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Extremal α-pseudocompact abelian groups

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Abstract. Let α be an infinite cardinal. Generalizing a recent result of Comfort and van Mill, we prove that every α-pseudocompact abelian group of weight $> \alpha$ has some proper dense α-pseudocompact subgroup and admits some strictly finer α-pseudocompact group topology.

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1 Introduction

In this paper all topological spaces and groups are supposed to be Tychonov.

The following notion was introduced by Kennison.

Definition 1.1 ([29]). Let α be an infinite cardinal. A topological space X is *α-pseudocompact* if $f(X)$ is compact in $\mathbb{R}^\alpha$ for every continuous function $f : X \to \mathbb{R}^\alpha$.

Note that ω-pseudocompactness coincides with pseudocompactness [29, Theorem 2.1]; so this definition generalizes that of pseudocompact space given by Hewitt [26]. As a direct consequence of the definition, a continuous image of an α-pseudocompact space is α-pseudocompact. Moreover, an α-pseudocompact space of weight $\leq \alpha$ is compact.

If $\alpha \geq \beta$ are infinite cardinals, α-pseudocompact implies β-pseudocompact and in particular pseudocompact. In [10, Theorem 1.1] it is proved that every pseudocompact group G is *precompact* (i.e., the completion $\widetilde{G}$ is compact [30]).

A pseudocompact group G is *s-extremal* if it has no proper dense pseudocompact subgroup and it is *r-extremal* if there exists no strictly finer pseudocompact group topology on G [2]. Recently Comfort and van Mill proved that a pseudocompact abelian group is either *s*- or *r*-extremal if and only if it is metrizable [13, Theorem 1.1]. The question of whether every either *s*- or *r*-extremal pseudocompact group is metrizable was posed in 1982 [7, 11] and many papers in the following twenty-five years were devoted to the study of this problem [2, 3, 4, 5, 6, 7, 9, 11, 12, 13, 18, 23].

In the survey [2] exposing the story of the solution of this problem there is a wish of extending [13, Theorem 1.1] to the more general case of not necessarily abelian groups. Moreover it is suggested to consider, for any pair of topological classes $\mathcal{P}$ and $\mathcal{Q}$, the problem of understanding whether every topological group $G \in \mathcal{P}$ admits a dense subgroup and/or a strictly larger group topology in $\mathcal{Q}$. This problem is completely solved by [13, Theorem

1.1] in the case $\mathcal{P} = \mathcal{Q} = \{$pseudocompact abelian groups$\}$. Here we consider and solve the case $\mathcal{P} = \mathcal{Q} = \{\alpha$-pseudocompact abelian groups$\}$.

We generalize to α-pseudocompact groups the definitions of the different levels of extremality given for pseudocompact groups in [2, 9, 18]. For $\alpha = \omega$ we find exactly the definitions of s-, r-, d-, c- and weak-extremality.

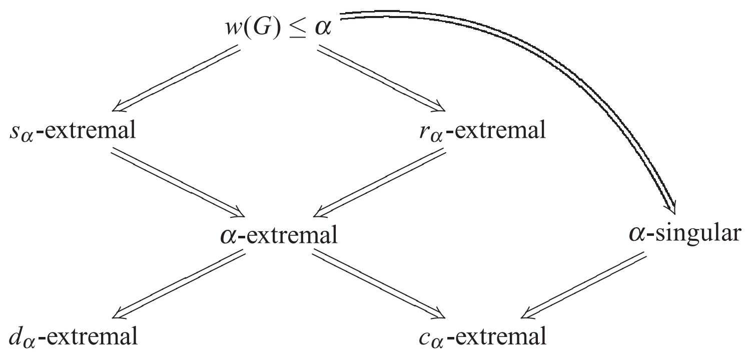
Definition 1.2. Let α be an infinite cardinal. An α-pseudocompact group G is:
- s_α*-extremal* if it has no proper dense α-pseudocompact subgroup;
- r_α*-extremal* if there exists no strictly finer α-pseudocompact group topology on G;
- d_α*-extremal* if G/D is divisible for every dense α-pseudocompact subgroup D of G;
- c_α*-extremal* if $r_0(G/D) < 2^\alpha$ for every dense α-pseudocompact subgroup D of G;
- α*-extremal* if it is both d_α- and c_α-extremal.

Moreover we extend for every infinite cardinal α the concept of singular group introduced in [18, Definition 1.2] and used later in [16, 17]:

Definition 1.3. Let α be an infinite cardinal. A topological group G is α*-singular* if there exists a positive integer m such that $w(mG) \leq \alpha$.

If $\beta \leq \alpha$ are infinite cardinals, then s_β- (respectively, r_β-, d_β-, c_β-, β-) extremality yields s_α- (respectively, r_α-, d_α-, c_α-, α-) extremality, for α-pseudocompact groups. Immediate examples of d_α- and c_α-extremal α-pseudocompact groups are divisible and torsion α-pseudocompact groups respectively. Intuitively, α-singular groups are those having “large” torsion subgroups (see Lemma 5.5).

In the following diagram we give an idea of the relations among these levels of extremality for α-pseudocompact abelian groups. The non-obvious implications in the diagram are proved in Proposition 2.9, Theorem 4.4 and Lemma 5.6.



The obvious symmetry of this diagram is “violated” by the property α-singular; but we show that it is equivalent to c_α-extremal (see Corollary 1.6).

The main theorem of this paper shows that four of the remaining properties in the diagram coincide:

Theorem 1.4. *Let α be an infinite cardinal. For an α-pseudocompact abelian group G the following conditions are equivalent:*

(a) *G is α-extremal;*
(b) *G is either s_α- or r_α-extremal;*
(c) $w(G) \leq \alpha$.

Our proof of Theorem 1.4 does not depend on the particular case $\alpha = \omega$, proved in [13, Theorem 1.1]. However many ideas used here are taken from previous proofs in [5, 6, 8, 9, 12, 13, 18]. Moreover we apply a set-theoretical lemma from [13] (see Lemma 5.9). In each of these cases we give references.

Example 5.10 shows that in general d_α- and c_α-extremality do not coincide with the other levels of extremality.

To prove Theorem 1.4 we generalize a lot of results which hold for pseudocompact abelian groups to α-pseudocompact abelian groups. We first establish in Section 2 basic properties of α-pseudocompact groups. Then we show that c_α-extremal α-pseudocompact abelian groups have "small" free rank (see Theorem 3.4). Moreover Theorem 5.4 proves Theorem 1.4 in the torsion case.

Then we prove that for compact abelian groups α-singularity is equivalent to c_α-extremality and to a third property of a completely different nature:

Theorem 1.5. *Let α be an infinite cardinal. For a compact abelian group K the following conditions are equivalent:*
(a) *K is c_α-extremal;*
(b) *K is α-singular;*
(c) *there exists no continuous surjective homomorphism of K onto S^I, where S is a metrizable compact non-torsion abelian group and $|I| > \alpha$.*

Using (c) we prove that the free rank of non-α-singular α-pseudocompact abelian groups is "large" (see Proposition 5.7). This allows us to extend the equivalence of (a) and (b) to the more general case of α-pseudocompact abelian groups:

Corollary 1.6. *Let α be an infinite cardinal. An α-pseudocompact abelian group is c_α-extremal if and only if it is α-singular.*

The last stage of the proof of Theorem 1.4 is to show that every α-singular α-extremal α-pseudocompact abelian group has weight $\leq \alpha$, applying the torsion case of the theorem.

Notation and terminology

The symbols $\mathbb{Z}$, $\mathbb{P}$, $\mathbb{N}$ and $\mathbb{N}_+$ are used for the set of integers, the set of primes, the set of non-negative integers and the set of positive integers, respectively. The circle group $\mathbb{T}$ is identified with the quotient group $\mathbb{R}/\mathbb{Z}$ of the reals $\mathbb{R}$ and carries its usual compact topology. For a cardinal α and a set X we denote by X^α the product of α many copies of X and by $X^{(\alpha)}$ the direct sum of α many copies of X, that is $\bigoplus_\alpha X$.

Let G be an abelian group. For $n \in \mathbb{N}_+$ let $nG = \{ng : g \in G\}$. We say that G is torsion if every element of G is torsion. Moreover G is torsion-free if no element of G is torsion. Finally, G is non-torsion if there exists at least one element of G, which is not torsion. The group G is said to be *of exponent* $n \in \mathbb{N}_+$ if n is such that $nG = 0$. Equivalently we say that it is bounded-torsion. If m is a positive integer, $G[m] = \{x \in G : mx = 0\}$ and $\mathbb{Z}(m)$ is the cyclic group of order m.

We denote by $r_0(G)$ the free rank of G and by $r_p(G)$ the p-rank of G, for $p \in \mathbb{P}$. Moreover $r(G) := r_0(G) + \sup_{p\in\mathbb{P}} r_p(G)$ is the rank of G. If H is a group and $h : G \to H$ is a homomorphism, then we denote by $\Gamma_h := \{(x, h(x)) : x \in G\}$ the graph of h.

We recall the definitions of some cardinal invariants. For a topological space X the *weight* $w(X)$ of X is the minimum cardinality of a base for the topology on X. Moreover, if $x \in X$,

- the *character* $\chi(x,X)$ *at* x of X is the minimal cardinality of a basis of the filter of the neighborhoods of x in X, and
- the *character* of X is $\chi(X) = \sup_{x\in X} \chi(x,X)$.

Analogously

- the *pseudocharacter* $\psi(x,X)$ *at* x of X is the minimal cardinality of a family $\mathscr{F}$ of neighborhoods of x in X such that $\bigcap_{U\in\mathscr{F}} U = \{x\}$, and
- the *pseudocharacter* of X is $\psi(X) = \sup_{x\in X} \psi(x,X)$.

In general $\psi(X) \leq \chi(X) \leq w(X) \leq 2^{|X|}$ and $|X| \leq 2^{w(X)}$.

The interior of a subset A of X is the union of all open sets within A and is denoted by $\mathrm{Int}_X(A)$ and $\overline{A}^X$ is the closure of A in X (sometimes we write only $\overline{A}$ when there is no possibility of confusion).

Let G be a topological group. If M is a subset of G, then $\langle M\rangle$ is the smallest subgroup of G containing M. We denote by $\widetilde{G}$ the two-sided completion of G; in case G is precompact $\widetilde{G}$ coincides with the Weil completion.

