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Fractional operators as traces of operator-valued curves

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A B S T R A C T

We relate non integer powers $\mathcal{L}^s$, $s > 0$ of a given (unbounded) positive self-adjoint operator $\mathcal{L}$ in a real separable Hilbert space $\mathcal{H}$ with a certain differential operator of order $2\lceil s\rceil$, acting on even curves $\mathbb{R} \to \mathcal{H}$. This extends the results by Caffarelli–Silvestre and Stinga–Torrea regarding the characterization of fractional powers of differential operators via an extension problem.

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1. Introduction

The study of the fractional powers of differential operators via their relations with generalized harmonic extensions and corresponding Dirichlet-to-Neumann operators began more than fifty years ago [14] and became popular thanks to the celebrated work [3] of Caffarelli and Silvestre, which stimulated a fruitful line of research. The idea of relating the operators $(-\Delta)^s$, $s \in (0,1)$, acting on $\mathbb{R}^n$ and $-\mathrm{div}(y^{1-2s}\nabla)$ acting on $\mathbb{R}^n \times \mathbb{R}_+$, has been adapted to cover much more general situations. The first contribution in this direction is due to Stinga and Torrea [18]; important generalizations were given in [1,9].

The case of higher order powers of $(-\Delta)^s$ has been investigated firstly in [6] via conformal geometry techniques. We also cite [5,8,10,12,17], the more recent papers [4,7] and references there-in.

Before describing our results, let us notice that any extension $w = w(\cdot, y)$ of a given $u = u(\cdot)$ can be related with the curve $y \mapsto w(\cdot, y)$ taking values in a suitable function space. In the present paper we use this interpretation to handle any non-integer power $s > 0$ of a linear operator $\mathcal{L}$ in quite a general framework.

Let $\mathcal{H}$ be a separable real Hilbert space with scalar product $(\cdot,\cdot)_{\mathcal{H}}$ and norm $\|\cdot\|_{\mathcal{H}}$. Let

$$\mathcal{L} : \mathcal{D}(\mathcal{L}) \to \mathcal{H} \ , \qquad \mathcal{D}(\mathcal{L}) \subseteq \mathcal{H}$$

be a given unbounded, self-adjoint operator. In order to simplify the exposition, we first assume that $\mathcal{L}$ is positive definite and has discrete spectrum (some generalizations are given in Section 5). We organize the spectrum of $\mathcal{L}$ in a nondecreasing sequence of eigenvalues $(\lambda_j)_{j\ge1}$, counting with their multiplicities, and denote by $\varphi_j \in \mathcal{D}(\mathcal{L})$ a complete orthonormal system of corresponding eigenvectors.

Given $s \in \mathbb{R}$, the s-th power of $\mathcal{L}$ in the sense of spectral theory is the operator

$$\mathcal{L}^s u = \sum_{j=1}^{\infty} \lambda_j^s u_j \, \varphi_j \ , \quad \text{where} \quad u_j = (u, \varphi_j)_{\mathcal{H}} \ , \tag{1.1}$$

so that $\mathcal{L}^0$ is the identity in $\mathcal{H}$. If $s > 0$, the natural domain of the quadratic form

$$u \mapsto (\mathcal{L}^s u, u)_{\mathcal{H}} = \sum_{j=1}^{\infty} \lambda_j^s u_j^2$$

is denoted by $\mathcal{H}^s_{\mathcal{L}}$. Clearly $\mathcal{H}^s_{\mathcal{L}}$ coincides with the domain of $\mathcal{L}^{\frac{s}{2}}$; it is a Hilbert space with scalar product and norm given by

$$(u, v)_{\mathcal{H}^s_{\mathcal{L}}} = (\mathcal{L}^{\frac{s}{2}} u, \mathcal{L}^{\frac{s}{2}} v)_{\mathcal{H}} \ , \quad \|u\|_{\mathcal{H}^s_{\mathcal{L}}} = \|\mathcal{L}^{\frac{s}{2}} u\|_{\mathcal{H}} \ . \tag{1.2}$$

We identify the dual space $(\mathcal{H}^s_{\mathcal{L}})'$ with $\mathcal{H}^{-s}_{\mathcal{L}} = \{\mathcal{L}^s u \mid u \in \mathcal{H}^s_{\mathcal{L}} \ \}$ via the identity

$$\langle \mathcal{L}^s u, v\rangle = (\mathcal{L}^{\frac{s}{2}} u, \mathcal{L}^{\frac{s}{2}} v)_{\mathcal{H}} \quad \text{for any } u, v \in \mathcal{H}^s_{\mathcal{L}}.$$

Notice that $\mathcal{L}^s$ is an isometry $\mathcal{H}^s_{\mathcal{L}} \to \mathcal{H}^{-s}_{\mathcal{L}}$ with inverse $\mathcal{L}^{-s}$.

In this paper we relate the operator $\mathcal{L}^s : \mathcal{H}^s_{\mathcal{L}} \to \mathcal{H}^{-s}_{\mathcal{L}}$ for $s > 0$ non-integer to certain linear operator acting on **even** curves $\mathbb{R} \to \mathcal{H}^s_{\mathcal{L}}$ (this simplifies the treatment in case of higher powers $s > 1$, compare with [7]).

Let $b \in (-1, 1)$. Denote by $L^{2;b}(\mathbb{R} \to \mathcal{H})$ the Hilbert space of curves $U : \mathbb{R} \to \mathcal{H}$ such that $\|U(y)\|^2_{\mathcal{H}}$ is integrable on $\mathbb{R}$ with respect to the measure $|y|^b dy$. Further, $L^{2;b}_{\mathbf{e}}(\mathbb{R} \to \mathcal{H})$ stands for the subspace of even curves.

For $U \in L^{2;b}_{\mathbf{e}}(\mathbb{R} \to \mathcal{H})$ we consider the (unbounded) operators

$$\mathbb{D}_b U = -\partial^2_{yy} U - b y^{-1} \partial_y U = -|y|^{-b} \partial_y (|y|^b \partial_y U) \ , \qquad \mathbb{L}_b U = \mathbb{D}_b U + \mathcal{L} U. \tag{1.3}$$

Denoting by $U_j(y) = (U_j(y), \varphi)_{\mathcal{H}}$ the coordinates of $U(y)$, we have

$$\mathbb{L}_b U = \sum_{j=1}^{\infty} \big((\mathbb{D}_b + \lambda_j) U_j\big)\, \varphi_j,$$

and the corresponding quadratic form reads

$$(\mathbb{L}_b U, U)_{L^{2,b}} = \int_{-\infty}^{+\infty} |y|^b \big(\|\partial_y U(y)\|^2_{\mathcal{H}} + \|\mathcal{L}^{\frac{1}{2}}(U(y))\|^2_{\mathcal{H}}\big) dy = \sum_{j=1}^{\infty} \int_{-\infty}^{+\infty} |y|^b (|\partial_y U_j|^2 + \lambda_j |U_j|^2) dy.$$

In Section 4 we study in detail the natural domain

$$H^{k;b}_{\mathcal{L},\mathbf{e}}(\mathbb{R} \to \mathcal{H}) \subset L^{2;b}_{\mathbf{e}}(\mathbb{R} \to \mathcal{H}) \ , \qquad k \in \mathbb{N},$$

of the quadratic form $U \mapsto (\mathbb{L}^k_b U, U)_{L^{2;b}}$. Lemma 4.2 provides explicit expressions for its Hilbertian scalar product and related norm, which are denoted by $(\cdot,\cdot)_{H^{k;b}_{\mathcal{L},\mathbf{e}}}$, $\|\cdot\|^2_{H^{k;b}_{\mathcal{L},\mathbf{e}}}$, respectively, and shows that the Dirac-type trace function $\delta_0(V) = V(0)$ is continuous from $H^{k;b}_{\mathcal{L},\mathbf{e}}(\mathbb{R} \to \mathcal{H})$ into $\mathcal{H}^{k-\frac{1+b}{2}}_{\mathcal{L}}$.

Our main results involve the linear transform

$$\mathcal{P}_s[u](y) = \frac{2^{1-s}}{\Gamma(s)} \sum_{j=1}^{\infty} (\sqrt{\lambda_j}|y|)^s K_s(\sqrt{\lambda_j}|y|)\, u_j \varphi_j \tag{1.4}$$

for $u = \sum_j u_j \varphi_j \in \mathcal{H}$ and $y \in \mathbb{R}$, where K_s is the modified Bessel function of the second kind (the Macdonald function; compare with [18], where $s \in (0, 1)$ is assumed).

Due to the regularity and decaying properties of the Bessel functions, in Lemma A.1 of Appendix A, we prove that for any $u \in \mathcal{H}$, $\mathcal{P}_s[u]$ is an even curve in $\mathcal{H}$; in addition $\mathcal{P}_s[u] \in \mathcal{C}^\infty(\mathbb{R}_+ \to \mathcal{H}^\sigma_{\mathcal{L}})$ for every $\sigma > 0$.

To state our main result we introduce the floor and ceiling notation: for $s > 0$ not integer,

$$\lfloor s \rfloor := \text{integer part of } s; \quad \lceil s \rceil := \lfloor s \rfloor + 1.$$

Theorem 1.1. *Let $s > 0$ be non-integer. We put*

$$\mathfrak{b} := 1 - 2(s - \lfloor s \rfloor) \in (-1, 1)\,.$$

For any $u \in \mathcal{H}^s_{\mathcal{L}}$ the following facts hold.

i)

$$\|\mathcal{P}_s[u]\|^2_{H^{\lceil s \rceil;\mathfrak{b}}_{\mathcal{L},\mathbf{e}}} = 2d_s \|u\|^2_{\mathcal{H}^s_{\mathcal{L}}} \quad where \quad d_s = 2^{\mathfrak{b}}\Gamma\Big(\frac{1+\mathfrak{b}}{2}\Big)\frac{\lfloor s \rfloor !}{\Gamma(s)}\,. \tag{1.5}$$

That is, the transform $\mathcal{P}_s$ is an isometry $\mathcal{H}^s_{\mathcal{L}} \to H^{\lceil s \rceil;\mathfrak{b}}_{\mathcal{L},\mathbf{e}}(\mathbb{R} \to \mathcal{H})$ (up to a constant);

ii) $\mathcal{P}_s[u]$ achieves

$$\min_{\substack{U \in H^{\lceil s \rceil;\mathfrak{b}}_{\mathcal{L},\mathbf{e}}(\mathbb{R}\to\mathcal{H}) \\ U(0)=u}} \|U\|^2_{H^{\lceil s \rceil;\mathfrak{b}}_{\mathcal{L},\mathbf{e}}} = 2d_s \|u\|^2_{\mathcal{H}^s_{\mathcal{L}}}; \tag{1.6}$$

iii) $(\mathcal{P}_s[u], V)_{H^{\lceil s \rceil;\mathfrak{b}}_{\mathcal{L},\mathbf{e}}} = 2d_s\, \langle \mathcal{L}^s u, V(0)\rangle$ for any $V \in H^{\lceil s \rceil;\mathfrak{b}}_{\mathcal{L},\mathbf{e}}(\mathbb{R} \to \mathcal{H})$;

iv) $\mathcal{P}_s[u]$ solves the differential equation

$$\mathbb{L}^{\lceil s \rceil}_{\mathfrak{b}} \mathcal{P}_s[u] = 0 \quad in \quad \mathbb{R}_+ \tag{1.7}$$

and satisfies

$$\begin{aligned} &\lim_{y \to 0^+} \mathcal{P}_s[u](0) = u && in\ \mathcal{H}^s_{\mathcal{L}}\,, \\ &\lim_{y \to 0^+} y^{\mathfrak{b}}\, \partial_y\big(\mathbb{L}^{\lfloor s \rfloor}_{\mathfrak{b}} \mathcal{P}_s[u]\big)(y) = -d_s\, \mathcal{L}^s u && in\ \mathcal{H}^{-s}_{\mathcal{L}}. \end{aligned} \tag{1.8}$$

Additional information on the regularity of $\mathcal{P}_s[u]$ and on its behavior at $\{y = 0\}$ is given in Appendix A, see in particular Theorems A.6 and A.9. Corollary A.2 improves the convergence in [18, Theorem 1.1], where $s \in (0, 1)$ is assumed; in Subsection A.2 we point out some isometric properties of the operator P_s in the spirit of [16].

We can also consider negative, non-integer orders.

Let $s > 0$. If $\zeta \in \mathcal{H}^{-s}_{\mathcal{L}}$ then $\mathcal{L}^{-s}\zeta \in \mathcal{H}^s_{\mathcal{L}}$, so that for any $y \in \mathbb{R}$ we can compute

$$\mathcal{P}_s[\mathcal{L}^{-s}\zeta](y) = \frac{2^{1-s}}{\Gamma(s)} \sum_{j=1}^{\infty} \lambda_j^{-s} (\sqrt{\lambda_j}|y|)^s K_s(\sqrt{\lambda_j}|y|)\ \zeta_j \varphi_j.$$

The next result is in fact a corollary of Theorem 1.1.

Theorem 1.2. *Let $s>0$, $\mathfrak{b}\in(-1,1)$ be as in Theorem 1.1. For any $\zeta\in\mathcal{H}_{\mathcal{L}}^{-s}$ the following facts hold.*

i)

$$\|\mathcal{P}_{-s}[\zeta]\|^2_{H^{\lceil s\rceil;\mathfrak{b}}_{\mathcal{L},\mathbf{e}}}=2d_s\|\zeta\|^2_{\mathcal{H}_{\mathcal{L}}^{-s}}\quad where\quad \mathcal{P}_{-s}:=\left(\mathcal{P}_s\circ\mathcal{L}^{-s}\right). \tag{1.9}$$

That is, the transform $\mathcal{P}_{-s}$ is an isometry $\mathcal{H}_{\mathcal{L}}^{-s}\to H^{\lceil s\rceil;\mathfrak{b}}_{\mathcal{L},\mathbf{e}}(\mathbb{R}\to\mathcal{H})$ (up to a constant);

ii) $\mathcal{P}_{-s}[\zeta]$ *achieves*

$$\min_{U\in H^{\lceil s\rceil;\mathfrak{b}}_{\mathcal{L},\mathbf{e}}(\mathbb{R}\to\mathcal{H})}\left(\|U\|^2_{H^{\lceil s\rceil;\mathfrak{b}}_{\mathcal{L},\mathbf{e}}}-4d_s\langle\zeta,U(0)\rangle\right)=-2d_s\|\zeta\|^2_{\mathcal{H}_{\mathcal{L}}^{-s}}. \tag{1.10}$$

iii) $(\mathcal{P}_{-s}[\zeta],V)_{H^{\lceil s\rceil;\mathfrak{b}}_{\mathcal{L},\mathbf{e}}}=2d_s\,\langle\zeta,V(0)\rangle$ *for any* $V\in H^{\lceil s\rceil;\mathfrak{b}}_{\mathcal{L},\mathbf{e}}(\mathbb{R}\to\mathcal{H})$*;*

iv) $\mathcal{P}_{-s}[\zeta]$ *solves the differential equation*

$$\mathbb{L}^{\lceil s\rceil}_{\mathfrak{b}}\mathcal{P}_{-s}[\zeta]=0\quad in\quad \mathbb{R}_+$$

and satisfies

$$\begin{aligned}&\lim_{y\to0^+}y^{\mathfrak{b}}\,\partial_y\big(\mathbb{L}^{\lfloor s\rfloor}_{\mathfrak{b}}\mathcal{P}_{-s}[\zeta]\big)(y)=-d_s\,\zeta &&in\quad \mathcal{H}_{\mathcal{L}}^{-s}\ ,\\ &\lim_{y\to0^+}\mathcal{P}_{-s}[\zeta](y)=\mathcal{L}^{-s}\zeta &&in\quad \mathcal{H}_{\mathcal{L}}^{s}\,.\end{aligned}$$

The paper is organized as follows. We start by introducing and studying, in Section 2, some Sobolev-type spaces $H^{k;b}_{\mathbf{e}}(\mathbb{R})$ depending on the integer $k\ge1$ and on the parameter $b\in(-1,1)$. In Section 3 we investigate the properties of the functions

$$\psi_s(y)=c_s|y|^sK_s(|y|),\quad c_s=\frac{2^{1-s}}{\Gamma(s)}, \tag{1.11}$$

which are involved in the definition of the operator $u\mapsto\mathcal{P}_s[u]$. The main result here is Theorem 3.3, which constitutes the basic tool in the proof of Theorem 1.1.

