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# Input/output-style approach to standardized traditional amortization plans

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**Abstract**
We apply an Input/Output approach to the analysis of the so-called traditional standardized amortization plans (STAP). The approach is based on a clear distinction between Input/trigger and Output of a STAP. The Output is a set of four vectors obtained by applying a set of transformation matrices to an appropriate Input/trigger vector. The main result of the paper is that, by this approach, any STAP reveals a dual nature through the introduction of the twin trigger theory.

**Keywords** Amortization plan · Input–output · Transformation matrices · Twin triggers

## 1 Motivation of the study

In the history of mathematical economics Input–Output analysis (henceforth IO) played a significant role. We recall here that at least three among all time top fundamental contributions to the economic sciences have been based explicitly or implicitly on IO. We refer to Quesnay (1766), considered the forerunner in this field, who used IO as the base to explain and clarify the central role of the primary sector in generating the surplus in an economic system; Walras (1874), who applied IO as the base of his pure theory of general equilibrium in markets with perfect competition; Leontief (1941), Nobel Prize in Economics 1973, who gave a formal definition of IO analysis and found it as the best way to analyse and understand the connection between different sectors of an economic system. Obviously, countless applications of the IO model have been gradually developed in various fields, also beyond economics, such as industrial, medical, and environmental. [for a recent exhaustive bibliography, see Giorgi (2022)].

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Here, still in the economic area, we propose an IO model-based technique, exploiting transformation matrices, with a much more modest goal: to shed some light on the issue of standardized traditional amortization plans (henceforth STAPs). This theme, to the best of our knowledge, has never been treated with this approach so far.

In greater detail, an amortization plan is described by a numerical table which is the result (Output) of a set of rules involving a series of parameters (Inputs). The idea is therefore to apply a matrix approach that guarantees, given the Input, a coherent Output from a contractual-financial point of view. This approach aims to provide a definitively clear reading of the structure of the STAP.

## 2 Introduction

In theory and in practice there is no univocal definition of amortization plan; from time to time and depending on the context, the term can take on a different meaning. In particular, the Plan can be understood as[1]: the set of rules which allow the elements of the plan itself to be specified starting from the contractual conditions; the summary table of the elements themselves; the set of three information: contractual conditions list, rules for constructing the plan starting from the conditions, final result or summary table of the plan items.

In any case, a Plan is the result of a negotiated agreement between two counter parties, the lender and the borrower. In general, the principle of contractual freedom should be accepted; however, such a freedom is constrained by the need of protecting the weak contractor. In this regulatory framework, Financial Mathematics must be able to translate the current legislation rules into a rigorous set of algebraic rules (to a set of contractually defined parameters) which definitively produce plans capable of achieving efficiency and fairness objectives in a frame of transparent clarity. However, in the absence of a clear methodological approach, the connection between parameters and rules may produce misunderstandings or interpretative difficulties. This, in turn, may generate uncertainty in the judgments on the adherence of a Plan to the contractual will of the parties and to the existing legislation.[2]

This paper intends to analyse the question in an alternative way following an Input/Output (IO) approach: Input, a well-defined set of contractual parameters which, through the rules synthesized by the so-called transformation matrices, provide the Output, that is the summary table of the items of the Plan. In this setting, the Output turns out to be a set of four $N$-dimensional column vectors; each of these output vectors is obtained by applying a specific transformation matrix to a given trigger; each transformation matrix embeds, in turn, rules and contractual parameters. It seems natural to define these matrices as the IO matrices[3] of the Plan. Two choices of Input/trigger

[1] As regards general definition and rules of mortgage amortization schemes from a mathematical point of view, we quote here only some classical books among many others: Bortot et al. (1993), Cacciafesta (2001), Daboni and De Ferra (1993) and Moriconi (1994).

[2] On this point, see for instance Faro (2014), Fersini and Olivieri (2015), Mari and Aretusi (2018), Medina (2011), Pressacco and Ziani (2020) and Saccardo et al. (2009).

[3] The literature in the field of linear algebra is huge. See for instance, without claiming to be exhaustive, De Finetti (1944), Giorgi (1991) and Lang (1987).

are possible: either a vector of principal repayments (**C**) or a vector of instalments (**R**). Whatever the vector type chosen, it must also satisfy a feasibility condition, specific to the type of trigger. We label the summary table, Output of this approach, respectively as STAPC (if the trigger is the vector **C**) and STAPR (trigger **R**).

Applying our approach, a first result is that the vector output **R** in the STAPC satisfies a specific condition which is natural to consider as the feasibility condition of the trigger **R** of a STAPR. Symmetrically, it is verified that the vector output **C** in the STAPR satisfies the feasibility condition of the trigger **C** of a STAPC.

Among the transformation matrices of a STAPC, a "key pivot" role is played by the matrix $F$, whose output is the instalments sequence; conversely, in a STAPR the pivot role is played by the matrix $\Phi$, whose output is the reimbursement sequence. The choice $\Phi = F^{-1}$ opens the way to a definition of "twin sequences", i.e. sequences **C** and **R** whose trigger/output role is exchangeable and of "twin STAPs", i.e. a couple of STAPC and STAPR whose triggers are twin sequences, which implies that the twin STAPs are perfectly coincident.

The plan of the paper is as follows: Sect. 3 presents a formalized definition of the STAP, Standardized Traditional Amortization Plan with a concise description of the rules that guide it and of the contractual parameters. These parameters are needed to move from a theoretical definition of a plan to a concrete version that specifies unequivocally times and amounts of the counterparties commitments and, in particular, those of the borrower; this allows the latter both to clearly distinguish between capital and interest payments and, also, to be able to verify how these obligations are determined according to the general rules. In this framework, the concept of transformation matrices and of plan triggering is introduced as well as the distinction between STAPs with principal quota trigger and those with instalment trigger and, also, the distinction between the respective trigger feasibility conditions.

Section 4 discusses the plan whose trigger is a sequence of principal reimbursements (STAPC) and the derivation, from the rules and from the trigger condition, of the four transformation matrices that characterize these plans. In particular, it is proved that the output sequence of the instalments of any STAPC satisfies a condition of financial equivalence which will be assumed as the admissibility condition of a STAPR.

Section 5 discusses the instalment trigger plan (STAPR) and the derivation, from the same rules and from the new trigger condition, of the four new transformation matrices. It is proved that the output sequence of the principal quotas of a STAPR constructed using such matrices, in addition to satisfying the closure condition of the STAPR, in turn satisfies the admissibility condition of the sequence of principal quotas of a STAPC.

Section 6 introduces and illustrates a theory of twin STAPs, one STAPC and one STAPR. It is based on the concept of twin pairs of principal and instalment sequences. The coincidence of the twin STAPs is proved when the respective triggers correspond to two twin sequences.

In Sect. 7, the twin STAP theory is applied to the particular case of the so-called French STAP with constant instalments. In the classic approach, it is obtained as STAPR with triggering a sequence of constant instalments whose amount is the only one that (given the STAP rules and the set of contractual parameters) satisfies the feasibility condition of the instalment trigger. It is verified, by our IO approach, that the

Plan obtained in this way coincides with the particular STAPC with triggering principal amounts increasing in geometric progression (coherent with the contractual effective periodic interest rate) which in turn satisfies the specific feasibility condition. The result, rather surprising at first glance, is a simple consequence of the general theory of twin STAPs. It is superfluous to underline the importance of this result for practical purposes. Although less intuitive, the trigger condition of principal quotas in geometric progression of proper common ratio (consistent with the effective periodic rate) of the STAP, together with their feasibility condition, produces the Output sequence of constant instalments that is twin of the Input sequence of principal quotas.

