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Interval temporal logics provide a natural framework for qualitative and quantitative temporal reasoning over interval structures, where the truth of formulae is defined over intervals rather than points. In this paper, we study the complexity of the satisfiability problem for Metric Propositional Neighborhood Logic (MPNL). MPNL features two modalities to access intervals "to the left" and "to the right" of the current one, respectively, plus an infinite set of length constraints. MPNL has been recently shown to be decidable by a doubly exponential procedure. We improve such a result by proving that MPNL is actually EXPSPACE-complete (even when length constraints are encoded in binary), when interpreted over finite structures, the naturals, and the integers. Moreover, we develop an optimal decision procedure for MPNL over the integers, which can be easily tailored to the cases of finite linear orders and of the naturals.

1 Introduction

Interval temporal logics provide a natural framework for temporal representation and reasoning about interval structures over linearly (or partially) ordered domains. They take time intervals as the primitive ontological entities and define truth of formulae with respect to them instead of to time instants. The modal operators of an temporal logic correspond to binary relations between pairs of intervals (in fact, an interval logic of ternary interval relations has been developed by Venema in [15]). A special role in the interval logic setting is commonly accorded to Halpern and Shoham's modal logic of time intervals, abbreviated HS, whose modalities make it possible to express all Allen's (binary) interval relations [1].

Interval-based formalisms have been extensively used in various areas of computer science and artificial intelligence, including hardware specification and verification, constraint processing, planning and plan validation, theories of action and change, and natural language understanding. However, in many applications, severe syntactic and semantic restrictions have been imposed that considerably weaken their expressive power. Interval temporal logics relax these restrictions, thus allowing one to express much more complex temporal properties. Unfortunately, most of them, including HS and the majority of its fragments, turn out to be undecidable (a comprehensive survey on interval logics can be found in [11]; an up-to-date picture of decidability and undecidability results about them is given in [9, 13]).

One of the few cases of a decidable interval logic with genuine interval semantics, that is, not reducible to point-based semantics, is the propositional logic of temporal neighborhood (Propositional Neighborhood Logic, PNL for short), interpreted over various classes of temporal structures, including all, dense, discrete, and finite linear orders, as well as rational, integer, and natural numbers [10]. PNL is the fragment of HS featuring two modalities corresponding to Allen's relations *meets* and *met by* (the

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one is the inverse of the other). Decidability of PNL with respect to various classes of linear orders has been proved in [3] via a reduction to the satisfiability problem for the two-variable fragment of first-order logic for binary relational structures over ordered domains [12]. Decidability of PNL with respect to other classes of linear orders via a model-theoretic direct argument has been recently shown in [7], where tableau-based optimal decision procedures for PNL, interpreted in the considered classes of linear orders, have also been developed.

Despite its seeming simplicity, PNL is well-suited for a number of concrete application domains. One of them is that of transaction-time databases (also called append-only databases), that keep track of the sequence of timestamped versions of the database, where information is never removed and new information is appended to existing information, respecting the temporal ordering. However, in such an application domain as well in various other ones, a metric dimension turns out to be a necessary ingredient. A metric extension of PNL has been developed by Bresolin et al. in [2]. The resulting logic, called Metric PNL (MPNL for short), pairs PNL modalities with a family of special atomic propositions expressing integer constraints (equalities and inequalities) on the length of the intervals over which they are evaluated. The authors show that the satisfiability problem for MPNL, interpreted over natural numbers, is decidable. However, they leave the precise characterization of its complexity as an open problem. Metric constraints in MPNL are expressed in terms of some $k \in \mathbb{N}$. When k is a constant of the formula or it is expressed in unary, MPNL is NEXPTIME-complete, but when k is expressed in binary, then the satisfiability problem for MPNL has been shown to be somewhere in between EXPSPACE and 2NEXPTIME only.

In this paper, we focus our attention on MPNL. We first provide an original model-theoretic proof of the decidability of the satisfiability problem for MPNL, with a binary encoding of metric constraints, over finite linear orders, natural numbers, and integer numbers. As a matter of fact, the proof gives us a doubly-exponential upper bound to the size of the (pseudo-)model for the input MPNL formula (if any), when interpreted in the linear orders under consideration. Then, we devise an EXPSPACE decision procedure for MPNL, interpreted over the integer numbers, and we show how to adapt it to the cases of finite linear orders and natural numbers. EXPSPACE-completeness immediately follows from the already known EXPSPACE-hardness of the problem. As a by-product, we solve the issue about the exact complexity of MPNL, interpreted over the natural numbers, when k is expressed in binary, which was left open in [2]. Moreover, since MPNL is expressively complete for a fragment of first-order logic with two variables and one successor function, interpreted over the same classes of linear orders [2], the proposed decision procedure can be used to check the satisfiability of formulae of such a logic as well.

The paper is organized as follows. In Section 2, we introduce the logic. Then, in Section 3, we provide some basic definitions and results to be used in the following. In Section 4, we prove the decidability of the satisfiability problem for MPNL over finite linear orders. In the following two sections, we generalize such a result to the cases of natural and integer numbers by showing that every satisfiable formula has a model that can be represented with a suitable small "generator". Finally, in Section 7, we outline an EXPSPACE decision procedure for satisfiability checking in the most general case of integer numbers, which can be easily tailored to the cases of finite linear orders and natural numbers.

2 The logic MPNL

The logic MPNL can be viewed as a natural metric extension of PNL. The language of PNL consists of a set $\mathcal{AP}$ of atomic propositions, the propositional connectives $\neg$ and $\vee$, and the modal operators $\Diamond_r$ and $\Diamond_l$ for Allen's relations *meets* and *met by*, respectively [1]. Representation theorems, axiomatic systems, and decidability results for PNL, interpreted over various classes of linear orders, have been given in [3, 10]. An optimal tableau-based method for deciding the satisfiability problem for the future fragment of PNL (RPNL) over the natural numbers has been presented in [8], and later extended to the full PNL over the integers in [5], while an optimal tableau system for RPNL over the class of all linear orders can be found in [6]. Optimal tableau-based decision procedures for PNL, interpreted over various classes of linear orders, can be found in [7].

An extension of PNL, interpreted over the natural numbers, with (a limited set of) metric constraints has been defined and systematically studied in [2] (as a matter of fact, a metric extension of RPNL was first considered in [4]). Let δ be the *distance* function over natural numbers defined as $\delta(x,y) = |x-y|$ (the same definition applies to any finite linear order and to the integers). Metric PNL (MPNL) is obtained from PNL by adding a set of (pre-interpreted) atomic propositions for length constraints. These propositions allow one to constrain the length of the current interval and can be viewed as the natural metric generalization of the modal constant π of propositional interval logics [10], which evaluates to true precisely over point-intervals. Formally, for each $\sim \in \{<, \leqslant, =, \geqslant, >\}$, MPNL features a length constraint $\mathsf{len}_{\sim k}$, whose semantics is defined as follows: $M,[x,y] \Vdash \mathsf{len}_{\sim k}$ iff $\delta(x,y) \sim k$. Hereafter, we limit ourselves to one type of metric constraints only, namely, $\mathsf{len}_{<k}$, as all the remaining ones can be expressed in terms of it. As an example, we have that $M,[x,y] \Vdash \mathsf{len}_{=k} \Leftrightarrow M,[x,y] \Vdash \mathsf{len}_{<k+1} \wedge \neg \mathsf{len}_{<k}$. Formulae of MPNL (denoted by $\varphi, \psi, \ldots$) are generated by the following grammar:

$$\varphi ::= \mathsf{len}_{<k} \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \Diamond_l \varphi \mid \Diamond_r \varphi, \text{ where } p \in \mathcal{AP} \text{ and } k \in \mathbb{N}.$$

The other propositional connectives, the logical constants $\top$ ($\mathtt{true}$) and $\bot$ ($\mathtt{false}$), and the dual modal operators $\Box_r$ and $\Box_l$ are defined as usual. Moreover, the modal constant π can be defined as $\mathsf{len}_{<1}$.

Given a linearly-ordered domain $\mathbb{D} = \langle D, < \rangle$, a (*non-strict*) *interval* over $\mathbb{D}$ is an ordered pair $[x,y]$, with $x \leqslant y$. We denote by $\mathbb{I}(\mathbb{D})$ the set of all intervals over $\mathbb{D}$. Moreover, we denote by y_{max} the greatest point in D (if there is not such a point, we put $y_{max} = +\infty$) and by y_{min} the least point in D (if there is not such a point, we put $y_{min} = -\infty$). The semantics of MPNL is given in terms of *models* of the form $M = \langle \mathbb{D}, V \rangle$, where $V : \mathcal{AP} \to 2^{\mathbb{I}(\mathbb{D})}$ is a valuation function assigning a set of intervals to every atomic proposition. From now on, we assume the domain D to be either $\mathbb{Z}$, $\mathbb{N}$, or a finite prefix of $\mathbb{N}$. We recursively define the truth relation $\Vdash$ as follows:

- $M,[x,y] \Vdash p$ iff $[x,y] \in V(p)$, for any $p \in \mathcal{AP}$;
- $M,[x,y] \Vdash len_{<k}$ iff $\delta(x,y) < k$;
- $M,[x,y] \Vdash \neg\varphi$ iff it is not the case that $M,[x,y] \Vdash \varphi$;
- $M,[x,y] \Vdash \varphi \vee \psi$ iff $M,[x,y] \Vdash \varphi$ or $M,[x,y] \Vdash \psi$;
- $M,[x,y] \Vdash \Diamond_l \varphi$ iff there exists $z \leqslant x$ such that $M,[z,x] \Vdash \varphi$;
- $M,[x,y] \Vdash \Diamond_r \varphi$ iff there exists $z \geqslant y$ such that $M,[y,z] \Vdash \varphi$.

Any MPNL-formula φ is said to be *satisfiable* if there exists a model M and an interval $[x,y]$ on it such that $M,[x,y] \Vdash \varphi$.

In [2], the satisfiability problem for MPNL has been shown to be decidable when interpreted over the set of natural numbers. More precisely, it has been shown that the satisfiability problem for MPNL over the set of natural numbers is NEXPTIME-complete when either the maximal k that occurs in metric constraints is a constant or the parameter k of metric constraints is represented in unary, and it is in between EXPSPACE and 2NEXPTIME when the parameter k is represented in binary. In the following, by a model-theoretic argument, we will show that the satisfiability problem for MPNL over finite linear orders, the natural numbers, and the integer numbers, with a binary representation of the parameter(s) k of metric constraints, is actually EXPSPACE-complete, and we develop an optimal decision procedure for it. It is worth noticing that the model-theoretic argument behaves, in a way, worse than the one in [2], as it provides a doubly-exponential upper bound on the size of (pseudo-)models regardless of the representation of k. Nevertheless, we will show that in the search for a model of a given formula, at any time, we need to keep track of a portion of the model that can be recorded in exponential space, thus leading to an EXPSPACE decision procedure.

