_{[v]}$ is true there. Since buttons can only be pushed and not unpushed, it follows that $[w] \leq [v]$ in the quotient lattice, and hence $w \leq v$ in F. □

The following extends [HL08, Lemma 6.4]:

Lemma 3.3.26 (Hamkins-Löwe, [HL08, Lemma 6.4])**.** *If M is a Kripke model with a frame F in $\mathcal{C}_{\mathsf{FG}}$ and a world w_0 and W is a model of set theory with a sufficiently large independent family of buttons and switches, then there is an assignment of the propositional variables q_i to set theoretical assertions ψ_i such that for any modal assertion φ we have*

$$(M, w_0) \vDash \varphi(q_0, \ldots, q_n) \text{ if and only if } W \vDash \varphi(\psi_0, \ldots, \psi_n).$$

Proof. Let w_0 be the $\leq$-least element of F. Let p_w be the statements from Lemma 3.3.25. Since w_0 is the initial world of F, we get $W \vDash p_{w_0}$. Define $\psi_i = \bigvee\{p_w : (M, w) \vDash q_i\}$. We show the following claim, from which the rest clearly follows:

$$(M, w) \vDash \varphi(q_0, \ldots, q_n) \text{ if and only if } W \vDash \Box(p_w \to \varphi(\psi_0, \ldots, \psi_n)).$$

The proof goes by induction on the complexity of formulas. We just need to prove the case involving black modalities. The rest is in the proof of [HL08, Lemma 6.4]. Assume first that $(M, w) \vDash \blacklozenge\varphi(q_0, \ldots, q_n)$. Then, there exists some $v \leq w$ such that $(M, v) \vDash \varphi(q_0, \ldots, q_n)$. By induction hypothesis, $W \vDash \Box(p_v \to \varphi(\psi_0, \ldots, \psi_n))$. Because of Lemma 3.3.25, if $v \leq w$, then $W \vDash \Box(p_w \to \blacklozenge p_v)$. It then follows that if $v \leq w$, $W \vDash \Box(p_w \to \blacklozenge\varphi(\psi_0, \ldots, \psi_n))$. Therefore, $W \vDash \Box(p_v \to \varphi(\psi_0, \ldots, \psi_n))$. By induction hypothesis, this holds if and only if $(M, v) \vDash \varphi(q_0, \ldots, q_n)$, if and only if, given that $v \leq w$, $(M, w) \vDash \blacklozenge\varphi(q_0, \ldots, q_n)$. For the converse, assume that $W \vDash \Box(p_w \to \blacklozenge\varphi(\psi_0, \ldots, \psi_n))$, so that $\varphi(\psi_0, \ldots, \psi_n)$ holds in every extension of W with p_w. Every of such extensions have p_v with $v \leq w$, so it must be the case that $W \vDash \Box(p_v \to \varphi(\psi_0, \ldots, \psi_n))$ for some $v \leq w$. By induction hypothesis, $(M, v) \vDash \varphi(q_0, \ldots, q_n)$ for some $v \leq w$, hence $(M, w) \vDash \blacklozenge\varphi(q_0, \ldots, q_n)$. □

Hamkins and Löwe proved that if $V = L$, then there exists an independent collection of infinitely many buttons and infinitely many switches [HL08, Lemma 6.1]. We state this result explicitly:

Lemma 3.3.27 (Hamkins-Löwe)**.** *If $V = L$, then there is an independent collection of infinitely many buttons and infinitely many switches.*

Although our proof of Theorem 3.3.28 does not require an explicit construction of such a collection, we provide here several of them for the sake of completeness[8]. The collection of buttons and switches proposed by Hamkins and Löwe in [HL08, Lemma 6.1] was $\{b_n, s_m\}_{n,m\in\omega}$, where $b_n =$ "ω_n^L is not a cardinal" and $s_m =$ "the GCH holds at $\aleph_{\omega+m}$" as an independent collection of buttons and switches. However, Rittberg pointed out in his Master's thesis that the independence of the collection of buttons in this family may fail. While each b_n is indeed a button and any finite pattern of them can be pushed by forcing over L, their independence must also be preserved in all forcing extensions of L, which is not guaranteed here. Specifically, although it is possible to

[8] This paragraph closely follows the discussion appearing after [HLL15, Theorem 16].

control any two buttons, complications arise with more. For instance, in a generic extension $L[G]$ of L where $2^\omega \geq \aleph_3$ and no cardinals have been collapsed, collapsing $\aleph_2^L$ to $\aleph_1^L$ would necessarily also collapse $\aleph_3^L$, making it unclear how to push b_2 without also affecting b_1 or b_3. This issue is not critical, as Hamkins and Löwe had also provided an alternative family of independent buttons and switches in [HL08]. Additionally, Rittberg himself proposed another independent family of buttons (see [Rit10, Section 2.4.2]), defined by:

$$c_n = \text{“at least one of } \aleph_{3n}^L, \aleph_{3n+1}^L, \text{ and } \aleph_{3n+2} \text{ is not a cardinal,} \\ \text{or } |\mathcal{P}(\aleph_{3n}^L)| > |\aleph_{3n+1}^L|\text{”},$$

for each $n < \omega$. Further, Friedman, Fuchino, and Sakai constructed another independent family: for each $n < \omega$, let T_n^L denote the L-least $\aleph_n$-Souslin tree. Then any subfamily of the assertions

$$d_n = \text{“}\aleph_n^L \text{ is not a cardinal or } T_n^L \text{ is not an } \aleph_n\text{-Souslin tree”}$$

form an independent family of buttons over L (see [FFS17, Section 5]). Finally, the correct family of independent buttons proposed by Hamkins and Löwe ([HL08, Theorem 29]) was given by

$$e_n = \text{“}S_n \text{ is no longer stationary”},$$

where $\omega_1^L = \bigsqcup S_n$ is the L-least partition of ω_1^L into ω-many disjoint stationary sets.