Numerical examples of French STAPs, and other STAPs with non-constant instalments generated by principal quotas not in geometric progression or in geometric progression of a non proper common ratio follow in Sect. 8. The conclusions then are given in Sect. 9.

## 3 Input and output of the STAP

The Output of a STAP is described by four $N$-dimensional vectors which represent the evolution of the relevant quantities of the plan at evenly spaced instants of time, i.e. principal reimbursements $\mathbf{C}$, interest quotas $\mathbf{I}$, instalments $\mathbf{R}$ and outstanding debt $\mathbf{D}$.[4]

As far as Inputs are concerned, the situation is more complex. It is worth distinguishing a first set of four contractual parameters $\mathbf{x} = (D_0, N, k, j)$ and a sequence (as the result of a negotiation agreement) that serves as an Input/trigger for the plan itself.

In the first set there are:

- $D_0$, principal amount at time $t_0 = 0$;
- $N$, total number of instalments;
- $k$, number of instalments in a year or annual frequency of the borrower's payments evenly spaced, with $1/k$ measure in years of the interval between two consecutive payments (and of the interval between the lender's payment and the first instalment);
- $j$, constant nominal interest rate stated in the loan agreement.

The following relations hold:

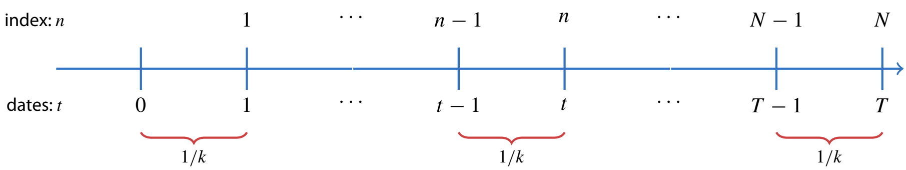
- $T = N \cdot (1/k)$, duration of the loan (measured in years) from the initial payment by the lender (time 0) to the last payment by the borrower (time $T$);
- $t_n = n \cdot (1/k)$, $n = 1, \ldots, N$ date of the $n$-th instalment of the borrower; $t_0 = 0$ date of the lender's obligation;
- $i_k = j \cdot (1/k)$, periodic effective interest rate, constant for the entire term of the loan, obtained by multiplying by $(1/k)$, or by dividing the interest rate of the Plan by $k$. In the jargon of financial mathematics, the rate $j$ of the plan is considered as a nominal rate convertible $k$ times a year.

[4] Sometimes a fifth vector, the extinguished loan principal, also appears in the STAPs; this is superfluous information (at least in our framework) for our discussion, which justifies its exclusion.

- $u_k = (1 + i_k)$, single period capitalization factor;
- $v_k = \frac{1}{(1+i_k)}$, associate single period discount factor;
- $u(0, t_n) = (1 + i_k)^n$, capitalization factor[5] of the period $(0, t_n)$ under compound interest capitalization regime (briefly, compound regime);
- $v(0, t_n) = \frac{1}{(1+i_k)^n}$, discount factor of the period $(0, t_n)$ under compound regime.

Picture 3 recaps symbols and relations.

$N$ periods of constant duration $1/k$ in the interval $[0, T]$




Let us remind that a STAP is governed by the following rules:

$$D_n = D_{n-1} - C_n \tag{1}$$
$$I_n = i_k \cdot D_{n-1} \tag{2}$$
$$R_n = C_n + I_n \tag{3}$$
$$D_N = 0 \tag{4}$$

Equations (1)–(4) are relations between the Output items of the Plan, and are valid for each $n = 1, \ldots, N$. In particular, Eq. (3) breaks down each borrower's instalment, $R_n$ into two distinct components: $C_n$, called principal repayment, has, as clarified by (1), the task of progressively decreasing (at least if positive) the principal amount of the loan; and $I_n$, called interest quota, is the compensation that the borrower owes to the lender for the availability, in the $n$-th period of duration $t_n - t_{n-1} = 1/k$, of the portion of the loan amount not yet repaid. Equation (2) shows that to calculate this compensation, it is sufficient to multiply the amount of the outstanding balance at the beginning of the period by the effective periodic interest rate $i_k$.[6]

Condition (4) is a closure condition of the plan. It requires that, at the final $T$ due date (with the payment of the last instalment), the debt is exactly extinguished (formally $D_N = 0$).

As far as the Input/trigger of the STAP is concerned, STAP with principal quotas trigger (STAPC) and STAP with instalments trigger (STAPR) are distinguished. In a STAPC, a sequence of principal repayments (a vector **C**) is added to the input **x**; in a STAPR a sequence of instalments (a vector **R**).

[5] For $k > 1$, i.e. for infra-annual periods, it is needed to distinguish between subscript $k$, related to the duration $1/k$ and the subscript $n$, which is the counter of the borrower's instalments.

[6] According to this rule, it is implicitly accepted that the interest accrued in each period is payable periodically (at the end of each period). For details on this point, see Pressacco and Ziani (2020) and Pressacco et al. (2022).

## 4 STAPC and its transform matrices

It was anticipated at the conclusion of the previous section that a STAPC is triggered by a **C** sequence of principal repayments. This sequence must satisfy a feasibility condition. It ensures that the STAPC generated by applying rules (1)–(3) satisfies the closure condition (4). More precisely, the feasibility condition guarantees that the rule (3) and the closure condition (4) are not contradictory, i.e. that there exist inputs $(\mathbf{x}, \mathbf{C})$ which, applying the three fundamental rules, generate Plans that satisfy the closure condition (4). It is shown that the feasibility condition of the **C** trigger of a STAPC is:

$$D_0 = \sum_{n=1}^{N} C_n \tag{5}$$

It imposes the equality between the lender's payment (left-hand-side) and the pure algebraic sum of the principal reimbursements of the borrower (right-hand-side).[7]

We now define the four transformation matrices of a STAPC. Each of them transforms an admissible trigger (vector **C**) into one of the four output vectors of the STAPC.

- Transformation matrix of the trigger **C** in the output **C**: it is obviously the identity matrix

$$\mathbf{C} = I\mathbf{C} \tag{6}$$

It has elements $e_{r,s}$ all equal to 1 on the main diagonal ($r = s$) and all zero for $r \neq s$:

$$e_{r,s} = 0 \quad \forall r \neq s; \quad e_{r,s} = 1 \quad \forall r = s$$
$$I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \tag{7}$$

- Transformation matrix of the **C** trigger into the outstanding balance output **D**: it is the $L$ matrix such that

$$\mathbf{D} = L\mathbf{C} \tag{8}$$

[7] In particular, the condition of non-negativity of the principal quotas and of strict positivity of the last quota is useful in order to adhere to current Italian legislation [see Pressacco and Ziani (2020) and Pressacco et al. (2022)].

