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A Distinguished Subgroup of Compact Abelian Groups

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Abstract: Here "group" means additive abelian group. A compact group G contains δ–subgroups, that is, compact totally disconnected subgroups Δ such that G/Δ is a torus. The canonical subgroup $\mathbf{\Delta}(G)$ of G that is the sum of all δ–subgroups of G turns out to have striking properties. Lewis, Loth and Mader obtained a comprehensive description of $\mathbf{\Delta}(G)$ when considering only finite dimensional connected groups, but even for these, new and improved results are obtained here. For a compact group G, we prove the following: $\mathbf{\Delta}(G)$ contains $\mathrm{tor}(G)$, is a dense, zero-dimensional subgroup of G containing every closed totally disconnected subgroup of G, and $G/\mathbf{\Delta}(G)$ is torsion-free and divisible; $\mathbf{\Delta}(G)$ is a functorial subgroup of G, it determines G up to topological isomorphism, and it leads to a "canonical" resolution theorem for G. The subgroup $\mathbf{\Delta}(G)$ appeared before in the literature as $\mathrm{td}(G)$ motivated by completely different considerations. We survey and extend earlier results. It is shown that td, as a functor, preserves proper exactness of short sequences of compact groups.

Keywords: full free subgroup; (locally) compact abelian group; Pontryagin Duality; totally disconnected; 0-dimensional; precompact; functorial subgroup; quasi-torsion element; minimal group; totally minimal group; exotic torus

MSC: 20K15; 22K45; 22C05

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1. Introduction

The topological groups studied in this paper are mainly the Pontryagin duals of discrete abelian groups with some emphasis on the duals of torsion-free groups. The latter are exactly the compact connected abelian groups. Non-compact topological groups prominently appear in Section 7.

The result ([1], Proposition 8.15, p. 416) deals with the existence of compact totally disconnected subgroups Δ of a compact group G such that G/Δ is a torus. These δ–***subgroups*** enter into the Resolution Theorem for compact abelian groups ([1], Theorem 8.20, p. 420, see also Section 6). The duals of the short exact sequences $\Delta \rightarrowtail G \twoheadrightarrow T$ where G is a compact group, Δ is a δ–subgroup of G and thus T is a torus, are precisely the exact sequences $F \rightarrowtail A \twoheadrightarrow D$ where A is a discrete group, F is a free subgroup of A and D is a torsion group. This suggests the study of the ***full free subgroups*** F of A, i.e., the free subgroups of A with torsion quotient. Let $\mathcal{F}(A)$ denote the family of all full free subgroups of A and let $\mathcal{D}(G)$ denote the family of all δ–subgroups of the compact group G. In Theorem 1, a comprehensive description of $\mathcal{F}(A)$ is established, and by duality a similarly comprehensive description of $\mathcal{D}(G)$ is obtained (Theorem 6). In fact, there is an anti-isomorphism of semi-lattices $\delta : \mathcal{F}(A) \to \mathcal{D}(G)$ where $G = A^\wedge$ (Theorem 5).

The canonical subgroup $\mathbf{\Delta}(G) := \sum \mathcal{D}(G)$ of G, referred to as "Fat Delta", has interesting properties:

(FD1) It contains $\mathrm{tor}(G)$, is dense in G, and $G/\mathbf{\Delta}(G)$ is torsion-free and divisible (Theorem 6(2),(4),(6) and Theorem 10(2)).
(FD2) If G is not totally disconnected, then $\mathbf{\Delta}(G)$ is a proper subgroup of G, and hence is not locally compact (Proposition 6(1)).
(FD3) $\mathbf{\Delta}(G)$ is zero-dimensional (Theorem 19), and contains every closed totally disconnected subgroup of G (Proposition 5).
(FD4) Fat Delta is a functorial subgroup in the sense that for any morphism $f : G \to H$ we have $f[\mathbf{\Delta}(G)] \subseteq \mathbf{\Delta}(H)$ (Corollary 3, Proposition 10(1)), moreover $f[\mathbf{\Delta}(G)] = \mathbf{\Delta}(H)$ if f is surjective (Proposition 10(2)).
(FD5) The Fat Delta of a product is the product of the Fat Deltas of the factors (Theorem 10(4), Proposition 10(4)).
(FD6) If $G = A^\wedge$ is a compact group, then $\mathbf{\Delta}(G) = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ (see Theorem 10(1) for a more rigorous formulation).
(FD7) $\mathbf{\Delta}(G)$ determines G up to topological isomorphism (Theorem 12).

The group $\mathbf{\Delta}(G)$ coincides with $\mathrm{tor}(G)$ if and only if $G = T \times E$ with T a finite dimensional torus and E a bounded group (Theorem 9). We obtain a "canonical" resolution theorem (Theorem 15) for a compact abelian group G where the canonical $\mathbf{\Delta}(G)$ replaces a random δ–subgroup.

In [2], the case of connected compact groups of finite dimension was studied; here we generalize to arbitrary compact abelian groups of any dimension, but even in the case of finite dimension, our results on Fat Delta surpass by far those in [2].

Furthermore, Fat Delta, defined differently, in greater generality, and called $\mathrm{td}(G)$, previously appeared in the literature ([3], pp. 127–128, [4]). We quote, elaborate, and extend results from earlier works as follows.

In Section 7, we provide a different 'projective' characterization of $\mathrm{td}(G)$ (see Proposition 9(1)) and various applications of $\mathbf{\Delta}(G) = \mathrm{td}(G)$. It is proved that td, as a functor, preserves proper exactness of short sequences of compact groups (Corollary 4). The interest in the subgroup $\mathrm{td}(G)$ of compact groups (see Definition 4) was triggered by the intensive research on the Open Mapping Theorem since the early seventies of the last century [3–15] (see Definition 5 for the relevant properties and Theorem 17 for criteria for the inheritance of these properties from dense subgroups). Section 7.3 is focused on the topological p-Sylow subgroups $\mathrm{td}_p(G)$ of $\mathrm{td}(G)$.

The characterization (FD6) of Fat Delta for compact groups first appeared in ([6], (2), p. 217) and ([3], Proposition 4.1.4).

In Section 8, we discuss some open problems.

In a forthcoming paper [16], we extend the characterization (FD6) to larger classes of topological abelian groups (e.g., subgroups of LCA groups). To this end, we introduce there a new series of functorial subgroups in TAG, related to $\mathrm{td}(G)$ and $\mathrm{td}_p(G)$, and consider alternative definitions of Fat Delta for non compact groups.

2. Notation and Background

Our reference on abelian groups is [17]. As a rule $A, B, C, D, E, \ldots$ denote discrete groups and $G, H, K, L, \ldots$ are used to denote topological groups. Unless otherwise stated, p is an arbitrary prime number. If $\mathcal{C}$ is a category of groups, then "A is a $\mathcal{C}$–group" and "$A \in \mathcal{C}$" means that A is an object of $\mathcal{C}$. By $A \leq B$ we mean that A is a sub-object of B when $A, B \in \mathcal{C}$. We will deal with the following categories:

- The category AG of discrete abelian groups with morphisms algebraic homomorphisms, $\cong$ denoting isomorphism in this category, also called algebraic isomorphism;
- TAG is the category of topological abelian groups with morphisms continuous algebraic homomorphisms, $\cong_t$ denoting isomorphism in this category;
- LCA is as usual the full subcategory of TAG consisting of locally compact Hausdorff groups.

We will use $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ while $\mathbb{P}$ denotes the set of all prime numbers. Furthermore, $\mathbb{R}$ denotes the additive group of real numbers, $\mathbb{Z}$ the integers and

$\mathbb{T}$ the additively written circle group $\mathbb{R}/\mathbb{Z}$ equipped with the compact quotient topology. A ***torus*** is a topological group isomorphic with a power $\mathbb{T}^{\mathfrak{m}}$ where $\mathfrak{m}$ is any cardinal.

The torsion subgroup (p-torsion subgroup) of an abelian group G is denoted by $\mathrm{tor}(G)$ ($\mathrm{tor}_p(G)$, respectively). We have $\mathrm{tor}(\mathbb{T}) = \mathbb{Q}/\mathbb{Z} \leq \mathbb{T}$ with the subspace topology, and $\mathrm{tor}_p(\mathbb{T}) \leq \mathbb{Q}/\mathbb{Z}$ with the subspace topology. We use $\mathbb{Z}(p^\infty) := \mathrm{tor}_p(\mathbb{T}) = \{m/p^n + \mathbb{Z} \mid m, n \in \mathbb{N}_0\}$ in agreement with ([1], p. 27).

The m–socle of a group X is $X[m] := \{x \in X \mid mx = 0\}$ and the socle of X is $\mathrm{Soc}(X) = \bigoplus_{p\in\mathbb{P}} X[p]$. By μ_m^X we denote multiplication by m in X. For a subgroup, Y of X and $m \in \mathbb{N}$, define

$$m_X^{-1}Y = \{x \in X \mid mx \in Y\}, \text{ equivalently, } m_X^{-1}Y/Y = (X/Y)[m].$$

This concept is used to construct larger full free subgroups from given full free subgroups.

In the following discussion of divisible hulls, $\mathbb{Z}(p^\infty)$ is the discrete quasi-cyclic group ([17], p. 16).

The group D is a ***divisible hull*** of A if D is divisible and A is an essential subgroup of D, equivalently, if D/A is a torsion group and $\bigoplus_{p\in\mathbb{P}} D[p] \subseteq A$. Divisible hulls exist for any group and divisible groups are direct sums of copies of $\mathbb{Q}$ and of $\mathbb{Z}(p^\infty)$, $p \in \mathbb{P}$ ([17], p. 136).

The $\mathbb{Z}$–adic topology of $\mathbb{Z}$ (having as a local base at 0 the filter base $\{n\mathbb{Z} : n \in \mathbb{N}\}$) will be denoted by $\nu_{\mathbb{Z}}$. We denote by $G^\wedge$ the Pontryagin dual of a TAG–group G, while $\widehat{G}$ is reserved for the completion of G. In particular, $\widehat{\mathbb{Z}}$ is the completion of $(\mathbb{Z}, \nu_{\mathbb{Z}})$ and $\widehat{\mathbb{Z}}_p$ is the completion of $\mathbb{Z}$ in the p-adic topology.

For topological groups G, H we will deal with $\mathrm{cHom}(G, H)$, the set of all continuous homomorphisms from G to H. Throughout, we assume that the groups of morphisms $\mathrm{cHom}(G, H)$ carry the compact-open topology. We will use the notation of ([1], p. 337), so recall that the sets $W(C, U) = \{f \in \mathrm{cHom}(G, H) \mid f[C] \subseteq U\}$ where C is compact in G and U is open in H, form a basis for the topology of $\mathrm{cHom}(G, H)$.

By $\mathrm{c}(G)$ we denote the 0–component of G and by $\mathrm{a}(G)$ the arc component of $0 \in G$. A Hausdorff topological group G is ***zero-dimensional*** if G has a base of clopen sets. Clearly, every linearly topologized group is zero-dimensional and every zero-dimensional group is totally disconnected. Recall that a group is linearly topologized if it possesses a neighborhood basis at 0 consisting of subgroups.

Lemma 1 ([1], E8.6, p. 414)**.** *Let G be a locally compact abelian group. Then G is totally disconnected if and only if G is zero-dimensional.*

A topological abelian group G is said to be ***precompact*** if its completion is compact. It is a well-known and deep fact that a topological abelian group G is precompact if and only if the topology of G is generated by its continuous characters, which means that the characters $\chi \in G^\wedge$ separate the points of G and the injective (continuous) diagonal map $G \to \prod_{\chi \in G^\wedge} \chi[G] \leq \mathbb{T}^{G^\wedge}$ is an embedding ([3]).

Proposition 1. *Let G be a topological abelian group and let G_i, $i \in I$, be a family of topological groups. Then*

$$\mathrm{cHom}(G, \textstyle\prod_{i\in I} G_i) \cong_{\mathrm{t}} \textstyle\prod_{i\in I} \mathrm{cHom}(G, G_i).$$

Proof. Let $\pi_j : \prod_{i\in I} G_i \to G_j$ be the projections. Then

$$\pi : \mathrm{cHom}(G, \textstyle\prod_{i\in I} G_i) \to \textstyle\prod_{i\in I} \mathrm{cHom}(G, G_i) : \pi(f) = (\ldots, \pi_i \circ f, \ldots).$$

is the restriction of the well-known algebraic isomorphism. Evidently, $\pi(f) = (\ldots, \pi_i \circ f, \ldots) \in \prod_{i\in I} \mathrm{cHom}(G, G_i)$, so π is well-defined.

To show that π is continuous, consider the generic open neighborhood $V = \prod_{i\in I} V_i$, $V_i = W(C, U_i)$, of $0 \in \prod_{i\in I} \mathrm{cHom}(G, G_i)$ where J is a finite subset of I, $C \subseteq G$ is compact, U_j, $j \in J$, is an open neighborhood of $0 \in G_j$, $\forall\, i \notin J : U_i = G_i$. Then $W := W(C, \prod_{i\in I} V_i)$ is an open neighborhood of $0 \in \prod_{i\in I} \mathrm{cHom}(G, \prod_{i\in I} G_i)$ and $\pi[W] \subseteq V$.

To show that π is open, we consider a basic open subset U of $\mathrm{Hom}(A, \prod_{i\in I} G_i)$, i.e., $U = W(C, \prod_i U_i)$ where C is compact in G and $\prod_i U_i$ is open in $\prod_{i\in I} G_i$, i.e., there is a finite subset J of I such that $\forall\, i \in J : U_i$ is open in G_i and $\forall\, i \notin J : U_i = G_i$. Then $\forall\, i \in J : W(C, U_i)$ is open in $\mathrm{Hom}(G, G_i)$ and $\forall\, i \notin J : W(C, U_i) = W(C, G_i) = G_i$. Hence, $\prod_{i\in I} W(C, U_i)$ is open in $\prod_{i\in I} \mathrm{Hom}(G, G_i)$ and it is easily checked that $\pi[W(C, \prod_i U_i)] = \prod_{i\in I} W(C, U_i)$ showing that π is an open map. □

Let A be a discrete group and G any topological group. Then, the compact open topology on $\mathrm{Hom}(A, G)$ coincides with the subspace topology of $\mathrm{Hom}(A, G) \subseteq G^A$ where G^A carries the product topology (=topology of point-wise convergence). This is well-known and is easily seen noting that the compact subsets of A are exactly the finite subsets.

Let G and H be topological groups. Recall ([18], p. 1) that $\alpha \in \mathrm{cHom}(G, H)$ is ***proper*** if α is open onto its range. A short exact sequence $K \rightarrowtail G \twoheadrightarrow H$ is ***proper*** if both maps are proper. Embeddings of subgroups are examples of proper monomorphisms, and proper epimorphisms are quotient maps. For a subgroup H of an abelian group G, we denote by $H \overset{\mathrm{ins}}{\rightarrowtail} G$ the inclusion homomorphism, a proper map.

In ([1], Proposition 1.17, p. 12) Proposition 2 is proved for $G = \mathbb{T}$ in which case $\prod_{i\in I} \mathrm{Hom}(A_i, G)$ is compact and it is easy to show that Φ is a quotient map.

Proposition 2. *Let A_i, $i \in I$, be a family of discrete abelian groups, G a topological abelian group. Then*

$$\Phi : \prod_{i\in I} \mathrm{Hom}(A_i, G) \overset{\rightarrow}{\cong}_{\mathrm{t}} \mathrm{Hom}\left(\bigoplus_{i\in I} A_i, G\right) : (\Phi((\ldots, f_i, \ldots)))\left(\sum_{i\in I} a_i\right) = \sum_{i\in I} f_i(a_i).$$

Proof. Let $\mathrm{ins}_i : A_i \to \bigoplus_{i\in I} A_i$ be the insertions belonging to the direct sum. The map Φ is the standard algebraic isomorphism and

$$\Phi^{-1} : \mathrm{Hom}(\textstyle\bigoplus_{i\in I} A_i, G) \to \prod_{i\in I} \mathrm{Hom}(A_i, G) : \Phi^{-1}(f) = (\ldots, f \circ \mathrm{ins}_i, \ldots).$$

We first show that Φ^{-1} is continuous. By definition of the product topology, Φ^{-1} is continuous if and only if $\pi_i \circ \Phi^{-1} : \mathrm{Hom}(\bigoplus_{i\in I} A_i, G) \to \mathrm{Hom}(A_i, G)$ is continuous where $\pi_i : \prod_{j\in I} \mathrm{Hom}(A_j, G) \to \mathrm{Hom}(A_i, G)$ is the projection belonging to the product. Let U be an open neighborhood of $0 \in G$ and let F be a finite subset of A_i. Then $W := W(F, U)$ is a generic neighborhood of $0 \in \mathrm{Hom}(A_i, G)$. As $F \subseteq \bigoplus_{i\in I} A_i$, the set $W' = \{f \in \mathrm{Hom}(\bigoplus_{i\in I} A_i, G) \mid f[F] \subseteq U\}$ is an open neighborhood of $0 \in \mathrm{Hom}(\bigoplus_{i\in I} A_i, G)$. Evidently $\Phi^{-1}[W'] \subseteq W$.

We show next that Φ is continuous. Let F be a finite subset of $\bigoplus_{i\in I} A_i$ and U an open neighborhood of $0 \in G$. Then $W = W(F, U)$ is a generic open neighborhood of $0 \in \mathrm{Hom}(\bigoplus_{i\in I} A_i, G)$. Then there is a finite subset J of I such that $F \subseteq \bigoplus_{i\in J} A_i$. Furthermore, for $j \in J$, there exist finite sets $B_j \subseteq A_j$ such that $F \subseteq \sum_{j\in J} B_j$. For $i \notin J$, let $B_j = \{0\}$. There exists an open neighborhood V of $0 \in G$ such that $V^{|J|} \subseteq U$. Then $\prod_{i\in I} W(B_i, V)$ is an open neighborhood of $0 \in \prod_{i\in I} \mathrm{Hom}(A_i, G)$. We claim that $\Phi[\prod_{i\in I} W(B_i, V)] \subseteq W(F, U)$. Let $f = (f_i) \in \prod_{i\in I} W(B_i, V)$ and $b = \sum_{j\in J} b_j \in F$. Then $(\Phi(f))(b) = \sum_{j\in J} f_j(b_j) \in V^{|J|} \subseteq U$. □

The following is surely well-known.

Lemma 2. *Let G and H be topological abelian groups and $\varphi : G \to H$ a surjective homomorphism with kernel K.*

(1) *Suppose that φ is continuous and K is dense in G. Then H is indiscrete.*
(2) *Suppose that φ is an open map and H is indiscrete. Then K is dense in G.*
(3) *Suppose that H is indiscrete and* $\mathrm{cHom}(G,H)$ *is endowed with the compact-open topology. Then* $\mathrm{cHom}(G,H)$ *is indiscrete.*

Proof. (1) Suppose that C is a non-empty closed subset of H. Then $\varphi^{-1}[C]$ is closed in G containing K. As K is dense in G it follows that $\varphi^{-1}[C] = G$. It follows that $C = H$. Hence, the only closed sets in H are H and $\emptyset$, so H is indiscrete.

(2) Let $x \in G$ and $U = -U$ a symmetric open neighborhood of $0 \in G$. Then $\varphi[U]$ is non-empty and open in H and as H is indiscrete, $\varphi[U] = H$. Hence, there is $y \in U$ such that $\varphi(y) = \varphi(x)$ and so $x - y = z \in K$ and $z \in x + U$ showing that K is dense in G.

(3) The open sets of $\mathrm{cHom}(G,H)$ are the sets of the form $W := W(C,U) = \{f \in \mathrm{cHom}(G,H) \mid f[C] \subseteq U\}$ where C is a compact subset of G and U is an open subset of H. By hypothesis $U = \emptyset$ or $U = H$. Whatever C may be, in the first case $W = \emptyset$ and in the second case $W = \mathrm{cHom}(G,H)$. □

3. The Meet Semi-Lattice $\mathcal{F}(A)$ of Full Free Subgroups in AG

The following notation relating an arbitrary group A with its torsion-free quotient $A_0 := A/\operatorname{tor}(A)$ will be used throughout.

Let $A \in \mathrm{AG}$ and let $\varphi_0 : A \to A_0$ be the natural epimorphism. For future use we record the short exact sequence

$$\mathrm{E}_0 : \operatorname{tor}(A) \overset{\mathrm{ins}}{\rightarrowtail} A \overset{\varphi_0}{\twoheadrightarrow} A_0.$$

It is well-known that $\mathbb{Q}A_0 := \mathbb{Q} \otimes_{\mathbb{Z}} A_0 \cong \mathbb{Q} \otimes_{\mathbb{Z}} A$ is a $\mathbb{Q}$–vector space containing $A_0 \cong \mathbb{Z} \otimes_{\mathbb{Z}} A_0$ as an essential subgroup. Thus $\mathbb{Q}A_0$ is a divisible hull of A_0. The ***rank*** of A is the dimension of $\mathbb{Q}A_0$: $\mathrm{rk}(A) := \mathrm{rk}(A_0) := \dim_{\mathbb{Q}}(\mathbb{Q}A_0)$.

