Hardy type inequalities with mixed weights in cones

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Abstract. We study Hardy type inequalities involving mixed cylindrical and spherical weights, for functions supported in cones. These inequalities are related to some singular or degenerate differential operators.

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1 Introduction

We deal with the best constant $m_{p,a,b}(\mathcal{C})$ in inequalities of the type

$$m_{p,a,b}(\mathcal{C}) \int\limits_{\mathcal{C}} |y|^a |z|^{-b-p} |u|^p \, dz \le \int\limits_{\mathcal{C}} |y|^a |z|^{-b} |\nabla u|^p \, dz \ , \quad u \in C_c^\infty(\mathcal{C}) \, , \tag{1.1}$$

where $\mathcal{C} \subseteq \mathbb{R}^d$ is a cone, that is a dilation-invariant open set, $p > 1$, $a, b \in \mathbb{R}$ and $z = (x, y)$ is the variable in $\mathbb{R}^d \equiv \mathbb{R}^{d-k} \times \mathbb{R}^k$. Inequality (1.1) includes the well known case of purely spherical weights; for this reason we assume that $1 \le k < d$ and $a \ne 0$.

Our starting motivation arose from the growing interest, inspired by [1], in differential operators of the form

$$\mathcal{L}u = -\mathrm{div}(|y|^a A(z) \nabla u) \, . \tag{1.2}$$

Indeed, in case $p = 2$, the right hand side in (1.1) is the quadratic form associated to the differential operator $\mathcal{L}$, where $A(z) = |z|^{-b} \mathcal{I}_d$. Starting with the seminal paper [5], large efforts have been spent to investigate degenerate/singular operators including (1.2) (see for instance [2, 3] and references therein). We refer also to the papers [13, 14, 16] by Susanna Terracini and collaborators. With respect to [13] (even solutions), and [14] (odd solutions), the relevant cones are $\mathbb{R}^d$ and $\mathbb{R}^d_+ = \mathbb{R}^{d-1} \times (0, \infty)$, respectively.

Remark 1 *If $\mathcal{C}$ is the union of two disjoint cones $\mathcal{C}_1, \mathcal{C}_2$, then $m_{p,a,b}(\mathcal{C}) = \min\{m_{p,a,b}(\mathcal{C}_1), m_{p,a,b}(\mathcal{C}_2)\}$. Thus it would be enough to consider connected cones.*

In dealing with (1.1), one is forced to assume that the weights involved are locally integrable on the cone $\mathcal{C}$. This leads us to distinguish different situations, depending on the position of $\mathcal{C}$ with respect to the singular set

$$\Sigma_0 := \{y = 0\}.$$

More precisely, the cone $\mathcal{C} = \mathbb{R}^d$ needs $k + a > 0$ and $d + a > p + b$; if $\mathcal{C} \subseteq \mathbb{R}^d \setminus \{0\}$ then we have to require that

$$\mathcal{C} \subseteq \mathbb{R}^d \setminus \Sigma_0 \ , \quad \text{or} \quad k + a > 0. \tag{1.3}$$

In any case, a special role is played by the real constant

$$\mathrm{H}_{p,a,b} = \frac{d+a}{p} - \frac{p+b}{p} \, . \tag{1.4}$$

We start with a simple result, that deals with the two largest cones (here, ∂_r stands for the radial derivative).

Theorem 1 *Let $k + a > 0$. Then the inequality*

$$|\mathrm{H}_{p,a,b}|^p \int\limits_{\mathbb{R}^d} |y|^a |z|^{-b-p} |u|^p \, dz \le \int\limits_{\mathbb{R}^d} |y|^a |z|^{-b} |\partial_r u|^p \, dz \tag{1.5}$$

holds for any $u \in C^\infty_c(\mathbb{R}^d \setminus \{0\})$, with a sharp constant in the left hand side.

In particular, $m_{p,a,b}(\mathbb{R}^d \setminus \{0\}) = |\mathrm{H}_{p,a,b}|^p$.

If $d + a > p + b$, then (1.5) holds for any $u \in C^\infty_c(\mathbb{R}^d)$, and $m_{p,a,b}(\mathbb{R}^d) = \mathrm{H}^p_{p,a,b}$.

Theorem 1 implies that the Hardy inequality (with mixed weights) in $\mathbb{R}^d \setminus \{0\}$ holds with a positive constant if and only if $d + a \neq p + b$. From the proof, see Section 3, it is evident that the corresponding best constant is not achieved on any reasonable function space.

We point out a remarkable case. It is well known that

$$\Big(\frac{k+a-p}{p}\Big)^p \int\limits_{\mathbb{R}^d} |y|^{a-p} |u|^p \, dz \le \int\limits_{\mathbb{R}^d} |y|^a |\nabla u|^p \, dz \quad \text{for any } u \in C^\infty_c(\mathbb{R}^d), \tag{1.6}$$

provided that $a > p - k$, which is needed for the local integrability of the weight in the left-hand side. Notice that in the threshold case $a = p - k$, the constant in (1.6) vanishes. In contrast, a Hardy inequality involving *mixed weights* holds, as pointed out in the next statement.

Corollary 1 *Let $d > k$. Then*

$$\Big(\frac{d-k}{p}\Big)^p \int\limits_{\mathbb{R}^d} |y|^{p-k}|z|^{-p}|u|^p\,dz \le \int\limits_{\mathbb{R}^d} |y|^{p-k}|\nabla u|^p\,dz \quad \textit{for any } u \in C^\infty_c(\mathbb{R}^d).$$

The constant in the left hand side can not be improved.

From now on we deal with general cones $\mathcal{C} \subset \mathbb{R}^d$. We start with the *superdegenerate case* (cf. [16]).

Theorem 2 *Let $\mathcal{C} \subseteq \mathbb{R}^d \setminus \{0\}$. If $k + a \ge p$, then $m_{p,a,b}(\mathcal{C} \setminus \Sigma_0) = m_{p,a,b}(\mathcal{C})$. In particular,*

$$m_{p,a,b}(\mathbb{R}^d \setminus \Sigma_0) = |\mathrm{H}_{p,a,b}|^p.$$

The next result is already known in case of purely spherical weights [11, Section 6]; see also [15] for related issues in case $p = 2$. Here we denote by Π the restriction of the orthogonal projection $\Pi : \mathbb{R}^{d-k} \times \mathbb{R}^k \to \{0\} \times \mathbb{R}^k$ to the unit sphere $\mathbb{S}^{d-1}$.