For any abelian group G let $\mathrm{Hom}(G,\mathbb{T})$ be the group of all homomorphisms of G to the circle group $\mathbb{T}$. When (G,τ) is an abelian topological group, the set of τ-continuous homomorphisms $\chi : G \to \mathbb{T}$ (*characters*) is a subgroup of $\mathrm{Hom}(G,\mathbb{T})$ and is denoted by $\widehat{G}$; endowed with the compact-open topology, $\widehat{G}$ is the Pontryagin dual of G.

For undefined terms see [19, 21, 27].

2 The α-pseudocompactness

For compact groups we recall the following results about cardinal invariants. (We use the second part of (a) without giving reference, because it is a well-known fact.)

Fact 2.1 ([27, 28]). *Let K be a compact group of weight $\geq \omega$. Then:*
(a) $d(K) = \log w(K)$ *and* $|K| = 2^{w(K)}$*;*
(b) $\psi(K) = \chi(K) = w(K)$*;*
(c) $w(K) = |\widehat{K}|$.

To begin studying extremal α-pseudocompact groups we need a characterization of α-pseudocompact groups similar to that of pseudocompact groups given by the Comfort and Ross theorem, that is Theorem 2.4 below. We find this characterization in Theorem 2.5 combining Theorem 2.4 with the following result.

Definition 2.2 ([24]). Let X be a topological space and let α be an infinite cardinal.
- A G_α*-set* of X is the intersection of α many open subsets of X.
- A subset of X is G_α*-dense* in X if it has non-empty intersection with every G_α-set of X.

The G_α-sets for $\alpha = \omega$ are the well known G_δ-sets. A topological space X has $\psi(X) \leq \alpha$ precisely when $\{x\}$ is a G_α-set of X for every $x \in X$.

The next result is a corollary of [24, Theorem 1.2]. If X is a topological space, then we indicate by βX its Čech-Stone compactification.

Corollary 2.3. *Let α be an infinite cardinal and let X be a topological space. Then X is α-pseudocompact if and only if X is G_α-dense in βX.*

The following theorem, due to Comfort and Ross, characterizes pseudocompact groups.

Theorem 2.4 ([10, Theorem 4.1]). *Let G be a precompact group. Then the following conditions are equivalent:*
(a) *G is pseudocompact;*
(b) *G is G_δ-dense in $\widetilde{G}$;*
(c) *$\widetilde{G} = \beta G$.*

The next theorem characterizes an α-pseudocompact group in terms of its completion.

Theorem 2.5. *Let α be an infinite cardinal and let G be a precompact group. Then the following conditions are equivalent:*
(a) *G is α-pseudocompact;*
(b) *G is G_α-dense in $\widetilde{G} = \beta G$.*

Proof. Note that (a) implies that G is pseudocompact. So in particular $\widetilde{G} = \beta G$ by Theorem 2.4. Then (a)⇔(b) is given precisely by Corollary 2.3. □

Corollary 2.6. *Let α be an infinite cardinal. Let G be a topological group and D a dense subgroup of G. Then D is α-pseudocompact if and only if D is G_α-dense in G and G is α-pseudocompact.*

Proof. Suppose that D is α-pseudocompact. It follows that $\widetilde{D}$ is compact and D is G_α-dense in $\widetilde{D}$ by Theorem 2.5. Since D is dense in G, $\widetilde{D} = \widetilde{G}$ and hence D is G_α-dense in G and G is G_α-dense in $\widetilde{G}$. By Theorem 2.5 G is α-pseudocompact.

Assume that G is α-pseudocompact and that D is G_α-dense in G. Then G is G_α-dense in $\widetilde{G}$ by Theorem 2.5 and D is G_α-dense in $\widetilde{G}$. So $\widetilde{G} = \widetilde{D}$ and hence D is α-pseudocompact by Theorem 2.5. □

Lemma 2.7. *Let α be an infinite cardinal. Let G be a topological group and H an α-pseudocompact subgroup of G such that $[G : H]$ is finite. Then G is α-pseudocompact.*

Proof. It suffices to note that each (of the finitely many) cosets xH is α-pseudocompact. □

The following lemma is the generalization to α-pseudocompact groups of [9, Theorem 3.2].

Lemma 2.8. *Let α be an infinite cardinal. If G is an α-pseudocompact group and $\psi(G) \leq \alpha$, then G is compact and so $w(G) = \psi(G) \leq \alpha$.*

Proof. Since $\psi(G) \leq \alpha$, it follows that $\{e_G\} = \bigcap_{\lambda<\alpha} U_\lambda$ for neighborhoods U_λ of e_G in G and by the regularity of G it is possible to choose every U_λ closed in G. Let $K = \widetilde{G}$. Then $\bigcap_{\lambda<\alpha} \overline{U_\lambda}^K$ contains a non-empty G_α-set W of K. Moreover $G \cap W \subseteq G \cap \bigcap_{\lambda<\alpha} \overline{U_\lambda}^K = \{e_G\}$ and $W \setminus \{e_G\}$ is a G_α-set of K. Since G is G_α-dense in K by Theorem 2.5, this is possible only if $W = \{e_G\}$. So $\psi(K) \leq \alpha$ and we can conclude that $G = K$ is compact. Moreover $w(G) = \psi(G)$ by Fact 2.1(b). □

The next proposition covers the implication (c)⇒(b) of Theorem 1.4, even for non-necessarily abelian groups.

Proposition 2.9. *Let α be an infinite cardinal and let (G, τ) be a compact group of weight $\leq \alpha$. Then (G, τ) is s_α- and r_α-extremal.*

Proof. First we prove that (G, τ) is s_α-extremal. Let D be a dense α-pseudocompact subgroup of (G, τ). Then $w(D) \leq \alpha$ and so D is compact. So D is closed in (G, τ) and therefore $D = G$.

Now we prove that (G, τ) is r_α-extremal. Let τ' be an α-pseudocompact group topology on G such that $\tau' \geq \tau$. Since $\psi(G, \tau) \leq \alpha$, it follows that also $\psi(G, \tau') \leq \alpha$. By Lemma 2.8 (G, τ') is compact. Then $\tau' = \tau$. □

2.1 The family $\Lambda_\alpha(G)$

Let G be a topological group and α an infinite cardinal. We define

$$\Lambda_\alpha(G) = \{N \lhd G : N \text{ closed } G_\alpha\}.$$

Usually $\Lambda_\omega(G)$ is denoted by $\Lambda(G)$ (see [9, 18]). For $\alpha \geq \beta$ infinite cardinals $\Lambda_\alpha(G) \supseteq \Lambda_\beta(G)$.

In Theorem 2.13 we prove that for α-pseudocompact groups the families in the following claim coincide.

Claim 2.10. *Let α be an infinite cardinal and let G be a topological group. Then*

$$\Lambda_\alpha(G) \supseteq \{N \lhd G : \textit{closed},\ \psi(G/N) \leq \alpha\} \supseteq \{N \lhd G : \textit{closed},\ w(G/N) \leq \alpha\}.$$

Proof. Let N be a closed normal subgroup of G and suppose that $w(G/N) \leq \alpha$. It follows that $\psi(G/N) \leq \alpha$. So N is a G_α-set of G and hence $N \in \Lambda_\alpha(G)$. □

For $\alpha = \omega$, the following lemma is [9, Lemma 1.6].

Lemma 2.11. *Let α be an infinite cardinal. Let G be a precompact group and W a G_α-set of G such that $e_G \in W$. Then W contains some $N \in \Lambda_\alpha(G)$ such that $\psi(G/N) \leq \alpha$.*

Proof. Let $W = \bigcap_{i\in I} U_i$, where U_i are open subsets of G and $|I| = \alpha$. Let $i \in I$. Since U_i is a G_δ-set of G containing e_G, then there exists $N_i \in \Lambda(G)$ such that $N_i \subseteq U_i$ and $\psi(G/N_i) \leq \omega$ [9, Lemma 1.6]. Let $N = \bigcap_{i\in I} N_i$. Then $N \in \Lambda_\alpha(G)$. Moreover $\psi(G/N) \leq \alpha$ because there exists a continuous injective homomorphism $G/N \to \prod_{i\in I} G/N_i$ and so $\psi(G/N) \leq \psi(\prod_{i\in I} G/N_i) \leq \alpha$. □

Corollary 2.12. *Let α be an infinite cardinal. If G is precompact and W is a G_α-set of G, then there exist $a \in W$ and $N \in \Lambda_\alpha(G)$ such that $aN \subseteq W$. So a subset H of G is G_α-dense in G if and only if $(xN) \cap H \neq \emptyset$ for every $x \in G$ and $N \in \Lambda_\alpha(G)$.*

The next theorem and (b) of its corollary were proved in the case $\alpha = \omega$ in [9, Theorem 6.1 and Corollary 6.2].

Theorem 2.13. *Let α be an infinite cardinal and let G be a precompact group. Then G is α-pseudocompact if and only if $w(G/N) \leq \alpha$ for every $N \in \Lambda_\alpha(G)$.*

Proof. Suppose that G is α-pseudocompact. Since $N \in \Lambda_\alpha(G)$, by Lemma 2.11 there exists $L \in \Lambda_\alpha(G)$ such that $L \leq N$ and $\psi(G/L) \leq \alpha$. Thanks to Lemma 2.8 we have that G/L is compact of weight $w(G/L) = \psi(G/L) \leq \alpha$. Since G/N is continuous image of G/L, it follows that $w(G/N) \leq \alpha$.

Suppose that $w(G/N) \leq \alpha$ for every $N \in \Lambda_\alpha(G)$. By Corollary 2.12 and Theorem 2.5 it suffices to prove that $xM \cap G \neq \emptyset$ for every $x \in \widetilde{G}$ and every $M \in \Lambda_\alpha(\widetilde{G})$. Let $\widetilde{\pi} : \widetilde{G} \to \widetilde{G}/M$ be the canonical projection, $\pi = \widetilde{\pi} \restriction_G$ and $N = G \cap M$. Hence $N \in \Lambda_\alpha(G)$. By the hypothesis $w(G/N) \leq \alpha$ and so G/N is compact. Since $\pi(G)$ is continuous image of $G/\ker\pi = G/N$, so $\pi(G)$ is compact as well. Since G is dense in $\widetilde{G}$, it follows that $\pi(G)$ is dense in $\widetilde{G}/M$ and so $\pi(G) = \widetilde{G}/M$. Consequently $xM \in \pi(G) = \{gM : g \in G\}$ and hence $xM = gM$ for some $g \in G$, that is $g \in xM \cap G \neq \emptyset$. □

Corollary 2.14. *Let α be an infinite cardinal. Let G be an α-pseudocompact abelian group and let $N \in \Lambda_\alpha(G)$. Then:*
(a) *if $L \in \Lambda_\alpha(N)$, then $L \in \Lambda_\alpha(G)$;*
(b) *N is α-pseudocompact;*
(c) *if L is a closed subgroup of G such that $N \subseteq L$, then $L \in \Lambda_\alpha(G)$.*

Proof. (a) Since N is closed in G and L is closed in N, it follows that L is closed in G. Moreover L is a G_α-set of G, because N is a G_α-set of G and L is a G_α-set of N.