For $F \in \mathcal{F}(A)$, set $F_0 := \varphi_0[F] = \frac{F \oplus \operatorname{tor}(A)}{\operatorname{tor}(A)} \cong F$. Then $\mathrm{rk}(A) = \mathrm{rk}(F) = \mathrm{rk}(F_0)$.

In the literature, the dimension of a compact abelian group is defined in several equivalent ways. The cardinal $\dim(G) = \mathrm{rk}(G^\wedge)$ will serve for the purposes of this article.

For every prime p we define the p-***rank of*** A by $\mathrm{rk}_p(A) := \dim_{\mathbb{Z}/p\mathbb{Z}}(A[p])$.

A discrete divisible group D is determined up to isomorphism by the invariants $\mathrm{rk}_p(D)$ counting the summands isomorphic to $\mathbb{Z}(p^\infty)$ and $\mathrm{rk}(D/\operatorname{tor}(D))$ counting the summands isomorphic to $\mathbb{Q}$. See ([17], Chapter 4) for details.

Lemma 3. *If A is a torsion-free group, then* $\mathrm{rk}_p(A/pA) \le \mathrm{rk}(A)$.

Proof. It suffices to check that if $\{b_1, \dots, b_n\}$ is a linearly independent subset of A/pA, where $b_i = a_i + pA$, $a_i \in A$, then $\{a_1, \dots, a_n\}$ is linearly independent in A. Assume that $m_1a_1 + \cdots + m_na_n = 0$, with $m_i \in \mathbb{Z}$ for $i = 1, 2, \dots, n$. As A is torsion-free, we can assume without loss of generality that $\gcd(p, m_j) = 1$ for some $j = 1, 2, \dots, n$. After projecting in A/pA we obtain $m_1b_1 + \cdots + m_nb_n = 0$. By the choice of $\{b_1, \dots, b_n\}$ this gives $m_i = p\mathbb{Z}$ for all i. This contradicts $\gcd(p, m_j) = 1$. □

Now we see that the p-ranks of a compact connected group G of finite dimension are bounded from above by $\dim(G)$.

Corollary 1. *Let A be a discrete torsion-free group of finite rank n. Then* $\mathrm{rk}_p(A^\wedge) = \mathrm{rk}_p(A/pA) \le n$.

Proof. Clearly, $G = A^\wedge$ is a compact connected group with $\dim(G) = n$. The socle $G[p]$ of G is the kernel of μ_p^G, the multiplication by p in G, and hence closed and therefore compact. We have the proper exact sequence $G[p] \rightarrowtail G \overset{\mu_p^G}{\twoheadrightarrow} G$ which gives the proper exact sequence

$G^\wedge \overset{\mu_p^{G^\wedge}}{\rightarrowtail} G^\wedge = A^{\wedge\wedge} = A \twoheadrightarrow A/pA \cong G[p]^\wedge$. Hence, $\mathrm{rk}_p(G[p]^\wedge) = \mathrm{rk}_p(A/pA) \leq \mathrm{rk}(A) = n < \infty$, by Lemma 3. Thus, $G[p] \cong G[p]^\wedge$ and $\mathrm{rk}_p(G) = \mathrm{rk}_p(G[p]) = \mathrm{rk}_p(G[p]^\wedge) \leq n$. □

We first illuminate the abundance of full free subgroups in a group.

Lemma 4. *Let* $\mathrm{tor}(A) \neq A \in \mathrm{AG}$. *Then the following hold.*

(1) $\{a_i \mid i \in I\}$ *is a linearly independent set in A if and only if* $\{a_i + \mathrm{tor}(A) \mid i \in I\}$ *is a linearly independent set in* A_0. *Moreover,* $\{a_i \mid i \in I\}$ *is maximal linearly independent if and only if* $\{a_i + \mathrm{tor}(A) \mid i \in I\}$ *is maximal linearly independent.*

(2) *If* $\{a_i \mid i \in I\}$ *is a (maximal) linearly independent set in A and* $\forall\, i \in I : t_i \in \mathrm{tor}(A)$, *then* $\{a_i + t_i \mid i \in I\}$ *is a (maximal) linearly independent subset of A.*

(3) *Every linearly independent set extends to a maximal linearly independent set. In particular, every torsion-free element in A is contained in a maximal linearly independent subset.*

(4) *If* $\{a_i \mid i \in I\}$ *is a maximal linearly independent subset of A, then* $F = \bigoplus_{i \in I} \mathbb{Z}a_i$ *is a full free subgroup of A. Conversely, if* $F = \bigoplus_{i \in I} \mathbb{Z}a_i$ *is a full free subgroup of A, then* $\{a_i \mid i \in I\}$ *is a maximal linearly independent subset of A.*

(5) *If* $F \in \mathcal{F}(A)$, *then* $F_0 \cong F$ *and* $A_0/F_0 \cong A/(F \oplus \mathrm{tor}(A))$, $F_0 \in \mathcal{F}(A_0)$, *and* $\varphi_0^{-1}[F_0] = F \oplus \mathrm{tor}(A)$.

(6) *Given* $F_0 \in \mathcal{F}(A_0)$, *there exists* $F \in \mathcal{F}(A)$ *such that* $\varphi_0[F] = F_0$ *and* $\varphi^{-1}[F_0] = F \oplus \mathrm{tor}(A)$. *If* $F, F' \in \mathcal{F}(A)$ *and* $F_0 = F_0'$, *then there is* $\varphi \in \mathrm{Hom}(F, \mathrm{tor}(A))$ *such that* $F' = \{\varphi(x) + x \mid x \in F\}$. *Note that* $\mathrm{Hom}(F, \mathrm{tor}(A)) \cong \mathrm{tor}(A)^{\mathrm{rk}(A)}$

(7) *A maximal linearly independent subset of* A_0 *is a* $\mathbb{Q}$*–basis of* $\mathbb{Q}A_0$.

(8) *If* $\{v_i \mid i \in I\}$ *is a* $\mathbb{Q}$*–basis of* $\mathbb{Q}A_0$, *then there exist positive integers* m_i *such that* $\forall\, i \in I$: $m_i v_i \in A$ *and* $F = \bigoplus_{i \in I} \mathbb{Z}(m_i v_i)$ *is a full free subgroup of A.*

Proof. Maximal linearly independent subsets exist by Zorn's Lemma.

(6) Suppose that $F, F' \in \mathcal{F}(A)$ and $F_0 = F_0'$. Then $F \oplus \mathrm{tor}(A) = F' \oplus \mathrm{tor}(A)$. By ([19], Lemma 1.1.3, p. 6) there exists $\varphi \in \mathrm{Hom}(F, \mathrm{tor}(A))$ such that $F' = \{\varphi(x) + x \mid x \in F\}$.

The rest consists of easy and well-known observations. □

We always assume that $A_0 \neq \{0\}$, i.e., we assume that A is not a torsion group. The dual $T^\wedge$ of a torsion group T is a compact totally disconnected group.

Theorem 1. *For* $A \in \mathrm{AG}$, *the family* $\mathcal{F} := \mathcal{F}(A)$ *has the following properties.*

(1) *Let* $F, F' \in \mathcal{F}$. *Then* $F \cap F' \in \mathcal{F}$.

(2) *If* $F \in \mathcal{F}$, $F' \leq F$ *and* F/F' *is a torsion group, then* $F' \in \mathcal{F}$.

(3) *If* $F \in \mathcal{F}$, *then* $\forall\, m \in \mathbb{N} : mF \in \mathcal{F}$ *and* $\bigcap_m mF = \{0\}$.

(4) $\bigcap \mathcal{F} = \{0\}$. *If* $A \neq \mathrm{tor}(A)$ *then* $\bigcup \mathcal{F} = A \setminus \mathrm{tor}(A)$ *and* $\sum \mathcal{F} = A$.

(5) $\mathcal{F}$ *is a meet semi-lattice with meet* $\cap$.

(6) *Let* $F \in \mathcal{F}$. *Then* $\forall\, m \in \mathbb{N} : m_A^{-1} F = F' \oplus A[m]$ *for some* $F' \in \mathcal{F}$, *and* $(F' \oplus A[m])/F = (A/F)[m]$. *If A is torsion-free, then* $m_A^{-1} F \in \mathcal{F}$.

Proof. (1) Certainly $F \cap F'$ is free as subgroup of free groups. The map $A/(F \cap F') \to A/F \oplus A/F' : a + (F \cap F') \mapsto (a + F, a + F')$ is well-defined and injective. Hence, $A/(F \cap F')$ is torsion.

(2) and (3) are trivial.

(4) It follows from (3) that $\bigcap \mathcal{F} = \{0\}$. If $A \neq \mathrm{tor}(A)$, then $\bigcup \mathcal{F} = A \setminus \mathrm{tor}(A)$ is evident, and it follows from Lemma 4(2) that $\sum \mathcal{F} = A$.

(5) Follows from (1).

(6.1) We first assume that A is torsion-free. Then the multiplication μ_m^A is injective, and $\mu_m^A : A \to mA$ is an isomorphism. Thus, $m^{-1}F = (\mu_m^A)^{-1}[mA \cap F]$ is free since F is free. As $F \subseteq m_A^{-1} F$ it follows that $m_A^{-1} F \in \mathcal{F}$.

(6.2) Recall that $F_0 \in \mathcal{F}(F_0)$ and by (6.1) $F_0 \subseteq m_{A_0}^{-1} F_0 \in \mathcal{F}(A_0)$ with $(m_{A_0}^{-1} F_0)/F_0 = (A_0/F_0)[m]$. It is straightforward to check that $\varphi_0[m_A^{-1} F] \subseteq m_{A_0}^{-1} F_0$ and it follows that

$\varphi_0[m_A^{-1}F]$ is free. Hence, the epimorphism $\varphi_0 : m_A^{-1}F \to \varphi_0[m_A^{-1}F]$ splits with kernel $m_A^{-1}F \cap \mathrm{tor}(A) = \mathrm{tor}(A)[m]$. Hence, $m_A^{-1}F = F' \oplus \mathrm{tor}(A)[m]$ for some free group $F' \cong \varphi_0[m_A^{-1}F]$.

It remains to show that A/F' is a torsion group. As A_0/F_0 is torsion and $F_0 \subseteq \varphi_0[m_A^{-1}F]$, we see that $A_0/\varphi_0[m_A^{-1}F]$ is a torsion group. Let $a \in A$. Then there is $k \in \mathbb{N}$ such that $k\varphi_0(a) \in \varphi_0[m_A^{-1}F] = \varphi_0[F']$. Hence, $k\varphi_0(a) = \varphi_0(ka) = \varphi_0(b)$ for some $b \in F'$ and $ka - b \in \mathrm{Ker}(\varphi_0) = \mathrm{tor}(A)$. Thus, there is $k' \in \mathbb{N}$ such that $k'ka = k'b \in F'$. Finally, $(F' \oplus A[m])/F = (m_A^{-1}F)/F = (A/F)[m]$. □

Remark 1. *In general, $\mathcal{F}(A)$ is not closed under finite sums, so $\mathcal{F}(A)$ may not be a lattice, and therefore, $A = \sum \mathcal{F}(A)$ may not be the directed union (direct limit) of its members. However, for $A = A_0$, using Theorem 1(6) (with $\mathrm{tor}(A) = \{0\}$), given $F \in \mathcal{F}(A)$, also the larger $m_A^{-1}F$ is a full free subgroup, and as A/F is a torsion group, we obtain an ascending chain*

$$F = (1!)_A^{-1}F \subseteq (2!)_A^{-1}F \subseteq \cdots \subseteq (m!)_A^{-1}F \subseteq ((m+1)!)_A^{-1}F \subseteq \cdots$$

of full free subgroups of A whose union is A.

In the case of a torsion-free group A of finite rank, the quotients A/F for $F \in \mathcal{F}(A)$ are somewhat alike ([2], Theorem 3.5(9)). For arbitrary rank there is a great variety of quotients A/F.

Proposition 3. *Let A be an abelian group of infinite rank $\mathfrak{m}$. Let $F \in \mathcal{F}(A)$. Then $\mathrm{rk}(F) = |F| = \mathfrak{m}$. Let T be any torsion group that is $\mathfrak{m}$–generated. Then there is an epimorphism $\varphi : F \twoheadrightarrow T$ with $F_\varphi := \mathrm{Ker}(\varphi) \in \mathcal{F}(A)$, and there is an exact sequence $T \rightarrowtail A/F_\varphi \twoheadrightarrow A/F$. Moreover, $\mathrm{rk}_p(A/F) \leq \mathfrak{m}$.*

Proof. Routine and simple. □

In the case of infinite rank, the sum of two full free subgroups need not be free, as shown by Jim Reid (([20], Theorem 2.2)):

Theorem 2. *Let A be a torsion-free group of infinite rank.*

(a) (([20], Theorem 2.2) and its proof) Given a free subgroup F of A with $\mathrm{rk}(F) = \mathrm{rk}(A)$, there is a second free subgroup F_1 such that $A = F + F_1$.

(b) ([20], Corollary 3.5) There exists a full free subgroup F_0 of A such that A/F_0 is divisible (A is "quotient divisible").

One can deduce from (a) that in a torsion-free group A of infinite rank every non-free subgroup of torsion index is the sum of two full free subgroups.

Definition 1. *An abelian group A is $\mathcal{F}$**-summable** if for any $F_1, F_2 \in \mathcal{F}(A)$ also $F_1 + F_2 \in \mathcal{F}(A)$.*

Theorem 2(a) yields:

Theorem 3. *$A \in \mathrm{AG}$ is $\mathcal{F}$-summable if and only if A is either torsion-free of finite rank or is free of arbitrary rank.*

Proof. If $\mathrm{tor}(A) \neq \{0\}$, then there are full free subgroups whose sum contains torsion elements (Lemma 4(2)). So a summable group must be torsion-free.

Suppose that A is torsion-free and $\mathcal{F}$-summable of infinite rank. Then A is the sum of two free subgroups and hence of two full free subgroups. As A is summable, it is free. The converse is clear.

If the torsion-free group A has finite rank, then full free subgroups are finitely generated and finitely generated torsion-free subgroups are free. $\square$

4. The Semi-Lattices $\mathcal{F}(A)$ and $\mathcal{D}(G)$

Let $A \in \mathrm{AG}$ and $G = A^\wedge$. Then G is compact, not necessarily connected. Let $F \in \mathcal{F}(A)$. Then $F \overset{\mathrm{ins}}{\rightarrowtail} A \overset{\alpha}{\twoheadrightarrow} A/F$ is exact where α is the natural epimorphism. Therefore,

$$(A/F)^\wedge \overset{\alpha^\wedge}{\rightarrowtail} G \overset{\mathrm{restr}}{\twoheadrightarrow} F^\wedge$$

is exact, where $F^\wedge$ is a torus isomorphic to $\mathbb{T}^{\mathrm{rk}(A)}$ and $\alpha^\wedge[(A/F)^\wedge]$ is a compact totally disconnected subgroup of G. Hence, $\alpha^\wedge[(A/F)^\wedge] \in \mathcal{D}(G)$. We obtained the mapping

$$\mathcal{F}(A) \to \mathcal{D}(G) : F \mapsto \alpha^\wedge\big[(A/F)^\wedge\big]. \tag{1}$$

Let G be a compact group and $A = G^\wedge$. Then A is a possibly mixed group. Let $\Delta \in \mathcal{D}(G)$. Then $\Delta \overset{\mathrm{ins}}{\rightarrowtail} G \overset{\beta}{\twoheadrightarrow} G/\Delta$ is exact where β is the natural epimorphism and G/Δ is a torus. Therefore,

$$(G/\Delta)^\wedge \overset{\beta^\wedge}{\rightarrowtail} A \overset{\mathrm{restr}}{\twoheadrightarrow} \Delta^\wedge$$

is exact, where $\Delta^\wedge$ is a torsion group and $\beta^\wedge[(G/\Delta)^\wedge]$ is a full free subgroup of A. Hence, $\beta^\wedge[(G/\Delta)^\wedge] \in \mathcal{F}(A)$. We obtained

$$\mathcal{D}(G) \to \mathcal{F}(A) : \Delta \mapsto \beta^\wedge\big[(G/\Delta)^\wedge\big]. \tag{2}$$

Lemma 5. *Let $G \in \mathrm{LCA}$ and H a closed subgroup of G. The sequence $H \overset{\mathrm{ins}}{\rightarrowtail} G \overset{\varphi}{\twoheadrightarrow} G/H$ is exact in* LCA. *Hence,*

$$(G/H)^\wedge \overset{\varphi^\wedge}{\rightarrowtail} G^\wedge \overset{\mathrm{restr}}{\twoheadrightarrow} H^\wedge \tag{3}$$

is exact in LCA, *and $\varphi^\wedge[(G/H)^\wedge] = (G^\wedge, H)$.*

Proof. Suppose first that $\chi \in (G^\wedge, H)$. Then $\chi[H] = \{0\}$, so $\chi\restriction_H = 0$. Hence, χ is in the kernel of the restriction map in (3), i.e., $\chi \in \mathrm{Ker}(\mathrm{restr}) = \alpha^\wedge[(A/F)^\wedge]$. Conversely, if $\chi \in \mathrm{Ker}(\mathrm{restr})$, then $\chi[H] = \{0\}$ and $\chi \in (G^\wedge, H)$. $\square$

For a general topological abelian group G, the family $\mathrm{Lat}(G)$ of closed subgroups is a lattice with the operations $C_1 \wedge C_2 = C_1 \cap C_2$ and $C_1 \vee C_2 = \overline{C_1 + C_2}$. There also exist greatest lower bounds and least upper bounds for infinite families: Let $\mathcal{C}$ be a family of closed subgroups of G. Then $\bigcap \mathcal{C}$ is a closed subgroup of G and $\bigcap \mathcal{C} = \bigwedge \mathcal{C}$. The subgroup $\overline{\sum \mathcal{C}}$ is closed and $\overline{\sum \mathcal{C}} = \bigvee \mathcal{C}$. See ([1], p. 361).

We will establish that $\mathcal{F}(A)$ and $\mathcal{D}(A^\wedge)$ are anti-isomorphic semi-lattices. To do so, we use results of ([1], p. 351) where we find annihilators $H^\perp$ defined as follows.

For $G \in \mathrm{LCA}$, we have the pairing $G^\wedge \times G \to \mathbb{T} : (\chi, g) \mapsto \chi(g)$. For a subset X of G, we define the ***annihilator*** $X^\perp$ ***of*** $X \subseteq G$ ***in*** $G^\wedge$ by $X^\perp := (G^\wedge, X) = \{\chi \in G^\wedge \mid \chi[X] = 0\}$ while for $Y \subseteq G^\wedge$, we define $Y^\perp = \{g \in G \mid \forall \rho \in Y : \rho(g) = 0\}$

Note that $X^{\perp\perp} \subseteq G$ is not the same as $(G^{\wedge\wedge}, X^\perp) = (G^{\wedge\wedge}, (G^\wedge, X))$. However, they are topologically isomorphic:

Lemma 6. *Let $A \in \mathrm{LCA}$. Then, for $X \subseteq A$, the natural evaluation isomorphism $\eta_A : A \to A^{\wedge\wedge}$ restricts to an isomorphism $X^{\perp\perp} \to (A^{\wedge\wedge}, X^\perp)$, $\eta_A\big[X^{\perp\perp}\big] = (A^{\wedge\wedge}, X^\perp)$. In particular, $X^{\perp\perp}$ is a full free subgroup of A if and only if $(A^{\wedge\wedge}, X^\perp)$ is a full free subgroup of $A^{\wedge\wedge}$.*

Proof. We need to check that $\eta_A[X^{\perp\perp}] = (A^{\wedge\wedge}, X^\perp)$. Let $a \in A$. Then
$\eta_A(a) \in (A^{\wedge\wedge}, X^\perp) \Longleftrightarrow \forall \chi \in X^\perp : \eta_A(a)(\chi) = \chi(a) = 0 \Longleftrightarrow a \in X^{\perp\perp}$. $\square$

We rely on the basic properties of annihilators ([1], pp. 351–362), in particular see ([1], Theorem 7.64, p. 392); alternatively see ([21], pp. 270–275).