Finally, we show that LFG is the correct modal logic for our interpretation of the $\Box$ and $\blacksquare$ modalities:

Theorem 3.3.28. *If* ZFC *is consistent, the* ZFC*-provable principles of forcing and grounds are exactly those in* LFG*.*

Proof. By Proposition 3.3.20, the valid principles of forcing and grounds are in LFG. We prove the other direction. By Theorem 3.3.16, if φ is not in LFG, it fails in a Kripke model M with a $\mathcal{C}_{\mathsf{FG}}$-frame. Let w be the world in M where φ fails and let w_0 be the $\leq$-least element of M. By Remark 3.3.7, since $\mathcal{C}_{\mathsf{FG}}$-frames are bidirectional, strong-directed and transitive, w_0 access w via a $\leq \cup \blacktriangleleft$-path of length at most 3. Consider the three following cases:

1. The path is of length 1. Then, $w = w_0$ and $w_0 \Vdash \neg\varphi$.

2. The path is of length 2. Since w_0 is the $\leq$-least element of M, it can only be the case that $w_0 \leq w$. Since $w \Vdash \neg\varphi$, then $w_0 \Vdash \Diamond\neg\varphi$.

3. The path is of length 3. Again, since w_0 is the $\leq$-least elemet of M and because of strong-directedness, it can only happen that $w_0 \leq v \leq w$ for some $v \in M$. Then, by transitivity, $w_0 \leq w$, so we are again in the previous case[9].

Let χ be either $\neg\varphi$ or $\Diamond\neg\varphi$. By Lemma 3.3.26, if W is a model of set theory with a sufficiently large independent family of buttons and switches, then there is an assignment of propositional variables of χ to sentences ψ_i of set theory such that $W \vDash \chi(\psi_0, \ldots, \psi_n)$. If ZFC is consistent, so is $\mathsf{ZFC} + V = L$. By Lemma 3.3.27, there is a model of set theory L with an independent family of infinitely many buttons and infinitely many switches. It then happens that $L \vDash \chi(\psi_0, \ldots, \psi_n)$. Let $L[G]$ be the forcing extension of L obtained after pushing buttons and activating switches so the configuration of pushed buttons and on switches is that which corresponds to w. Then, $L[G] \vDash \neg\varphi(\psi_0, \ldots, \psi_n)$, so φ is not a valid principle of FG, hence it is not a ZFC-provable principle of FG. $\square$

[9]Note that this also shows that the $\leq$-least element of a bidirectional, transitive and strong-directed frame access every other element in the frame via a $\leq$-path of lenght 2.

3.3.2 Final remarks

In [HL08, Section 4], Hamkins and Löwe observed that if W is a model of set theory with arbitarily large finite independent families of buttons and switches, such as for instance is any model of $V = L$, then the ZFC-provable principles of forcing[10] in W are exactly those in S4.2 ([HL08, Theorem 15]). The valid principles of forcing and the valid principles of forcing in a model of set theory are not in principle the same. Indeed, Hamkins and Löwe showed that there are models of set theory whose valid principles of forcing go beyond S4.2, as it happens for instance with every model of ZFC+MP, where MP stands for Hamkins' Maximiality Principle. These models satisfy S4.2 and all instances of the scheme $\Diamond\Box\varphi \to \varphi$. We are here in a similar situation. It follows from the proof of Theorem 3.3.28 that there are models of set theory, for instance those that satisfy $V = L$, whose ZFC-provable principles of FG are exactly the general ZFC-provable principles of FG. It is enough that they admit arbitrarily large finite independent families of buttons and switches.

Corollary 3.3.29. *If W is a model of set theory with an arbitrarily large finite independent family of buttons and switches, then the valid principles of* FG *in W are exactly those in* LFG.

However, there are models of set theory whose ZFC-provable principles of FG go beyond LFG. An example is Inamdar and Löwe's model from [IL16, section 5]. It can be shown that this is a model of LFG + Top(■), where Top(■) is the axiom

$$\blacklozenge((\blacksquare\varphi \leftrightarrow \varphi) \wedge (\blacksquare\neg\varphi \leftrightarrow \neg\varphi))$$

introduced by the authors in the cited paper.

3.4 The modal logic of forcing and inner models

In this section we interpret $\Box$ as before, but interpret $\blacksquare$ as *holding in every inner model of* ZFC. Recall that given M, N two transitive models of set theory, N is an inner model of M if $N \subseteq M$ and $Ord^N = Ord^M$. As before, our intention is provide a modal logic that is suitable for the given interpretation, characterise the logic in terms of frames, and, finally, show the completeness with respect to the interpretation. However, this work is, at the moment of writing this dissertation, unfinished. At the end of the section, we describe the challenges in finding a suitable class of frames for the logic, though we do suggest what this class might look like. We also give a brief idea of how we expect to approach proving completeness with respect to the interpretation.

We suggest the following modal logic as a candidate for the forcing and inner models interpretation:

Definition 3.4.1. FIM is the minimal set of $\mathcal{L}_{\Box\blacksquare}$-formulas closed under modus ponens and substitution, and containing classical tautologies and

1. S4.2($\Box$),
2. S4.2Top(■),
3. Conf: $\blacklozenge\Diamond\varphi \to \Diamond\blacklozenge\varphi$,
4. $\mathsf{Inv}_{\mathsf{FIM}}$: $\varphi \to \Box\blacklozenge\varphi$.