3 Atoms, types, dependencies, and compass structures

In this section, we introduce the basic machinery to be used in the following sections. Let $M = \langle \mathbb{D}, V \rangle$ be a model for an MPNL-formula φ. In the sequel, we relate every interval in M to the set of sub-formulae of φ it satisfies. To do that, we introduce the key notions of φ-*atom* and φ-*type*. First of all, we define the *closure* $\mathcal{C}l(\varphi)$ of φ as the set of all sub-formulae of φ and of their negations (we identify $\neg\neg\alpha$ with α, $\neg\Diamond_r\alpha$ with $\Box_r\neg\alpha$, and so on), and we define $\mathcal{K}_\varphi = \{k \mid \mathtt{len}_{<k} \in \mathcal{C}l(\varphi)\}$ as the set of all metric parameters that appear in φ.

Definition 1. *A* φ-atom *is any non-empty set* $F \subseteq \mathcal{C}l(\varphi)$ *such that:*

1. *for every* $\alpha \in \mathcal{C}l(\varphi)$*, we have* $\alpha \in F$ *iff* $\neg\alpha \notin F$*,*
2. *for every* $\gamma = \alpha \vee \beta \in \mathcal{C}l(\varphi)$*, we have* $\gamma \in F$ *iff* $\alpha \in F$ *or* $\beta \in F$*, and*
3. *for every* k, k' *in* $\mathcal{K}_\varphi$ *such that* $k < k'$*, we have that* $\mathtt{len}_{<k} \in A$ *implies* $\mathtt{len}_{<k'} \in A$.

Intuitively, a φ-*atom* is a maximal *locally consistent* set of formulas chosen from $\mathcal{C}l(\varphi)$. Note that the cardinality of $\mathcal{C}l(\varphi)$ is *linear* in the length $|\varphi|$ of φ, while the number of φ-atoms is *at most exponential* in $|\varphi|$ (precisely, we have that $|\mathcal{C}l(\varphi)|$ is at most $2|\varphi|$ and there are at most $2^{|\varphi|}$ distinct atoms). We define $\mathcal{A}_\varphi$ as the set of all possible atoms that can be built over $\mathcal{C}l(\varphi)$. For every model M and every interval $[x,y] \in \mathbb{I}(\mathbb{D})$, we associate the set of all formulas $\psi \in \mathcal{C}l(\varphi)$ such that $M,[x,y] \Vdash \psi$ with $[x,y]$. We call such a set the φ-*type* of $[x,y]$ and we denote it by $\mathcal{T}ype_M([x,y])$. We have that every φ-type is a φ-atom, but not vice versa. Hereafter, φ-atoms (resp., φ-types) will be simply called atoms (resp., types). Given an atom F, we denote by $\mathcal{O}bs_r(F)$ (resp., $\mathcal{O}bs_l(F)$) the set of all *future* (resp., *past*) *observable formulae* of F, namely, the set of formulae $\psi \in F$ such that $\Diamond_r\psi \in \mathcal{C}l(\varphi)$ (resp., $\Diamond_l\psi \in \mathcal{C}l(\varphi)$). Similarly, given an atom F, we denote by $\mathcal{R}eq_r(F)$ (resp., $\mathcal{R}eq_l(F)$) the set of all $\Diamond_r$-*requests* (resp., $\Diamond_l$-*requests*) of F, namely, the set of formulae $\psi \in \mathcal{C}l(\varphi)$ such that $\Diamond_r\psi \in F$ (resp., $\Diamond_l\psi \in F$), and we use the shorthand $\mathcal{R}eq(F)$ for $\mathcal{R}eq_r(F) \cup \mathcal{R}eq_l(F)$. Making use of the above notions, we can define the following relation between two atoms F and G:

$$F \xrightarrow{R} G \quad \text{iff} \quad \mathcal{O}bs_r(G) \subseteq \mathcal{R}eq_r(F) \text{ and } \mathcal{O}bs_l(F) \subseteq \mathcal{R}eq_l(G)$$

The relation $\xrightarrow{R}$ satisfies a *view-to-type dependency*, that is, for every pair of intervals $[x,y],[x',y']$ in $\mathbb{I}(\mathbb{D})$, we have that $y = x'$ implies $\mathcal{T}ype_M([x,y]) \xrightarrow{R} \mathcal{T}ype_M([x',y'])$.

We provide now a natural interpretation of MPNL over grid-like structures (*compass structures*) by exploiting the existence of a natural bijection between the intervals $[x,y]$ and the points (x,y) of a $D \times D$ grid with $x \leqslant y$. Such an interpretation was originally proposed by Venema in [14], and it can be given for HS and all its fragments as well. As an example, Figure 1 shows four intervals $[x_0,y_0],...,[x_3,y_3]$ such that (i) $y_0 = x_1$, (ii) $x_0 = y_2$, (iii) the length of $[x_2,y_2]$ is less than k, and (iv) the length of $[x_3,y_3]$ is

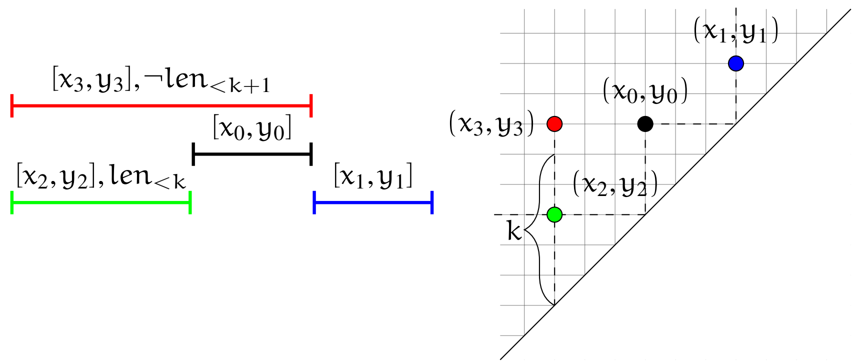

Figure 1: Correspondence between intervals and the points of the compass structure.

greater than k, together with the corresponding points $(x_0,y_0),...,(x_3,y_3)$ of the grid (notice that Allen's interval relations *meets* and *met by* are mapped into the corresponding spatial relations between pairs of points). Such an alternative interpretation of MPNL over compass structures will be exploited in the decidability proofs to make them easier to understand.

Definition 2. *Given an* MPNL *formula* φ*, a* compass φ-structure *is a pair* $\mathcal{G} = (\mathbb{P}_{\mathbb{D}}, \mathcal{L})$*, where* $\mathbb{P}_{\mathbb{D}}$ *is the set of points of the form* (x,y)*, with* $x,y \in D$ *and* $x \leqslant y$*, and* $\mathcal{L}$ *is a function that maps any point* $(x,y) \in \mathbb{P}_{\mathbb{D}}$ *to a* φ*-atom* $\mathcal{L}(x,y)$ *in such a way that:*

- *for every pair of points* $(x,y),(x',y') \in \mathbb{P}_{\mathbb{D}}$ *, if* $y = x'$ *then* $\mathcal{L}(x,y) \xrightarrow{R} \mathcal{L}(x',y')$ *(***temporal consistency***);*
- *for every point* $(x,y) \in \mathbb{P}_{\mathbb{D}}$*, and every* $\mathrm{len}_{<k} \in \mathcal{L}(x,y)$*,* $y - x < k$ *(***length consistency***).*

We say that a compass φ-structure $\mathcal{G} = (\mathbb{P}_{\mathbb{D}}, \mathcal{L})$ *features* a formula ψ if there exists a point $(x,y) \in \mathbb{P}_{\mathbb{D}}$ such that $\psi \in \mathcal{L}(x,y)$. Fulfilling compass structures are defined as follows.

Definition 3. *Given an* MPNL *formula* φ *and compass* φ*-structure* $\mathcal{G} = (\mathbb{P}_{\mathbb{D}}, \mathcal{L})$ *for it, we say that* $\mathcal{G}$ *is* ***fulfilling*** *if and only if for every point* $(x,y) \in \mathbb{P}_{\mathbb{D}}$ *and every formula* $\psi \in \mathcal{R}eq_r(\mathcal{L}(x,y))$ *(resp.,* $\psi \in \mathcal{R}eq_l(\mathcal{L}(x,y))$*), there exists a point* $(x',y') \in \mathbb{P}_{\mathbb{D}}$ *such that* $x' = y$ *(resp.,* $y' = x$*) and* $\psi \in \mathcal{L}(x',y')$*.*

The following proposition proves that the satisfiability problem for MPNL is reducible to the problem of deciding, for any given formula φ, whether there exists a φ-compass structure featuring φ. Its easy proof is left to the reader.

Proposition 1. *An* MPNL*-formula* φ *is satisfiable if and only if there exists a fulfilling* φ*-compass structure that features* φ*.*

Without loss of generality, we will assume φ to be satisfied by the initial point-interval 0 (resp., to belong to $\mathcal{L}(0,0)$) [13].

Given an MPNL-formula φ, we denote by k_φ the maximum k occurring in φ. If there is not any k in φ, we simply put $k_\varphi = 0$. We assume k_φ, as well as any length constraint occurring in φ, to be encoded in binary, and thus it immediately follows that $k_\varphi \leqslant 2^{|\varphi|}$.

Given a compass φ-structure $\mathcal{G} = (\mathbb{P}_{\mathbb{D}}, \mathcal{L})$, we define a *marking function* $\mathcal{M} : \mathbb{P}_{\mathbb{D}} \to \mathcal{A}_\varphi \times 2^{\mathcal{C}l(\varphi)} \times \{0,\ldots,k_\varphi\}$ such that, for every $(x,y) \in \mathbb{P}_{\mathbb{D}}$, $\mathcal{M}(x,y) = (F,\Psi,h)$, where (i) $F = \mathcal{L}(x,y)$, (ii) $\Psi = \{\psi \in \mathcal{C}l(\varphi) \mid \psi \in \mathcal{R}eq_r(x,x) \wedge \forall x \leqslant y' \leqslant y(\psi \notin \mathcal{L}(x,y'))\}$, and (iii) h is defined as follows:

$$h = \begin{cases} y - x & \text{if } y - x < k_\varphi; \\ k_\varphi & \text{otherwise.} \end{cases}$$

Notice that, for every point (x,y), Ψ is the set of formulae that must belong to the labeling of points (x,y'), with $y' > y$ (points "above" (x,y)), to guarantee the fulfilling of all $\Diamond_r$-requests in $\mathcal{L}(x,x)$, that is, for each $\psi \in \Psi$, there must exist at least one point (x,y') such that $\psi \in \mathcal{L}(x,y')$).