Theorem 4 ([1], Theorem 7.64(iv),(v), (vi), p. 392)**.** *Let $G \in \mathrm{LCA}$. Then $H \mapsto H^{\perp} = (G^{\wedge}, H)$ with $H^{\perp\perp} = H$, is a lattice anti-isomorphism between* $\mathrm{Lat}(G)$ *and* $\mathrm{Lat}(G^{\wedge})$*. In particular, $H \subseteq K$ if and only if $K^{\perp} \subseteq H^{\perp}$.*

If $H \rightarrowtail G \twoheadrightarrow G/H$ is proper exact in LCA, *then*

$$(G/H)^{\wedge} \cong_{t} H^{\perp} \quad \text{and} \quad H^{\wedge} \cong_{t} G^{\wedge}/H^{\perp}.$$

Theorem 5. *Let $A \in \mathrm{AG}$ and $G = A^{\wedge}$. The lattice anti-isomorphism $H \mapsto H^{\perp}$ of Theorem 4 restricts to an anti-isomorphism of semi-lattices $\delta : \mathcal{F}(A) \to \mathcal{D}(G)$. In particular we have:*

- *$\mathcal{D}(G)$ is a join semi-lattice with join $+$.*
- *$\forall\, F, F_1, F_2 \in \mathcal{F}(A) : \delta(F) = F^{\perp} \in \mathcal{D}(G)$; if $F_1 \subseteq F_2$, then $\delta(F_2) \subseteq \delta(F_1)$; $\delta(F_1 \cap F_2) = \delta(F_1) + \delta(F_2)$.*

Proof. By Theorem 4 we only need to show that $\delta(\mathcal{F}(A)) = \mathcal{D}(G)$.

Let $F \in \mathcal{F}(A)$. Then $F^{\perp} = (G, F)$ by definition, and $(G, F) = \alpha^{\wedge}[(A/F)^{\wedge}] \in \mathcal{D}(G)$ by Lemma 5 and (1). So δ is well defined.

Let $\Delta \in \mathcal{D}(G)$. By (2) $\beta^{\wedge}[(G/\Delta)^{\wedge}] \in \mathcal{F}(A^{\wedge\wedge})$ and by Lemma 6 $\Delta^{\perp} \in \mathcal{F}(A)$ and $\delta(\Delta^{\perp}) = \Delta^{\perp\perp} = \Delta$. □

We now establish, for a compact group $G = A^{\wedge}$, the properties of $\mathcal{D}(G)$ corresponding to the properties of $\mathcal{F}(A)$. Recall that for any $m \in \mathbb{N}$ and any subgroup Y of X, we have $m_X^{-1}Y = \{x \in X \mid mx \in Y\}$.

Definition 2. *For a compact abelian group G set $\mathbf{\Delta}(G) := \sum \mathcal{D}(G)$.*

We collect here some properties of the subgroup $\mathbf{\Delta}(G)$, "Fat Delta".

Theorem 6. *Let $G = A^{\wedge}$, $A \in \mathrm{AG}$. The family $\mathcal{D} := \mathcal{D}(G)$ has the following properties.*

(1) $\mathcal{D}$ is a join semi-lattice with join $+$. Hence, $\mathbf{\Delta}(G) = \bigcup \mathcal{D}$.
(2) $\mathbf{\Delta}(G)$ is dense in G, while $\bigcap \mathcal{D} = \{0\}$ if $c(G) \neq \{0\}$, otherwise $\bigcap \mathcal{D} = G$.
(3) Let $\Delta = \delta(F)$ and $\Delta' = \delta(F')$ and assume that $\Delta \subseteq \Delta'$. Then $F' \subseteq F$ and $\Delta'/\Delta \cong_{t} (F/F')^{\wedge}$.
(4) If $\Delta \in \mathcal{D}$, then $m_G^{-1}\Delta \in \mathcal{D}$ for any $m \in \mathbb{N}$. Hence, $\mathrm{tor}(G/\Delta) \subseteq \mathbf{\Delta}/\Delta$ and $\mathrm{tor}(G) \subseteq \mathbf{\Delta}(G)$.
(5) Let $\Delta \in \mathcal{D}$ and $m \in \mathbb{N}$. Then there is $\Delta' \in \mathcal{D}$ such that $m\Delta = \Delta' \cap mG$. If A is torsion-free, then $m\Delta \in \mathcal{D}$.
(6) $G/\mathbf{\Delta}(G)$ is torsion-free.

Proof. (1) Theorem 5 establishes the semi-lattice property. As $\mathcal{D}$ is closed under finite sums, we have $\sum \mathcal{D} = \bigcup \mathcal{D}$.

(2) By ([1], Theorem 7.64(vii), p. 392) $\overline{\bigcup \mathcal{D}} = (G, \bigcap \mathcal{F}(A)) = (G, 0) = G$ and $\bigcap \mathcal{D} = (G, \sum \mathcal{F}(A))$. If $A \neq \mathrm{tor}(A)$, then $\sum \mathcal{F}(A) = A$, by Theorem 1(4). So, $\overline{\bigcup \mathcal{D}} = (G, A) = \{0\}$ in this case. If G is totally disconnected, then $\mathcal{D} = \{G\}$, so $\bigcap \mathcal{D} = G$.

(3) We have the following commutative diagram with natural maps and exact rows

$$\begin{array}{ccccc} & & & & F/F' \\ & & & & \big\downarrow{\scriptstyle \mathrm{ins}} \\ F' & \xrightarrow{\mathrm{ins}} & A & \xrightarrow{\varphi'} & A/F' \\ \big\downarrow{\scriptstyle \mathrm{ins}} & & \big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle \psi} \\ F & \xrightarrow{\mathrm{ins}} & A & \xrightarrow{\varphi} & A/F \end{array} \quad \text{and its dual} \quad \begin{array}{ccccc} (A/F)^{\wedge} & \xrightarrow{\varphi^{\wedge}} & G & \xrightarrow{\mathrm{restr}} & F^{\wedge} \\ \big\downarrow{\scriptstyle \psi^{\wedge}} & & \big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle \mathrm{restr}} \\ (A/F')^{\wedge} & \xrightarrow{(\varphi')^{\wedge}} & G & \xrightarrow{\mathrm{restr}} & (F')^{\wedge} \\ \big\downarrow{\scriptstyle \mathrm{restr}} & & & & \\ (F/F')^{\wedge} & & & & \end{array}$$

We conclude that

$$\frac{\Delta'}{\Delta} = \frac{(\varphi')^\wedge[(A/F')^\wedge]}{\varphi^\wedge[(A/F)^\wedge]} \cong_{\mathrm{t}} \frac{(A/F')^\wedge}{\psi^\wedge[(A/F)^\wedge]} \cong_{\mathrm{t}} \left(\frac{F}{F'}\right)^\wedge$$

(4) Let $\Delta = F^\perp = (G, F) \in \mathcal{D}$ with $F \in \mathcal{F}(A)$. If $m \in \mathbb{N}$, then $m_G^{-1}\Delta = m_G^{-1}(G, F) = (G, mF)$ (cf. [21], Lemma 6.4.14, p. 274), so since $mF \in \mathcal{F}(A)$ we have $m_G^{-1}\Delta \in \mathcal{D}$.

(5) Let $\Delta = \delta(F)$ for some $F \in \mathcal{F}(A)$. By Theorem 1(3) we know that $m_A^{-1}F = F' \oplus A[m]$ for some $F' \in \mathcal{F}(A)$. Using ([21], Lemma 6.4.13, p. 27) and ([21], Lemma 6.4.15, p. 27) we obtain $m\Delta = m\delta(F) = \overline{m(G,F)} = (G, m_A^{-1}F) = (G, F' + A[m]) = (G, F') \cap (G, A[m]) = \delta(F') \cap \overline{mG}$. Furthermore, $\delta(F)$ and G are both compact and hence, so are $m\delta(F)$ and mG, therefore closed, and equal to the closures.

(6) Let $x \in G$. If $mx \in \mathbf{\Delta}(G)$ for some $m \in \mathbb{N}$, then $mx \in \Delta$ for some $\Delta \in \mathcal{D}$. Then $x \in m_G^{-1}\Delta \in \mathcal{D}$ by (4), thus $x \in \mathbf{\Delta}(G)$. Therefore $G/\mathbf{\Delta}(G)$ is torsion-free. □

The fact that linearly independent sets can be enlarged to maximal linearly independent sets has the following dual.

Proposition 4. *Let $G = A^\wedge$ be a compact abelian group of infinite dimension.*

Suppose that Θ is a subgroup of G such that G/Θ is a torus of dimension $\mathfrak{m}$. Then Θ contains some $\Delta \in \mathcal{D}(G)$ and $\mathfrak{m} \leq \dim(G)$.

Proof. $\Theta = E^\perp$ for some subgroup E of A (Theorem 4). We claim that E is a free subgroup of A. From $\Theta \overset{\mathrm{ins}}{\rightarrowtail} G \twoheadrightarrow T$ where T is a torus of dimension $\mathfrak{m}$, we conclude the exact sequence $T^\wedge \rightarrowtail A^{\wedge\wedge} \overset{\mathrm{restr}}{\twoheadrightarrow} \Theta^\wedge$. By Lemma 5 $T^\wedge \cong (A^{\wedge\wedge}, \Theta)$ and by Lemma 6 $(A^{\wedge\wedge}, \Theta) \cong (A, \Theta) = E^{\perp\perp} = E$. As $T^\wedge$ is free of rank $\mathfrak{m}$ as the dual of a torus, so is E and $\mathfrak{m} \leq \mathrm{rk}(A)$. Let F be a full free subgroup containing E. Then $\Delta := F^\perp \in \mathcal{D}(G)$ and $\Theta = E^\perp \supseteq F^\perp = \Delta$. □

Let $G = A^\wedge$. We next study the connection between $\mathcal{D}(G)$, $\mathcal{D}(\mathrm{c}(G))$, $\mathbf{\Delta}(G)$, and $\mathbf{\Delta}(\mathrm{c}(G))$. Given A, let $T = \mathrm{tor}(A)$ and let $F \in \mathcal{F}(A)$. Then, we obtain the following commutative diagram with exact rows and its dual.

$$\begin{array}{ccccc} & & F & \overset{\varphi_0}{\rightarrowtail\!\!\!\twoheadrightarrow} & F_0 \\ & & \downarrow{\scriptstyle \mathrm{ins}} & & \downarrow{\scriptstyle \mathrm{ins}} \\ T & \overset{\mathrm{ins}}{\rightarrowtail} & A & \overset{\varphi_0}{\twoheadrightarrow} & A_0 \\ \downarrow{\scriptstyle \psi_T} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi_0} \\ \frac{T\oplus F}{F} & \overset{\mathrm{ins}}{\rightarrowtail} & \frac{A}{F} & \overset{\varphi_0}{\twoheadrightarrow} & \frac{A_0}{F_0} \end{array} \quad \text{with dual} \quad \begin{array}{ccccc} \left(\frac{A_0}{F_0}\right)^\wedge & \overset{\varphi_0^\wedge}{\rightarrowtail} & \left(\frac{A}{F}\right)^\wedge & \overset{\mathrm{restr}}{\twoheadrightarrow} & \left(\frac{T\oplus F}{F}\right)^\wedge \\ \downarrow{\scriptstyle \psi_0^\wedge} & & \downarrow{\scriptstyle \psi^\wedge} & & \downarrow{\scriptstyle \psi_T^\wedge} \\ A_0^\wedge & \overset{\varphi_0^\wedge}{\rightarrowtail} & G & \overset{\mathrm{restr}}{\twoheadrightarrow} & T^\wedge \\ \downarrow{\scriptstyle \mathrm{restr}} & & \downarrow{\scriptstyle \mathrm{restr}} & & \\ F_0^\wedge & \overset{\varphi_0^\wedge}{\rightarrowtail\!\!\!\twoheadrightarrow} & F^\wedge & & \end{array}$$

We now set

- $\Delta := \psi^\wedge[\left(\frac{A}{F}\right)^\wedge]$
- $G_0 := \varphi_0^\wedge[A_0^\wedge]$
- $\Delta_0 := (\varphi_0^\wedge \circ \psi_0^\wedge)[\left(\frac{A_0}{F_0}\right)^\wedge] = (\psi^\wedge \circ \varphi_0^\wedge))[\left(\frac{A_0}{F_0}\right)^\wedge] \subseteq G$

and obtain

$$\begin{array}{ccccc} \Delta_0 & \overset{\mathrm{ins}}{\rightarrowtail} & \Delta & \overset{\mathrm{restr}_\Delta}{\twoheadrightarrow} & T^\wedge \\ \downarrow{\scriptstyle \mathrm{ins}} & & \downarrow{\scriptstyle \mathrm{ins}_\Delta} & & \downarrow{\scriptstyle =} \\ G_0 & \overset{\mathrm{ins}}{\rightarrowtail} & G & \overset{\mathrm{restr}}{\twoheadrightarrow} & T^\wedge \\ \downarrow{\scriptstyle \mathrm{restr}} & & \downarrow{\scriptstyle \mathrm{restr}_G} & & \\ F^\wedge & \overset{=}{\rightarrowtail\!\!\!\twoheadrightarrow} & F^\wedge & & \end{array} \tag{4}$$

We have the following easy consequences.

Theorem 7. *Let $G = A^\wedge$ where $A \in \mathrm{AG}$ and consider (4), where $T = \mathrm{tor}(A)$.*

(1) $G_0 = \varphi_0^\wedge[A_0^\wedge] = \mathrm{tor}(A)^\perp$ *coincides with the* 0*–component* $\mathrm{c}(G)$ *of* G.

(2) $\mathrm{c}(G)$ *is divisible and so, algebraically,* $G \cong \mathrm{c}(G) \oplus T^\wedge$ *and* $G \cong_{\mathrm{t}} \mathrm{c}(G) \oplus T^\wedge$ *if and only if* A *splits, i.e.,* $A \cong A_0 \oplus T$.

(3) $\Delta_0 = \Delta \cap \mathrm{c}(G)$. *Thus* $\mathcal{D}(\mathrm{c}(G)) = \{D \cap \mathrm{c}(G) \mid D \in \mathcal{D}(G)\}$, $\mathbf{\Delta}(\mathrm{c}(G)) = \mathbf{\Delta}(G) \cap \mathrm{c}(G)$, *and* $\mathbf{\Delta}(\mathrm{c}(G))$ *is closed in* $\mathbf{\Delta}(G)$.

(4) $G = \Delta + \mathrm{c}(G)$ *and* $\mathbf{\Delta}(G) = \Delta + \mathbf{\Delta}(\mathrm{c}(G))$.

(5) $\mathrm{c}(G)/\Delta_0 \cong_{\mathrm{t}} G/\Delta$ *and* $\Delta/\Delta_0 \cong_{\mathrm{t}} G/\mathrm{c}(G) \cong_{\mathrm{t}} T^\wedge$.

(6) *With the established notation* $\mathbf{\Delta}(\mathrm{c}(G))$ *is divisible and hence algebraically a direct summand of* $\mathbf{\Delta}(G)$.

(7) *There is a topological isomorphism* $\mathrm{tor}(A)^\wedge \cong_{\mathrm{t}} \frac{\Delta}{\Delta_0} \to \frac{\mathbf{\Delta}(G)}{\mathbf{\Delta}(\mathrm{c}(G))}$.

Proof. (1) As A_0 is torsion-free, its dual G_0 is connected and G/G_0 is totally disconnected. Hence, G_0 is the 0–component of G: $\mathrm{c}(G) = G_0$. The equality $\varphi_0^\wedge[A_0^\wedge] = \mathrm{tor}(A)^\perp$ follows from Lemma 5.

(2) $\mathrm{c}(G)$ is divisible as the dual of a torsion-free group. The rest is evident.

(3) It follows from the definition that $\Delta_0 \subseteq \Delta \cap \mathrm{c}(G)$. On the other hand, let $x \in \Delta \cap \mathrm{c}(G)$. Then $0 = \mathrm{restr}_G(x) = (\mathrm{restr}_G \circ \mathrm{ins}_\Delta)(x) = \mathrm{restr}_\Delta(x)$, hence, $x \in \Delta_0$. This proves the equality $\Delta_0 = \Delta \cap \mathrm{c}(G)$.

The topological isomorphism $A_0^\wedge \to \mathrm{c}(G) = \varphi_0^\wedge[A_0^\wedge]$ maps the family $\mathcal{D}(A_0^\wedge)$ onto $\mathcal{D}(\mathrm{c}(G))$. Thus, the annihilator $(A_0^\wedge, (F \oplus T)/T)$, a typical member of $\mathcal{D}(A_0^\wedge)$, is mapped onto $\Delta_0 = (G, F) \cap \mathrm{c}(G)$, a typical member of $\mathcal{D}(\mathrm{c}(G))$. Therefore, $\mathcal{D}(\mathrm{c}(G)) = \{D \cap \mathrm{c}(G) \mid D \in \mathcal{D}(G)\}$ and $\mathbf{\Delta}(\mathrm{c}(G)) = \mathbf{\Delta}(G) \cap \mathrm{c}(G)$.

(4) We have $\Delta + \mathrm{c}(G) = \psi^\wedge[(A/F)^\wedge] + \varphi_0^\wedge[A_0^\wedge] = (G, F) + (G, T) = (G, F \cap T) = G$. By (4) we have $\Delta + \mathbf{\Delta}(\mathrm{c}(G)) = \Delta + (\mathrm{c}(G) \cap \mathbf{\Delta}(G)) = (\Delta + \mathrm{c}(G)) \cap \mathbf{\Delta}(G) = \mathbf{\Delta}(G)$.

(5) Follows immediately from (3) and (4).

(6) $\mathrm{c}(G)$ is divisible and $\mathbf{\Delta}(\mathrm{c}(G))$ is pure in $\mathrm{c}(G)$. Hence, $\mathbf{\Delta}(\mathrm{c}(G))$ is divisible.

(7) We have the following commutative diagram with exact row and natural maps:

$$
\begin{array}{ccccc}
\Delta_0 & \xrightarrow{\mathrm{ins}} & \Delta & \xrightarrow{\alpha} & \frac{\Delta}{\Delta_0} \\
\downarrow{\scriptstyle \mathrm{ins}} & & \downarrow{\scriptstyle \mathrm{ins}} & & \downarrow{\scriptstyle \xi} \\
\mathbf{\Delta}(\mathrm{c}(G)) & \xrightarrow{\mathrm{ins}} & \mathbf{\Delta}(G) & \longrightarrow & \frac{\mathbf{\Delta}(G)}{\mathbf{\Delta}(\mathrm{c}(G))}
\end{array}
$$

The map ξ is injective because $\Delta \cap \mathbf{\Delta}(\mathrm{c}(G)) = \Delta_0$. By (5), $\mathbf{\Delta}(G) = \Delta + \mathbf{\Delta}(\mathrm{c}(G))$, so ξ is surjective. To show that ξ is continuous, let U be open in $\mathbf{\Delta}(G)/\mathbf{\Delta}(\mathrm{c}(G))$. By commutativity of the right square in the diagram, $W := \{x \in \Delta \mid \xi(\alpha(x)) \in U\}$ is open in Δ, thus $\xi^{-1}[U] = \alpha[W]$ is open in Δ/Δ_0. Therefore, ξ is continuous. Since Δ/Δ_0 is compact, we conclude that ξ is a topological isomorphism. □

A number of results on free subgroups are worth dualizing.

Theorem 8. *Let $G = A^\wedge$ be a compact abelian group of infinite dimension.*

(1) *Suppose that* D *is a closed subgroup of* G *such that* G/D *is a torus. Then there exists* $\Delta \in \mathcal{D}(G)$ *such that* $D \cap \Delta \cap \mathrm{c}(G) = 0$. *In particular, for every* $\Delta \in \mathcal{D}(G)$, *there is* $\Delta' \in \mathcal{D}(G)$ *such that* $\Delta \cap \Delta' \cap \mathrm{c}(G) = 0$.

(2) *There exists* $\Delta \in \mathcal{D}(G)$ *such that* $\Delta_0 = \Delta \cap \mathrm{c}(G) \in \mathcal{D}(\mathrm{c}(G))$ *is torsion-free.*

Proof. (1) Let $F = D^\perp$. Then $F \cong_{\mathrm{t}} (G/D)^\wedge$ is a free subgroup of A. So $\varphi_0[F]$ is free in A_0. By Theorem 2(a), there exists a free subgroup F_1 of A_0 such that $A_0 = F + F_1$. By enlarging F_1 if necessary we may assume that F_1 is maximal, i.e., full free. There exists a (full) free subgroup F_2 of A such that $\varphi_0[F_2] = F_1$. Then $A = F + F_2 + \mathrm{tor}(A)$ and $0 = F^\perp \cap F_2^\perp \cap \mathrm{tor}(A)^\perp$. The claim is established by setting $\Delta' = F_2^\perp$.