Theorem 3 *Let $\mathcal{C}$ satisfy (1.3) and put $\omega_{\mathcal{C}} = \mathbb{S}^{d-1} \cap \mathcal{C}$. Then $m_{p,a,b}(\mathcal{C}) = \mathcal{M}_{p,a,b}(\omega_{\mathcal{C}})$, where*

$$\mathcal{M}_{p,a,b}(\omega_{\mathcal{C}}) = \inf_{\substack{\varphi \in C^\infty_c(\omega_{\mathcal{C}})\\ \varphi \ne 0}} \frac{\displaystyle\int_{\omega_{\mathcal{C}}} |\Pi\sigma|^a(|\nabla_\sigma\varphi|^2 + \mathrm{H}^2_{p,a,b}|\varphi|^2)^{\frac{p}{2}}\,d\sigma}{\displaystyle\int_{\omega_{\mathcal{C}}} |\Pi\sigma|^a|\varphi|^p d\sigma}\,.$$

Under the assumption (1.3), we can define the following weighted Sobolev spaces:

*S*1) $\mathcal{D}^{1,p}_0(\mathcal{C}; |y|^a|z|^{-b}dz)$ is the completion of $C^\infty_c(\mathcal{C})$ with respect to the norm

$$\|u\|^p = \int\limits_{\mathcal{C}} |y|^a|z|^{-b}\big(|\nabla u|^p + |z|^{-p}|u|^p\big)\,dz\,;$$

*S*2) $W^{1,p}_0(\omega_{\mathcal{C}}; |\Pi\sigma|^a d\sigma)$ is the completion of $C^\infty_c(\omega_{\mathcal{C}})$ with respect to the norm

$$\|\varphi\|^p = \int\limits_{\omega_{\mathcal{C}}} |\Pi\sigma|^a\big(|\nabla_\sigma\varphi|^p + |\varphi|^p\big)\,d\sigma.$$

The best constant $\mathcal{M}_{p,a,b}(\omega_{\mathcal{C}})$ is attained in $W^{1,p}_0(\omega_{\mathcal{C}};|\Pi\sigma|^a d\sigma)$, due to the compactness of the embedding $W^{1,p}_0(\omega_{\mathcal{C}};|\Pi\sigma|^a d\sigma) \hookrightarrow L^p(\omega_{\mathcal{C}};|\Pi\sigma|^a d\sigma)$ given by Corollary 3 in Section 2. In contrast, the next result holds.

Theorem 4 *Let $\mathcal{C}$ satisfy (1.3). The infimum $m_{p,a,b}(\mathcal{C})$ is not achieved on $\mathcal{D}^{1,p}_0(\mathcal{C};|y|^a|z|^{-b}dz)$.*

Theorem 3 shows, in particular, that $m_{p,a,b}(\mathcal{C}) \geq |\mathrm{H}_{p,a,b}|^p$ (notice that for $k+a>0$, this evidently follows from Theorem 1) and that equality might occur, compare with Theorem 2. It is natural to look for conditions that guarantee the validity of the strict inequality.

Since $\mathcal{M}_{p,a,b}(\omega_{\mathcal{C}})$ is achieved, we can easily infer conditions to have that

$$m_{p,a,b}(\mathcal{C}) > |\mathrm{H}_{p,a,b}|^p. \tag{1.7}$$

Consider for instance the following situations:

i) Σ_0 does not intersect $\overline{\omega}_{\mathcal{C}}$. Then $|\Pi\sigma|^a$ is bounded and bounded away from 0, so we are in fact in the case $a=0$. We have that the inequality (1.7) holds if the set $\mathbb{S}^{d-1}\setminus\omega_{\mathcal{C}}$ is not p-negligible in $\mathbb{S}^{d-1}$, which means that the corresponding p-capacity is positive, see [9, Theorem 14.1.2].

ii) $k+a<kp$. Take $q\in[1,p)$ such that $q(k+a)<kp$ (recall that $k+a$ can be negative). Then (1.7) holds if the set $\mathbb{S}^{d-1}\setminus\omega_{\mathcal{C}}$ is not q-negligible in $\mathbb{S}^{d-1}$, as $W^{1,p}_0(\omega_{\mathcal{C}};|\Pi\sigma|^a d\sigma)$ is embedded into $W^{1,q}_0(\omega_{\mathcal{C}})$ by Hölder inequality.

Larger exponents a are included in the next statement.

Theorem 5 *Let $k+a\geq kp$. If $k=1$, assume in addition that $\mathcal{C}\subset\mathbb{R}^d_+=\{y>0\}$ (otherwise, Theorem 2 and Remark 1 can be applied). Then (1.7) holds provided that $\partial\omega_{\mathcal{C}}\setminus\Sigma_0$ is not p-negligible in $\mathbb{S}^{d-1}$.*

It not easy to calculate $m_{p,a,b}(\omega_{\mathcal{C}})$ for general cones and exponents. If $a=0$ then Theorem 3 gives

$$m_{2,0,b}(\mathcal{C}) = \inf_{\substack{\varphi\in H^1_0(\omega_{\mathcal{C}})\\ \varphi\neq 0}} \frac{\displaystyle\int_{\omega_{\mathcal{C}}}(|\nabla_\sigma\varphi|^2+\mathrm{H}^2_{2,0,b}|\varphi|^2)d\sigma}{\displaystyle\int_{\omega_{\mathcal{C}}}|\varphi|^2 d\sigma} = \lambda_1(\omega_{\mathcal{C}})+|\mathrm{H}_{2,0,b}|^2,$$

where $\lambda_1(\omega_{\mathcal{C}})$ is the first eigenvalue of the Laplace-Beltrami operator on $\omega_{\mathcal{C}}$ with Dirichlet boundary conditions.

For the cone $\mathbb{R}^d \setminus \Sigma_0$ we have the following result (here $s^+ = \max\{s, 0\}$).

Theorem 6 *Let $p = 2$. Then $m_{2,a,b}(\mathbb{R}^d \setminus \Sigma_0) > |\mathrm{H}_{2,a,b}|^2$ if and only if $k + a < 2$. More precisely,*

$$m_{2,a,b}(\mathbb{R}^d \setminus \Sigma_0) = (d-k)(2-(k+a))^+ + |\mathrm{H}_{2,a,b}|^2.$$

In case $k = 1$, the singular set $\Sigma_0 = \{y = 0\}$ is a hyperplane which disconnects $\mathbb{R}^d$. Thus from Theorems 2, 6 we immediately obtain the next statement.

Corollary 2 *i) If $a \geq p - 1$, then $m_{p,a,b}(\mathbb{R}^d_+) = |\mathrm{H}_{p,a,b}|^p$.*

ii) If $p = 2$, then $m_{2,a,b}(\mathbb{R}^d_+) = (d-1)(1-a)^+ + |\mathrm{H}_{2,a,b}|^2$ for any $a \in \mathbb{R}$.

In the Caffarelli-Silvestre [1] setting we have $d = n + 1$, $a = 1 - 2s \in (-1, 1)$ and $b = 0$. In case $n > 2s$ (which is a restriction only if $n = 1$), the equality

$$\inf_{\substack{u \in C^\infty_c(\mathbb{R}^{n+1}) \\ \varphi \neq 0}} \frac{\displaystyle\int_{\mathbb{R}^{n+1}} |y|^{1-2s} |\nabla u|^2 dz}{\displaystyle\int_{\mathbb{R}^{n+1}} |y|^{1-2s} |z|^{-2} |u|^2 dz} = \Big(\frac{n-2s}{2}\Big)^2$$

(see for instance [6, Lemma 2.4], [10, formula (11)]), follows via Corollary 2 as well. On the half space $\mathbb{R}^{n+1}_+$ we have

$$\inf_{\substack{u \in C^\infty_c(\mathbb{R}^{n+1}_+) \\ \varphi \neq 0}} \frac{\displaystyle\int_{\mathbb{R}^{n+1}_+} |y|^{1-2s} |\nabla u|^2 dz}{\displaystyle\int_{\mathbb{R}^{n+1}_+} |y|^{1-2s} |z|^{-2} |u|^2 dz} = \Big(\frac{n+2s}{2}\Big)^2.$$

Notice that no restriction on s in case $n = 1$ is needed.