(b) Let $L \in \Lambda_\alpha(N)$. By (a) L is a G_α-set of G and so there exists $M \in \Lambda_\alpha(G)$ such that $M \subseteq L$ by Lemma 2.11. By Theorem 2.13 $w(G/M) \leq \alpha$ and consequently $w(N/M) \leq w(G/M) \leq \alpha$. Since N/L is continuous image of N/M, it follows that $w(N/L) \leq w(N/M) \leq \alpha$. Hence N is α-pseudocompact by Theorem 2.13.

(c) Since $w(G/N) \leq \alpha$ by Theorem 2.13 and G/L is continuous image of G/N, it follows that $w(G/L) \leq \alpha$. Hence $L \in \Lambda_\alpha(G)$ by Claim 2.10. □

Items (a) and (c) of this corollary and the following lemmas were proved in the pseudocompact case in [18, Section 2].

Lemma 2.15. *Let α be an infinite cardinal, G an α-pseudocompact abelian group and D a subgroup of G. Then:*
(a) *D is G_α-dense in G if and only if $D + N = G$ for every $N \in \Lambda_\alpha(G)$;*
(b) *if D is G_α-dense in G and $N \in \Lambda_\alpha(G)$, then $D \cap N$ is G_α-dense in N and G/D is algebraically isomorphic to $N/(D \cap N)$.*

Proof. (a) follows from Corollary 2.12 and (b) follows from (a) and Corollary 2.14. □

Lemma 2.16. *Let α be an infinite cardinal, G an α-pseudocompact abelian group, L a closed subgroup of G and $\pi : G \to G/L$ the canonical projection.*
(a) *If $N \in \Lambda_\alpha(G)$ then $\pi(N) \in \Lambda_\alpha(G/L)$.*
(b) *If D is a G_α-dense subgroup of G/L then $\pi^{-1}(D)$ is a G_α-dense subgroup of G.*

Proof. (a) By Theorem 2.13 we have $w(G/N) \leq \alpha$. Since $(G/L)/\pi(N) = (G/L)/((N + L)/L)$ and $(G/L)/((N+L)/L)$ is topologically isomorphic to $G/(N+L)$, it follows that $w((G/L)/\pi(N)) \leq \alpha$. Hence $\pi(N) \in \Lambda_\alpha(G/L)$ by Claim 2.10.

(b) Let $N \in \Lambda_\alpha(G)$. Since $\pi(N + \pi^{-1}(D)) = \pi(N) + D$ and by (a) $\pi(N) \in \Lambda_\alpha(G/L)$, Lemma 2.15(a) implies that $\pi(N) + D = G/L$. Then $\pi(N + \pi^{-1}(D)) = G/L$ and so $N + \pi^{-1}(D) = G$. By Lemma 2.15(a) $\pi^{-1}(D)$ is G_α-dense in G. □

The next proposition shows the stability under taking quotients of d_α- and c_α-extremality.

Proposition 2.17. *Let α be an infinite cardinal. Let G be an α-pseudocompact abelian group and let L be a closed subgroup of G. If G is d_α- (respectively, c_α-) extremal, then G/L is d_α- (respectively, c_α-) extremal.*

Proof. Let $\pi : G \to G/L$ be the canonical projection. If D is a G_α-dense subgroup of G/L, by Lemma 2.16(b) $\pi^{-1}(D)$ is G_α-dense in G. Moreover $G/\pi^{-1}(D)$ is algebraically isomorphic to $(G/L)/D$.

Suppose that G/L is not d_α-extremal. Then there exists a G_α-dense subgroup D of G/L such that $(G/L)/D$ is not divisible. Therefore $G/\pi^{-1}(D)$ is not divisible, hence G is not d_α-extremal. If G/L is not c_α-extremal. Then there exists a G_α-dense subgroup D of G/L such that $r_0((G/L)/D) \geq 2^\alpha$. Consequently $r_0(G/\pi^{-1}(D)) \geq 2^\alpha$, so G is not c_α-extremal. □

In this proposition we consider only d_α- and c_α-extremality. Indeed Theorem 1.4 and Example 5.10 prove that, for α-pseudocompact abelian groups, these are the only levels of extremality that are not equivalent to having weight $\leq \alpha$.

2.2 The P_α-topology

A topological space X is of *first category* if it can be written as the union of countably many nowhere dense subsets of X. Moreover X is of *second category (Baire)* if it is not of first category, i.e., if for every family $\{U_n\}_{n\in\mathbb{N}}$ of open dense subsets of X, also $\bigcap_{n\in\mathbb{N}} U_n$ is dense in X.

Let α be an infinite cardinal and let τ be a topology on a set X. Then $P_\alpha\tau$ denotes the topology on X generated by the G_α-sets of X, which is called *P_α-topology*. Obviously $\tau \leq P_\alpha\tau$. If X is a topological space, we denote by $P_\alpha X$ the set X endowed with the P_α-topology.

The following theorem is the generalization for topological groups of [8, Lemma 2.4] to the P_α-topology.

Theorem 2.18. *Let α be an infinite cardinal and let G be an α-pseudocompact group. Then $P_\alpha G$ is Baire.*

Proof. Let K be the completion of G. Then K is compact by Theorem 2.4. Let $\mathcal{B} = \{xN : x \in K, N \in \Lambda_\alpha(K)\}$. By Corollary 2.12 $\mathcal{B}$ is a base of $P_\alpha K$. Consider a family $\{U_n\}_{n\in\mathbb{N}}$ of open dense subsets of $P_\alpha K$. We can chose them so that $U_n \supseteq U_{n+1}$ for every $n \in \mathbb{N}$. Let $A \in P_\alpha \tau, A \neq \emptyset$. Then $A \cap U_0$ is a non-empty element of $P_\alpha \tau$. Therefore there exists $B_0 \in \mathcal{B}$ such that $B_0 \subseteq A \cap U_0$. We proceed by induction. If $B_n \in \mathcal{B}$ has been defined, then $B_n \cap U_{n+1}$ is a non-empty open set in $P_\alpha K$ and so there exists $B_{n+1} \in \mathcal{B}$ such that $B_{n+1} \subseteq B_n \cap V_{n+1}$. Then

$$A \cap \bigcap_{n\in\mathbb{N}} U_n = \bigcap_{n\in\mathbb{N}} (A \cap U_n) \supseteq \bigcap_{n\in\mathbb{N}} B_n.$$

Moreover $\bigcap_{n\in\mathbb{N}} B_n \neq \emptyset$, because $\{B_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of closed subsets of K, which is compact.

By [8, Lemma 2.4(b)] a G_δ-dense subspace of a Baire space is Baire. Being α-pseudocompact, G is G_α-dense in K by Theorem 2.5. Then G is G_α-dense in $P_\alpha K$, which is Baire by the previous part of the proof, and so $P_\alpha G$ is Baire. □

Let G be an abelian group. In what follows $G^\#$ denotes G endowed with the *Bohr topology*, i.e., the initial topology of all elements of $\mathrm{Hom}(G, \mathbb{T})$.

The following lemma is the generalization to the P_α-topology of [9, Theorem 5.16].

Lemma 2.19. *Let α be an infinite cardinal and let (G, τ) be a precompact abelian group such that every $h \in \mathrm{Hom}(G, \mathbb{T})$ is $P_\alpha\tau$-continuous. Then $(G, P_\alpha\tau) = P_\alpha G^\#$.*

Proof. Let $\tau_G^\#$ be the Bohr topology on G, that is $G^\# = (G, \tau_G^\#)$. By the hypothesis $\tau_G^\# \leq P_\alpha\tau$. Then $P_\alpha\tau_G^\# \leq P_\alpha P_\alpha\tau = P_\alpha\tau$. Moreover $\tau \leq \tau_G^\#$ yields $P_\alpha\tau \leq P_\alpha\tau_G^\#$. □

In the first part of the theorem we give a more detailed description of the topology $P_\alpha G^\#$, while the equivalence of (a), (b) and (c) is the counterpart of [9, Theorem 5.17] for the P_α-topology. Some ideas in the proof of this second part are similar to those in the proof of [9, Theorem 5.17], but ours is shorter and simpler, thanks to the description of the topology $P_\alpha G^\#$ in algebraic terms.

Theorem 2.20. *Let α be an infinite cardinal and let G be an abelian group. Then $\Lambda'_\alpha(G^\#) := \{N \leq G : |G/N| \leq 2^\alpha\} \subseteq \Lambda_\alpha(G^\#)$ is a local base at 0 of $P_\alpha G^\#$. Consequently the following conditions are equivalent:*
(a) $|G| \leq 2^\alpha$;
(b) *$P_\alpha G^\#$ is discrete;*
(c) *$P_\alpha G^\#$ is Baire.*

Proof. We prove first that $\Lambda''_\alpha(G^\#) := \{\bigcap_{\lambda<\alpha} \ker\chi_\lambda : \chi_\lambda \in \mathrm{Hom}(G, \mathbb{T})\} \subseteq \Lambda_\alpha(G^\#)$ is a local base at 0 of $P_\alpha G^\#$ and then the equality $\Lambda'_\alpha(G^\#) = \Lambda''_\alpha(G^\#)$.

