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Monopoly in Bilateral Exchange: Some Historico-Analytical Remarks

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ABSTRACT

In this paper, we characterize, under the assumption that the aggregate demand of the atomless part for the commodity held by the monopolist is invertible, the monopoly quantity setting solution proposed by Kats (1974a) as the monopolist's most preferred commodity bundle in the complement of the atomless part's offer curve which is feasible with respect to the aggregate initial endowments. Then, we show that the monopoly solution in bilateral exchange introduced by Busetto et al. (2023) coincides with the monopoly quantity setting solution proposed by Kats (1974a). Finally, we reformulate, the Shitovitz paradox, first exhibited by Busetto et al. (2025), in terms of the Kats monopoly solution. In particular we show that, for any allocation corresponding to a Kats' monopoly solution, there is an allocation in the core, which is, utility-wise, advantageous for the monopolist and nonadvantageous for the small traders, and which is not in the feasible complement of the atomless part's offer curve.

JEL Classification: D42, D51

1 | Introduction

To the best of our knowledge, Vilfredo Pareto was the first who gave a formalized treatment of the problem of monopoly for a general pure exchange economy with any finite number of commodities, in the first volume of his *Cours d'économie politique*, published in 1896, pp. 62–71 (henceforth just Pareto 1896). His monopoly quantity-setting solution rests on the assumption that the monopolist gets no utility from the only commodity he is endowed with, and only cares about the revenue he can obtain by selling it.

Seventy years later, Schydrowsky and Siamwalla (1966) proposed a formulation of the problem of monopoly without any mention of the previous work by Pareto (1896). In the context of a pure exchange economy, they considered a bilateral framework where one commodity is held by one trader behaving as a monopolist while the other is held by a “competitors' community.” In

contrast to Pareto's analysis, the monopolist desires both commodities. The authors provided a geometrical representation of the monopoly solution as the point of tangency between the monopolist's indifference curve passing through the equilibrium allocation and the offer curve of the competitors' community. They did not mention the geometrical treatment of the monopoly problem previously given, at a very embryonic stage, by Edgeworth (1881, p. 114).

A few years later, Kats (1974a), again without mentioning (Pareto 1896), analyzed a pure exchange economy where one trader behaves as a monopolist, “calling the game” and maximizing his utility, whereas all the other traders in the economy behave competitively. He claimed that the monopoly quantity-setting solution must correspond to the monopolist's most preferred commodity bundle compatible with the aggregate initial endowments and with the offer curve of the competitive traders.

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Busetto et al. (2023) provided a theoretical foundation of the monopoly solution by formalizing an explicit trading process inspired to that first sketched by Pareto (1896).

In particular, they considered the mixed version of a monopolistic two-commodity exchange economy introduced by Shitovitz (1973) in his Example 1, in which one commodity is held only by the monopolist, represented as an atom, and the other is held only by small traders, represented by an atomless part. Busetto et al. (2023) assumed that the monopolist acts strategically, making a bid of the commodity he holds in exchange for the other commodity, while the atomless part behaves *à la* Walras; given the monopolist's bid, prices adjust to equate it to the aggregate net demand of the atomless part. Each trader belonging to the atomless part then obtains his Walrasian demand whereas the monopolist's final holding is determined as the difference between his endowment and his bid, for the commodity he holds, and as the value of his bid in terms of relative prices, for the other commodity. They defined a monopoly equilibrium as a strategy played by the monopolist, represented by a positive bid of the commodity he holds, which guarantees him to obtain, via the trading process described above, a most preferred final holding among those he can achieve through his bids.

The theoretical framework proposed by Busetto et al. (2023) to define and analyze a monopoly equilibrium in bilateral exchange can be simplified, under the assumption that the aggregate demand of the atomless part for the commodity held by the monopolist is invertible, and compared with the standard analysis of monopoly in a partial equilibrium context.

In this paper, we characterize, under this assumption, the monopoly quantity setting solution proposed by Kats (1974a). In particular, we call Kats maximization problem the mathematical program whose solution is the monopolist's most preferred commodity bundle in the complement of the atomless part's offer curve which is feasible with respect to the aggregate initial endowments. Then, we prove a proposition establishing that an allocation is a monopoly allocation if and only if it is a solution to the Kats maximization problem, thereby providing an explicit economic theoretical foundation of the monopoly solution proposed by Kats (1974a).

Shitovitz (1997) proposed a comparison, in terms of the monopolist's utility, between the core and the monopoly solution in mixed exchange economies. Busetto et al. (2025) provided a version of the Shitovitz argument adapted to the model of monopoly in bilateral exchange through which they exhibited a new kind of paradox: for any monopoly allocation there is an allocation in the core, which is neither a monopoly allocation nor a Walras allocation, and which is, utility-wise, advantageous for the monopolist and non-advantageous for the small traders. In this paper, under the assumption that the aggregate demand of the atomless part for the commodity held by the monopolist is invertible, we reformulate the Shitovitz paradox in terms of the Kats monopoly solution. In particular we show that, for any allocation corresponding to a solution to the Kats maximization problem, there is an allocation in the core, which is, utility-wise, advantageous for the monopolist and nonadvantageous

for the small traders, and which is not in the feasible complement of the atomless part's offer curve.