Let $\mathcal{A}_{\varphi}^{\mathcal{M}}$ be the image of $\mathcal{M}$. We call any triplet in $\mathcal{A}_{\varphi}^{\mathcal{M}}$ a *marked atom*. It can be easily shown that $|\mathcal{A}_{\varphi}^{\mathcal{M}}| \leqslant 2^{3|\varphi|}$ ($|A_{\varphi}| \leqslant 2^{|\varphi|}$, $|\mathcal{R}eq_r(\mathcal{L}(x,x))| \leqslant |\varphi|$, and $k_{\varphi} \leqslant 2^{|\varphi|}$).

Definition 4. *Given a* MPNL *formula* φ*, a compass* φ*-structure* $\mathcal{G} = (\mathbb{P}_{\mathbb{D}}, \mathcal{L})$ *for* φ*, and a point* $y \in D$*, we define the* horizontal configuration *of* y *in* $\mathcal{G}$ *as a counting function* $\mathcal{C}_y : \mathcal{A}_{\varphi}^{\mathcal{M}} \to \mathbb{N} \cup \{\omega\}$ *such that for every* $(F,\Psi,h) \in \mathcal{A}_{\varphi}^{\mathcal{M}}$*,* $\mathcal{C}_y(F,\Psi,h) = |\{x \mid \mathcal{M}(x,y) = (F,\Psi,h)\}|$.

It is worth noticing that, for any given y, (i) there exists a unique marked atom of the form $(F,\Psi,0)$, with $\mathcal{C}_y(F,\Psi,0) = 1$, and (ii) for every $0 < h < k_{\varphi}$, there exists at most 1 marked atom of the form (F,Ψ,h), and if for every marked atom (F,Ψ,h), $\mathcal{C}(F,\Psi,h) = 0$, then $\mathcal{C}(F',\Psi',h') = 0$ for every marked atom (F',Ψ',h') with $h' > h$. On the contrary, there is not a bound on the number of occurrences of a marked node of the form (F,Ψ,k_{φ}) (it can be equal to ω).

Finally, we define the following equivalence relation on the set of horizontal configurations, where p and f are defined as $p = |\{\Diamond_l\psi \in \mathcal{C}l(\varphi)\}|$ and $f = |\{\Diamond_r\psi \in \mathcal{C}l(\varphi)\}|$, respectively.

Definition 5. *Given an* MPNL *formula* φ *and a compass* φ*-structure* $\mathcal{G} = (\mathbb{P}_{\mathbb{D}}, \mathcal{L})$ *for it, we say that two horizontal configurations* $\mathcal{C}_y$ *and* $\mathcal{C}_{y'}$ *are* equivalent *(written* $\mathcal{C}_y \equiv \mathcal{C}_{y'}$*) if and only if for every* $(F,\Psi,h) \in \mathcal{A}_{\varphi}^{\mathcal{M}}$*, either* $\mathcal{C}_{y'}(A,\Psi,h) = \mathcal{C}_y(F,\Psi,h)$ *or* $h = k_{\varphi}$ *and both* $\mathcal{C}_y(F,\Psi,k_{\varphi}) \geqslant p \cdot f + p$ *and* $\mathcal{C}_{y'}(F,\Psi,k_{\varphi}) \geqslant p \cdot f + p$.

It can be easily shown that the relation of Definition 5 is an equivalence relation of finite index. For every marked atom $(F,\Psi,h) \in \mathcal{A}_{\varphi}^{\mathcal{M}}$, we do not distinguish between two configurations $\mathcal{C}_y$ and $\mathcal{C}_{y'}$ such that $\mathcal{C}_y(F,\Psi,h)$ and $\mathcal{C}_{y'}(F,\Psi,h)$ are different, but both greater than $p \cdot f + p$. Hence, the number of equivalence classes in $\equiv$ is bounded by

$$\Big(p \cdot f + p + 1\Big)^{|\mathcal{A}_{\varphi}^{\mathcal{M}}|} \leqslant \left(\frac{|\varphi|^2}{4} + \frac{|\varphi|}{2} + 1\right)^{2^{3|\varphi|}},$$

since $p \cdot f + p \leqslant \frac{|\varphi|^2}{4} + \frac{|\varphi|}{2}$ and $\left|\mathcal{A}_{\varphi}^{\mathcal{M}}\right| \leqslant 2^{3|\varphi|}$.

4 Decidability of MPNL over finite linear orders

In this section, we show that if there exists a finite fulfilling compass structure $\mathcal{G}$ for an MPNL formula φ, then there exists a finite fulfilling compass structure $\mathcal{G}'$ whose size is doubly exponential in the length of φ. To prove this result, we will make use of the following lemma, which states that we can always shrink the size of a fulfilling compass structure, provided that we can find two points with the same horizontal configuration.

Lemma 1. *Let* φ *be an* MPNL *formula and let* $\mathcal{G} = (\mathbb{P}_{\mathbb{D}}, \mathcal{L})$ *be a finite fulfilling* φ*-compass structure which features* φ*. If there exist two distinct points* $\overline{y} < \overline{y}'$ *in* D *such that* $\mathcal{C}_{\overline{y}} \equiv \mathcal{C}_{\overline{y}'}$*, then it is possible to build a finite fulfilling compass structure* $\mathcal{G}' = (\mathbb{P}_{\mathbb{D}'}, \mathcal{L}')$ *featuring* φ *with* $|D'| = |D| - (\overline{y}' - \overline{y})$.

Proof. Suppose that $\mathcal{G} = (\mathbb{P}_{\mathbb{D}}, \mathcal{L})$ is a finite fulfilling φ-compass structure which features φ and such that there exist two distinct points $\overline{y} < \overline{y}'$ in D with $\mathcal{C}_{\overline{y}} \equiv \mathcal{C}_{\overline{y}'}$. We build the required compass structure $\mathcal{G}' = (\mathbb{P}_{\mathbb{D}'}, \mathcal{L}')$, with $|D'| = |D| - (\overline{y}' - \overline{y})$, by executing the following procedure.

1. For every $(x,y) \in \mathbb{P}_{\mathbb{D}'}$, with $y \leqslant \overline{y}$, we put $\mathcal{L}'(x,y) = \mathcal{L}(x,y)$.
2. For every $(x,y) \in \mathbb{P}_{\mathbb{D}'}$, with $y > \overline{y}$ and $\overline{y} - k_\varphi < x \leqslant y$, we put $\mathcal{L}'(x,y) = \mathcal{L}(x + (\overline{y}' - \overline{y}), y + (\overline{y}' - \overline{y}))$.
3. For every $(A,\Psi,k_\varphi) \in \mathcal{A}^{\mathcal{M}}_\varphi$, we define a partial injective function $g : \{0,\ldots,\overline{y}-k_\varphi\} \to \{0,\ldots,\overline{y}'-k_\varphi\}$ as follows:
$$g(x) = \begin{cases} x' \text{ with } \mathcal{M}(x',\overline{y}') = (A,\Psi,k_\varphi) & \text{if } \mathcal{M}(x,\overline{y}) = (A,\Psi,k_\varphi) \text{ and} \\ & \mathcal{C}_{\overline{y}}(A,\Psi,k_\varphi) = \mathcal{C}_{\overline{y}'}(A,\Psi,k_\varphi) \\ \mathtt{undefined} & \text{otherwise} \end{cases}$$
By the injectivity of g, every x (where g is defined) is associated with a distinct x'. Moreover, since $\mathcal{C}_{\overline{y}}(A,\Psi,k_\varphi) = \mathcal{C}_{\overline{y}'}(A,\Psi,k_\varphi)$, for every x' such that $\mathcal{M}(x',\overline{y}') = (A,\Psi,k_\varphi)$, there exists (a unique) x such that $g(x) = x'$. Now, for every $0 \leqslant x \leqslant \overline{y} - k_\varphi$ such that $g(x)$ is defined, we put $\mathcal{L}'(x,\overline{y}+i) = \mathcal{L}(g(x),\overline{y}'+i)$ for every $1 \leqslant i \leqslant y_{max} - \overline{y}'$.
4. For every $(A,\Psi,k_\varphi) \in \mathcal{A}^{\mathcal{M}}_\varphi$ such that $\mathcal{C}_{\overline{y}'}(A,\Psi,k_\varphi) \geqslant p \cdot f + p$, we choose a "witness" $w_{(A,\Psi)}$ such that $\mathcal{M}(w_{(A,\Psi)},\overline{y}') = (A,\Psi,k_\varphi)$. Then, we identify a set $\mathcal{ES}^{\overline{y}'}_{(A,\Psi)} = \{y'_1,\ldots,y'_m\}$ (*essential elements*) which is a minimal set such that, for every $\psi \in \Psi$, there exists a point $y'_j \in \mathcal{ES}^{\overline{y}'}_{(A,\Psi)}$ with $\psi \in \mathcal{L}(w_{(A,\Psi)},y'_j)$. By the definition of marked atom (in particular, by definition of Ψ), we have that $y'_i > \overline{y}'$ for every $1 \leqslant i \leqslant m$. It is easy to see that $m \leqslant f$. We define the set $\mathcal{B}\mathtt{locked}^{\overline{y}}_{(A,\Psi)} = \{x'_1,\ldots,x'_{m'}\}$ of blocked elements as the minimal set for which for every $1 \leqslant i \leqslant m$ and for every $\psi \in \mathcal{R}eq_l(y'_i,y'_i)$ if there exists x' with $\psi \in \mathcal{L}(x,y'_i)$ and $\mathcal{M}(x',\overline{y}') = (A,\Psi,k_\varphi)$ then there exists $1 \leqslant j \leqslant m'$ with $\psi \in \mathcal{L}(x'_j,y'_i)$ and $\mathcal{M}(x'_j,\overline{y}') = (A,\Psi,k_\varphi)$.