If W is a G_α-set of G such that $0 \in W$, then $W \supseteq \bigcap_{\lambda<\alpha} U_\lambda$, where each U_λ is a neighborhood of 0 in $G^\#$ belonging to the base. This means that $U_\lambda = \chi_\lambda^{-1}(V_\lambda)$, where $\chi_\lambda \in \mathrm{Hom}(G, \mathbb{T})$ and V_λ is a neighborhood of 0 in $\mathbb{T}$. Therefore $W \supseteq \bigcap_{\lambda<\alpha} \chi_\lambda^{-1}(0) = \bigcap_{\lambda<\alpha} \ker\chi_\lambda$. Note that each $\chi_\lambda^{-1}(0)$ is a G_δ-set of $G^\#$, since $\{0\}$ is a G_δ-set of $\mathbb{T}$ and hence $\bigcap_{\lambda<\alpha} \chi_\lambda^{-1}(0)$ is a G_α-set of $G^\#$. Until now we have proved that $\Lambda''_\alpha(G^\#)$ is a local base at 0 of $P_\alpha G^\#$. Moreover it is contained in $\Lambda_\alpha(G^\#)$, because each $\bigcap_{\lambda<\alpha} \ker\chi_\lambda$, where $\chi_\lambda \in \mathrm{Hom}(G, \mathbb{T})$, is a closed G_α-subgroup of $G^\#$.

It remains to prove that $\Lambda'_\alpha(G^\#) = \Lambda''_\alpha(G^\#)$. Let $N = \bigcap_{\lambda<\alpha} \ker\chi_\lambda$, where every $\chi_\lambda \in \mathrm{Hom}(G,\mathbb{T})$. Since for every $\lambda < \alpha$ there exists an injective homomorphism $G/\ker\chi_\lambda \to \mathbb{T}$, it follows that there exists an injective homomorphism $G/N \to \mathbb{T}^\alpha$. Then $|G/N| \leq 2^\alpha$. To prove the converse inclusion let $N \in \Lambda'_\alpha(G^\#)$. Then N is closed in $G^\#$, because every subgroup of G is closed in $G^\#$. Moreover, since $r(G/N) \leq 2^\alpha$, there exists an injective homomorphism $i : G/N \to \mathbb{T}^\alpha$; in fact, G/N has an essential subgroup $B(G/N)$ (i.e., $B(G/N)$ non-trivially intersects each non-trivial subgroup of G/N) algebraically isomorphic to $\mathbb{Z}^{(r_0(G/N))} \oplus \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p)^{(r_p(G/N))}$. Since $r_0(\mathbb{T}^\alpha) = 2^\alpha$ and $r_p(\mathbb{T}^\alpha) = 2^\alpha$ for every $p \in \mathbb{P}$, there exists an injective homomorphism $B(G/N) \to \mathbb{T}^\alpha$ and by the divisibility of $\mathbb{T}^\alpha$ this homomorphism can be extended to G/N (the extended homomorphism is still injective by the essentiality of $B(G/N)$ in G/N). Let $\pi : G \to G/N$ be the canonical projection and let $\pi_\lambda : \mathbb{T}^\alpha \to \mathbb{T}$ be the canonical projection for every $\lambda < \alpha$. Then $\chi_\lambda = \pi_\lambda \restriction_{i(G)} \circ i \circ \pi : G \to \mathbb{T}$ is a homomorphism. Moreover $N = \bigcap_{\lambda<\alpha} \ker\chi_\lambda$.

It is clear that (a)⇒(b) by the first part of the proof and that (b)⇒(c).

(c)⇒(a) Suppose for a contradiction that $|G| > 2^\alpha$. We prove that $P_\alpha G^\#$ is of first category. Let $D(G) = \mathbb{Q}^{(r_0(G))} \oplus \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)^{(r_p(G))}$ be the divisible hull of G. Moreover G has a subgroup algebraically isomorphic to $B(G) = \mathbb{Z}^{(r_0(G))} \oplus \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p)^{(r_p(G))}$. We can think $B(G) \leq G \leq D(G) \leq \mathbb{T}^I$, where $|I| = r(G) = r_0(G) + \sup_{p\in\mathbb{P}} r_p(G)$, because $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ are algebraically isomorphic to subgroups of $\mathbb{T}$ for every $p \in \mathbb{P}$. Since $|G| > 2^\alpha$, it follows that $|I| > 2^\alpha$.

For $x \in G$ let $s(x) = \{i \in I : x_i \neq 0\}$ and for $n \in \mathbb{N}$ we set

$$A(n) = \{x \in G : |s(x)| \leq n\}.$$

Then $G = \bigcup_{n\in\mathbb{N}} A(n)$.

For every $n \in \mathbb{N}$ we have $A(n) \subseteq A(n+1)$. We prove that $A(n)$ is closed in the topology τ induced on G by $\mathbb{T}^I$ for every $n \in \mathbb{N}$: it is obvious that $A(0)$ is compact and $A(1)$ is compact, because every open neighborhood of 0 in (G,τ) contains all but a finite number of elements of $A(1)$. Moreover, for every $n \in \mathbb{N}$ with $n > 1$, $A(n)$ is the sum of n copies of $A(1)$ and so it is compact.

To conclude the proof we have to prove that each $A(n)$ has empty interior in $P_\alpha G^\#$. Since $A(n) \subseteq A(n+1)$ for every $n \in \mathbb{N}$, it suffices to prove that $\mathrm{Int}_{P_\alpha G^\#}(A(n))$ is empty for sufficiently large $n \in \mathbb{N}$; we consider $n \in \mathbb{N}_+$. By the first part of the proof, it suffices to show that if $x \in G$ and N is a subgroup of G such that $|G/N| \leq 2^\alpha$, then $x \in x + N \not\subseteq A(n)$ for all $n \in \mathbb{N}_+$. Moreover we can suppose that $x = 0$. In fact, if there exist $x \in G$ and $N \leq G$ with $|G/N| \leq 2^\alpha$, such that $x + N \subseteq A(n)$, then $x = x + 0 \in A(n)$ and $0 \in N \subseteq -x + A(n) \subseteq A(n) + A(n) \subseteq A(2n)$. So let $N \leq G$ be such that $|G/N| \leq 2^\alpha$. Let $\{I_\xi : \xi < (2^\alpha)^+\}$ be a family of subsets of I such that $|I_\xi| = n$ and $I_\xi \cap I_{\xi'} = \emptyset$ for every $\xi < \xi' < (2^\alpha)^+$. For every $\xi < (2^\alpha)^+$ there exists $x_\xi \in B(G) \leq G$ such that $s(x_\xi) = I_\xi$. Let $\pi : G \to G/N$ be the canonical projection. Since $|\{x_\xi : \xi < (2^\alpha)^+\}| = (2^\alpha)^+ > 2^\alpha \geq |G/N|$, it follows that there exist $\xi < \xi' < (2^\alpha)^+$ such that $\pi(x_\xi) = \pi(x_{\xi'})$. Then $x_\xi - x_{\xi'} \in \ker\pi = N$. But $s(x_\xi - x_{\xi'}) = I_\xi \cup I_{\xi'}$ and $|I_\xi \cup I_{\xi'}| = 2n$. Hence $x_\xi - x_{\xi'} \notin A(n)$. □

3 Construction of G_α-dense subgroups

The following lemma is a generalization to α-pseudocompact abelian groups of [12, Lemma 2.13]. The construction is the same.

Lemma 3.1. *Let α be an infinite cardinal. Let G be an α-pseudocompact abelian group and $G = \bigcup_{n\in\mathbb{N}} A_n$, where all A_n are subgroups of G. Then there exist $n \in \mathbb{N}$ and $N \in \Lambda_\alpha(G)$ such that $A_n \cap N$ is G_α-dense in N.*

Proof. Since $(G, P_\alpha\tau)$ is Baire by Theorem 2.18 and since $G = \bigcup_{n\in\mathbb{N}} \overline{A_n}^{P_\alpha\tau}$, there exists $n \in \mathbb{N}$ such that $\mathrm{Int}_{P_\alpha\tau}\overline{A_n}^{P_\alpha\tau} \neq \emptyset$. The family $\{x+N : x \in G,\ N \in \Lambda_\alpha(G)\}$ is a base of $P_\alpha\tau$ by Corollary 2.12; consequently there exist $x \in G$ and $N \in \Lambda_\alpha(G)$ such that $x+N \subseteq \overline{A_n}^{P_\alpha\tau}$. Since $x+N$ is open and closed in $P_\alpha\tau$, then $\overline{A_n \cap (x+N)}^{P_\alpha\tau} = x+N$, i.e., $A_n \cap (x+N)$ is G_α-dense in $x+N$.

We can suppose without loss of generality that $x \in A_n$, because we can choose $a \in A_n$ such that $a+N = x+N$. In fact, since $A_n \cap (x+N) \neq \emptyset$, because $A_n \cap (x+N)$ is G_α-dense in $x+N$, it follows that there exists $a \in (x+N) \cap A_n$. In particular $a \in x+N$ and so $a+N = x+N$.

We can choose $x = 0$ because all $A_n \leq G$: since $A_n \cap (x+N)$ is G_α-dense in $x+N$, it follows that $(A_n - x) \cap N$ is G_α-dense in N and $A_n - x = A_n$ since $x \in A_n$. □

For $\alpha = \omega$ the next lemma is [6, Lemma 4.1(b)].

Lemma 3.2. *Let α be an infinite cardinal and let G be an α-pseudocompact abelian group. If $N \in \Lambda_\alpha(G)$ and D is G_α-dense in N, then there exists a subgroup E of G such that $|E| \leq 2^\alpha$ and $D+E$ is G_α-dense in G.*

Proof. Since D is G_α-dense in $N \in \Lambda_\alpha(G)$, it follows that $x+D$ is G_α-dense in $x+N$ for every $x \in G$. By Theorem 2.13 G/N is compact of weight α and so $|G/N| \leq 2^\alpha$, i.e., there exists $X \subseteq G$ with $|X| \leq 2^\alpha$ such that $G/N = \{x+N : x \in X\}$. We set $E = \langle X \rangle$; then $|E| \leq 2^\alpha$ and $D+E$ is G_α-dense in G. □

Lemmas 3.1 and 3.2 imply that in case G is an α-pseudocompact abelian group such that $G = \bigcup_{n\in\mathbb{N}} A_n$, where all A_n are subgroups of G, then there exist $n \in \mathbb{N}$, $N \in \Lambda_\alpha(G)$ and $E \leq G$ with $|E| \leq 2^\alpha$ such that $(A_n \cap N)+E$ is G_α-dense in G. In particular we have the following useful result.

Corollary 3.3. *Let α be an infinite cardinal. Let G be an α-pseudocompact abelian group such that $G = \bigcup_{n\in\mathbb{N}} A_n$, where all $A_n \leq G$. Then there exist $n \in \mathbb{N}$ and a subgroup E of G such that $|E| \leq 2^\alpha$ and $A_n + E$ is G_α-dense in G.*

Thanks to the previous results we prove the following theorem, which gives a first restriction for extremal α-pseudocompact abelian groups, i.e., the free rank cannot be too big. The case $\alpha = \omega$ of this theorem is [18, Theorem 3.6]. That theorem in its own terms was inspired by [5, Theorem 5.10 (b)] and used ideas from the proof of [6, Proposition 4.4].