The paper is organized as follows. In Section 2, we introduce the mathematical model and we define the notion of a monopoly equilibrium when the aggregate demand of the atomless part for the commodity held by the monopolist is invertible. In Section 3, we provide an economic theoretical foundation of the monopoly solution proposed by Kats (1974a). In Section 4, we reformulate the Shitovitz paradox with respect to this solution. In Section 5, we conclude.

2 | Mathematical Model

We consider a pure exchange economy with large traders, represented as atoms, and small traders, represented by an atomless part. The space of traders is denoted by the measure space (T, \mathcal{T}, μ) , where T is the set of traders, \mathcal{T} is the σ -algebra of all μ -measurable subsets of T , and μ is a real valued, non-negative, countably additive measure defined on \mathcal{T} . We assume that (T, \mathcal{T}, μ) is finite, that is, $\mu(T) < \infty$. Let T_0 denote the atomless part of T . We assume that $\mu(T_0) > 0$.¹ Moreover, we assume that $T \setminus T_0 = \{m\}$, that is, the measure space (T, \mathcal{T}, μ) contains only one atom, the “monopolist.” A null set of traders is a set of measure 0. Null sets of traders are systematically ignored throughout the paper. Thus, a statement asserted for “each” trader in a certain set is to be understood to hold for all such traders except possibly for a null set of traders. A coalition is a nonnull element of \mathcal{T} . The word “integrable” is to be understood in the sense of Lebesgue.

In the exchange economy, there are two different commodities. A commodity bundle is a point in R_+^2 . An assignment (of commodity bundles to traders) is an integrable function $\mathbf{x}: T \rightarrow R_+^2$. There is a fixed initial assignment \mathbf{w} , satisfying the following assumption.

Assumption 1. $\mathbf{w}^1(m) > 0, \mathbf{w}^2(m) = 0$ and $\mathbf{w}^1(t) = 0, \mathbf{w}^2(t) > 0$, for each $t \in T_0$.

An allocation is an assignment \mathbf{x} such that $\int_T \mathbf{x}(t) d\mu = \int_T \mathbf{w}(t) d\mu$. The preferences of each trader $t \in T$ are described by a utility function $u_t: R_+^2 \rightarrow R$, satisfying the following assumptions.

Assumption 2. $u_t: R_+^2 \rightarrow R$ is continuous, strongly monotone, and strictly quasi-concave, for each $t \in T$.

Let \mathcal{B} denote the Borel σ -algebra of R_+^2 . Moreover, let $\mathcal{T} \otimes \mathcal{B}$ denote the σ -algebra generated by the sets $D \times F$ such that $D \in \mathcal{T}$ and $F \in \mathcal{B}$.

Assumption 3. $u: T \times R_+^2 \rightarrow R$, given by $u(t, x) = u_t(x)$, for each $t \in T$ and for each $x \in R_+^2$, is $\mathcal{T} \otimes \mathcal{B}$ -measurable.

Assumption 4. There is a coalition \bar{T} in T_0 such that $u_t(\cdot)$ is differentiable, additively separable, that is, $u_t(x) = v_t^1(x^1) + v_t^2(x^2)$, for each $x \in R_+^2$, and $\frac{dv_t^1(0)}{dx^1} = +\infty$, for each $t \in \bar{T}$.²

A price vector is a non-null vector $p \in R_+^2$. By Assumption 2, it is possible to define the Walrasian demand of traders of the atomless part as the function $\mathbf{x}_0 : T_0 \times R_{++}^2 \rightarrow R_+^2$ such that, for each $t \in T_0$ and for each $p \in R_{++}^2$, $\mathbf{x}_0(t, p) = \operatorname{argmax} \{u_t(x) : x \in R_+^2 \text{ and } px \leq p\mathbf{w}(t)\}$. The function $\mathbf{x}_0(\cdot, p)$ is integrable, for each $p \in R_{++}^2$, by Proposition 1 in Busetto et al. (2023).

We now introduce an abridged version of the model of monopoly proposed by Busetto et al. (2023) which keeps its essential features and allows us to define the notion of monopoly equilibrium.

Let $E = \{e \in R_+ : e \leq \mathbf{w}^1(m)\}$ denote the strategy set of atom m . We denote by $e \in E$ a strategy of atom m , where e represents the amount of commodity 1 that atom m offers in exchange for commodity 2.

We then provide the following definition.

Definition 1. Given a strategy $e \in E$, a price vector p is said to be market clearing if $p \in R_{++}^2$ and

$$\int_{T_0} \mathbf{x}_0^1(t, p) \, d\mu = e\mu(m) \tag{1}$$

$$\int_{T_0} \mathbf{x}_0^2(t, p) \, d\mu + e\mu(m) \frac{p^1}{p^2} = \int_{T_0} \mathbf{w}^2(t) \, d\mu$$

Market clearing price vectors can be normalized by Proposition 2 in Busetto et al. (2023). Henceforth, we say that a price vector p is normalized if $p \in \Delta$ where $\Delta = \{p \in R_+^2 : \sum_{i=1}^2 p^i = 1\}$. Moreover, we denote by $\partial\Delta$ the boundary of the unit simplex Δ .

Busetto et al. (2023), in their Proposition 5, proved that, under Assumptions 1, 2, 3, and 4, given a strategy $e \in E$, there exists a market clearing price $p \in \Delta \setminus \partial\Delta$ if and only if $e > 0$. We denote by $\pi(\cdot)$ a correspondence which associates, with each strategy $e \in E$, the set of prices p satisfying (1), if $e > 0$, and is equal to $\{0\}$, otherwise. A price selection $p(\cdot)$ is a function which associates, with each strategy selection $e \in E$, a price vector $p \in \pi(e)$.