We later justify in this section the choice of this logic as a possible option. Let us start with a few observations regarding the mono-modal fragment S4.2Top(■) of FIM.

[10] This is the equivalent notion of the ZFC-provable principles of FG given in Definition 3.3.19. See [HL08, Main Definition 1].

3.4.1 An observation about S4.2Top($\blacksquare$)

In [IL16], Inamdar and Löwe provided several frame characterisations for this logic. Namely, they showed that this system is frame complete for the class of finite topped pre-orders, the class of sharp pre-orders and, finally, the class of inverted lollipops ([IL16, Theorem 2.6]). However, the original proof contains some flaws. In Lemma 3.4.3, we provide a corrected version of the proof (which remains quite similar in structure) and discuss the issues in the original argument in the remark below.

Recall that a pre-order is *topped* if it has a top cluster[11] consisting exclusively of one node. It is *sharp* if is topped and remains directed after removing the largest element. An *inverted lollipop* is a pre-order that is topped and a pre-Boolean algebra once the largest element has been removed. Inamdar and Löwe's frame completeness result for S4.2Top relied on the following lemmata:

Lemma 3.4.2 (Inamdar-Löwe, [IL16, Lemmas 2.3, 2.4 and 2.5])**.**

1. *Every Kripke model on a finite rooted directed pre-order is bisimilar to a Krikpe model on a finite pre-Boolean algebra.*
2. *Every Kripke model on a frame that is a finite topped pre-order is bisimilar to a Kripke model on a frame that is a finite sharp pre-order.*
3. *Every Krikpe model on a frame that is a finite rooted sharp pre-order is bisimilar to a Kripke model on a frame that is a finite inverted lollipop.*

Because of (2) and (3), it is enough to prove that S4.2Top is complete with respect to the class of topped pre-orders.

Lemma 3.4.3 (Inamdar-Löwe)**.** S4.2Top *is complete with respect to the class of finite topped pre-orders.*

Proof. Let φ be a formula consistent with S4.2Top and $\mathcal{M} = (W, \trianglelefteq, v)$ be a canonical model of S4.2Top and let $w \in \mathcal{M}$ be such that $\varphi \in w$. Let $\mathcal{M}[w]$ be the submodel of $\mathcal{M}$ generated by w and note that it is reflexive, transitive, connected and strong-directed. Finally, let $\mathcal{M}_\Phi$ be the transitive filtration of $\mathcal{M}[w]$ with respect to the set

$$\Phi = \{\psi : \psi \text{ is a subformula of } \varphi\} \cup \{\blacklozenge((q \leftrightarrow \blacksquare q) \wedge (\neg q \leftrightarrow \blacksquare\neg q) : q \in \mathsf{var}(\varphi)\}.$$

Since Φ is finite and the transitive filtration preserves reflexivity, transitivity, as well as strong-directedness, $\mathcal{M}_\Phi$ is a finite, connected and directed pre-order. Now, towards a contradiction, assume $\mathcal{M}_\Phi$ is not topped, so that there are two distinct elements S, T in the top cluster. Since all the elements of the top cluster see each other, there must be a variable $p \in \text{sub}(\varphi)$ with $S \Vdash p$ and $T \Vdash \neg p$, for otherwise S and T agree on the valuation on formulas in Φ. Since Top is valid in $\mathcal{M}$, $\mathsf{Top}(p)$ is valid in $\mathcal{M}_\Phi$. By directedness, there is $U \in \mathcal{M}_\Phi$ with

$$U \Vdash (p \leftrightarrow \blacksquare p) \wedge (\neg p \leftrightarrow \blacksquare\neg p).$$

But since $U \trianglelefteq_\Phi S$ as well as $U \trianglelefteq_\Phi T$, S and T necessarily agree on the valuation of p. A contradiction. □

Remark 3.4.4. In the original proof, it appears, though not explicitly stated, that Inamdar and Löwe consider the minimal filtration, which in general does not preserve transitivity. They assert that $\mathcal{M}_\Phi$ satisfies S4.2Top. This in particular implies the transitivity of $\mathcal{M}_\Phi$ but provide no justification for this claim. Instead, we consider the transitive filtration. The transitive filtration of a generated submodel of the canonical model of S4.2 satisfies the theorems of S4.2. As for the Top condition, we do not need to prove that $\mathcal{M}_\Phi$ preserves it, since the statement of the theorem already follows from extending their original subformula-closed set of formulas, which in the original proof was simply the collection of subformulas of φ, with the collection of stances $\{\blacklozenge((q \leftrightarrow \blacksquare q) \wedge (\neg q \leftrightarrow \blacksquare\neg q) : q \in \mathsf{var}(\varphi)\}$, as we do.

[11] See the paragraph after Definition 3.3.1.

3.4.2 Justification for FIM: soundness

The modal logic FIM appears well-suited to our interpretation, and in fact, all of its axioms are sound. Ultimate problems with the completeness theorems may suggest that more axioms are needed, though, but this will be discussed in the next section.