Theorem 3.4. *Let α be an infinite cardinal and let G be an α-pseudocompact abelian group. If G is c_α-extremal, then $r_0(G) \leq 2^\alpha$.*

Proof. Let $\kappa = r_0(G)$ and let M be a maximal independent subset of G consisting of non-torsion elements. Then $|M| = \kappa$ and there exists a partition $M = \bigcup_{n\in\mathbb{N}_+} M_n$ such that $|M_n| = \kappa$ for each $n \in \mathbb{N}_+$. Let $U_n = \langle M_n \rangle$, $V_n = U_1 \oplus \ldots \oplus U_n$ and $A_n = \{x \in G : n!x \in V_n\}$ for every $n \in \mathbb{N}_+$. Then $G = \bigcup_{n\in\mathbb{N}_+} A_n$. By Corollary 3.3 there exist $n \in \mathbb{N}_+$ and a subgroup E of G such that $D = A_n + E$ is G_α-dense in G and $|E| \leq 2^\alpha$. Hence $|E/(A_n \cap E)| \leq 2^\alpha$. Since $D/A_n = (A_n + E)/A_n$ is algebraically isomorphic to $E/(A_n \cap E)$, it follows that $|D/A_n| \leq 2^\alpha$. Since G is c_α-extremal, it follows that $r_0(G/D) < 2^\alpha$ and so $r_0(G/A_n) \leq 2^\alpha$ because $(G/A_n)/(D/A_n)$ is algebraically isomorphic to G/D. On the other hand, $r_0(G/A_n) \geq \kappa$, as U_n embeds into G/A_n. Hence $\kappa \leq 2^\alpha$. □

4 The dense graph theorem

For $\alpha = \omega$ the following lemma is [18, Lemma 3.7]. The idea of this lemma comes from the proof of [9, Theorem 4.1].

Lemma 4.1. *Let α be an infinite cardinal. Let G be a topological abelian group and H a compact abelian group with $|H| > 1$ and $w(H) \leq \alpha$. Let $h : G \to H$ be a surjective homomorphism. Then Γ_h is G_α-dense in $G \times H$ if and only if* $\ker h$ *is G_α-dense in G.*

Proof. Suppose that Γ_h is G_α-dense in $G \times H$. Let W be a non-empty G_α-set of G. Since $W \times \{0\}$ is a G_α-set of $G \times H$, then $\Gamma_h \cap (W \times \{0\}) \neq \emptyset$. But $\Gamma_h \cap (W \times \{0\}) = (W \cap \ker h) \times \{0\}$ and so $W \cap \ker h \neq \emptyset$. This proves that $\ker h$ is G_α-dense in G.

Suppose that $\ker h$ is G_α-dense in G. Every non-empty G_α-set of $G \times H$ contains a G_α-set of $G \times H$ of the form $W \times \{y\}$, where W is a non-empty G_α-set of G and $y \in H$. Since h is surjective, there exists $z \in G$ such that $h(z) = y$. Also $z + \ker h$ is G_α-dense in G and so $W \cap (z + \ker h) \neq \emptyset$. Consequently there exists $x \in W \cap (z + \ker h)$. From $x \in z + \ker h$ it follows that $x - z \in \ker h$. Therefore $h(x - z) = 0$ and so $h(x) = h(z) = y$. Since $x \in W$, it follows that $(x, y) \in (W \times \{y\}) \cap \Gamma_h$. This proves that $(W \times \{y\}) \cap \Gamma_h \neq \emptyset$. Hence Γ_h is G_α-dense in $G \times H$. □

In this lemma we conclude that $\ker h$ is proper in G, from the hypotheses that h is surjective and H is not trivial.

The next remark explains the role of the graph of a homomorphism in relation to the topology of the domain (see [18, Remark 2.12] for more details).

Remark 4.2. Let (G, τ) and H be topological groups and $h : (G, \tau) \to H$ a homomorphism. Consider the map $j : G \to \Gamma_h$ such that $j(x) = (x, h(x))$ for every $x \in G$. Then j is an open isomorphism. Endow Γ_h with the group topology induced by the product $(G, \tau) \times H$. The topology τ_h is *the weakest group topology on G such that $\tau_h \geq \tau$ and for which j is continuous.* Then $j : (G, \tau_h) \to \Gamma_h$ is a homeomorphism. Moreover τ_h is the weakest group topology on G such that $\tau_h \geq \tau$ and for which h is continuous. Clearly h is τ-continuous if and only if $\tau_h = \tau$.

The following theorem gives a necessary condition for an α-pseudocompact group to be either s_α- or r_α-extremal. We call it "dense graph theorem" because of the nature of this necessary condition. Moreover it is the generalization to α-pseudocompact groups of [9, Theorem 4.1].

Theorem 4.3. *Let α be an infinite cardinal and let (G,τ) be an α-pseudocompact group such that there exists a homomorphism $h: G\to H$ where H is an α-pseudocompact abelian group with $|H|>1$ and Γ_h is G_α-dense in $(G,\tau)\times H$. Then:*
(a) *there exists an α-pseudocompact group topology $\tau' > \tau$ on G such that $w(G,\tau') = w(G,\tau)$;*
(b) *there exists a proper G_α-dense subgroup D of (G,τ) such that $w(D)=w(G,\tau)$.*

Proof. We first prove that H can be chosen compact of weight $\leq\alpha$ and that in such a case h is surjective. There exists a continuous character $\chi: H\to\mathbb{T}$ such that $\chi(H)\neq\{0\}$. Let $H'=\chi(H)\subseteq\mathbb{T}$ and $h'=\chi\circ h$. Then H' is compact and metrizable. So H' is either $\mathbb{T}$ or $\mathbb{Z}(n)\leq\mathbb{T}$ for some integer $n>1$. Since $1_G\times\chi:(G,\tau)\times H\to(G,\tau)\times H'$ is a continuous surjective homomorphism such that $(1_G\times\chi)(\Gamma_h)=\Gamma_{h'}$ and Γ_h is G_α-dense in $(G,\tau)\times H$, it follows that $\Gamma_{h'}$ is G_α-dense in $(G,\tau)\times H'$. Let $p_2: G\times H'\to H'$ be the canonical projection. Then $p_2(\Gamma_{h'})=h'(G)$ is G_α-dense in H', which is metrizable. Hence $h'(G)=H'$ and h' is surjective.

(a) Since G is G_α-dense in $\widetilde{(G,\tau)}$ by Theorem 2.5 and since Γ_h is G_α-dense in $(G,\tau)\times H$, it follows that Γ_h is G_α-dense in $\widetilde{(G,\tau)}\times H$. Consequently Γ_h with the topology inherited from $\widetilde{(G,\tau)}\times H$ is α-pseudocompact in view of Corollary 2.6. As in Remark 4.2 let τ_h be the coarsest group topology on G such that $\tau_h\geq\tau$ and h is τ_h-continuous; then (G,τ_h) is topologically isomorphic to Γ_h and so it is α-pseudocompact. If $\tau_h=\tau$, then h is continuous and the closed graph theorem yields that Γ_h is closed in $(G,\tau)\times H$. This is not possible because Γ_h is dense in $(G,\tau)\times H$ by the hypothesis. Hence $\tau_h\gneq\tau$. By the hypothesis $w(G,\tau)>\omega$ and since H is metrizable, then

$$w(G,\tau_h)=w(\Gamma_h)=w((G,\tau)\times H)=w(G,\tau)\cdot w(H)=w(G,\tau).$$

(b) Let $D=\ker h$. By Lemma 4.1 D is G_α-dense in (G,τ). Moreover D is proper in G. Clearly $w(D)=w(G,\tau)$. □

The next theorem shows that α-extremality "puts together" s_α- and r_α-extremality. It is the generalization to α-pseudocompact abelian groups of [18, Theorem 3.12].

Theorem 4.4. *Let α be an infinite cardinal and let G be an α-pseudocompact abelian group which is either s_α- or r_α-extremal. Then G is α-extremal.*

Proof. Suppose looking for a contradiction that G is not α-extremal. Then there exists a dense α-pseudocompact subgroup D of G such that either G/D is not divisible or $r_0(G/D)\geq 2^\alpha$. In both cases D has to be a proper dense α-pseudocompact subgroup of G. Then G is not s_α-extremal. We prove that G is not r_α-extremal as well. Note that D is G_α-dense in G by Corollary 2.6. Let $\pi: G\to G/D$ be the canonical projection.

Now we build a surjective homomorphism $h: G/D\to H$, where H is compact $|H|>1$ and $\ker h$ is G_α-dense in G. By assumption D is a G_α-dense subgroup of G such that either G/D is not divisible or $r_0(G/D)\geq 2^\alpha$. In the first case G/D admits a non-trivial finite quotient H, while in the second case we can find a surjective homomorphism $G/D\to\mathbb{T}=H$ as $|\mathbb{T}|\leq r_0(G/D)$. Since $\ker h$ contains D in both cases, $\ker h$ is G_α-dense in G. Apply Lemma 4.1 and Theorem 4.3 to conclude that G is not r_α-extremal. □

The following proposition and lemma are the generalizations to the α-pseudocompact case of [9, Theorems 5.8 and 5.9] respectively. The ideas used in the proofs are similar. The next claim is needed in the proofs of both.

Claim 4.5. *Let $p \in \mathbb{P}$, let G be an abelian group of exponent p and $h : G \to \mathbb{Z}(p) \leq \mathbb{T}$ a continuous surjective homomorphism. Then Γ_h has index p in $G \times \mathbb{Z}(p)$.*

Proof. Consider $\xi : G \times \mathbb{Z}(p) \to \mathbb{Z}(p)$, defined by $\xi(g,y) = h(g) - y$ for all $(g,y) \in G \times \mathbb{Z}(p)$. Then ξ is surjective and $\ker\xi = \Gamma_h$. Therefore $G \times \mathbb{Z}(p)/\ker\xi = G \times \mathbb{Z}(p)/\Gamma_h$ is algebraically isomorphic to $\mathbb{Z}(p)$ and so they have the same cardinality p. □

The following proposition shows that for α-pseudocompact abelian groups of prime exponent s_α-extremality is equivalent to r_α-extremality.