$L$ has elements all equal to 1 above the main diagonal and all others equal to 0. Formally:

$$l_{r,s} = 1 \quad \forall r < s; \quad l_{r,s} = 0 \quad \forall r \geq s$$
$$L = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \tag{9}$$

- Transformation matrix of the $\mathbf{C}$ trigger in the interest quotas output $\mathbf{I}$: it is the $G$ matrix such that

$$\mathbf{I} = G\mathbf{C} \tag{10}$$

$G$ has elements all equal to $i_k$ on or above the main diagonal and all others equal to 0. Formally:

$$g_{r,s} = 0 \quad \forall r > s; \quad g_{r,s} = i_k \quad \forall r \leq s$$
$$G = \begin{bmatrix} i_k & i_k & i_k & \dots & i_k \\ 0 & i_k & i_k & \dots & i_k \\ 0 & 0 & i_k & \dots & i_k \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & i_k \end{bmatrix} \tag{11}$$

- Transformation matrix of the $\mathbf{C}$ trigger in the instalments output $\mathbf{R}$: it is the $F$ matrix such that

$$\mathbf{R} = F\mathbf{C} \tag{12}$$

It must be $F = I + G$. Consequently $F$ has elements all equal to $u_k$ on the main diagonal, all equal to $i_k$ above the main diagonal, all equal to 0 below it. Formally:

$$f_{r,s} = 0 \quad \forall r > s; \quad f_{r,s} = u_k = 1 + i_k \quad \forall r = s; \quad f_{r,s} = i_k \ \forall r < s$$
$$F = \begin{bmatrix} u_k & i_k & i_k & \dots & i_k \\ 0 & u_k & i_k & \dots & i_k \\ 0 & 0 & u_k & \dots & i_k \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & u_k \end{bmatrix} \tag{13}$$

***Proof*** Obvious for $I$. □

***Proof*** For the $L$: let us start from (8) it gives $D_r$, $r$-th component of the output column vector $\mathbf{D}$ as the product of the $r$-th row of the $L$ matrix by the trigger column vector

**C**. It therefore results:

$$D_r = (0C_1 + \cdots + 0C_r + 1C_{r+1} + \cdots + 1C_N) \\ = (C_{r+1} + \cdots + C_N) \tag{14}$$

In turn, expressing $D_{r-1}$ as the product of the $(r-1)$- th row of the $L$ by the trigger we have:

$$D_{r-1} = (0C_1 + \cdots + 0C_{r-1} + 1C_r + \cdots + 1C_N) \\ = (C_r + \cdots + C_N) \tag{15}$$

and therefore $D_{r-1} = C_r + (C_{r+1} + \cdots + C_N) = C_r + D_r$ equivalent to (1). □

Let's move on to (10). It returns $I_r$, the $r$-th component of the output column vector **I** as the product of the $r$-th row of $G$ and the trigger **C**. It therefore results:

$$I_r = 0C_1 + \cdots + 0C_{r-1} + i_k \cdot C_r + \cdots + i_k \cdot C_N \\ = i_k \cdot (C_r + \cdots + C_N) = i_k \cdot D_{r-1} \tag{16}$$

and Eq. (2) is then satisfied. □

Finally we consider the (12). It gives $R_r$, the $r$-th component of the output column vector **R** as the product of the $r$-th row of $F$ and the trigger **C**. It therefore results:

$$R_r = 0C_1 + \cdots + 0C_{r-1} + (1 + i_k)C_r + i_k \cdot C_{r+1} + \cdots + i_k \cdot C_N \\ = C_r + i_k \cdot (C_r + \cdots + C_N) = C_r + i_k \cdot D_{r-1} \\ = C_r + I_r \tag{17}$$

and Eq. (3) is then satisfied. □

***Remark 1*** The matrix $L$ also guarantees compliance with the closing condition (5); indeed, it turns out that $D_N$ (last component of the output **D**), product of the $N$-th row of $L$ times the trigger, is equal to:

$$D_N = 0C_1 + \cdots + 0C_N = 0 \tag{18}$$

Very important is also the following

**Theorem 1** *The vector* **R** *obtained from* (12) *satisfies:*

$$D_0 = \sum_{n=1}^{N} v_k^n \cdot R_n \tag{19}$$

The proof of Theorem 1 is given in the Appendix.

***Remark 2*** The left-hand-side of (19) is the lender's initial payment related to epoch 0; at the right-hand-side, the discounted value at that time, under compound regime, of the borrower's payments. Each instalment $R_n$ is in fact multiplied by the discount factor $v(0, t_n) = v_k^n = \frac{1}{(1+i_k)^n}$ characteristic of the compound regime at the effective periodic rate $i_k$.[8]

***Remark 3*** Note that the relation (19) is not imposed a priori; it is a consequence of the (6)–(8)–(10)–(12) transformation rules and of the feasibility condition of the input **C**.

Summing up, given the triad $(i_k, N, D_0)$ a STAP can be uniquely defined by choosing an admissible vector **C** as input to which to apply the (6)–(8)–(10)–(12)[9].

## 5 STAPR and its transform matrices

In this section we will apply the approach used for the construction of the STAPC with the only variant of substituting the **R** trigger in place of the **C** trigger, maintaining the rules (including the closing condition, still expressed by the $D_N = 0$) and, obviously, replacing the trigger feasibility condition which must now be referred to the sequence of trigger instalments.

If we want to maintain the fundamental rules governing a STAP, it seems logical to impose that the feasibility condition on the trigger sequence **R** satisfies the relation (19) of Theorem 1. The meaning of this condition is that the rules of a STAP do not depend on the trigger; in other words, a STAPC and a STAPR are not bearers of different logics in the relationship that is established between the parties, but only in the trigger starting point.

Therefore, given the feasibility condition on the **R** trigger:

$$D_0 = \sum_{n=1}^{N} v_k^n \cdot R_n \tag{20}$$

formally identical to (19), this results in the following choices of the four transformation matrices:

$$\mathbf{R} = I\mathbf{R} \tag{21}$$

$$\mathbf{C} = \Phi\mathbf{R} = F^{-1}\mathbf{R} \tag{22}$$

$$\mathbf{I} = \Gamma\mathbf{R} = (I - \Phi)\mathbf{R} = (I - F^{-1})\mathbf{R} \tag{23}$$

$$\mathbf{D} = \Lambda\mathbf{R} = \frac{u_k}{i_k}\Gamma\mathbf{R} - I\mathbf{R} = \frac{u_k}{i_k}(I - F^{-1})\mathbf{R} - I\mathbf{R} \tag{24}$$

8 For the mere purpose of reporting, we point out that this has led some Italian scholars to embrace (in our opinion unwisely) the thesis of the presence of interests on interests in the STAPs and deduce the presence of illegitimate anatocism in these plans. We do not enter here in such juridical problems.

9 Moriconi (1994, p. 73) (authors translation): "In some cases, the distinction between principal and interest is so important that these quantities are chosen as starting data in determining the instalments... and take on the role of real project specifications."

Equation (21) is obvious: it translates the request that output $\mathbf{R}$ corresponds to the trigger of STAPR.

Equation (22) corresponds to the idea that, with unchanged $\mathbf{x}$ contractual parameters and rules, if $\mathbf{R}^*$ is output of STAPC with trigger $\mathbf{C}^*$, then $\mathbf{C}^*$ must be output of STAPR with trigger $\mathbf{R}^*$ and vice versa. More formally:

$$\mathbf{C}^* = \Phi\mathbf{R}^* \text{ iff } \mathbf{R}^* = F\mathbf{C}^* \tag{25}$$

this implies:

$$\mathbf{C}^* = \Phi F\mathbf{C}^* \tag{26}$$

$$\mathbf{R}^* = F\Phi\mathbf{R}^* \tag{27}$$

which, in turn, imply:

$$\Phi F = F\Phi = I \tag{28}$$

and therefore:

$$\Phi = F^{-1} \tag{29}$$

Equation (23) then follows from the rule (3) in the version $\mathbf{I} = \mathbf{R} - \mathbf{C}$ and from (21), (22) and (29).