Recall that, while before $\blacksquare$ was interpreted as *it holds in every ground*, we now read it as *it holds in every inner model*. A key difference between the ground and inner model interpretation is that it requires us to reduce the axiom $\mathsf{Inv}_{\mathsf{FG}}$ to one direction only, since there exist inner models that are not grounds. Moreover, this new interpretation brings about other kind of difficulties that we did not have in the previous section. In Definition 3.3.17, we assigned to each modal assertion a sentence in the language of set theory $\mathcal{L}_\in$. We could do this because of Laver's theorem, which secures that grounds are first-order definable in their forcing extensions, which in turn implies that sentences of the form $\exists M(M \text{ is a ground } \wedge M \vDash \varphi)$ are first-order, too (provided that φ is a first-order forumula, of course). In contrast, inner models, considered in full generality, are not first-order definable in their superstructures. Then, if one interprets the $\Box$ and $\blacksquare$ modalities by simply writing *inner model* where *ground* appears in Definition 3.3.17, one would eventually be quantifying over second-order objects. Indeed, by the previous, the interpretation of every formula φ of the form $\blacklozenge\psi$ with ψ a $\mathcal{L}_\in$-formula is already a $\mathcal{L}^2_\in$-formula, and, still in the fashion of Definition 3.3.17,

$$v(\Diamond\varphi) = \exists\mathbb{P}\exists p \in \mathbb{P}(\mathbb{P} \text{ is a poset } \wedge p \Vdash_\mathbb{P} v(\blacklozenge\psi)).$$

To make full sense of this, we thus need to consider forcing in second-order logic. We need to consider a second-order forcing relation $p \Vdash \phi(\vec{a}, \vec{A})$, where $\phi(\vec{a}, \vec{A}) \in \mathcal{L}^2_\in$ is a second-order formula, possibly with second-order (i.e., relation) parameters $A_0, \dots, A_k$, where each A_i is a relation over the ground model. We will nonetheless only consider forcing with partial orders $\mathbb{P}$ belonging to the ground model, i.e., second-order set-forcing (and not class forcing).

For the rest of the chapter, we use the following notational convention. Existential and universal quantifiers with the super-index 1 ($\exists^1, \forall^1$) denote second-order existential and universal quantifiers. Existential and universal quantifiers with no index denote first-order existential and universal quantifiers.

Definition 3.4.5. Let $v : \mathsf{var} \to \mathcal{L}^2_\in$ assign a $\mathcal{L}^2_\in$-formula to each propositional variable. Recursively, we extend v to arbitrary bimodal $\mathcal{L}_{\Box,\blacksquare}$-formulas as follows:

- $v(\varphi \wedge \psi) = v(\varphi) \wedge v(\psi)$;
- $v(\varphi \vee \psi) = v(\varphi) \vee v(\psi)$;
- $v(\neg\varphi) = \neg v(\varphi)$;
- $v(\blacklozenge\varphi) = \exists^1 M(M \text{ is an inner model } \wedge M \vDash v(\varphi))$;
- $v(\Diamond\varphi) = \exists\mathbb{P}\exists p \in \mathbb{P}(\mathbb{P} \text{ is a poset } \wedge p \Vdash_\mathbb{P} v(\varphi))$;

In Remark 3.3.18, we observe that the Forcing Theorem allows us to interpret $\Diamond\varphi$ as "φ holds in some forcing extension". For the same to hold in this context, we need a suitable version of the forcing theorem for the forcing notions we are currently considering, that is, forcings over second-order formulas. This is given by the next proposition.

From now on, we interpret $\mathcal{L}_{\Box,\blacksquare}$-formulas over some model $M \in H(\omega_1)$.

Proposition 3.4.6 (Forcing Theorem for forcings over second-order formulas)**.** *Let $\mathbb{P} \in M \in H(\omega_1)$ be a poset, $\vec{A} = A_0, \dots, A_k$ be a finite sequence of relations on M, and let X be a countable elementary substructure of $H(\omega_1)$ containing the transitive closure of $\{M, A_0, \dots, A_k\}$ and H be the transitive collapse of X. Let φ be a formula in $\mathcal{L}^2_\in$ and let $p \in G \subseteq \mathbb{P}$ where G is H-generic. Then,*

$$M[G] \vDash \varphi(\vec{a}, \vec{A}) \textit{ if and only if } \exists p \in G(p \Vdash^M \varphi(\vec{a}, \vec{A})),$$

where $\vec{a}$ and $\vec{A}$ are a sequence of first-order names and a sequence of second-order names, respectively.

Proof. First, we observe the following. Let $\varphi(\vec{a}, \vec{A})$ be an arbitrary formula in $\mathcal{L}^2_\in$ and let D be the set $\{q \leq p : q \text{ decides } \varphi(\vec{a}, \vec{A})\}$, that is,

$$D = \{q \leq p : q \Vdash^M \varphi(\vec{a}, \vec{A}) \vee q \Vdash^M \neg\varphi(\vec{a}, \vec{A})\}.$$

Clearly, D is dense in $\mathbb{P}$. Moreover, D is second-order definable over M. Therefore, D is first-order definable over $H(\omega_1)$. The structure $H(\omega_1)$ is a model of ZFC^-, so in particular it is a model of Separation, so $D \in H(\omega_1)$. Since H is the transitive collapse of a countable elementary substructure of $H(\omega_1)$ and D and $\vec{A}$ are contained in H, we have $D \in H$. Since G is H-generic, $D \cap G$ is non-empty.