Section 4 contains the description of the Hilbert space $H^{k;b}_{\mathcal{L},\mathbf{e}}(\mathbb{R}\to\mathcal{H})$ of even curves in $\mathcal{H}$ mentioned above, and the proofs of Theorems 1.1 and 1.2.

Generalizations and examples are given in Section 5.

As already mentioned, the Appendix contains several results about the operator $\mathcal{P}_s$.

Notation. Let X be a Hilbert space with scalar product $(\cdot,\cdot)_X$ and norm $\|\cdot\|_X$. For any $b\in(-1,1)$ and any open interval $I\subseteq\mathbb{R}$, the space

$$L^{2;b}(I\to X):=L^2(I\to X;|y|^bdy)$$

is endowed with the Hilbertian scalar product

$$(U,V)_{L^{2;b}} = \int_{-\infty}^{+\infty} |y|^b (U(y),V(y))_X \, dy \qquad U,V \in L^{2;b}(I \to X)$$

and corresponding norm $\|\cdot\|_{L^{2;b}}$ (notice that we avoid the longer notation $\|\cdot\|_{L^{2;b}(I\mapsto X)}$).

Let $k \geq 0$ be an integer. We denote by $\mathcal{C}^k(I \to X)$ the space of curves $I \to X$ which are continuously differentiable up to the order k. If $U \in \mathcal{C}^k(I \to X)$, then $\partial_y^\ell U$ is the derivative of order $\ell = 0, \ldots, k$ (however, we will often write ∂^2_{yy} instead of ∂^2_y). Further, $\mathcal{C}^\infty(I \to X) = \bigcap\limits_{k\geq 0} \mathcal{C}^k(I \to X)$.

Accordingly with a commonly used notation, curves in $\mathcal{C}^{k,\sigma}(I \to X) \subset \mathcal{C}^k(I \to X)$ have Hölder continuous derivatives of order k. For our purposes, it is convenient to put

$$\widetilde{\mathcal{C}}^\alpha(I \to X) = \begin{cases} \mathcal{C}^{\lfloor\alpha\rfloor,\alpha-\lfloor\alpha\rfloor}(I \to X) & \text{if } \alpha > 0 \text{ is not an integer} \\ \mathcal{C}^{\lfloor\alpha\rfloor-1,1}(I \to X) & \text{if } \alpha \geq 1 \text{ is an integer.} \end{cases} \tag{1.12}$$

Also, for $U \in \widetilde{\mathcal{C}}^\alpha(I \to X)$ we put

$$[\![U]\!]_{\widetilde{\mathcal{C}}^\alpha} = \begin{cases} \sup\limits_{\substack{y_1,y_2\in\mathbb{R}\\ y_1\neq y_2}} \dfrac{\|\partial_y^{\lfloor\alpha\rfloor}U(y_1) - \partial_y^{\lfloor\alpha\rfloor}U(y_2)\|_X}{|y_1-y_2|^{\alpha-\lfloor\alpha\rfloor}} & \text{if } \alpha \notin \mathbb{N}, \\ \sup\limits_{\substack{y_1,y_2\in\mathbb{R}\\ y_1\neq y_2}} \dfrac{\|\partial_y^{\alpha-1}U(y_1) - \partial_y^{\alpha-1}U(y_2)\|_X}{|y_1-y_2|} & \text{if } \alpha \in \mathbb{N}. \end{cases}$$

Notice that $\widetilde{\mathcal{C}}^\alpha(I \to X) \subset \mathcal{C}^{\lfloor\alpha\rfloor}(I \to X)$ if and only if α is not an integer.

Let $k \in \mathbb{N} \cup \{\infty\}$. The spaces of even curves in $L^{2;b}(\mathbb{R} \to X), \mathcal{C}^k(\mathbb{R} \to X)$ are denoted by $L^{2;b}_{\mathbf{e}}(\mathbb{R} \to X)$, $\mathcal{C}^k_{\mathbf{e}}(\mathbb{R} \to X)$, respectively, and $\mathcal{C}^k_{c,\mathbf{e}}(\mathbb{R} \to X)$ is the space of compactly supported functions in $\mathcal{C}^k_{\mathbf{e}}(\mathbb{R} \to X)$.

We write $L^{2;b}_{\mathbf{e}}(\mathbb{R}), \mathcal{C}^k_{\mathbf{e}}(\mathbb{R}), \mathcal{C}^\infty_{c,\mathbf{e}}(\mathbb{R})$ instead of $L^{2;b}(\mathbb{R} \to \mathbb{R}), \mathcal{C}^k(\mathbb{R} \to \mathbb{R}), \mathcal{C}^\infty_{c,\mathbf{e}}(\mathbb{R} \to \mathbb{R})$.

2. Spaces of real valued functions

In this section, for any parameter $b \in (-1,1)$ and any integer $k \geq 0$ we introduce the Sobolev-type space $H^{k;b}_{\mathbf{e}}(\mathbb{R})$, which is related to the differential operators $(\mathbb{D}_b + \lambda)^k$, $\lambda > 0$.

The choice of working with even functions has been inspired by [7]. This strategy is needed in case $b \neq 0$ to overcome some technical difficulties produced by the singularity of the operator $\mathbb{D}_b$ in (1.3) at $y = 0$.

In fact, as noticed in [7], if $\psi \in \mathcal{C}^2_{\mathbf{e}}(\mathbb{R})$, then $y^{-1}\partial_y\psi(y) = \partial^2_{yy}\psi(0) + o(1)$ as $y \to 0$, which implies $\mathbb{D}_b\psi \in \mathcal{C}^0_{\mathbf{e}}(\mathbb{R})$. More generally,

$$(\mathbb{D}_b + \lambda)^m\psi \in \mathcal{C}^{k-2m}_{\mathbf{e}}(\mathbb{R}) \quad \textit{for any integer } m \leq k/2 \textit{ and any } \psi \in \mathcal{C}^k_{\mathbf{e}}(\mathbb{R}). \tag{2.1}$$

Our definition of $H^{k;b}_{\mathbf{e}}(\mathbb{R})$ is based on induction procedure, starting from the lower order cases $k = 1, 2$.

First order. For $\lambda > 0$, we endow the weighted Hilbert space

$$H^{1;b}(\mathbb{R}) := H^1(\mathbb{R}; |y|^b dy) = \{\psi \in L^{2;b}(\mathbb{R}) \mid \partial_y \psi \in L^{2;b}(\mathbb{R})\}$$

with the scalar product

$$(\psi, \eta)_{\lambda, H^{1;b}} = \int_{-\infty}^{+\infty} |y|^b (\partial_y \psi \partial_y \eta + \lambda \psi \eta)\, dy$$

and the corresponding norm $\|\psi\|_{\lambda, H^{1;b}}$. If $\lambda = 1$ we drop it and simply write $(\psi, \eta)_{H^{1;b}}$ and $\|\psi\|_{H^{1;b}}$. Clearly, the norms $\|\cdot\|_{\lambda, H^{1;b}}$ are equivalent for all $\lambda > 0$ and moreover

$$\|\psi(\cdot\sqrt{\lambda})\|^2_{\lambda, H^{1;b}_{\mathbf{e}}} = \lambda^{1-\frac{1+b}{2}} \|\psi(\cdot)\|^2_{H^{1;b}_{\mathbf{e}}}. \tag{2.2}$$

Lemma 2.1.

i) $\mathcal{C}^\infty_c(\mathbb{R})$ *is dense in* $H^{1;b}(\mathbb{R})$*;*
ii) $H^{1;b}(\mathbb{R}) \subset H^1_{\mathrm{loc}}(\mathbb{R})$ *if* $b \in (-1, 0]$ *and* $H^{1;b}(\mathbb{R}) \subset W^{1,p}_{\mathrm{loc}}(\mathbb{R})$ *for arbitrary* $p \in [1, \frac{2}{1+b})$ *if* $b \in (0, 1)$*;*
iii) $H^{1;b}(\mathbb{R}) \subset \mathcal{C}^{0, \frac{1-b_+}{2}}_{\mathrm{loc}}(\mathbb{R})$*;*
iv) *There exists* $m_b > 0$ *such that* $\|\psi\|^2_{H^{1;b}} \ge m_b |\psi(0)|^2$ *for any* $\psi \in H^{1;b}(\mathbb{R})$*.*

Proof. For *i*) see [13]. The first part of *ii*) is trivial; to prove the second one use Hölder's inequality.

If $b \le 0$ then *ii*) implies *iii*) immediately. Assume $b \in (0, 1)$ and take $\psi \in \mathcal{C}^\infty_c(\mathbb{R})$. Since

$$\psi(y_2) - \psi(y_1) = \int_{y_1}^{y_2} |t|^{\frac{-b}{2}} (|t|^{\frac{b}{2}} \partial_t \psi(t))\, dt$$

for any $y_1, y_2 \in \mathbb{R}$, then Hölder's inequality and the density result in *i*) imply that

$$\begin{aligned} |\psi(y_2) - \psi(y_1)|^2 &\le \frac{1}{1-b} \|\partial_y \psi\|^2_{L^{2;b}} \left| y_2 |y_2|^{-b} - y_1 |y_1|^{-b} \right| \\ &\le \frac{2^{1-b}}{1-b} \|\partial_y \psi\|^2_{L^{2;b}} \left| y_2 - y_1 \right|^{1-b} \end{aligned} \tag{2.3}$$

for any $\psi \in H^{1;b}(\mathbb{R})$, $y_1, y_2 \in \mathbb{R}$. Since $\mathcal{C}^\infty_c(\mathbb{R})$ is dense in $H^{1;b}(\mathbb{R})$ and since ψ was arbitrarily chosen in $H^{1;b}(\mathbb{R})$, the inclusion in *iii*) easily follows.

Lastly, given $\psi \in H^{1;b}(\mathbb{R})$ we use (2.3) to get the existence of a constant $c > 0$ depending only on b such that

$$c|y|^b|\psi(0)|^2 \le |y|\|\partial_y \psi\|^2_{L^{2;b}} + |y|^b|\psi(y)|^2$$

for any $y \in \mathbb{R}$. Then iii) follows via integration over $(0,1)$. □

Remark 2.2. It follows from Theorem 3.3 in Section 3 that the best constant in iv) is

$$m_b = 2^{1+b}\Gamma\Big(\frac{1+b}{2}\Big)\Gamma\Big(\frac{1-b}{2}\Big)^{-1},$$

and it is achieved by the function ψ_s, see (1.11), for $s = \frac{1-b}{2}$.

We will be mainly concerned with $H^{1;b}_{\mathbf{e}}(\mathbb{R})$, the subspace of even functions in $H^{1;b}(\mathbb{R})$. For future convenience, we notice that the proof of Lemma 2.1 gives

$$|\psi(y_2) - \psi(y_1)|^2 \le \frac{1}{1-b}\|\partial_y\psi\|^2_{L^{2;b}}\,\big||y_2|^{1-b} - |y_1|^{1-b}\big| \tag{2.4}$$

for any $\psi \in H^{1;b}_{\mathbf{e}}(\mathbb{R})$, $y_1, y_2 \in \mathbb{R}$.

Second order. If $\psi \in H^{1;b}_{\mathbf{e}}(\mathbb{R})$ then $|y|^b\partial_y\psi \in L^{2;-b}(\mathbb{R}) \subset L^1_{\mathrm{loc}}(\mathbb{R})$. We put

$$H^{2;b}_{\mathbf{e}}(\mathbb{R}) = \big\{\psi \in H^{1;b}_{\mathbf{e}}(\mathbb{R}) \mid |y|^b\partial_y\psi \in H^{1;-b}(\mathbb{R})\big\}.$$

Let $\psi \in \mathcal{C}^2_{c,\mathbf{e}}(\mathbb{R})$. Then $\partial_y(|y|^b\partial_y\psi) = -|y|^b\mathbb{D}_b\psi$, which implies $\psi \in H^{2;b}_{\mathbf{e}}(\mathbb{R})$ by (2.1). We extend the pointwise defined operator $\mathbb{D}_b$ to $H^{2;b}_{\mathbf{e}}(\mathbb{R})$ by putting

$$\mathbb{D}_b\psi := -|y|^{-b}\partial_y(|y|^b\partial_y\psi) \qquad \text{for } \psi \in H^{2;b}_{\mathbf{e}}(\mathbb{R}),$$

so that $\mathbb{D}_b : H^{2;b}_{\mathbf{e}}(\mathbb{R}) \to L^{2;b}_{\mathbf{e}}(\mathbb{R})$.

Lemma 2.3. *Let* $\psi \in H^{2;b}_{\mathbf{e}}(\mathbb{R})$. *Then*

$$(\mathbb{D}_b\psi, \eta)_{L^{2;b}} = (\partial_y\psi, \partial_y\eta)_{L^{2;b}} \qquad \textit{for any } \eta \in H^{1;b}_{\mathbf{e}}(\mathbb{R}); \tag{2.5}$$

$$(\mathbb{D}_b\psi, \eta)_{L^{2;b}} = (\,\psi\,, \mathbb{D}_b\eta)_{L^{2;b}} \qquad \textit{for any } \eta \in H^{2;b}_{\mathbf{e}}(\mathbb{R}). \tag{2.6}$$

Proof. Let $\eta \in \mathcal{C}^\infty_{c,\mathbf{e}}(\mathbb{R})$. We can use integration by parts to compute

$$\int_{-\infty}^{+\infty} |y|^b(\mathbb{D}_b\psi)\eta\,dy = -\int_{-\infty}^{+\infty} \partial_y(|y|^b\partial_y\psi)\eta\,dy = \int_{-\infty}^{+\infty} |y|^b\partial_y\psi\,\partial_y\eta\,dy.$$

Thus i) follows, thanks to the density result in Lemma 2.1. Clearly ii) is an immediate consequence of i). □

It remains to introduce a Hilbertian structure on $H^{2;b}_{\mathbf{e}}(\mathbb{R})$. Given $\lambda > 0$, we put

$$(\psi, \eta)_{\lambda, H^{2;b}_{\mathbf{e}}} = ((\mathbb{D}_b + \lambda)\psi, (\mathbb{D}_b + \lambda)\eta)_{L^{2;b}} \ , \quad \|\psi\|_{\lambda, H^{2;b}_{\mathbf{e}}} = \|(\mathbb{D}_b + \lambda)\psi\|_{L^{2;b}}.$$

If $\lambda = 1$ we drop it and simply write $(\psi, \eta)_{H^{2;b}_{\mathbf{e}}}$ and $\|\psi\|_{H^{2;b}_{\mathbf{e}}}$. Notice that

$$(\mathbb{D}_b + \lambda)\psi(\cdot\sqrt{\lambda}) = \lambda\left[(\mathbb{D}_b + 1)\psi\right](\cdot\sqrt{\lambda}) \ , \tag{2.7}$$

which implies

$$\|\psi(\cdot\sqrt{\lambda})\|^2_{\lambda, H^{2;b}_{\mathbf{e}}} = \lambda^{2-\frac{1+b}{2}}\|\psi(\cdot)\|^2_{H^{2;b}_{\mathbf{e}}} \quad \text{for any } \psi \in H^{2;b}_{\mathbf{e}}(\mathbb{R}). \tag{2.8}$$

Lemma 2.4. *Let $\lambda > 0$, $\psi \in H^{2;b}_{\mathbf{e}}(\mathbb{R})$. Then*

$$\|\psi\|^2_{\lambda, H^{2;b}_{\mathbf{e}}} \geq \lambda\|\psi\|^2_{\lambda, H^{1;b}}.$$

Therefore, $H^{2;b}_{\mathbf{e}}(\mathbb{R})$ is a Hilbert space, and is continuously embedded in $H^{1;b}_{\mathbf{e}}(\mathbb{R})$.