The paper is organized as follows. The preliminary Section 2 is mostly devoted to the properties of the weighted Sobolev space $W^{1,p}_0(\omega; |\Pi\sigma|^a d\sigma)$. The proofs of the theorems stated above are collected in Section 3.

Notation. We denote by Π both the orthogonal projection $\Pi : \mathbb{R}^d \to \{0\} \times \mathbb{R}^k$, and its restriction to the unit sphere $\mathbb{S}^{d-1}$. So, if $r > 0, \sigma \in \mathbb{S}^{d-1}$ are the spherical coordinates of $z = (x, y) \in \mathbb{R}^{d-k} \times \mathbb{R}^k$, then $|y| = r|\Pi\sigma|$.

For $R > 0$ we denote by B_R the ball of radius R about the origin.

Through the paper, any positive constant whose value is not important is denoted by c. It may take different values at different places. To indicate that a constant depends on some parameters we list them in parentheses.

2 Preliminaries

We start by pointing out the local integrability properties of the weights involved.

Lemma 1 *Let $a, \beta \in \mathbb{R}$. Then*

i) $|y|^a|z|^{-\beta} \in L^1_{\rm loc}(\mathbb{R}^d \setminus \{0\})$ *if and only if* $k + a > 0$*;*

ii) $|\Pi\sigma|^a \in L^1(\mathbb{S}^{d-1})$ *if and only if* $k + a > 0$*;*

iii) $|y|^a|z|^{-\beta} \in L^1_{\rm loc}(\mathbb{R}^d)$ *if and only if* $k + a > 0$ *and* $d + a > \beta$*.*

Proof. Evidently, $|y|^a|z|^{-\beta} \in L^1_{\rm loc}(\mathbb{R}^d \setminus \{0\})$ if and only if $|y|^a \in L^1_{\rm loc}(\mathbb{R}^k)$, that is, $k + a > 0$. Since

$$\int\limits_{B_R} |y|^a|z|^{-\beta}\, dz = \int\limits_0^R r^{d+a-\beta-1} dr \int\limits_{\mathbb{S}^{d-1}} |\Pi\sigma|^a\, d\sigma$$

for any $R > 0$, then *ii)* and *iii)* readily follow. □

The main results in this section deal with the weighted Sobolev spaces $W^{1,p}_0(\omega; |\Pi\sigma|^a d\sigma)$ for $\omega \subseteq \mathbb{S}^{d-1}$ open, under the assumption

$$\omega \cap \Sigma_0 = \emptyset \qquad \text{if } k + a \le 0. \tag{2.1}$$

Since $|\Pi\sigma|^a \in L^1_{\rm loc}(\omega)$, the space $W^{1,p}_0(\omega; |\Pi\sigma|^a d\sigma)$ is plainly defined, accordingly with $S2)$ in the Introduction.

Lemma 2 *Let ω satisfy (2.1). If $\mathbb{S}^{d-1} \cap \Sigma_0 \not\subset \overline{\omega}$, then there exist $t > 0$ depending only on a, k and p, and $c(\omega) > 0$ such that*

$$\int\limits_{\omega\cap\{|\Pi\sigma|<\varepsilon\}} |\Pi\sigma|^a|\varphi|^p\, d\sigma \le c(\omega)\varepsilon^t \int\limits_{\omega} |\Pi\sigma|^a|\nabla_\sigma\varphi|^p\, d\sigma \tag{2.2}$$

for any $\varphi \in W^{1,p}_0(\omega; |\Pi\sigma|^a d\sigma)$ and any $\varepsilon > 0$.

Proof. We can assume that the south pole $\mathrm{e} = (-1, 0 \dots, 0)$ does not belong to $\overline{\omega}$.

Let $\mathrm{P} : \mathbb{R}^{d-1} \equiv \mathbb{R}^{d-1-k} \times \mathbb{R}^k \to \mathbb{S}^{d-1} \setminus \{\mathrm{e}\} \subset \mathbb{R}^d$ be the inverse of the stereographic projection from e. More explicitly,

$$\mathrm{P}(\xi, y) = (\mu - 1, \mu\xi; \mu y) \in \mathbb{S}^{d-1} \subset (\mathbb{R} \times \mathbb{R}^{d-k-1}) \times \mathbb{R}^k \ , \quad \mu(\xi, y) = \frac{2}{1 + |\xi|^2 + |y|^2}$$

(the variable ξ has to be omitted if $k = d - 1$).

Let $\Omega := \mathrm{P}^{-1}(\omega)$. Then $\Omega \subset \mathbb{R}^{d-1}$ is open and bounded. Moreover, if $k + a \le 0$ then $\Omega \cap \Sigma_0 = \emptyset$ because $\omega \cap \Sigma_0 = \emptyset$ by (2.1).

Next, for $\varphi \in C^\infty_c(\omega)$ we put $\tilde{\varphi} := \varphi \circ \mathrm{P} \in C^\infty_c(\Omega)$. Since $\mu = \mu(\xi, y)$ is bounded and bounded away from 0 on Ω, we have that there exist $c > 1, R > 0$ depending only on ω such that

$$\begin{gathered}
\int\limits_\omega |\Pi\sigma|^a |\nabla_\sigma \varphi|^p \, d\sigma = \int\limits_\Omega \mu^{d+a-p-1} |y|^a |\nabla \tilde{\varphi}|^p \, d\xi dy \ge c^{-1} \int\limits_\Omega |y|^a |\nabla \tilde{\varphi}|^p \, d\xi dy \\
\int\limits_{\omega \cap \{|\Pi\sigma| < \varepsilon\}} |\Pi\sigma|^a |\varphi|^p d\sigma \le \int\limits_{\Omega \cap \{|y| < R\varepsilon\}} \mu^{d+a-1} |y|^a |\tilde{\varphi}|^p d\xi dy \le c \int\limits_{\Omega \cap \{|y| < R\varepsilon\}} |y|^a |\tilde{\varphi}|^p d\xi dy.
\end{gathered} \tag{2.3}$$

The conclusion of the proof follows via the Hardy-Maz'ya inequalities in [9, Chapter 2].

Let $k + a > 0$. We fix any $0 < t < \min\{k + a, p\}$ and use

$$\int\limits_{\mathbb{R}^{d-1}} |y|^{a-t} |\tilde{\varphi}|^p \, d\xi dy \le \left(\tfrac{p}{k+a-t}\right)^p \int\limits_{\mathbb{R}^{d-1}} |y|^{a+p-t} |\nabla \tilde{\varphi}|^p \, d\xi dy \, ,$$

to estimate

$$\int\limits_{\Omega \cap \{|y| < R\varepsilon\}} |y|^a |\tilde{\varphi}|^p \, d\xi dy \le c\varepsilon^t \int\limits_{\mathbb{R}^{d-1}} |y|^{a-t} |\tilde{\varphi}|^p \, d\xi dy \le c\varepsilon^t \int\limits_{\mathbb{R}^{d-1}} |y|^{a+p-t} |\nabla \tilde{\varphi}|^p \, d\xi dy \le c\varepsilon^t \int\limits_\Omega |y|^a |\nabla \tilde{\varphi}|^p \, d\xi dy \, .$$

Thus (2.2) is proved in this case, thanks to (2.3).