   Since $m \leqslant f$, we have that $|\mathcal{B}\mathtt{locked}^{\overline{y}'}_{(A,\Psi)}| \leqslant p \cdot f$. We can choose a set $\mathcal{B}\mathtt{locked}^{\overline{y}}_{(A,\Psi)} = \{x_1,\ldots,x_{m'}\}$ such that, for every $1 \leqslant i \leqslant m'$, we have $\mathcal{M}(x_i,\overline{y}) = \mathcal{M}(w_{(A,\Psi)},\overline{y}')(= (A,\Psi,k_\varphi))$. Then, we put $\mathcal{L}'(x_i,\overline{y}+j) = \mathcal{L}(x'_i,\overline{y}'+j)$ for every $1 \leqslant i \leqslant m'$ and every $1 \leqslant j \leqslant y_{max} - \overline{y}'$. As a result, the labeling of all points (x_i,y) in $\mathcal{G}'$, with $1 \leqslant i \leqslant m'$, is defined and all $\Diamond_r$-requests of points (x_i,x_i) are fulfilled. Finally, we select p elements $\mathcal{W}\mathtt{it}_{(A,\Psi)} = \{x_{m'+1},\ldots,x_{m'+p}\}$ not belonging to $\mathcal{B}\mathtt{locked}^{\overline{y}}_{(A,\Psi)}$ and such that $\mathcal{M}(x_i,\overline{y}) = \mathcal{M}(w_{(A,\Psi)},\overline{y}')(= (A,\Psi,k_\varphi))$ (the existence of these points is guaranteed by the fact that $\mathcal{C}_{\overline{y}}(A,\Psi,k_\varphi) \geqslant p \cdot f + p$ and $|\mathcal{B}\mathtt{locked}^{\overline{y}}_{(A,\Psi)}| \leqslant p \cdot f$), and we put $\mathcal{L}'(x_{m'+i},\overline{y}) = \mathcal{L}(w_{(A,\Psi)},\overline{y}'+j)$ for every $1 \leqslant i \leqslant p$ and every $1 \leqslant j \leqslant y_{max} - \overline{y}'$.
5. Unfortunately, the previous steps do not guarantee that all $\Diamond_l$-requests are fulfilled in $\mathcal{G}'$. Consider a point $y > \overline{y}$ and a formula $\psi \in \mathcal{R}eq_l(y,y)$ which is not fulfilled in $\mathcal{G}'$. By construction, we have that $\mathcal{L}'(y,y) = \mathcal{L}(y + (\overline{y}' - \overline{y}), y + (\overline{y}' - \overline{y}))$ and thus, since $\mathcal{G}$ is fulfilling, there exists a point $(x'_\psi, y + (\overline{y}' - \overline{y}))$ such that $\psi \in \mathcal{L}(x'_\psi, y + (\overline{y}' - \overline{y}))$. Two cases may arise:

   a) for every $(A,\Psi,k_\varphi) \in \mathcal{A}^{\mathcal{M}}_\varphi$ we have that $y + (\overline{y}' - \overline{y}) \notin \mathcal{ES}^{\overline{y}'}_{(A,\Psi)}$. Let $\mathcal{M}(x'_\psi,\overline{y}') = (A,\Psi,k_\varphi)$ the marked atoms associated with the point $(x'_\psi,\overline{y}')$ in $\mathcal{G}$. We have that $\mathcal{C}_{\overline{y}'}(\mathcal{M}(x'_\psi,\overline{y}')) \geqslant p \cdot f + p$ (if this was not the case, x'_ψ would not belong to the range of g, thus violating the properties we impose to it at step 3). Then we have defined at step 4 the set $\mathcal{W}\mathtt{it}_{(A,\Psi)} = \{x_{m'+1},\ldots,x_{m'+p}\}$. Since $|\mathcal{R}eq_l(\mathcal{L}(y,y))| \leqslant p$ and the formula ψ has not yet been fulfilled in $\mathcal{G}'$ for (y,y) there exists $1 \leqslant j \leqslant p$ for which for every $\psi' \in \mathcal{R}eq_l(\mathcal{L}(y,y)) \cap \mathcal{L}(x_{m+j},y)$ there exists $0 \leqslant l \leqslant p$ with $l \neq j$ and $\psi' \in \mathcal{L}(x_{m+l},y)$. This means that (x_{m+j},y) is "useless" for the fulfilling of the $\Diamond_l$ requests of (y,y), moreover since $y+(\overline{y}'-\overline{y}) \notin \mathcal{ES}^{\overline{y}'}_{(A,\Psi)}$ we have that for every $\psi \in \Psi$ there exists $y' \neq y$ with $\psi \in \mathcal{L}(x_j,y')$ and thus (x_j,y) is "useless" for the fulfilling of the $\Diamond_r$-request of (x_j,x_j) in $\mathcal{G}'$. For this reasons $\mathcal{L}'(x_j,y)$ can be safely rewritten as $\mathcal{L}'(x_j,y) = \mathcal{L}(x'_\psi, y+(\overline{y}'-\overline{y}))$;

b) there exists $(\overline{A},\overline{\Psi},k_\varphi) \in \mathcal{A}^{\mathcal{M}}_\varphi$ with $y+(\overline{y}'-\overline{y}) \in \mathcal{ES}^{\overline{y}'}_{(A,\Psi)}$. Let $\mathcal{M}(x'_\psi,\overline{y}') = (A,\Psi,k_\varphi)$ the marked atoms associated with the point $(x'_\psi,\overline{y}')$ in $\mathcal{G}$. We have that $\mathcal{C}_{\overline{y}'}(\mathcal{M}(x'_\psi,\overline{y}')) \geqslant p\cdot f+p$ (if this was not the case, x'_ψ would not belong to the range of g, thus violating the properties we impose to it at step 3). Then we have defined at step 4 the set $\mathcal{W}\mathtt{it}_{(A,\Psi)} = \{x_{m'+1},\dots,x_{m'+p}\}$. Since $|\mathcal{R}eq_l(\mathcal{L}(y,y))| \leqslant p$ and the formula ψ has not yet been fulfilled in $\mathcal{G}'$ for (y,y) there exists $1 \leqslant j \leqslant p$ for which for every $\psi' \in \mathcal{R}eq_l(\mathcal{L}(y,y)) \cap \mathcal{L}(x_{m+j},y)$ there exists $0 \leqslant l \leqslant p$ with $l \neq j$ and $\psi' \in \mathcal{L}(x_{m+l},y)$. This means that (x_{m+j},y) is "useless" for the fulfilling of the $\Diamond_l$ requests of (y,y). In addition we can prove that $y+(\overline{y}'-\overline{y}) \notin \mathcal{ES}^{\overline{y}'}_{(A,\Psi)}$. Suppose by contradiction that $y+(\overline{y}'-\overline{y}) \in \mathcal{ES}^{\overline{y}'}_{(A,\Psi)}$ then by the procedure applied in step 4 there exists $x_{j'} \in \mathcal{B}\mathtt{locked}^{\overline{y}}_{(A,\Psi)}$ for which $\psi \in \mathcal{L}(x_{j'},y)$ and thus ψ is fulfilled for (y,y) (contradiction). Then we have that for every $\psi \in \Psi$ there exists $y' \neq y$ with $\psi \in \mathcal{L}(x_j,y')$ and thus (x_j,y) is "useless" for the fulfilling of the $\Diamond_r$-request of (x_j,x_j) in $\mathcal{G}'$. For this reasons $\mathcal{L}'(x_j,y)$ can be safely rewritten as $\mathcal{L}'(x_j,y) = \mathcal{L}(x'_\psi, y+(\overline{y}'-\overline{y}))$.

6. The previous step fulfills all $\Diamond_l$-requests in $\mathcal{G}'$. However, there may exist some point $(x,y) \in \mathbb{P}_{\mathbb{D}'}$ with $y > \overline{y}$ whose labeling is still undefined. Let (x,y) be one of such points. By the very same argument of step 5, we can assume that $\mathcal{C}_{\overline{y}}(\mathcal{M}(x,\overline{y})) \geqslant p\cdot f+p$. Now, let $\mathcal{M}(x,\overline{y}) = (A,\Psi,k_\varphi)$ and let $w_{(A,\Psi)}$ be the witness defined at step 4. We can safely complete the labeling of $\mathcal{G}'$ by putting $\mathcal{L}'(x,y) = \mathcal{L}(w_{(A,\Psi)}, y+(\overline{y}'-\overline{y})$.

At the end of the procedure, $\mathcal{G}'$ turns out to be a fulfilling compass structure for φ. □

By exploiting the above lemma, we can prove that a formula φ is satisfiable in a finite compass structure if and only if it is satisfiable by a finite compass structure where all horizontal configurations are pairwise non-equivalent.

Theorem 1. *Let φ be an* MPNL*-formula, if there exists a finite fulfilling φ-compass structure $\mathcal{G} = (\mathbb{P}_{\mathbb{D}},\mathcal{L})$ which features φ, then there exists a finite fulfilling φ-compass structure $\mathcal{G}' = (\mathbb{P}_{\mathbb{D}'},\mathcal{L}')$ which features φ such that* $|D'| \leqslant \left(\frac{|\varphi|^2}{4}+\frac{|\varphi|}{2}+1\right)^{2^{3|\varphi|}}$.

Proof. Let $\mathcal{G} = (\mathbb{P}_{\mathbb{D}},\mathcal{L})$ be a finite fulfilling compass structure featuring φ and suppose that $|D| > \left(\frac{|\varphi|^2}{4}+\frac{|\varphi|}{2}+1\right)^{2^{3|\varphi|}}$. Since the index of $\equiv$ is finite and smaller than $|D|$, there exist two distinct points $\overline{y} < \overline{y}'$ in D such that $\mathcal{C}_y \equiv \mathcal{C}'_y$. Then, we can exploit Lemma 1 to build a smaller compass structure $\mathcal{G}_1 = (\mathbb{P}_{\mathbb{D}_1},\mathcal{L}_1)$ such that $|D_1| = |D|-(\overline{y}'-\overline{y})$. By iterating the application of Lemma 1, we eventually obtain a compass structure $\mathcal{G}_n = (\mathbb{P}_{\mathbb{D}_n},\mathcal{L}_n)$ such that all horizontal configurations are pairwise non-equivalent. Since the number of equivalence classes in $\equiv$ is less than or equal to $\left(\frac{|\varphi|^2}{4}+\frac{|\varphi|}{2}+1\right)^{2^{3|\varphi|}}$, the thesis immediately follows. □

5 Decidability of MPNL over the naturals

In this section, we extend the results of the previous section in order to deal with satisfiability of MPNL over $\mathbb{N}$. First, we identify a subset of finite compass structures, called *compass generators*, which turn out to be crucial for decidability.