(2) By Theorem 2(b), there exists a full free subgroup F_0 of A_0 such that A_0/F_0 is divisible. There exists $F \in \mathcal{F}(A) : \varphi_0[F] = F_0$ and $\frac{\mathrm{tor}(A)\oplus F}{F} \rightarrowtail \frac{A}{F} \twoheadrightarrow \frac{A_0}{F_0}$ is exact. By duality $(A_0/F_0)^\wedge \rightarrowtail (A/F)^\wedge \twoheadrightarrow \mathrm{tor}(A)^\wedge$. As A_0/F_0 is divisible, its dual is torsion-free ([1], corollary 8.5, p. 410), so $\Delta_0 := (A_0/F_0)^\wedge \in \mathcal{D}(A_0^\wedge)$ is torsion-free. Modulo embeddings $\Delta_0 \subseteq \Delta = (A/F)^\wedge \in \mathcal{D}(A)$ and $\Delta_0 = \Delta \cap \mathrm{c}(G)$. $\square$

Corollary 2. *Let $G = A^\wedge$ be a compact connected abelian group of infinite dimension, i.e., A is torsion-free of infinite torsion-free rank.*

(1) Suppose that D is a subgroup of G such that G/D is a torus. Then there exists a subgroup D' of G such that $D \cap D' = 0$ and G/D' is a torus. In particular, for every $\Delta \in \mathcal{D}(G)$, there is $\Delta' \in \mathcal{D}(G)$ such that $\Delta \cap \Delta' = 0$.

(2) There exists a torsion-free $\Delta \in \mathcal{D}(G)$.

We can easily settle the question when $\mathbf{\Delta}(G)$ is as small as possible, i.e., $\mathbf{\Delta}(G) = \mathrm{tor}(G)$.

Theorem 9. *Let $G = A^\wedge$ be a compact abelian group. Then $\mathbf{\Delta}(G) = \mathrm{tor}(G)$ if and only if $G \cong_t \mathbb{T}^n \times E$ where $n \in \mathbb{N}_0$ and E is bounded.*

Proof. We only need to consider the consequences of $\mathbf{\Delta}(G)$ being a torsion group. As $\mathbf{\Delta}(G) = \sum \mathcal{D}(G)$, this occurs if and only if every $\Delta \in \mathcal{D}(G)$ is a torsion group. Since Δ is compact, it must be bounded torsion. Furthermore, we use that for every $F \in \mathcal{F}(A)$, the dual $(A/F)^\wedge$ is topologically isomorphic to some $\Delta \in \mathcal{D}(G)$, so a bounded torsion group.

(a) Assume first that A is torsion-free. By Corollary 2(2) we have $n := \mathrm{rk}(A) < \infty$. Now pick an arbitrary $F \in \mathcal{F}(A)$. Since $(A/F)^\wedge$ is a bounded torsion group, so is A/F, hence, $mA \subseteq F$ for some $m \in \mathbb{N}$, so $A \cong mA$ is free of rank n and $G \cong_t \mathbb{T}^n$.

(b) In the general situation, by Theorem 7(3), $\mathbf{\Delta}(\mathrm{c}(G))$ must be a torsion group and hence by (b), $A/\mathrm{tor}(A)$ must be free of finite rank. So $A = F \oplus \mathrm{tor}(A)$ for some finite rank free subgroup F of A, thus $G \cong_t H \times E$ with $H \cong_t \mathbb{T}^n$ and $E \cong_t \mathrm{tor}(A)^\wedge \cong_t (A/F)^\wedge$. For the latter group to be torsion, it must be bounded. $\square$

Remark 2. *The dual concept (in the category sense of reversing arrows) of $\mathcal{F}(A)$ is the family $\mathcal{K}(A) := \{\mathrm{Ker}(\psi) \mid \psi \in \mathrm{Hom}(A, F), F \text{ free}\}$. It is easy to see that $\mathcal{K}(A)$ is closed under finite intersections and $\mathfrak{K}(A) = \bigcap \mathcal{K}(A) = \bigcap\{\mathrm{Ker}(\psi) \mid \psi \in \mathrm{Hom}(A, \mathbb{Z})\}$ is a fully invariant subgroup of A that has no free direct summands. Mostly we have $\mathfrak{K}(A) = A$, e.g., if A is divisible or torsion. We call A* ***free-reduced*** *if $\mathrm{Hom}(A, \mathbb{Z}) = \{0\}$, equivalently, if A has no free direct summands. Evidently, A* is free-reduced if and only if *$G = A^\wedge$* is torus-free. *For a compact group G, let $\mathcal{T}(G) = \{T \mid T \text{ is a torus subgroup of } G\}$.*

For any $A \in \mathrm{AG}$, and a short exact sequence $K \overset{\mathrm{ins}}{\rightarrowtail} A \overset{\varphi_F}{\twoheadrightarrow} F$ where F is free, it follows that $F^\wedge \overset{\varphi_F^\wedge}{\rightarrowtail} G := A^\wedge \overset{\mathrm{restr}}{\twoheadrightarrow} K^\wedge$ is exact in LCA. *Hence, $\varphi_F^\wedge[F^\wedge] = K^\perp$ is a torus subgroup of G and we have a map*

$$\kappa : \mathcal{K}(A) \to \mathcal{T}(G) : \kappa(K) = K^\perp.$$

As for $\mathcal{F}(A)$ and $\mathcal{D}(G)$ it follows that κ is a bijective map satisfying $\kappa(K_1 \cap K_2) = \kappa(K_1) + \kappa(K_2)$ and $K_1 \subseteq K_2$ if and only if $\kappa(K_2) \subseteq \kappa(K_1)$. In particular, $\mathfrak{T}(G) := \sum \mathcal{T}(G) = \bigcup \mathcal{T}(G)$ and $\kappa(\mathfrak{K}(A)) = (\mathfrak{K}(A))^\perp = \mathfrak{T}(G)$. This recaptures most of the results of ([1], p.p 440, 441). The group $A/\mathfrak{K}(A)$ need not be free and the dual group $\mathfrak{T}(G)$ need not be a torus.

A theorem of K. Stein ([17], Corollary 8.3, p. 114) says that every countable torsion-free group A_0 has a decomposition $A_0 = F \oplus \mathfrak{K}(A_0)$ where F is free. The duals of countable torsion-free groups are exactly the compact connected metric groups ([1], pp. 447–450).

Remark 3. *One can ask further whether a compact group has other connected factors of dimension 1 (so-called solenoids of which $\mathbb{T}$ is an example). For finite dimensional connected compact groups this leads to the "Main Decomposition" that was derived in [22].*

5. The Fat Delta of Compact Groups

So far we know from Theorem 6 that for any compact abelian group G,

- $\mathbf{\Delta}(G) = \sum \mathcal{D}(G) = \bigcup \mathcal{D}(G)$,
- $\mathbf{\Delta}(G)$ is dense in G,
- $\mathrm{tor}(G) \subseteq \mathbf{\Delta}(G)$ and $G/\mathbf{\Delta}(G)$ is torsion-free.
- $\mathbf{\Delta}(\mathrm{c}(G))$ is divisible.

In this section, we will establish further properties of Fat Delta. We start with a preliminary observation.

Lemma 7. *Let G and H be compact abelian groups and let $\alpha : G \to H$ be a continuous epimorphism. Then we have:*

(1) If G is totally disconnected, then so is H.
(2) If G is a torus, then so is H.

Proof. (1) Since α is surjective, the adjoint map $\alpha^{\wedge} : H^{\wedge} \to G^{\wedge}$ is injective ([23], (24.38), p. 392). Assume that G is totally disconnected. Then $G^{\wedge}$ is torsion, thus $H^{\wedge}$ is torsion and therefore H is totally disconnected (see [23], (24.26), p. 385).

(2) Now suppose G is a torus. Then $G^{\wedge}$ is free, so since subgroups of free groups are free, $H^{\wedge}$ is free. Thus, H is a torus. □

In general, Fat Delta does not contain every totally disconnected subgroup. However, it contains all closed totally disconnected subgroups:

Proposition 5. *Let G be a compact abelian group and D a closed totally disconnected subgroup of G. Then $D \subseteq \mathbf{\Delta}(G)$. Thus $\mathbf{\Delta}(G)$ is the subgroup of G generated by all closed totally disconnected subgroups of G.*

Proof. Choose $\Delta \in \mathcal{D}(G)$. Then $\Delta + D$ is compact ([23], (4.4), p. 17), and the natural map $\alpha : \Delta \times D \to \Delta + D$ is a continuous epimorphism. By Lemma 7(1), $\Delta + D$ is totally disconnected. Now consider the continuous epimorphism $\beta : G/\Delta \to (G/\Delta)/[(\Delta + D)/\Delta] \cong_{\mathrm{t}} G/(\Delta + D)$ (see [23], (5.35), p. 45). Since G/Δ is a torus, so is $G/(\Delta + D)$ by Lemma 7(2). This means that $\Delta + D \in \mathcal{D}(G)$, so $D \subseteq \Delta + D \subseteq \mathbf{\Delta}$ as claimed. □

The significance of Proposition 5 is that it shows that for a compact group G Fat Delta $\mathbf{\Delta}(G)$ coincides with the subgroup $\mathrm{td}(G)$ that is defined and motivated by totally different considerations (see Definition 4 and Proposition 9(2)). For the sake of easy reference, we list the results that could be proved easily in the present context but are proved in greater generality in the exhaustive study of $\mathrm{td}(G)$ in Section 7.

Proposition 6. *Let G be a compact abelian group. Then the following are true.*

(1) $\mathbf{\Delta}(G)$ is zero-dimensional, in particular totally disconnected (Theorem 19). Consequently, if G is not totally disconnected, then $G \neq \mathbf{\Delta}(G)$ and hence $\mathbf{\Delta}(G)$ is not a locally compact subgroup of G.
(2) Any countable extension of $\mathbf{\Delta}(G)$ is zero-dimensional (in particular totally disconnected) as well (Proposition 11).

$\mathbf{\Delta}(A^{\wedge}) = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$

We first establish some background.

Lemma 8. *Let G, H, K be topological abelian groups.*

(1) *Suppose that H is a topological subgroup of K such that for all* $f \in \mathrm{cHom}(G,K)$ *we have* $f[G] \subseteq H$. *Let* $\mathrm{ins} : H \to K$ *be the insertion. Then* $\mathrm{ins}_* : \mathrm{cHom}(G,H) \to \mathrm{cHom}(G,K) : \mathrm{ins}_*(f) = \mathrm{ins} \circ f$ *is a topological isomorphism.*

(2) *Suppose that* $H \overset{\alpha}{\rightarrowtail} K \overset{\beta}{\twoheadrightarrow} L$ *is a short exact sequence in* TAG, α *is proper, and G is some other topological group. Then*

$$\mathrm{cHom}(G,H) \overset{\alpha_*}{\rightarrowtail} \mathrm{cHom}(G,K) \overset{\beta_*}{\to} \mathrm{cHom}(G,L), \text{ where } \alpha_*(f) = \alpha \circ f,\ \beta_*(f) = \beta \circ f,$$

is an exact sequence in TAG *and* α_* *is proper. The map* β_* *is not claimed to be surjective.*

(3) *Let* $A \overset{\alpha}{\rightarrowtail} B \overset{\beta}{\twoheadrightarrow} C$ *be a short exact sequence of discrete groups and let G be a divisible topological group. Then*

$$\mathrm{cHom}(C,G) \overset{\beta^*}{\rightarrowtail} \mathrm{cHom}(B,G) \overset{\alpha^*}{\twoheadrightarrow} \mathrm{cHom}(A,G), \text{ where } \beta^*(f) = f \circ \beta,\ \alpha^*(f) = f \circ \alpha,$$

is an exact sequence of topological groups. In addition, β^* *is proper.*

(4) *For a discrete torsion group* $T = \bigoplus_{p \in \mathbb{P}} \mathrm{tor}_p(T)$, *we have* $\mathrm{cHom}(T, \mathbb{Q}/\mathbb{Z}) \cong_t T^\wedge$, *the topological isomorphism being* ins_*, *and* $T^\wedge \cong_t \prod_{p \in \mathbb{P}} (\mathrm{tor}_p(T))^\wedge$ *where* $(\mathrm{tor}_p(T))^\wedge \cong_t \mathrm{Hom}(\mathrm{tor}_p(T), \mathbb{Z}(p^\infty))$.

Proof. (1) It is evident that ins_* is bijective and maps $W(C, H \cap V)$ onto $W(C,V)$ where C is compact in G and V is open in K.

(2) By standard discrete homological algebra

$$\mathrm{Hom}(G,H) \overset{\alpha_*}{\rightarrowtail} \mathrm{Hom}(G,K) \overset{\beta_*}{\to} \mathrm{Hom}(G,L) \to \mathrm{Ext}(G,H)$$

is exact in AG. Let $f \in \mathrm{cHom}(G,H)$. Then $\alpha_*(f) = \alpha \circ f$ is continuous and α_* is well-defined. Similarly, $\beta_* : \mathrm{cHom}(G,K) \to \mathrm{cHom}(G,L)$ is well-defined.

To show that α_* is continuous, let C be a compact subset of G and let U_K be an open neighborhood of $0 \in K$. Then $V := W(C, U_K)$ is a basic open neighborhood of $0 \in \mathrm{cHom}(G,K)$. It follows that $U := W(C, \alpha^{-1}[U_K])$ is an open neighborhood of $0 \in \mathrm{cHom}(G,H)$ and α_* maps U into V as is easily checked.

We show next that our sequence is exact at $\mathrm{cHom}(G,K)$. As $\beta_* \circ \alpha_* = (\beta \circ \alpha)_* = 0$ we have $\mathrm{Im}(\alpha_*) \subseteq \mathrm{Ker}(\beta_*)$. To show that $\mathrm{Ker}(\beta_*) \subseteq \mathrm{Im}(\alpha_*)$, let $f \in \mathrm{Ker}(\beta_*)$. By the discrete exactness there exist $g \in \mathrm{Hom}(G,H)$ such that $f = \alpha \circ g$. To conclude, we need to show that g is continuous. To do so let U be open in H. By assumption α is proper, hence, there is an open set $V \subseteq K$ such that $\alpha[U] = \alpha[H] \cap V$. Then $U \subseteq \alpha^{-1}[\alpha[H] \cap V] = \alpha^{-1}[V]$ and actually $U = \alpha^{-1}[V]$. In fact, let $x \in H$ such that $\alpha(x) \in V \cap \alpha[H] = \alpha[U]$. So there exists $x' \in U$ such that $\alpha(x) = \alpha(x')$ and as α is injective, $x = x' \in U$. We now get that $g^{-1}[U] = g^{-1}[\alpha^{-1}[V]] = f^{-1}[V]$ is open in G, showing that g is continuous.

It remains to show that α_* is proper. Let C be compact in G and U open in H. Then $W(C,U)$ is a generic open set in $\mathrm{cHom}(G,H)$ and $W(C,V)$, where $\alpha[U] = \alpha[H] \cap V$, is open in $\mathrm{cHom}(G,K)$. We claim that

$$\alpha_*[W(C,U)] = \alpha_*[\mathrm{cHom}(G,H)] \cap W(C,V).$$

Let $f \in W(C,U)$. Then $f[C] \subseteq U$ and hence $\alpha_*(f)[W(C,U)] \subseteq \alpha_*[\mathrm{cHom}(G,H)] \cap W(C,V)$ because $(\alpha \circ f)[C] \subseteq \alpha[U] \subseteq V$.

Now let $g \in \alpha_*[\mathrm{cHom}(G,H)] \cap W(C,V)$. Then there is $f \in \mathrm{cHom}(G,H)$ such that $g = \alpha \circ f$. We show that $f \in W(C,U)$. In fact, $(g[C] \subseteq V) \implies ((\alpha[f[C]] \subseteq \alpha[H] \cap V = \alpha[U]$. As α is injective it follows that $f[C] \subseteq U$, i.e., $f \in W(C,U)$.

(3) By discrete abelian group theory we have that $\mathrm{Hom}(C,G) \overset{\beta^*}{\rightarrowtail} \mathrm{Hom}(B,G) \overset{\alpha^*}{\twoheadrightarrow} \mathrm{Hom}(A,G) \to \mathrm{Ext}(C,G)$ is exact and $\mathrm{Ext}(C,G) = \{0\}$ as G is divisible. We have $\mathrm{cHom} = \mathrm{Hom}$ as A, B, C are discrete, so the exactness of the claimed sequence is clear.

We need to show that β^* and α^* are continuous when the Hom groups are given the compact-open topology.

To show that β^* is continuous, let K be a compact (=finite) subset of B and let U_G be an open neighborhood of $0 \in G$. Then $V := W(K, U_G)$ is a generic open neighborhood of $0 \in \mathrm{cHom}(B, G)$. Hence, $U := W(\beta[K], U_G)$ is an open neighborhood of $0 \in \mathrm{cHom}(C, G)$. Let $f \in U$. Then $f[\beta[K]] = \beta^*(f)[K] \subseteq U_G$, i.e., $\beta^*(f) \in V$ showing that β^* is continuous. Similarly, α^* is continuous.

It remains to show that β^* is proper. The set $V := W(K, U_G)$, where K is compact (=finite) in C and U_G open in G, is a generic open subset of $\mathrm{cHom}(C, G)$. As β is surjective, there is a finite subset K' of B such that $\beta[K'] = K$. Then $U := W(K', U_G)$ is an open subset of $\mathrm{cHom}(B, G)$. We claim that $\beta^*[V] = U \cap \beta^*[\mathrm{cHom}(C, G)]$. In fact, $f \in V$ means that $f[K] \subseteq U_G$ and hence $\beta^*(f)[K'] = f[\beta[K']] = f[K] \subseteq U_G$, so $\beta^*[V] \subseteq U \cap \beta^*[\mathrm{cHom}(C, G)]$. To show equality, let $g \in U \cap \beta^*[\mathrm{cHom}(C, G)]$. Then there exists $f \in \mathrm{cHom}(C, G)$ such that $g = f \circ \beta$ and $U_G \supset g[K'] = (f \circ \beta)[K'] = f[\beta[K']] = f[K]$, so $f \in V$.

(4) By Lemma 8(1) we have $\mathrm{cHom}(T, \mathbb{Q}/\mathbb{Z}) \cong_t \mathrm{cHom}(T, \mathbb{T}) \cong_t T^\wedge$. By Proposition 2 $T^\wedge \cong_t \prod_{p \in \mathbb{P}} (\mathrm{tor}_p(T))^\wedge$, and again Lemma 8(1) entails $(\mathrm{tor}_p(T))^\wedge \cong_t \mathrm{Hom}(\mathrm{tor}_p(T), \mathbb{Z}(p^\infty))$. □

Lemma 9. *Let K, X, Y, K', X', Y' be topological abelian groups. It is assumed that the diagram*

$$\begin{array}{ccccc} K & \xhookrightarrow{\text{ins}} & X & \xrightarrow{\alpha} & Y \\ \downarrow{\scriptstyle \xi_K} & & \downarrow{\scriptstyle \xi_X} & & \downarrow{\scriptstyle \eta} \\ K' & \xhookrightarrow{\text{ins}} & X' & \xrightarrow{\beta} & Y' \end{array}$$

is commutative, all maps are continuous, its rows are exact, ξ_X is proper, β is a quotient map, i.e., β is open, and ξ_K is an isomorphism. Then α is a quotient map.

Proof. Let U be open in X. As ξ is proper, there is an open set V of X' such that $\xi_X[U] = \xi_X[X] \cap V$. We claim that $\alpha[U] = \eta^{-1}[\beta[V]]$ that is open in Y.

First let $x \in U$. Then $\eta(\alpha(x)) = \beta(\xi_X(x)) \in \beta[V]$, hence, $\alpha(x) \in \eta^{-1}\beta[V]$.

Now suppose that $y \in \eta^{-1}[\beta[V]] \subseteq Y$. Then $\eta(y) = \beta(v)$ for some $v \in V$. There exists $x \in X$ such that $\alpha(x) = y$. Hence, $\beta(\xi_X(x)) = \eta(\alpha(x)) = \beta(v)$, and thus, $v - \xi_X(x) \in \mathrm{Ker}(\beta)$. It follows that there exists $k \in K$ such that $\xi_K(k) = v - \xi_X(x)$ and so $\xi_X(k + x) = v \in \xi[X] \cap V = \xi[U]$. As ξ is injective it follows that $k + x \in U$ and $\alpha(k + x) = \alpha(x) = y$. □

We have the proper short exact sequence of topological groups

$$\mathrm{E} : \mathbb{Q}/\mathbb{Z} \overset{\text{ins}}{\rightarrowtail} \mathbb{T} \overset{\gamma}{\twoheadrightarrow} \mathbb{R}/\mathbb{Q}$$

where, as usual, $\mathbb{T}$ is the quotient group of $\mathbb{R}$, $\mathbb{Q}/\mathbb{Z}$ the subgroup of $\mathbb{T}$, and $\mathbb{R}/\mathbb{Q}$ carries the quotient topology which is indiscrete as $\mathbb{Q}$ is dense in $\mathbb{R}$ (Lemma 2(1)).