Finally (24), on the basis of rule (2), implies that the $(n-1)$-th row of $\mathbf{D}$ is obtained by dividing the $n$-th row of $I$ by $i_k$.

The following theorem is then provided (see Appendix for the proof):

**Theorem 2** *The transformation matrix $\Phi$, inverse of $F$, has all zero elements below the main diagonal; all equal to $v_k$ on the main diagonal; all equal to $-i_k \cdot v_k^s$ in each upper diagonal parallel to the main one and which starts from the column $s$.*

Formally:

$$\phi_{r,s} = 0 \quad \forall r > s; \quad \phi_{r,s} = v_k \quad \forall r = s; \quad \phi_{r,s} = -i_k \cdot v^{s-(r-1)} \quad \forall r < s$$

$$\Phi = \begin{bmatrix} v_k & -i_k v_k^2 & -i_k v_k^3 & -i_k v_k^4 & \dots & -i_k v_k^N \\ 0 & v_k & -i_k v_k^2 & -i_k v_k^3 & \dots & -i_k v_k^{N-1} \\ 0 & 0 & v_k & -i_k v_k^2 & \dots & -i_k v_k^{N-2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & v_k \end{bmatrix} \tag{30}$$

***Remark 4*** The first row of the $\Phi$ transformation matrix in the $\mathbf{C}$ output is enlightening! It works (in the product of this row by the trigger column) for the coherent calculation of the first principal repayment which turns out to be $C_1 = v_k^1 R_1 - i_k \sum_{s=2}^N v_k^n R_n$.[10]

[10] Note that it is clear that in STAPR one cannot expect to impose that the first principal repayment is equal to $C_1 = v_k^1 R_1$: this implies neglecting the interest accrued in the first period on the remaining part of the debt. For details see Pressacco and Ziani (2020) and Pressacco et al. (2022).

The transformation matrix $\Gamma$, which transforms the trigger $\mathbf{R}$ into the output $\mathbf{I}$, is now the difference between the matrices $I$ and $F^{-1}$, which in turn transform the trigger, respectively, into the output $\mathbf{R}$ and $\mathbf{C}$. It turns out that:

$$\gamma_{r,s} = 0 - \phi_{r,s} = 0 - 0 \quad \forall r > s;$$
$$\gamma_{r,s} = 1 - \phi_{r,s} = 1 - v_k = i_k v_k \quad \forall r = s;$$
$$\gamma_{r,s} = 0 - \phi_{r,s} = i_k \cdot v^{s-r+1} \quad \forall r < s.$$
$$\Gamma = (I - \Phi) = (I - F^{-1}) = \begin{bmatrix} i_k v_k & i_k v_k^2 & i_k v_k^3 & i_k v_k^4 & \dots & i_k v_k^N \\ 0 & i_k v_k & i_k v_k^2 & i_k v_k^3 & \dots & i_k v_k^{N-1} \\ 0 & 0 & i_k v_k & i_k v_k^2 & \dots & i_k v_k^{N-2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & i_k v_k \end{bmatrix} \tag{31}$$

As for the transformation matrix $\Lambda$, which transform the trigger $\mathbf{R}$ into the outstanding debt $\mathbf{D}$, we show that (see Appendix for the proof):

$$\Lambda = \frac{u_k}{i_k}(I - \Phi) - I = \frac{u_k}{i_k}\Gamma - I \tag{32}$$

Formally:

$$\lambda_{r,s} = 0 \quad \forall r \geq s; \quad \lambda_{r,s} = v_k^{s-r} \quad \forall r < s;$$
$$\Lambda = \begin{bmatrix} 0 & v_k^1 & v_k^2 & v_k^3 & \dots & v_k^{N-1} \\ 0 & 0 & v_k^1 & v_k^2 & \dots & v_k^{N-2} \\ 0 & 0 & 0 & v_k^1 & \dots & v_k^{N-3} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \tag{33}$$

In particular, the last row of $\Lambda$ confirms that the plan satisfies the closing condition $D_N = 0$.

## 6 A theory of twin STAPs

We now give the following:

**Definition 1** A pair of sequences $\mathbf{C}^*, \mathbf{R}^*$ are twin sequences of a STAP if and only if $\mathbf{R}^*$ is output of the STAPC with feasible trigger $\mathbf{C}^*$ and $\mathbf{C}^*$ is output of the STAPR with feasible trigger $\mathbf{R}^*$.

The following Corollary holds:

**Corollary 1** *Given a couple $\mathbf{C}^*, \mathbf{R}^*$ of twin sequences, the STAPC with trigger $\mathbf{C}^*$ and the STAPR with trigger $\mathbf{R}^*$ are twin plans, i.e. they have the same values for all* *four outputs. Besides the same values of the reimbursement quotas* $\mathbf{C}^*$*, and of the instalments* $\mathbf{R}^*$*, which come by hypothesis from the twin character of the triggers of the two plans, they have common values also for the other two outputs. Indeed* $\mathbf{I}^*$ *is the sequence of the interest output common to the STAPC and the STAPR and,* $\mathbf{D}^*$ *is the sequence of the debts output of the STAPC and the STAPR.*

***Proof*** Immediate for $\mathbf{I}^*$ as, in STAPC, it is $\mathbf{R}^* = \mathbf{C}^* + \mathbf{I}$ and, in STAPR it is $\mathbf{I} = \mathbf{R}^* - \mathbf{C}^*$, while for $\mathbf{D}$ it is, in both plans, $D_n = D_0 - (C_1^* + C_2^* + \cdots + C_n^*)$. □

## 7 The French STAP and the theory of twin STAPs

The approach introduced in the previous sections is quite interesting as it offers two alternative ways to derive the STAP universally known as (traditional) French amortization plan with constant instalments.

Given the set $\mathbf{x}$, it is usual to select the trigger $\mathbf{R}$ as the starting point to build the plan. The feasibility condition of the trigger, jointly with the choice of constant instalments, give a well-known closed formula of the amount $R_n = R$ of each instalment[11] then, exploiting the fundamental rules, it is possible to derive recursively the whole STAPR.

As an alternative to the recursive procedure, we may apply the Input-Output approach and use the transformation matrices $(I, \Phi, \Gamma, \Lambda)$ to obtain directly all the output vectors of the STAPR with trigger $\mathbf{R}^*$ and, in particular, the reimbursement output vector $\mathbf{C}^*$.

In turn, we could also use $\mathbf{C}^*$ as the twin trigger of a STAPC, so that, exploiting Corollary 1, this STAPC is the twin of the former STAPR. Then this twin STAPC (with trigger $\mathbf{C}^*$) has constant instalments equal to $\mathbf{R}^*$. It could be checked that also the trigger $\mathbf{C}^*$ is characterized by a regularity property: the sequence of reimbursement quotas follows the rule $C_{n+1} = C_n u_k$, for any $n = 1, \ldots, N-1$. In the jargon of mathematics this sequence is geometrically increasing at a common ratio $u_k$. This too is nothing new: it is a well-known result that the traditional French amortization plan is characterized by a sequence of reimbursement quotas that satisfies this property. We quickly recall the traditional reasoning behind this result.

The starting point is the condition $R_{n+1} = R_n$ of equality between any couple of two consecutive instalments (without knowledge of the exact common value of each instalment); then, we apply the three basic rules as follows:

- by (1), $C_{n+1} + I_{n+1} = C_n + I_n$,
- by (3), $C_{n+1} + i_k D_n = C_n + i_k D_{n-1} \Rightarrow C_{n+1} = C_n + i_k(D_{n-1} - D_n)$,
- by (2), $C_{n+1} = C_n + i_k C_n = C_n u_k$.