If $k + a \le 0$ we use

$$\int\limits_{\mathbb{R}^{d-1}} |y|^{a-p} |\tilde{\varphi}|^p \, d\xi dy \le \left(\tfrac{p}{p-(k+a)}\right)^p \int\limits_{\mathbb{R}^{d-1}} |y|^a |\nabla \tilde{\varphi}|^p \, d\xi dy$$

to get

$$\int\limits_{\Omega \cap \{|y| < R\varepsilon\}} |y|^a |\tilde{\varphi}|^p \, d\xi dy \le c\varepsilon^p \int\limits_{\mathbb{R}^{d-1}} |y|^{a-p} |\tilde{\varphi}|^p \, d\xi dy \le c\varepsilon^p \int\limits_{\mathbb{R}^{d-1}} |y|^a |\nabla \tilde{\varphi}|^p \, d\xi dy,$$

and (2.2) again follows. The proof is complete. □

Lemma 3 *Let ω satisfy (2.1). If $\mathbb{S}^{d-1} \cap \Sigma_0 \not\subset \overline{\omega}$, then*

$$\int_\omega |\Pi\sigma|^a |\varphi|^p \, d\sigma \le c \int_\omega |\Pi\sigma|^a |\nabla_\sigma \varphi|^p \, d\sigma \quad \textit{for any} \quad \varphi \in W_0^{1,p}(\omega; |\Pi\sigma|^a d\sigma) \tag{2.4}$$

and the embedding $W_0^{1,p}(\omega; |\Pi\sigma|^a d\sigma) \hookrightarrow L^p(\omega; |\Pi\sigma|^a d\sigma)$ is compact.

Proof. Trivially, inequality (2.4) follows from (2.2), by choosing $\varepsilon = 1$.

The embedding operator $W_0^{1,p}(\omega; |\Pi\sigma|^a d\sigma) \hookrightarrow L^p(\omega \cap \{|\Pi\sigma| > \varepsilon\}; |\Pi\sigma|^a d\sigma)$ is compact by the Rellich theorem. Lemma 2 shows that the operator $W_0^{1,p}(\omega; |\Pi\sigma|^a d\sigma) \hookrightarrow L^p(\omega; |\Pi\sigma|^a d\sigma)$ can be approximated in norm by compact operators. Thus it is compact itself. □

Corollary 3 *i) The embedding $W_0^{1,p}(\mathbb{S}^{d-1} \setminus \Sigma_0; |\Pi\sigma|^a d\sigma) \hookrightarrow L^p(\mathbb{S}^{d-1}; |\Pi\sigma|^a d\sigma)$ is compact;*

ii) If $k + a > 0$, the embedding $W^{1,p}(\mathbb{S}^{d-1}; |\Pi\sigma|^a d\sigma) \hookrightarrow L^p(\mathbb{S}^{d-1}; |\Pi\sigma|^a d\sigma)$ is compact.

Proof. Take a point $\mathrm{e} \in \mathbb{S}^{d-1} \cap \Sigma_0$ and a cut-off function $\eta \in C_c^\infty(\mathbb{S}^{d-1} \setminus \{\mathrm{e}\})$ such that $\eta \equiv 1$ in a neighborhood of $-\mathrm{e}$. The operators

$$\varphi \mapsto \eta\varphi \, , \quad \varphi \mapsto (1-\eta)\varphi \, , \qquad W_0^{1,p}(\mathbb{S}^{d-1} \setminus \Sigma_0; |\Pi\sigma|^a d\sigma) \to L^p(\mathbb{S}^{d-1}; |\Pi\sigma|^a d\sigma)$$

are compact by Lemma 3, which proves *i*). For *ii*) repeat the same argument. □

3 Proofs

Proof of Theorem 1. Notice that $|\Pi\sigma|^a \in L^1(\mathbb{S}^{d-1})$, see Lemma 1. Let $u \in C_c^\infty(\mathbb{R}^d)$. If $d+a \le p+b$ assume in addition that $u \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$.

We use the classical Hardy inequality for functions of one variable, which holds with a sharp and not achieved constant, see [7, Theorem 330], to estimate

$$\int_0^\infty r^{d+a-b-1} |(\partial_r u)(r\sigma)|^p \, dr \ge |\mathrm{H}_{p,a,b}|^p \int_0^\infty r^{d+a-b-p-1} |u(r\sigma)|^p \, dr$$

for any $\sigma \in \mathbb{S}^{d-1}$. It follows that

$$\int\limits_{\mathbb{R}^d} |y|^a |z|^{-b} |\partial_r u|^p \, dz = \int\limits_{\mathbb{S}^{d-1}} |\Pi\sigma|^a \, d\sigma \int\limits_0^\infty r^{d+a-b-1} |(\partial_r u)(r\sigma)|^p \, dr$$

$$\geq |\mathrm{H}_{p,a,b}|^p \int\limits_{\mathbb{S}^{d-1}} |\Pi\sigma|^a \, d\sigma \int\limits_0^\infty r^{d+a-b-p-1} |u(r\sigma)|^p \, dr = |\mathrm{H}_{p,a,b}|^p \int\limits_{\mathbb{R}^d} |y|^a |z|^{-b-p} |u|^p \, dz,$$

which concludes the proof. □

Proof of Theorem 2. Clearly $m_{p,a,b}(\mathcal{C} \setminus \Sigma_0) \geq m_{p,a,b}(\mathcal{C})$. To prove the opposite inequality fix any nontrivial function $u \in C^\infty_c(\mathcal{C})$. Our aim is to suitably approximate u by a sequence of functions in $C^\infty_c(\mathcal{C} \setminus \Sigma_0)$. We take a function $\eta \in C^\infty_c(\mathbb{R})$ such that $\eta \equiv 1$ on $[0, 1]$, $\eta \equiv 0$ on $[2, \infty)$ and $\|\eta\|_\infty \leq 1$. For any integer $h \geq 1$ we put

$$u_h(x, y) = \eta\Big(\frac{-\log |y|}{h}\Big) u(x, y).$$

Then $u_h \in C^\infty(\mathbb{R}^k \setminus \Sigma_0)$, $u_h \to u$ pointwise and

$$u_h(x, y) = 0 \ \text{ if } |y| \leq e^{-2h} \ , \qquad u_h(x, y) = u(x, y) \ \text{ if } |y| \geq e^{-h}.$$

We claim that

$$\begin{aligned} \int\limits_{\mathcal{C}} |y|^a |z|^{-b-p} |u_h|^p \, dz &= \int\limits_{\mathcal{C}} |y|^a |z|^{-b-p} |u|^p \, dz + o_h(1) \\ \int\limits_{\mathcal{C}} |y|^a |z|^{-b} |\nabla u_h|^p \, dz &= \int\limits_{\mathcal{C}} |y|^a |z|^{-b} |\nabla u|^p \, dz + o_h(1) \end{aligned} \tag{3.1}$$

as $h \to \infty$. The first limit in (3.1) plainly follows by Lebesgue's theorem. To prove the second one, it suffices to show that

$$I_h := \int\limits_{S_h} |y|^a |z|^{-b} |\nabla u_h|^p \, dz = o_h(1) \ , \qquad S_h := \mathbb{R}^{d-k} \times \{e^{-2h} < |y| < e^{-h}\}.$$