Proposition 4.6. *Let α be an infinite cardinal and let (G,τ) be an α-pseudocompact abelian group of exponent $p \in \mathbb{P}$. Then the following conditions are equivalent:*
(a) *there exist an α-pseudocompact abelian group H with $|H| > 1$ and a homomorphism $h : G \to H$ such that Γ_h is G_α-dense in $(G,\tau) \times H$;*
(b) *(G,τ) is not s_α-extremal;*
(c) *(G,τ) is not r_α-extremal.*

Proof. (a)⇒(b) and (a)⇒(c) follows from Theorem 4.3.

(b)⇒(c) Suppose that (G,τ) is not s_α-extremal. Then there exists a proper dense α-pseudocompact subgroup D of (G,τ). We can suppose without loss of generality that D is maximal and so that $|G/D| = p$. Let τ' be the coarsest group topology such that $\tau' \supseteq \tau \cup \{x + D : x \in G\}$. Since $D \notin \tau$ but $D \in \tau'$, so $\tau' > \tau$. Since $(D, \tau' \restriction_D) = (D, \tau \restriction_D)$ and D is an α-pseudocompact subgroup of (G,τ), it follows that D is an α-pseudocompact subgroup of (G,τ'). Hence (G,τ') is α-pseudocompact by Lemma 2.7.

(c)⇒(a) Suppose that G is not r_α-extremal. Then there exists an α-pseudocompact group topology τ' on G such that $\tau' > \tau$. Since both topologies are precompact, there exists an homomorphism $h : G \to \mathbb{T}$ such that h is τ'-continuous but not τ-continuous. Note that $h(G) \neq \{0\}$. Being G of exponent p, so $h(G) = \mathbb{Z}(p) \leq \mathbb{T}$. Let

$$H = \mathbb{Z}(p).$$

Since h is not τ-continuous, by the closed graph theorem Γ_h is not closed in $(G,\tau) \times \mathbb{Z}(p)$. Moreover $|(G,\tau) \times \mathbb{Z}(p)/\Gamma_h| = p$ by Claim 4.5. Since Γ_h is a subgroup of index p in $(G,\tau) \times \mathbb{Z}(p)$ and it is not closed, then Γ_h is dense in $(G,\tau) \times \mathbb{Z}(p)$.

Endow G with the topology τ_h, that is the coarsest group topology on G such that $\tau_h \geq \tau$ and h is τ_h-continuous (see Remark 4.2). Then (G,τ_h) is α-pseudocompact, because h is τ'-continuous and so $\tau_h \leq \tau'$ and τ' is α-pseudocompact. By Remark 4.2 Γ_h is topologically isomorphic to (G,τ_h) and so Γ_h is α-pseudocompact. Since Γ_h is dense and α-pseudocompact in $(G,\tau) \times \mathbb{Z}(p)$, Corollary 2.6 yields that Γ_h is G_α-dense in $(G,\tau) \times \mathbb{Z}(p) = (G,\tau) \times H$. □

Lemma 4.7. *Let α be an infinite cardinal. Let (G,τ) be an α-pseudocompact abelian group of exponent $p \in \mathbb{P}$ such that G is either s_α- or r_α-extremal. Then every $h \in \mathrm{Hom}(G,\mathbb{T})$ is $P_\alpha\tau$-continuous (i.e., $\mathrm{Hom}(G,\mathbb{T}) \subseteq \widehat{(G,P_\alpha\tau)}$).*

Proof. If $h \equiv 0$, then h is $P_\alpha\tau$-continuous. Suppose that $h \not\equiv 0$. Then $h(G) = \mathbb{Z}(p) \leq \mathbb{T}$. Since G is either s_α- or r_α-extremal, by Proposition 4.6 Γ_h is not G_α-dense in $(G,\tau)\times\mathbb{Z}(p)$, that is Γ_h is not dense in $P_\alpha((G,\tau)\times\mathbb{Z}(p)) = P_\alpha(G,\tau)\times P_\alpha\mathbb{Z}(p) = (G,P_\alpha\tau)\times\mathbb{Z}(p)$. Claim 4.5 implies that $|G\times\mathbb{Z}(p)/\Gamma_h| = p$. Since Γ_h is not dense and is of prime index in $(G,P_\alpha\tau)\times\mathbb{Z}(p)$, it follows that Γ_h is closed in $(G,P_\alpha\tau)\times\mathbb{Z}(p)$. By the closed graph theorem h is $P_\alpha\tau$-continuous. □

5 The α-singular groups

5.1 The torsion abelian groups and extremality

The following definition and two lemmas are the generalization to the α-pseudocompact case of [9, Notation 5.10, Theorem 5.11 and Lemma 5.13]. The constructions are almost the same.

Definition 5.1. Let α be an infinite cardinal, let X be a topological space and $Y \subseteq X$. The *α-closure* of Y in X is $\alpha\text{-cl}_X(Y) = \bigcup\{\overline{Y}^X : A \subseteq Y,\ |A| \leq \alpha\}$.

For $Y \subseteq X$, the set $\alpha\text{-cl}_X(Y)$ is *α-closed* in X, i.e., $\alpha\text{-cl}_X(\alpha\text{-cl}_X(Y)) = \alpha\text{-cl}_X(Y)$.

Lemma 5.2. *Let α be an infinite cardinal. Let G be an α-pseudocompact abelian group and let $h \in \mathrm{Hom}(G,\mathbb{T})$. Then the following conditions are equivalent:*
(a) $h \in \alpha\text{-}cl_{\mathrm{Hom}(G,\mathbb{T})}\widehat{G}$*;*
(b) *there exists $N \in \Lambda_\alpha(G)$ such that $N \subseteq \ker h$.*

Proof. (a)⇒(b) Suppose that $h \in \alpha\text{-cl}_{\mathrm{Hom}(G,\mathbb{T})}\widehat{G}$. Let $A \subseteq \widehat{G}$ such that $|A| \leq \alpha$ and $h \in \overline{A}^{\mathrm{Hom}(G,\mathbb{T})}$. We set $N = \bigcap\{\ker f : f \in A\}$. Then $N \in \Lambda_\alpha(G)$. Moreover $N \subseteq \ker h$ as $h \in \overline{A}^{\mathrm{Hom}(G,\mathbb{T})}$.

(b)⇒(a) Let $N \in \Lambda_\alpha(G)$ and let $\pi : G \to G/N$ be the canonical projection. The group G/N is compact of weight $\leq \alpha$ and so $|\widehat{G/N}| = w(G/N) \leq \alpha$ by Fact 2.1(c). We enumerate the elements of $\widehat{G/N}$ as $\widehat{G/N} = \{\chi_\lambda : \lambda < \alpha\}$ and define $A = \{\chi_\lambda \circ \pi : \lambda < \alpha\} \leq \widehat{G}$. We prove that $h \in \overline{A}^{\mathrm{Hom}(G,\mathbb{T})}$. Suppose that $h \notin \overline{A}^{\mathrm{Hom}(G,\mathbb{T})}$. Since A is a closed subgroup of the compact group $\mathrm{Hom}(G,\mathbb{T})$, there exists $\xi \in \widehat{\mathrm{Hom}(G,\mathbb{T})}$ such that $\xi(h) \neq 0$ and $\xi(f) = 0$ for every $f \in A$. By the Pontryagin duality there exists $x \in G$ such that $f(x) = \xi(f)$ for every $f \in \widehat{G}$. Then $\chi_\lambda(\pi(x)) = \xi(\chi_\lambda \circ \pi) = 0$ for every $\chi_\lambda \in \widehat{G/N}$. Then $\pi(x) = x + N = N$ and so $x \in N \subseteq \ker h$, i.e., $h(x) = 0$. But $h(x) = \xi(h) \neq 0$, a contradiction. □

Lemma 5.3. *Let α be an infinite cardinal. Let (G,τ) be an α-pseudocompact abelian group of exponent $p \in \mathbb{P}$ such that G is either s_α- or r_α-extremal and $|G| = \beta \geq \alpha$. Then for the completion K of (G,τ):*
(a) *for every $h \in \mathrm{Hom}(G,\mathbb{T})$ there exists $h' \in Hom(K,\mathbb{T})$ such that $h'\restriction_G = h$ and h' is $P_\alpha K$-continuous;*
(b) $\mathrm{Hom}(G,\mathbb{T}) \subseteq \alpha\text{-}cl_{\mathrm{Hom}(G,\mathbb{T})}\widehat{(G,\tau)}$*;*
(c) $\psi(G,\tau) \leq \alpha\cdot\log\beta$.

Proof. (a) Since G is G_α-dense in K by Theorem 2.5, it follows that G is dense in $P_\alpha K$. By Lemma 4.7 every $h \in \mathrm{Hom}(G,\mathbb{T})$ is $P_\alpha\tau$-continuous. Therefore h can be extended to $h' \in \mathrm{Hom}(K,\mathbb{T})$, such that h' is $P_\alpha K$-continuous, because $(G, P_\alpha\tau)$ is dense in $P_\alpha K$.

(b) Since $h \in \mathrm{Hom}(G,\mathbb{T})$, by (a) there exists $h' \in \mathrm{Hom}(K,\mathbb{T})$ such that $h' \restriction_G = h$ and h' is $P_\alpha K$-continuous. By Lemma 5.2 $h' \in \alpha\text{-cl}_{\mathrm{Hom}(K,\mathbb{T})}\widehat{K}$. Therefore $h \in \alpha\text{-cl}_{\mathrm{Hom}(G,\mathbb{T})}\widehat{(G,\tau)}$. Indeed, let $A' \subseteq \widehat{K}$ be such that $|A'| \leq \alpha$ and $h' \in \overline{A'}^{\mathrm{Hom}(K,\mathbb{T})}$. For $f' \in A'$ we set $f = f' \restriction_G \in \widehat{(G,\tau)}$ and $A = \{f' \restriction_G : f' \in A'\}$. There exists a net $\{f'_\lambda\}_\lambda$ in A' such that $f'_\lambda \to h'$ in $\mathrm{Hom}(K,\mathbb{T})$; since the topology on $\mathrm{Hom}(K,\mathbb{T})$ is the point-wise convergence topology, this means that $f'_\lambda(x) \to h'(x)$ for every $x \in K$. Then $f_\lambda(x) \to h(x)$ for every $x \in G$. Hence $f_\lambda \to h$ in $\mathrm{Hom}(G,\mathbb{T})$ and so $h \in \overline{A}^{\mathrm{Hom}(G,\mathbb{T})} \subseteq \alpha\text{-cl}_{\mathrm{Hom}(G,\mathbb{T})}\widehat{(G,\tau)}$.