Let A be a discrete group and F a full free subgroup of A of rank $\mathfrak{m} := \mathrm{rk}(A)$. We have exact sequences $F \overset{\text{ins}}{\rightarrowtail} A \overset{\varphi_F}{\twoheadrightarrow} A/F$ where A/F is a torsion group. We obtain a diagram as follows.

$$\begin{array}{ccccc} \mathrm{Hom}(A/F, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\varphi_F^*} & \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\text{ins}^*} & \mathrm{Hom}(F, \mathbb{Q}/\mathbb{Z}) \\ \downarrow{\scriptstyle \text{ins}_*} & & \downarrow{\scriptstyle \text{ins}_*} & & \downarrow{\scriptstyle \text{ins}_*} \\ \mathrm{Hom}(A/F, \mathbb{T}) & \xrightarrow{\varphi_F^*} & \mathrm{Hom}(A, \mathbb{T}) & \xrightarrow{\text{ins}^*} & \mathrm{Hom}(F, \mathbb{T}) \\ & & \downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle \gamma_*} \\ & & \mathrm{Hom}(A, \mathbb{R}/\mathbb{Q}) & \xrightarrow{\text{ins}^*} & \mathrm{Hom}(F, \mathbb{R}/\mathbb{Q}) \end{array} \tag{5}$$

(1) By standard discrete homological algebra the diagram is commutative and rows and columns are exact.
(2) All the domains of the Hom groups carry the discrete topology, hence cHom = Hom in all cases.
(3) All Hom groups in the diagram carry the compact-open topology. It follows from Lemma 8(2) that all the maps $(\cdot)_*$ are continuous. It follows from Lemma 8(3) that all the maps $(\cdot)^*$ are continuous.
(4) By Lemma 8(1) the left most ins_* is a topological isomorphism.
(5) By Lemma 8(2) columns 2 and 3 are exact in TAG.
(6) By Lemma 8(3) the three rows are exact in TAG.
(7) The situation of Lemma 9 matches the top part of (5) and we conclude that $\mathrm{ins}^* : \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(F, \mathbb{Q}/\mathbb{Z})$ is a quotient map. It is easy to see that $\mathrm{ins}^* : \mathrm{Hom}(A, \mathbb{R}/\mathbb{Q}) \to \mathrm{Hom}(F, \mathbb{R}/\mathbb{Q})$ is an isomorphism. Since both groups are indiscrete (by Lemma 2(3)), this is a topological isomorphism.

Theorem 10. *Let $G = A^\wedge$ where $A \in \mathrm{AG}$ has torsion-free rank $\mathfrak{m}$. Then:*
(1) $\mathbf{\Delta}(G) = \mathrm{ins}_*[\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})] \subseteq G$ *where* $\mathrm{ins} : \mathbb{Q}/\mathbb{Z} \to \mathbb{T}$.
(2) $G/\mathbf{\Delta}(G) \cong \mathbb{R}^{\mathfrak{m}}$. *Algebraically,* $\mathrm{c}(G) = \mathbf{\Delta}(\mathrm{c}(G)) \oplus K$ *where* $K \cong \mathbb{R}^{\mathfrak{m}}$.
(3) If $G_i = A_i^\wedge$ where $A_i \in \mathrm{AG}$ $(i \in I)$, then $\mathbf{\Delta}(\prod_{i\in I} G_i) \cong_{\mathrm{t}} \prod_{i\in I} \mathbf{\Delta}(G_i)$.

Proof. (1) Row 2 of (5) implies that $\varphi_F^*(\mathrm{Hom}(A/F, \mathbb{T})) \subseteq \mathrm{ins}_*[\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})]$ where $\Delta = \varphi_F^*(\mathrm{Hom}(A/F, \mathbb{T}))$ is a delta subgroup of G. As F was arbitrary it follows that $\mathbf{\Delta}(G) \subseteq \mathrm{ins}_*[\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})]$. It remains to show that $\mathbf{\Delta}(G) \supset \mathrm{ins}_*[\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})]$.

Let $f \in \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ and set $K = \mathrm{Ker}(f)$. Then $f[A] \subseteq \mathbb{Q}/\mathbb{Z}$ is a torsion group, so A/K is a torsion group and any full free subgroup F of K is a full free subgroup of A. Let F be so given. Then $g : A/F \to \mathbb{Q}/\mathbb{Z} : g(a + F) = f(a)$ is a well-defined homomorphism and $f = g \circ \varphi_F = \varphi_F^*(g)$, so $f \in \Delta \subseteq \mathbf{\Delta}(G)$.

(2) We have algebraic isomorphisms $G/\mathbf{\Delta}(G) \cong \mathrm{Hom}(A, \mathbb{R}/\mathbb{Q}) \cong \mathrm{Hom}(F, \mathbb{R}/\mathbb{Q}) \cong (\mathbb{R}/\mathbb{Q})^{\mathfrak{m}} \cong \mathbb{R}^{\mathfrak{m}}$, the first isomorphism granted by exactness of column 2 of (5), the second isomorphism by (7), and the remaining isomorphism is easy to see. The group $\mathbf{\Delta}(\mathrm{c}(G))$ is divisible as observed earlier (see Theorem 6(6)), so it is algebraically a direct summand of $\mathrm{c}(G)$. Since $\mathrm{rk}(A_0) = \mathfrak{m}$, applying the above argument to $c(G) = A_0^\wedge$ we deduce that $c(G)/\mathbf{\Delta}(c(G)) \cong (\mathbb{R}/\mathbb{Q})^{\mathfrak{m}}$. Therefore, we have $\mathrm{c}(G) = \mathbf{\Delta}(\mathrm{c}(G)) \oplus K$, with $K \cong c(G)/\mathbf{\Delta}(c(G)) \cong (\mathbb{R}/\mathbb{Q})^{\mathfrak{m}} \cong \mathbb{R}^{\mathfrak{m}}$.

(3) Set $A = \bigoplus_{i\in I} A_i$ and $G = \prod_{i\in I} G_i$. Then we have $G = \prod_{i\in I} A_i^\wedge \cong_{\mathrm{t}} A^\wedge$, so $G \cong_{\mathrm{t}} A^\wedge$ and $\mathbf{\Delta}(G) \cong_{\mathrm{t}} \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \cong_{\mathrm{t}} \prod_{i\in I} \mathrm{Hom}(A_i, \mathbb{Q}/\mathbb{Z}) \cong_{\mathrm{t}} \prod_{i\in I} \mathbf{\Delta}(G_i)$. □

From now on, we will identify $\mathbf{\Delta}(G)$ with $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ if $G = A^\wedge$ is compact. The next corollary, establishing that $\mathbf{\Delta}$ is a functorial subgroup and showing that $\mathbf{\Delta}$, as a functor, preserves exactness of short sequences of compact groups, will be reproved in greater generality in Proposition 10.

Corollary 3. *(1) Let G and H be compact abelian groups and $g \in \mathrm{cHom}(G, H)$. Then $g(\mathbf{\Delta}(G)) \subseteq \mathbf{\Delta}(H)$; in particular $\mathbf{\Delta}(G)$ is fully invariant in G and if $G \leq H$, then $\mathbf{\Delta}(G) \leq \mathbf{\Delta}(H)$.*
(2) Let G, H, K be compact abelian groups. Suppose that $G \rightarrowtail H \twoheadrightarrow K$ is a short exact sequence in TAG. Then $\mathbf{\Delta}(G) \rightarrowtail \mathbf{\Delta}(H) \twoheadrightarrow \mathbf{\Delta}(K)$ is a short exact sequence in TAG.

Proof. (1) Without loss of generality $G = A^\wedge$, $H = B^\wedge$ and $g = f^\wedge = f^*$ for some $f \in \mathrm{Hom}(B, A)$. Then $\mathbf{\Delta}(G) = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \xrightarrow{g} \mathrm{Hom}(B, \mathbb{Q}/\mathbb{Z}) = \mathbf{\Delta}(H)$.

(2) Without loss of generality $G = A^\wedge, H = B^\wedge, K = C^\wedge$ and $G \rightarrowtail H \twoheadrightarrow K$ is the dual of $C \rightarrowtail B \twoheadrightarrow A$. Then $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \rightarrowtail \mathrm{Hom}(B, \mathbb{Q}/\mathbb{Z}) \twoheadrightarrow \mathrm{Hom}(C, \mathbb{Q}/\mathbb{Z})$ is an exact sequence of topological groups (Lemma 8(3)). □

The next proposition shows how Fat Delta can be used to recognize finite-dimensional compact groups.

Proposition 7. *Let G be a compact abelian group and $\Delta \in \mathcal{D}(G)$ with $G/\Delta \cong_t \mathbb{T}^\kappa$, where $\kappa = \dim(G) = \mathrm{rk}(G^\wedge)$. Then $\mathbf{\Delta}(G)/\Delta \cong (\mathbb{Q}/\mathbb{Z})^\kappa$. In particular, $\mathbf{\Delta}(G)/\Delta = \mathrm{tor}(G/\Delta)$ if and only if G is finite-dimensional.*

Proof. To the proper short exact sequence $\Delta \rightarrowtail G \twoheadrightarrow G/\Delta$ apply Corollary 3(2) to deduce that $\mathbf{\Delta}(G)/\Delta \cong \mathbf{\Delta}(G/\Delta)$, due to the fact that $\Delta = \mathbf{\Delta}(\Delta)$. Since $G/\Delta \cong_t \mathbb{T}^\kappa$, we have $\mathbf{\Delta}(G)/\Delta \cong \mathbf{\Delta}(\mathbb{T}^\kappa) \cong_t (\mathbf{\Delta}(\mathbb{T}))^\kappa = (\mathbb{Q}/\mathbb{Z})^\kappa$ by Theorem 10(3). Since $(\mathbb{Q}/\mathbb{Z})^\kappa$ is torsion precisely when $\kappa < \infty$, we are done. □

We will use the following well-known result below.

Theorem 11 ([24], page 86, Corollary 8.48)**.** *Let G, C be Hausdorff abelian groups, assume that C is complete, H is a dense subgroup of G. Then every morphism $f : H \to C$ has a unique extension $\overline{f} : G \to C$.*

Theorem 12. *Let G and H be compact abelian groups. Then $G \cong_t H$ if and only if $\mathbf{\Delta}(G) \cong_t \mathbf{\Delta}(H)$.*

Proof. (a) Suppose $\phi : G \to H$ is an isomorphism of topological groups. By Corollary 3 applied to ϕ and ϕ^{-1}, we obtain $\phi(\mathbf{\Delta}(G)) = \mathbf{\Delta}(H)$, hence $\mathbf{\Delta}(G) \cong_t \mathbf{\Delta}(H)$.

(b) Let $f : \mathbf{\Delta}(G) \to \mathbf{\Delta}(H)$ be an isomorphism of topological groups. The group H is compact, hence complete, and $\mathbf{\Delta}(G)$ is dense in G. Hence, there is a unique extension morphism $\overline{f} : G \to H$ of f. Similarly, we have the unique continuous extension $\overline{f^{-1}} : H \to G$ of $f^{-1} : \mathbf{\Delta}(H) \to \mathbf{\Delta}(G)$. The morphism $\overline{f^{-1}} \circ \overline{f} : G \to G$ extends $\mathrm{id}_{\mathbf{\Delta}(G)} : \mathbf{\Delta}(G) \to \mathbf{\Delta}(G)$ which is also extended by id_G, hence, by uniqueness we have $\overline{f^{-1}} \circ \overline{f} = \mathrm{id}_G$. Similarly, $\overline{f} \circ \overline{f^{-1}} = \mathrm{id}_H$. Hence, $(\overline{f})^{-1} = \overline{f^{-1}}$ is continuous. □

Theorem 12 and Corollary 3 imply that $G \mapsto \mathbf{\Delta}(G)$ is a category equivalence on the category of compact abelian groups to the category of all $\mathbf{\Delta}(G)$. This calls for *a useful characterization of the class of topological groups that appear as $\mathbf{\Delta}(G)$ for some compact abelian group G.* So far we can say the following. If D is a topological group such that $D \cong_t \mathbf{\Delta}(G)$ for some compact group G, then the following are true.

(1) D is totally disconnected and zero-dimensional (Proposition 11).
(2) The completion $\widehat{D}$ of D is compact (i.e., D is precompact), $D = \mathbf{\Delta}(\widehat{D})$ and $\mathrm{tor}(D) = \mathrm{tor}(\widehat{D})$.
(3) D contains a directed family $\mathcal{D}$ of compact totally disconnected subgroups such that $D = \bigcup \mathcal{D}$.
(4) $D \in \mathrm{LCA}$ if and only if $\widehat{D}$ is totally disconnected, and, if so, $D = \widehat{D}$ is compact.
(5) D is totally minimal (Theorem 19).

Given a group D with all the required properties, we would have $\mathbf{\Delta}(\widehat{D}) \cong_t D$, i.e., the completion functor is the inverse of the functor $\mathbf{\Delta}$.

Theorem 12 and the preceding discussion suggest to study the structure of $\mathbf{\Delta}(G)$ for a given compact group G. We will attempt this below in the simplest possible case of solenoids. A ***solenoid*** is a compact connected group of dimension 1, i.e., the dual of a torsion-free group of rank 1. To do so, we will use a simple result on divisible hulls of discrete groups and Lemma 10 on divisible hulls of certain products of groups.

Lemma 10. *Let P be a set of prime numbers, X_p be discrete groups and $X = \prod_{p\in P} X_p$. For each $p \in P$, let D_p be a divisible hull of X_p. Let $D := \prod_{p\in P} D_p$. Assume that each D_p/X_p is a p-primary group. Let $\mathrm{D}(X)$ be a subgroup of D containing X such that $\mathrm{D}(X)/X = \mathrm{tor}(D/X)$. Then*

(1) $\mathrm{D}(X)$ *is a divisible hull of* X*,*
(2) $\mathrm{D}(X) = \prod^{\mathrm{loc}}_{p\in P}(D_p, X_p) := \{(d_p) \in D \mid d_p \in X_p \text{ for almost all } p \in P\}$*,*

(3) $\mathrm{D}(X)/X \cong \bigoplus_{p\in P} D_p/X_p$.

Proof. (1) Clearly D is divisible as a product of divisible groups and

$$D/\mathrm{D}(X) \cong (D/X)/(\mathrm{D}(X)/X) \cong (D/X)/(\mathrm{tor}(D/X))$$

is torsion-free, hence $\mathrm{D}(X)$ is pure in D and therefore divisible. It remains to show that X is essential in $\mathrm{D}(X)$. For any prime q, we have $D[q] = \prod_{p\in P} D_p[q] \leq \prod_{p\in P} X_p = X$. Indeed, for $q \neq p \in P$ we have $D_p[q] \leq X_p$ because D_p/X_p is p-primary, while $D_q[q] \subseteq X_q$ because D_q is the divisible hull of X_q.

(2) Let $(d_p) \in \mathrm{D}(X)$. Then $m(d_p) \in X$ for some $m \neq 0$ which requires that $\forall\, p \in P : md_p \in X_p$. Our hypotheses imply that $d_p \in X_p$ for all those p that do not divide m. So $\mathrm{D}(X) \subseteq \prod^{\mathrm{loc}}_{p\in P}(D_p, X_p)$ and equality is evident.

(3) The map $\xi : \mathrm{D}(X) \to \bigoplus_{p\in P} D_p/X_p : \xi((d_p)) = \sum_{p\in P} d_p + X_p$ is evidently well-defined, surjective, and $\mathrm{Ker}(\xi) = X$. □

Torsion-free groups A with $\mathrm{rk}(A) = 1$, ***rank-one groups*** for short, are discussed and classified in ([17], Chapter 12, Section 1). These are exactly the groups isomorphic with additive subgroups of $\mathbb{Q}$ containing $\mathbb{Z}$. Types are equivalence classes $[(h_p)_{p\in\mathbb{P}}]$ of "height sequences" $(h_p)_{p\in\mathbb{P}}$ where $0 \leq h_p \leq \infty$. Two height sequences are equivalent if they differ only at finitely many places where both sequences have finite entries. For the precise definition of type see Lemma 11(1) or ([17], p. 409, 411).

Two rank-one groups are isomorphic if and only if their types are equal.

Lemma 11 displays a representative rank-one group, its type, and dual solenoid. For a prime p, we define $\frac{1}{p^\infty}\mathbb{Z} := \left\langle \frac{1}{p^k}\mathbb{Z} \mid k \in \mathbb{N}\right\rangle$.

Lemma 11. *(1) Let $\mathbb{Z} \leq A \leq \mathbb{Q}$. Then there exist values h_p such that*

$$A = \left\langle \frac{1}{p^{h_p}}\mathbb{Z} \mid p \in \mathbb{P}, 0 \leq h_p \leq \infty \right\rangle, \frac{A}{\mathbb{Z}} \cong \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^{h_p}), \text{ and } \mathrm{tp}(A) = [(h_p)_{p\in\mathbb{P}}].$$

For $\mathbb{P}_\infty := \{p \mid h_p = \infty\}$, one has $p \in \mathbb{P}_\infty$ if and only if $pA = A$.

(2) Let $\Sigma = A^\wedge$ and $\Delta := (A/\mathbb{Z})^\wedge$. Then (with a harmless identification) $\Delta \in \mathcal{D}(\Sigma)$, and $\Delta \cong_{\mathrm{t}} \prod_{p\in\mathbb{P}} \widehat{\mathbb{Z}}(p^{h_p})$ where $\widehat{\mathbb{Z}}(p^\infty) = \widehat{\mathbb{Z}}_p$ is the group of p-adic integers and $\widehat{\mathbb{Z}}(p^{h_p}) = \mathbb{Z}(p^{h_p})$ is the cyclic group of order p^{h_p} for $h_p < \infty$. Furthermore, Σ and $\mathbf{\Delta}(\Sigma)$ are divisible, $\mathbf{\Delta}(\Sigma)/\Delta \cong \mathbb{Q}/\mathbb{Z}$, and $\mathrm{tor}(\Sigma) \subseteq \mathbf{\Delta}(\Sigma)$.

(3) $\mathrm{Soc}(\Sigma) = \bigoplus_{p\notin\mathbb{P}_\infty} \mathbb{Z}(p)$ and $\mathrm{tor}(\Sigma) = \bigoplus_{p\notin\mathbb{P}_\infty} \mathbb{Z}(p^\infty)$.

Proof. (1) Given $p \in \mathbb{P}$ either A contains every fraction $1/p^k$ (in which case $h_p = \infty$) or A contains a smallest fraction $1/p^{h_p}$. These fractions generate A and determine the type of A. (The h_p are the "p-heights" of $1 \in A$.)

To prove the last assertion, note that $pA = A$ implies $h_p = \infty$. Conversely, if $h_p = \infty$, then $\frac{1}{p^{h_p}}\mathbb{Z} = \langle \frac{p}{p^k}\mathbb{Z} \mid k \in \mathbb{N}\rangle = p\left(\frac{1}{p^{h_p}}\mathbb{Z}\right)$, thus $pA = A$.

(2) Σ is divisible by ([1], Corollary 8.5, p. 410). By Theorem 6(6) it follows that $\mathbf{\Delta}(\Sigma)$ is pure in Σ and hence is also divisible. By Proposition 7 $\mathbf{\Delta}(\Sigma)/\Delta \cong \mathbb{Q}/\mathbb{Z}$ and by Theorem 6(4) $\mathrm{tor}(\Sigma) \subseteq \mathbf{\Delta}(\Sigma)$. The rest is clear.

(3) By Corollary 1, $\mathrm{rk}_p(\Sigma) = \mathrm{rk}_p(A/pA) \leq 1$.

According to (1), $\mathrm{rk}_p(A/pA) > 0$ if and only if $A \neq pA$, i.e., when $h_p < \infty$. Hence, $\mathrm{rk}_p(\Sigma) > 0$ if and only if $h_p < \infty$ (i.e., when $p \notin \mathbb{P}_\infty$) and in this case $\mathrm{rk}_p(\Sigma) = 1$. This proves that $\mathrm{Soc}(\Sigma) = \bigoplus_{p\notin\mathbb{P}_\infty} \mathbb{Z}(p)$. As $\mathrm{tor}(\Sigma)$ is divisible, it is the divisible hull of $\bigoplus_{p\notin\mathbb{P}_\infty} \mathbb{Z}(p)$ and so $\mathrm{tor}(\Sigma) = \bigoplus_{p\notin\mathbb{P}_\infty} \mathbb{Z}(p^\infty)$. □

We illustrate the situation with some special cases.

Example 1. *(1) For a first concrete example, let* $A_1 = \sum_{p\in\mathbb{P}} \frac{1}{p}\mathbb{Z}$ *and* $\Sigma_1 = A_1^\wedge$. *Then* $\mathrm{tp}(A_1) = [(1,1,\ldots)]$, *and* $\mathbf{\Delta}(\Sigma_1)$ *is the divisible hull of* $\Delta = \prod_{p\in\mathbb{P}} \mathbb{Z}(p)$ *and* $\mathbf{\Delta}(\Sigma_1) = \prod^{\mathrm{loc}}_{p\in\mathbb{P}}(\mathbb{Z}(p^\infty), \mathbb{Z}(p))$.