Let $\delta \in (0, 1)$ be such that $\mathrm{supp}(u) \subset \{\delta < |z| < \delta^{-1}\}$. Since

$$|\nabla u_h(x, y)| \leq \|\nabla \eta\|_\infty \frac{|u(x, y)|}{h|y|} + |\nabla u(x, y)| \quad \text{on } S_h,$$

we can estimate

$$I_h \le \frac{c}{h^p} \int\limits_{S_h} |y|^{a-p}|z|^{-b}|u|^p \, dz + o_h(1) \le \frac{c(\delta,u)}{h^p} \int\limits_{e^{-2h}}^{e^{-h}} r^{k+a-p-1} dr + o_h(1) = o_h(1).$$

In conclusion, we have

$$m_{p,a,b}(\mathcal{C}\setminus\Sigma_0) \le \frac{\displaystyle\int\limits_{\mathcal{C}} |y|^a|z|^{-b}|\nabla u_h|^p \, dz}{\displaystyle\int\limits_{\mathcal{C}} |y|^a|z|^{-b-p}|u_h|^p \, dz} = \frac{\displaystyle\int\limits_{\mathcal{C}} |y|^a|z|^{-b}|\nabla u|^p \, dz}{\displaystyle\int\limits_{\mathcal{C}} |y|^a|z|^{-b-p}|u|^p \, dz} + o_h(1).$$

Since $u \in C^\infty_c(\mathcal{C})$ was arbitrarily chosen, the inequality $m_{p,a,b}(\mathcal{C}\setminus\Sigma_0) \le m_{p,a,b}(\mathcal{C})$ follows. The proof of Theorem 2 is complete. □

Remark 2 *The same argument shows that* $W^{1,p}_0(\omega\setminus\Sigma_0; |\Pi\sigma|^a d\sigma) = W^{1,p}_0(\omega; |\Pi\sigma|^a d\sigma)$ *for any open* $\omega \subseteq \mathbb{S}^{d-1}$*, provided that* $k+a \ge p$*.*

Proof of Theorem 3. We have to compare the infima

$$m(\mathcal{C}) = \inf_{\substack{u\in C^\infty_c(\mathcal{C})\\ u\neq 0}} \frac{\displaystyle\int_{\mathcal{C}} |y|^a|z|^{-b}|\nabla u|^p \, dz}{\displaystyle\int_{\mathcal{C}} |y|^a|z|^{-b-p}|u|^p \, dz} \quad , \quad \mathcal{M}(\omega_{\mathcal{C}}) = \inf_{\substack{\varphi\in W^{1,p}_0(\omega_{\mathcal{C}};|\Pi\sigma|^a d\sigma)\\ \varphi\neq 0}} \frac{\displaystyle\int_{\omega_{\mathcal{C}}} |\Pi\sigma|^a(|\nabla_\sigma\varphi|^2 + \mathrm{H}^2|\varphi|^2)^{\frac{p}{2}} \, d\sigma}{\displaystyle\int_{\omega_{\mathcal{C}}} |\Pi\sigma|^a|\varphi|^p \, d\sigma},$$

where $\mathrm{H} := \mathrm{H}_{p,a,b}$ is given by (1.4) (in this proof we omit the indexes p, a, b).

Thanks to Theorem 2, we can assume that the following stronger hypothesis hold:

$$\mathcal{C} \subset \mathbb{R}^d \setminus \Sigma_0 \ , \quad \text{or} \quad 0 < k+a < p. \tag{3.2}$$

In the first case, the weight $|y|^a|z|^{-b}$ is bounded and bounded away from zero on any compact set in $\mathcal{C}$. If $0 < k+a < p$ then $|y|^a$ belongs to the Muckenhoupt class A_p. It follows that the weighted space $W^{1,p}_{\mathrm{loc}}(\mathcal{C}; |y|^a|z|^{-b}dz) \subset L^1_{\mathrm{loc}}(\mathcal{C})$ is well defined (see for instance [8, Subsection 1.9], where the notation $H^{1,p}_{\mathrm{loc}}(\Omega; w(z)dz)$ is used). In addition, thanks to [8, Theorems 3.51, 3.66] we have that any nonnegative and nontrivial function $u \in W^{1,p}_{\mathrm{loc}}(\mathcal{C}; |y|^a|z|^{-b}dz)$ satisfying

$$-\mathrm{div}\big(|y|^a|z|^{-b}\,|\nabla u|^{p-2}\nabla u\big) \ge 0 \qquad \text{in } \mathcal{C} \tag{3.3}$$

is lower semicontinuous and positive in $\mathcal{C}$.

The best constant $\mathcal{M}(\omega_{\mathcal{C}})$ is attained by a function $\Phi \in W^{1,p}_0(\omega_{\mathcal{C}}; |\Pi\sigma|^a d\sigma)$, due to the compactness of embedding $W^{1,p}_0(\omega_{\mathcal{C}}; |\Pi\sigma|^a d\sigma) \hookrightarrow L^p(\omega_{\mathcal{C}}; |\Pi\sigma|^a d\sigma)$ given by Corollary 3. By a standard argument, Φ can not change sign in $\omega_{\mathcal{C}}$. Thus, we can assume that Φ is nonnegative.

We use spherical coordinates to define

$$U(r\sigma) = r^{-\mathrm{H}}\Phi(\sigma) \tag{3.4}$$

for $r > 0$ and $\sigma \in \omega_{\mathcal{C}}$. Since

$$|\nabla U|^2 = (\partial_r U)^2 + r^{-2}|\nabla_\sigma U|^2 = r^{-2(\mathrm{H}+1)}\big(\mathrm{H}^2\Phi^2 + |\nabla_\sigma \Phi|^2\big),$$

we have that $U \in W^{1,p}_{\mathrm{loc}}(\mathcal{C}; |y|^a|z|^{-b}dz)$. Moreover, for any fixed $v \in C^\infty_c(\mathcal{C})$ it holds that

$$\nabla U \cdot \nabla v = \partial_r U \partial_r v + r^{-2}\nabla_\sigma U \cdot \nabla_\sigma v = r^{-(\mathrm{H}+2)}\big(- \mathrm{H}\Phi r \partial_r v + \nabla_\sigma \Phi \cdot \nabla_\sigma v\big)$$

Notice that $d + a - b = p(\mathrm{H} + 1)$, see (1.4). We have

$$\begin{aligned}
&\int\limits_{\mathcal{C}} |y|^a|z|^{-b}|\nabla U|^{p-2}\nabla U \cdot \nabla v\, dz \\
&\qquad = \int\limits_{\omega_{\mathcal{C}}} d\sigma \int\limits_0^\infty |\Pi\sigma|^a \big(\mathrm{H}^2\Phi^2 + |\nabla_\sigma \Phi|^2\big)^{\frac{p-2}{2}} (-\mathrm{H}\Phi\, r^{\mathrm{H}}\, \partial_r v + r^{\mathrm{H}-1}\nabla_\sigma \Phi \cdot \nabla_\sigma v)\, dr \\
&\qquad\qquad = \int\limits_0^\infty r^{\mathrm{H}-1} dr \int\limits_{\omega_{\mathcal{C}}} |\Pi\sigma|^a \big(\mathrm{H}^2\Phi^2 + |\nabla_\sigma \Phi|^2\big)^{\frac{p-2}{2}} (\mathrm{H}^2\Phi v + \nabla_\sigma \Phi \cdot \nabla_\sigma v)\, d\sigma,
\end{aligned}$$

where we used integration by parts and then Fubini's theorem.