Then, it is for any $n = 1, \ldots, N$: $C_n = C_1\,(1 + i_k)^{n-1} = C_1\, u_k^{n-1}$.

Finally, exploiting the feasibility condition on the reimbursement sequence, we have:

$$D_0 = C_1 + C_1\, u_k^1 + C_1\, u_k^2 + \cdots + C_1\, u_k^{N-1} = C_1 \sum_{n=0}^{N-1} u_k^n \tag{34}$$

[11] In particular it is: $R = \dfrac{D_0 \cdot i_k}{1 - (1 + i_k)^{-N}}$.

which gives a closed formula to derive $C_1$ (for any given $\mathbf{x}$) and the whole unique (given $\mathbf{x}$) feasible sequence $\mathbf{C}$ of the reimbursement quotas coherent with the constant instalment condition (under the STAP rules). It is just this sequence that may play the role of trigger of the STAPC with constant instalments.

Our IO approach clarifies that this STAPC is just the twin of the STAPR with trigger $\mathbf{R}$ derived from the feasibility condition on a set of constant instalments (at first sight linked to the compound capitalization regime). Neither of the two approaches has a logical primacy; yet in our opinion, the one with trigger $\mathbf{C}$ has the advantage of no sophisticated idea of financial mathematics (as compound capitalization) is needed to back the trigger.[12]

This is just an example, even if the most important in the real world applications. More generally, the theory of twin STAPs may be applied to derive any STAP generated by a trigger $\mathbf{R}^*$ as the twin STAP plan generated by the twin trigger $\mathbf{C}^* = F^{-1}\mathbf{R}^*$ or, conversely, to derive any STAP generated by a trigger $\mathbf{C}^*$ as the twin STAP generated by the twin trigger $\mathbf{R}^* = F\mathbf{C}^*$.

Going back to the French amortization plan, we stress the following remarks:

***Remark 5*** Given the set of contractual parameters $\mathbf{x} = (D_0, N, k, j)$ and the rules (1)–(5), there is a unique trigger vector $\mathbf{R}^*$ which generates the STAPR with $R_n = R$ for any $n = 1, \ldots, N$, coherent with rules and parameters.

***Remark 6*** Given the same set of contractual parameters $\mathbf{x} = (D_0, N, k, j)$ and the rules (1)–(5), there is a unique trigger vector $\mathbf{C}^*$ which generates the STAPC with constant instalments $R$, coherent with rules and parameters. This trigger is the twin of $\mathbf{R}^*$.

We underline that in both cases, given $\mathbf{x}$, there is no free choice of the trigger. In other words, the condition of constant instalments places a binding constraint on the choice of one or the other trigger; there are no degrees of freedom in such a choice at all.

The constant instalment condition is equivalent to the condition of increasing geometric sequence of principal reimbursements at a common ratio coherent with the contractual periodic interest rate (ratio $u_k = 1 + i_k = 1 + j/k$).

Clearly, the constant instalment condition is more friendly and immediately understood by any debtor and this is why this terminology is currently used to shortly describe the plan in real world transactions. Nevertheless, from a logical point of view there is no conceptual difference between the two.

In the more general case of STAP with deterministically variable instalments/reimbursment quotas a well-defined plan needs the specification of the trigger sequence either as a sequence of instalments $\mathbf{R}$ or of reimbursement quotas $\mathbf{C}$. Given the set of contractual parameters such specifications should satisfy the trigger specific feasibility condition. Nevertheless, to obtain the same plan, just one of the sequences may be freely chosen (within the feasibility condition). The other trigger is necessarily the twin one.

[12] Our approach makes clear that it is the rule of periodicity in the collectability of the interest payments that drives the whole process.

## 8 Some examples of twin STAPs

We present here just three examples of STAPs sharing the same $\mathbf{x} = (D_0 = 1000.00; N = 4; k = 1; j = i = 0.10)$. We apply the IO approach to get the four vectors of fundamental quantities of the twin STAPs.

***Example 1*** trigger $\mathbf{C}$ with constant reimbursement quotas and corresponding twin.

The trigger $\mathbf{C}$ has all elements equals to $C = \frac{1000}{4} = 250$.
According to (8), the output $\mathbf{D}$ is given by $L\,\mathbf{C}$:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 250 \\ 250 \\ 250 \\ 250 \end{bmatrix} = \begin{bmatrix} 750 \\ 500 \\ 250 \\ 0 \end{bmatrix}$$

According to (10), the output $\mathbf{I}$ is given by $G\,\mathbf{C}$:

$$\begin{bmatrix} 0.10 & 0.10 & 0.10 & 0.10 \\ 0 & 0.10 & 0.10 & 0.10 \\ 0 & 0 & 0.10 & 0.10 \\ 0 & 0 & 0 & 0.10 \end{bmatrix} \cdot \begin{bmatrix} 250 \\ 250 \\ 250 \\ 250 \end{bmatrix} = \begin{bmatrix} 100 \\ 75 \\ 50 \\ 25 \end{bmatrix}$$

According to (12), the output $\mathbf{R}$ is given by $F\,\mathbf{C}$:

$$\begin{bmatrix} 1.10 & 0.10 & 0.10 & 0.10 \\ 0 & 1.10 & 0.10 & 0.10 \\ 0 & 0 & 1.10 & 0.10 \\ 0 & 0 & 0 & 1.10 \end{bmatrix} \cdot \begin{bmatrix} 250 \\ 250 \\ 250 \\ 250 \end{bmatrix} = \begin{bmatrix} 350 \\ 325 \\ 300 \\ 275 \end{bmatrix}$$

Let us now take the output $\mathbf{R}$ of the STAPC as input twin trigger $\mathbf{R}^*$ in order to get the four vectors of the twin STAPR.

So, according to (21) $\mathbf{R} = I\mathbf{R}$ it is of course:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 350 \\ 325 \\ 300 \\ 275 \end{bmatrix} = \begin{bmatrix} 350 \\ 325 \\ 300 \\ 275 \end{bmatrix}$$

According to (24), the output $\mathbf{D}$ is given by $\Lambda\,\mathbf{R}$:

$$\begin{bmatrix} 0 & 0.909 & 0.826 & 0.751 \\ 0 & 0 & 0.909 & 0.826 \\ 0 & 0 & 0 & 0.909 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 350 \\ 325 \\ 300 \\ 275 \end{bmatrix} = \begin{bmatrix} 750 \\ 500 \\ 250 \\ 0 \end{bmatrix}$$

According to (23), the output $\mathbf{I}$ is given by $\Gamma\,\mathbf{R}$:

$$\begin{bmatrix} 0.091 & 0.083 & 0.075 & 0.068 \\ 0 & 0.091 & 0.083 & 0.075 \\ 0 & 0 & 0.091 & 0.083 \\ 0 & 0 & 0 & 0.091 \end{bmatrix} \cdot \begin{bmatrix} 350 \\ 325 \\ 300 \\ 275 \end{bmatrix} = \begin{bmatrix} 100 \\ 75 \\ 50 \\ 25 \end{bmatrix}$$

According to (22), the output $\mathbf{C}$ is given by $\Phi\,\mathbf{R}$:

$$\begin{bmatrix} 0.909 & -0.083 & -0.075 & -0.068 \\ 0 & 0.909 & -0.083 & -0.075 \\ 0 & 0 & 0.909 & -0.083 \\ 0 & 0 & 0 & 0.909 \end{bmatrix} \cdot \begin{bmatrix} 350 \\ 325 \\ 300 \\ 275 \end{bmatrix} = \begin{bmatrix} 250 \\ 250 \\ 250 \\ 250 \end{bmatrix}$$

So $\mathbf{C}^*$ and $\mathbf{R}^*$ are twin triggers.