For our proof, we proceed by induction on the complexity of formulas. The only case we have to pay attention to is that in which φ is of the form $\exists^1 X \varphi(\vec{a}, \vec{A}, X)$. Assume first that $\exists p \in G(p \Vdash^M \exists^1 X \varphi(\vec{a}, \vec{A}, X))$. Then,

$$\exists p \in G \forall q \leq p \exists r \leq q \exists^1 \dot{X}(r \Vdash^M \varphi(\dot{X})),$$

which implies that

$$\exists r \in G \exists^1 \dot{X}(r \Vdash^M \varphi(\dot{X})).$$

By induction hypothesis, we get $M[G] \vDash \varphi(\dot{X})$. Conversely, assume that $\forall p \in G(p \nVdash^M \exists^1 X \varphi(X))$. Since $D \cap G \neq \emptyset$, it follows that

$$\exists p \in G(p \Vdash^M \forall^1 X \neg\varphi(X)).$$

Then,

$$\exists p \in G \forall^1 \dot{X}(p \Vdash^M \neg\varphi(\dot{X})).$$

By induction hypothesis, we have

$$\forall^1 \dot{X}(M[G] \vDash \varphi(\dot{X}),$$

hence

$$M[G] \vDash \forall^1 X \neg\varphi(X),$$

so we are done. □

We prove now that FIM is sound with respect to our interpretation. That the axioms of S4.2 are valid for both $\Box$ and $\blacksquare$ is clear. It is easy to see that $\mathsf{Inv}_{\mathsf{FIM}}$ is valid, too: every model where φ holds is a ground, hence an inner model, of all its forcing extensions, so it also satisfies $\Box\blacklozenge\varphi$. Note that the validity of $\Box\blacklozenge\varphi \to \varphi$ is no longer true. Indeed, since L is an inner model of every model, every model satisfies $\Box\blacklozenge\mathsf{CH}$. However, no model of $\neg\mathsf{CH}$, satisfies the instance of $\Box\blacklozenge\varphi \to \varphi$ where $\varphi = \mathsf{CH}$.

Proposition 3.4.7. *Axiom $\mathsf{Inv}_{\mathsf{FIM}}$ is sound with respect to the forcing and inner models interpretation. Axiom $\mathsf{Inv}_{\mathsf{FG}}$, the invertibility axiom for the modal logic of forcing and grounds, is not sound for this interpretation.*

Axiom Conf is also valid:

Proposition 3.4.8. *The confluence axiom $\blacklozenge\Diamond\varphi \to \Diamond\blacklozenge\varphi$ is sound with respect to the set-theoretic interpretation.*

Proof. Let M be a model such that there is a submodel $N \subseteq M$ with the same ordinals and $N \models \exists \mathbb{P} \exists p \in \mathbb{P}(p \Vdash_{\mathbb{P}} \Phi)$ for some $\Phi \in \mathcal{L}^2_{\in}$. Let $\check{N} = \{(\check{x}, 1_{\mathbb{P}}) : x \in N\}$ be the canonical name for N in $M^{\mathbb{P}}$. Then we define $\dot{W}$, the canonical $M^{\mathbb{P}}$-name for $N[G]$, by letting

$$\dot{W} = \{(\dot{x}^G, 1_{\mathbb{P}}) : \dot{x} \in \check{N}^{\mathbb{P}}\}.$$

Then, for some p, $p \Vdash^M_{\mathbb{P}} \check{N}[\dot{G}] \models \Phi$. For suppose otherwise and assume that for any p there is $q \leqslant p$ such that $q \Vdash^M_{\mathbb{P}} \check{N}[\dot{G}] \models \neg\Phi$ but then $N[\dot{G}] \models \neg\Phi$. So it follows follows that

$$p \Vdash^M_{\mathbb{P}} \exists^1 \dot{W}(\dot{W} \models \Phi).$$

It is left to verify that $\mathrm{Ord}^{M[G]} = \mathrm{Ord}^{\dot{W}}$. We show that for each $\dot{x} \in \mathrm{Ord}^{M[G]}$, we have $1_{\mathbb{P}} \Vdash^M \dot{x} = \mathrm{Ord}^{\dot{W}_{\dot{x}}}$. This can be verified by the induction on von Neumann rank using the fact that $1_{\mathbb{P}} \Vdash^M \check{N} \cap \mathrm{Ord} = \mathrm{Ord}$. □

Thus, soundness follows:

Theorem 3.4.9. FIM *is sound with respect to the given set-theoretic interpretation.*

It is worth mentioning that in the context of our intended interpretation we may want to express the existence of a minimal inner model. The assertion

$$\Diamond\blacklozenge\blacksquare\varphi \to \blacklozenge\blacksquare\varphi$$

does the job, but it is already a theorem of FIM.

Remark 3.4.10. $\vdash_{\mathsf{FIM}} \Diamond\blacklozenge\blacksquare\varphi \to \blacklozenge\blacksquare\varphi$.

As a last observation, we note that $\Diamond\blacklozenge\varphi \to \blacklozenge\Diamond\varphi$, the inverse of Conf, is not valid in our interpretation:

Proposition 3.4.11. *There is a model W of set theory and an interpretation v as in Definition 3.4.5 such that $W \not\models v(\Diamond\blacklozenge\varphi \to \blacklozenge\Diamond\varphi)$.*