Proof. Thanks to (2.8) we can assume that $\lambda = 1$. By Lemma 2.3 with $\eta = \psi$ we have $(\mathbb{D}_b\psi, \psi)_{L^{2;b}} = \|\partial_y\psi\|^2_{L^{2;b}}$. Thus

$$\begin{aligned}\int_{-\infty}^{+\infty} |y|^b|(\mathbb{D}_b + 1)\psi|^2\, dy &= \int_{-\infty}^{+\infty} |y|^b|\mathbb{D}_b\psi|^2\, dy + 2\int_{-\infty}^{+\infty} |y|^b(\mathbb{D}_b\psi)\psi\, dy + \int_{-\infty}^{+\infty} |y|^b|\psi|^2\, dy \\ &\geq 2\int_{-\infty}^{+\infty} |y|^b|\partial_y\psi|^2\, dy + \int_{-\infty}^{+\infty} |y|^b|\psi|^2\, dy\,,\end{aligned}$$

which implies $\|\psi\|^2_{H^{2;b}_{\mathbf{e}}} \geq \|\psi\|^2_{H^{1;b}}$. The conclusion of the proof is standard. □

Higher order. If $k > 2$ and $\lambda > 0$ we use induction to define

$$\begin{aligned} H^{k;b}_{\mathbf{e}}(\mathbb{R}) &= \left\{\psi \in H^{k-1;b}_{\mathbf{e}}(\mathbb{R}) \mid \mathbb{D}_b\psi \in H^{k-2,b}_{\mathbf{e}}(\mathbb{R})\right\} \ , \\ (\psi, \eta)_{\lambda, H^{k;b}_{\mathbf{e}}} &= ((\mathbb{D}_b + \lambda)\psi, (\mathbb{D}_b + \lambda)\eta)_{\lambda, H^{k-2;b}_{\mathbf{e}}} \ , \\ \|\psi\|_{\lambda, H^{k;b}_{\mathbf{e}}} &= \|(\mathbb{D}_b + \lambda)\psi\|_{\lambda, H^{k-2;b}_{\mathbf{e}}} \ . \end{aligned}$$

As usual, if $\lambda = 1$ we drop it and simply write $(\psi, \eta)_{H^{k;b}_{\mathbf{e}}}$ and $\|\psi\|_{H^{k;b}_{\mathbf{e}}}$.

Notice that $\mathcal{C}^k_{c,\mathbf{e}}(\mathbb{R}) \subset H^{k;b}_{\mathbf{e}}(\mathbb{R})$ by (2.1). In the next lemma we collect the main properties of the spaces $H^{k;b}_{\mathbf{e}}(\mathbb{R})$ for $k \geq 1$. In particular it implies that $\|\cdot\|_{\lambda, H^{k;b}_{\mathbf{e}}}$, for different λ's, define the same Hilbertian structure on $H^{k;b}_{\mathbf{e}}(\mathbb{R})$. We omit its easy proof, which is based on previous results and induction.

Lemma 2.5. *Let $k \geq 1$, $b \in (-1,1)$, $\psi \in H^{k;b}_{\mathbf{e}}(\mathbb{R})$ and $\lambda > 0$. The following facts hold.*

i) $\|\psi\|^2_{\lambda, H^{k;b}_{\mathbf{e}}} = \begin{cases} \|\partial_y(\mathbb{D}_b+\lambda)^{\frac{k-1}{2}}\psi)\|^2_{L^{2;b}} + \lambda\|(\mathbb{D}_b+\lambda)^{\frac{k-1}{2}}\psi\|^2_{L^{2;b}} & \text{if } k \text{ is odd,} \\ \|(\mathbb{D}_b+\lambda)^{\frac{k}{2}}\psi\|^2_{L^{2;b}} & \text{if } k \text{ is even;} \end{cases}$

ii) $\|\psi(\cdot\sqrt{\lambda})\|^2_{\lambda, H^{k;b}_{\mathbf{e}}} = \lambda^{k-\frac{1+b}{2}}\|\psi(\cdot)\|^2_{H^{k;b}_{\mathbf{e}}}$;

iii) $(\mathbb{D}_b+\lambda)^m\psi \in H^{k-2m;b}_{\mathbf{e}}(\mathbb{R})$ *for any positive integer* $m < k/2$;

iv) $\|\psi\|^2_{\lambda, H^{k;b}_{\mathbf{e}}} \geq \lambda^{k-j}\|\psi\|^2_{\lambda, H^{j;b}} \geq \lambda^k\|\psi\|^2_{L^{2;b}}$ *for any* $j = 1, \ldots, k$;

v) $\|\psi\|^2_{\lambda, H^{k;b}_{\mathbf{e}}} \geq m_b\lambda^{k-\frac{1+b}{2}}|\psi(0)|^2$, *where m_b is the constant in Lemma 2.1.*

We now establish some integration by parts formulae. It suffices to take $\lambda = 1$.

Lemma 2.6. *Let $k \geq 2$, $\psi \in H^{2(k-1);b}_{\mathbf{e}}(\mathbb{R})$, $\eta \in H^{k;b}_{\mathbf{e}}(\mathbb{R})$. Then*

$$(\psi, \eta)_{H^{k;b}_{\mathbf{e}}} = ((\mathbb{D}_b+1)^{k-1}\psi, (\mathbb{D}_b+1)\eta)_{L^{2;b}}.$$

Proof. Notice that $H^{2(k-1);b}_{\mathbf{e}}(\mathbb{R}) \subset H^{k;b}_{\mathbf{e}}(\mathbb{R})$.

If $k = 2$, the equality in the lemma holds by definition.

If $k = 2m \geq 4$ is even, we use (2.6) with $(\mathbb{D}_b+1)^m\psi \in H^{2(m-1);b}_{\mathbf{e}}(\mathbb{R})$ instead of ψ and $(\mathbb{D}_b+1)^{m-1}\eta \in H^{2;b}_{\mathbf{e}}(\mathbb{R})$ instead of η to get

$$(\psi, \eta)_{H^{2m;b}_{\mathbf{e}}} = ((\mathbb{D}_b+1)^m\psi, (\mathbb{D}_b+1)^m\eta)_{L^{2;b}} = ((\mathbb{D}_b+1)^{m+1}\psi, (\mathbb{D}_b+1)^{m-1}\eta)_{L^{2;b}}.$$

If $m = 2$ we are done. Otherwise, repeat the same procedure $m-1$ times to get

$$(\psi, \eta)_{H^{2m;b}_{\mathbf{e}}} = ((\mathbb{D}_b+1)^{2m-1}\psi, (\mathbb{D}_b+1)\eta)_{L^{2;b}}\ , \tag{2.9}$$

which concludes the proof in the even case.

If $k = 2m+1 \geq 3$ is odd we apply (2.5) with $(\mathbb{D}_b+1)^m\psi \in H^{2m;b}_{\mathbf{e}}(\mathbb{R})$ instead of ψ and $(\mathbb{D}_b+1)^m\eta \in H^{1;b}_{\mathbf{e}}(\mathbb{R})$ instead of η to get

$$(\,\partial_y((\mathbb{D}_b+1)^m\psi), \partial_y((\mathbb{D}_b+1)^m\eta))_{L^{2;b}} = (\mathbb{D}_b(\mathbb{D}_b+1)^m\psi, (\mathbb{D}_b+1)^m\eta)_{L^{2;b}}.$$

It follows that

$$\begin{aligned}(\psi, \eta)_{H^{k;b}_{\mathbf{e}}} &= (\,\partial_y((\mathbb{D}_b+1)^m\psi), \partial_y((\mathbb{D}_b+1)^m\eta))_{L^{2;b}} + ((\mathbb{D}_b+1)^m\psi, (\mathbb{D}_b+1)^m\eta)_{L^{2;b}} \\ &= ((\mathbb{D}_b+1)^{m+1}\psi, (\mathbb{D}_b+1)^m\eta)_{L^{2;b}} = ((\mathbb{D}_b+1)\psi, \eta)_{H^{2m;b}_{\mathbf{e}}}.\end{aligned}$$

To conclude the proof, use (2.9) with ψ replaced by $(\mathbb{D}_b+1)\psi$. □

Remark 2.7. It is well known that smooth, compactly supported functions are dense in $H^k(\mathbb{R})$ for any $k > 0$. Recall that $\mathcal{C}^\infty_c(\mathbb{R})$ is dense in $H^{1;b}(\mathbb{R})$ for any $b \in (-1,1)$ by [13]. It would be of interest to prove the density of $\mathcal{C}^\infty_{c,\mathbf{e}}(\mathbb{R})$ in $H^{k;b}_{\mathbf{e}}(\mathbb{R})$ in case $b \neq 0$, $k > 1$.

3. Bessel functions and related issues

The basic properties of the Bessel function K_α can be found for instance [11, Sections 8.4, 8.5]. For any $\alpha \in \mathbb{R}$ the standard modified Bessel function of the second kind $K_\alpha = K_{-\alpha}$ solves

$$\partial^2_{yy} K_\alpha(y) + y^{-1}\partial_y K_\alpha(y) - (1+\alpha^2 y^{-2})K_\alpha(y) = 0 \quad \text{on } \mathbb{R}_+$$

and decays exponentially as $y \to +\infty$. If $\alpha \neq 0$ then

$$K_\alpha(y) = 2^{|\alpha|-1}\Gamma(|\alpha|)y^{-|\alpha|} + o(y^{-|\alpha|}) \quad \text{as } y \to 0^+.$$

Bessel functions of different orders are related by the formulae

$$\partial_y(y^\alpha K_\alpha(y)) = -y^\alpha K_{\alpha-1}(y)\,, \qquad K_\alpha(y) - K_{\alpha-2}(y) = 2(\alpha-1)y^{-1}K_{\alpha-1}(y).$$

Next, for $s > 0$ and $\lambda > 0$ we put

$$\psi_{s,\lambda}(y) := \psi_s(\sqrt{\lambda}\,y) = c_s(\sqrt{\lambda}|y|)^s K_s(\sqrt{\lambda}|y|)\,, \tag{3.1}$$

see (1.11). Notice that

$$\psi_{s,\lambda} \in \mathcal{C}^0_{\mathbf{e}}(\mathbb{R})\,, \quad \psi_{s,\lambda}(0) = 1\,, \quad \psi_{s,\lambda} \in \mathcal{C}^\infty(\mathbb{R}_+)\,,$$

and $\psi_{s,\lambda}$ decays exponentially at infinity together with its derivatives of any order. Further, (2.7) readily implies

$$(\mathbb{D}_b + \lambda)^m \psi_{s,\lambda}(y) = \lambda^m \left[(\mathbb{D}_b + 1)^m \psi_s\right](\sqrt{\lambda}\,y) \tag{3.2}$$

for any $y \neq 0$ and any integer $m \geq 1$.

Lemma 3.1. *Let $s > 0$ be non-integer and put $\mathfrak{b} = 1 - 2(s - \lfloor s \rfloor)$. Then ψ_s solves the following differential equations on $\mathbb{R}_+$:*

i) $\partial_y \psi_s(y) = \begin{cases} -d_s\, y^{2s-1}\psi_{1-s}(y) & \text{if } 0 < s < 1, \\ -\frac{1}{2(s-1)}\, y\psi_{s-1}(y) & \text{if } s > 1; \end{cases}$

ii) $-\partial^2_{yy}\psi_s(y) + \psi_s(y) = \begin{cases} d_s(2s-1)y^{2(s-1)}\psi_{1-s}(y) & \text{if } 0 < s < 1, \\ \frac{2s-1}{2(s-1)}\psi_{s-1} & \text{if } s > 1; \end{cases}$

iii) $(\mathbb{D}_\mathfrak{b} + 1)^{\lceil s \rceil}\psi_s = 0$;

iv) If $s > 1$ then for any $m = 1, \ldots, \lfloor s \rfloor$

$$(\mathbb{D}_\mathfrak{b} + 1)^m \psi_s = \frac{d_s}{d_{s-m}}\,\psi_{s-m} = \frac{\lfloor s \rfloor!}{\lfloor s-m \rfloor!}\,\frac{\Gamma(s-m)}{\Gamma(s)}\,\psi_{s-m}. \tag{3.3}$$

Proof. Let $s \in (0,1)$. By the properties of the Bessel functions we get

$$\partial_y \psi_s(y) = -c_s y^s K_{1-s}(y) = -c_s y^{2s-1}(y^{1-s}K_{1-s}(y)) = -d_s y^{2s-1}\psi_{1-s}(y).$$

This gives the first equality in i). Now we notice that we can compute $\partial_y \psi_{1-s}$ via the first equality in i), where s is replaced by $1-s$. The proofs of ii), iii) readily follow. This completes the proof in this case.

Now let $s > 1$. We compute

$$\partial_y \psi_s(y) = c_s \partial_y (y^s K_s(y)) = -c_s y (y^{s-1}K_{s-1}(y)) = -\frac{c_s}{c_{s-1}}\psi_{s-1}(y)$$

which gives the second equality in i). Also, we get

$$\begin{aligned}\partial^2_{yy}\psi_s(y) &= -c_s\partial_y(y^s K_{s-1}(y)) = -c_s y^s(-K_{s-2}(y) + y^{-1}K_{s-1}(y)) \\ &= c_s y^s\big((1-2s)y^{-1}K_{s-1}(y) + K_s(y)\big)\end{aligned}$$

by the recurrence formula for K_s. Hence

$$\partial^2_{yy}\psi_s(y) = \frac{c_s}{c_{s-1}}(1-2s)\psi_{s-1} + \psi_s(y),$$

which gives ii) for $s > 1$. To prove iv) we notice that this last equality implies

$$(\mathbb{D}_{\mathfrak{b}} + 1)\psi_s = -\partial^2_{yy}\psi_s - (1-2s+2\lfloor s\rfloor)\partial_y\psi_s + \psi_s = \frac{\lfloor s\rfloor}{s-1}\psi_{s-1} = \frac{d_s}{d_{s-1}}\psi_{s-1}\,.$$

Thus (3.3) holds for $m = 1$. To conclude the proof of iv) repeat the same argument a finite number of times.

It remains to prove iii) in this case. We use iv) with $m = \lfloor s\rfloor$ and then iii) with s replaced by $s - \lfloor s\rfloor \in (0,1)$ to get

$$(\mathbb{D}_{\mathfrak{b}} + 1)^{\lceil s\rceil}\psi_s = \frac{d_s}{d_{1-\lfloor s\rfloor}}(\mathbb{D}_{\mathfrak{b}} + 1)\psi_{s-\lfloor s\rfloor} = 0\,.$$

The lemma is completely proved. □

Remark 3.2. Since $K_s > 0$ on $\mathbb{R}_+$, from i) in Lemma 3.1 it readily follows that the positive function ψ_s achieves its maximum at the origin.

The next theorem contains our main result on the functions ψ_s (recall our non-standard definition of Hölder spaces in (1.12)).