***Example 2*** trigger $\mathbf{R}$ with constant instalments and corresponding twin

The trigger $\mathbf{R}$ is now equals to $R_n = R = 1000 \cdot \dfrac{0.10}{1-(1.10)^{-4}} = 315.47$, according to its feasibility condition.

According to (24), the output $\mathbf{D}$ is given by $\Lambda\,\mathbf{R}$:

$$\begin{bmatrix} 0 & 0.909 & 0.826 & 0.751 \\ 0 & 0 & 0.909 & 0.826 \\ 0 & 0 & 0 & 0.909 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 315.47 \\ 315.47 \\ 315.47 \\ 315.47 \end{bmatrix} = \begin{bmatrix} 784.53 \\ 547.51 \\ 286.79 \\ 0 \end{bmatrix}$$

According to (23), the output $\mathbf{I}$ is given by $\Gamma\,\mathbf{R}$:

$$\begin{bmatrix} 0.091 & 0.083 & 0.075 & 0.068 \\ 0 & 0.091 & 0.083 & 0.075 \\ 0 & 0 & 0.091 & 0.083 \\ 0 & 0 & 0 & 0.091 \end{bmatrix} \cdot \begin{bmatrix} 315.47 \\ 315.47 \\ 315.47 \\ 315.47 \end{bmatrix} = \begin{bmatrix} 100 \\ 78.45 \\ 54.75 \\ 28.68 \end{bmatrix}$$

According to (22), the output $\mathbf{C}$ is given by $\Phi\,\mathbf{R}$:

$$\begin{bmatrix} 0.909 & -0.083 & -0.075 & -0.068 \\ 0 & 0.909 & -0.083 & -0.075 \\ 0 & 0 & 0.909 & -0.083 \\ 0 & 0 & 0 & 0.909 \end{bmatrix} \cdot \begin{bmatrix} 315.47 \\ 315.47 \\ 315.47 \\ 315.47 \end{bmatrix} = \begin{bmatrix} 215.47 \\ 237.02 \\ 260.72 \\ 286.79 \end{bmatrix}$$

Let us now take the output $\mathbf{C}$ of the STAPR as input twin trigger $\mathbf{C}^*$ in order to get the four vectors of the twin STAPC.

According to (8), the output $\mathbf{D}$ is given by $L\,\mathbf{C}$:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 215.47 \\ 237.02 \\ 260.72 \\ 286.79 \end{bmatrix} = \begin{bmatrix} 784.53 \\ 547.51 \\ 286.79 \\ 0 \end{bmatrix}$$

According to (10), the output $\mathbf{I}$ is given by $G\,\mathbf{C}$:

$$\begin{bmatrix} 0.10 & 0.10 & 0.10 & 0.10 \\ 0 & 0.10 & 0.10 & 0.10 \\ 0 & 0 & 0.10 & 0.10 \\ 0 & 0 & 0 & 0.10 \end{bmatrix} \cdot \begin{bmatrix} 215.47 \\ 237.02 \\ 260.72 \\ 286.79 \end{bmatrix} = \begin{bmatrix} 100 \\ 78.45 \\ 54.75 \\ 28.68 \end{bmatrix}$$

According to (12), the output $\mathbf{R}$ is given by $F\,\mathbf{C}$:

$$\begin{bmatrix} 1.10 & 0.10 & 0.10 & 0.10 \\ 0 & 1.10 & 0.10 & 0.10 \\ 0 & 0 & 1.10 & 0.10 \\ 0 & 0 & 0 & 1.10 \end{bmatrix} \cdot \begin{bmatrix} 215.47 \\ 237.02 \\ 260.72 \\ 286.79 \end{bmatrix} = \begin{bmatrix} 315.47 \\ 315.47 \\ 315.47 \\ 315.47 \end{bmatrix}$$

So, still, $\mathbf{C}^*$ and $\mathbf{R}^*$ are twin triggers.

***Example 3*** trigger $\mathbf{C}$ with increasing geometric sequence of principal reimbursements at a common ratio $q = 5\%$ and corresponding twin.

Now, according to (34), the first component of the trigger is given by:

$$C_1 = 1000\,\frac{0.05}{(1.05)^4 - 1} = 232.01$$

and the others are respectively: $C_2 = C_1\,1.05 = 243.61$; $C_3 = C_1\,(1.05)^2 = 255.79$; $C_4 = C_1\,(1.05)^3 = 268.58$.

According to (8), the output $\mathbf{D}$ is given by $L\,\mathbf{C}$:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 232.01 \\ 243.61 \\ 255.79 \\ 268.58 \end{bmatrix} = \begin{bmatrix} 767.99 \\ 524.38 \\ 268.58 \\ 0 \end{bmatrix}$$

According to (10), the output $\mathbf{I}$ is given by $G\,\mathbf{C}$:

$$\begin{bmatrix} 0.10 & 0.10 & 0.10 & 0.10 \\ 0 & 0.10 & 0.10 & 0.10 \\ 0 & 0 & 0.10 & 0.10 \\ 0 & 0 & 0 & 0.10 \end{bmatrix} \cdot \begin{bmatrix} 232.01 \\ 243.61 \\ 255.79 \\ 268.58 \end{bmatrix} = \begin{bmatrix} 100.00 \\ 76.80 \\ 52.44 \\ 26.86 \end{bmatrix}$$

According to (12), the output $\mathbf{R}$ is given by $F\,\mathbf{C}$:

$$\begin{bmatrix} 1.10 & 0.10 & 0.10 & 0.10 \\ 0 & 1.10 & 0.10 & 0.10 \\ 0 & 0 & 1.10 & 0.10 \\ 0 & 0 & 0 & 1.10 \end{bmatrix} \cdot \begin{bmatrix} 232.01 \\ 243.61 \\ 255.79 \\ 268.58 \end{bmatrix} = \begin{bmatrix} 332.01 \\ 320.41 \\ 308.23 \\ 295.44 \end{bmatrix}$$

Let us now take the output $\mathbf{R}$ of the STAPC as input twin trigger $\mathbf{R}^*$ in order to get the four vectors of the twin STAPR.