Proof. Let G be L-generic for the forcing adding a Cohen subset of every successor κ with Easton support so that $L[G] \models$ "every successor κ has a Cohen subset". We show that $L[G]$ is the W in the statement. Let $H \subseteq \mathsf{Coll}(\omega_1, \omega)$ be mutually generic with G over L. In $L[G][H](= L[H][G])$, consider the inner model M of the form $L[\langle c_{\lambda^{+n}} : \lambda \in \mathrm{Sing} \wedge n \in H \rangle]$, where c_γ denotes the Cohen subset of γ given by G. M is a class generic extension of $L[H]$. In particular, it is a model of ZFC. Let now φ be the sentence saying "there is a well-ordering X of ω_1^L such that κ has a Cohen subset if and only if $\kappa = \lambda^{+n}$ for some $\lambda \in \mathrm{Sing}$ and $n \in X$". Since $M \models \varphi$, we get that $L[G] \models \Diamond\blacklozenge\varphi$. Let N be an arbitrary inner model of $L[G]$ with $\omega_1^L = \omega_1^N$, so φ fails in N for all intervals $(\lambda, \lambda^{+\omega})$, since set-forcing can only add a bounded amount of Cohen subsets. Then, $N \not\models \Diamond\varphi$, hence $L[G] \not\models \blacklozenge\Diamond\varphi$, that is, $L[G] \not\models \Diamond\blacklozenge\varphi \to \blacklozenge\Diamond\varphi$. □

3.4.3 Final remarks

We still do not have completness results for FIM. Here, we discuss some observations done when trying to get to Kripke completeness. We begin by proving a few canonicity results. The following shows that the axiom $\mathsf{Inv}_{\mathsf{FIM}}$ is canonical with respect to the property $\leq^{-1} \subseteq \blacktriangleleft$:

Lemma 3.4.12. *Axiom $\mathsf{Inv}_{\mathsf{FIM}}$ is valid in a frame $\mathcal{F} = (W, \leq, \blacktriangleleft)$ if and only if $\leq^{-1} \subseteq \blacktriangleleft$.*

Proof. For the right to left direction, let $u, v \in W$ be such that $u \leq v$ and $u \Vdash \varphi$. Since $v \blacktriangleleft u$, we have $v \Vdash \blacklozenge\varphi$, hence $u \Vdash \Box\blacklozenge\varphi$. We prove the left to right direction by contraposition. Assume $\leq^{-1} \not\subseteq \blacktriangleleft$ and let $u, v \in W$ such that $(u, v) \in \leq$ but $(v, u) \notin \blacktriangleleft$. Let V be a valuation in $\mathcal{F}$ such that $V(\varphi) = \{u\}$. Then $(\mathcal{F}, V), u \not\Vdash \varphi \to \Box\blacklozenge\varphi$. □

Definition 3.4.13. We say that $\mathcal{F} := (W, \leq, \blacktriangleleft)$ is *confluent* if whenever $u, v, w \in W$ are such that $u \blacktriangleleft v \leq w$, then there exists $w' \in W$ such that $u \leq w' \blacktriangleleft w$.

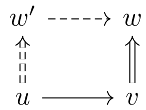

Figure 3.2: Confluence.

Axiom Conf is canonical for confluence:

Lemma 3.4.14. *Axiom* Conf *is valid in a frame if and only if it is confluent.*

Proof. For the right to left direction, assume that $\mathcal{F}$ is confluent and let $u \in W$ such that $u \Vdash \blacklozenge\Diamond\varphi$. Let $v, w \in W$ such that $u \mathrel{\underline{\blacktriangleleft}} v \leq w$ with $v \Vdash \Diamond\varphi$ and $w \Vdash \varphi$. By confluence, there is $w' \in W$ such that $u \leq w' \mathrel{\underline{\blacktriangleleft}} w$. Since $w \Vdash \varphi$, then $w' \Vdash \blacklozenge\varphi$, and since $u \leq w'$, $u \Vdash \Diamond\blacklozenge\varphi$. For the other direction, let $\mathcal{F} = (W, \leq, \mathrel{\underline{\blacktriangleleft}})$ be a non-confluent frame and let $u, v, w \in W$ with $u \mathrel{\underline{\blacktriangleleft}} v \leq w$ such that there is no $w'' \in W$ satisfying $u \leq w'' \mathrel{\underline{\blacktriangleleft}} v$. Let now V be a valuation on $\mathcal{F}$ such that $V(\varphi) = \{w\}$. Then, $(\mathcal{F}, V), u \Vdash \blacklozenge\Diamond\varphi$ but $(\mathcal{F}, V), u \nVdash \Diamond\blacklozenge\varphi$, for otherwise there would exist a world $w' \in W$ such that $u \leq w' \mathrel{\underline{\blacktriangleleft}} v$. □

Remark 3.4.15. An analogous argument to the one above shows that $\Diamond\blacklozenge\varphi \to \blacklozenge\Diamond\varphi$ is valid in a frame $(W, \leq, \mathrel{\underline{\blacktriangleleft}})$ if and only if for every $u, v, w \in W$ such that $u \leq v \mathrel{\underline{\blacktriangleleft}} w$ there exists $w' \in W$ such that $u \mathrel{\underline{\blacktriangleleft}} w' \leq w$. However, as shown in Proposition 3.4.11, axiom $\Diamond\blacklozenge\varphi \to \blacklozenge\Diamond\varphi$ is not valid in our intended interpretation.