Theorem 3.3. *Let $s > 0$ be non-integer, put $\mathfrak{b} = 1 - 2(s - \lfloor s \rfloor)$ and let $\lambda > 0$. Then*

$$\psi_{s,\lambda} \in H^{\lceil s \rceil;\mathfrak{b}}_{\mathbf{e}}(\mathbb{R}) \cap \widetilde{\mathcal{C}}^{2s}(\mathbb{R}) \ ; \tag{3.4}$$

$$\lim_{y \to 0^+} y^{\mathfrak{b}} \partial_y \big((\mathbb{D}_{\mathfrak{b}} + \lambda)^{\lfloor s \rfloor} \psi_{s,\lambda} \big) = -\, d_s \lambda^s \tag{3.5}$$

where d_s is the constant in (1.5). Moreover $\psi_{s,\lambda}$ satisfies

$$(\psi_{s,\lambda}, \eta)_{\lambda, H^{\lceil s \rceil;\mathfrak{b}}_{\mathbf{e}}} = 2d_s \, \lambda^s \, \eta(0) \quad \textit{for any} \quad \eta \in H^{\lceil s \rceil;\mathfrak{b}}_{\mathbf{e}}(\mathbb{R}). \tag{3.6}$$

Finally, $\psi_{s,\lambda}$ admits the following variational characterization,

$$\|\psi_{s,\lambda}\|^2_{\lambda, H^{\lceil s \rceil;\mathfrak{b}}_{\mathbf{e}}} = \inf_{\substack{\eta \in H^{\lceil s \rceil;\mathfrak{b}}_{\mathbf{e}}(\mathbb{R}) \\ \eta(0)=1}} \| \, \eta \, \|^2_{\lambda, H^{\lceil s \rceil;\mathfrak{b}}_{\mathbf{e}}} = 2d_s \lambda^s \, . \tag{3.7}$$

Proof. Thanks to (3.2), we assume that $\lambda = 1$. We divide the proof in two steps.

Step 1. Let $\lfloor s \rfloor = 0$. Then $\mathfrak{b} = 1 - 2s$ and

$$\partial_y \psi_s(y) = -d_s \, y^{-\mathfrak{b}} \psi_{1-s}(y) = -d_s \, y^{-\mathfrak{b}} + o(y^{-\mathfrak{b}}) \quad \text{as } y \to 0^+, \tag{3.8}$$

which proves (3.5). Since in addition ψ_s decays exponentially at infinity, from (3.8) we first infer that $\psi_s \in H^{1;1-2s}_{\mathbf{e}}(\mathbb{R})$.

To prove that $\psi_s \in \widetilde{\mathcal{C}}^{2s}(\mathbb{R})$ we fix two points $y_1, y_2 \in \mathbb{R}$. By the symmetry of ψ_s, we can assume that $y_1, y_2 \ge 0$.

Let $0 < 2s \le 1$. For $y > 0$ we have $|\partial_y \psi_s(y)| = d_s y^{2s-1} \psi_{1-s}(y) \le d_s y^{2s-1}$. Thus $\psi_s \in \widetilde{\mathcal{C}}^{2s}(\mathbb{R})$ follows from

$$|\psi_s(y_1) - \psi_s(y_2)| \le d_s \Big| \int_{y_1}^{y_2} y^{2s-1} dy \Big| = \frac{d_s}{2s} |y_1^{2s} - y_2^{2s}| \le \frac{d_s}{2s} |y_1 - y_2|^{2s}.$$

If $1 < 2s < 2$ we use *ii*) in Lemma 3.1 to estimate

$$|\partial^2_{yy} \psi_s(y)| = |\psi_s(y) - d_s(2s-1) y^{2(s-1)} \psi_{1-s}(y)| \le 1 + c y^{2(s-1)}$$

for $y > 0$. Using integration as before, we plainly get

$$|\partial_y \psi_s(y_1) - \partial_y \psi_s(y_2)| \le |y_1 - y_2| + c|y_1 - y_2|^{2s-1}.$$

Since $\partial_y \psi_s$ decays exponentially at infinity, we infer that there exists a constant $c > 0$ depending only on s, such that

$$|\partial_y \psi_s(y_1) - \partial_y \psi_s(y_2)| \le c|y_1 - y_2|^{2s-1}$$

which, in turns concludes the proof of (3.4).

Next, by *iii*) in Lemma 3.1 we have that

$$\partial_y(y^{\mathfrak{b}}\partial_y\psi_s) = y^{\mathfrak{b}}\psi_s \quad \text{on } \mathbb{R}_+. \tag{3.9}$$

We test (3.9) with an arbitrary $\eta \in \mathcal{C}^\infty_{c,\mathbf{e}}(\mathbb{R})$. Taking (3.8) into account we obtain

$$\int_0^\infty y^{\mathfrak{b}}\psi_s\eta\, dy = \int_0^\infty \partial_y(y^{\mathfrak{b}}\partial_y\psi_s)\eta\, dy = d_s\eta(0) - \int_0^\infty y^{\mathfrak{b}}\partial_y\psi_s\partial_y\eta\, dy.$$

By the evenness of ψ_s and η, this implies that $(\psi_s,\eta)_{H^{1;\mathfrak{b}}_{\mathbf{e}}} = 2d_s\eta(0)$. Thus (3.6) holds in case $\lfloor s\rfloor = 0$, thanks to the density result in Lemma 2.1.

From (3.6) it follows that $(\psi_s,\eta-\psi_s)_{H^{1;\mathfrak{b}}_{\mathbf{e}}} = 0$ for any $\eta \in H^{1;\mathfrak{b}}_{\mathbf{e}}(\mathbb{R})$ such that $\eta(0) = 1$. Thus, ψ_s is the minimal distance projection of 0 on the hyperplane $\{\eta(0) = 1\} \subset H^{1;\mathfrak{b}}_{\mathbf{e}}(\mathbb{R})$, that is, ψ_s is the unique solution to the minimization problem in (3.7). This completes the proof in the case $s \in (0,1)$.

Step 2: Let $\lfloor s\rfloor \ge 1$. Thanks to *i*) in Lemma 3.1 we see that

$$\partial_y\psi_s(y) = -\frac{1}{2(s-1)}y\ \psi_{s-1}(y) = -\frac{1}{2(s-1)}y + o(y) \quad \text{as } y \to 0^+,$$

hence $\psi_s \in \mathcal{C}^2(\mathbb{R})$. Next, as in case $2s \in (1,2)$ we use *ii*) in Lemma 3.1 to infer that $\partial^2_{yy}\psi_s$ has the same regularity as ψ_{s-1}. If $s \in (1,2)$ we obtain $\psi_s \in \widetilde{\mathcal{C}}^{2s}(\mathbb{R})$ by Step 1; if $s > 2$ one can use a bootstrap argument to prove that $\psi_s \in \widetilde{\mathcal{C}}^{2s}(\mathbb{R})$. By the decaying of ψ_s at infinity we also infer that

$$\psi_s \in H^{2\lfloor s\rfloor;\mathfrak{b}}_{\mathbf{e}}(\mathbb{R}) \subset H^{\lceil s\rceil;\mathfrak{b}}_{\mathbf{e}}(\mathbb{R}),$$

which concludes the proof of (3.4).

To prove (3.5) it suffices to notice that (3.3) and Step 1 give

$$\lim_{y\to 0^+} y^{\mathfrak{b}}\partial_y\big((\mathbb{D}_{\mathfrak{b}}+1)^{\lfloor s\rfloor}\psi_s\big) = \frac{d_s}{d_{s-\lfloor s\rfloor}}\lim_{y\to 0^+} y^{\mathfrak{b}}\partial_y\psi_{s-\lfloor s\rfloor} = -d_s.$$

We now prove (3.6). Take any $\eta \in H^{\lceil s\rceil;\mathfrak{b}}_{\mathbf{e}}(\mathbb{R})$. We apply Lemma 2.6 with $k = \lceil s\rceil$ and $\psi = \psi_s$ to obtain

$$(\psi_s,\eta)_{H^{\lceil s\rceil;\mathfrak{b}}_{\mathbf{e}}} = ((\mathbb{D}_{\mathfrak{b}}+1)^{\lfloor s\rfloor}\psi_s, (\mathbb{D}_{\mathfrak{b}}+1)\eta)_{L^{2;\mathfrak{b}}}.$$

Therefore, (3.3), (2.5) and Step 1 with s replaced by $s - \lfloor s\rfloor \in (0,1)$ give

$$(\psi_s,\eta)_{H^{\lceil s\rceil;\mathfrak{b}}_{\mathbf{e}}} = \frac{d_s}{d_{s-\lfloor s\rfloor}}(\psi_{s-\lfloor s\rfloor}, (\mathbb{D}_{\mathfrak{b}}+1)\eta)_{L^{2;\mathfrak{b}}} = \frac{d_s}{d_{s-\lfloor s\rfloor}}(\psi_{s-\lfloor s\rfloor},\eta)_{H^{1;\mathfrak{b}}_{\mathbf{e}}} = 2d_s\eta(0),$$

and (3.6) follows. For (3.7) argue as in Step 1. □

Remark 3.4. The recurrence formulae (3.3) plainly imply the identities

$$(\mathbb{D}_{\mathfrak{b}}+1)^m\psi_s(0)=\frac{d_s}{d_{s-m}}\ ,\qquad m=1,\dots,\lfloor s\rfloor$$
$$\lim_{y\to0^+}y^{-1}\partial_y\big((\mathbb{D}_{\mathfrak{b}}+1)^m\psi_s\big)=-\frac{d_s}{d_{s-m}}\,\frac{1}{2(s-m-1)}\ ,\qquad m=0,\dots,\lfloor s\rfloor-1$$

We conclude this section with a corollary of Theorem 3.3, which might be of independent interest.

Corollary 3.5. *Let $\lfloor s\rfloor$ be even. Then the following virial-type formulae hold:*

$$\int\limits_{-\infty}^{+\infty}|y|^{\mathfrak{b}}|(\mathbb{D}_{\mathfrak{b}}+1)^{\frac{\lfloor s\rfloor}{2}}\psi_s|^2=\frac{s}{\lceil s\rceil}\,2d_s\ ,$$
$$\int\limits_{-\infty}^{+\infty}|y|^{\mathfrak{b}}|\partial_y((\mathbb{D}_{\mathfrak{b}}+1)^{\frac{\lfloor s\rfloor}{2}}\psi_s)|^2=\frac{\lceil s\rceil-s}{\lceil s\rceil}\,2d_s\ .$$

Proof. We use (3.7) with s replaced by $s+1$ and then (3.3) to get

$$2d_{s+1}=\|\psi_{s+1}\|^2_{H^{2+\lfloor s\rfloor;\mathfrak{b}}_{\mathbf{e}}}=\int\limits_{-\infty}^{+\infty}|y|^{\mathfrak{b}}|(\mathbb{D}_{\mathfrak{b}}+1)^{\frac{\lfloor s\rfloor}{2}+1}\psi_{s+1}|^2\,dy$$
$$=\frac{d^2_{s+1}}{d^2_s}\int\limits_{-\infty}^{+\infty}|y|^{\mathfrak{b}}|(\mathbb{D}_{\mathfrak{b}}+1)^{\frac{\lfloor s\rfloor}{2}}\psi_s|^2\,dy,$$

and the first equality follows. For the second one, recall that $2d_s=\|\psi_s\|^2_{H^{\lceil s\rceil;\mathfrak{b}}_{\mathbf{e}}}$. □

4. Spaces of curves in $\mathcal{H}$; proof of the main results

We start this section by studying the (unbounded) operators $\mathbb{L}^k_bU=(\mathbb{D}_b+\mathcal{L})^kU$ on $L^{2;b}_{\mathbf{e}}(\mathbb{R}\to\mathcal{H})$, for any $b\in(-1,1)$ and any integer $k\ge0$.

Any function $U\in L^{2;b}(\mathbb{R}\to\mathcal{H})$ can be decomposed as follows,

$$U(y)=\sum_{j=1}^{\infty}U_j(y)\varphi_j,$$

where $U_j=(U,\varphi_j)_{\mathcal{H}}\in L^{2;b}_{\mathbf{e}}(\mathbb{R})$ for any $j\ge1$, and

$$\|U\|^2_{L^{2;b}} = \sum_{j=1}^{\infty} \int_{-\infty}^{+\infty} |y|^b |U_j|^2\, dy = \sum_{j=1}^{\infty} \|U_j\|^2_{L^{2;b}}.$$

Recall that

$$\mathbb{L}_b U = (\mathbb{D}_b + \mathcal{L})U = -\partial^2_{yy} U - by^{-1}\partial_y U + \mathcal{L}U \ , \quad \mathcal{L}\varphi_j = \lambda_j \varphi_j$$

and that we are assuming $\lambda_j \geq \lambda_1 > 0$. Thus, at least formally we have

$$\mathbb{L}^k_b U = \sum_{j=1}^{\infty} [(\mathbb{D}_b + \lambda_j)^k U_j]\, \varphi_j.$$

We define

$$H^{k;b}_{\mathcal{L},\mathbf{e}}(\mathbb{R} \to \mathcal{H}) = \Big\{ U \in L^{2;b}_{\mathbf{e}}(\mathbb{R} \to \mathcal{H}) \ \Big|\ U_j = (U, \varphi_j)_{\mathcal{H}} \in H^{k;b}_{\mathbf{e}}(\mathbb{R}) \text{ and } \|U\|_{H^{k;b}_{\mathcal{L},\mathbf{e}}} < \infty \Big\},$$

where

$$\|U\|^2_{H^{k;b}_{\mathcal{L},\mathbf{e}}} := \sum_{j=1}^{\infty} \ \|U_j\|^2_{\lambda_j, H^{k;b}_{\mathbf{e}}}$$

(we recall that $\|\cdot\|_{\lambda_j, H^{k;b}_{\mathbf{e}}}$ are equivalent norms in the space $H^{k;b}_{\mathbf{e}}(\mathbb{R})$, see Section 2). Thanks to Lemma 2.5, it is easily checked that $H^{k;b}_{\mathcal{L},\mathbf{e}}(\mathbb{R} \to \mathcal{H})$ is a Hilbert space with scalar product

$$(U, V)_{H^{k;b}_{\mathcal{L},\mathbf{e}}} = \sum_{j=1}^{\infty} \ (U_j, V_j)_{\lambda_j, H^{k;b}_{\mathbf{e}}}.$$

For future convenience we provide another definition of $H^{k;b}_{\mathcal{L},\mathbf{e}}(\mathbb{R} \to \mathcal{H})$. Consider the standard weighted Sobolev space

$$H^{1;b}(\mathbb{R} \to \mathcal{H}) := H^1(\mathbb{R} \to \mathcal{H}; |y|^b dy) = \{U \in L^{2;b}(\mathbb{R} \to \mathcal{H}) \mid \partial_y U \in L^{2;b}(\mathbb{R} \to \mathcal{H})\},$$

and denote by $H^{1;b}_{\mathbf{e}}(\mathbb{R} \to \mathcal{H})$ the space of even curves in $H^{1;b}(\mathbb{R} \to \mathcal{H})$. Then we let

$$H^{2;b}_{\mathbf{e}}(\mathbb{R} \to \mathcal{H}) = \{U \in H^{1;b}_{\mathbf{e}}(\mathbb{R} \to \mathcal{H}) \mid |y|^b \partial_y U \in H^{1;-b}(\mathbb{R} \to \mathcal{H})\}$$

so that

$$\mathbb{D}_b U := -|y|^{-b}\partial_y(|y|^b \partial_y U) \in L^{2;b}_{\mathbf{e}}(\mathbb{R} \to \mathcal{H}) \quad \text{for any } U \in H^{2;b}_{\mathbf{e}}(\mathbb{R} \to \mathcal{H}).$$

Finally, for $k \geq 3$ we use induction to define

$$H^{k;b}_{\mathbf{e}}(\mathbb{R}\to\mathcal{H}) = \{U \in H^{k-1;b}_{\mathbf{e}}(\mathbb{R}\to\mathcal{H}) \mid \mathbb{D}_b U \in H^{k-2;b}_{\mathbf{e}}(\mathbb{R}\to\mathcal{H})\}.$$

The proof of the next lemma is simple but boring, and we omit it.

Lemma 4.1. *Let $k \geq 1$ be an integer, $b \in (-1,1)$. Then*

$$H^{k;b}_{\mathcal{L},\mathbf{e}}(\mathbb{R}\to\mathcal{H}) = H^{k;b}_{\mathbf{e}}(\mathbb{R}\to\mathcal{H}) \cap L^{2;b}(\mathbb{R}\to\mathcal{H}^k_{\mathcal{L}}).$$

The next lemma will be useful for the proof of our main results.