So, according to (21) $\mathbf{R} = I\mathbf{R}$ it is of course:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 332.01 \\ 320.41 \\ 308.23 \\ 295.44 \end{bmatrix} = \begin{bmatrix} 332.01 \\ 320.41 \\ 308.23 \\ 295.44 \end{bmatrix}$$

According to (24), the output $\mathbf{D}$ is given by $\Lambda\,\mathbf{R}$:

$$\begin{bmatrix} 0 & 0.909 & 0.826 & 0.751 \\ 0 & 0 & 0.909 & 0.826 \\ 0 & 0 & 0 & 0.909 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 332.01 \\ 320.41 \\ 308.23 \\ 295.44 \end{bmatrix} = \begin{bmatrix} 767.99 \\ 524.38 \\ 268.58 \\ 0 \end{bmatrix}$$

According to (23), the output $\mathbf{I}$ is given by $\Gamma\,\mathbf{R}$:

$$\begin{bmatrix} 0.091 & 0.083 & 0.075 & 0.068 \\ 0 & 0.091 & 0.083 & 0.075 \\ 0 & 0 & 0.091 & 0.083 \\ 0 & 0 & 0 & 0.091 \end{bmatrix} \cdot \begin{bmatrix} 332.01 \\ 320.41 \\ 308.23 \\ 295.44 \end{bmatrix} = \begin{bmatrix} 100.00 \\ 76.80 \\ 52.44 \\ 26.86 \end{bmatrix}$$

According to (22), the output $\mathbf{C}$ is given by $\Phi\,\mathbf{R}$:

$$\begin{bmatrix} 0.909 & -0.083 & -0.075 & -0.068 \\ 0 & 0.909 & -0.083 & -0.075 \\ 0 & 0 & 0.909 & -0.083 \\ 0 & 0 & 0 & 0.909 \end{bmatrix} \cdot \begin{bmatrix} 332.01 \\ 320.41 \\ 308.23 \\ 295.44 \end{bmatrix} = \begin{bmatrix} 232.01 \\ 243.61 \\ 255.79 \\ 268.58 \end{bmatrix}$$

Also this case, $\mathbf{C}^*$ and $\mathbf{R}^*$ are twin triggers.

## 9 Conclusions

We found that the IO analysis approach is able to shed some light on the somehow fascinating and controversial (at least in the Italian current debate) STAP issue. The main idea behind this approach is the clear distinction between input/trigger and outputs of the amortization plan. It helps to clarify that, given the set of contractual parameters

$\mathbf{x}$, any STAP has a dual nature. There are two perfectly equivalent twin STAPs: the STAPC with trigger/input a feasible sequence of principal reimbursements and a set of four matrices $(I, L, G, F)$, which transform the trigger into the four output vectors of the STAPC and, the STAPR with trigger/input a feasible sequence of instalments, and a set of four matrices $(\Phi, \Lambda, \Gamma, I)$, which in turn transform the trigger into the four output vectors of the STAPR.

The approach makes clear that the two STAPs are coincident, i.e., twin, if and only if two independent conditions are satisfied: the instalment trigger of the STAPR must be the instalment output of the STAPC and the matrix $\Phi$, which transforms the instalment trigger in the reimbursement output must be the inverse of $F$, which transforms the reimbursement trigger in the instalment output. A symmetric reasoning may be applied keeping as starting point a STAPR with trigger/input a sequence of instalments. From a logical point of view, neither of the two starting points is privileged and there are no reasons to think that the two STAPs should follow different rules depending on the choice of the trigger.

The IO approach introduced here may be applied to any STAP, but it is of particular utility once it is applied to the case of the plan currently labelled French STAP. It helps us to understand that the contractual agreement on a sequence of constant instalments is not necessarily obtained as the outcome of a STAPR. It may be obtained also as the outcome of the STAPC with agreed trigger a proper sequence of reimbursement quotas. It should be noted that the corresponding feasibility condition does not directly involve either any *a priori* condition of constancy of instalments, or any financial regime framework. Rather these two conditions are merely the outcome of the STAPC.

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## Declarations

**Conflict of interest** The authors declare that they have no conflict of interest.



## Appendix

Let us now provide the proof of Theorem 1.

***Proof*** The right-hand-side of (19) can be rewritten using the rules as follows:

$$\sum_{n=1}^{N} v_k^n R_n = \sum_{n=1}^{N} v_k^n (C_n + I_n)$$

$$
\begin{aligned}
&= \sum_{n=1}^{N} v_k^n C_n + \sum_{n=1}^{N} v_k^n I_n \\
&= \sum_{n=1}^{N} v_k^n C_n + \sum_{n=1}^{N} v_k^n i_k D_{n-1} \\
&= \sum_{n=1}^{N} v_k^n C_n + \sum_{n=1}^{N} v_k^n i_k (D_0 - (C_1 + \cdots + C_{n-1})) \\
&= \sum_{n=1}^{N} v_k^n C_n + i_k \sum_{n=1}^{N} v_k^n (D_0 - (C_1 + \cdots + C_{n-1})) \\
&= \sum_{n=1}^{N} v_k^n C_n + i_k \cdot D_0 \cdot \sum_{n=1}^{N} v_k^n - i_k \cdot \sum_{n=1}^{N} v_k^n \cdot (C_1 + \cdots + C_{n-1}) \qquad (35)
\end{aligned}
$$

The second addendum is equal to $i_k \cdot D_0 \frac{(1-v_k^N)}{i_k} = D_0 \cdot (1 - v_k^N)$.

We will prove that the sum of the first and third addendum is equal to $v_k^N (C_1 + C_2 + \cdots + C_N) = v_k^N D_0$.

To this purpose after rewriting the third addendum in the form

$$
-i_k \cdot \sum_{n=1}^{N-1} C_n \, (v_k^{n+1} + v_k^{n+2} + \cdots v_k^N)
$$

we add and subtract $i_k \cdot \sum_{n=1}^{N-1} C_n \cdot (v_k^1 + \cdots + v_k^n)$.

The sum of the first and third addends and the two additional addends is given by:

$$
\begin{aligned}
&\sum_{n=1}^{N} C_n \, v_k^n - i_k \sum_{n=1}^{N-1} C_n \, (v_k^{n+1} + \cdots + v_k^N) - i_k \sum_{n=1}^{N-1} C_n \, (v_k^1 + \cdots + v_k^n) \\
&\quad + i_k \sum_{n=1}^{N-1} C_n (v_k^1 + \cdots + v_k^n) \\
&= \sum_{n=1}^{N} C_n \, v_k^n - i_k \sum_{n=1}^{N-1} C_n \, (v_k^1 + \cdots + v_k^N) + i_k \sum_{n=1}^{N-1} C_n (v_k^1 + \cdots + v_k^n) \\
&= \sum_{n=1}^{N} C_n \, v_k^n - \sum_{n=1}^{N-1} C_n \, (1 - v_k^N) + \sum_{n=1}^{N-1} C_n (1 - v_k^n) \\
&= \sum_{n=1}^{N-1} C_n \, v_k^n + C_N v_k^N - \sum_{n=1}^{N-1} C_n + \sum_{n=1}^{N-1} C_n \, v_k^N + \sum_{n=1}^{N-1} C_n - \sum_{n=1}^{N-1} C_n v_k^n \\
&= v_k^N \sum_{n=1}^{N} C_n = v_k^N \, D_0 \qquad (36)
\end{aligned}
$$

At the end, the overall sum of the right-hand-side is:

$$D_0 \cdot (1 - v_k^N) + k^N D_0 = D_0 \tag{37}$$

□

Let us now prove Theorem 2.

***Proof*** Recall that $\phi_{r,s}$, the element at the intersection of the row $r$ and the column $s$ of the $\Phi$:

$$\phi_{r,s} = 0 \quad \forall r > s; \quad \phi_{r,s} = v_k \quad \forall r = s; \quad \phi_{r,s} = -i_k \cdot v^{s-(r-1)} \quad \forall r < s$$

$$F = \begin{bmatrix} u_k & i_k & i_k & \dots & i_k \\ 0 & u_k & i_k & \dots & i_k \\ 0 & 0 & u_k & \dots & i_k \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & u_k \end{bmatrix}$$

$$\Phi = F^{-1} = \begin{bmatrix} v_k & -i_k v_k^2 & -i_k v_k^3 & -i_k v_k^4 & \dots & -i_k v_k^N \\ 0 & v_k & -i_k v_k^2 & -i_k v_k^3 & \dots & -i_k v_k^{N-1} \\ 0 & 0 & v_k & -i_k v_k^2 & \dots & -i_k v_k^{N-2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & v_k \end{bmatrix} \tag{38}$$

For the demonstration, instead of constructing $\Phi$ starting from $F$, we verify that $\Phi$, defined on the basis of (38), satisfies (28): $F\Phi = I$.