Definition 6. *Let φ be an MPNL formula. An* $\mathbb{N}$-compass generator *for φ is a finite φ-compass structure $\mathcal{G} = (\mathbb{P}_{\mathbb{D}}, \mathcal{L})$, which features φ, that satisfies the following conditions:*

1. *all $\Diamond_l$-requests of every point $(x, y) \in \mathbb{P}_{\mathbb{D}}$ are fulfilled;*
2. *there exists $y_{inf} < y_{max}$ such that:*
   (a) *for every $(F, \Psi, h) \in \mathcal{A}_{\varphi}^{\mathcal{M}}$, if $\mathcal{C}_{y_{max}}(F, \Psi, h) > 0$, then $\mathcal{C}_{y_{inf}}(F, \Psi, h) > 0$, and*
   (b) *$\mathcal{M}(x, y_{max}) = (F, \emptyset, h)$, for every $0 \leqslant x \leqslant y_{inf}$.*

Theorem 2. *An MPNL formula φ is satisfiable over $\mathbb{N}$ if and only if there exists an $\mathbb{N}$-compass generator which features φ.*

Proof. To prove the left-to-right direction, suppose φ to be satisfiable over $\mathbb{N}$, and let $\mathcal{G} = (\mathbb{P}_{\mathbb{N}}, \mathcal{L})$ be a fulfilling compass structure which features φ. Since the index of the equivalence relation $\equiv$ over the set of configurations is finite, there must exist an infinite sequence $\mathcal{S} = y_1 < y_2 < \ldots$ in $\mathbb{N}$ such that $\mathcal{C}_{y_i} \equiv \mathcal{C}_{y_j}$ for every $i, j \in \mathbb{N}$. Consider now the first element y_1 in the sequence $\mathcal{S}$, and let $(x, y_1) \in \mathbb{P}_{\mathbb{N}}$ be a point on the row y_1. Suppose $\mathcal{M}(x, y_1) = (F, \Psi, k_{\varphi})$. Since $\mathcal{G}$ is fulfilling, for every $\psi \in \Psi$, there exists $y_{\psi} > y_1$ such that $\psi \in \mathcal{L}(x, y_{\psi})$. Let $\overline{y}$ be the maximum of such y_{ψ} with respect to all $x \leqslant y_1$ and all $\psi \in \Psi$, with $\mathcal{M}(x, y_1) = (F, \Psi, k_{\varphi})$, and let y_j be the smallest element in $\mathcal{S}$ such that $\overline{y} < y_j$. By the definition of the marking function $\mathcal{M}$, we have that $\mathcal{M}(x, y_j) = (F, \emptyset, h)$, for every $0 \leqslant x \leqslant y_1$. Consider now the restriction $\mathcal{G}'$ of $\mathcal{G}$ to $D = \{0, 1, \ldots, y_j\}$. It is straightforward to check that, given $y_{max} = y_j$, y_1 satisfies the conditions for y_{inf} of Definition 6, and thus $\mathcal{G}'$ is an $\mathbb{N}$-compass generator featuring φ ($(0, 0)$ belongs to $\mathcal{G}'$).

To prove the right-to-left direction, suppose that $\mathcal{G} = (\mathbb{P}_{\mathbb{D}}, \mathcal{L})$ is an $\mathbb{N}$-compass generator for φ. We build a fulfilling compass structure $\mathcal{G}_{\omega} = (\mathbb{P}_{\mathbb{N}}, \mathcal{L}_{\omega})$ as the (infinite) union of an appropriate sequence of $\mathbb{N}$-compass generators $\mathcal{G}_0 \subset \mathcal{G}_1 \subset \ldots$. First, we take $\mathcal{G}_0 = \mathcal{G}$. Then, for every $i \geqslant 0$, we build $\mathcal{G}_{i+1} = (\mathbb{P}_{\mathbb{D}_{i+1}}, \mathcal{L}_{i+1})$ starting from $\mathcal{G}_i = (\mathbb{P}_{\mathbb{D}_i}, \mathcal{L}_i)$ as follows. Let $y_{inf} \in D_i$ satisfy the conditions of Definition 6, and let $gap = y_{max} - y_{inf}$. We put $D_{i+1} = \{0, 1, \ldots, y_{max}, \ldots, y_{max} + gap\}$ and we define $\mathcal{L}_{i+1}$ as follows:

1. for every $(x, y) \in \mathbb{P}_{\mathbb{D}_i}$, we put $\mathcal{L}_{i+1}(x, y) = \mathcal{L}_i(x, y)$;
2. for every $(x, y) \in \mathbb{P}_{\mathbb{D}_{i+1}}$ such that $x > y_{max} - k_{\varphi}$ and $y > y_{max}$, we put $\mathcal{L}_{i+1}(x, y) = \mathcal{L}_i(x - gap, y - gap)$;
3. for every $(x, y) \in \mathbb{P}_{\mathbb{D}_{i+1}}$ such that $y_{max} - k_{\varphi} \geqslant x > y_{inf} - k_{\varphi}$ and $y > y_{max}$, we put $\mathcal{L}_{i+1}(x, y) = \mathcal{L}_i(x', y - gap)$, for some x' such that $\mathcal{M}(x', y_{inf}) = \mathcal{M}(x, y_{max})$ (the existence of such an x' is guaranteed by property (a) of Definition 6);
4. for every $(x, y) \in \mathbb{P}_{\mathbb{D}_{i+1}}$ such that $y_{inf} - k_{\varphi} \geqslant x \geqslant 0$ and $y > y_{max}$, we put $\mathcal{L}_{i+1}(x, y) = \mathcal{L}_i(x, y - gap)$.

By construction, it holds that for every $(F, \Psi, h) \in \mathcal{A}_{\varphi}^{\mathcal{M}}$, if $\mathcal{C}_{y_{max}+gap}(F, \Psi, h) > 0$, then $\mathcal{C}_{y_{max}}(F, \Psi, h) > 0$, Moreover, $\mathcal{M}(x, y_{max} + gap) = (A, \emptyset, h)$, for every $0 \leqslant x \leqslant y_{max}$, and thus $\mathcal{G}_{i+1}$ is a $\mathbb{N}$-compass generator for φ.

The fulfilling compass structure satisfying φ on $\mathbb{N}$ we were looking for is $\mathcal{G}_{\omega} = \bigcup_{i \geqslant 0} \mathcal{G}_i$. $\square$

Theorem 3. *Let φ be an MPNL formula. If there exists an $\mathbb{N}$-compass generator $\mathcal{G} = (\mathbb{P}_{\mathbb{D}}, \mathcal{L})$ that features φ, then there exists an $\mathbb{N}$-compass generator $\mathcal{G}' = (\mathbb{P}_{\mathbb{D}'}, \mathcal{L}')$, that features φ, with $|D'| \leqslant \left(2^{3|\varphi|}+2\right) \cdot \left(\frac{|\varphi|^2}{4} + \frac{|\varphi|}{2} + 1\right)^{2^{3|\varphi|}} + 1$.*

Proof. Let $\mathcal{G} = (\mathbb{P}_{\mathbb{D}}, \mathcal{L})$ be an $\mathbb{N}$-compass generator which features φ, and let $y_{inf} \in D$ satisfy the conditions of Definition 6. We define a minimal set $S = \{\overline{y}_0, \ldots, \overline{y}_m\}$ of elements in D such that (i) $\overline{y}_0 = 0$, (ii) $\overline{y}_j < \overline{y}_{j+1}$, for each $0 \leqslant j < m$, (iii) $\overline{y}_{m-1} = y_{inf}$, (iv) $\overline{y}_m = y_{max}$, and (v) for every $(F, \Psi, h) \in \mathcal{A}_{\varphi}^{\mathcal{M}}$, if $\mathcal{C}_{y_{inf}}(F, \Psi, h) > 0$, then there exists $\overline{y}_j$ such that $\mathcal{M}(\overline{y}_j, y_{inf}) = (F, \Psi, h)$. From the minimality requirement, it follows that $m \leqslant 2^{3|\varphi|} + 3$.

We build a finite sequence of $\mathbb{N}$-compass generators $\mathcal{G}_0 \supset \mathcal{G}_1 \supset \ldots \supset \mathcal{G}_n$, whose last element is a small enough $\mathbb{N}$-compass generator $\mathcal{G}_n$, as follows. We start with $\mathcal{G}_0 = \mathcal{G}$. Now, let $\mathcal{G}_i = (\mathbb{P}_{\mathbb{D}_i}, \mathcal{L}_i)$ be the i-th compass generator in the sequence, and let $S_i = \{\overline{y}_0, \ldots, \overline{y}_m\}$ be the above-defined minimal set of elements in D_i. If there exist no y, y', with $\overline{y}_j \leqslant y < y' < \overline{y}_{j+1}$ for some $0 \leqslant j < m$, such that $\mathcal{C}_y \equiv \mathcal{C}_{y'}$, we terminate the construction and put $n = i$, that is, $\mathcal{G}_i$ is the last $\mathbb{N}$-compass generator in the sequence. Otherwise, we must distinguish two cases. If $y_{inf} \leqslant y, y' < y_{max}$, then the application of (the construction of) Lemma 1 to the pair of positions y and y' produces an $\mathbb{N}$-compass generator $\mathcal{G}_{i+1} = (\mathbb{P}_{\mathbb{D}_{i+1}}, \mathcal{L}_{i+1})$, with $|D_{i+1}| = |D_i| - (y' - y)$. It can be easily checked that the resulting structure satisfies the conditions of Definition 6 (notice that some triples may disappear from y_{max}, that is, $\mathcal{C}_{y_{max}}(F, \Psi, h)$ may become equal to 0 for some triple (F, Ψ, h)). If $\overline{y}_j \leqslant y, y' < \overline{y}_{j+1}$ for some $j \leqslant \overline{y}_{m-2}$, we can still apply (the construction of) Lemma 1 to the pair of positions y and y', but we must guarantee that all triples belonging to the row y_{inf} in D_i are preserved. It is possible to show this can be done (whenever necessary) by an appropriate choice of the witnesses at step 4 of (the construction of) Lemma 1. It is worth noticing that in both cases, while the positions between $\overline{y}_{j+1}$ and $\overline{y}_{m-2}$ (if any) remain unchanged, those between $\overline{y}_1$ and $\overline{y}_j$ may change from S_i to S_{i+1}.

At the end of the procedure, all the horizontal configurations in between two consecutive elements $\overline{y}_j, \overline{y}_{j+1} \in S$ are pairwise non-equivalent. From this, it immediately follows that the final $\mathbb{N}$-compass generator $\mathcal{G}_n = (\mathbb{P}_{\mathbb{D}_n}, \mathcal{L}_n)$ is such that $|D_n| \leqslant \left(2^{3|\varphi|}+2\right) \cdot \left(\frac{|\varphi|^2}{4} + \frac{|\varphi|}{2} + 1\right)^{2^{3|\varphi|}} + 1$. □

6 Decidability of MPNL over the integers

In this section, we extend the notion of *compass generator* in order to prove the decidability of the satisfiability problem for MPNL over $\mathbb{Z}$.