Lemma 4.2.

i) If $U \in H^{k;b}_{\mathcal{L},\mathbf{e}}(\mathbb{R}\to\mathcal{H})$ then the following facts hold,

$$\|U\|^2_{H^{k;b}_{\mathcal{L},\mathbf{e}}} = \begin{cases} \|\partial_y(\mathbb{L}_b^{\frac{k-1}{2}} U)\|^2_{L^{2;b}} + \|\mathcal{L}^{\frac{1}{2}}(\mathbb{L}_b^{\frac{k-1}{2}} U)\|^2_{L^{2;b}} & \textit{if } k \textit{ is odd,} \\ \|\mathbb{L}_b^{\frac{k}{2}} U\|^2_{L^{2;b}} & \textit{if } k \textit{ is even;} \end{cases}$$

$$\|U\|^2_{H^{k;b}_{\mathcal{L},\mathbf{e}}} \geq \lambda_1^{k-j}\|U\|^2_{H^{j;b}_{\mathcal{L},\mathbf{e}}} \ \geq\ \lambda_1^k\|U\|^2_{L^{2;b}} \quad \textit{for any } j = 1,\ldots,k; \tag{4.1}$$

ii) the Dirac delta-type function

$$\delta_0 : H^{k;b}_{\mathcal{L},\mathbf{e}}(\mathbb{R}\to\mathcal{H}) \to \mathcal{H}_{\mathcal{L}}^{k-\frac{1+b}{2}}, \qquad \delta_0(V) = V(0)$$

is well defined and continuous.

Proof. To prove $i)$ use Lemma 2.5. Next, let $U = \sum\limits_{j=1}^{\infty} U_j\varphi_j$ be any curve in $H^{k;b}_{\mathcal{L},\mathbf{e}}(\mathbb{R}\to\mathcal{L})$. Thanks to $v)$ in Lemma 2.5 we can estimate

$$\|U\|^2_{H^{k;b}_{\mathcal{L},\mathbf{e}}} = \sum_{j=1}^{\infty} \|U_j\|^2_{\lambda_j, H^{k;b}_{\mathbf{e}}} \geq m_b \sum_{j=1}^{\infty} \lambda_j^{k-\frac{1+b}{2}} |U_j(0)|^2 = m_b\|U(0)\|^2_{\mathcal{H}_{\mathcal{L}}^{k-\frac{1+b}{2}}},$$

which concludes the proof. $\square$

Remark 4.3. It turns out that $H^{k;b}_{\mathcal{L},\mathbf{e}}(\mathbb{R}\to\mathcal{H}) \subset \mathcal{C}^{0,\frac{1-b_+}{2}}_{\rm loc}(\mathbb{R}\to\mathcal{H})$. For the proof, let $U = \sum\limits_{j=1}^{\infty} U_j\varphi_j \in H^{1;b}_{\mathcal{L},\mathbf{e}}(\mathbb{R}\to\mathcal{H})$ and $y_1, y_2 \in \mathbb{R}$. We use (2.4) with $\psi = U_j \in H^{k;b}_{\mathbf{e}}(\mathbb{R})$ to estimate

$$\|U(y_2) - U(y_1)\|^2_{\mathcal{H}} = \sum_{j=1}^{\infty} |U_j(y_2) - U_j(y_1)|^2 \leq \frac{1}{1-b}\|U\|^2_{H^{1;b}_{\mathcal{L},\mathbf{e}}} \big||y_2|^{1-b} - |y_1|^{1-b}\big|.$$

Since $H^{k;b}_{\mathcal{L},\mathbf{e}}(\mathbb{R}\to\mathcal{H})$ is continuously embedded in $H^{1;b}_{\mathcal{L},\mathbf{e}}(\mathbb{R}\to\mathcal{H})$ by (4.1), the claim follows.

Proof of Theorem 1.1 Recall that $\mathfrak{b}=1-2(s-\lfloor s\rfloor)\in(-1,1)$. For $u=\sum_j u_j\varphi_j\in\mathcal{H}$, we use the notation introduced in (3.1) to rewrite (1.4) as

$$\mathcal{P}_s[u](y)=\sum_{j=1}^{\infty}\psi_{s,\lambda_j}(y)\,u_j\varphi_j\,. \tag{4.2}$$

Take $u\in\mathcal{H}^s_{\mathcal{L}}$. We have $\psi_{s,\lambda_j}\in H^{\lceil s\rceil;\mathfrak{b}}_{\mathbf{e}}(\mathbb{R})$ and $\|\psi_{s,\lambda_j}\|^2_{\lambda_j,H^{\lceil s\rceil;\mathfrak{b}}_{\mathbf{e}}}=2d_s\lambda_j^s$ by Theorem 3.3. Thus

$$\|\mathcal{P}_s[u]\|^2_{H^{\lceil s\rceil;\mathfrak{b}}_{\mathcal{L},\mathbf{e}}}=\sum_{j=1}^{\infty}u_j^2\|\,\psi_{s,\lambda_j}\|^2_{\lambda_j,H^{\lceil s\rceil;\mathfrak{b}}_{\mathbf{e}}}=2d_s\sum_{j=1}^{\infty}\lambda_j^s u_j^2=2d_s\|\mathcal{L}^{\frac{s}{2}}u\|^2_{\mathcal{H}}=2d_s\|u\|^2_{\mathcal{H}^s_{\mathcal{L}}}\,,$$

and (1.5) is proved.

Next, take any $V\in H^{\lceil s\rceil;\mathfrak{b}}_{\mathcal{L},\mathbf{e}}(\mathbb{R}\to\mathcal{H})$ and put $V_j(y)=(V(y),\varphi_j)_{\mathcal{H}}$. We have

$$(\mathcal{P}_s[u],V)_{H^{\lceil s\rceil;\mathfrak{b}}_{\mathcal{L},\mathbf{e}}}=\sum_{j=1}^{\infty}u_j(\psi_{s,\lambda_j},V_j)_{\lambda_j,H^{\lceil s\rceil;\mathfrak{b}}_{\mathbf{e}}}=2d_s\sum_{j=1}^{\infty}\lambda_j^s u_jV_j(0)=2d_s\langle\mathcal{L}^s u,V(0)\rangle$$

by (3.6), which proves *iii*).

Evidently *iii*) implies that $\mathcal{P}_s[u]$ is a weak solution to (1.7). Since $\mathcal{P}_s[u]$ is smooth on $\mathbb{R}_+$ by Lemma A.1, we see that in fact $\mathcal{P}_s[u]$ solves (1.7) pointwise. The first equality in (1.8) is satisfied by *iii*) in Lemma A.1.

To conclude the proof of (1.8), we first compute

$$\mathbb{L}_{\mathfrak{b}}^{\lfloor s\rfloor}\mathcal{P}_s[u](y)=\sum_{j=1}^{\infty}\lambda_j^{\lfloor s\rfloor}\big((\mathbb{D}_{\mathfrak{b}}+1)^{\lfloor s\rfloor}\psi_s\big)(\sqrt{\lambda_j}y)\,u_j\varphi_j\,.$$

Now we use two items in Lemma 3.1, namely, *iv*) (with $m=\lfloor s\rfloor$) and then *i*) (with $s-\lfloor s\rfloor$ instead of s). This gives

$$\begin{aligned}y^{\mathfrak{b}}(\partial_y\mathbb{L}_{\mathfrak{b}}^{\lfloor s\rfloor}\mathcal{P}_s[u])(y)&=\frac{d_s}{d_{s-\lfloor s\rfloor}}\sum_{j=1}^{\infty}\lambda_j^{\lfloor s\rfloor}y^{\mathfrak{b}}\big(\partial_y\psi_{s-\lfloor s\rfloor}\big)(\sqrt{\lambda_j}y)\,u_j\varphi_j\\&=-d_s\sum_{j=1}^{\infty}\psi_{\lceil s\rceil-s}(\sqrt{\lambda_j}y)\,\lambda_j^s u_j\varphi_j=-d_s\mathcal{P}_{s-\lfloor s\rfloor}[\mathcal{L}^s u](y).\end{aligned} \tag{4.3}$$

The second limit in (1.8) follows from *iii*) in Lemma A.1, and *iv*) is proved.

It remains to prove *ii*). Let $V\in H^{\lceil s\rceil;\mathfrak{b}}_{\mathcal{L},\mathbf{e}}(\mathbb{R}\to\mathcal{H})$ be such that $V(0)=u$. Then $V_j(0)=u_j$ for any $j\geq 1$. Thus (3.7) gives

$$u_j^2\|\psi_{s,\lambda_j}\|^2_{\lambda_j,H_{\mathbf{e}}^{\lceil s\rceil;\mathfrak{b}}} \le \|V_j\|^2_{\lambda_j,H_{\mathbf{e}}^{\lceil s\rceil;\mathfrak{b}}}$$

for any $j \ge 1$. Thus

$$\|\mathcal{P}_s[u]\|^2_{H_{\mathcal{L},\mathbf{e}}^{\lceil s\rceil;\mathfrak{b}}} = \sum_{j=1}^{\infty} u_j^2\|\,\psi_{s,\lambda_j}\|^2_{\lambda_j,H_{\mathbf{e}}^{\lceil s\rceil;\mathfrak{b}}} \le \sum_{j=1}^{\infty}\|V_j\|^2_{\lambda_j,H_{\mathbf{e}}^{\lceil s\rceil;\mathfrak{b}}} = \|V\|^2_{H_{\mathcal{L},\mathbf{e}}^{\lceil s\rceil;\mathfrak{b}}},$$

and *ii*) follows. The theorem is completely proved. □

Proof of Theorem 1.2 Recall that $\mathcal{P}_s[u] : \mathcal{H}^s_{\mathcal{L}} \to H_{\mathcal{L},\mathbf{e}}^{\lceil s\rceil;\mathfrak{b}}(\mathbb{R}\to\mathcal{H})$ is, up to the constant $2d_s$, an isometry by item *i*) in Theorem 1.1; in addition, $\mathcal{L}^{-s} : \mathcal{H}^{-s}_{\mathcal{L}} \to \mathcal{H}^s_{\mathcal{L}}$ is an isometry. Thus for any $\zeta \in \mathcal{H}^{-s}_{\mathcal{L}}$ we have that

$$\|\mathcal{P}_{-s}[\zeta]\|^2_{H_{\mathcal{L},\mathbf{e}}^{\lceil s\rceil;\mathfrak{b}}} = 2d_s\|\mathcal{L}^{-s}\zeta\|_{\mathcal{H}^s_{\mathcal{L}}} = 2d_s\|\zeta\|_{\mathcal{H}^{-s}_{\mathcal{L}}},$$

and (1.9) is proved. The conclusions in *iii*), *iv*) are immediate consequences of Theorem 1.1 (with $u := \mathcal{L}^{-s}\zeta$).

Finally, notice that the strictly convex minimization problem in (1.10) has a unique solution $\widehat{U} \in H_{\mathcal{L},\mathbf{e}}^{\lceil s\rceil;\mathfrak{b}}(\mathbb{R}\to\mathcal{H})$, and that $\widehat{U}$ satisfies

$$(\widehat{U},V)_{H_{\mathcal{L},\mathbf{e}}^{\lceil s\rceil;\mathfrak{b}}} = 2d_s\langle\zeta,V(0)\rangle = 2d_s\langle\mathcal{L}^s u,V(0)\rangle \quad \text{for any} \quad V \in H_{\mathcal{L},\mathbf{e}}^{\lceil s\rceil;\mathfrak{b}}(\mathbb{R}\to\mathcal{H}).$$

Thus $\widehat{U} = \mathcal{P}_s[u] = \mathcal{P}_{-s}[\zeta]$ by *iii*) in Theorem 1.1. □

5. Generalizations and examples

First we notice that the case of a complex Hilbert space $\mathcal{H}$ can be managed as well, with minor modifications in notation. Below we provide some more significant generalizations of our main result. They are based on Theorem 3.3.

5.1. Nonnegative operators

Assume that $\mathcal{L}$ is self-adjoint, with a discrete spectrum, nonnegative and with a nontrivial kernel. Trivially, for any $s > 0$ we have $\ker\mathcal{L}^s = \ker\mathcal{L}$, hence

$$\mathcal{L}^s u = \mathcal{L}^s(u - \Pi u),$$

where $\Pi : \mathcal{H} \to \ker\mathcal{L}$ is the orthogonal projection on $\ker\mathcal{L}$. The domain of the quadratic form $u \mapsto (\mathcal{L}^s u, u)_{\mathcal{H}}$ is

$$H^s_{\mathcal{L}} = \ker\mathcal{L} \oplus H^s_{\mathcal{L}_\perp}, \quad \mathcal{L}_\perp := \mathcal{L}|_{(\ker\mathcal{L})^\perp} : (\ker\mathcal{L})^\perp \to (\ker\mathcal{L})^\perp.$$

Notice that $\mathcal{L}_\perp$ is self-adjoint, with a discrete spectrum and positive. Thus Theorem 1.1 provides a full characterization of $\mathcal{L}_\perp^s$ and of the corresponding quadratic form on $\mathcal{H}^s_{\mathcal{L}_\perp}$. This gives, in turn, corresponding results for $\mathcal{L}^s$ and for its quadratic form on $\mathcal{H}^s_{\mathcal{L}}$.

In particular, the operator $u \mapsto \mathcal{P}_s[u]$ in (1.4) is the identity on $\ker \mathcal{L}$ and

$$\mathcal{P}_s[u](y) = \Pi[u] + \mathcal{P}_s^\perp[u - \Pi u](y), \tag{5.1}$$

where $\mathcal{P}_s^\perp$ is the isometry given by Theorem 1.1 for the operator $\mathcal{L}_\perp$. Since $\mathcal{P}_s[u]$ differs from $\mathcal{P}_s^\perp[u - \Pi u]$ by a constant curve, then $\mathcal{P}_s[u]$, $\mathcal{P}_s^\perp[u]$ enjoy the same regularity properties in the Appendix.

5.2. Non-discrete spectrum

Let $\mathcal{L}$ be a nonnegative, self-adjoint operator in the Hilbert space $\mathcal{H}$. Then there exists a unique projector-valued spectral measure E on $\mathbb{R}$ supported on the spectrum $\sigma(\mathcal{L}) \subset [0, \infty)$, such that

$$\mathcal{L} = \int\limits_{[\Lambda,\infty)} \lambda \, dE(\lambda),$$

where $\Lambda \geq 0$ is the bottom of $\sigma(\mathcal{L})$ (see e.g., [2, Ch. 6]).

For $s > 0$, the s-power of $\mathcal{L}$ is formally defined via

$$\mathcal{L}^s = \int\limits_{[\Lambda,\infty)} \lambda^s \, dE(\lambda).$$

We denote by $\mathcal{H}^s_{\mathcal{L}}$ the domain of the corresponding quadratic form, which is a Hilbert space with norm $\|\cdot\|^2_{\mathcal{H}^s_{\mathcal{L}}} = \|\mathcal{L}^{\frac{s}{2}} \cdot\|^2_{\mathcal{H}} + \|\cdot\|^2_{\mathcal{H}}$.

Let us first assume that $\mathcal{L}$ be positive definite, i.e. $\Lambda > 0$. Then $\|\mathcal{L}^{\frac{s}{2}} \cdot\|_{\mathcal{H}}$ is an equivalent norm in $\mathcal{H}^s_{\mathcal{L}}$.

For $s > 0$ non-integer and $u \in \mathcal{H}$ we consider the curve

$$\mathcal{P}_s[u](y) = \int\limits_{[\Lambda,\infty)} \psi_s(\sqrt{\lambda} y) \, dE(\lambda) u, \tag{5.2}$$

where ψ_s is the function in (1.11). As in the discrete case, we have that $\mathcal{P}_s$ maps any $u \in \mathcal{H}$ into an even curve in $\mathcal{H}$; in addition $\mathcal{P}_s[u] \in \mathcal{C}^\infty(\mathbb{R}_+ \to \mathcal{H}^\sigma_{\mathcal{L}})$ for every $u \in \mathcal{H}, \sigma > 0$.