Since Φ achieves $\mathcal{M}(\omega_{\mathcal{C}})$, we infer that

$$\begin{aligned}
\int\limits_{\mathcal{C}} |y|^a|z|^{-b}|\nabla U|^{p-2}\nabla U \cdot \nabla v\, dz &= \mathcal{M}(\omega_{\mathcal{C}}) \int\limits_0^\infty r^{\mathrm{H}-1} dr \int\limits_{\omega_{\mathcal{C}}} |\Pi\sigma|^a \Phi^{p-1} v\, d\sigma \\
&= \mathcal{M}(\omega_{\mathcal{C}}) \int\limits_0^\infty r^{p\mathrm{H}-1} dr \int\limits_{\omega_{\mathcal{C}}} |\Pi\sigma|^a U^{p-1} v\, d\sigma = \mathcal{M}(\omega_{\mathcal{C}}) \int\limits_{\mathcal{C}} |y|^a|z|^{-b-p} U^{p-1} v\, dz\,.
\end{aligned}$$

We proved that U is a nonnegative local solution to

$$-\mathrm{div}\big(|y|^a|z|^{-b}\, |\nabla u|^{p-2}\nabla u\big) = \mathcal{M}(\omega_{\mathcal{C}})\ |y|^a|z|^{-b-p}\, |u|^{p-2}u \quad \text{in } \mathcal{C}, \tag{3.5}$$

in the sense that

$$\int_{\mathcal{C}} |y|^a|z|^{-b}|\nabla U|^{p-2}\nabla U \cdot \nabla v \, dz = \mathcal{M}(\omega_{\mathcal{C}}) \int_{\mathcal{C}} |y|^a|z|^{-b-p}U^{p-1}v \, dz \tag{3.6}$$

for any $v \in C^\infty_c(\mathcal{C})$. Since $U \in W^{1,p}_{\mathrm{loc}}(\mathcal{C}; |y|^a|z|^{-b}dz)$, then (3.6) holds for any $v \in \mathcal{D}^{1,p}_0(\mathcal{C}; |y|^a|z|^{-b}dz)$ with compact support.

To prove the inequality $\mathcal{M}(\omega_{\mathcal{C}}) \le m(\mathcal{C})$ fix $v \in C^\infty_c(\mathcal{C})$. Since $U \in W^{1,p}_{\mathrm{loc}}(\mathcal{C}; |y|^a|z|^{-b}dz)$ solves (3.3), then U is bounded away from zero on the support of v. It follows that $U^{1-p}|v|^p \in \mathcal{D}^{1,p}_0(\mathcal{C}; |y|^a|z|^{-b}dz)$, hence it can be used as test function in (3.6). We infer the equality

$$\int_{\mathcal{C}} |y|^a|z|^{-b}|\nabla U|^{p-2}\nabla U \cdot \nabla (U^{1-p}|v|^p) \, dz = \mathcal{M}(\omega_{\mathcal{C}}) \int_{\mathcal{C}} |y|^a|z|^{-b-p}|v|^p \, dz. \tag{3.7}$$

To handle the first integral in (3.7) we notice that

$$\nabla U \cdot \nabla (U^{1-p}|v|^p) = pU^{1-p}(\nabla U \cdot \nabla v)|v|^{p-2}v - (p-1)U^{-p}|\nabla U|^2|v|^p.$$

Thus

$$|\nabla U|^{p-2}\nabla U \cdot \nabla (U^{1-p}|v|^p) \le p\Big(\frac{|\nabla U||v|}{U}\Big)^{p-1}|\nabla v| - (p-1)\Big(\frac{|\nabla U||v|}{U}\Big)^{p} \le |\nabla v|^p$$

by the elementary Young inequality $p|s|^{p-1}|t| \le |t|^p + (p-1)|s|^p$. Thus we conclude that

$$\int_{\mathcal{C}} |y|^a|z|^{-b}|\nabla v|^p \, dz \ge \mathcal{M}(\omega_{\mathcal{C}}) \int_{\mathcal{C}} |y|^a|z|^{-b-p} \, |v|^p \, dz.$$

Since v was arbitrarily chosen in $C^\infty_c(\mathcal{C})$, we proved that $\mathcal{M}(\omega_{\mathcal{C}}) \le m(\mathcal{C})$.

To prove the opposite inequality we adopt a standard strategy. Consider the sequence

$$u_\delta(r, \sigma) = r^{-\mathrm{H} \pm \delta}\Phi(\sigma) \qquad \text{on } \mathcal{C}_\pm,$$

where we have set $\mathcal{C}_+ = \mathcal{C} \cap B_1$, $\mathcal{C}_- = \mathcal{C} \setminus B_1$. Clearly, $u_\delta \in \mathcal{D}^{1,p}_0(\mathcal{C}; |y|^a|z|^{-b}dz)$ and moreover

$$\int_{\mathcal{C}} |y|^a|z|^{-b-p}|u_\delta|^p \, dz = \int_{\mathcal{C}_+} |y|^a|z|^{-b-p}|u_\delta|^p \, dz + \int_{\mathcal{C}_-} |y|^a|z|^{-b-p}|u_\delta|^p \, dz = \frac{2}{p\delta} \int_{\omega_{\mathcal{C}}} |\Pi\sigma|^a|\Phi|^p \, d\sigma. \tag{3.8}$$

It is easy to see that $|\nabla u_\delta| = r^{-\mathrm{H}-1\pm\delta}\Big[(\mathrm{H}\mp\delta)^2\Phi^2 + |\nabla_\sigma\Phi|^2\Big]^{\frac{1}{2}}$ on $\mathcal{C}_\pm$, from which one easily infer

$$\int\limits_{\mathcal{C}} |y|^a|z|^{-b}|\nabla u_\delta|^p\,dz = \frac{1}{p\delta}\int\limits_{\omega_\mathcal{C}} |\Pi\sigma|^a\big\{\big[(\mathrm{H}-\delta)^2\Phi^2 + |\nabla_\sigma\Phi|^2\big]^{\frac{p}{2}} + \big[(\mathrm{H}+\delta)^2\Phi^2 + |\nabla_\sigma\Phi|^2\big]^{\frac{p}{2}}\big\}\,d\sigma$$
$$= \frac{2}{p\delta}\Big(\int\limits_{\omega_\mathcal{C}} |\Pi\sigma|^a\big[\mathrm{H}^2\Phi^2 + |\nabla_\sigma\Phi|^2\big]^{\frac{p}{2}}\,d\sigma + O(\delta^2)\Big)$$

as $\delta \to 0$. Since Φ achieves $\mathcal{M}(\omega_\mathcal{C})$, we infer that

$$\int\limits_{\mathcal{C}} |y|^a|z|^{-b}|\nabla u_\delta|^p\,dz = \frac{2}{p\delta}\Big(\mathcal{M}(\omega_\mathcal{C})\int\limits_{\omega_\mathcal{C}} |\Pi\sigma|^a|\Phi|^p\,d\sigma + O(\delta^2)\Big)\,.$$

Taking into account the definition of $m(\mathcal{C})$ and (3.8), we see that

$$m(\mathcal{C}) \;\le\; \frac{\displaystyle\int_\mathcal{C} |y|^a|z|^{-b}|\nabla u_\delta|^p\,dz}{\displaystyle\int_\mathcal{C} |y|^a|u_\delta|^p\,dz} = \mathcal{M}(\omega_\mathcal{C}) + O(\delta^2),$$

which concludes the proof. □

Proof of Theorem 4. As in the proof of Theorem 3 we omit the indexes p, a, b.