Now, towards a Kripke completeness result, we could, for instance, proceed as in the previous section:

Let $\mathcal{M} = (W, \leq, \mathrel{\underline{\blacktriangleleft}})$ be the canonical model for FIM. Let φ be a formula consistent with FIM and let $w \in W$ be such that $\varphi \in w$. By well-known canonicity arguments, the canonical model $(M, \leq, \mathrel{\underline{\blacktriangleleft}})$ of FIM is a reflexive, transitive and weak-directed frame. By Lemma 3.4.12 and Lemma 3.4.14, it satisfies $\leq^{-1} \subseteq \mathrel{\underline{\blacktriangleleft}}$ and it is confluent. Let $(\mathcal{M}[w], \leq_{\restriction \mathcal{M}[w]}, \mathrel{\underline{\blacktriangleleft}}_{\restriction \mathcal{M}[w]})$ be the submodel of $\mathcal{M}$ generated by w. It is clear that $\mathcal{M}[w]$ is also reflexive, transitive, weak-direced, confluent and satisfies $\leq^{-1}_{\restriction \mathcal{M}[w]} \subseteq \mathrel{\underline{\blacktriangleleft}}_{\restriction \mathcal{M}[w]}$. Moreover, it is connected. Axiom $\mathsf{Inv}_{\mathsf{FIM}}$ is a fragment of the axiom $\mathsf{Inv}_{\mathsf{GF}}$ from the previous section, and it leaves free room for the existence of pairs of nodes u and v such that $u \mathrel{\underline{\blacktriangleleft}} v$ but $v \nleq v$. That is, in general, FIM-frames no longer present bidirectionality. If we intend follow an approach similar to the one in the previous section, we need to use different filtration relations. Because of the presence of the axiom Top($\blacksquare$) in FIM, we also choose a different filtration set, guided by the reasoning in the proof of Lemma 3.4.3. Let Φ then be as in the mentioned proof, i.e.,

$$\Phi = \{\psi : \psi \text{ is a subformula of } \varphi\} \cup \{\blacklozenge((q \leftrightarrow \blacksquare q) \wedge (\neg q \leftrightarrow \blacksquare\neg q) : q \in \mathsf{var}(\varphi)\},$$

and let $\mathcal{M}_\Phi$ denote the filtration $(W/\Phi, \leq^*_t, \mathrel{\underline{\blacktriangleleft}}_t)$ of the generated submodel $\mathcal{M}[w]$, where $\leq^*_t$ is as defined in the previous section and $\mathrel{\underline{\blacktriangleleft}}_t$ is the transitive filtration relation, that is, $[u]_\Phi \leq^*_t [v]_\Phi$ if and only if

$$[(\Diamond\varphi \in \Phi \wedge v \Vdash \varphi \vee \Diamond\varphi) \to u \Vdash \Diamond\varphi] \ \wedge \ [(\blacklozenge\varphi \in \Phi \wedge u \Vdash \varphi \vee \blacklozenge\varphi) \to v \Vdash \blacklozenge\varphi],$$

and $[u]_\Phi \mathrel{\underline{\blacktriangleleft}}_t [v]_\Phi$ if and only if

$$(\blacklozenge\varphi \in \Phi \wedge M, v \Vdash \varphi \vee \blacklozenge\varphi) \to M, u \Vdash \blacklozenge\varphi.$$

The filtrations $\leq^*_t$ and $\mathrel{\underline{\blacktriangleleft}}_t$ preserve reflexivity and transitivity. Furthermore, by construction, the filtration still satisfies semi-invertibility:

Lemma 3.4.16. $(\leq^*_t)^{-1} \subseteq \mathrel{\underline{\blacktriangleleft}}_t$.

However, we do not know whether these filtrations preserve confluence or even weak directedness. Other approaches, such as a variation of the construction described in [AP24, pp. 10–13] or an unravelling strategy similar to the one discussed in the previous section, have also proven unsuccessful at this point.

Bibliography

[Ago22] Claudio Agostini. *Generalized Descriptive Set Theory at uncountable cardinals & Actions of monoids in combinatorics.* PhD thesis, Università degli studi di Torino, 2022.

[AMR] Claudio Agostini and Luca Motto Ros. Trees and topology. Manuscript in preparation.

[AMR22] Alessandro Andretta and Luca Motto Ros. Souslin quasi-orders and bi-embeddability of uncountable structures. *Memoirs of the American Mathematical Society*, 277(1365):vii+189, 2022.

[AMRS23] Claudio Agostini, Luca Motto Ros, and Philipp Schlicht. Generalized Polish spaces at regular uncountable cardinals. *Journal of the London Mathematical Society. Second Series*, 108(5):1886–1929, 2023.

[AP24] Juan Pablo Aguilera and Fedor Pakhomov. The logic of correct models. *arXiv preprint arXiv:2402.15382*, 2024.

[BDM25] Fernando Barrera, Vincenzo Dimonte, and Sandra Müller. The λ-PSP at λ-coanalytic sets. *arXiv preprint arXiv:2504.15675*, 2025.

[BDRV01] Patrick Blackburn, Maarten De Rijke, and Yde Venema. *Modal logic: graph. Darst*, volume 53. Cambridge University Press, 2001.

[BNL25] Omer Ben-Neria and Philipp Lücke. On Σ_1-definable closed unbounded sets. *Canadian Journal of Mathematics*, pages 1–33, 2025.

[CS19] Shani Cohen and Saharon Shelah. Generalizing random real forcing for inaccessible cardinals. *Israel Journal of Mathematics*, 234(2):547–580, 2019.

[Dev17] Keith J Devlin. *Constructibility*, volume 6. Cambridge University Press, 2017.

[DIL23] Vincenzo Dimonte, Martina Iannella, and Philipp Lücke. Descriptive properties of I2-embeddings. *The Journal of Symbolic Logic*, pages 1–26, 2023.

[DMRon] Vincenzo Dimonte and Luca Motto Ros. Generalised descriptive set theory at singular cardinals of countable cofinality. In preparation.

[DPT24] Vincenzo Dimonte, Alejandro Poveda, and Sebastiano Thei. The baire and perfect set properties at singulars cardinals. *arXiv preprint arXiv:2408.05973*, 2024.

[DW96] Harold G Dales and W Hugh Woodin. *Super-real fields: totally ordered fields with additional structure*, volume 14. Oxford University Press, 1996.