Further, for $b \in (-1, 1)$ we introduce the following (unbounded) operators acting on even curves $U \in L^{2;b}_{\mathbf{e}}(\mathbb{R} \to \mathcal{H})$,

$$\mathbb{L}_b U = \int\limits_{[\Lambda,\infty)} (\mathbb{D}_b + \lambda) dE(\lambda) U \,, \qquad \mathbb{D}_b U = -\partial^2_{yy} U - b y^{-1} \partial_y U,$$

compare with (1.3).

For any integer $k \geq 1$ we introduce the space

$$H^{k;b}_{\mathcal{L},\mathbf{e}}(\mathbb{R} \to \mathcal{H}) = \Big\{ U \in L^{2;b}_{\mathbf{e}}(\mathbb{R} \to \mathcal{H}) \ \big| \ \|U\|_{H^{k;b}_{\mathbf{e}}} < \infty \ \Big\}.$$

Here $\|\cdot\|_{H^{k;b}_{\mathbf{e}}}$ is defined similarly as we did in the discrete case. More precisely, if k is even then

$$\|U\|^2_{H^{k;b}_{\mathcal{L},\mathbf{e}}} := \int\limits_{\mathbb{R}} |y|^b \Big[\int\limits_{[\Lambda,\infty)} d\big(E(\lambda)V(y,\lambda), V(y,\lambda)\big) \Big] \, dy,$$

where $V(y,\lambda) = (\mathbb{D}_b + \lambda)^{\frac{k}{2}} U(y)$. If k is odd then

$$\|U\|^2_{H^{k;b}_{\mathcal{L},\mathbf{e}}} := \int\limits_{\mathbb{R}} |y|^b \Big[\int\limits_{[\Lambda,\infty)} d\big(E(\lambda)\partial_y V(y,\lambda), \partial_y V(y,\lambda)\big) + \int\limits_{[\Lambda,\infty)} \lambda \, d\big(E(\lambda)V(y,\lambda), V(y,\lambda)\big) \Big] \, dy,$$

where $V(y,\lambda) = (\mathbb{D}_b + \lambda)^{\frac{k-1}{2}} U(y)$.

With the above definitions, Theorem 1.1 holds true, and its proof can be carried out with no essential modifications.

If $\Lambda = 0$ is an eigenvalue of $\mathcal{L}$ one can use a decomposition similar to (5.1) and the above remarks in the present subsection for the restriction of $\mathcal{L}$ to $\ker \mathcal{L}^{\perp}$.

A more complicated case is when $0 \in \sigma(\mathcal{L})$ is not an eigenvalue but a point of continuous spectrum. Clearly $\|\mathcal{L}^{\frac{s}{2}} \cdot \|_{\mathcal{H}}$ cannot bound $\|\cdot\|_{\mathcal{H}}$ and therefore it is only a seminorm in $\mathcal{H}^s_{\mathcal{L}}$. Denote by $\widehat{\mathcal{H}}^s_{\mathcal{L}}$ the completion of $\mathcal{H}^s_{\mathcal{L}}$ with respect to $\|\mathcal{L}^{\frac{s}{2}} \cdot \|_{\mathcal{H}}$.

To avoid additional difficulties, we assume that $\|\mathcal{L}^{\frac{s}{2}} \cdot \|_{\mathcal{H}}$ is a norm in $\widehat{\mathcal{H}}^s_{\mathcal{L}}$. In this case one can define a suitable space of curves, and prove a result similar to Theorem 1.1.

5.3. Examples

The approach proposed in the present paper can be used, for instance, to recover non-integer powers of a large class of differential operators.

The case of the Dirichlet Laplacian in a bounded, smooth domain $\Omega \subset \mathbb{R}^n$ is included in Theorem 1.1. Any curve $y \mapsto U(y) \in L^2(\Omega) = \mathcal{H}$ is identified with the function $(x,y) \mapsto U(y)(x)$, $\Omega \times \mathbb{R} \to \mathbb{R}$, so that $L^{2;b}(\mathbb{R} \to L^2(\Omega)) \equiv L^2(\Omega \times \mathbb{R}; |y|^b dxdy)$, and

$$\|U\|^2_{L^{2;b}(\mathbb{R}\to L^2(\Omega))} = \int\limits_{-\infty}^{+\infty} |y|^b \|U(y)\|^2_{L^2(\Omega)} \, dy = \iint\limits_{\Omega\times\mathbb{R}} |y|^b |U(x,y)|^2 \, dxdy \, .$$

Further, $L^{2;b}_{\mathbf{e}}(\mathbb{R} \to L^2(\Omega))$ is identified with the space of functions in $L^2(\Omega \times \mathbb{R}; |y|^b dxdy)$ which are even in the y-variable, that is denoted by $L^2_{\mathbf{e}}(\Omega \times \mathbb{R}; |y|^b dxdy)$.

We choose $\mathcal{L} = -\Delta_D$, the Laplace operator with domain $H^1_0(\Omega) \cap H^2(\Omega)$. Its eigenvalues λ_j and corresponding eigenfunctions φ_j solve the Dirichlet problem

$$\begin{cases} -\Delta\varphi_j = \lambda_j\varphi_j & \text{in } \Omega \\ \varphi_j = 0 & \text{on } \partial\Omega, \end{cases} \qquad \int\limits_{\Omega} \varphi_j\varphi_h \, dx = \delta_{jh}.$$

The natural domain $\mathcal{H}^s_{-\Delta_D}(\Omega)$ of the quadratic form $u \mapsto ((-\Delta_D)^s u, u)_{L^2}$ can be described by the results in [19, Section 1], see also [15, Lemma 3]:

$$\mathcal{H}^s_{-\Delta_D}(\Omega) = \left\{ u \in H^s(\Omega) \;\Big|\; (-\Delta)^m u\big|_{\partial\Omega} = 0 \;\text{ if }\; m \in \mathbb{N}_0,\; 2m < s - \frac{1}{2} \right\}$$

(recall that functions in $H^s(\Omega)$ have a trace on $\partial\Omega$ if and only if $s > \frac{1}{2}$).

We see that

$$\mathbb{L}_{\mathfrak{b}} U = -\boldsymbol{\Delta} U - \mathfrak{b} y^{-1}\partial_y U = -|y|^{-\mathfrak{b}}\text{div}(|y|^{\mathfrak{b}}\nabla U), \tag{5.3}$$

where $-\boldsymbol{\Delta}$ is the Dirichlet Laplacian in $\Omega \times \mathbb{R}$.

For s non-integer, Theorem 1.1 relates the nonlocal operator $(-\Delta_D)^s$, with the local operator $\mathbb{L}^{\lceil s \rceil}_{\mathfrak{b}}$ acting on $H^{\lceil s \rceil;\mathfrak{b}}_{-\Delta_D,\mathbf{e}}(\mathbb{R} \to L^2(\Omega)) \equiv H^{\lceil s \rceil;\mathfrak{b}}_{-\Delta_D,\mathbf{e}}(\Omega \times \mathbb{R})$. For instance, with obvious notation, we have

$$\begin{aligned} H^{1;\mathfrak{b}}_{-\Delta_D,\mathbf{e}}(\Omega \times \mathbb{R}) &= \big\{ U \in H^1_{\mathbf{e}}(\Omega \times \mathbb{R}; |y|^{\mathfrak{b}} dxdy) \mid U(\cdot, y) \in H^1_0(\Omega) \;\text{ for } y \neq 0 \;\big\}, \\ \|U\|^2_{H^{1;\mathfrak{b}}_{-\Delta_D,\mathbf{e}}} &= \iint\limits_{\Omega\times\mathbb{R}} |y|^{\mathfrak{b}} |\nabla U|^2 \, dxdy\,; \\ H^{2;\mathfrak{b}}_{-\Delta_D,\mathbf{e}}(\Omega \times \mathbb{R}) &= \big\{ U \in H^{1;\mathfrak{b}}_{-\Delta_D,\mathbf{e}}(\Omega \times \mathbb{R}) \mid |y|^{\mathfrak{b}}\nabla U \in H^1(\Omega \times \mathbb{R}; |y|^{-\mathfrak{b}} dxdy) \;\big\}, \\ \|U\|^2_{H^{2;\mathfrak{b}}_{-\Delta_D,\mathbf{e}}} &= \iint\limits_{\Omega\times\mathbb{R}} |y|^{\mathfrak{b}} |\mathbb{L}_{\mathfrak{b}} U|^2 \, dxdy = \iint\limits_{\Omega\times\mathbb{R}} |y|^{-\mathfrak{b}} |\text{div}(|y|^{\mathfrak{b}}\nabla U)|^2 \, dxdy. \end{aligned}$$

The Neumann Laplacian in Ω fits in the situation described in Subsection 5.1. Now we choose $\mathcal{L} = -\Delta_N$. It is an unbounded operator on $L^2(\Omega)$ with eigenvalues $\lambda_j \geq 0$ and eigenfunctions φ_j solving

$$\begin{cases} -\Delta\varphi_j = \lambda_j\varphi_j & \text{in } \Omega \\ \partial_\nu\varphi_j = 0 & \text{on } \partial\Omega, \end{cases} \qquad \int\limits_{\Omega} \varphi_j\varphi_h \, dx = \delta_{jh}.$$

The natural domain $\mathcal{H}^s_{-\Delta_N}(\Omega)$ of the quadratic form $u \mapsto ((-\Delta_N)^s u, u)_{L^2}$ is

$$\mathcal{H}^s_{-\Delta_N}(\Omega) = \left\{ u \in H^s(\Omega) \;\Big|\; \partial_\nu(-\Delta)^m u\big|_{\partial\Omega} = 0 \;\text{ if }\; m \in \mathbb{N}_0,\; 2m < s - \frac{3}{2} \right\},$$

see [19, Section 1].

In this case, the operator $\mathbb{L}_{\mathfrak{b}}$ is pointwise defined as in (5.3). For $s \notin \mathbb{N}$, the nonlocal operator $(-\Delta_N)^s$ is related to $\mathbb{L}_{\mathfrak{b}}^{\lceil s \rceil}$, acting on a different domain

$$H^{\lceil s \rceil;\mathfrak{b}}_{-\Delta_N,\mathbf{e}}(\mathbb{R} \to L^2(\Omega)) \equiv H^{\lceil s \rceil;\mathfrak{b}}_{-\Delta_N,\mathbf{e}}(\Omega \times \mathbb{R}).$$

The approach described in Subsection 5.1 covers this example, as $\lambda_1 = 0$.

Lastly, if $n > 2s$ then the fractional Laplacian $(-\Delta)^s$ on $\mathbb{R}^n$ fits into the general approach in Subsection 5.2. In this case, thanks to Hardy inequality the space $\widehat{\mathcal{H}}^s_{-\Delta}$ can be identified with the standard homogeneous Sobolev space $\mathcal{D}^s(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n; |x|^{-2s}dx)$. The resulting space of curves can be identified with the space $\mathcal{D}^{\lceil s \rceil;\mathfrak{b}}_{\mathbf{e}}(\mathbb{R}^{n+1})$ in [7].

Declaration of competing interest

None.

Data availability

No data was used for the research described in the article.

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Appendix A. On the transforms $\mathcal{P}_s$

Here we assume that $s > 0$ is non-integer and study the transform $\mathcal{P}_s[\cdot]$, see (4.2). We start by noticing that formulae (1.1) and (1.2) hold and that $\mathcal{H}^s_{\mathcal{L}}$ is the domain of the quadratic form of $\mathcal{L}^s$, for negative orders s as well.

Lemma A.1. *Let $s > 0$, $\sigma \in \mathbb{R}$.*

i) For any $u \in \mathcal{H}$, we have $\mathcal{P}_s[u] \in \mathcal{C}^\infty(\mathbb{R}_+ \to \mathcal{H}^\sigma_{\mathcal{L}})$, and $\|\partial_y^k \mathcal{P}_s[u](y)\|_{\mathcal{H}^\sigma_{\mathcal{L}}}$ decays exponentially as $y \to \infty$, for any order $k \geq 0$;

ii) The linear operator $u \mapsto \mathcal{P}_s[u](y)$ is nonexpansive in $H^\sigma_{\mathcal{L}}$, that is,

$$\|\mathcal{P}_s[u](y)\|_{\mathcal{H}^\sigma_{\mathcal{L}}} \leq \|u\|_{\mathcal{H}^\sigma_{\mathcal{L}}} \quad \text{for any } y \in \mathbb{R}; \tag{A.1}$$

iii) If $u \in \mathcal{H}^\sigma_{\mathcal{L}}$ then $\mathcal{P}_s[u] \in \mathcal{C}^0(\mathbb{R} \to \mathcal{H}^\sigma_{\mathcal{L}})$ and $\mathcal{P}_s[u](0) = u$;

iv) The operator $u \mapsto \mathcal{P}_s[u](y)$ commutes with the fractional powers of $\mathcal{L}$, that is,

$$\mathcal{P}_s[\mathcal{L}^\sigma u](y) = \mathcal{L}^\sigma\big(\mathcal{P}_s[u](y)\big). \tag{A.2}$$

Proof. By the properties of the Bessel functions, for any integer $k \geq 0$ and any $\delta > 0$ we have $|(\partial_y^k \psi_s)(y)| \leq c(\delta)e^{-y}$ for $y > \sqrt{\lambda_1}\delta$, where the constant $c(\delta)$ depends on δ, s and k but not on y. Thus, for $y > \delta$ we have

$$\lambda_j^{k+\sigma} |(\partial_y^k \psi_s)(\sqrt{\lambda_j} y)|^2 \leq c(\delta)^2 \lambda_j^{k+\sigma} e^{-2\sqrt{\lambda_j} y} \leq C(\delta) e^{-\sqrt{\lambda_1} y}$$

because $\lambda_j \geq \lambda_1 > 0$, where the new constant $C(\delta)$ depends only on δ, s, σ and k. It readily follows that

$$\|\partial_y^k \mathcal{P}_s[u](y)\|^2_{\mathcal{H}^\sigma_\mathcal{L}} = \sum_{j=1}^\infty \lambda_j^{k+\sigma} u_j^2 |(\partial_y^k \psi_s)(\sqrt{\lambda_j} y)|^2 \leq C(\delta) \|u\|^2_{\mathcal{H}}\, e^{-\sqrt{\lambda_1} y}$$

for any $u \in \mathcal{H}$, provided that $y > \delta$, and $i)$ is proved.

Now we take $u \in \mathcal{H}^\sigma_\mathcal{L}$. By Remark 3.2, we have $0 < \psi_{s,\lambda_j}(y) \leq \psi_{s,\lambda_j}(0) = 1$. Thus

$$\|\mathcal{P}_s[u](y)\|^2_{\mathcal{H}^\sigma_\mathcal{L}} = \sum_{j=1}^\infty \lambda_j^\sigma u_j^2 (\psi_{s,\lambda_j}(y))^2 \leq \sum_{j=1}^\infty \lambda_j^\sigma u_j^2 = \|u\|^2_{\mathcal{H}^\sigma_\mathcal{L}}\,,$$

which proves $ii)$. Further, we have

$$\|u - \mathcal{P}_s[u](y)\|^2_{\mathcal{H}^\sigma_\mathcal{L}} = \sum_{j=1}^\infty \lambda_j^\sigma u_j^2 (\psi_{s,\lambda_j}(0) - \psi_{s,\lambda_j}(y))^2 \leq \sum_{j=1}^\infty \lambda_j^\sigma u_j^2. \tag{A.3}$$

The first series in (A.3) is dominated by a convergent number series and converges to zero termwise as $y \to 0$. We infer that $\|u - \mathcal{P}_s[u](y)\|^2_{\mathcal{H}^\sigma_\mathcal{L}} \to 0$ as $y \to 0$, which implies $\mathcal{P}_s[u] \in \mathcal{C}^0(\mathbb{R} \to \mathcal{H}^\sigma_\mathcal{L})$, and $iii)$ is proved.