(c) By Fact 2.1(a),(c) $d(\mathrm{Hom}(G,\mathbb{T})) = \log w(\mathrm{Hom}(G,\mathbb{T})) \leq \log\beta$. Then there exists a dense subset S of $\mathrm{Hom}(G,\mathbb{T})$ such that $|S| \leq \log\beta$. By (b) for every $h \in S$ there exists $A(h) \subseteq \widehat{(G,\tau)}$ such that $|A(h)| \leq \alpha$ and $h \in \overline{A(h)}^{\mathrm{Hom}(G,\mathbb{T})}$. Then $A := \bigcup\{A(h) : h \in S\}$ is dense in $\mathrm{Hom}(G,\mathbb{T})$, because $S \subseteq \overline{A}^{\mathrm{Hom}(G,\mathbb{T})}$. Moreover $A \subseteq \widehat{(G,\tau)}$ and $|A| \leq \alpha \cdot \log\beta$. Let $x \in G \setminus \{0\}$ and let $\{V_n : n \in \mathbb{N}\}$ be a local base at 0 of $\mathbb{T}$. Since A is dense in $\mathrm{Hom}(G,\mathbb{T})$, it separates the points of G and so there exists $f \in A$ such that $f(x) \neq 0$. Then there exists $n \in \mathbb{N}$ such that $f(x) \notin V_n$. Therefore $\bigcap_{n\in\mathbb{N}, f\in A} f^{-1}(V_n) = \{0\}$ and hence $\psi(G,\tau) \leq |A| \leq \alpha \cdot \log\beta$. □

Now we prove Theorem 1.4 in the torsion case. For $\alpha = \omega$ it implies [9, Corollary 7.5] which proof inspires our proof. Since every torsion α-pseudocompact group is c_α-extremal, we observe that a torsion α-pseudocompact abelian group is α-extremal if and only if it is d_α-extremal.

Theorem 5.4. *Let α be an infinite cardinal and let G be an α-pseudocompact torsion abelian group. Then G is α-extremal if and only if $w(G) \leq \alpha$.*

Proof. If $w(G) \leq \alpha$, then G is α-extremal by Proposition 2.9 and Theorem 4.4.

Suppose that $w(G) > \alpha$. We prove that there exists $p \in \mathbb{P}$ such that $w(G/\overline{pG}) > \alpha$. Since G is torsion, then it is bounded-torsion by [9, Lemma 7.4]. Therefore $K = \widetilde{G}$ is bounded-torsion. Consequently K is topologically isomorphic to $\prod_{p\in\mathbb{P}} t_p(K)$, where $t_p(K) = \{x \in K : p^n x = 0 \text{ for some } n \in \mathbb{N}_+\}$. Since $w(K) = \max_{p\in\mathbb{P}} w(t_p(K))$ and $w(K) = w(G) > \alpha$, there exists $p \in \mathbb{P}$ such that $w(t_p(K)) > \alpha$. Moreover for this p we have $w(t_p(K)) = w(K_{(p)})$, where $K_{(p)} = t_p(K)/pt_p(K)$, by [16, Lemma 4.1(b)]. Consider the composition φ_p of the canonical projections $K \to t_p(K)$ and $t_p(K) \to K_{(p)}$. Since G is dense in K, it follows that $\varphi_p(G)$ is dense in $K_{(p)}$ and so $w(\varphi_p(G)) = w(K_{(p)}) > \alpha$. Moreover there exists a continuous isomorphism $G/(\ker\varphi_p \cap G) \to \varphi_p(G)$. Since $\ker\varphi_p = pK$ and $pK \cap G = \overline{pG}$, there exists a continuous isomorphism $G/\overline{pG} \to \varphi_p(G)$. Hence $w(G/\overline{pG}) \geq w(\varphi_p(G)) > \alpha$.

Let $G_1 = G/\overline{pG}$. We prove that G_1 is not s_α-extremal. Suppose for a contradiction that G_1 is s_α-extremal. By Lemma 4.7 every $h \in \mathrm{Hom}(G_1,\mathbb{T})$ is $P_\alpha\tau$-continuous, and so $P_\alpha G_1 = P_\alpha G_1^\#$ by Lemma 2.19. By Theorem 2.18 $P_\alpha G_1$ is Baire, hence $|G_1| \leq 2^\alpha$ by Theorem 2.20. By Lemma 5.3(c) $\psi(G_1) \leq \alpha \cdot \log 2^\alpha = \alpha$ and so Lemma 2.8 implies $w(G_1) = \psi(G_1) \leq \alpha$; this contradicts our assumption.

Then there exists a proper dense α-pseudocompact subgroup D of G_1. By Corollary 2.6 D is G_α-dense in G_1. Let $\pi : G \to G_1$ be the canonical projection. By Lemma 2.16(b) $\pi^{-1}(D)$ is a proper G_α-dense subgroup of G, then dense α-pseudocompact in G by Corollary 2.6. Since $G/\pi^{-1}(D)$ is algebraically isomorphic to G_1/D, it follows that $G/\pi^{-1}(D)$ is of exponent p and hence not divisible. Therefore G is not d_α-extremal and so not α-extremal. □

The next lemma gives some conditions equivalent to α-singularity for α-pseudocompact abelian groups. For $\alpha = \omega$ we find [25, Lemma 2.5], which generalized [18, Lemma 4.1].

Lemma 5.5. *Let α be an infinite cardinal and let G be an α-pseudocompact abelian group. Then the following conditions are equivalent:*
(a) *G is α-singular;*
(b) *there exists $m \in \mathbb{N}_+$ such that $G[m] \in \Lambda_\alpha(G)$;*
(c) *G has a torsion closed G_α-subgroup;*
(d) *there exists $N \in \Lambda_\alpha(G)$ such that $N \subseteq t(G)$;*
(e) *$\widetilde{G}$ is α-singular.*

Proof. Let $m \in \mathbb{N}_+$ and let $\varphi_m : G \to G$ be the continuous homomorphism defined by $\varphi_m(x) = mx$ for every $x \in G$. Then $\ker \varphi_m = G[m]$ and $\varphi_m(G) = mG$. Let $i : G/G[m] \to mG$ be the continuous isomorphism such that $i \circ \pi = \varphi_m$, where $\pi : G \to G/G[m]$ is the canonical homomorphism.

(a)⇒(b) There exists $m \in \mathbb{N}_+$ such that $w(mG) \leq \alpha$. Then $\psi(mG) \leq \alpha$. Since $i : G/G[m] \to mG$ is a continuous isomorphism, so $\psi(G/G[m]) \leq \alpha$. This implies that $G[m]$ is a G_α-set of G; then $G[m] \in \Lambda_\alpha(G)$.

(b)⇒(a) Suppose that $G[m] \in \Lambda_\alpha(G)$. Then the quotient $G/G[m]$ has weight $\leq \alpha$, hence it is compact. By the open mapping theorem the isomorphism $i : G/G[m] \to mG$ is also open and consequently it is a topological isomorphism. Then $w(mG) \leq \alpha$, that is G is α-singular.

(b)⇒(c) and (c)⇔(d) are obvious.

(d)⇒(b) By Corollary 2.14 N is α-pseudocompact and so N is bounded-torsion by [9, Lemma 7.4]. Therefore there exists $m \in \mathbb{N}_+$ such that $mN = \{0\}$. Thus $N \subseteq G[m]$ and so $G[m] \in \Lambda_\alpha(G)$ by Corollary 2.14(a).

(a)⇔(e) It suffices to note that $w(mG) = w(m\widetilde{G})$ for every $m \in \mathbb{N}$, because mG is dense in $m\widetilde{G}$ for every $m \in \mathbb{N}$. □

The following lemma proves one implication of Corollary 1.6 and it is the generalization of [18, Proposition 4.7].

Lemma 5.6. *Let α be an infinite cardinal and let G be an α-singular α-pseudocompact abelian group. Then $r_0(G/D) = 0$ for every G_α-dense subgroup D of G. In particular G is c_α-extremal.*

Proof. By definition there exists a positive integer m such that $w(mG) \leq \alpha$. Let D be a G_α-dense subgroup of G. Since mD is a G_α-dense subgroup of mG and $w(mG) \leq \alpha$, so $mD = mG$. Therefore $mG \leq D$ and hence the quotient G/D is bounded-torsion. In particular $r_0(G/D) = 0$.

Moreover G is c_α-extremal by the previous part, noting that a G_α-dense subgroup D of G is dense α-pseudocompact by Corollary 2.6. □

For a product $G = \prod_{i \in I} G_i$ of topological groups, let $\Sigma_\alpha G = \{x = (x_i) \in G : |\mathrm{supp}(x)| \leq \alpha\}$ be the Σ_α*-product* of G [17]. Then $\Sigma_\alpha G$ is G_α-dense in G.

Proof of Theorem 1.5. (a)⇒(c) Suppose that there exists a continuous surjective homomorphism φ of K onto S^β, where S is a metrizable compact non-torsion abelian group and $\beta > \alpha$. Then φ is also open. Note that $S^\beta = (S^\alpha)^\beta$. Let $T = S^\alpha$. Then $D = \Sigma_\alpha T^\beta$ is a G_α-dense subgroup of T^β, so dense α-pseudocompact by Corollary 2.6, and D trivially intersects the diagonal subgroup $\Delta T^\beta = \{x = (x_i) \in T^\beta : x_i = x_j \text{ for every } i, j < \beta\}$, which is topologically isomorphic to $T = S^\alpha$. Consequently $r_0(T^\beta/D) \geq 2^\alpha$. Therefore T^β is not c_α-extremal and K is not c_α-extremal as well by Proposition 2.17.

(b)⇒(a) is Lemma 5.6 and (b)⇔(c) is [17, Theorem 1.6]. □

5.2 Proof of the main theorem

Proposition 5.7. *Let α be an infinite cardinal. If G is a non-α-singular α-pseudocompact abelian group, then $r_0(N) = r_0(G) \geq 2^\alpha$ for every $N \in \Lambda_\alpha(G)$.*

Proof. First we prove that $r_0(G) \geq 2^\alpha$. Since G is non-α-singular, then $K = \widetilde{G}$ is non-α-singular as well by Lemma 5.5. By Theorem 1.5 there exists a continuous surjective homomorphism $\varphi : K \to S^I$, where S is a metrizable compact non-torsion abelian group and $|I| > \alpha$. Let J be a subset of I of cardinality α and consider the composition ϕ of φ with the canonical projection $S^I \to S^J$. The group S^J has weight α and free rank 2^α. Since G is dense in K, it follows that $\phi(G)$ is dense in S^J. But G is α-pseudocompact and so $\phi(G)$ is compact. Therefore $\phi(G) = S^J$ and hence $r_0(G) \geq 2^\alpha$.