Definition 7. *Let φ be an MPNL formula. A $\mathbb{Z}$-*compass generator *for φ is a finite φ-compass structure $\mathcal{G} = (\mathbb{P}_{\mathbb{D}}, \mathcal{L})$ such that there exist $y_{fut}, y_{past} \in D$, with $y_{min} < y_{past} < 0 < y_{fut} < y_{max}$, which satisfies the following conditions:*

1. *all $\Diamond_l$-requests of every point $(y, y) \in \mathbb{P}_{\mathbb{D}}$, with $y_{past} \leqslant y \leqslant y_{max}$ are fulfilled;*
2. *for every $(F, \Psi, h) \in \mathcal{A}_{\varphi}^{\mathcal{M}}$, if $\mathcal{C}_{y_{max}}(F, \Psi, h) > 0$, then $\mathcal{C}_{y_{fut}}(F, \Psi, h) > 0$, and $\mathcal{M}(x, y_{max}) = (F, \emptyset, h)$, for every $y_{min} \leqslant x \leqslant y_{fut}$;*
3. *for every $(F, \Psi, h) \in \mathcal{A}_{\varphi}^{\mathcal{M}}$, if $\mathcal{C}_{y_{past}}(F, \Psi, h) > 0$, then there exists $y_{past} \leqslant x \leqslant 0$ with $\mathcal{M}(x, 0) = (F, \Psi, h)$.*

Theorem 4. *An MPNL formula φ is satisfiable over $\mathbb{Z}$ if and only if there exists a $\mathbb{Z}$-compass generator for it.*

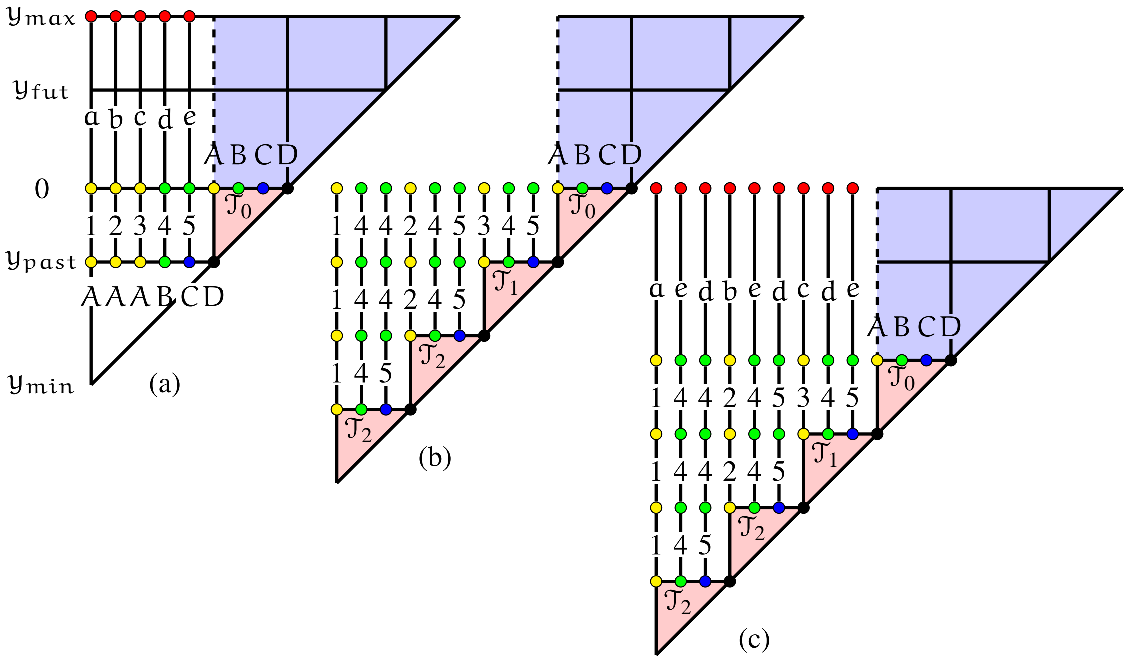

Figure 2: From a $\mathbb{Z}$-compass generator to a compass structure over $\mathbb{Z}$.

Proof. We start with the left-to-right direction. From the satisfiability of φ over $\mathbb{Z}$, it follows that there exists a fulfilling compass structure $\mathcal{G} = \langle \mathbb{P}_{\mathbb{Z}}, \mathcal{L} \rangle$ which features φ. It suffices to show that there exist five elements $y_{min} < y_{past} < 0 < y_{fut} < y_{max}$ that satisfy the conditions of Definition 7.

Since the index of the equivalence relation $\equiv$ over configurations is finite, there exists an infinite-to-the-past sequence of elements $\mathcal{S} = y_{-1} > y_{-2} > \ldots$ such that, for every $i, j \in \mathbb{N}$, $\mathcal{C}_{y_i} \equiv \mathcal{C}_{y_j}$. Without loss of generality, we can assume that $y_{-1} = 0$. Since $\mathcal{S}$ is infinite to the past, there exists $j < -1$ such that, for every $(F, \Psi, h) \in \mathcal{A}_{\varphi}^{\mathcal{M}}$, with $\mathcal{C}_{y_j}(F, \Psi, h) > 0$, there exists $y_j \leqslant x \leqslant y_{-1}$, with $\mathcal{M}(x, y_{-1}) = (F, \Psi, h)$. We put $y_{past} = y_j$. The elements y_{max} and y_{fut} can be selected using the very same argument of the proof of Theorem 2 guaranteeing that $0 < y_{fut} < y_{max}$. Next, we take an element $\overline{y} < y_{past}$ such that, for every $y_{past} \leqslant y \leqslant y_{max}$ and every $\psi \in \mathcal{R}eq_l(\mathcal{L}(y, y))$, there exists an element $\overline{y} \leqslant x \leqslant y$ with $\psi \in \mathcal{L}(x, y)$. We put $y_{min} = \overline{y}$. Finally, we define a compass structure $\mathcal{G}' = \langle \mathbb{P}_{\mathbb{D}}, \mathcal{L}' \rangle$ such that $D = \{y_{min}, \ldots, y_{max}\}$ and, for every $(x, y) \in \mathbb{P}_{\mathbb{D}}$, the condition $\mathcal{L}'(x, y) = \mathcal{L}(x, y)$ holds. $\mathcal{G}'$ is a $\mathbb{Z}$-compass generator for φ.

The right-to-left direction is much more involved with respect to the case of natural numbers. We give a sketch of the proof only, taking advantage of the pictorial representation given in Figure 6. Figure 6.a depicts a $\mathbb{Z}$-compass generator $\mathcal{G} = \langle \mathbb{P}_{\mathbb{D}}, \mathcal{L} \rangle$ for some MPNL formula φ. The vertical segments that are used to fill in the gaps that appear during the construction of the infinite suffix are suitably numbered; lowercase letters are used to identify the vertical segments that will be exploited to fill in the new vertical lines in between 0 and y_{max}; Finally, upper case letters identify the marked atoms.

We start with the definition of the labeling of the infinite prefix of $\mathbb{Z}$. First, we remove all portions of the compass structure consisting of the vertical lines starting at $x \leqslant y_{past}$ (Figure 6.b). As an effect, we have that the $\Diamond_l$ requests of the points (x, x), with $y_{past} \leqslant x \leqslant 0$, may not be fulfilled. To make the argument more concrete, suppose that they need all the vertical segments $1, \ldots, 5$ in order to fulfill all their $\Diamond_l$ requests. To fix such a problem, we must create a sufficient number of "rooms" for the correct copy of these five vertical segments. To do that, we copy the triangle $\mathcal{T}_0$ three times on the diagonal (Figure 6.b). Moreover, we fill in the emerging verticals by using the segments $1,\ldots,5$. Notice that, since the marked atom on the lower end of the segments $1,2$, and 3 is the same, we can copy each of them above the other. Then, we have three available rooms on the left side of $\mathcal{T}_0$ for the segments $1,2$, and 3, and we can copy them fixing the defects for the $\Diamond_l$ requests of the points (x,x), with $y_{past} \leqslant x \leqslant 0$. The $\Diamond_l$ requests of points (x,x), with $0 \leqslant x \leqslant y_{max}$, are satisfied by copying the verticals denoted by lower case letters above the appropriate verticals (Figure 6.c), possibly duplicating some of them. Now, we can use the very same procedure to fix the defects for the points (x,x), with $2 \cdot y_{past} \leqslant x < y_{past})$ (the points on the edge of $\mathcal{T}_1$). By repeating this procedure infinitely many times, we can correctly label all the points in the infinite prefix of $\mathbb{Z}$. Then, we apply the procedure of Theorem 2 to y_{fut} and y_{max} in order to guarantee that the $\Diamond_r$-requests of all points (x,x), with $x \in \mathbb{Z}$, are fulfilled. The resulting compass structure $\mathcal{G}' = \langle \mathbb{P}_{\mathbb{Z}}, \mathcal{L}' \rangle$ is a fulfilling compass structure which features φ. $\square$

Theorem 5. *Let φ be an MPNL formula. If there exists a $\mathbb{Z}$-compass generator $\mathcal{G} = (\mathbb{P}_{\mathbb{D}}, \mathcal{L})$ that features φ, then there exists a $\mathbb{Z}$-compass generator $\mathcal{G}' = (\mathbb{P}_{\mathbb{D}'}, \mathcal{L}')$, that features φ, with $|D'| \leqslant \left(2^{3|\varphi|+1} + 8\right) \cdot \left(\frac{|\varphi|^2}{4} + \frac{|\varphi|}{2} + 1\right)^{2^{3|\varphi|}} + 1$.*

Proof. Let $\mathcal{G} = (\mathbb{P}_{\mathbb{D}}, \mathcal{L})$ be a $\mathbb{Z}$-compass generator, that features φ, and let y_{fut} and $y_{past} \in D$ satisfy the conditions of Definition 7. We define a minimal set $S = \{\overline{y}_0, \ldots, \overline{y}_m\}$ of elements in D such that (i) $\overline{y}_0 = y_{min}$, (ii) $\overline{y}_j < \overline{y}_{j+1}$, for each $0 \leqslant j < m$, (iii) $\overline{y}_2 = y_{past}$, (iv) $\overline{y}_j = 0$, for some $j > 2$, and for every $(F, \Psi, h) \in \mathcal{A}_{\varphi}^{\mathcal{M}}$ with $\mathcal{C}_{y_{past}} \geqslant 0$, there exists $2 \leqslant l \leqslant j$ such that $\mathcal{M}(\overline{y}_l, 0) = (F, \Psi, h)$, (v) $\overline{y}_{m-1} = y_{fut}$, (vi) $\overline{y}_m = y_{max}$, and (vii) for every $(F, \Psi, h) \in \mathcal{A}_{\varphi}^{\mathcal{M}}$, if $\mathcal{C}_{y_{inf}}(F, \Psi, h) > 0$, then there exists $\overline{y}_j \leqslant \overline{y}_l \leqslant \overline{y}_{m-1}$ such that $\mathcal{M}(\overline{y}_l, y_{fut}) = (F, \Psi, h)$. From the minimality requirement, it follows that $m \leqslant 2^{3|\varphi|+1} + 9$.