If $m(\mathcal{C}) = 0$ the result is trivial, as nonzero constant functions are not in $\mathcal{D}^{1,p}_0(\mathcal{C}; |y|^a|z|^{-b}dz)$. Thus, let $m(\mathcal{C}) > 0$.

Arguing as in the proof of Theorem 3, we can assume that (3.2) is satisfied. Let $U(r\sigma) = r^{-\mathrm{H}}\Phi(\sigma)$ be the function in (3.4). We already proved that $U \in W^{1,p}_{\mathrm{loc}}(\mathcal{C}; |y|^a|z|^{-b}dz)$ is a lower semicontinuous and positive local solution to

$$-\mathrm{div}\big(|y|^a|z|^{-b}\,|\nabla u|^{p-2}\nabla u\big) = m(\mathcal{C})\;|y|^a|z|^{-b-p}\,|u|^{p-2}u \quad \text{in } \mathcal{C}, \tag{3.9}$$

compare with (3.5) and recall that $m(\mathcal{C}) = \mathcal{M}(\omega_\mathcal{C})$. From (3.9) we infer that U is locally bounded outside Σ_0 by [12], therefore it is of class $C^{1,\alpha}$ on $\mathcal{C} \setminus \Sigma_0$ by [4, 17].

By contradiction, assume that $u \in \mathcal{D}^{1,p}_0(\mathcal{C}; |y|^a|z|^{-b}dz)$ achieves $m(\mathcal{C})$. Up to a change of sign, u is nonnegative and is a weak solution to (3.9). Thus, just as U, the function u is positive on $\mathcal{C}$ and of class $C^{1,\alpha}$ outside the singular set Σ_0.

Take a domain A, compactly contained in $\mathcal{C}\setminus\Sigma_0$. The main step in the proof consists in showing that the open sets

$$A_1=\left\{z\in A\ |\ \ \nabla u\cdot\nabla U\lneq|\nabla u||\nabla U|\right\}\,,\quad A_2=\left\{z\in A\ |\ \ \frac{|\nabla u|}{u}\neq\frac{|\nabla U|}{U}\ \right\}$$

are empty. This easily imply that there exists a constant $\lambda>0$ such that $u=\lambda U$ on A. Since A was arbitrarily chosen, the constant λ does not depend on A. Thus u is proportional to U on $\mathcal{C}\setminus\Sigma_0$, which is impossible as $U\notin\mathcal{D}^{1,p}_0(\mathcal{C};|y|^a|z|^{-b}dz)$.

To show that the sets A_1,A_2 are empty we refine the calculations in the proof of Theorem 3. Take a sequence of nonnegative functions $u_h\in C^\infty_c(\mathcal{C})$ such that $u_h\to u$ in $\mathcal{D}^{1,p}_0(\mathcal{C};|y|^a|z|^{-b}dz)$. We have

$$\begin{aligned}m(\mathcal{C})\int\limits_{\mathcal{C}}|y|^a|z|^{-b-p}u^p\,dz&=m(\mathcal{C})\int\limits_{\mathcal{C}}|y|^a|z|^{-b-p}\,u_h^p\,dz+o_h(1)\\&\le\int\limits_{\mathcal{C}}|y|^a|z|^{-b}|\nabla U|^{p-2}\Big(p\,\frac{\nabla U\cdot\nabla u_h}{U^{p-1}}u_h^{p-1}-(p-1)\frac{|\nabla U|^2u_h^p}{U^p}\Big)dz+o_h(1)\,.\end{aligned}\tag{3.10}$$

Now, as $h\to\infty$ we have

$$\begin{aligned}\nu_1:=p\int\limits_{A_1}|y|^a|z|^{-b}|\nabla U|^{p-2}\Big(\frac{|\nabla U|\,|\nabla u|}{U^{p-1}}-\frac{\nabla U\cdot\nabla u}{U^{p-1}}\Big)\,u^{p-1}\,dz\\=p\int\limits_{A_1}|y|^a|z|^{-b}|\nabla U|^{p-2}\frac{|\nabla U|\,|\nabla u_h|-\nabla U\cdot\nabla u_h}{U^{p-1}}\,u_h^{p-1}\,dz+o_h(1)\,.\end{aligned}$$

Notice that the nonnegative constant ν_1 is positive if and only if A_1 has positive measure. From (3.10) it follows that

$$\begin{aligned}&m(\mathcal{C})\int\limits_{\mathcal{C}}|y|^a|z|^{-b-p}u^p\,dz\\&\qquad\le\int\limits_{\mathcal{C}}|y|^a|z|^{-b}\Big[p\Big(\frac{|\nabla U|u_h}{U}\Big)^{p-1}|\nabla u_h|-(p-1)\Big(\frac{|\nabla U|u_h}{U}\Big)^p\Big]dz-\nu_1+o_h(1).\end{aligned}$$

Next, as $h\to\infty$ we have

$$\begin{aligned}\nu_2:=\int\limits_{A_2}|y|^a|z|^{-b}\Big(|\nabla u|^p+(p-1)\Big(\frac{|\nabla U|u}{U}\Big)^p-p\Big(\frac{|\nabla U|u}{U}\Big)^{p-1}|\nabla u|\Big)dz\\=\int\limits_{A_2}|y|^a|z|^{-b}\Big(|\nabla u_h|^p+(p-1)\Big(\frac{|\nabla U|u_h}{U}\Big)^p-p\Big(\frac{|\nabla U|u_h}{U}\Big)^{p-1}|\nabla u_h|\Big)dz+o_h(1)\,.\end{aligned}$$

Thanks to Young's inequality, we have that $\nu_2 > 0$ if and only if A_2 has positive measure.