Let $N \in \Lambda_\alpha(G)$. Then N is α-pseudocompact by Corollary 2.14. Since G is non-α-singular, so N is non-α-singular. In fact, by Lemma 5.5 if $m \in \mathbb{N}_+$ then $G[m] \notin \Lambda_\alpha(G)$. By Corollary 2.14(a) if $m \in \mathbb{N}_+$ then $N[m] \notin \Lambda_\alpha(G)$, so $N[m] \notin \Lambda_\alpha(N)$ by Corollary 2.14(b) and hence N is non-α-singular by Lemma 5.5. Then $r_0(N) \geq 2^\alpha$ by the first part of the proof. Moreover $r_0(G) = r_0(G/N) \cdot r_0(N)$. By Theorem 2.13 $w(G/N) \leq \alpha$ and so G/N is compact with $|G/N| \leq 2^\alpha$. Hence $r_0(G) = r_0(N)$. □

The next lemma shows that c_α-extremality is hereditary for α-pseudocompact subgroups that are sufficiently large. This is the generalization to α-pseudocompact groups of [18, Theorem 4.11].

Lemma 5.8. *Let α be an infinite cardinal and let G be a c_α-extremal pseudocompact abelian group. Then every α-pseudocompact subgroup of G of index $\leq 2^\alpha$ is c_α-extremal. In particular, every $N \in \Lambda_\alpha(G)$ is c_α-extremal.*

Proof. Aiming for a contradiction, assume that there exists an α-pseudocompact subgroup N of G with $|G/N| \leq 2^\alpha$ such that N is not c_α-extremal. Then there exists a dense α-pseudocompact subgroup D of N with $r_0(N/D) \geq 2^\alpha$. Therefore $|G/N| \leq r_0(N/D)$. By [18, Corollary 4.9] there exists a subgroup L of G/D such that $L + N/D = G/D$ and $r_0((G/D)/L) \geq r_0(N/D)$. Let $\pi : G \to G/D$ be the canonical projection and $D_1 = \pi^{-1}(L)$. Then $N + D_1 = G$. Since D is G_α-dense in N by Corollary 2.6, so $\overline{D_1}^{P_\alpha G} \supseteq N + D_1 = G$ and so D_1 is G_α-dense in G; equivalently D_1 is dense α-pseudocompact in G by Corollary

2.6. Since $G/\pi^{-1}(L)$ is algebraically isomorphic to $(G/D)/L$, it follows that $r_0(G/D_1) = r_0((G/D)/L) \geq r_0(N/D) \geq 2^\alpha$. We have produced a G_α-dense subgroup D_1 of G with $r_0(G/D_1) \geq 2^\alpha$, a contradiction.

If $N \in \Lambda_\alpha(G)$ then $w(G/N) \leq \alpha$ and so G/N is compact. Hence $|G/N| \leq 2^\alpha$. So N is c_α-extremal by the previous part of the proof. □

Lemma 5.9 ([13, Lemma 3.2]). *Let α be an infinite cardinal and suppose that $\mathcal{A}$ is a family of subsets of 2^α such that:*
(1) *for $\mathcal{B} \subseteq \mathcal{A}$ such that $|\mathcal{B}| \leq \alpha$, $\bigcap_{B\in\mathcal{B}} B \in \mathcal{A}$ and*
(2) *each element of $\mathcal{A}$ has cardinality 2^α.*
Then there exists a countable infinite family $\mathcal{B}$ of subsets of 2^α such that:
(a) *$B_1 \cap B_2 = \emptyset$ for every $B_1, B_2 \in \mathcal{B}$ and*
(b) *if $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then $|A \cap B| = 2^\alpha$.*

Now we can prove our main results.

Proof of Corollary 1.6. If G is α-singular then G is c_α-extremal by Lemma 5.6.

Suppose that G is c_α-extremal and assume for a contradiction that G is not α-singular. By Theorem 3.4 $r_0(G) \leq 2^\alpha$ and by Proposition 5.7 $r_0(G) \geq 2^\alpha$. Hence $r_0(G) = 2^\alpha$. Let $D(G_1) = \mathbb{Q}^{(S)}$, with $|S| = 2^\alpha$, be the divisible hull of the torsion-free quotient $G_1 = G/t(G)$, where $t(G)$ denotes the subgroup of all torsion elements of G. Let $\pi : G \to D(G_1)$ be the composition of the canonical projection $G \to G_1$ and the inclusion map $G_1 \hookrightarrow D(G_1)$.

For a subset A of S let

$$G(A) = \pi^{-1}\left(\mathbb{Q}^{(A)}\right) \quad \text{and} \quad \mathcal{A} = \{A \subseteq S : G(A) \text{ contains some } N \in \Lambda_\alpha(G)\}.$$

Then $\mathcal{A}$ has the property that for $\mathcal{B} \subseteq \mathcal{A}$ such that $|\mathcal{B}| \leq \alpha$, $\bigcap_{B\in\mathcal{B}} B \in \mathcal{A}$; and $|A| = 2^\alpha$ for all $A \in \mathcal{A}$, as $r_0(N) = 2^\alpha$ for every $N \in \Lambda_\alpha(G)$ by Proposition 5.7 and $r_0(G) = 2^\alpha$. By Lemma 5.9 there exists a partition $\{P_n\}_{n\in\mathbb{N}}$ of S such that $|A \cap P_n| = 2^\alpha$ for every $A \in \mathcal{A}$ and for every $n \in \mathbb{N}$. Define $V_n = G(P_0 \cup \ldots \cup P_n)$ for every $n \in \mathbb{N}$ and note that $G = \bigcup_{n\in\mathbb{N}} V_n$. By Lemma 3.1 there exist $m \in \mathbb{N}$ and $N \in \Lambda(G)$ such that $D = V_m \cap N$ is G_α-dense in N, so dense α-pseudocompact in N by Corollary 2.6. By Lemma 5.8, to get a contradiction it suffices to show that $r_0(N/D) = 2^\alpha$.

Let F be a torsion-free subgroup of N such that $F \cap D = \{0\}$ and maximal with this property. Suppose for a contradiction that $|F| = r_0(N/D) < 2^\alpha$. So $\pi(F) \subseteq \mathbb{Q}^{(S_1)}$ for some $S_1 \subseteq S$ with $|S_1| < 2^\alpha$ and $W = P_0 \cup \ldots \cup P_m \cup S_1$ has $|W \cap P_{m+1}| < 2^\alpha$. Consequently $W \notin \mathcal{A}$ and so $N \not\subseteq G(W)$. Note that $G/G(W)$ is torsion-free, because if $x \in G$ and $mx \in G(W)$ for some $m \in \mathbb{N}_+$, then $m\pi(x) = \pi(mx) \in \mathbb{Q}^{(W)}$, so $\pi(x) \in \mathbb{Q}^{(W)}$ and hence $x \in G(W)$. Take $x \in N \setminus G(W)$. Since $G/G(W)$ is torsion-free, $\langle x\rangle \cap G(W) = \{0\}$ and x has infinite order. But $D + F \subseteq G(W)$ and so $\langle x\rangle \cap (D + F) = \{0\}$, that is $(F + \langle x\rangle) \cap D = \{0\}$. This contradicts the maximality of F. □

Proof of Theorem 1.4. (a)⇒(c) If G is α-extremal, in particular it is c_α-extremal and so α-singular by Corollary 1.6.

Suppose that $w(G) > \alpha$. Since G is α-singular, there exists $m \in \mathbb{N}_+$ such that $w(mG) \leq \alpha$; in particular mG is compact and so closed in G. Since $w(mG) \leq \alpha$ and $w(G) = w(G/mG) \cdot$

$w(mG)$, it follows that $w(G/mG) = w(G) > \alpha$. Then G/mG is not α-extremal by Theorem 5.4 and so G is not α-extremal by Proposition 2.17.

(c)⇒(b) is Proposition 2.9 and (b)⇒(a) is Theorem 4.4. □

The following example shows that c_α- and d_α-extremality cannot be equivalent conditions in Theorem 1.4. Item (a) shows that α-singular α-pseudocompact abelian groups need not be d_α-extremal and also that there exists a c_α-extremal (non-compact) α-pseudocompact abelian group of weight $> \alpha$, which is not d_α-extremal. It is the analogous of [18, Example 4.4]. In item (b) we give an example of a non-c_α-extremal d_α-extremal α-pseudocompact abelian group of weight $> \alpha$.

Example 5.10. Let α be an infinite cardinal.
(a) Let $p \in \mathbb{P}$ and let H be the subgroup of $\mathbb{Z}(p)^{2^\alpha}$ defined by $H = \Sigma_\alpha \mathbb{Z}(p)^{2^\alpha}$. Then H is α-pseudocompact by Corollary 2.6, because it is G_α-dense in $\mathbb{Z}(p)^{2^\alpha}$. The group $G = \mathbb{T}^\alpha \times H$ is an α-singular (so c_α-extremal by Lemma 5.6) α-pseudocompact abelian group with $r_0(G) = 2^\alpha$ and $w(G) = 2^\alpha > \alpha$. Thus G is not d_α-extremal by Theorem 1.4.
(b) Let $G = \mathbb{T}^{2^\alpha}$. Then G is an α-pseudocompact divisible abelian group G of weight $> \alpha$. So G is d_α-extremal of weight $> \alpha$. We can prove that G is not c_α-extremal as in the proof of Theorem 1.5, that is, noting that $G = (\mathbb{T}^\alpha)^{2^\alpha} = T^{2^\alpha}$, where $T = \mathbb{T}^\alpha$, and $\Sigma_\alpha T^{2^\alpha}$ is a G_α-dense (so dense α-pseudocompact by Corollary 2.6) subgroup of T^{2^α} such that $r_0(T^{2^\alpha}/\Sigma T^{2^\alpha}) \geq 2^\alpha$. That G is not c_α-extremal follows also from Theorem 3.4, because G has free rank $> 2^\alpha$.

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