As pointed out in Busetto et al. (2023), the analysis of the monopoly problem in bilateral exchange can be simplified by introducing the assumption that the aggregate demand of the atomless part for commodity 1 is invertible.

The following proposition states that the aggregate demand of the atomless part for commodity 1 is strictly positive for each strictly positive price vector.

Proposition 1. Under Assumptions 1, 2, 3, and 4, $\int_{T_0} \mathbf{x}_0^1(t, p) \, d\mu > 0$ for each $p \in \Delta \setminus \partial\Delta$.

Proof. Suppose that $\int_{T_0} \mathbf{x}_0^1(t, p) \, d\mu = 0$, for some $p \in \Delta \setminus \partial\Delta$. Then, we have that $\int_T \mathbf{x}_0^1(t, p) \, d\mu = 0$ as $\mu(\bar{T}) > 0$. However, it must be that $\int_T \mathbf{x}_0^1(t, p) \, d\mu > 0$, by the lemma in Busetto et al. (2025), a contradiction. \square

In their Proposition 7, Busetto et al. (2023) proved that the function $\int_{T_0} \mathbf{x}_0^1(t, \cdot) \, d\mu$ is invertible if and only if, for each $x \in R_{++}$, there is a unique $p \in \Delta \setminus \partial\Delta$ such that $x = \int_{T_0} \mathbf{x}_0^1(t, p) \, d\mu$. In their Proposition 8, Busetto et al. (2023) also proved that, when the aggregate demand of the atomless part for commodity 1 is invertible, there exists a unique price selection $\hat{p}(\cdot)$. By analogy with partial equilibrium analysis, $\hat{p}(\cdot)$ can be interpreted as the inverse demand function of the monopolist.

Henceforth, we assume that the aggregate demand of the atomless part for commodity 1 is invertible.

Assumption 5. The function $\int_{T_0} \mathbf{x}_0^1(t, \cdot) \, d\mu$ is invertible.

Given a strategy $e \in E$ and a price vector p , consider the assignment determined as follows:

$$\mathbf{x}(m, e, p) = \left(\mathbf{w}^1(m) - e, e \frac{p^1}{p^2} \right), \text{ if } p \in \Delta \setminus \partial\Delta,$$

$$\mathbf{x}(m, e, p) = \mathbf{w}(m), \text{ otherwise,}$$

and

$$\mathbf{x}(t, p) = \mathbf{x}_0(t, p), \text{ if } p \in \Delta \setminus \partial\Delta,$$

$$\mathbf{x}(t, p) = \mathbf{w}(t), \text{ otherwise,}$$

for each $t \in T_0$.

Given a strategy $e \in E$ and the price selection $\hat{p}(\cdot)$, traders' final holdings are determined according to the above assignment as follows

$$\mathbf{x}(m) = \mathbf{x}(m, e, \hat{p}(e)),$$

$$\mathbf{x}(t) = \mathbf{x}(t, \hat{p}(e)),$$

for each $t \in T_0$. Traders' final holdings constitute an allocation, by Proposition 6 in Busetto et al. (2023). Moreover, it is straightforward to verify that $\hat{p}(e)\mathbf{x}(m, e, \hat{p}(e)) = \hat{p}(e)\mathbf{w}(m)$.

When the aggregate demand of the atomless part for commodity 1 is invertible, the general definition of a monopoly equilibrium provided by Busetto et al. (2023) in their Definition 4 can be reformulated with respect to the monopolist's inverse demand function $\hat{p}(\cdot)$.

Definition 2. A strategy $\tilde{e} \in E$ such that $\tilde{e} > 0$ is a monopoly equilibrium if

$$u_m(\mathbf{x}(m, \tilde{e}, \hat{p}(\tilde{e}))) \geq u_m(\mathbf{x}(m, e, \hat{p}(e))),$$

for each $e \in E$.

A monopoly allocation is an allocation $\tilde{\mathbf{x}}$ such that $\tilde{\mathbf{x}}(m) = \mathbf{x}(m, \tilde{e}, \hat{p}(\tilde{e}))$ and $\tilde{\mathbf{x}}(t) = \mathbf{x}_0(t, \hat{p}(\tilde{e}))$, for each $t \in T_0$, where \tilde{e} is a monopoly equilibrium.

3 | Kats' Monopoly Solution

When the aggregate demand of the atomless part for commodity 1 is invertible, the monopoly equilibrium can be characterized by means of the notion of offer curve of the atomless part, defined as the set $\{x \in R_+^2 : x = \int_{T_0} \mathbf{x}_0(t, p) d\mu \text{ for some } p \in \Delta \setminus \partial\Delta\}$, and of the notion of feasible complement of the offer curve of the atomless part, defined as the set $\{x \in R_+^2 : x\mu(m) + \int_{T_0} \mathbf{x}_0(t, p) d\mu = \int_T \mathbf{w}(t) d\mu \text{ for some } p \in \Delta \setminus \partial\Delta\}$.