We build a finite sequence of $\mathbb{Z}$-compass generators $\mathcal{G}_0 \supset \mathcal{G}_1 \supset \ldots \supset \mathcal{G}_n$, whose last element is a small enough $\mathbb{Z}$-compass generator $\mathcal{G}_n$. We start with $\mathcal{G}_0 = \mathcal{G}$. Now, let $\mathcal{G}_i = (\mathbb{P}_{\mathbb{D}_i}, \mathcal{L}_i)$ be the i-th compass generator in the sequence and let $S_i = \{\overline{y}_0, \ldots, \overline{y}_m\}$ be the above-defined minimal set of elements in D_i. If there exist no y, y', with $\overline{y}_j \leqslant y < y' < \overline{y}_{j+1}$ for some $0 \leqslant j < m$, such that $\mathcal{C}_y \equiv \mathcal{C}_{y'}$, we terminate the sequence and put $n = i$. Otherwise, as in Theorem 3, we apply (the construction of) Lemma 1 to y and y' to obtain the compass generator $\mathcal{G}_{i+1} = (\mathbb{P}_{\mathbb{D}_{i+1}}, \mathcal{L}_{i+1})$, with $|D_{i+1}| = |D_i| - (y' - y)$.

At the end of the procedure, all the horizontal configurations in between two consecutive elements $\overline{y}_j, \overline{y}_{j+1} \in S$ are pairwise non-equivalent. From this, it immediately follows that the final $\mathbb{Z}$-compass generator $\mathcal{G}_n = (\mathbb{P}_{\mathbb{D}_n}, \mathcal{L}_n)$ is such that $|D_n| \leqslant \left(2^{3|\varphi|+1} + 8\right) \cdot \left(\frac{|\varphi|^2}{4} + \frac{|\varphi|}{2} + 1\right)^{2^{3|\varphi|}} + 1$. $\square$

7 Decision procedure

In this section, we give a decision procedure that solves the satisfiability problem for MPNL interpreted over the integers. Both the procedure for the finite case and that for the natural numbers can be easily tailored from it. Given an MPNL formula, it is indeed possible to encode a finite model into $\mathbb{Z}$ by means of the following formula:

$$\psi_{fin} = \#_{all} \wedge \# \wedge \Box_r \neg \#_{all} \Box_r \Box_r \neg \#_{all} \wedge \Box_l \neg \#_{all} \Box_l \Box_l \neg \#_{all} \wedge$$
$$[G](\# \leftrightarrow \#_{all} \vee \Diamond_r \Diamond_l \#_{all} \vee \Diamond_l \Diamond_r \#_{all} \vee (\Diamond_r \Diamond_r \Diamond_l \#_{all} \wedge \Diamond_l \Diamond_l \Diamond_r \#_{all}))$$

Under the assumption that $\#_{all}$ and $\#$ do not appear in φ, this formula can be translated inductively as follows: (i) if $\varphi = p$ or $\varphi = len_{<k}$, then $tr(\varphi) = \varphi \wedge \#$, (ii) if $\varphi = \neg\psi$, then $tr(\varphi) = \neg\# \vee \neg tr(\psi)$, (iii) if $\varphi = \psi_1 \vee \psi_2$, then $tr(\varphi) = (\psi_1 \wedge \#) \vee (\psi_2 \wedge \#)$, (iv) if $\varphi = \Diamond_r \psi$, then $tr(\varphi) = \Diamond_r(\# \wedge \psi)$, (v) if $\varphi = \Diamond_l \psi$, then $tr(\varphi) = \Diamond_l(\# \wedge \psi)$. It is easy to prove that φ has a finite model if and only if $\psi_{fin} \wedge tr(\varphi)$ has a model on $\mathbb{Z}$. Moreover, φ has a model in the linear order of natural numbers if and only if $\psi_{nat} \wedge tr(\varphi)$ has a model on $\mathbb{Z}$, where ψ_{nat} is defined as follows:

$$\# \wedge \Box_l \neg\# \wedge \Box_l \Box_l \neg\# \wedge [G]((\neg\# \wedge \Diamond_r \#) \rightarrow (\Box_r \# \wedge \Box_r \Box_r \#))$$

In Figure 7, the detailed code of a procedure for checking whether an MPNL formula φ is satisfiable is given. The procedure builds a candidate model for φ starting from y_{min} and exploring two consecutive horizontal configurations at every step. Every configuration is represented using an exponential number of counters, bounded by the maximum size of a $\mathbb{Z}$-compass structure given in Theorem 5 (doubly exponential in the size of $|\varphi|$). However, assuming that the values of all counters are encoded in binary, the maximum value for each counter takes an exponential storage space. The very same argument can be used to give an exponential space bound for the $steps$ counter. Moreover, the procedure needs to keep track of a constant number of horizontal configurations only ($\mathcal{C}^{min}, \mathcal{C}^{past}, \mathcal{C}^0, \mathcal{C}^{fut}, \mathcal{C}^{max}, \overline{\mathcal{C}}, \mathcal{C}, \mathcal{C}'$, $\mathcal{C}^{right}$, and $\mathcal{C}^{left}$). Pairing this result with the EXPSPACE-hardness proved in [4], we can state the following theorem.

Theorem 6. *The satisfiability problem for MPNL interpreted over (any subsets of) the integers is EXPSPACE-complete.*

References

[1] J.F. Allen (1983): *Maintaining Knowledge About Temporal Intervals*. Communications of the Association for Computing Machinery 26(11), pp. 832–843.

[2] D. Bresolin, D. Della Monica, V. Goranko, A. Montanari & G. Sciavicco (2011): *Metric Propositional Neighborhood Logics on Natural Numbers*. Journal of Software and System Modeling (in press) doi:10.1007/s10270-011-0195-y.

[3] D. Bresolin, V. Goranko, A. Montanari & G. Sciavicco (2009): *Propositional Interval Neighborhood Logics: Expressiveness, Decidability, and Undecidable Extensions*. Annals of Pure and Applied Logic 161(3), pp. 289–304.

[4] D. Bresolin, V. Goranko, A. Montanari & G. Sciavicco (2009): *Right Propositional Neighborhood Logic over Natural Numbers with Integer Constraints for Interval Lengths*. In: Proc. of the 7th IEEE International Conference on Software Engineering and Formal Methods (SEFM), IEEE Comp. Society Press, pp. 240–249.

[5] D. Bresolin, A. Montanari & P. Sala (2007): *An optimal tableau-based decision algorithm for Propositional Neighborhood Logic*. In: Proc. of the 24th Annual Symposium on Theoretical Aspects of Computer Science (STACS), LNCS 4393, Springer, pp. 549–560.

[6] D. Bresolin, A. Montanari, P. Sala & G. Sciavicco (2008): *Optimal Tableaux for Right Propositional Neighborhood Logic over Linear Orders*. In: Proc. of the 11th European Conference on Logics in AI (JELIA), LNAI 5293, Springer, pp. 62–75.

[7] D. Bresolin, A. Montanari, P. Sala & G. Sciavicco (2010): *Tableau-based decision procedures for Propositional Neighborhood Logic*. Technical Report 01, Dipartimento di Matematica e Informatica, Università di Udine, Italy.

[8] D. Bresolin, A. Montanari & G. Sciavicco (2007): *An optimal decision procedure for Right Propositional Neighborhood Logic*. Journal of Automated Reasoning 38(1-3), pp. 173–199.

proc GuessConfiguration ( )
for all $(F,\Psi,h) \in \mathcal{A}^{\mathcal{M}}_{\varphi}$, $\mathcal{C}(F,\Psi,h) \leftarrow 0$;
let $S_r \subseteq \{\psi \in \mathcal{C}l(\varphi \mid \Diamond_r\psi \in \mathcal{C}l(\varphi))\}$;
let $S_l \subseteq \{\psi \in \mathcal{C}l(\varphi \mid \Diamond_l\psi \in \mathcal{C}l(\varphi))\}$;
for all $1 \leqslant i < k_\varphi$
  let F **an atom s.t.** $\mathcal{R}eq_r(F) = S_r$ **and** $\mathtt{Len}(F) = i$;
  let $\Psi \subseteq \{\psi \in \mathcal{C}l(\varphi) \mid \Diamond_r\psi \in \mathcal{C}l(\varphi))\}$;
  $\overline{\mathcal{C}}(F,\Psi,i) \leftarrow 1$;
for all $(F,\Psi,k_\varphi) \in \mathcal{A}^{\mathcal{M}}_{\varphi}$ **s.t.** $\mathcal{R}eq_r(F) = S_r$
  let $0 \leqslant i \leqslant k_\varphi$, $\overline{\mathcal{C}}(F,\Psi,h) \leftarrow i$
return $\overline{\mathcal{C}}$;

proc Merge $(\mathcal{C},\mathcal{C}')$
for all $(F,\Psi,h) \in \mathcal{A}^{\mathcal{M}}_{\varphi}$
  $\overline{\mathcal{C}}(F,\Psi,h) \leftarrow \mathcal{C}(F,\Psi,h) + \mathcal{C}'(F,\Psi,h)$;
return $\overline{\mathcal{C}}$;

proc Len (F)
if $\exists 1 \leqslant h < k_\varphi$ **s.t.** $\neg \mathtt{len}_{<h} \in F \wedge \mathtt{len}_{<h+1} \in F$
  then return h
  else return k_φ

proc NC_MinToPast $(\mathcal{C}^{\mathtt{current}})$
let $S_r \subseteq \{\psi \in \mathcal{C}l(\varphi \mid \Diamond_r\psi \in \mathcal{C}l(\varphi))\}$;
let $S_l \subseteq \{\psi \in \mathcal{C}l(\varphi \mid \Diamond_l\psi \in \mathcal{C}l(\varphi))\}$;
let F_π **an atom with** $\mathtt{len}_{<1} \in F_\pi$, $\mathcal{R}eq_r(F_\pi) = S_r$, **and** $\mathcal{R}eq_l(F_\pi) = S_l$;
for all $(F,\Psi,h) \in \mathcal{A}^{\mathcal{M}}_{\varphi} \mathcal{C}(F,\Psi,h) \leftarrow 0$;
$\mathcal{C}(F_\pi, \mathcal{R}eq_r(F) \setminus F_\pi, 1) \leftarrow 1$;
for all $(F,\Psi,h) \in \mathcal{A}^{\mathcal{M}}_{\varphi}$
  for $(1 \leqslant i \leqslant \mathcal{C}^{\mathtt{current}}(G,\Psi,h))$
    if $h = k_\varphi$
      then $k \leftarrow k_\varphi$
      else $k \leftarrow h+1$
    let G **s.t.** $\mathtt{Len}(G) = k$, $\mathcal{R}eq_r(G) = S_r$, **and** $\mathcal{R}eq_l(G) = \mathcal{R}eq_l(F)$;
    $\mathcal{C}(G,\Psi\setminus G,k) \leftarrow \mathcal{C}'(G,\Psi\setminus G,k)+1$;
return $\mathcal{C}$;