[Eng89] Ryszard Engelking. *General topology*, volume 6 of *Sigma Series in Pure Mathematics.* Heldermann Verlag, Berlin, second edition, 1989. Translated from the Polish by the author.

[FFS17] Sy-David Friedman, Sakaé Fuchino, and Hiroshi Sakai. On the set-generic multiverse. In *The Hyperuniverse Project and Maximality*, pages 109–124. Springer, 2017.

[FHK14] Sy-David Friedman, Tapani Hyttinen, and Vadim Kulikov. Generalized descriptive set theory and classification theory. *Mem. Amer. Math. Soc.*, 230(1081):vi+80, 2014.

[FL17] Sy-David Friedman and Giorgio Laguzzi. A null ideal for inaccessibles. *Archive for Mathematical Logic*, 56(5):691–697, 2017.

[FM08] Antongiulio Fornasiero and Marcello Mamino. Arithmetic of Dedekind cuts of ordered abelian groups. *Annals of Pure and Applied Logic*, 156(2-3):210–244, 2008.

[Fuc63] L. Fuchs. *Partially ordered algebraic systems.* Pergamon Press, Oxford; Addison-Wesley Publishing Co., Inc., Reading, Mass.-Palo Alto, Calif.-London, 1963.

[Gal19] Lorenzo Galeotti. *The theory of generalised real numbers and other topics in logic.* PhD thesis, Universität Hamburg, 2019.

[HL08] Joel Hamkins and Benedikt Löwe. The modal logic of forcing. *Transactions of the American Mathematical Society*, 360(4):1793–1817, 2008.

[HL13] Joel David Hamkins and Benedikt Löwe. Moving up and down in the generic multiverse. In *Logic and Its Applications: 5th Indian Conference, ICLA 2013, Chennai, India, January 10-12, 2013. Proceedings 5*, pages 139–147. Springer, 2013.

[HLL15] Joel David Hamkins, George Leibman, and Benedikt Löwe. Structural connections between a forcing class and its modal logic. *Israel Journal of Mathematics*, 207(2):617–651, 2015.

[Ianed] Martina Iannella. PhD Thesis. From real-life to very strong axioms. Classification problems in Descriptive Set Theory & regularity properties in Generalized Descriptive Set Theory. 2023. Unpublished.

[IL16] Tanmay Inamdar and Benedikt Löwe. The modal logic of inner models. *The Journal of Symbolic Logic*, 81(1):225–236, 2016.

[Jec03] Thomas Jech. *Set Theory*, volume 14. Springer, 2003.

[JS13] Ronald Jensen and John Steel. K without the measurable. *The Journal of Symbolic Logic*, 78(3):708–734, 2013.

[Kan08] Akihiro Kanamori. *The Higher Infinite: large cardinals in set theory from their beginnings.* Springer Science & Business Media, 2008.

[Kec95] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1995.

[Koe88] Peter Koepke. Some applications of short core models. *Annals of pure and applied logic*, 37(2):179–204, 1988.

[Lav07] Richard Laver. Certain very large cardinals are not created in small forcing extensions. *Annals of Pure and Applied Logic*, 149(1-3):1–6, 2007.

[Mac37] Holbrook Mann MacNeille. Partially ordered sets. *Transactions of the American Mathematical Society*, 42(3):416–460, 1937.

[Mea21] Toby Meadows. Two arguments against the generic multiverse. *The Review of Symbolic Logic*, 14(2):347–379, 2021.

[Mit09] William J Mitchell. Beginning Inner Model Theory. In *Handbook of set theory*, pages 1449–1495. Springer, 2009.

[MV93] Alan Mekler and Jouko Väänänen. Trees and Π^1_1-subsets of ${}^{\omega_1}\omega_1$. *J. Symbolic Logic*, 58(3):1052–1070, 1993.

[Nyi99] Peter J. Nyikos. On some non-Archimedean spaces of Alexandroff and Urysohn. *Topology Appl.*, 91(1):1–23, 1999.

[Rei79] Hans-Christian Reichel. Towards a unified theory of semimetric and metric spaces. In *Topological structures, II (Proc. Sympos. Topology and Geom., Amsterdam, 1978), Part 2*, volume 116 of *Math. Centre Tracts*, pages 209–241. Math. Centrum, Amsterdam, 1979.

[Rit10] Colin Jakob Rittberg. Diploma thesis on the modal logic of forcing. 2010.

[Rud87] Walter Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, 3 edition, 1987.

[Sch14] Ralf Schindler. *Set theory: exploring independence and truth*. Springer, 2014.

[Ste95] John Steel. Projectively well-ordered inner models. *Annals of Pure and Applied Logic*, 74(1):77–104, 1995.

[Ste16] John Steel. An Introduction to Iterated Ultrapowers. In *Forcing, iterated ultrapowers, and Turing degrees*, pages 123–174. World Scientific, 2016.

[Sto62] Arthur Harold Stone. Non-separable borel sets. *General Topology and its Relations to Modern Analysis and Algebra*, pages 341–342, 1962.

[Usu17] Toshimichi Usuba. The downward directed grounds hypothesis and very large cardinals. *Journal of Mathematical Logic*, 17(02):1750009, 2017.

[Vau74] Robert Vaught. Invariant sets in topology and logic. *Fund. Math*, 82(269-294):75, 1974.

[Won23] Ned J. H. Wontner. *Views from a Peak: Generalisations and Descriptive Set Theory*. PhD thesis, Institute for Logic, Language and Computation, Universiteit van Amsterdam, 2023.