We now show that, when the aggregate demand of the atomless part for commodity 1 is invertible, our model can provide an economic theoretical foundation of the solution proposed by Kats (1974a). The following proposition shows that, when the aggregate demand of the atomless part for commodity 1 is invertible, the feasible complement of the atomless part's offer curve is a subset of the set of the monopolist's final holdings.

Proposition 2. *Under Assumptions 1, 2, 3, 4, and 5, the set $\{x \in R_+^2 : x\mu(m) + \int_{T_0} \mathbf{x}_0(t, p) d\mu = \int_T \mathbf{w}(t) d\mu \text{ for some } p \in \Delta \setminus \partial\Delta\}$, is a subset of the set $\{x \in R_+^2 : x = \mathbf{x}(m, e, \hat{p}(e)) \text{ for some } e \in E\}$.*

Proof. Consider a commodity bundle $\bar{x} \in \{x \in R_+^2 : x\mu(m) + \int_{T_0} \mathbf{x}_0(t, p) d\mu = \int_T \mathbf{w}(t) d\mu \text{ for some } p \in \Delta \setminus \partial\Delta\}$. Then, there is unique a price vector $\bar{p} \in \Delta \setminus \partial\Delta$ such that

$$\bar{x}^1\mu(m) + \int_{T_0} \mathbf{x}_0^1(t, \bar{p}) d\mu = \int_T \mathbf{w}^1(t) d\mu$$

as the function $\int_{T_0} \mathbf{x}_0^1(t, \cdot) d\mu$ is invertible. It must be that $0 \leq \bar{x}^1 < \mathbf{w}^1(m)$ as $\int_{T_0} \mathbf{x}_0^1(t, \bar{p}) d\mu > 0$. Let $\bar{e} \in E$ be such that $\bar{e} = \mathbf{w}^1(m) - \bar{x}^1$. Then, we have that

$$\begin{aligned} & \bar{x}^1\mu(m) + \int_{T_0} \mathbf{x}_0^1(t, \hat{p}(\bar{e})) d\mu \\ &= (\mathbf{w}^1(m) - \bar{e})\mu(m) + \int_{T_0} \mathbf{x}_0^1(t, \hat{p}(\bar{e})) d\mu = \mathbf{w}^1(m)\mu(m), \end{aligned}$$

as $\int_{T_0} \mathbf{x}_0^1(t, \hat{p}(\bar{e})) d\mu = \bar{e}\mu(m)$. But then, it must be that $\bar{p} = \hat{p}(\bar{e})$. Moreover, it must be that

$$\bar{x}^2\mu(m) + \int_{T_0} \mathbf{x}_0^2(t, \bar{p}) d\mu = \int_T \mathbf{w}^2(t) d\mu,$$

as $\bar{x} \in \{x \in R_+^2 : x\mu(m) + \int_{T_0} \mathbf{x}_0(t, p) d\mu = \int_T \mathbf{w}(t) d\mu \text{ for some } p \in \Delta \setminus \partial\Delta\}$. But then, we have that

$$\bar{x}^2 = \bar{e} \frac{\bar{p}^1}{\bar{p}^2},$$

as \bar{p} is market clearing, by Proposition 3 in Busetto et al. (2023). Therefore, we conclude that

$$\bar{x} = \mathbf{x}(m, \bar{e}, \bar{p}) = \mathbf{x}(m, \bar{e}, \hat{p}(\bar{e})).$$

Hence, the set $\{x \in R_+^2 : x\mu(m) + \int_{T_0} \mathbf{x}_0(t, p) d\mu = \int_T \mathbf{w}(t) d\mu \text{ for some } p \in \Delta \setminus \partial\Delta\}$, is a subset of the set $\{x \in R_+^2 : x = \mathbf{x}(m, e, \hat{p}(e)) \text{ for some } e \in E\}$. \square

Kats (1974a) considered both the cases of a quantity setting and a price setting monopoly in a pure exchange economy where one trader behaves as a monopolist, "calling the game" and maximizing his utility, whereas all the other traders in the economy behave competitively. He claimed that the monopoly quantity setting solution must correspond to the monopolist's most preferred commodity bundle compatible with both the aggregate initial endowments and the offer curve of the competitive traders. We can restate the Kats maximization problem as follows

$$\begin{aligned} & \max_x u_m(\mathbf{x}(m)) \\ & \text{subject to} \\ & u_m(\mathbf{x}(m)) \geq u_m(\mathbf{w}(m)), \\ & \mathbf{x}(t) = \mathbf{x}_0(t, p), \text{ for some } p \in \Delta \setminus \partial\Delta, \\ & \mathbf{x}(m) \in \left\{ x \in R_+^2 : x\mu(m) + \int_{T_0} \mathbf{x}_0(t, p) d\mu = \int_T \mathbf{w}(t) d\mu \text{ for some } p \in \Delta \setminus \partial\Delta \right\} \end{aligned}$$

However, he did not propose any explicit trading process which could lead to the monopoly solution. The following proposition establishes that an allocation is a monopoly allocation if and only if it is a solution to the Kats maximization problem. Thereby, it provides an explicit economic theoretical foundation of the monopoly solution proposed by Kats (1974a).