(2) Next let $A_2 = \mathbb{Q}$. *Then* $\mathrm{tp}(A_2) = [(\infty,\infty,\ldots)]$, $\Sigma_2 = \mathbb{Q}^\wedge$ *is torsion-free,* $\Delta = \prod_{p\in\mathbb{P}} \widehat{\mathbb{Z}}_p$ *and* $\mathbf{\Delta}(\Sigma_2)$ *is the divisible hull of* Δ, *so* $\mathbf{\Delta}(\Sigma_2) = \prod^{\mathrm{loc}}_{p\in\mathbb{P}}(\widehat{\mathbb{Q}}_p, \widehat{\mathbb{Z}}_p)$ *where* $\widehat{\mathbb{Q}}_p = \frac{1}{p^\infty}\widehat{\mathbb{Z}}_p$ *is the additive group of p-adic numbers.*

(3) For $A_3 = \mathbb{Z}$, $\mathrm{tp}(A_3) = [(0,0,\ldots)]$, $\Sigma_3 = \mathbb{Z}^\wedge = \mathbb{T}$, $\Delta = \{0\}$, *but* $\mathrm{Soc}(\Sigma_3) = \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p) \subseteq \mathbf{\Delta}(\Sigma_3)$, $\mathbf{\Delta}(\Sigma_3)$ *is the divisible hull of* $\mathrm{Soc}(\Sigma_3)$, *so* $\mathbf{\Delta}(\Sigma_3) = \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty) = \mathbb{Q}/\mathbb{Z} = \mathrm{tor}(\Sigma_3)$.

Note that Δ is a particular δ–subgroup of Σ. Sometimes (e.g., (1), (2)), but not always (e.g., (3)), $\mathbf{\Delta}(\Sigma)$ is the divisible hull of Δ. In the general case additional δ–subgroups must be employed.

Proof. (1) In this case, $\forall\, p \in \mathbb{P} : h_p = 1$. By Lemma 11(1) $\mathbf{\Delta}(\Sigma_1)$ is the divisible hull of Δ, and the rest follows from Lemma 10.

(2) $\Sigma_2 = \mathbb{Q}^\wedge$ is torsion-free, $\mathrm{Soc}(\Sigma_2) = \{0\} \subseteq \Delta$, $A/\mathbb{Z} = \mathbb{Q}/\mathbb{Z} = \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)$, $\Delta := (A/\mathbb{Z})^\wedge = \prod_{p\in\mathbb{P}} \widehat{\mathbb{Z}}_p$, and $\mathbf{\Delta}(\Sigma)$ is the divisible hull of Δ.

(3) Clear. □

The next theorem deals with the general case. The relevance of the final assertion will become clear in Section 7.3 (see Definition 9 and Example 4, see also Problem 2).

Theorem 13. *Let* $A = \sum_{p\in\mathbb{P}} \frac{1}{p^{h_p}}\mathbb{Z}$. *Define* $\Sigma = A^\wedge$ *and* $\mathbb{P}_\infty$ *as above, and let*

$$\mathbb{P}_{\mathrm{fin}} := \{p \mid 0 < h_p < \infty\} \text{ and } \mathbb{P}_0 := \{p \mid h_p = 0\}.$$

Then $\mathbf{\Delta}(\Sigma)$ *is the divisible hull of* $\prod_{p\in\mathbb{P}_\infty} \widehat{\mathbb{Z}}_p \oplus \prod_{p\in\mathbb{P}_{\mathrm{fin}}} \mathbb{Z}(p^{h_p}) \oplus \bigoplus_{p\in\mathbb{P}_0} \mathbb{Z}(p)$, *so*

$$\mathbf{\Delta}(\Sigma) = \prod^{\mathrm{loc}}_{p\in\mathbb{P}_\infty}(\mathbb{Q}_p, \widehat{\mathbb{Z}}_p) \oplus \prod^{\mathrm{loc}}_{p\in\mathbb{P}_{\mathrm{fin}}}(\mathbb{Z}(p^\infty), \mathbb{Z}(p^{h_p})) \oplus \bigoplus_{p\in\mathbb{P}_0} \mathbb{Z}(p^\infty). \tag{6}$$

Moreover, $\mathrm{Soc}(\Sigma)$ *is dense in* Σ *if and only if* $\mathbb{P}_0$ *is infinite.*

Proof. $\Delta = \prod_{p\in\mathbb{P}_\infty} \widehat{\mathbb{Z}}_p \oplus \prod_{p\in\mathbb{P}_{\mathrm{fin}}} \mathbb{Z}(p^{h_p}) \subseteq \mathbf{\Delta}(\Sigma)$ and $\mathbf{\Delta}(\Sigma)$ is not the divisible hull of Δ if $\mathbb{P}_0 \neq \emptyset$. However, (Lemma 11) $D := \Delta \oplus \bigoplus_{p\in\mathbb{P}_0} \mathbb{Z}(p) \subseteq \mathbf{\Delta}(\Sigma)$ and $\mathrm{Soc}(\mathbf{\Delta}(\Sigma)) \subseteq \mathrm{Soc}(\Sigma) \subseteq D$. Hence, $\mathbf{\Delta}(\Sigma)$ is the divisible hull of D. Apply Lemma 10.

Recall that $\mathrm{Soc}(\Sigma) = \bigoplus_{p\in\mathbb{P}_0} \mathbb{Z}(p) \oplus \bigoplus_{p\in\mathbb{P}_{\mathrm{fin}}} \mathbb{Z}(p)$, where $\mathbb{Z}(p) = \Sigma[p]$ when the latter is non-trivial. Let $\phi : \Sigma \to \Sigma/\Delta = \mathbb{T}$. We claim that

$$\phi(\bigoplus_{p\in\mathbb{P}_0} \mathbb{Z}(p)) = \bigoplus_{p\in\mathbb{P}_0} \mathbb{T}[p]. \tag{7}$$

Indeed, if $t = \phi(x) \in (\Sigma/\Delta)[p]$ for some $x \in \Sigma$ and $p \in \mathbb{P}_0$, then $h_p = 0$ and $pt = 0$ in Σ/Δ, so $px \in \Delta$. It follows from the above description of Δ that Δ is p-divisible for $p \in \mathbb{P}_0$. Hence, $px = pz$ where $z \in \Delta$. Then $px - pz = 0$, so $x - z = \Sigma[p] = \mathbb{Z}(p)$. Therefore, $t = \phi(x) = \phi(x - z)$. This proves (7).

If $\mathbb{P}_0$ is infinite, $\bigoplus_{p\in\mathbb{P}_0} \mathbb{T}[p]$ is dense in $\mathbb{T}$, hence (7) implies that the compact subgroup $\Sigma_1 := \overline{\bigoplus_{p\in\mathbb{P}_0} \Sigma[p]}$ of Σ satisfies $\phi(\Sigma_1) = \overline{\bigoplus_{p\in\mathbb{P}_0} \mathbb{T}[p]} = \mathbb{T}$. Hence, $1 = \dim\mathbb{T} \leq \dim\Sigma_1 \leq \dim\Sigma = 1$ and consequently, $\dim\Sigma_1 = \dim\Sigma = 1$, hence $\dim\Sigma/\Sigma_1 = \dim\Sigma - \dim\Sigma_1 = 0$. Since Σ/Σ_1 is connected, this implies $\Sigma_1 = \Sigma$.

If $\mathbb{P}_0$ is finite, then Σ_1 is finite, while $\Sigma_2 = \overline{\bigoplus_{p\in\mathbb{P}_{\text{fin}}} \Sigma[p]} \leq \Delta$. Therefore, using again Lemma 11(3),

$$\overline{\text{Soc}(\Sigma)} = \overline{\bigoplus_{p\in\mathbb{P}_0} \Sigma[p] + \bigoplus_{p\in\mathbb{P}_{\text{fin}}} \Sigma[p]} = \Sigma_1 + \overline{\bigoplus_{p\in\mathbb{P}_{\text{fin}}} \Sigma[p]} = \Sigma_1 + \Sigma_2 \leq \Sigma_1 + \Delta \neq \Sigma,$$

since $\Sigma_1 + \Delta$ is a totally disconnected, while Σ is connected. □

Remark 4. *By Theorem 12, two compact groups are isomorphic if and only if their Fat Deltas are isomorphic as topological groups. A classification of Fat Deltas amounts to a classification of compact groups. A compact group is just the completion of its Fat Delta. Solenoids indicate the problems ahead.*

For any solenoid Σ, we have $\text{rk}_p(\Sigma) = 0$ for $p \in \mathbb{P}_\infty$, $\text{rk}_p(\Sigma) = 1$ for $p \in \mathbb{P}_{fin} \cup \mathbb{P}_0$. For the concrete examples $\text{rk}(\mathbf{\Delta}(\Sigma_1)) = 2^{\aleph_0}$, $\text{rk}(\mathbf{\Delta}(\Sigma_2)) = 2^{\aleph_0}$, while $\text{rk}(\mathbf{\Delta}(\Sigma_3)) = 0$. In general $\text{rk}(\mathbf{\Delta}(\Sigma)) = 2^{\aleph_0}$ except that $\text{rk}(\mathbf{\Delta}(\Sigma)) = 0$ for $\Sigma = \mathbb{Z}^\wedge = \mathbb{T}$. The algebraic invariants of $\mathbf{\Delta}(\Sigma)$ are the same for many non-isomorphic solenoids Σ. So the topological isomorphism class of $\mathbf{\Delta}(\Sigma)$, and hence of Σ, is in no way determined by these invariants. To distinguish between two Fat Deltas that are algebraically isomorphic one needs to know their topology. The description (6) involves the types of Σ. It may help in determining the topology of $\mathbf{\Delta}(\Sigma)$. Conversely, knowing the topology of $\mathbf{\Delta}(\Sigma)$ should make it possible to recapture the type of Σ.

6. Resolutions

The Resolution Theorem, a structure theorem for compact abelian groups, first appeared in [25] and later in an extended form in ([1], Theorem 8.20, p. 420), where it got its name.

Definition 3. *Recall that the "Lie algebra" of G, $\mathfrak{L}(G)$, defined as $\mathfrak{L}(G) = \text{cHom}(\mathbb{R}, G)$, is a real topological vector space via the stipulation $(rf)(x) := f(rx)$ where $f \in \mathfrak{L}(G)$ and $r, x \in \mathbb{R}$, and carries the topology of uniform convergence on compact sets ([1], Definition 5.7, p. 117, Proposition 7.36, p. 373). For every morphism $\varphi : G \to H$ in* TAG*, one obtains a morphism $\mathfrak{L}(\varphi) : \mathfrak{L}(G) \to \mathfrak{L}(H)$ in the category $\text{TAG}_\mathbb{R}$ of real topological vector spaces by letting $\mathfrak{L}(\varphi)(f) := \varphi \circ f$ for $f \in \mathfrak{L}(G)$. This defines a functor $\mathfrak{L} : \text{TAG} \to \text{TAG}_\mathbb{R}$ with the following useful properties:*

- *(i) ([1], Proposition 7.38(i), p. 374) $\mathfrak{L}(G) = \mathfrak{L}(c(G))$ and $\mathfrak{L}$ commutes with products, i.e., $\mathfrak{L}(\prod_i G_i) \cong_t \prod_i \mathfrak{L}(G_i)$.*
- *(ii) ([1], Proposition 7.38(ii), p. 374) If $\varphi : G \to H$ is a morphism in* TAG*, then $\mathfrak{L}(\varphi)$ is injective, whenever* Ker *φ is totally disconnected;*
- *(iii) ([1], Corollary 8.19, p. 419) if G is a compact group and $\Delta \in \mathcal{D}(G)$ with $G/\Delta = \mathbb{T}^\mathfrak{m}$, then, with $\varphi : G \to G/\Delta$, $\mathfrak{L}(\varphi) : \mathfrak{L}(G) \to \mathfrak{L}(G/\Delta) = \mathbb{R}^\mathfrak{m}$ is a topological isomorphism. The last equality is in fact a topological isomorphism obtained as composition of two others. The first one is the isomorphism $\mathfrak{L}(\mathbb{T}^\mathfrak{m}) \cong_t \mathfrak{L}(\mathbb{T})^\mathfrak{m}$ from (i). The second one is $\mathfrak{L}(\mathbb{T}) \cong_t \mathbb{R}$, that can be obtained from the obvious equality $\mathfrak{L}(\mathbb{T}) = \mathbb{R}^\wedge$, by letting $\rho : \mathbb{R} \to \mathfrak{L}(\mathbb{T}) : \rho(r)(x) = rx + \mathbb{Z}$ for $r, x \in \mathbb{R}$.*

The exponential map *is the morphism $\exp_G : \mathfrak{L}(G) \to G$ defined by $\exp(\chi) = \chi(1)$ ([1], p. 372). It "commutes" with morphisms $\varphi : G \to H$ in* TAG*, i.e., $\varphi \circ \exp_G = \exp_H \circ \mathfrak{L}(\varphi)$. This means that $\exp = (\exp_G)_{G\in\text{TAG}}$ is a natural transformation from the functor $\mathfrak{L}$ to the identity functor of* TAG*. For further properties of the "Lie algebra" $\mathfrak{L}(G)$ and the "exponential morphism" see ([1] Proposition 7.38, p. 374, Theorem 7.66, p. 395)). In particular, $\exp_\mathbb{T} : \mathfrak{L}(\mathbb{T}) \to \mathbb{T}$ is defined by $\exp_\mathbb{T}(\rho(r)) = r + \mathbb{Z}$ for $r \in \mathbb{R}$ and ρ as in (iii) above.*

We can now recall the original Resolution Theorem.

Proposition 8 ([25], Proposition 2.2)**.** *For a compact abelian group G there is a compact zero-dimensional subgroup* Δ *of G such that the homomorphism*

$$\varphi : \Delta \times \mathfrak{L}(G) \to G : \varphi((d,\chi)) = d + \exp(\chi)$$

satisfies the following conditions:

(1) φ *is continuous, surjective, and open, i.e., is a quotient morphism.*
(2) $\mathrm{Ker}(\varphi)$ *is algebraically and topologically isomorphic to* $\Gamma := \exp^{-1}[\Delta]$*, and* Γ *is a closed totally disconnected subgroup of* $\mathfrak{L}(G)$*. In particular, it does not contain any nonzero vector spaces.*
(3) $\varphi[\{0\} \times \mathfrak{L}(G)] = \exp[\mathfrak{L}(G)]$ *is dense in* $\mathrm{c}(G)$*, the identity component of G.*

In the above notation, one can prove also that $\exp[\mathfrak{L}(G)] = \mathrm{a}(G)$, the path connected component of 0, while $\mathrm{c}(G) = \overline{\mathrm{a}(G)}$ ([1], Theorem 8.30, p. 430 and Theorem 8.4, p. 409).

We first revisit the classical Resolution Theorem for compact connected groups of finite dimension with substantial additions as we determine the kernel of the resolution map φ explicitly up to topological isomorphism (see (4)).

Theorem 14 (Resolution Theorem)**.** *Let G be a compact abelian group of finite dimension* $n := \dim(G)$*. For* $\Delta \in \mathcal{D}(G)$ *define* $\varphi : \Delta \times \mathfrak{L}(G) \to G$ *by* $\varphi(d,\chi) = d + \exp(\chi)$ *for* $(d,\chi) \in \Delta \times \mathfrak{L}(G)$*. Then:*

(1) φ *is surjective, continuous, and open.*
(2) $\Gamma := \mathrm{Ker}(\varphi) = \{(-\exp(\chi),\chi) \mid \chi \in \exp^{-1}[\Delta]\}$*. The projection* $\Delta \times \mathfrak{L}(G) \to \mathfrak{L}(G)$ *maps* Γ *isomorphically onto* $\exp^{-1}[\Delta]$*, so* $\Gamma \cong_{\mathrm{t}} \exp^{-1}[\Delta]$*. Furthermore,* $\exp^{-1}[\Delta]$ *is a closed totally disconnected subgroup of* $\mathfrak{L}(G)$*.*
(3) $\mathfrak{L}(G) \cong_{\mathrm{t}} \mathbb{R}^n$*, in particular* $\dim_{\mathbb{R}}(\mathfrak{L}(G)) = n$*;*
(4) $\Gamma \cong_{\mathrm{t}} \mathbb{Z}^n$ *where* $\mathbb{Z}^n$ *carries the discrete topology, i.e., the subspace topology in* $\mathbb{R}^n$*.*
(5) $\exp[\exp^{-1}[\Delta]] = \Delta \cap \mathrm{a}(G)$ *is dense in* Δ*.*

Proof. (1) and (2) are part of ([1], Theorem 8.20, p. 420).

(3) Follows from (iii).

(4) By (2) the projection $\Delta \times \mathfrak{L}(G) \to \mathfrak{L}(G)$ induces a continuous epimorphism

$$G \cong_{\mathrm{t}} \frac{\Delta \times \mathfrak{L}(G)}{\Gamma} \twoheadrightarrow \frac{\mathfrak{L}(G)}{\exp^{-1}[\Delta]}$$

hence $\frac{\mathfrak{L}(G)}{\exp^{-1}[\Delta]}$ is compact. By ([1], Theorem A1.12.(i), p. 715) and (3), there is a basis $\{e_i\}$ of $\mathfrak{L}(G) \cong_{\mathrm{t}} \mathbb{R}^n$, i.e., $\mathfrak{L}(G) = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_n$, such that $\exp^{-1}[\Delta] \cong_{\mathrm{t}} \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_p \oplus \mathbb{Z}e_{p+1} \oplus \cdots \oplus \mathbb{Z}e_{p+q}$ and $\frac{\mathfrak{L}(G)}{\exp^{-1}[\Delta]} \cong_{\mathrm{t}} \mathbb{T}^q \oplus \mathbb{R}^{n-p-q}$. As $\exp^{-1}[\Delta]$ is totally disconnected we have $p = 0$, and as $\mathfrak{L}(G)/\exp^{-1}[\Delta]$ is compact, $0 = n - p - q = n - q$ and it follows that $q = n$.

(5) It is routine to verify that $\exp[\exp^{-1}[\Delta]] = \Delta \cap \mathrm{a}(G)$. Set $\mathbb{Z}_\Delta := \Delta \cap \mathrm{a}(G)$. It is easily seen that $\Gamma \subset \mathbb{Z}_\Delta \times \mathfrak{L}(G) \subset \overline{\mathbb{Z}_\Delta} \times \mathfrak{L}(G) \subset \Delta \times \mathfrak{L}(G)$. We obtain the exact sequence

$$\frac{\overline{\mathbb{Z}_\Delta} \times \mathfrak{L}(G)}{\Gamma} \overset{\mathrm{ins}}{\rightarrowtail} \frac{\Delta \times \mathfrak{L}(G)}{\Gamma} \overset{f}{\twoheadrightarrow} \frac{\Delta \times \mathfrak{L}(G)}{\overline{\mathbb{Z}_\Delta} \times \mathfrak{L}(G)} \cong_{\mathrm{t}} \frac{\Delta}{\overline{\mathbb{Z}_\Delta}}$$

Here $\frac{\Delta \times \mathfrak{L}(G)}{\Gamma} \cong_{\mathrm{t}} G$ is connected, hence $\Delta/\overline{\mathbb{Z}_\Delta}$ is connected as well, by the surjectivity of f. On the other hand, $\Delta/\overline{\mathbb{Z}_\Delta}$ is totally disconnected because Δ, being compact and totally disconnected is profinite ([1], Theorem 1.34, p. 22), and quotients of profinite groups are profinite ([26], Proposition 2.2.1(a), p. 28), and in particular totally disconnected. This is possible only when the quotient $\Delta/\overline{\mathbb{Z}_\Delta}$ is trivial. Therefore, $\overline{\mathbb{Z}_\Delta} = \Delta$. □

Remark 5. *(a) For the torus $G = \mathbb{T}^n$ one has $\mathfrak{L}(G) = \mathbb{R}^n$, so the Resolution theorem applied to G is simply the covering homomorphism $\varphi : \mathbb{R}^n \to \mathbb{T}^n$ if one takes $\Delta = 0$ (in general Δ must be a finite subgroup of $\mathbb{T}^n$).*

(b) Using the fact that $\exp[\mathfrak{L}(G)] = \mathrm{a}(G)$, the covering map φ could be replaced by the surjective, continuous, and open map $\psi : \Delta \times \mathrm{a}(G) \to G : \varphi(d, x) = d + x$, for $(d, x) \in \Delta \times \mathrm{a}(G)$ which has the advantage that now both groups Δ and $\mathrm{a}(G)$ are subgroups of G. One has to take into account that the map $\exp_G : \mathfrak{L}(G) \to \mathrm{a}(G)$ need not be injective. More precisely, $\mathfrak{K}(G) = \mathrm{Ker}(\exp)$ is trivial precisely when G is torus-free. However, even when G is torus-free, this map is only a continuous isomorphism that need not be a homeomorhism.