Since the equality in $iv)$ is trivial, the proof is complete. □

Thanks to Lemma A.1 and (4.3), we can improve the convergences in (1.8) as follows.

Corollary A.2. *Let $s > 0$ be non-integer, $\mathfrak{b} = 1 - 2(s - \lfloor s \rfloor)$. Assume that $u \in \mathcal{H}^\sigma_\mathcal{L}$ for some $\sigma \in \mathbb{R}$. Then $\mathcal{P}_s[u]$ solves the differential equation (1.7) and satisfies the boundary conditions*

$$\lim_{y \to 0^+} \mathcal{P}_s[u](0) = u \quad in \;\; \mathcal{H}^\sigma_\mathcal{L}$$

$$\lim_{y \to 0^+} y^\mathfrak{b}\, \partial_y \big(\mathbb{L}_\mathfrak{b}^{\lfloor s \rfloor} \mathcal{P}_s[u]\big)(y) = -d_s\, \mathcal{L}^s u \quad in \;\; \mathcal{H}^{\sigma - 2s}_\mathcal{L}.$$

Remark A.3. For any integer $k \geq 0$ we have

$$\psi_{k+\frac{1}{2}}(y) = \frac{1}{(k+1)!} |y|^k e^{-|y|}, \qquad \mathcal{P}_{k+\frac{1}{2}}[u](y) = \frac{1}{(k+1)!} |y|^k\, \mathcal{P}_{\frac{1}{2}}[\mathcal{L}^{\frac{1}{2}} u](y)\,.$$

A.1. Derivatives

The regularity of the curve $\mathcal{P}_s[u]$ given in Lemma A.1 improves as s increases. We start by proving a technical result which involves the Beta function

$$\mathrm{B}(\tau,t)=\int_0^1 x^{\tau-1}(1-x)^{t-1}\,dx=\frac{\Gamma(t)\Gamma(\tau)}{\Gamma(t+\tau)}\,.$$

The coefficients in the next lemma are computed by taking inspiration from [7, Section 4].

Lemma A.4. *Let $\sigma\in\mathbb{R}$, $u\in\mathcal{H}^\sigma_{\mathcal{L}}$, $y>0$.*

i) If $s\in(0,1)$ then $\partial_y\mathcal{P}_s[u](y)=-d_s y^{2s-1}\mathcal{P}_{1-s}[\mathcal{L}^s u](y)$;
ii) If $s>1$ then for any $m=1,\dots,\lfloor s\rfloor$ it holds that

$$\partial_y^{2m}\mathcal{P}_s[u](y)=\frac{1}{\mathrm{B}\big(s,\frac12\big)}\sum_{\ell=0}^m\binom{m}{\ell}(-1)^\ell\mathrm{B}\big(s-\ell,\tfrac12\big)\cdot\mathcal{P}_{s-\ell}[\mathcal{L}^m u](y) \tag{A.4}$$

$$\partial_y^{2m-1}\mathcal{P}_s[u](y)=y\cdot\frac{1}{\mathrm{B}\big(s,\frac12\big)}\sum_{\ell=1}^m\binom{m-1}{\ell-1}(-1)^\ell\mathrm{B}\big(s-\ell,\tfrac32\big)\cdot\mathcal{P}_{s-\ell}[\mathcal{L}^m u](y). \tag{A.5}$$

Proof. If $s\in(0,1)$ then $\partial_y\psi_{s,\lambda}(y)=-d_s y^{2s-1}\lambda^s\psi_{1-s,\lambda}(y)$ by Lemma 3.1. Thus

$$\partial_y\mathcal{P}_s[u](y)=-d_s y^{2s-1}\cdot\sum_{j=1}^\infty\partial_y^{2m}\psi_{1-s,\lambda_j}(y)\,\lambda_j^s u_j\varphi_j,$$

and the identity in $i)$ follows.

To handle the case $s>1$ we put $\gamma_{s,\ell}=\dfrac{\mathrm{B}(s-\ell,\frac12)}{\mathrm{B}(s,\frac12)}=\dfrac{\Gamma\big(s+\frac12\big)}{\Gamma(s)}\,\dfrac{\Gamma(s-\ell)}{\Gamma\big(s+\frac12-\ell\big)}$. Using $ii)$ in Lemma 3.1 and induction one gets

$$\partial_y^{2m}\psi_s(y)=\sum_{\ell=0}^m\binom{m}{\ell}(-1)^\ell\gamma_{s,\ell}\;\psi_{s-\ell}(y), \tag{A.6}$$

for any integer $m=1,\dots,\lfloor s\rfloor$. Since $\partial_y^{2m}\psi_{s,\lambda}(y)=\lambda^m\big(\partial_y^{2m}\psi\big)(\sqrt{\lambda}y)$, we infer that

$$\partial_y^{2m}\mathcal{P}_s[u](y)=\sum_{j=1}^\infty\partial_y^{2m}\psi_{s,\lambda_j}(y)\,u_j\varphi_j=\sum_{\ell=0}^m\binom{m}{\ell}(-1)^\ell\gamma_{s,\ell}\sum_{j=1}^\infty\psi_{s-\ell,\lambda_j}(y)\,\lambda_j^m u_j\varphi_j,$$

which proves (A.4).

Arguing as for $i)$ we obtain

$$\partial_y \mathcal{P}_s[u](y) = -\frac{y}{2(s-1)} \mathcal{P}_{s-1}[\mathcal{L}u](y),$$

i.e. (A.5) holds if $m = 1$. If $m > 1$ we use (A.6) for $m - 1$ and then $i)$ in Lemma 3.1 to compute

$$\begin{aligned}\partial_y^{2m-1} \psi_s(y) = \partial_y\, \partial_y^{2(m-1)} \psi_s &= \sum_{\ell=0}^{m-1} \binom{m-1}{\ell} (-1)^\ell \gamma_{s,\ell}\; \partial_y \psi_{s-\ell}(y) \\ &= y \cdot \sum_{\ell=0}^{m-1} \binom{m-1}{\ell} (-1)^{\ell+1} \frac{\gamma_{s,\ell}}{2(s-\ell-1)}\; \psi_{s-\ell-1}(y) \\ &= y \cdot \sum_{\ell=1}^{m} \binom{m}{\ell} (-1)^\ell \frac{\ell\, \gamma_{s,\ell}}{2m(s+\frac{1}{2}-\ell)}\; \psi_{s-\ell}(y)\,.\end{aligned}$$

Then (A.5) follows by arguing as in the “even” case. □

Theorem A.5. *Let $2s \ge 1$, $\sigma \in \mathbb{R}$ and let k be an integer, with $1 \le k \le \lfloor 2s \rfloor$.*

i) Let $u \in \mathcal{H}^\sigma_{\mathcal{L}}$. Then

$$\|\partial_y^k \mathcal{P}_s[u](y)\|_{\mathcal{H}^{\sigma-k}_{\mathcal{L}}} \le c_k \|u\|_{\mathcal{H}^\sigma_{\mathcal{L}}} \quad \textit{for any } y > 0, \tag{A.7}$$

where the constant c_k depends only on s and k. Thus, for any $y > 0$ the linear operator $u \mapsto \partial_y^k \mathcal{P}_s[u](y)$ is continuous $\mathcal{H}^\sigma_{\mathcal{L}} \to \mathcal{H}^{\sigma-k}_{\mathcal{L}}$;

ii) If in addition[2] $k < 2s$ then $\partial_y^k \mathcal{P}_s[u] \in \mathcal{C}^0(\mathbb{R} \to \mathcal{H}^{\sigma-k}_{\mathcal{L}})$ for any $u \in \mathcal{H}^\sigma_{\mathcal{L}}$.

Proof. It is convenient to define

$$M_{\alpha,\beta} = \max_{y\ge 0} y^{2\beta} \psi_\alpha(y)^2\ , \quad \alpha, \beta > 0.$$

If $\frac{1}{2} \le s < 1$, then $i)$ in Lemma A.4 gives

$$\|\partial_y \mathcal{P}_s[u](y)\|^2_{\mathcal{H}^{\sigma-1}_{\mathcal{L}}} = d_s^2 \|y^{2s-1} \mathcal{P}_{1-s}[\mathcal{L}^s u](y)\|^2_{\mathcal{H}^{\sigma-1}_{\mathcal{L}}}.$$

The conclusion in $i)$ follows, because

[2] This is a restriction only if s is a half integer.

$$\|y^{2s-1}\mathcal{P}_{1-s}[\mathcal{L}^s u](y)\|^2_{\mathcal{H}^{\sigma-1}_{\mathcal{L}}} = \sum_{j=1}^{\infty} \lambda_j^{\sigma-1} y^{2(2s-1)} \lambda_j^{2s} u_j^2 \psi_{1-s}(\sqrt{\lambda_j}y)^2$$
$$= \sum_{j=1}^{\infty} \lambda_j^{\sigma} u_j^2 (\sqrt{\lambda_j}y)^{2(2s-1)} \psi_{1-s}(\sqrt{\lambda_j}y)^2 \le M_{1-s,2s-1}\|u\|^2_{\mathcal{H}^{\sigma-1}_{\mathcal{L}}}. \tag{A.8}$$

If $2s > 1$, then the series in (A.8) are dominated by a convergent number series and converge to zero termwise as $y \to 0$. We infer that $\partial_y \mathcal{P}_s[u](y) \to 0$ in $\mathcal{H}^{\sigma-1}_{\mathcal{L}}$ as $y \to 0$, which proves $ii)$ in this case.

Next, let $s > 1$. We first face the case when $k \le 2\lfloor s\rfloor$ is even. Take integers ℓ, m with $0 \le \ell \le m \le \lfloor s\rfloor$. By Lemma A.1 we have

$$\|\mathcal{P}_{s-\ell}[\mathcal{L}^m u](y)\|_{\mathcal{H}^{\sigma-2m}_{\mathcal{L}}} \le \|\mathcal{L}^m u\|_{\mathcal{H}^{\sigma-2m}_{\mathcal{L}}} = \|u\|_{\mathcal{H}^{\sigma}_{\mathcal{L}}}\ , \quad \mathcal{P}_{s-\ell}[\mathcal{L}^m u] \in \mathcal{C}^0(\mathbb{R} \to \mathcal{H}^{\sigma-2m}_{\mathcal{L}}).$$

Taking also (A.4) into account, we see that the conclusions hold in this case.

Let now $k \le 2\lfloor s\rfloor - 1$ be odd. For $1 \le \ell \le m$ we estimate

$$\|y\mathcal{P}_{s-\ell}[\mathcal{L}^m u](y)\|^2_{\mathcal{H}^{\sigma-2m+1}_{\mathcal{L}}} = \|y\mathcal{L}^m(\mathcal{P}_{s-\ell}[u](y))\|^2_{\mathcal{H}^{\sigma-2m+1}_{\mathcal{L}}} = \|y\mathcal{P}_{s-\ell}[u](y)\|^2_{\mathcal{H}^{\sigma+1}_{\mathcal{L}}}$$
$$= \sum_{j=1}^{\infty} \lambda_j^{\sigma} u_j^2 (\sqrt{\lambda_j}y)^2 \psi_{s-\ell}(\sqrt{\lambda_j}y)^2 \le M_{s-\ell,1}\|u\|^2_{\mathcal{H}^{\sigma}_{\mathcal{L}}}. \tag{A.9}$$

In view of (A.5), we see that (A.7) holds also in this case. By repeating the argument we used for $\frac{1}{2} \le s < 1$ one plainly conclude the proof also in this case.

It remains to discuss the case $\lfloor s\rfloor + \frac{1}{2} \le s < \lceil s\rceil$ and $k = 2\lfloor s\rfloor + 1 = \lfloor 2s\rfloor$. We differentiate formula (A.4) for $m = \lfloor s\rfloor$. To compute $\partial_y \mathcal{P}_{s-\ell}[\mathcal{L}^{\lfloor s\rfloor}u](y)$, we use (A.5) for $\ell = 1, \dots, \lfloor s\rfloor - 1$ and $i)$ in Lemma A.4 for the last ℓ. It gives

$$\partial_y^{2\lfloor s\rfloor+1}\mathcal{P}_s[u](y) = -\sum_{\ell=1}^{\lfloor s\rfloor} a_{s,\ell}\cdot\big(y\mathcal{P}_{s-\ell}[\mathcal{L}^{\lceil s\rceil}u](y)\big) - a_s\cdot\big(y^{2(s-\lfloor s\rfloor)-1}\,\mathcal{P}_{\lceil s\rceil-s}[\mathcal{L}^s u](y)\big), \tag{A.10}$$

where the coefficients $a_{s,\ell}, a_s \in \mathbb{R}$ depend only on s and ℓ. One can easily adapt the arguments we used for (A.9), (A.8). In this way one proves $i)$ if $\lfloor s\rfloor + \frac{1}{2} \le s < \lceil s\rceil$, and $ii)$ if $\lfloor s\rfloor + \frac{1}{2} < s < \lceil s\rceil$. □

Theorem A.6. *Let $s > 1$, $\sigma \in \mathbb{R}$, $u \in \mathcal{H}^{\sigma}_{\mathcal{L}}$. Then, for any $k = 1, \dots, \lfloor s\rfloor$ we have*

$$\mathcal{P}_s[u](y) = \frac{1}{\Gamma(s)}\sum_{m=1}^{k}\frac{\Gamma(s-m)}{2^{2m}m!}\cdot\mathcal{L}^m u\cdot y^{2m} + o(y^{2k}) \qquad \text{as } y \to 0 \tag{A.11}$$

with convergence in $\mathcal{H}^{\sigma-2k}_{\mathcal{L}}$.

Proof. Take an integer $k = 1, \dots, \lfloor s \rfloor$. By *ii)* in Theorem A.5 we have that $\mathcal{P}_s[u] \in \mathcal{C}^{2k}(\mathbb{R} \to \mathcal{H}_{\mathcal{L}}^{\sigma-2k})$. Further, for any $m = 1, \dots, k$, Lemma A.4 gives

$$\partial_y^{2m}\mathcal{P}_s[u](0) = \frac{1}{\mathrm{B}(s, \frac{1}{2})} \sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^{\ell} \mathrm{B}(s-\ell, \tfrac{1}{2}) \cdot \mathcal{L}^m u$$

and $\partial_y^{2m-1}\mathcal{P}_s[u](0) = 0$. Then (A.11) follows via Taylor expansion formula, thanks to Lemma A.7 below. □

Lemma A.7. *Let $m \le \lfloor s \rfloor$ be a positive integer. Then*

$$\kappa_{s,m} := \frac{1}{B(s, \frac{1}{2})} \sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^{\ell} \mathrm{B}(s-\ell, \tfrac{1}{2}) = (-1)^m \frac{\Gamma(s-m)}{\Gamma(s)} \frac{1}{2^{2m} m!} (2m)!\,.$$

Proof. We compute

$$\begin{aligned}\sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^{\ell} \mathrm{B}\big(s-\ell, \tfrac{1}{2}\big) &= \int_0^1 x^{-\frac{1}{2}} (1-x)^{s-m-1} \Big(\sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^{\ell} (1-x)^{m-\ell} \Big) dx \\ &= (-1)^m \int_0^1 x^{m-\frac{1}{2}} (1-x)^{s-m-1} dx = (-1)^m \mathrm{B}(s-m, m+\tfrac{1}{2}).\end{aligned}$$

Recalling the Legendre duplication formula, we infer that

$$\begin{aligned}\kappa_{s,m} &= (-1)^m \frac{\mathrm{B}(s-m, m+\frac{1}{2})}{\mathrm{B}(s, \frac{1}{2})} = (-1)^m \frac{\Gamma(s-m)}{\Gamma(s)} \frac{\Gamma(m+\frac{1}{2})}{\sqrt{\pi}} \\ &= (-1)^m \frac{\Gamma(s-m)}{\Gamma(s)} \frac{2^{1-2m}\Gamma(2m)}{\Gamma(m)} = (-1)^m \frac{\Gamma(s-m)}{\Gamma(s)} \frac{1}{2^{2m} m!} (2m)!,\end{aligned}$$

which completes the proof. □

Corollary A.8. *Let $s > 1$, $u \in \mathcal{H}_{\mathcal{L}}^{s}$. Then for any integer $m = 1, \dots, \lfloor s \rfloor$ we have that*

$$\mathbb{L}_b^m \mathcal{P}_s[u](y) = \frac{d_s}{d_{s-m}} \mathcal{P}_{s-m}[\mathcal{L}^m u](y)\ , \quad y \in \mathbb{R}.$$

$$\lim_{y \to 0} y^{-1} \partial_y^{2m-1} \mathcal{P}_s[u] = \lim_{y \to 0} \partial_y^{2m} \mathcal{P}_s[u] = \kappa_{s,m} \mathcal{L}^m u$$

where $\kappa_{s,m}$ is the constant in Lemma A.7. The limits are taken in the $\mathcal{H}_{\mathcal{L}}^{s-2m}$ topology.