Proposition 3. *Under Assumptions 1, 2, 3, 4, and 5, an allocation $\bar{\mathbf{x}}$ is a monopoly allocation if and only if it is a solution to the Kats maximization problem.*

Proof. Let $\bar{\mathbf{x}}$ be a monopoly allocation. Then, there exists a strategy $\bar{e} \in E$ which is a monopoly equilibrium. Let $\bar{p} = \hat{p}(\bar{e})$. Then, we have that $\bar{\mathbf{x}}(t) = \mathbf{x}_0(t, \bar{p})$, for each $t \in T_0$. Moreover, We have that

$$\begin{aligned} & \bar{\mathbf{x}}^1(m, \bar{e}, \bar{p})\mu(m) + \int_{T_0} \mathbf{x}_0^1(t, \bar{p}) d\mu \\ &= (\mathbf{w}^1(m) - \bar{e})\mu(m) + \int_{T_0} \mathbf{x}_0^1(t, \bar{p}) d\mu = \mathbf{w}^1(m)\mu(m), \end{aligned}$$

and

$$\begin{aligned} & \bar{\mathbf{x}}^2(m, \bar{e}, \bar{p})\mu(m) + \int_{T_0} \mathbf{x}_0^2(t, \bar{p}) d\mu \\ &= \bar{e}\mu(m) \frac{\bar{p}^1}{\bar{p}^2} + \int_{T_0} \mathbf{x}_0^2(t, \bar{p}) d\mu = \int_T \mathbf{w}^2(t) d\mu, \end{aligned}$$

as \bar{p} is market clearing. Then, we have shown that $\bar{\mathbf{x}}(m) \in \{x \in R_+^2 : x\mu(m) + \int_{T_0} \mathbf{x}_0(t, p) d\mu = \int_T \mathbf{w}(t) d\mu$

for some $p \in \Delta \setminus \partial\Delta$. But then, we have that $u_m(\tilde{\mathbf{x}}(m)) \geq u_m(\mathbf{w}(m))$ and $u_m(\tilde{\mathbf{x}}(m))$ is maximal in the set $\{x \in R_+^2 : x\mu(m) + \int_{T_0} \mathbf{x}_0(t, p) d\mu = \int_T \mathbf{w}(t) d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$ as $u_m(\mathbf{x}(m, \tilde{e}, \tilde{p}(\tilde{e}))) \geq u_m(\mathbf{x}(m, e, \tilde{p}(e)))$, for each $e \in E$, and $\{x \in R_+^2 : x\mu(m) + \int_{T_0} \mathbf{x}_0(t, p) d\mu = \int_T \mathbf{w}(t) d\mu$ for some $p \in \Delta \setminus \partial\Delta\} \subset \{x \in R_+^2 : x = \mathbf{x}(m, e, \tilde{p}(e))$ for some $e \in E\}$, by Proposition 2. Hence, $\tilde{\mathbf{x}}$ is a solution to the Kats maximization problem.

Let $\tilde{\mathbf{x}}$ be a solution to the Kats maximization problem. Then, we have that $\tilde{\mathbf{x}}(m) \in \{x \in R_+^2 : x\mu(m) + \int_{T_0} \mathbf{x}_0(t, p) d\mu = \int_T \mathbf{w}(t) d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$. But then, there is a unique price vector $\tilde{p} \in \Delta \setminus \partial\Delta$ such that

$$\tilde{\mathbf{x}}^1(m)\mu(m) + \int_{T_0} \mathbf{x}_0^1(t, \tilde{p}) d\mu = \int_T \mathbf{w}^1(t) d\mu$$

as the function $\int_{T_0} \mathbf{x}_0^1(t, \cdot) d\mu$ is invertible. It must be that $0 \leq \tilde{\mathbf{x}}^1(m) < \mathbf{w}^1(m)$ as $\int_{T_0} \mathbf{x}_0^1(t, \tilde{p}) d\mu > 0$. Let $\tilde{e} \in E$ be such that $\tilde{e} = \mathbf{w}^1(m) - \tilde{\mathbf{x}}^1(m)$. Then, we have that

$$\begin{aligned} &\tilde{\mathbf{x}}^1(m)\mu(m) + \int_{T_0} \mathbf{x}_0^1(t, \tilde{p}(\tilde{e})) d\mu \\ &= (\mathbf{w}^1(m) - \tilde{e})\mu(m) + \int_{T_0} \mathbf{x}_0^1(t, \tilde{p}(\tilde{e})) d\mu = \mathbf{w}^1(m)\mu(m), \end{aligned}$$

as $\int_{T_0} \mathbf{x}_0^1(t, \tilde{p}(\tilde{e})) d\mu = \tilde{e}\mu(m)$. But then, it must be that $\tilde{p} = \tilde{p}(\tilde{e})$. Moreover, it must be that

$$\tilde{\mathbf{x}}^2(m)\mu(m) + \int_{T_0} \mathbf{x}_0^2(t, \tilde{p}) d\mu = \int_{T_0} \mathbf{w}^2(t) d\mu,$$

as $\tilde{\mathbf{x}}(m) \in \{x \in R_+^2 : x\mu(m) + \int_{T_0} \mathbf{x}_0(t, p) d\mu = \int_T \mathbf{w}(t) d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$. But then, we have that

$$\tilde{\mathbf{x}}^2(m) = \tilde{e} \frac{\tilde{p}^1}{\tilde{p}^2},$$

as \tilde{p} is market clearing, by Proposition 3 in Busetto et al. (2023). Therefore, we conclude that

$$\tilde{\mathbf{x}}(m) = \mathbf{x}(m, \tilde{e}, \tilde{p}) = \mathbf{x}(m, \tilde{e}, \tilde{p}(\tilde{e}))$$