(c) As an application of Theorem 14 we obtain a nice presentation of the solenoid $\Sigma_2 = \mathbb{Q}^\wedge$ from Example 1 (2). As shown there, Σ_2 has a delta subgroup $\Delta = \widehat{\mathbb{Z}} = \prod_{p\in\mathbb{P}} \widehat{\mathbb{Z}}_p$ and $\Sigma_2/\Delta \cong \mathbb{T}$. So by Definition 3 (iii), $\mathfrak{L}(\Sigma_2) \cong_t \mathbb{R}$. Hence, Theorem 14 gives a resolution $\varphi : \Delta \times \mathbb{R} \to \Sigma_2$, with $\Gamma = \ker \varphi \cong_t \mathbb{Z}$ and $\Delta \cap a(\Sigma_2) = \langle \chi_1 \rangle \cong \mathbb{Z}$, where $\chi_1 : \mathbb{Q} \to \mathbb{T}$ is defined by $\chi_1(x) = x + \mathbb{Z}$ for $x \in \mathbb{Q}$.

The same representation can also be obtained directly by standard use of Pontryagin duality. Indeed, let $\mathbf{1} = (1_p)_{p\in\mathbb{P}} \in \Delta$ and $u = (\mathbf{1}, -1) \in \Delta \times \mathbb{R}$. Then $\langle u \rangle \cong_t \mathbb{Z}$ and $K = (\Delta \times \mathbb{R})/\langle u \rangle$ is a compact connected torsion-free group of dimension one, so its dual $K^\wedge$ is a discrete divisible torsion-free group of rank one. Therefore, $K^\wedge \cong \mathbb{Q}$ and $K \cong_t \mathbb{Q}^\wedge$.

We also obtain a "canonical resolution", where the arbitrary $\Delta \in \mathcal{D}(G)$ is replaced by the canonical subgroup $\mathbf{\Delta}(G)$.

Theorem 15 (Canonical Resolution Theorem)**.** *Let G be a compact abelian group and $\mathbf{\Delta}(G) = \bigcup \mathcal{D}(G)$. Then*

(1) the map $\varphi : \mathbf{\Delta}(G) \times \mathfrak{L}(G) \to G : \varphi((d, \chi)) = d + \exp(\chi) = d + \chi(1)$ is surjective, continuous, and open;

(2) $\Gamma := \mathrm{Ker}(\varphi) = \{(\exp(\chi), -\chi) \mid \chi \in \exp^{-1}[\mathbf{\Delta}(G)]\} \cong_t \exp^{-1}[\mathbf{\Delta}(G)] \subset \mathfrak{L}(G)$ is torsion-free and φ induces an isomorphism $(\mathbf{\Delta}(G) \times \mathfrak{L}(G))/\Gamma \cong_t G$;

(3) If G is connected of finite dimension $\dim(G) = n$, then $\Gamma \cong_t \mathbb{Q}^n$.

(4) $\exp[\exp^{-1}[\mathbf{\Delta}(G)]] = \mathrm{a}(G) \cap \mathbf{\Delta}(G)$ is dense in G.

Proof. (1) The map φ is clearly homomorphic, continuous and surjective. To show that it is open, let W be an open set in $\mathbf{\Delta}(G) \times \mathfrak{L}(G)$. We can assume without loss of generality that it is a basic open set, i.e., $W = U \times U'$, where U is open in $\mathbf{\Delta}(G)$ and U' is open in $\mathfrak{L}(G)$. Then $\forall\, \Delta \in \mathcal{D}(G) : \Delta \cap U$ is an open set of Δ, so $(\Delta \cap U) \times U'$ is an open set of $\Delta \times \mathfrak{L}(G)$. By the ordinary Resolution Theorem $O_\Delta := \varphi[(\Delta \cap U) \times U']$ is open in G. Hence, so is

$$\varphi[U \times U'] = \varphi\left[\left(\bigcup_{\Delta\in\mathcal{D}(G)} (\Delta \cap U)\right) \times U'\right] = \varphi\left[\left(\bigcup_{\Delta\in\mathcal{D}(G)} (\Delta \cap U) \times U'\right)\right] = \bigcup_{\Delta\in\mathcal{D}(G)} O_\Delta.$$

(2) The map $\Gamma \to \exp^{-1}[\mathbf{\Delta}] : (\exp(\chi), -\chi) \to \chi$ clearly is bijective, homomorphic, continuous and open. Being isomorphic to a subgroup of $\mathfrak{L}(G)$, the group Γ is torsion-free. The last assertion is obvious.

(3) Fix arbitrarily $\Delta \in \mathcal{D}(G)$ and let

$$\varphi_\Delta : \Delta \times \mathfrak{L}(G) \to G, \text{ defined by } \varphi_\Delta(d, \chi) = d + \exp(\chi), \text{ and } \Gamma_\Delta = \mathrm{Ker}(\varphi_\Delta).$$

By (2) Γ is torsion-free. We will show that Γ is divisible and Γ/Γ_Δ is a torsion group. This says that Γ is the usual algebraic divisible hull of $\Gamma_\Delta \cong_t \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n \subset \mathfrak{L}(G)$ (see the proof of item (4) of Theorem 14). Hence, $\exp^{-1}[\mathbf{\Delta}(G)] \cong_t \Gamma$ is the divisible hull of $\exp^{-1}[\Delta]$ and is $\mathbb{Q}e_1 \oplus \cdots \oplus \mathbb{Q}e_n \subset \mathfrak{L}(G)$ with the subspace topology. This shows that $\Gamma \cong_t \mathbb{Q}^n$.

To show that $\exp^{-1}[\mathbf{\Delta}]$, and hence Γ, is divisible, suppose that $x \in \mathfrak{L}(G)$ and $\exp(x) \in \mathbf{\Delta}(G)$. As $\mathfrak{L}(G)$ is divisible, given $m \in \mathbb{N}$, there is $y \in \mathfrak{L}(G)$ such that $my = x$. Hence, $m\exp(y) = \exp(x)$ and as $\mathbf{\Delta}(G)$ is divisible (Proposition 7(4)) there is $d \in \mathbf{\Delta}(G)$ such that

$m\exp(y) = md$. It follows that $\exp(y) - d \in \mathrm{tor}(G) \subset \mathbf{\Delta}(G)$, hence $\exp(y) \in \mathbf{\Delta}(G)$ and $y \in \exp^{-1}(\mathbf{\Delta})$ which establishes the claim.

Finally, to show that Γ/Γ_Δ is a torsion group let $\chi \in \exp^{-1}[\mathbf{\Delta}(G)]$, i.e., $\exp(\chi) \in \mathbf{\Delta}(G)$. By Proposition 7, there is $m \in \mathbb{N}$ such that $m\chi \in \Delta$. It follows that $m(\exp(\chi), -\chi) = (\exp(m\chi), -m\chi) \in \Gamma_\Delta$.

(4) Write $\mathbf{\Delta}(G) = \bigcup_{\Delta\in\mathcal{D}(G)} \Delta$ and use the fact that $\mathrm{a}(G) \cap \Delta$ is dense in Δ for every $\Delta \in \mathcal{D}(G)$, by Theorem 14(5). Then

$$\overline{\mathrm{a}(G) \cap \mathbf{\Delta}(G)} = \overline{\mathrm{a}(G) \cap \bigcup_{\Delta\in\mathcal{D}(G)} \Delta} = \overline{\bigcup_{\Delta\in\mathcal{D}(G)} \mathrm{a}(G) \cap \Delta} \supseteq \bigcup_{\Delta\in\mathcal{D}(G)} \overline{\mathrm{a}(G) \cap \Delta} \supseteq \bigcup_{\Delta\in\mathcal{D}(G)} \Delta = \mathbf{\Delta}(G).$$

Since $\mathbf{\Delta}(G)$ is dense in G, this proves that $\mathrm{a}(G) \cap \mathbf{\Delta}(G)$ is dense in G. □

In the next example, we apply the canonical resolution theorem 15 to two solenoids. The first one is $\mathbb{T} = \mathbb{Z}^\wedge$ and its canonical resolution adds nothing essentially new.

Example 2. *(a) For the solenoid, $\mathbb{T} = \mathbb{Z}^\wedge$ there is an isomorphism $\rho : \mathbb{R} \to \mathfrak{L}(G)$ and $\exp(\rho(r)) = r + \mathbb{Z}$, where $r \in \mathbb{R}$, by Definition 3(iii). Since $\mathbf{\Delta}(T) = \mathrm{tor}(\mathbb{T}) = \mathbb{Q}/\mathbb{Z}$, we obtain the canonical resolution $\varphi : \mathbb{Q}/\mathbb{Z} \times \mathbb{R} \to \mathbb{T} : \varphi((a + \mathbb{Z}, r)) = (a + \mathbb{Z}) + (r + \mathbb{Z}) = a + r + \mathbb{Z}$ with $\Gamma = \{(r + \mathbb{Z}, -r) \mid r \in \mathbb{Q}\}$ and evidently $\Gamma \cong_{\mathrm{t}} \mathbb{Q}$.*

(b) For the solenoid $\Sigma_2 = \mathbb{Q}^\wedge$ from Example 1 (2) $\Delta(\Sigma_2)$ is the divisible hull of its delta subgroup $\Delta = \prod_p \widehat{\mathbb{Z}}_p$. Moreover, $\mathfrak{L}(\Sigma_2) \cong_{\mathrm{t}} \mathbb{R}$ (see Remark 5(c)). Theorem 15 gives the canonical resolution $\varphi : \Delta(\Sigma_2) \times \mathbb{R} \to \Sigma_2$ with $\Gamma = \ker\varphi \cong_{\mathrm{t}} \mathbb{Q}$ as in (a) and $\mathrm{a}(\Sigma_2) \cap \mathbf{\Delta}(\Sigma_2) \cong \mathbb{Q}$ dense in Σ_2.

Denote by $\widetilde{\mathbb{Q}}$ the group $\Delta(\Sigma_2)$ equipped with the finer topology obtained by taking Δ as an open topological subgroup of $\widetilde{\mathbb{Q}}$. Then $\widetilde{\mathbb{Q}}$ is a locally compact ring and $\mathbb{A} := \widetilde{\mathbb{Q}} \times \mathbb{R}$ is the adele ring of $\mathbb{Q}$. Composing φ with the identity $\mathbb{A} \to \Delta(\Sigma_2) \times \mathbb{R}$ we obtain a continuous surjective homomorphism $\varphi : \mathbb{A} \to \Sigma_2$ which is again open by the Open Mapping Theorem (as $\mathbb{A}$ is σ-compact). Hence, Σ_2 is a quotient of $\mathbb{A}$.

7. Fat Delta Through the Looking Glass of Quasi-Torsion Elements

Fat Delta existed previously in the literature in a rather different form and in greater generality. In Section 7.1 we recall the definition of quasi-torsion element and the subgroup $\mathrm{td}(G)$ of quasi-torsion elements, showing that $\mathrm{td}(G) = \mathbf{\Delta}(G)$ for compact groups (Proposition 9).

7.1. Quasi-Torsion Elements

Definition 4 (([3], p. 127), [4])**.** *Let G be a Hausdorff abelian topological group. Define $\mathrm{td}(G)$ to be the set of all quasi-torsion elements of G, where $x \in G$ is* ***quasi-torsion*** *if $\langle x\rangle$ is either finite or its subspace topology is non-discrete and linear.*

This definition was given by [4] for arbitrary, not necessarily abelian, topological groups. Then $\mathrm{td}(G)$ need not be a subgroup of G, as the following example shows.

Example 3. *Take the compact group $G = SL_3(\mathbb{R})$ of rotations of $\mathbb{R}^3$. Then $\mathrm{td}(G) = \mathrm{tor}(G)$ is the set of all torsion elements of G, while the subgroup $\langle \mathrm{td}(G)\rangle$ generated by $\mathrm{td}(G)$ is the whole G since $\mathrm{td}(G)$ is invariant under conjugations and G is a simple group. A geometric proof of the equality $\langle \mathrm{td}(G)\rangle = G$ is based on the well-known fact that every rotation can be presented as a composition of two symmetries (known to have order 2).*

Remark 6. *If every convergent sequence is eventually constant in a topological abelian group G, then $\mathrm{td}(G) = \mathrm{tor}(G)$ (the assumption $\mathrm{td}(G) \neq \mathrm{tor}(G)$ leads to a contradiction: if $x \in \mathrm{td}(G) \setminus \mathrm{tor}(G)$, then the group $\langle x\rangle$ is non-discrete and metrizable, so $\langle x\rangle$ has convergent sequences that are not eventually constant).*

Infinite compact groups always have convergent sequences that are not eventually constant (since they contain copies of the Cantor set $\{0,1\}^\omega$*). An example of an infinite precompact abelian group where every convergent sequence is eventually constant can be obtained as follows. For a* TAG*–group* (G,τ) *the Bohr topology of* (G,τ) *is the initial topology* τ^+ *of all* $\chi \in (G,\tau)^\wedge$ *(that can be obtained by the diagonal embedding* $G \to \mathbb{T}^{G^\wedge}$*). For the sake of brevity we also write* G^+ *for* (G,τ^+)*. In case* τ *is discrete,* G^+ *is usually denoted by* $G^\#$*. It is a well-known fact that in* $G^\#$ *every convergent sequence is eventually constant ([3]), so* $\mathrm{td}(G^\#) = \mathrm{tor}(G^\#)$*.*

Proposition 9. *Let G be a topological abelian group.*

1. *If* $x \in G$*, then* $x \in \mathrm{td}(G)$ *if and only if there exists a continuous homomorphism* $f : (\mathbb{Z},\nu_\mathbb{Z}) \to G$ *with* $f(1) = x$*;*
2. $\mathrm{td}(G)$ *is a subgroup of G containing every compact totally disconnected subgroup of G;*
3. *If G is complete (in particular, locally compact), then* $\mathrm{td}(G)$ *coincides with the union of all compact, totally disconnected subgroups of G.*

Proof. (1) Assume that $x \in \mathrm{td}(G)$. If $\langle x\rangle$ is finite, then $\langle x\rangle$ is isomorphic to a quotient group of $(\mathbb{Z},\nu_\mathbb{Z})$, so the desired homomorphism f is easy to obtain. If $\langle x\rangle$ is infinite and carries a non-discrete linear topology, then the homomorphism $f : (\mathbb{Z},\nu_\mathbb{Z}) \to G$ with $f(1) = x$ is obviously continuous. On the other hand, if there exists a continuous homomorphism $f : (\mathbb{Z},\nu_\mathbb{Z}) \to G$ with $f(1) = x$, then the subgroup $\langle x\rangle$ is either finite or has linear precompact topology, so $x \in \mathrm{td}(G)$.

(2) If $x, y \in \mathrm{td}(G)$, then by (1) there exist continuous homomorphisms $f, g : (\mathbb{Z},\nu_\mathbb{Z}) \to G$ with $f(1) = x$ and $g(1) = y$. This gives a continuous homomorphism $h = f \oplus g : (\mathbb{Z},\nu_\mathbb{Z}) \times (\mathbb{Z},\nu_\mathbb{Z}) \to G$ defined by $h(n,m) = nx + my$. The restriction $h\restriction_{\Delta_\mathbb{Z}}: \Delta_\mathbb{Z} \to G$ satisfies $h(1,1) = x + y$ and since $\Delta_\mathbb{Z} \cong (\mathbb{Z},\nu_\mathbb{Z})$, witnesses $x + y \in \mathrm{td}(G)$ by (1).

If N is a compact, totally disconnected subgroup of G, then N has a linear topology. Therefore, for every $x \in N$, the subgroup $\langle x\rangle$ is either finite or its subspace topology is linear and non-discrete (as otherwise $\langle x\rangle$ it would be a closed (so compact) discrete subgroup of N, a contradiction). Therefore, $x \in \mathrm{td}(G)$.

(3) Assume now that G complete and $x \in \mathrm{td}(G)$. Then x is quasi-torsion and $\langle x\rangle$ is either finite or its subspace topology is non-discrete and linear. Hence, its closure $\overline{\langle x\rangle}$ is the completion of $\langle x\rangle$, and thus, compact and totally disconnected. □

For a compact group $G = A^\wedge$, by Proposition 5 and Proposition 9(2), we have $\mathrm{td}(G) = \mathbf{\Delta}(G)$, and by Theorem 10 $\mathbf{\Delta}(G) = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$. We summarize:

Theorem 16. *Let* $G = A^\wedge$ *where* $A \in \mathrm{AG}$*. Then*

$$\mathbf{\Delta}(G) = \mathrm{td}(G) = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}).$$

We quote from previous papers reconfirming foregoing results.

Proposition 10. *(1) ([3], Theorem 4.1.7(a)) If* $f : G \to H$ *is a continuous homomorphism of topological abelian groups, then* $f[\mathrm{td}(G)] \subseteq \mathrm{td}(H)$*, i.e.,* td *is a functorial subgroup; in particular* $\mathrm{td}(G)$ *is fully invariant in G.*

(2) ([27], Theorem 11) If G and H in (1) are compact and f is surjective, then $f[\mathrm{td}(G)] = \mathrm{td}(H)$*.*

(3) (([4], Proposition 1.3(a)) and ([3], Theorem 4.1.7(b))) If G is a topological abelian group and H is a subgroup of G, then $\mathrm{td}(H) = H \cap \mathrm{td}(G)$*;*

(4) (([4], Proposition 1.4(a)) and ([3], Theorem 4.1.7(e))) Let $\{G_i : i \in I\}$ *be a family of topological abelian groups. Then* $\mathrm{td}(\prod_{i\in I} G_i) = \prod_{i\in I} \mathrm{td}(G_i)$*.*

Remark 7. *Comments on the various items of Proposition 10.*

(a) Items (1), (3) and (4) follow from Proposition 9 and reinforce Corollary 3(1) by showing that td *is a functorial subgroup in the* **larger** *category* TAG*.*

(b) *In (2) "compact" cannot be replaced by "locally compact" (take $G = \mathbb{R}$, $H = \mathbb{T}$ and f the canonical quotient map, then $\mathrm{td}(\mathbb{R}) = \{0\}$, while $\mathrm{td}(\mathbb{T}) = \mathbb{Q}/\mathbb{Z} \neq \{0\}$).*

(c) *Item (4) reinforces Theorem 10(3) showing that it remains valid in the* **larger** *category* TAG.

Now we use item (1) from Proposition 10 to show that the subgroup $\mathrm{td}(G)$ is zero-dimensional when G is precompact, i.e., a subgroup of a compact group. We shall see in [16] that this remains true under the weaker assumption that G is locally precompact, i.e., a subgroup of a locally compact group.

Proposition 11. *Let G be a precompact abelian group. Then every subgroup H of G with $[H : (H \cap \mathrm{td}(G))] < \mathfrak{c}$ is zero-dimensional. In particular, $\mathrm{td}(G)$ is zero-dimensional.*

Proof. The following folklore fact will be needed in the sequel:

Claim 1. *Every proper subgroup H of $\mathbb{T}$ is zero-dimensional.*

Proof. H is either finite of dense. If H is finite then it is clearly zero-dimensional. If H is dense, then for any fixed $a \in \mathbb{T} \setminus H$ also $a + H$ is dense and disjoint from H. Hence, $\{\Gamma_{b,c} \cap H \colon b, c \in a + H\}$, where $\Gamma_{b,c}$ is an open arc in $\mathbb{T}$ with ends b and c, is a base of the induced topology on H consisting of clopen sets of H. □

First, we show that $\chi[\mathrm{td}(G)] \subseteq \mathbb{Q}/\mathbb{Z}$ for any $\chi \in G^\wedge$. Assume that $x \in \mathrm{td}(G)$, to check that $\chi(x) \in \mathbb{Q}/\mathbb{Z}$ pick an arbitrary $\chi \in G^\wedge$. Then $\chi(x) \in \mathrm{td}(\mathbb{T})$, by Proposition 10(1). By Example 1(3), $\mathrm{td}(\mathbb{T}) = \mathbb{Q}/\mathbb{Z}$, so $\chi(x) \in \mathbb{Q}/\mathbb{Z}$.

Since $H/(H \cap \mathrm{td}(G)) \cong (H + \mathrm{td}(G))/\mathrm{td}(G)$, our hypothesis implies that $[(H + \mathrm{td}(G)) : \mathrm{td}(G)] < \mathfrak{c}$. Hense, for every $\chi \in G^\wedge$ the subgroup $\chi[H + \mathrm{td}(G)]$ contains the countable subgroup $\chi[\mathrm{td}(G)] \subseteq \mathbb{Q}/\mathbb{Z}$ as a subgroup of index $< \mathfrak{c}$, so $|\chi[H]| \leq |\chi[H + \mathrm{td}(G)]| < \mathfrak{c}$ too. Consequently $\chi[H] \neq \mathbb{T}$, so $\chi[H]$ is zero-dimensional for every $\chi \in G^\wedge$. Since zero-dimensionality is preserved under taking direct products, $\prod_{\chi \in G^\wedge} \chi[H]$ is zero-dimensional, by Claim 1. Since H is precompact (as a subgroup of G), H isomorphic to a subgroup of $\prod_{\chi \in G^\wedge} \chi[H]$ by ([3], Theorem 2.3.2). Since zero-dimensionality is preserved under taking subgroups, we deduce that H is zero-dimensional. □

7.2. The Subgroup $\mathrm{td}(G)$ of Compact Groups and Minimality

The Open Mapping Theorem can be reached in two steps:

Definition 5. *A Hausdorff topological group G is:*

(a) ***minimal*** *if every continuous isomorphism $f : G \to H$ onto a Hausdorff topological group H is open.*

(b) ***totally minimal*** *if G satisfies the (full) Open Mapping Theorem, i.e., every continuous homomorphism $f : G \to H$ onto a Hausdorff topological group H is open.*

Compact groups are well-known to be totally minimal. On the other hand, a Hausdorff topological group G is totally minimal if and only if all Hausdorff quotients of G are minimal.