We proved that

$$m(\mathcal{C}) \int\limits_{\mathcal{C}} |y|^a |z|^{-b-p}\, u^p \, dz \le \int\limits_{\mathcal{C}} |y|^a |z|^{-b} |\nabla u_h|^p \, dz - (\nu_1 + \nu_2) + o_h(1),$$

and letting $h \to \infty$ we infer that

$$\begin{aligned} m(\mathcal{C}) \int\limits_{\mathcal{C}} |y|^a |z|^{-b-p}\, u^p \, dz &\le \int\limits_{\mathcal{C}} |y|^a |z|^{-b} |\nabla u|^p \, dz - (\nu_1 + \nu_2) \\ &= m(\mathcal{C}) \int\limits_{\mathcal{C}} |y|^a |z|^{-b-p}\, u^p \, dz - (\nu_1 + \nu_2)\,, \end{aligned}$$

as u achieves $m(\mathcal{C})$. This implies that $\nu_1 = \nu_2 = 0$, which is equivalent to say that the sets A_1, A_2 are empty, as claimed. □

Proof of Theorem 5. We can find an open geodesic ball $\mathcal{B}_\theta \subset \mathbb{S}^{d-1}$ such that $\overline{\mathcal{B}_\theta} \cap \Sigma_0 = \emptyset$ and the set $\mathcal{B}_\theta \cap \partial\omega_{\mathcal{C}}$ is not p-negligible in the sphere.

By [9, Theorem 14.1.2] and since $|\Pi\sigma|$ is bounded and bounded away from 0 on $\mathcal{B}_\theta$, we have that

$$\int\limits_{B_\theta} |\varphi|^p \, d\sigma \le c(\theta, a) \int\limits_{B_\theta} |\Pi\sigma|^a |\nabla_\sigma \varphi|^p \, d\sigma$$

for any $\varphi \in C^\infty_0(\omega_{\mathcal{C}}; |\Pi\sigma|^a d\sigma)$.

By density, we infer that $W^{1,p}_0(\omega_{\mathcal{C}}; |\Pi\sigma|^a d\sigma)$ does not contain constant functions, which is enough, thanks to Theorem 3 and since the best constant $\mathcal{M}_{p,a,b}(\omega_{\mathcal{C}})$ is attained in $W^{1,p}_0(\omega_{\mathcal{C}}; |\Pi\sigma|^a d\sigma)$. □

Proof of Theorem 6. The last claim in Theorem 3 gives $m_{2,a,b}(\mathbb{R}^d \setminus \Sigma_0) = \mathrm{H}^2_{2,a,b}$ if $k + a \ge 2$. Thus we can assume $k + a < 2$. In view of Theorem 3, we have

$$m_{2,a,b}(\mathbb{R}^d \setminus \Sigma_0) = \mathrm{H}^2_{2,a,b} \; + \min_{\substack{\varphi \in W^{1,2}_0(\mathbb{S}^{d-1} \setminus \Sigma_0; |\Pi\sigma|^a d\sigma) \\ \varphi \ne 0}} \frac{\displaystyle\int_{\mathbb{S}^{d-1}} |\Pi\sigma|^a |\nabla_\sigma \varphi|^2 \, d\sigma}{\displaystyle\int_{\mathbb{S}^{d-1}} |\Pi\sigma|^a |\varphi|^2 \, d\sigma}\,. \tag{3.11}$$

Since the embedding $W_0^{1,2}(\mathbb{S}^{d-1} \setminus \Sigma_0; |\Pi\sigma|^a d\sigma) \to L^2(\mathbb{S}^{d-1}; |\Pi\sigma|^a d\sigma)$ is compact, see Corollary 3, the minimum in (3.11) is the first eigenvalue λ_1 of the problem

$$\begin{cases} -\mathrm{div}_\sigma(|\Pi\sigma|^a \nabla_\sigma \varphi) = \lambda |\Pi\sigma|^a \varphi & \text{in } \mathbb{S}^{d-1} \setminus \Sigma_0 \\ \varphi \in W_0^{1,2}(\mathbb{S}^{d-1} \setminus \Sigma_0; |\Pi\sigma|^a d\sigma). \end{cases} \tag{3.12}$$

By known facts, λ_1 is simple and the corresponding eigenfunction φ_1 is the only nonnegative one. By direct computations based on the remarks above, one can check that

$$\varphi_1(\sigma) = |\Pi\sigma|^{2-(k+a)}\ , \qquad \lambda_1 = (d-k)(2-(k+a)).$$

In fact, if we write $|y| = r\cos(\theta)$, $|x| = r\sin(\theta)$, $0 < \theta < \frac{\pi}{2}$, then for φ depending only on θ, problem (3.12) is rewritten as follows:

$$-(\cos^{k+a-1}(\theta)\sin^{d-k-1}(\theta)\varphi_\theta)_\theta = \lambda \cos^a(\theta)\sin^{d-k-1}(\theta)\varphi. \tag{3.13}$$

So, it is evident that $\varphi_1 = \cos^{2-a-k}(\theta)$ satisfies (3.13) with $\lambda = \lambda_1$, which concludes the proof. □

References

[1] L. A. Caffarelli and L. E. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), no. 7-9, 1245–1260.

[2] G. R. David, J. Feneuil and S. Mayboroda, *Elliptic theory for sets with higher co-dimensional boundaries*, Mem. Amer. Math. Soc. **274** (2021), no. 1346, vi+123 pp.

[3] G. R. David and S. Mayboroda, *Approximation of Green functions and domains with uniformly rectifiable boundaries of all dimensions*, Adv. Math. **410** (2022), part A, Paper No. 108717, 52 pp.

[4] E. DiBenedetto, *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. **7** (1983), no. 8, 827–850.

[5] E. B. Fabes, C. E. Kenig and R. P. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations **7** (1982), no. 1, 77–116.

[6] M. M. Fall and V. Felli, *Unique continuation property and local asymptotics of solutions to fractional elliptic equations*, Comm. Partial Differential Equations **39** (2014), no. 2, 354–397.

[7] G. H. Hardy, J. E. Littlewood and G. Pólya, "Inequalities", Cambridge, at the University Press, 1952.

[8] J. Heinonen, T. Kilpeläinen and O. Martio, "Nonlinear potential theory of degenerate elliptic equations", Dover Publications, Inc., Mineola, NY, 2006.

[9] V. G. Maz'ya, "Sobolev spaces with applications to elliptic partial differential equations", second, revised and augmented edition, Grundlehren der mathematischen Wissenschaften, 342, Springer, Heidelberg, 2011.

[10] R. Musina and A. I. Nazarov, *On fractional Laplacians—2*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **33** (2016), no. 6, 1667–1673.

[11] A. I. Nazarov, *Dirichlet and Neumann problems to critical Emden-Fowler type equations*, J. Global Optim. **40** (2008), no. 1-3, 289–303.

[12] J. B. Serrin Jr., *Local behavior of solutions of quasi-linear equations*, Acta Math. **111** (1964), 247–302.

[13] Y. Sire, S. Terracini and S. Vita, *Liouville type theorems and regularity of solutions to degenerate or singular problems part I: even solutions*, Comm. Partial Differential Equations **46** (2021), no. 2, 310–361.

[14] Y. Sire, S. Terracini and S. Vita, *Liouville type theorems and regularity of solutions to degenerate or singular problems part II: odd solutions*, Math. Eng. **3** (2021), no. 1, Paper No. 5, 50 pp.

[15] S. Terracini, G. Tortone and S. Vita, *On s-harmonic functions on cones*, Anal. PDE **11** (2018), no. 7, 1653–1691.

[16] S. Terracini, G. Tortone and S. Vita, *Higher order boundary Harnack principle via degenerate equations*, preprint arXiv:2301.00227, https://doi.org/10.48550/arXiv.2301.00227.

[17] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations **51** (1984), no. 1, 126–150.

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