Proof. The first equality follows from formulae (3.2) and (3.3):

$$\mathbb{L}_b^m \mathcal{P}_s[u](y) = \sum_{j=1}^{\infty} u_j (\mathbb{D}_b + \lambda_j)^m \psi_{s,\lambda_j}(y)\, u_j\varphi_j = \sum_{j=1}^{\infty} \lambda_j^m [(\mathbb{D}_b + 1)^m \psi_s](\sqrt{\lambda_j} y)\, u_j\varphi_j$$
$$= \frac{d}{d_{s-m}} \sum_{j=1}^{\infty} \psi_{s-m,\lambda_j}(y)\, \lambda_j^m u_j \varphi_j .$$

To conclude the proof, use *ii*) in Lemma A.4 and then *iii*) in Lemma A.1. □

Our last result in this section involves the Hölder-type spaces $\widetilde{\mathcal{C}}^\alpha$ in (1.12).

Theorem A.9. *Let $s > 0$ non-integer, $\sigma \in \mathbb{R}$, $u \in \mathcal{H}^\sigma_{\mathcal{L}}$, $\alpha \in (0, 2s]$. Then*

$$\mathcal{P}_s[u] \in \widetilde{\mathcal{C}}^\alpha(\mathbb{R} \to \mathcal{H}^{\sigma-\alpha}_{\mathcal{L}}) , \qquad [\![\mathcal{P}_s[u]]\!]_{\widetilde{\mathcal{C}}^\alpha} \le c\|u\|_{\mathcal{H}^\sigma_{\mathcal{L}}}. \tag{A.12}$$

Proof. Thanks to *ii*) in Theorem A.5, we only have to investigate the Hölderianity of $\partial_y^{\lfloor\alpha\rfloor}\mathcal{P}_s[u]$ if $\alpha > \lfloor\alpha\rfloor$, and the Lipschitz properties of $\partial_y^{\alpha-1}\mathcal{P}_s[u]$ if α is integer.

Theorem 3.3 already gives $\psi_s \in \widetilde{\mathcal{C}}^{2s}(\mathbb{R})$. Since ψ_s decays exponentially at infinity together with its derivatives of any order, we infer that $\psi_s \in \widetilde{\mathcal{C}}^\alpha(\mathbb{R})$ for any $\alpha \in (0, 2s]$. Since trivially $\partial_y^k \psi_{s,\lambda}(y) = \lambda^{\frac{k}{2}} (\partial_y^k \psi_s)(\sqrt{\lambda} y)$ for any integer k and any $\lambda > 0$, then $[\![\psi_{s,\lambda}]\!]_{\widetilde{\mathcal{C}}^\alpha} = \lambda^{\frac{\alpha}{2}} [\![\psi_s]\!]_{\widetilde{\mathcal{C}}^\alpha}$ for any $\alpha \in (0, 2s]$.

Take arbitrary points $y_1, y_2 \in \mathbb{R}$. Without loss of generality, we can assume that $y_1, y_2 \ge 0$. If α is not an integer, then

$$\|\partial_y^{\lfloor\alpha\rfloor}\mathcal{P}_s[u](y_1) - \partial_y^{\lfloor\alpha\rfloor}\mathcal{P}_s[u](y_2)\|^2_{\mathcal{H}^{\sigma-\alpha}_{\mathcal{L}}} = \sum_{j=1}^{\infty} \lambda_j^{\sigma-\alpha} u_j^2 |\partial_y^{\lfloor\alpha\rfloor}\psi_{s,\lambda_j}(y_1) - \partial_y^{\lfloor\alpha\rfloor}\psi_{s,\lambda_j}(y_2)|^2$$
$$\le [\![\psi_s]\!]^2_{\widetilde{\mathcal{C}}^\alpha} \sum_{j=1}^{\infty} \lambda_j^{\sigma-\alpha} u_j^2 \lambda^\alpha |y_1 - y_2|^{2(\alpha-\lfloor\alpha\rfloor)} = [\![\psi_s]\!]^2_{\widetilde{\mathcal{C}}^\alpha} \|u\|^2_{\mathcal{H}^\sigma_{\mathcal{L}}} |y_1 - y_2|^{2(\alpha-\lfloor\alpha\rfloor)}.$$

If α is integer, with a similar computation we get

$$\|\partial_y^{\alpha-1}\mathcal{P}_s[u](y_1) - \partial_y^{\alpha-1}\mathcal{P}_s[u](y_2)\|^2_{\mathcal{H}^{\sigma-\alpha}_{\mathcal{L}}} \le c \sum_{j=1}^{\infty} \lambda_j^\sigma u_j^2 |y_1 - y_2|^2 = c\|u\|^2_{\mathcal{H}^\sigma_{\mathcal{L}}} |y_1 - y_2|^2.$$

In both cases, this concludes the proof. □

A.2. Isometric properties

From Theorem 1.1 we already know that the linear transform $u \mapsto \mathcal{P}_s[u]$ is, up to a constant, an isometry $\mathcal{H}^s_{\mathcal{L}} \to H^{\lceil s\rceil;\mathfrak{b}}_{\mathcal{L},\mathbf{e}}(\mathbb{R} \to \mathcal{H})$ for $\mathfrak{b} := 1 - 2(s - \lfloor s\rfloor)$. In this section we point out more isometric properties of $\mathcal{P}_s$. We stress the fact that $s > 0$ might be an integer number.

Theorem A.10. *Let $s > 0$, $b \in (-1, 1)$ and $\sigma \in \mathbb{R}$. Up to a constant (not depending on σ), the operator $\mathcal{P}_s$ is an isometry $\mathcal{H}^\sigma_\mathcal{L} \to L^{2;b}_\mathbf{e}(\mathbb{R} \to \mathcal{H}^{\sigma+\frac{1+b}{2}}_\mathcal{L})$. More precisely,*

$$\|\mathcal{P}_s[u]\|_{L^{2;b}(\mathbb{R}\to\mathcal{H}^{\sigma+\frac{1+b}{2}}_\mathcal{L})} = \|\psi_s\|_{L^{2;b}(\mathbb{R})}\, \|u\|_{\mathcal{H}^\sigma_\mathcal{L}} \qquad \textit{for any } u \in \mathcal{H}^\sigma_\mathcal{L}. \tag{A.13}$$

Proof. For $u \in \mathcal{H}^\sigma_\mathcal{L}$ we compute

$$\int\limits_{-\infty}^{+\infty} |y|^b \|\mathcal{P}_s[u](y)\|^2_{\mathcal{H}^{\sigma+\frac{1+b}{2}}} dy = \sum_{j=1}^{\infty} \lambda_j^{\sigma+\frac{1+b}{2}} u_j^2 \int\limits_{-\infty}^{+\infty} |y|^b |\psi_s(\sqrt{\lambda_j} y)|^2 \, dy$$
$$= \Big(\int\limits_{-\infty}^{+\infty} |y|^b |\psi_s(y)|^2 \, dy \Big) \sum_{j=1}^{\infty} \lambda_j^\sigma u_j^2 = \Big(\int\limits_{-\infty}^{+\infty} |y|^b |\psi_s(y)|^2 \, dy \Big) \|u\|^2_{\mathcal{H}^\sigma_\mathcal{L}},$$

and the Lemma is proved. □

Let $\alpha > 0$. We recall the definition of the Sobolev–Slobodetskii spaces and corresponding seminorms

$$H^\alpha(\mathbb{R}) = \{\psi \in L^2(\mathbb{R}) \mid [\![\psi]\!]^2_{H^\alpha} < \infty\}$$
$$[\![\psi]\!]^2_{H^\alpha} = \int\limits_{-\infty}^{+\infty} |(-\partial^2_{yy})^{\frac{\alpha}{2}} \psi(y)|^2 dy = \int\limits_{-\infty}^{+\infty} |\xi|^{2\alpha} |\widehat{\psi}(\xi)|^2 d\xi\,,$$

where $\widehat{\psi}$ stands for the unitary Fourier transform of $\psi \in L^2(\mathbb{R})$, namely,

$$\widehat{\psi}(\xi) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{-ixy} \psi(y) dy\,.$$

We first compute the Fourier transform of the function ψ_s in (1.11).

Proposition A.11. *Let $s > 0$ (possibly integer). Then*

$$\widehat{\psi_s}(\xi) = \frac{\sqrt{2}\Gamma\big(s+\frac{1}{2}\big)}{\Gamma(s)} (1+\xi^2)^{-\frac{1+2s}{2}}.$$

In particular, $\psi_s \in H^\alpha(\mathbb{R})$ if and only if $\alpha < 2s + \frac{1}{2}$, and in this case

$$[\![\psi_s]\!]^2_{H^\alpha} = \frac{\Gamma\big(s+\frac{1}{2}\big)^2}{s\Gamma(2s)\Gamma(s)^2}\, \Gamma\big(\alpha+\tfrac{1}{2}\big)\Gamma\big(2s-\alpha+\tfrac{1}{2}\big). \tag{A.14}$$

Proof. It is well known, see for instance [7, Lemma 4.2] for a simple proof, that

$$\widehat{(1+|\cdot|^2)^{-\frac{1+2s}{2}}}(y) = \frac{\Gamma(s)}{\sqrt{2}\Gamma\big(s+\frac{1}{2}\big)}\,\psi_s(y).$$

To conclude, use the symmetry of ψ_s and make direct computations. $\square$

For $\alpha > 0$ we introduce a Sobolev-type space of curves $\mathbb{R} \to \mathcal{H}$ and corresponding seminorm as follows:

$$H^\alpha(\mathbb{R}\to\mathcal{H}) = \Big\{U \in L^2(\mathbb{R}\to\mathcal{H}) \;\Big|\; [\![U]\!]^2_{H^\alpha} := \sum_{j=1}^{\infty}\int_{-\infty}^{+\infty} |\xi|^{2\alpha}|\widehat{U}_j(\xi)|^2 d\xi < \infty\Big\}.$$

It is evident that $H^\alpha(\mathbb{R}\to\mathcal{H})$ is a Hilbert space with norm

$$\|U\|^2_{H^\alpha} = [\![U]\!]^2_{H^\alpha} + \|U\|^2_{L^2} = \sum_{j=1}^{\infty}\int_{-\infty}^{+\infty} (|\xi|^{2\alpha}+1)|\widehat{U}_j(\xi)|^2\,d\xi.$$

Theorem A.12. *Let $s > 0$, $\sigma \in \mathbb{R}$ and let $\alpha \in (-\frac{1}{2}, 2s)$. Then $\mathcal{P}_s$ is a continuous transform $\mathcal{H}^\sigma_{\mathcal{L}} \to H^{\alpha+\frac{1}{2}}(\mathbb{R}\to\mathcal{H}^{\sigma-\alpha}_{\mathcal{L}})$. Moreover,*

$$[\![\mathcal{P}_s[u]]\!]^2_{H^{\alpha+\frac{1}{2}}(\mathbb{R}\to\mathcal{H}^{\sigma-\alpha}_{\mathcal{L}})} = \frac{\Gamma(s+\frac{1}{2})^2}{\Gamma(s)^2}\,\frac{\Gamma(\alpha+1)\Gamma(2s-\alpha)}{s\Gamma(2s)}\,\|u\|^2_{\mathcal{H}^\sigma_{\mathcal{L}}}\,. \tag{A.15}$$

Proof. Thanks to (A.13) we already know that

$$\|\mathcal{P}_s[u]\|^2_{L^2(\mathbb{R}\to\mathcal{H}^{\sigma-\alpha}_{\mathcal{L}})} \le \|\psi_s\|^2_{L^2(\mathbb{R})}\,\|u\|^2_{\mathcal{H}^{\sigma-\alpha-\frac{1}{2}}_{\mathcal{L}}} \le \lambda_1^{-\alpha-\frac{1}{2}}\|\psi_s\|^2_{L^2(\mathbb{R})}\,\|u\|^2_{\mathcal{H}^\sigma_{\mathcal{L}}}$$

for any $u \in \mathcal{H}^\sigma_{\mathcal{L}}$, which gives the continuity of $\mathcal{P}_s : \mathcal{H}^\sigma_{\mathcal{L}} \to L^2(\mathbb{R}\to\mathcal{H}^{\sigma-\alpha}_{\mathcal{L}})$, as $\lambda_1 > 0$.

Next, take $u = \sum_j u_j\varphi_j \in \mathcal{H}^\sigma_{\mathcal{L}}$. By the rescaling properties of the Fourier transform we have

$$\widehat{\psi_{s,\lambda_j}}(\xi)\,u_j\varphi_j = \lambda_j^{-\frac{1}{2}}\widehat{\psi_s}(\lambda_j^{-\frac{1}{2}}\xi)\,u_j\varphi_j.$$

This readily gives

$$\begin{aligned}
[\![\mathcal{P}_s[u]]\!]^2_{H^{\alpha+\frac{1}{2}}(\mathbb{R}\to\mathcal{H}^{\sigma-\alpha})} &= \sum_{j=1}^{\infty}\lambda_j^{\sigma-\alpha-1}u_j^2\int_{-\infty}^{+\infty}|\xi|^{2\alpha+1}\big|\widehat{\psi_s}(\lambda_j^{-\frac{1}{2}}\xi)\big|^2\,d\xi \\
&= \Big(\int_{-\infty}^{+\infty}|\xi|^{2\alpha+1}\big|\widehat{\psi_s}(\xi)\big|^2\,d\xi\Big)\sum_{j=1}^{\infty}\lambda_j^\sigma u_j^2,
\end{aligned}$$

which proves (A.15). This ends the proof by Proposition A.11 and Lemma A.1. □

We conclude by stating the next immediate consequence of Theorems A.10 and A.12, which is related to some results in [16].

Corollary A.13. *Let $s > 0$. For any $u \in \mathcal{H}^s_{\mathcal{L}}$ it holds that*

$$\begin{aligned} \|\mathcal{P}_s[u]\|^2_{L^2(\mathbb{R}\to\mathcal{H}^{s+\frac{1}{2}}_{\mathcal{L}})} &= \frac{\sqrt{\pi}\Gamma\big(2s+\frac{1}{2}\big)\Gamma\big(s+\frac{1}{2}\big)^2}{s\Gamma(2s)\Gamma(s)^2}\,\|u\|^2_{\mathcal{H}^s_{\mathcal{L}}} \\ [\![\mathcal{P}_s[u]]\!]^2_{H^{s+\frac{1}{2}}(\mathbb{R}\to\mathcal{H})} &= \frac{\Gamma(s+\frac{1}{2})^2}{\Gamma(2s)}\,\|u\|^2_{\mathcal{H}^s_{\mathcal{L}}}. \end{aligned}$$

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