and

$$\tilde{\mathbf{x}}(t) = \mathbf{x}_0(t, \tilde{p}) = \mathbf{x}_0(t, \tilde{p}(\tilde{e})),$$

for each $t \in T_0$. Suppose that there is a strategy $e' \in E$ such that $u_m(\mathbf{x}(m, \tilde{e}, \tilde{p}(\tilde{e}))) < u_m(\mathbf{x}(m, e', \tilde{p}(e')))$. Suppose that $e' = 0$. Then, it must be that $u_m(\tilde{\mathbf{x}}(m)) < u_m(\mathbf{w}(m))$, a contradiction. Suppose that $e' > 0$. Let $p' = p(e')$, $\mathbf{x}'(m) = \mathbf{x}(m, e', \tilde{p}(e'))$, and $\mathbf{x}'(t) = \mathbf{x}(t, p')$, for each $t \in T_0$. Then, we have that $\mathbf{x}'(m) \in \{x \in R_+^2 : x\mu(m) + \int_{T_0} \mathbf{x}_0(t, p) d\mu = \int_T \mathbf{w}(t) d\mu$

for some $p \in \Delta \setminus \partial\Delta\}$, as $\mathbf{x}'(m) + \int_{T_0} \mathbf{x}_0(t, p') d\mu = \int_T \mathbf{w}(t) d\mu$, and $u_m(\tilde{\mathbf{x}}(m)) < u_m(\mathbf{x}'(m))$, a contradiction. Hence, $\tilde{\mathbf{x}}$ is a monopoly allocation. \square

4 | Shitovitz' Paradox Revisited

Shitovitz (1997) proposed a comparison, in terms of the monopolist's utility, between the core and the monopoly solution in mixed exchange economies.

Busetto et al. (2025) provided a version of the Shitovitz argument adapted to the model of monopoly in bilateral exchange through which they exhibited a new kind of paradox: for any monopoly allocation there is an allocation in the core, which is neither a monopoly allocation nor a Walras allocation, and which is, utility-wise, advantageous for the monopolist and nonadvantageous for the small traders.

We shall now reformulate the Shitovitz paradox in terms of the Kats monopoly solution. In particular we shall show that, for any allocation corresponding to a solution to the Kats maximization problem, there is an allocation in the core, which is, utility-wise, advantageous for the monopolist and nonadvantageous for the small traders, and which is not in the feasible complement of the atomless part's offer curve.

We say that an assignment \mathbf{y} dominates an allocation \mathbf{x} via a coalition S if $u_t(\mathbf{y}(t)) \geq u_t(\mathbf{x}(t))$, for each $t \in S$, $u_t(\mathbf{y}(t)) > u_t(\mathbf{x}(t))$ for a non-null subset of traders t in S , and $\int_S \mathbf{y}(t) d\mu = \int_S \mathbf{w}(t) d\mu$. The core is the set of all allocations which are not dominated via any coalition.

Busetto et al. (2025) restated a maximization problem introduced by Shitovitz (1997), henceforth referred to as the Shitovitz maximization problem, as follows

$$\begin{aligned} &\max_{\mathbf{x}} u_m(\mathbf{x}(m)) \\ &\text{subject to} \\ &u_t(\mathbf{x}(t)) \geq u_t(\mathbf{w}(t)), \text{ for each } t \in T_0, \\ &\int_T \mathbf{x}(t) d\mu \leq \int_T \mathbf{w}(t) d\mu. \end{aligned}$$

The following proposition shows that the constraints of the Kats maximization problem are more restrictive than those of the Shitovitz maximization problem.

Proposition 4. *If the assignment $\hat{\mathbf{x}}$ satisfies the constraints of the Kats maximization problem, then it satisfies those of the Shitovitz maximization problem.*

Proof. Suppose that the assignment $\hat{\mathbf{x}}$ satisfies the constraints of the Kats maximization problem. Then, we have that $u_m(\hat{\mathbf{x}}(m)) \geq u_m(\mathbf{w}(m))$ and there is a $\hat{p} \in \Delta \setminus \partial\Delta$ such that $\hat{\mathbf{x}}\mu(m) + \int_{T_0} \hat{\mathbf{x}}_0(t, \hat{p}) d\mu = \int_T \mathbf{w}(t) d\mu$. Then, it must be that $\hat{\mathbf{x}}(t) = \hat{\mathbf{x}}_0(t, \hat{p})$ and $u_t(\hat{\mathbf{x}}(t)) \geq u_t(\mathbf{w}(t))$, for each $t \in T_0$.

Hence, we have that $\hat{\mathbf{x}}$ satisfies the constraints of the Shitovitz maximization problem. \square

It is immediate to verify that the converse of Proposition 4 does not hold as the initial assignment \mathbf{w} satisfies the constraints of the Shitovitz maximization problem but not those of the Kats maximization problem.

The following proposition restates the Shitovitz paradox exhibited by Busetto et al. (2025) in terms of the Kats monopoly solution and the feasible complement of the atomless part's offer curve.