proc NC_LeftRight $(\mathcal{C}^{\mathtt{left}},\mathcal{C}^{\mathtt{right}})$
let $S_r \subseteq \{\psi \in \mathcal{C}l(\varphi \mid \Diamond_r\psi \in \mathcal{C}l(\varphi))\}$;
let $S_l \subseteq \{\psi \in \mathcal{C}l(\varphi \mid \Diamond_l\psi \in \mathcal{C}l(\varphi))\}$;
let F_π **an atom with** $\mathtt{len}_{<1} \in F_\pi$, $\mathcal{R}eq_r(F_\pi) = S_r$, **and** $\mathcal{R}eq_l(F_\pi) = S_l$;
for all $(F,\Psi,h) \in \mathcal{A}^{\mathcal{M}}_{\varphi} \overline{\mathcal{C}}^{\mathtt{right}}(F,\Psi,h) \leftarrow 0$;
for all $(F,\Psi,h) \in \mathcal{A}^{\mathcal{M}}_{\varphi} \overline{\mathcal{C}}^{\mathtt{left}}(F,\Psi,h) \leftarrow 0$;
$\overline{\mathcal{C}}^{\mathtt{right}}(F_\pi, \mathcal{R}eq_r(F) \setminus F_\pi, 1) \leftarrow 1$;
for all $(F,\Psi,h) \in \mathcal{A}^{\mathcal{M}}_{\varphi}$
  for $(1 \leqslant i \leqslant \mathcal{C}^{\mathtt{right}}(G,\Psi,h))$
    if $h = k_\varphi$
      then $k \leftarrow k_\varphi$
      else $k \leftarrow h+1$
    let G **s.t.** $\mathtt{Len}(G) = k$, $\mathcal{R}eq_r(G) = S_r$, **and** $\mathcal{R}eq_l(G) = \mathcal{R}eq_l(F)$;
    $\overline{\mathcal{C}}^{\mathtt{right}}(G,\Psi\setminus G,k) \leftarrow \overline{\mathcal{C}}^{\mathtt{right}}(G,\Psi\setminus G,k)+1$;
  for $(1 \leqslant i \leqslant \mathcal{C}^{\mathtt{left}}(G,\Psi,h))$
    if $h = k_\varphi$
      then $k \leftarrow k_\varphi$
      else $k \leftarrow h+1$
    let G **s.t.** $\mathtt{Len}(G) = k$, $\mathcal{R}eq_r(G) = S_r$, **and** $\mathcal{R}eq_l(G) = \mathcal{R}eq_l(F)$;
    $\overline{\mathcal{C}}^{\mathtt{left}}(G,\Psi\setminus G,k) \leftarrow \overline{\mathcal{C}}^{\mathtt{left}}(G,\Psi\setminus G,k)+1$;
if $\left(\exists\psi \in S_l$ **s. t.** $\forall (F,\Psi,h) \in \mathcal{A}^{\mathcal{M}}_{\varphi}$ **with** $\psi \in A$ **we have** $\overline{\mathcal{C}}^{\mathtt{left}}(F,\Psi,h) = \overline{\mathcal{C}}^{\mathtt{right}}(F,\Psi,h) = 0 \right)$
  then return false ;
return $(\overline{\mathcal{C}}^{\mathtt{left}},\overline{\mathcal{C}}^{\mathtt{right}})$;

proc NC_ZeroToFut $(\mathcal{C}^{\mathtt{current}})$
let $S_r \subseteq \{\psi \in \mathcal{C}l(\varphi \mid \Diamond_r\psi \in \mathcal{C}l(\varphi))\}$;
let $S_l \subseteq \{\psi \in \mathcal{C}l(\varphi \mid \Diamond_l\psi \in \mathcal{C}l(\varphi))\}$;
let F_π **an atom with** $\mathtt{len}_{<1} \in F_\pi$, $\mathcal{R}eq_r(F_\pi) = S_r$, **and** $\mathcal{R}eq_l(F_\pi) = S_l$;
for all $(F,\Psi,h) \in \mathcal{A}^{\mathcal{M}}_{\varphi} \mathcal{C}(F,\Psi,h) \leftarrow 0$;
$\mathcal{C}(F_\pi, \mathcal{R}eq_r(F) \setminus F_\pi, 1) \leftarrow 1$;
for all $(F,\Psi,h) \in \mathcal{A}^{\mathcal{M}}_{\varphi}$
  for $(1 \leqslant i \leqslant \mathcal{C}^{\mathtt{current}}(G,\Psi,h))$
    if $h = k_\varphi$
      then $k \leftarrow k_\varphi$
      else $k \leftarrow h+1$
    let G **s.t.** $\mathtt{Len}(G) = k$, $\mathcal{R}eq_r(G) = S_r$, **and** $\mathcal{R}eq_l(G) = \mathcal{R}eq_l(F)$;
    $\mathcal{C}(G,\Psi\setminus G,k) \leftarrow \mathcal{C}(G,\Psi\setminus G,k)+1$;
if $\left(\exists\psi \in S_l$ **s. t.** $\forall (F,\Psi,h) \in \mathcal{A}^{\mathcal{M}}_{\varphi}$ **with** $\psi \in A$ **we have** $\mathcal{C}(F,\Psi,h) = 0 \right)$
  then return false ;
return $\mathcal{C}$;

proc MPNL-INTEGER-SAT (φ)
let $S_r \subseteq \{\psi \in \mathcal{C}l(\varphi \mid \Diamond_r\psi \in \mathcal{C}l(\varphi))\}$;
let $S_l \subseteq \{\psi \in \mathcal{C}l(\varphi \mid \Diamond_l\psi \in \mathcal{C}l(\varphi))\}$;
let F_π **an atom with** $\mathtt{len}_{<1} \in F_\pi$, $\mathcal{R}eq_r(F_\pi) = S_r$, **and** $\mathcal{R}eq_l(F_\pi) = S_l$;
for all $(F,\Psi,h) \in \mathcal{A}^{\mathcal{M}}_{\varphi}$, $\mathcal{C}^{\mathtt{min}}(F,\Psi,h) \leftarrow 0$;
$\mathcal{C}^{\mathtt{min}}(F_\pi, \mathcal{R}eq_r(F) \setminus F_\pi, 1) \leftarrow 1$;
$\mathcal{C}^{\mathtt{past}} \leftarrow \mathtt{GuessConfiguration}()$;
$\mathcal{C} \leftarrow \mathcal{C}^{\mathtt{min}}$;
$\mathtt{steps} \leftarrow 0$;
while $(\mathcal{C} \not\equiv \mathcal{C}^{\mathtt{past}})$
  if $\mathtt{steps} > \mathtt{BOUND\_MIN\_PAST}$
    then return false
  $\mathcal{C} \leftarrow \mathtt{NC\_MinToPast}(\mathcal{C})$;
  $\mathtt{steps} \leftarrow \mathtt{steps}+1$;
$\mathcal{C}^0 \leftarrow \mathcal{C}$;
$\mathcal{C}^{\mathtt{left}} \leftarrow \mathcal{C}$;
for all $(F,\Psi,h) \in \mathcal{A}^{\mathcal{M}}_{\varphi}$, $\mathcal{C}^{\mathtt{right}}(F,\Psi,h) \leftarrow 0$;
$\mathtt{steps} \leftarrow 0$;
while $(\mathcal{C}^{\mathtt{right}} \not\equiv \mathcal{C}^0)$
  if $\mathtt{steps} > \mathtt{BOUND\_PAST\_ZERO}$
    then return false
  $(\mathcal{C}^{\mathtt{left}},\mathcal{C}^{\mathtt{right}}) \leftarrow \mathtt{NC\_LeftRight}(\mathcal{C}^{\mathtt{left}},\mathcal{C}^{\mathtt{right}})$;
  $\mathtt{steps} \leftarrow \mathtt{steps}+1$;
$\mathcal{C}^{\mathtt{fut}} \leftarrow \mathtt{GuessConfiguration}()$;
$\mathcal{C} \leftarrow \mathtt{Merge}(\mathcal{C}^{\mathtt{left}},\mathcal{C}^{\mathtt{right}})$;
$\mathtt{steps} \leftarrow 0$;
while $(\mathcal{C} \not\equiv \mathcal{C}^{\mathtt{fut}})$
  if $\mathtt{steps} > \mathtt{BOUND\_ZERO\_FUT}$
    then return false
  $\mathcal{C} \leftarrow \mathtt{NC\_ZeroToFut}(\mathcal{C})$;
  $\mathtt{steps} \leftarrow \mathtt{steps}+1$;
$\mathcal{C}^{\mathtt{max}} \leftarrow \mathcal{C}$;
$\mathcal{C}^{\mathtt{left}} \leftarrow \mathcal{C}$;
for all $(F,\Psi,h) \in \mathcal{A}^{\mathcal{M}}_{\varphi}$, $\mathcal{C}^{\mathtt{right}}(F,\Psi,h) \leftarrow 0$;
$\mathtt{steps} \leftarrow 0$;
while $\left(\mathtt{Merge}(\mathcal{C}^{\mathtt{left}},\mathcal{C}^{\mathtt{right}}) \not\equiv \mathcal{C}^0 \vee \exists (F,\Psi,h) \in \mathcal{A}^{\mathcal{M}}_{\varphi}\right.$ **with** $\left.\mathcal{C}^{\mathtt{left}}(F,\Psi,h) > 0 \wedge \Psi \neq \emptyset \right)$
  if $\mathtt{steps} > \mathtt{BOUND\_FUT\_MAX}$
    then return false
  $(\mathcal{C}^{\mathtt{left}},\mathcal{C}^{\mathtt{right}}) \leftarrow \mathtt{NC\_LeftRight}(\mathcal{C}^{\mathtt{left}},\mathcal{C}^{\mathtt{right}})$;
  $\mathtt{steps} \leftarrow \mathtt{steps}+1$;
return true ;

Figure 3: The procedure for checking the satisfiability of ϕ over the integers.

[9] D. Della Monica (2011): *Expressiveness, Decidability, and Undecidability of Interval Temporal Logic*. Ph.D. thesis, Department of Mathematics and Computer Science, University of Udine, Udine, Italy.

[10] V. Goranko, A. Montanari & G. Sciavicco (2003): *Propositional interval neighborhood temporal logics*. Journal of Universal Computer Science 9(9), pp. 1137–1167.

[11] V. Goranko, A. Montanari & G. Sciavicco (2004): *A road map of interval temporal logics and duration calculi*. Journal of Applied Non-Classical Logics 14(1–2), pp. 9–54.

[12] M. Otto (2001): *Two Variable First-Order Logic over Ordered Domains*. Journal of Symbolic Logic 66(2), pp. 685–702.

[13] P. Sala (2010): *Decidability of Interval Temporal Logics*. Ph.D. thesis, Department of Mathematics and Computer Science, University of Udine, Udine, Italy.

[14] Y. Venema (1990): *Expressiveness and Completeness of an Interval Tense Logic*. Notre Dame Journal of Formal Logic 31(4), pp. 529–547.

[15] Y. Venema (1991): *A Modal Logic for Chopping Intervals*. Journal of Logic and Computation 1(4), pp. 453–476.