The first supply of non-compact (totally) minimal groups was obtained by means of the following notions of "strong" density:

Definition 6 ([28])**.** *A subgroup H of a topological abelian group G is **totally dense** if $\overline{N \cap H} = N$ for every closed subgroup N of G.*

Clearly, totally dense subgroups are dense (while $\mathbb{Z}(p^\infty)$ is dense in $\mathbb{T}$, but not totally dense). Obviously, the totally dense subgroups have the following weaker property:

Definition 7 ([5,11,15])**.** *A subgroup H of a topological abelian group G is* ***topologically essential*** *if $N \cap H \neq \{0\}$ for every non-trivial closed subgroup N of G.*

The term used for this property in [5,11,15] and in the remaining literature on the Open Mapping Theorem is "essential", but we prefer the more precise term "topologically essential" to avoid possible confusion.

Theorem 17. *Let H be a dense subgroup of a compact abelian group G.*

(a) ([11,15]) *H is minimal if and only if H is topologically essential in G.*
(b) ([28]) *H is totally minimal if and only if H is totally dense in G.*

Banaschewski [5] found the following general criterion: *if H is a dense subgroup of a topological abelian group G, then H is minimal if and only if G is minimal and H is topologically essential in G.* These criteria match perfectly the following remarkable result of Prodanov and Stoyanov [14] proved at a later stage, but conjectured by Prodanov in 1972 (see [13] for an earlier partial result in the totally minimal case):

Theorem 18 (Prodanov–Stoyanov Theorem)**.** *Minimal abelian groups are precompact.*

This theorem allows one to use exclusively the form of the criteria given in Theorem 17, so to reduce the study of the (totally) minimal abelian groups to that of the dense topologically essential (resp., totally dense) subgroups of the compact abelian groups. This explains the interest in topologically essential or totally dense subgroups of the compact abelian groups.

Proposition 12 ([11])**.** *The minimal topologies on $\mathbb{Z}$ are precisely the p-adic topologies.*

It was proved in [9] that the 2–adic topology of $\mathbb{Z}$ is minimal.

Proof. Assume that τ is a minimal topology on $\mathbb{Z}$ and let K be the completion of $(\mathbb{Z}, \tau)$. By the Prodanov–Stoyanov Theorem the group K is compact. By Theorem 17(a), $\mathbb{Z}$ is essential in K, hence K is torsion-free. Therefore, the dual of K is a discrete divisible group [1,3,23,29], hence a direct sums of copies of $\mathbb{Q}$ and of $\mathbb{Z}(p^\infty)$, $p \in \mathbb{P}$. Therefore, $K = (\mathbb{Q}^\wedge)^\alpha \times \prod_p \widehat{\mathbb{Z}}_p^{\beta_p}$. Again by Theorem 17(a), $\mathbb{Z}$ must be essential in this product, hence only one of these cardinals α, β_p can be non-zero, and it must be equal to 1. Since $\mathbb{Q}^\wedge$ has a Delta subgroup isomorphic to $\prod_p \widehat{\mathbb{Z}}_p$, again Theorem 17(a) implies that $\alpha = 0$. In other words, $K \cong \widehat{\mathbb{Z}}_p$ for some prime p, therefore, τ coincides with the p-adic topology on $\mathbb{Z}$. To conclude, the minimality of the p-adic topology follows from Theorem 17(a), since $\mathbb{Z}$ is essential in $K = \widehat{\mathbb{Z}}_p$, as all non-trivial closed subgroups of K are open. □

A similar argument shows that $\mathbb{Q}^n$ *admits no minimal topologies for* $0 < n < \infty$.

The functorial subgroup $\mathrm{td}(G)$ of a compact abelian group G is not only dense in G (Theorem 6(2)), but it is totally dense in G, as the next proposition shows.

Proposition 13. *Let G be a compact abelian group. Then* $\mathrm{td}(G)$ *is totally dense in G.*

Proof. Let N be a closed subgroup of G. Then $N \cap \mathrm{td}(G) = \mathrm{td}(N)$ by Proposition 10. Therefore, it suffices to check that $\mathrm{td}(G)$ is dense in G for every compact group G. This follows from Theorem 6, but we prefer to give an independent proof here.

Let $N := \overline{\mathrm{td}(G)}$. Applying to the closed subgroup N of G the exactness of td in the sense of Proposition 10(2), we deduce that $\mathrm{td}(G/N) = \{0\}$. To see that this implies $G/N = \{0\}$ and so $N = G$, consider the discrete dual $X = (G/N)^\wedge$ and assume by way of contradiction that $X \neq \{0\}$. Then there exists a subgroup Y of X such that $X/Y \neq \{0\}$ is torsion. Then $Y^\perp \cong (X/Y)^\wedge$ is a non-trivial compact totally disconnected subgroup of G/N, so $\mathrm{td}(G/N) \neq \{0\}$, a contradiction. □

We obtain the following theorem which, among other things, reconfirms that $\mathbf{\Delta}(G)$ is dense in G when G is compact.

Theorem 19. *Let G be a compact abelian group. Then $\mathrm{td}(G)$ is a dense totally minimal zero-dimensional subgroup of G.*

Proof. Proposition 13 ensures the total density (hence, density as well) of $\mathrm{td}(G)$. Total minimality of $\mathrm{td}(G)$ is then an immediate consequence of Theorem 17. To prove that $\mathrm{td}(G)$ is zero-dimensional, apply Proposition 11. $\square$

Since $\mathrm{td}(G) \neq G$ when G is not totally disconnected, this theorem provides a universal example of a non-compact totally minimal (and zero-dimensional) abelian group. This explains why it is not surprising that most of the first known examples of non-compact totally minimal groups known in the seventies were just $\mathbb{Q}/\mathbb{Z} = \mathrm{td}(\mathbb{T})$ ([15]), $(\mathbb{Q}/\mathbb{Z})^n = \mathrm{td}(\mathbb{T}^n)$ ([9]), $(\mathbb{Q}/\mathbb{Z})^{\mathbb{N}} = \mathrm{td}(\mathbb{T}^{\mathbb{N}})$ ([10]), and $(\mathbb{Q}/\mathbb{Z})^{\alpha} = \mathrm{td}(\mathbb{T}^{\alpha})$ ([27,30]).

Corollary 4. *Let G be a compact abelian group and H be a closed subgroup of G. Then $\mathrm{td}(H) \overset{\mathrm{ins}}{\rightarrowtail} \mathrm{td}(G) \twoheadrightarrow \mathrm{td}(G/H)$ is a proper short exact sequence in* TAG*.*

Proof. By Proposition 10 $\mathrm{td}(H) = \mathrm{td}(G) \cap H$ and $q[\mathrm{td}(G)] = \mathrm{td}(G/H)$ for the quotient homomorphism $q : G \to G/H$. This proves the exactness of the short exact sequence $\mathrm{td}(H) \overset{\mathrm{ins}}{\rightarrowtail} \mathrm{td}(G) \overset{f}{\twoheadrightarrow} \mathrm{td}(G/H)$, where $f = q \restriction_{\mathrm{td}(G)}$. The openness of f follows from the fact that $\mathrm{td}(G)$ is totally minimal, in view of Theorem 19. $\square$

7.3. Sylow Subgroups of td(G) *for* $G \in$ TAG

The characterization in Theorem 9 of the compact abelian groups G with $\mathrm{td}(G) = \mathrm{tor}(G)$ gives a very narrow class (practically rather close to the class of Lie groups). This shows that the restraint $\mathrm{td}(G) = \mathrm{tor}(G)$ is too stringent, or from another point of view, the subgroup $\mathrm{td}(G)$ is too large to be useful in certain circumstances. This is why here we recall a smaller subgroup of $\mathrm{td}(G)$ containing $\mathrm{tor}(G)$ that still keeps the advantages of $\mathrm{td}(G)$, but it is closer to $\mathrm{tor}(G)$. This subgroup is simply the subgroup generated by all topologically p-Sylow subgroups $\mathrm{td}_p(G)$ of $\mathrm{td}(G)$ defined as follows:

Definition 8 ([3,31])**.** *An element x of a topological abelian group G is **topologically p-torsion** if $p^n x \to 0$. Let*

$$G_p := \{x \in G \mid x \text{ is topologically } p\text{-torsion}\}$$

and let $\mathrm{td}_p(G) := (\mathrm{td}(G))_p$.

Then G_p is a subgroup of G. In case G is a profinite group, G_p is usually called the ***topological p-Sylow subgroup*** of G. We shall also keep this terminology when G is not necessarily profinite. Clearly, $H_p = G_p \cap H$ for a subgroup H of G.

Obviously, $\mathrm{tor}_p(G) \leq \mathrm{td}_p(G) \leq G_p$ for every G.

The notation $\mathrm{td}_p(G)$ used in Definition 8 is borrowed from [4,27], where $\mathrm{td}_p(G)$ denotes the subgroup of all elements $x \in G$ (called ***quasi-p-torsion*** in [4]) such that $\langle x \rangle$ is either a cyclic p-group, or $\langle x \rangle$ is isomorphic to $\mathbb{Z}$ equipped with the p-adic topology.

The equivalence of both definitions follows from: if $\langle x \rangle \cong \mathbb{Z}$ is equipped with a Hausdorff linear topology such that $p^n x \to 0$, then this linear topology necessarily coincides with the p-adic topology.

The sum $\sum_p \mathrm{td}_p(G)$ is direct ([4]). Following [4], we write $\mathrm{wtd}(G) = \bigoplus_{p \in \mathbb{P}} \mathrm{td}_p(G)$ in the sequel. Clearly,

$$\mathrm{tor}(G) \leq \mathrm{wtd}(G) \leq \mathrm{td}(G),$$

but these subgroups need not coincide in general. It is proved in [4] that, when G is compact, even the smaller subgroup $\mathrm{wtd}(G)$ is still totally dense in G. Since both total density and topological essentiality are transitive properties, a dense subgroup G of a compact abelian group K is totally dense (resp., topologically essential) in K if and only if $\mathrm{td}(G) = G \cap \mathrm{td}(K)$ is totally dense (resp., topologically essential) in $\mathrm{td}(K)$ if and only if $\mathrm{wtd}(G)$ is totally dense (resp., topologically essential) in $\mathrm{wtd}(K)$.

The next theorem from [29] shows that one can characterize the totally disconnected compact abelian groups in the class of all compact abelian groups G by specifying whether the subgroups $\mathrm{td}_p(G)$ of G are closed (compact) or not:

Theorem 20 ([3,29])**.** *For a compact abelian group G and every prime p the subgroup $\mathrm{td}_p(\mathrm{c}(G))$ is dense in $\mathrm{c}(G)$. In particular, the following conditions are equivalent:*

(1) *G is totally disconnected;*
(2) $\mathrm{td}(G) = G$*, i.e.,* $\mathrm{td}(G)$ *is compact;*
(3) $\mathrm{td}_p(G)$ *is compact for every prime p;*
(4) $\mathrm{td}_p(G)$ *is compact for some prime p;*
(5) $\mathrm{td}_p(G)$ *is closed in G for some prime p (equivalently, for all primes p);*
(6) *the topology induced from G on* $\mathrm{wtd}(G) = \bigoplus_{p\in\mathbb{P}} \mathrm{td}_p(G)$ *coincides with the topology induced by the product topology of* $\prod_{p\in\mathbb{P}} \mathrm{td}_p(G)$.

In case these conditions hold, then $G \cong_t \prod_{p\in\mathbb{P}} \mathrm{td}_p(G)$.

In Theorem 9, we determined the compact groups for which $\mathbf{\Delta}(G) = \mathrm{tor}(G)$. Using the smaller subgroup $\mathrm{wtd}(G)$ instead of $\mathbf{\Delta}(G)$, we impose the condition $\mathrm{wtd}(G) = \mathrm{tor}(G)$ instead of collapsing the whole chain $\mathrm{tor}(G) \leq \mathrm{wtd}(G) \leq \mathrm{td}(G)$. This leads to a concept introduced in [32]:

Definition 9 ([32])**.** *A compact abelian groups G is an **exotic torus**, if* $\mathrm{wtd}(G) = \mathrm{tor}(G)$.

Clearly, the usual tori are also exotic tori, but the solenoid Σ_1 defined in Example 1(1) is an exotic torus that is not a torus. The next theorem from [32] giving eleven equivalent characterizations of exotic tori (of those (2) was used in [32] as the original definition) provides further examples of exotic tori (see also Example 4 (3), (4)).

Theorem 21 ([32])**.** *For a compact abelian group $G = A^\wedge$ the following are equivalent:*

(1) $\mathrm{wtd}(G)$ *is torsion;*
(2) $\mathrm{Soc}(G)$ *is topologically essential;*
(3) *G contains copies of the p-adic integers $\widehat{\mathbb{Z}}_p$ for no prime p;*
(4) *$n = \dim(G) < \infty$ and for every continuous surjective homomorphism $f : G \to \mathbb{T}^n$ we have* $\mathrm{Ker}\, f = \prod_p B_p$*, where each B_p is a (bounded) compact p-group;*
(5) *$n = \dim(G) < \infty$ and there exists a homomorphism $f : G \to \mathbb{T}^n$ as in (3);*
(6) $\mathrm{wtd}(G) \cong (\mathbb{Q}/\mathbb{Z})^n \times \bigoplus_{p\in\mathbb{P}} B_p$ *algebraically, where each B_p is a (bounded) compact p-group;*
(7) *A is **strongly non-divisible**, i.e., all non-trivial quotients of A are non-divisible;*
(8) *every proper subgroup of A is contained in some maximal subgroup of A;*
(9) *A admits a surjective homomorphism $A \to \mathbb{Z}(p^\infty)$ for no prime p;*
(10) *$n = \mathrm{rk}(A) < \infty$ and $A/F \cong \bigoplus T_p$, where each T_p is a bounded p-group, for every $F \in \mathcal{F}(A)$;*
(11) *$n = \mathrm{rk}(A) < \infty$ and there exists $F \in \mathcal{F}(A)$ as in (10).*

Corollary 5. *If G is a non-trivial connected exotic torus, then $n = \dim(G) < \infty$ and* $\mathrm{wtd}(G) = \mathrm{tor}(G) \cong (\mathbb{Q}/\mathbb{Z})^n$*, i.e., all p-ranks of G coincide and equal* $\dim(G)$.

It was deduced from this corollary that the only divisible torsion abelian group that may carry minimal topologies are the groups $(\mathbb{Q}/\mathbb{Z})^n$, $n \in \mathbb{N}$ ([32]).

Following [12], call a compact group ***almost countable*** if it is the completion of countable minimal abelian group. This class of compact groups was described by Prodanov [12] as follows: a compact abelian group G is almost countable if and only if $n = \dim(G) < \infty$ and there exists a homomorphism $f : G \to \mathbb{T}^n$ such that $\operatorname{Ker} f = \prod_p (\widehat{\mathbb{Z}}_p^{e_p} \times F_p)$, where F_p is a finite p group and $e_p \in \{0,1\}$ for every prime p. These are the compact abelian groups G such that $\operatorname{td}(G)$ has a countable essential subgroup.

The larger class $\mathcal{K}$ of compact abelian groups, that contain copies of the group $\widehat{\mathbb{Z}}_p^{\mathbb{N}}$ for no prime p was studied in [6]. It is stable under extension and contains all almost countable compact groups, as well as all exotic tori. Its subclass of compact groups G that contain copies of the group $\widehat{\mathbb{Z}}_p^2$ for no prime p coincides with the completions of minimal abelian groups of countable rank, or equivalently, these are the compact abelian groups G such that $\operatorname{td}(G)$ has an essential subgroup of countable rank (see [3] or [6]).

Example 4. *Let $G = A^\wedge$ where $\mathbb{Z} \le A \le \mathbb{Q}$, be a solenoid, as in Lemma 11 and Theorem 13. It follows from $\mathbb{Z} \rightarrowtail A \twoheadrightarrow A/\mathbb{Z} \cong \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^{h_p})$ that*

$$(A/\mathbb{Z})^\wedge = \operatorname{Hom}(A/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \to G = A^\wedge \overset{\phi}{\twoheadrightarrow} \mathbb{T}$$

is exact and $(A/\mathbb{Z})^\wedge \to \boldsymbol{\Delta}(G) = \operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z}) \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$ is exact, with $(A/\mathbb{Z})^\wedge \cong_{\mathrm{t}} \prod_{p\in\mathbb{P}} \Delta_p$ where $\Delta_p \cong_{\mathrm{t}} \mathbb{Z}(p^{h_p})^\wedge$, so $\Delta_p \cong_{\mathrm{t}} \widehat{\mathbb{Z}}_p$ when $p \in \mathbb{P}_\infty$ and Δ_p is a cyclic p-group otherwise.

(1) G is an exotic torus if and only if $\mathbb{P}_\infty = \emptyset$ (i.e., $\operatorname{tp}(A)$ has no entries ∞).

(2) It follows from (1) that there are $\mathfrak{c}$ many pairwise non-isomorphic connected one-dimensional exotic tori G; they all have $\operatorname{wtd}(G) \cong \mathbb{Q}/\mathbb{Z}$, according to Corollary 5. Nevertheless, for these exotic tori G the subgroups $\operatorname{wtd}(G)$ remain pairwise non isomorphic (since, similarly to Theorem 12, if $\operatorname{wtd}(G) \cong_t \operatorname{wtd}(H)$, then $G \cong_t H$ for every pair of compact abelian groups G, H).

(3) According to Theorem 13, if G is an exotic torus, then $\operatorname{Soc}(G)$ is dense in G if and only if $\mathbb{P}_0$ is infinite (see ([32], Proposition 2.5) for a more general result in the case of connected exotic tori of arbitrary dimension). According to Theorem 21, in this case, $\operatorname{Soc}(G)$ is the smallest dense topologically essential subgroups of G.

(4) The second assertion in (3) is related to the following more general fact proved in ([33], Theorem 5.1) justifying the interest in dense socles: a connected compact abelian group G contains a smallest dense topologically essential (i.e., smallest dense minimal) subgroup of G if and only if G is an exotic torus with dense $\operatorname{Soc}(G)$.

8. Final Comments and Open Problems

One can deduce from Lemma 11(2) that for a solenoid Σ all delta subgroups Δ of Σ have the property that all subgroups of finite index of Δ are open.

Problem 1. *Classify the compact abelian groups whose delta subgroups have the property that all their subgroups of finite index are open.*

If $G = A^\wedge$ is a finite-dimensional compact connected abelian group, one can easily extend the argument in the proof of Theorem 13 and prove that $\operatorname{Soc}(G)$ is dense in G if $\mathbb{P}_0(G)$ is infinite, where $\mathbb{P}_0(G)$ is defined in this more general case as follows (a different proof in case G is an exotic torus can be found in ([32], Proposition 2.5)). Let $n = \dim G$, then there exists a short exact sequence $\mathbb{Z}^n \rightarrowtail A \twoheadrightarrow A/\mathbb{Z}^n$, where $A/\mathbb{Z}^n$ is torsion (actually, isomorphic to a subgroup of $(\mathbb{Q}/\mathbb{Z})^n$). In this notation, $\mathbb{P}_0(G) = \{p \in \mathbb{P} : \operatorname{rk}_p(A/\mathbb{Z}^n) = 0\}$. Obviously, $\mathbb{P}_0(G) = \mathbb{P}_0$, as defined in Theorem 13, when $n = 1$. The following example shows that when $\dim G > 1$, infinity of $\mathbb{P}_0(G)$ is not a necessary condition for the density of $\operatorname{Soc}(G)$.

Example 5. *Split $\mathbb{P} = \pi_1 \sqcup \pi_2$ in two disjoint infinite subsets π_1, π_2 (e.g., take π_1 to be the set of all primes of the form $4k+1$). For $i = 1, 2$ define the rational group $A_i = \langle 1/p : p \in \pi_i \rangle$ and the* *solenoid* $\Sigma_i = A_i^\wedge$. *Then both* Σ_1 *and* Σ_2 *have dense socles, by Theorem 13, so* $G = \Sigma_1 \times \Sigma_2$ *has dense socle as well. Nevertheless,* $\mathbb{P}_0(G) = \emptyset$.

Problem 2. *Find a criterion for density of* $\mathrm{Soc}(G)$ *for a finite-dimensional compact connected abelian group G.*

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