Proposition 5. *Under Assumptions 1, 2, 3, 4, and 5, if $\hat{\mathbf{x}}$ is a solution to the Kats maximization problem, then there is a solution to the Shitovitz maximization problem $\check{\mathbf{x}}$ which is in the core and is such that $u_m(\check{\mathbf{x}}(m)) > u_m(\hat{\mathbf{x}}(m))$ and $u_t(\check{\mathbf{x}}(t)) \leq u_t(\hat{\mathbf{x}}(t))$, for each $t \in T_0$, and $\check{\mathbf{x}} \notin \{x \in R_+^2: x\mu(m) + \int_{T_0} \mathbf{x}_0(t, p) d\mu = \int_T \mathbf{w}(t) d\mu \text{ for some } p \in \Delta \setminus \partial\Delta\}$.*

Proof. Let $\hat{\mathbf{x}}$ be a solution to the Kats maximization problem. Then, $\hat{\mathbf{x}}$ is a monopoly allocation by Proposition 3. There exists a solution $\check{\mathbf{x}}$ to the Shitovitz maximization problem by Proposition 2.1 in Shitovitz (1997), which is in the core, by Proposition 4 in Busetto et al. (2025), and is such that $u_m(\check{\mathbf{x}}(m)) > u_m(\hat{\mathbf{x}}(m))$ and $u_t(\check{\mathbf{x}}(t)) \leq u_t(\hat{\mathbf{x}}(t))$, for each $t \in T_0$, by Theorem 2 in Busetto et al. (2025). Suppose that $\check{\mathbf{x}} \in \{x \in R_+^2: x\mu(m) + \int_{T_0} \mathbf{x}_0(t, p) d\mu = \int_T \mathbf{w}(t) d\mu \text{ for some } p \in \Delta \setminus \partial\Delta\}$. Then, $\check{\mathbf{x}}$ satisfies the constraints of the Kats maximization problem as $u_m(\check{\mathbf{x}}(m)) \geq u_m(\mathbf{w}(m))$. But then, $\check{\mathbf{x}}$ is a solution to the Kats maximization problem, by Proposition 4. Therefore, we have that $\check{\mathbf{x}}$ is a monopoly allocation, by Proposition 3. But it cannot be a monopoly allocation, by Proposition 3 in Busetto et al. (2025), a contradiction. \square

Proposition 5 shows that the assumption requiring that aggregate demand of the atomless part for commodity 1 is invertible allows to refine the Shitovitz paradox. Indeed, the monopolist, to achieve an allocation in the core at which he is better off, while utility exploiting the small traders, with respect to the monopoly solution, has to get out of the feasible complement of the atomless part's offer curve.

The following example reproduces Example 2 in Busetto et al. (2025) in order illustrate this new facet of the Shitovitz paradox.

Example Consider the following specification of an exchange economy satisfying Assumptions 1, 2, 3, 4, and 5. $T_0 = [0, 1]$, $T \setminus T_0 = \{m\}$, $\mu(m) = 1$, $\mathbf{w}(m) = (1, 0)$, $u_m(x) = \frac{1}{2}x^1 + \sqrt{x^2}$, T_0 is taken with Lebesgue measure, $\mathbf{w}(t) = (0, 1)$, $u_t(x) = \sqrt{x^1} + x^2$, for each $t \in T_0$. Then, there is a unique solution to the Kats maximization problem $\hat{\mathbf{x}}$ and a unique solution to the Shitovitz maximization problem $\check{\mathbf{x}}$ which is in the core and is such that $u_m(\check{\mathbf{x}}(m)) > u_m(\hat{\mathbf{x}}(m))$ and $u_t(\check{\mathbf{x}}(t)) < u_t(\hat{\mathbf{x}}(t))$, for each $t \in T_0$, and $\check{\mathbf{x}} \notin \{x \in R_+^2: x\mu(m) + \int_{T_0} \mathbf{x}_0(t, p) d\mu = \int_T \mathbf{w}(t) d\mu \text{ for some } p \in \Delta \setminus \partial\Delta\}$.

Proof. The allocation $\hat{\mathbf{x}}$ such that $\hat{\mathbf{x}}(m) = (\frac{3}{4}, \frac{1}{4})$ and $\hat{\mathbf{x}}(t) = (\frac{1}{4}, \frac{3}{4})$, for each $t \in T_0$, is the unique monopoly

solution. Then, $\hat{\mathbf{x}}$ is the unique solution to the Kats maximization problem, by Proposition 3. The allocation $\check{\mathbf{x}}$ such that $\check{\mathbf{x}}(m) = (1 - (\frac{1}{4})^{\frac{2}{3}}, (\frac{1}{4})^{\frac{1}{3}})$ and $\check{\mathbf{x}}(t) = ((\frac{1}{4})^{\frac{2}{3}}, 1 - (\frac{1}{4})^{\frac{1}{3}})$, for each $t \in T_0$, is the unique solution to the Shitovitz maximization problem. The core consists of all the allocations \mathbf{x} of the form $(\mathbf{x}^1(m), \mathbf{x}^2(m)) = (\mathbf{x}^1(m), \frac{1}{4(1-\mathbf{x}^1(m))})$ and $(\mathbf{x}^1(t), \mathbf{x}^2(t)) = (1 - \mathbf{x}^1(m), 1 - \frac{1}{4(1-\mathbf{x}^1(m))})$, for each $t \in T_0$, where $1 - (\frac{1}{4})^{\frac{1}{3}} \leq \mathbf{x}^1(m) \leq 1 - (\frac{1}{4})^{\frac{2}{3}}$. Then, the allocation $\check{\mathbf{x}}$ is in the core. Moreover, we have that