In detail, called $f_{r,n}$ the generic element at the intersection of the row $r$ and the column $n$ of the $F$, and $\phi_{n,s}$ the one at the intersection of the row $n$ and the column $c$ of the $\Phi$, we verify that:

(a) for any $r > s$ the product of the $r$-th row of the $F$ by the $s$-th column of the $\Phi$ is null or formally that $\sum_{n=1}^{N} f_{r,n}\phi_{n,s} = 0$.
(b) for each $r = s$ the product of the $r$-th row of the $F$ by the $s$-th column of the $\Phi$ is 1; i.e. that $\sum_{n=1}^{N} f_{r,n}\phi_{n,r} = 1$.
(c) for each $r < s$ the product of the $r$-th row of the $F$ by the $s$-th column of the $\Phi$ is zero, i.e. again $\sum_{n=1}^{N} f_{r,n}\phi_{n,s} = 0$.

To check that (a)–(c) hold, the following properties of the $F$ and $\Phi$ matrices are used:

Property 1. Row $r$ of $F$ has $r - 1$ null elements starting from the first column, and the remaining $N - (r - 1)$ positive starting from $u_k$ (column $s = r$) and then all $i_k$ (from column $s = r + 1$ onwards).

Property 2. Column $s$ of $\Phi$ has $s$ nonzero elements starting from the first row. For $s = 1$ the first and only nonzero element is $v_k$. For $s > 1$ the first element is $-i v_k^s$; the subsequent $s - 2$ elements are equal to $-i v_k^p$ with $p$ integer power decreasing until $p = 2$. The last positive element is equal to $v_k$.

We now distinguish three cases:

1. $r = s$. For Property 1 the first $r-1$ addends of the sum $\sum_{n=1}^{N} f_{r,n}\phi_{n,s}$ are zero; for Property 2 the last $N-r$ are null. The addend $f_{r,r}\phi_{r,r} = u_k v_k = 1$ remains unique nonzero.
2. $r > s$ implies $r-1$ greater than or at least equal to $s$. For Property 2 only the first $s$ addends of the sum $\sum_{n=1}^{N} f_{r,n}\phi_{n,s}$ can be nonzero. For Property 1 the first $r-1$ are null. But $r-1 \geq s$ and therefore all the addends are zero. The sum is equal to zero.
3. $s > r$. For $n = 1, \ldots, r-1$ or $n = s+1, \ldots, N$ either $f$ or $\phi$ are null and, of course, their product; both factors are nonzero for any $n = r, \ldots, s$. In particular, for $n = s$ $f_{r,s} = i_k$ and $\phi_{s,s} = v_k$ and so the product is equal to $i_k\, v_k$; for $n = r$, $f_{r,r} = u_k$ and $\phi_{r,s} = -i_k\, v_k^{s-r+1}$, and the product is $-u_k(i_k\, v_k^{s-r+1}) = -i_k\, v_k^{s-r}$. We now distinguish two sub-cases: $s = r+1$ and $s = r+z$ with $z > 1$.
   In the first sub-case, the order addendum $r$ reduces to $u_k\,(-i_k\, v_k^2) = -i_k\, v_k$. There are no other nonzero addends and the sum of the two addends $i_k\, v_k - i_k\, v_k$ is equal to zero.
   In the second sub-case ($s = r+z$ with $z > 1$). The addends of order $r$ and $s$ remain unchanged. The addends of order $r+1, \ldots, s-1$ in number of $z-1$ are equal to $i_k(-i_k v_k^h)$ with $h = 2 \ldots (s-r)$. Taking into account that $u_k(-i_k v_k^{s-r+1}) = -i_k v_k^{s-r}$, the sum reduces to: $-i_k v_k^{s-r} \sum_{h=2}^{s-r} i_k^2 v_k^h + i_k v_k$. We add and subtract $i_k^2 v_k - i_k^2 v_k$ and rewrite the sum in the form:

$$
\begin{aligned}
&-i_k v_k^{s-r} - \sum_{h=1}^{s-r} i_k^2 v_k^h + i_k^2 v_k + i_k v_k \\
&= -i_k v_k^{s-r} - i_k^2 \sum_{h=1}^{s-r} v_k^h + i_k^2 v_k + i_k v_k \\
&= -i_k v_k^{s-r} - i_k^2 (1 - v_k^{s-r})/i_k + i_k v_k (1 + i_k) \\
&= -i_k v_k^{s-r} - i_k (1 - v_k^{s-r}) + i_k = 0
\end{aligned} \tag{39}
$$

It has thus been demonstrated that the product of $F$ by $\Phi$ is the identity matrix and therefore that $\Phi$ is precisely the inverse matrix of $F$. □

Let us now give the proof of the matrix $\Lambda$.

***Proof*** For the matrix $\Lambda$, the rule (1) is used starting from $D_1 = D_0 - C_1$ and, at the right-hand side, the feasibility condition on the trigger $\mathbf{R}$ for the expansion of $D_0$ and, as for $C_1$, it is obtained from the product of the first row of $\Phi$ and the trigger column. We therefore have:

$$
\begin{aligned}
D_1 &= \sum_{n=1}^{N} v_k^n R_n - v_k^1 R_1 + \sum_{n=2}^{N} i_k v_k^n R_n = v_k^1 R_1 + \sum_{n=2}^{N} v_k^n R_n - v_k^1 R_1 + \sum_{n=2}^{N} i_k v_k^n R_n \\
&= \sum_{n=2}^{N} v_k^n R_n + \sum_{n=2}^{N} i_k v_k^n R_n = \sum_{n=2}^{N} v_k^n R_n (1 + i_k) = \sum_{n=2}^{N} v_k^{n-1} R_n
\end{aligned} \tag{40}
$$

adding $+0R_1$ we have that the first row of the $\Lambda$ is $(0, v_k^1, v_k^2, \ldots, v_k^{N-1})$. Thus, the residual debt at epoch 1 was obtained by discounting, at that epoch, the instalments still to be paid starting from the second.

By iterating the procedure, it can be verified that the $r$-th row of the $\Lambda$ has exactly $r$ zero initial values, followed by increasing powers of the $v_k$, i.e. $(0, \ldots, 0, v_k^1, \ldots, v_k^{N-r})$. In general, therefore, the residual debt at the time $r$ is the discounted value, at that time, of the instalments still to be paid starting from the $(r+1)$-th.

Finally, since the last row has all null elements, the result is $D_N = 0$ and the closure condition is thus verified also for the STAPR and it is also verified formally $D_n = D_0 - (C_1 + C_2 + \cdots + C_n)$.

Summing up, the $\Lambda$ transformation matrix of the **R** trigger in the residual debt output is as follows:

$$\Lambda = \begin{bmatrix} 0 & v_k^1 & v_k^2 & v_k^3 & \ldots & v_k^{N-1} \\ 0 & 0 & v_k^1 & v_k^2 & \ldots & v_k^{N-2} \\ 0 & 0 & 0 & v_k^1 & \ldots & v_k^{N-3} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

$\square$

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