$$u_m(\check{\mathbf{x}}(m)) = u_m\left(1 - \left(\frac{1}{4}\right)^{\frac{2}{3}}, \left(\frac{1}{4}\right)^{\frac{1}{3}}\right) = \frac{1}{2} + \frac{3}{8}\left(\frac{1}{4}\right)^{-\frac{1}{3}} > \frac{7}{8} = u_m\left(\frac{3}{4}, \frac{1}{4}\right) = u_m(\hat{\mathbf{x}}(m)),$$

and

$$u_t(\check{\mathbf{x}}(t)) = u_t\left(\left(\frac{1}{4}\right)^{\frac{2}{3}}, 1 - \left(\frac{1}{4}\right)^{\frac{1}{3}}\right) = 1 < \frac{5}{4} = u_t\left(\frac{1}{4}, \frac{3}{4}\right) = u_t(\hat{\mathbf{x}}(t)),$$

for each $t \in T_0$. Finally, we have that $\check{\mathbf{x}} \notin \{x \in R_+^2: x\mu(m) + \int_{T_0} \mathbf{x}_0(t, p) d\mu = \int_T \mathbf{w}(t) d\mu \text{ for some } p \in \Delta \setminus \partial\Delta\}$ as there is no $p \in \Delta \setminus \partial\Delta$ such that $\check{\mathbf{x}}(t) = \mathbf{x}_0(t, p)$, for each $t \in T_0$. \square

5 | Conclusion

In the bilateral monopolistic framework of Shitovitz' Example 1 (1973, pp. 486–487), Aumann (1973) provided three examples, which show that monopoly may be, according to his terminology, “disadvantageous.” In fact, let us provisionally subscribe, for discussion's sake, to Shitovitz' assumption that in a “monopolistic market” a core allocation can suitably represent the market outcome, as regards the monopolist, too. Under such assumption, Aumann's examples are able to refute the following “conjecture,” implicit in Shitovitz' reasoning, which embodies the idea that a monopolist can always gain a definite advantage over competitive traders: “Conjecture. In a monopolistic market, for each core allocation \mathbf{x} there is a competitive allocation \mathbf{y} whose utility to the monopolist is \leq that of \mathbf{x} ” (see p. 1).

Kats (1974b) quoted the following passage from Aumann (1973) in which this author explained what is wrong in the preceding “conjecture” in spite of its intuitive appeal: “One feels on an intuitive, common sense level that the monopolist has a distinct advantage; but economic theory, rather than explaining this phenomenon, simply states it in a specific form. For an explanation, one looks to game theory; but evidently, the game-theoretic notion of core is not the proper vehicle for such an explanation” (see p. 10).

Kats (1974b) explicitly addressed the problem raised by Aumann (1973), proposing a variant of the model of strategic

exchange introduced by Debreu (1952) in which only the monopolist is allowed to manipulate the strategy sets of other traders. Nevertheless, we can apply to this model the same criticism raised by Shapley and Shubik (1977) at the model proposed by Debreu (1952): “[...] as a *descriptive* model his game shares the defect of the Walrasian model of being ill defined, or unrealistically defined, away from equilibrium” (see p. 939, fn. 1). For this reason, we have focused here on the monopoly solution proposed by Kats (1974a), which perfectly fits in the model of monopoly proposed by Busetto et al. (2023), since this model has a game-theoretical foundation.

In this paper, our focus is on the equivalence of the set of the monopoly allocations and the set of the allocations which are solutions to the Kats maximization problem and the invertibility of the aggregate demand of the atomless part for the commodity held by the monopolist is simply assumed. For the implications on the preferences condition, we refer the reader to Theorem 1 in Cheng (1985), where a set of sufficient conditions under which a system of demand functions satisfying gross substitutability can be inverted is proposed, while the main theorem in Fisher (1972) provides a set of necessary and sufficient conditions on preferences which generates a system of demand functions satisfying gross substitutability. Recent work by Diasakos and Gerasimou (2022) provides a different set of necessary and sufficient conditions on individual preferences compatible with invertible demand. We propose to develop in a future paper an analysis of the relationships between the studies on invertible demand functions mentioned above and our model of monopoly to obtain a specification of the preference conditions which guarantee invertibility in our context. This further analysis could also deal with the very issue of invertibility as a necessary condition for our equivalence result.

In a seminal paper, Gabszewicz and Grazzini (1999) investigated the effectiveness of tax and transfer policies in correcting market distortions in the framework of the bilateral oligopoly model introduced by Gabszewicz and Michel (1997). A similar analysis for our model of monopoly in bilateral exchange seems to us a worthy next step in our research agenda.

Here, we have considered a quantity-setting monopolist. We leave for future research addressing the problem of a price-setting monopolist, in the same bilateral framework as that used in this paper, drawing inspiration from another pioneering work by Vilfredo Pareto (see Pareto 1909, pp. 210–211, 594–605).

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Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Endnotes

¹The symbol 0 denotes the origin of R_+^2 as well as the real number zero: no confusion will result.

²Differentiability is to be understood as continuous differentiability and includes the case of infinite partial derivatives along the boundary of the consumption set (for a discussion of this case, see, for instance, Kreps 2012, p. 58). For instance, the utility function $u(x) = \sqrt{x^1} + \sqrt{x^2}$ satisfies the conditions imposed by Assumption 4.

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